Plasma Response to Symmetry Breaking Perturbations
in the Reversed Field Geometry

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Field reversal does not insure closure of the reversed field geometry. The closure is critically dependent on the shape of the toroidal field $\vec{B}_1$. The plasma diamagnetic currents are shown to establish a spacial scale for the field $\vec{B}_1$ which is $\lambda = a/\sqrt{\beta_1}$ with a the plasma radius and β_1 the plasma beta relative to the $\vec{B}_1$ field.

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28

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I. INTRODUCTION

The reversed field geometry is being actively investigated as a plasma containment device as part of the mirror [1] and θ pinch [2-4] programs and also in the Astron [5] configuration. The magnetic field in the geometry is generally taken to be approximately $\vec{B}_0 = \vec{\nabla}\phi \times \vec{\nabla}\psi$ with ϕ the toroidal angle and ψ the poloidal flux. One might think that if the constant ψ surfaces were closed the magnetic geometry would be closed, for $\vec{B}_0 \cdot \vec{\nabla}\psi = 0$. However, it has been shown [6] that if one writes the exact magnetic field as $\vec{B} = \vec{B}_0 + \vec{B}_1$, then the field $\vec{B}_1$ determines the closure of the geometry, no matter how small $\vec{B}_1$ may be (provided the ψ surfaces are closed). The fact that the topology of a magnetic geometry can be radically changed by an infinitesimal magnetic field presents a difficult problem in interpretation when the plasma is high beta compared to the perturbation, $\beta_1 = \bar{P}/(B_1/8\pi) \gg 1$. This paper will examine that problem.

II. GENERAL DISCUSSION

To understand the implication of an infinitesimal field $\vec{B}_1$ changing the topology, we will consider the usual plasma equilibrium equations $\vec{\nabla}P = (1/c)\vec{j} \times \vec{B}$ and $\vec{\nabla} \cdot \vec{j} = 0$. The first of these equations implies

$$\vec{\nabla} \cdot \vec{j}_1 = \frac{c}{B} \vec{\nabla}P \cdot [\hat{b} \times (\hat{b} \cdot \vec{\nabla}\hat{b})] \quad (1)$$

with $B = |\vec{B}|$, $\hat{b} = \vec{B}/B$, and $\vec{j}_\perp$ the current perpendicular to the magnetic field. The quantity $\hat{b} \cdot \vec{\nabla} \hat{b}$, which can be written $d\hat{b}/d\ell$, is the curvature vector for the field lines. Hence, $\hat{b} \times (\hat{b} \cdot \vec{\nabla} \hat{b})$ points in the so-called binormal direction.

Let us consider the zeroth-order current. That is zeroth-order quantities are used on the right side of Eq. (1). The zeroth-order magnetic field is, of course, $\vec{B}_0$. However, the zeroth-order pressure involves a subtlety. The equation $\vec{B} \cdot \vec{\nabla} P = 0$ implies to first order in the field $\vec{B}_1$

$$\vec{B}_0 \cdot \vec{\nabla} P_0 = 0 \quad (2)$$

$$\vec{B}_0 \cdot \vec{\nabla} P_1 + \vec{B}_1 \cdot \vec{\nabla} P_0 = 0 \quad . \quad (3)$$

To solve these equations, we will use flux coordinates ψ, θ, ϕ with θ a poloidal angle. In these coordinates

$$\vec{B}_0 \cdot \vec{\nabla} = \vec{B}_0 \cdot \vec{\nabla}_\theta \frac{\partial}{\partial \theta}$$

for $\vec{B}_0 = \vec{\nabla} \phi \times \vec{\nabla} \psi$ is orthogonal to the $\vec{\nabla} \psi$ and $\vec{\nabla} \phi$ directions. Equation (2) just implies that P_0 is a function of ψ and ϕ only. Equation (3) can be written

$$\frac{\partial P_1}{\partial \theta} = - \frac{1}{\vec{B}_0 \cdot \vec{\nabla}_\theta} \vec{B}_1 \cdot \vec{\nabla} P_0(\psi, \phi) \quad . \quad (4)$$

Integrating over θ , the left hand side of Eq. (4) will vanish because of the periodicity of θ . One is left with a differential

equation that P_0 must satisfy which involves the geometric structure of the $\vec{B}_1$ field but not its magnitude. For more discussion of Eq. (4), see Ref. 6.

With the zeroth-order quantities defined, a more intuitive form for $\hat{b}_0 \times (\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0)$ is needed so the implications of Eq. (1) can be investigated. Now $\hat{b}_0 = \alpha \vec{\nabla} \phi \times \vec{\nabla} \psi$ while $\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0 = \beta \vec{\nabla} \phi \times \vec{\nabla} \theta$ with α and β functions of position. The form for $\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0$ can be derived from the fact that $\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0$ is orthogonal to $\hat{b}_0$, and the toroidal angle is chosen so $\hat{\phi} = \vec{\nabla} \phi / |\vec{\nabla} \phi|$ satisfies $\partial \hat{\phi} / \partial \theta = 0$. (The ordinary definition of ϕ as the angle in a cylindrical coordinate system satisfies both $\partial \hat{\phi} / \partial \theta = 0$ and $\partial \hat{\phi} / \partial \psi = 0$, but $\partial \hat{\phi} / \partial \phi \neq 0$.) From the form of $\hat{b}_0$ and $\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0$ we have $\hat{b}_0 \times (\hat{b}_0 \cdot \vec{\nabla} \hat{b}_0) = \hat{\phi} / a$ with $a(\psi, \theta, \phi)$ the local minor radius of the plasma. Consequently, in the reversed field geometry with $\vec{B}_0 = \vec{\nabla} \phi \times \vec{\nabla} \psi$,

$$\vec{\nabla} \cdot \vec{j}_{\perp 0} = \frac{c}{a B_0} \hat{\phi} \cdot \vec{\nabla} P_0 \quad (5)$$

If the toroidal symmetry of P_0 is broken either by $\vec{B}_1$ or by ψ not being toroidally symmetric, then the perpendicular current is not divergence free and a parallel current is driven with $\vec{\nabla} \cdot \vec{j}_{\parallel 0} = -\vec{\nabla} \cdot \vec{j}_{\perp 0}$. Using $\vec{j}_{\parallel 0} = j_{\parallel 0} \vec{B} / B$ and defining the average of a function f over the zeroth-order field lines to be

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \frac{d\theta}{\hat{B}_0 \cdot \vec{\nabla} \theta} \quad (6)$$

we have

$$\langle \vec{B}_1 \cdot \vec{\nabla} \left(\frac{j_{||0}}{B_0} \right) \rangle = - \langle \vec{\nabla} \cdot \vec{j}_{||0} \rangle \quad (7)$$

Unless $\langle \vec{\nabla} \cdot \vec{j}_{||0} \rangle$ vanishes identically, which in general it will not do unless P_0 is toroidally symmetric, Eq. (7) gives the peculiar result $j_{||0} \propto 1/B_1$. This means the weaker the perturbation the stronger the plasma's response to it. The large parallel current, of course, changes $\vec{B}_1$. Indeed, one has $\vec{\nabla} \times \vec{B}_1 = 4\pi \vec{j}_{||0}/c$ or using Eqs. (5) and (7), one finds

$$\vec{\nabla} \times \vec{B}_1 \approx \frac{4\pi}{a} \frac{\lambda}{a} \frac{P_0}{B_1} \hat{b}_0 \quad (8)$$

where the pressure gradient is taken to be P_0/a and the constant pressure surfaces are assumed to differ by an amount λ/a from being toroidally symmetric. Since we know it is the structure of $\vec{B}_1$ and not its magnitude that determines the form of the constant pressure surfaces, $\vec{B}_1$ must change over a distance scale λ or $|\vec{\nabla} \times \vec{B}_1| \approx B_1/\lambda$. From Eq. (8) this implies

$$\lambda \approx \frac{a}{\sqrt{\beta_1}} \quad , \quad \beta_1 = \frac{8\pi P_0}{B_1^2} \quad (9)$$

The naive interpretation of Eq. (9) is that a topology changing magnetic field $\vec{B}_1$ penetrates only a distance $\lambda \approx a/\sqrt{\beta_1}$ into the plasma. Although this interpretation can be shown to be correct for many physical problems, in some situations λ may represent a structure scale for the field B_1 . In these

situations $\vec{B}_1$ changes the topology of the entire configuration despite the smallness of λ . To examine the question of the meaning of λ , a simpler system will be considered; the θ symmetric cylindrical plasma or Z pinch. This simple system can be viewed as the infinite aspect ratio approximation to the reversed field geometry.

III. Z PINCH MODEL

In the last section we found that a small perturbation magnetic field $\vec{B}_1$ could change, in a fundamental way, the topology of a reversed field configuration. However, the plasma diamagnetic current enforces a spacial scale on the perturbation field $\lambda \approx a/\sqrt{\beta_1}$ with β_1 the plasma beta value relative to the perturbation. To understand the phenomenon better, we will consider the simplest system in which this behavior arises, the θ symmetric Z pinch (Fig. 1). In this geometry the magnetic field is to zeroth order in the $\hat{\theta}$ direction and can in general be written as

$$\vec{B} = I \vec{\nabla} \theta + \vec{\nabla} \theta \times \vec{\nabla} \chi \quad . \quad (10)$$

This form for the magnetic field is identical to the general form for toroidally symmetric systems by replacing θ with ϕ and χ with ψ . Consequently, the well-known Grad-Shafranov theory of toroidal equilibria can be applied to the Z pinch with a trivial notation change. In particular, the pressure P and

the current I are functions of χ alone. In other words χ , no matter what its magnitude, determines the shape of the constant pressure surfaces. Using $\vec{\nabla}P = \vec{j} \times \vec{B}/c$ and $\vec{\nabla} \times \vec{B} = 4\pi \vec{j}/c$, one can show that

$$\Delta_* \chi = \frac{4\pi}{c} r j_\theta, \quad (11)$$

$$j_\theta = -c \left[\frac{1}{r} \frac{1}{8\pi} \frac{dr^2}{d\chi} + r \frac{dP}{d\chi} \right].$$

The definition of the Δ_* operator is

$$\Delta_* \chi \equiv r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \frac{\partial^2 \chi}{\partial z^2}. \quad (12)$$

In order to solve Eq. (11), we will take special forms for the pressure and B_z profiles. The special forms may seem artificial unless one first considers the steady-state equilibria (including resistivity) of a Z pinch which is not only θ symmetric but also has no z dependence. The simplest steady-state equilibrium has j_z uniform in the plasma, $B_\theta \propto r$, $P = P_0(1 - r^2/a^2)$, and B_z uniform. It is this equilibrium that we will be tacitly assuming the plasma would go to if it were unperturbed.

The mathematical form for χ which will be studied is

$$\chi = -\frac{1}{2} B_1 r^2 \left[1 + \frac{1}{r} F(r, z) \right] \quad (13)$$

with B_1 a constant. Since $B_z = -(\partial\chi/\partial r)/r$, the z component of

the magnetic field is B_1 when F is zero. Instead of retaining χ while discussing geometry, a radius r_0 is defined so $\chi = -B_1 r_0^2/2$. To have a definite problem, an assumption must be made about the current. This assumption determines in effect the fraction of the plasma pressure held by the B_z field. We will assume that if $F = 0$ that fraction is zero. This is equivalent to taking

$$\frac{1}{8\pi} \frac{dI^2}{d\chi} = r_0^2 \frac{dP}{d\chi} \quad (14)$$

which implies

$$r j_\theta = -c r F \frac{dP}{d\chi} \quad (15)$$

Substituting the form for χ given by Eq. (13) into Eq. (11) which is the Grad-Shafranov equation, one obtains

$$\frac{1}{r} \frac{d}{dr} r \frac{dF}{dr} + \frac{\partial^2 F}{\partial z^2} - \frac{1}{r^2} F = \frac{1}{\lambda^2} F \quad (16)$$

with

$$\frac{1}{\lambda^2} \equiv \frac{8\pi}{B_1} \frac{dP}{d\chi} \quad (17)$$

Equation (16) is of a form that is quite well known in plasma physics. For example, the equation for an electromagnetic wave of frequency ω in a plasma is

$$\nabla^2 \vec{E} = \frac{\omega_p^2 - \omega^2}{c^2} \vec{E} \quad (18)$$

with ω_p the plasma frequency. From our knowledge of the wave equation, we can immediately conclude that if $\lambda^2 > 0$ then F will decay exponentially in the plasma with λ the e-folding distance. However, if $\lambda^2 < 0$ the function F will have an oscillatory form in the plasma with $2\pi i \lambda$ the wavelength. Unfortunately, λ^2 is not a positive definite quantity.

Although λ^2 is not positive definite, it is a positive constant for a simple model of the plasma. In this model we let $P = P_0(1 - r_0^2/a^2)$ with a the plasma radius and $\chi = -B_1 r_0^2/2$. This is the simple equilibrium pressure profile mentioned in connection with the Z pinch without Z variation. With this form for $P(\chi)$, λ^2 is

$$\lambda^2 = \frac{a^2}{4\beta_1} \quad , \quad \beta_1 = \frac{4\pi P_0}{B_1^2} \quad . \quad (19)$$

The simplest problem is to study the effect of a weak toroidal ripple ($F \ll \lambda$) on the plasma. The ripple will be assumed to have a $\cos k z$ dependence so $F(r, z) = f(r) \cos k z$ then one finds inside the plasma

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \left(\frac{1}{r^2} + k^2 + \frac{1}{\lambda^2} \right) f = 0 \quad . \quad (20)$$

Outside the plasma, in the vacuum, the appropriate equation is identical to Eq. (20) except the $1/\lambda^2$ term is absent. The assumption $F \ll \lambda$ allows us to take $r = a$ as the plasma surface. Assuming $\lambda \ll a$ and $ka \ll 1$, the solution is

$$f(r) = \begin{cases} \Delta_v \left[\frac{2}{k} I_1(kr) - \left(1 - \frac{2\lambda}{a}\right) k a^2 K_1(kr) \right] , & r > a \\ 2\Delta_v \left(\frac{2\pi a}{\lambda} \right)^{1/2} e^{-a/\lambda} \frac{\lambda}{r} I_1(r/\lambda) , & r < a \end{cases} \quad (21)$$

with I_1 and K_1 the first-order modified Bessel functions of the first and second kind and Δ_v a constant. The term $\Delta_v 2I_1(kr)/k$ in the solution becomes exponentially large for $kr \gg 1$ and represents the ripple imposed by currents in wires well away from the plasma. This term by itself would give $B_z = B_1(1 + \Delta_v \cos kz)$; so the constant, Δ_v , is the magnitude of the ripple in the absence of the plasma. Inside the plasma, one finds that solution given by Eq. (21) is for $r \gg \lambda$

$$B_z = B_1 \left[1 + \left(\frac{a}{r} \right)^{1/2} \Delta_v e^{-(a-r)/\lambda} \cos kz \right] .$$

Interestingly, the ripple in B_z at the plasma edge ($r = a$) is the same as it would have been in the absence of the plasma. However, the ripple field is shielded out of the plasma, as expected, with a shielding distance λ .

If one extrapolates the solution given by Eq. (21) to $\Delta_v = 1$ (it is strictly valid only for $\Delta_v \ll 1$), then one finds the χ surfaces are open or closed depending on whether the χ surfaces would have been open or closed without the plasma. Unfortunately, the evaluation of the solution for the case in which an X point occurs on the plasma surface is difficult. The problem is the

boundary condition (that $\vec{\nabla}\chi$ match on the two sides of the constant χ surface which gives the plasma surface). If an X point occurs in the plasma volume, this implies the z component of the magnetic field changes sign as the X point is crossed radially. Since $dP/d\chi \approx - (dP/dr)/(rB_z)$, this means $dP/d\chi$ has opposite signs on opposite radial sides of the X point. A sign change in $dP/d\chi$ makes λ imaginary and changes exponential shielding into a sinusoidal dependence on position.

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782009

Fig. 1. The Cylindrical Coordinate System