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TRANSPORT PROPERTIES IN PARTIALLY AND FULLY IONIZED PLASMAS

Dermot Kelleher
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ABSTRACT

Transport equations are developed for the number densities, mean velocities, temperatures, stresses and heat fluxes of the constituents of an ionized gas by taking velocity moments of the Boltzmann equation. These equations are closed out by utilizing the Grad 13-moment approximation for each distribution function. This approximation provides a representation for the distribution function of each constituent which has as its first term a Maxwellian about the constituent mean velocity, and remaining terms which describe the deviation of the distribution function from this local constituent Maxwellian and which have as their coefficients the constituent traceless stress tensor and the constituent heat flux vector defined relative to the mean constituent velocity. These equations, coupled with Maxwell's equations to describe the electric and magnetic fields, constitute a closed set. This set of equations is anticipated to have validity for plasma systems of arbitrary degree of ionization, the constituents of which need not be close to thermodynamic equilibrium with each other. By limiting attention to systems with slowly varying flows, transport expressions, which reflect the influence of electric and magnetic fields, have been extracted and exhibited.

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I. INTRODUCTION

Many authors have developed methods of obtaining transport properties for plasma systems. Generally, however, most work concerns itself with a particular property or properties in certain specific situations. It was felt that the development of a set of equations from which transport properties of a plasma could be determined in a wide variety of situations would be of value.

The Boltzmann equation was accepted as providing an adequate mathematical description of a plasma system, and an adaptation of the Grad^(1, 2) 13-moment method was used to solve the equation.

Basically, this method differs from the more familiar Chapman-Enskog method, a perturbation type solution detailed by Chapman and Cowling,⁽³⁾ in that the distribution function is expanded in an infinite set of three-dimensional Hermite polynomials, the infinite set is truncated at an appropriate point, and the differential equations describing the coefficients of the truncated expansion are generated by taking velocity moments of the Boltzmann equation. The coefficients of the expansion are identified with such physically observable quantities as the number density, the temperature or energy, the pressure tensor and the heat flux vector of the constituents of the plasma system. Grad⁽¹⁾ originally developed this method for the problem of neutral gases. Kolodner⁽⁴⁾ later used it to describe fully ionized plasma systems which were close to equilibrium.

Everett⁽⁵⁾ modified this approach to develop magnetohydrodynamic equations for a fully ionized gas which was not necessarily close to equilibrium. Everett's modification to the original development consisted of taking the particle velocity relative to the flow velocity of each constituent as the independent variable rather than a velocity relative to the plasma flow velocity. Thus the form of the distribution function for each specie in a plasma is dependent only upon the physical observables associated with that constituent; the effect of other constituents upon the constituent of interest appears only in the collision integral. An examination of the relaxation times in a plasma shows that electrons relax to equilibrium with themselves on the order of $43A^{1/2}$ (where A is the mass number of the heavy particles) times faster than the heavy particles relax to equilibrium with themselves. Also the electrons relax to equilibrium with themselves on the order of $1836A$ times faster than electrons and heavy particles come to equilibrium with each other. Thus this modification, which is essentially an expansion about a local constituent Maxwellian instead of the plasma Maxwellian, would be expected to provide a more accurate description of plasma systems in which the constituents are not necessarily close to equilibrium with each other.

The approach used in this paper is an extension of the work of Everett; that is, the same expansion technique is used; however, the effect of neutral particles is included, and thus generalized transfer equations applicable to plasma systems of all degrees of ionization are developed. Transport properties of plasma systems are then extracted from these transfer equations and exhibited for a variety of situations.

In Section II the general theory and method of approach are set forth. The Grad 13-moment approximation to the distribution function is discussed in Section III. Section IV consists of a discussion of the method of evaluating the collision integrals and the results of the calculations.

The collision integrals for charged particle-charged particle elastic interactions have been calculated by Everett. These were recomputed and the results, with some minor corrections, are presented. The neutral particle-neutral particle and charged particle-neutral particle collision integrals, under the assumption of elastic collisions and a $\frac{K_{12}}{r^5}$ force law between interacting particles, have been calculated by many authors. However, the velocity space integrals which occur have not, to the knowledge of this author, been calculated using this particular coordinate system. Therefore, the details of the calculations and the results are presented. The collision transfer terms reduced to a more manageable form are presented in Section V. This reduction is accomplished by utilizing the large mass difference that exists between particles in a plasma system and by keeping terms only up to order ϵ^3 where ϵ is equal to the difference between the flow velocities of two constituents divided by the square root of the sum of the squares of the constituent thermal velocities.

In Section VI the generalized transfer equations are presented; and in Section VII transport relations extracted from these equations are exhibited. The transport coefficients of a plasma system are presented in Section VIII. In Appendix A a brief review of some of the equations used to describe plasma systems is presented.

Gaussian units are used in the presentation and symbols are defined as they occur.

II. GENERAL THEORY

The Boltzmann equation for charged particles may be written:

$$\frac{\partial f^a(\bar{x}, \bar{v}, t)}{\partial t} + v_i \frac{\partial f^a}{\partial r_i} + \frac{e^a}{ma} \left[E_i + \frac{1}{c} \epsilon_{ijk} v_j H_k \right] \frac{\partial f^a}{\partial v_i} = \sum_{\gamma} \left(\frac{\partial f^a}{\partial t} \right)_c^{\gamma} \quad (\text{II-1})$$

where a denotes either ions or electrons and γ can denote electrons, ions, or the various species of neutral particles present.

$$\left(\frac{\partial f^a}{\partial t} \right)_c^{\gamma}$$

represents the change in f^a , the distribution function of a type particles, per unit time due to collisions of a type particles with γ type particles.

For neutral particles the Boltzmann equation may be written:

$$\frac{\partial f^N}{\partial t} + v_i \frac{\partial f^N}{\partial r_i} + a_i \frac{\partial f^N}{\partial v_i} = \sum_{\gamma} \left(\frac{\partial f^N}{\partial t} \right)_c^{\gamma} \quad (\text{II-2})$$

where a_i is the acceleration experienced by neutrals, N denotes the various types of neutral particles present and γ is defined as above.

The electric and magnetic fields in II-1 are governed by Maxwell's equations.

The current density, J_k , charge density, q , number densities, n^{γ} , and constituent flow velocities, w_k^{γ} are defined in the usual manner.

$$\text{a. } J_k = n^i z e w_k^i - n^e e w_k^e$$

$$\text{b. } q = n^i z e - n^e e$$

$$\text{c. } n^{\gamma} = \int f^{\gamma} d^3 v$$

$$\text{d. } w_k^{\gamma} = \frac{1}{n^{\gamma}} \int f^{\gamma} v_k d^3 v \quad (\text{II-3})$$

It is convenient to transfer independent variables in Equations II-1 and II-2 from the particle velocity, v_i , to a random velocity, u_i ; $u_i = v_i - w_i^y$. With this transformation Equation II-1 becomes:

$$\begin{aligned} \frac{D_a f^a}{Dt} + u_i \frac{\partial f^a}{\partial r_i} + \left[\frac{e^a}{m^a} E_i + \frac{e^a}{m^a c} \epsilon_{ijk} w_j^a H_k - \frac{D_a w_i^a}{Dt} \right] \frac{\partial f^a}{\partial u_i} \\ + \frac{e^a}{m^a c} \epsilon_{ijk} u_j H_k \frac{\partial f^a}{\partial u_i} - \frac{\partial f^a}{\partial u_j} u_i \frac{\partial w_j^a}{\partial r_i} = \sum_y \left(\frac{\partial f^a}{\partial t} \right)_c^y \end{aligned} \quad (\text{II-4})$$

where

$$\frac{D_a}{Dt} = \frac{\partial}{\partial t} + w_i^a \frac{\partial}{\partial r_i}$$

Equation II-2 becomes:

$$\frac{D_N f^N}{Dt} + u_i \frac{\partial f^N}{\partial r_i} - \frac{D_N w_i^N}{Dt} \frac{\partial f^N}{\partial u_i} - \frac{\partial f^N}{\partial u_j} u_i \frac{\partial w_j^N}{\partial r_i} + a_i \frac{\partial f^N}{\partial u_i} = \sum_y \left(\frac{\partial f^N}{\partial t} \right)_c^y \quad (\text{II-5})$$

Transport equations for parameters of interest are generated by taking velocity moments of Boltzmann's equation. The procedure is to multiply Equations II-4 and II-5 by $\psi^a(\bar{u})$, where ψ^a is an arbitrary function of $\bar{u}$, and then to integrate over $\bar{u}$. With $\langle \psi^a \rangle \equiv \frac{1}{n^a} \int f^a \psi^a d^3 u$, the generated equations of transfer are:

$$\begin{aligned} \frac{D_a \{n^a \langle \psi^a \rangle\}}{Dt} + \frac{\partial}{\partial r_i} \{n^a \langle u_i \psi^a \rangle\} - \left\{ \frac{e^a}{m^a} E_i + \frac{e^a}{m^a c} \epsilon_{ijk} w_j^a H_k \right. \\ \left. - \frac{D_a w_i^a}{Dt} \right\} n^a \left\langle \frac{\partial \psi^a}{\partial u_i} \right\rangle - \frac{e^a n^a}{m^a c} \epsilon_{ijk} H_k \left\langle \frac{\partial (u_j \psi^a)}{\partial u_i} \right\rangle \\ + n^a \left\langle \frac{\partial (u_j \psi^a)}{\partial u_i} \right\rangle \frac{\partial w_i^a}{\partial r_j} = \sum_y \int \left(\frac{\partial f^a}{\partial t} \right)_c^y \psi^a d^3 u \end{aligned} \quad (\text{II-6})$$

and

$$\begin{aligned} \frac{D_N \{n^N \langle \psi^N \rangle\}}{Dt} + \frac{\partial}{\partial r_i} \{n^N \langle u_i \psi^N \rangle\} + \frac{D_N w_i^N}{Dt} n^N \left\langle \frac{\partial \psi^N}{\partial t} \right\rangle + n^N \left\langle \frac{\partial (u_j \psi^N)}{\partial u_i} \right\rangle \frac{\partial w_i^N}{\partial r_j} \\ - n^N \left\langle \frac{\partial (\psi^N)}{\partial u_i} \right\rangle a_i = \sum_y \int \left(\frac{\partial f^N}{\partial t} \right)_c^y \psi^N d^3 u \end{aligned} \quad (\text{II-7})$$

Taking $\psi^\gamma = 1$, $m^\gamma u_k$, $\frac{1}{2} m^\gamma u^2$, $m^\gamma u_k^o u_l$ and $\frac{1}{2} m^\gamma u^2 u_k$ transport equations for number densities, flow velocities, temperatures, traceless stress tensors, and heat fluxes are obtained, respectively.

For the charged particles these are:

$$a. \frac{D_a n^\alpha}{Dt} + n^\alpha \frac{\partial w_i^\alpha}{\partial r_i} + N^\alpha = 0$$

$$b. \frac{D_a w_i^\alpha}{Dt} = \frac{e^\alpha}{m^\alpha} \left[E_i + \frac{1}{c} \epsilon_{ijk} w_j^\alpha H_k \right] - \frac{1}{\rho^\alpha} \frac{\partial P_{ij}^\alpha}{\partial r_j} + \frac{1}{\rho^\alpha} S_i^\alpha$$

$$c. \frac{D_a T^\alpha}{Dt} = \frac{2}{3n^\alpha k} \left\{ - \frac{\partial Q_i^\alpha}{\partial r_i} - P_{ij}^\alpha \frac{\partial w_j^\alpha}{\partial r_i} + R^\alpha \right\}$$

$$d. \frac{D_a P_{rs}^{o\alpha}}{Dt} = - \frac{\partial}{\partial r_i} \rho^\alpha \langle u_i u_r^o u_s \rangle^\alpha + \frac{2e^\alpha}{m^\alpha c} \epsilon_{rjk} \overline{P_{js}^{o\alpha} H_k} - P_{rs}^{o\alpha} \frac{\partial w_i^\alpha}{\partial r_i} - 2P_{jr} \frac{\partial w_s^\alpha}{\partial r_j} + L_{rs}^{o\alpha}$$

$$e. \frac{D_a Q_r^\alpha}{Dt} = - \frac{\partial}{\partial r_i} \left\{ \frac{\rho^\alpha}{2} \langle u^2 u_i u_r \rangle^\alpha \right\} + \frac{5}{2} \left[\frac{e^\alpha}{m^\alpha} E_r + \frac{e^\alpha}{m^\alpha c} \epsilon_{rjk} w_j^\alpha H_k - \frac{D_a w_r^\alpha}{Dt} \right] p^\alpha$$

$$+ \left[\frac{e^\alpha}{m^\alpha} E_i + \frac{e^\alpha}{m^\alpha} \epsilon_{ijk} w_j^\alpha H_k - \frac{D_a w_i^\alpha}{Dt} \right] P_{ir}^{o\alpha} + \frac{e^\alpha}{m^\alpha c} \epsilon_{rjk} H_k Q_j - Q_r^\alpha \frac{\partial w_i^\alpha}{\partial r_i}$$

$$- Q_j^\alpha \frac{\partial w_r^\alpha}{\partial r_j} - \frac{\partial w_i^\alpha}{\partial r_j} p^\alpha \langle u_i u_j u_k \rangle^\alpha + M_r^\alpha$$

(II-8)

For neutral particles these are:

$$a. \frac{D_N n^N}{Dt} + n^N \frac{\partial w_i^N}{\partial r_i} + N^N = 0$$

$$b. \frac{D_N w_i^N}{Dt} = - \frac{1}{\rho^N} \frac{\partial P_{ij}^N}{\partial r_j} + a_i + \frac{1}{\rho^N} S_i^N$$

$$c. \frac{D_N T^N}{Dt} = \frac{2}{3n^N k} \left\{ - \frac{\partial Q_i^N}{\partial r_i} - P_{ij}^N \frac{\partial w_j^N}{\partial r_i} + R^N \right\}$$

$$d. \frac{D_N P_{rs}^{oN}}{Dt} = - \frac{\partial}{\partial r_i} \left\{ \rho^N \langle u_i u_r^o u_s \rangle^N \right\} - P_{rs}^{oN} \frac{\partial w_j^N}{\partial r_j} - 2P_{jr} \frac{\partial w_s^N}{\partial r_j} + L_{rs}^{oN}$$

$$e. \quad \frac{D_N Q_r^N}{Dt} = - \frac{\partial}{\partial r_i} \left\{ \frac{\rho^N}{2} \langle u_i^2 u_r \rangle^N \right\} - \frac{5}{2} \rho^N \frac{D_N w_r^N}{Dt} - P_{rs}^N \frac{D_N w_s^N}{Dt} - \frac{\partial w_i^N}{\partial r_i} Q_r^N - Q_j^N \frac{\partial w_r^N}{\partial r_j} - \frac{\partial w_i^N}{\partial r_j}.$$

$$\rho^N \langle u_i u_j u_r \rangle^N + \frac{3}{2} a_k^N P_{ir}^N + a_1^N P_{ir}^N + M_r^N \quad (II-9)$$

where:

$$N^\nu = \sum_\gamma \int \left(\frac{\partial f^\nu}{\partial t} \right)_c^\gamma d^3 u, \text{ the collision transfer of number densities;}$$

$$S_i^\nu = \sum_\gamma \int m^\nu u_i \left(\frac{\partial f^\nu}{\partial t} \right)_c^\gamma d^3 u,$$

the collision transfer of momentum density;

$$R^\nu = \sum_\gamma \int \frac{1}{2} m^\nu u^2 \left(\frac{\partial f^\nu}{\partial t} \right)_c^\gamma d^3 u,$$

the collision transfer of energy density;

$$L_{rs}^{o\nu} = \sum_\gamma \int m^\nu u_r^o u_s^o \left(\frac{\partial f^\nu}{\partial t} \right)_c^\gamma d^3 u,$$

the collision transfer of traceless stress;

$$M_r^\nu = \sum_\gamma \int \frac{1}{2} m^\nu u^2 u_r \left(\frac{\partial f^\nu}{\partial t} \right)_c^\gamma d^3 u,$$

the collision transfer of heat flux density; and $\rho^\nu = n^\nu m^\nu$, the mass density of constituent type ν .

Also traceless tensors and symmetric tensors are defined, respectively:

$$a. \quad A_{rs}^o = A_{rs} - \frac{1}{3} \delta_{rs} A_{ii}$$

$$b. \quad \overline{\overline{A}}_{rs} = \frac{1}{2} (A_{rs} + A_{sr}) \quad (II-10)$$

Temperature, stress, and heat flow of each constituent are defined:

$$a. \quad T^\nu = \frac{2}{3k} \left\langle \frac{m^\nu}{2} u^2 \right\rangle$$

$$\text{b. } P_{rs}^{\nu} = n^{\nu} \langle m^{\nu} u_r u_s \rangle \quad (\text{II-11})$$

$$\text{c. } Q_r^{\nu} = n^{\nu} \langle \frac{m^{\nu}}{2} u_r^2 \rangle$$

The temperature, pressure tensor, and the static pressure are observed to be related by the expression:

$$n^{\nu} kT^{\nu} = \frac{1}{3} P_{ss}^{\nu} = p^{\nu} \quad (\text{II-12})$$

The electric and magnetic fields are governed by Maxwell's equations:

$$\text{a. } \epsilon_{ijk} \frac{\partial H_k}{\partial r_j} = \frac{1}{c} \frac{\partial E_i}{\partial t} + \frac{4\pi}{c} J_i$$

$$\text{b. } \epsilon_{ijk} \frac{\partial E_k}{\partial r_j} = -\frac{1}{c} \frac{\partial H_i}{\partial t}$$

$$\text{c. } \frac{\partial H_i}{\partial r_i} = 0$$

$$\text{d. } \frac{\partial E_i}{\partial r_i} = 4\pi q \quad (\text{II-13})$$

where the electric field, $\bar{E}$, is considered to be the sum of the externally applied field and the field generated by particles outside of the Debye sphere of the particle being considered. That is:

$$\bar{E} = \int_{|r| > L_D}^{\infty} \frac{-q}{r^3} \bar{r} d^3 r \quad (\text{II-14})$$

It should be noted that Equations II-8, II-9, and II-13 do not constitute a closed set, since N^{ν} , S_r^{ν} , R_{rs}^{ν} , P_{rs}^{ν} and M_r^{ν} require a knowledge of f^{ν} for their determination. If, however, f^{ν} can be appropriately represented in terms of the unknown macroscopic quantities, then the collision transfer terms can be determined in terms of these macroscopic quantities, and the set of equations will constitute a closed set. A representation of the distribution function of this type is the Grad^(1,2) 13-moment representation.

III. THE GRAD 13-MOMENT REPRESENTATION OF THE DISTRIBUTION FUNCTIONS

The Grad 13-moment representation is obtained by expanding f^ν in three-dimensional Hermite polynomials and terminating the expansion with a contraction of the third rank polynomial.

The three-dimensional Hermite polynomials are defined by:⁽²⁾

$$H_{i_1, i_2, \dots, i_n} = \frac{(-1)^n}{W} \frac{\partial^n W}{\partial c_{i_1} \dots \partial c_{i_n}}, \quad (\text{III-1})$$

where

$$W = \frac{1}{(2\pi)^{3/2}} e^{-\frac{c^2}{2}}.$$

These polynomials have been shown by Grad to constitute a complete set which is orthogonal with respect to W .

Letting

$$c_k = \left(\frac{m^\nu}{kT^\nu} \right)^{1/2} u_k = \beta^\nu{}^{1/2} u_k,$$

the exact representation for f^ν is:

$$f^\nu = f_m^\nu \sum_{n=0}^{\infty} \frac{1}{n!} a_i^{(n)} H_i^{(n)}, \quad (\text{III-2})$$

where

$$f_m^\nu = n^\nu \left(\frac{\beta^\nu}{2\pi} \right)^{3/2} e^{-\beta^\nu \frac{u^2}{2}},$$

the Maxwellian distribution. The coefficients, $a_i^{(n)}$, are determined in the usual way, and the contraction of the third rank polynomial is performed. The approximate distribution function is then written:

$$f^\nu = n^\nu \left(\frac{\beta^\nu}{2\pi} \right)^{3/2} e^{-\beta^\nu \frac{u^2}{2}} \left\{ 1 + \frac{\beta^\nu}{2\rho^\nu} P_{ij}^{o\nu} (u_i u_j) + \frac{\beta^\nu}{5\rho^\nu} Q_i^\nu (u^2 u_i - \frac{5}{\beta^\nu} u_i) \right\} \quad (\text{III-3})$$

In this form it is seen that the distribution function is written as the local constituent Maxwellian plus terms which represent the deviation from this local equilibrium distribution. The distribution function is noted to be expressed in terms of the macroscopic properties n^ν , w_k^ν , T^ν , $P_{rs}^{o\nu}$ and Q_r^ν which permits the determination of $\langle u_i u_r^o u_s \rangle^\nu$, $\langle u^2 u_i u_r \rangle^\nu$ and the collision transfer terms in terms of these properties.

With regard to the convergence of the approximate form of the distribution function, it may be shown from Equation III-2 that the nth term has as its coefficient the (n-1) moment of the distribution function; hence, to the extent that moments of the distribution function beyond the heat flux are considered unimportant, confidence exists that terms in the expansion beyond the contraction of the third rank polynomial are unimportant.

IV. COLLISION TRANSFER TERMS

In order to develop a complete set of transfer equations, it is necessary to evaluate the collision transfer integrals for the following types of collisions.

- a. atom-atom
- b. ion-ion
- c. electron-electron
- d. atom-ion
- e. atom-electron
- f. ion-electron

The collision transfer terms may be written in the following form:⁽³⁾

$$\int \psi^\gamma \frac{\partial f^\gamma}{\partial t} d^3 u = - \int f^\nu f^\gamma (\psi^\gamma - \psi^{\gamma'}) v_r b db d\phi d^3 u d^3 \nu \quad (\text{IV-1})$$

where the primes denote post-collision variables, v_r is the absolute value of the relative velocity, $v_r = |v^\gamma - v^\nu|$, and b and ϕ are the variables which describe the geometry of an encounter, the impact parameter between two particles and the angular orientation of an interaction, respectively. Also it should be noted that Chapman and Cowling⁽³⁾ have shown that the form of the collision transfer term given in IV-1 is mathematically identical to the more familiar expression. That is:

$$- \int \psi^\gamma (f^\gamma f^\nu - f^{\gamma'} f^{\nu'}) v_r b db d\phi d^3 u d^3 \nu = \int \psi^\gamma \frac{\partial f^\gamma}{\partial t} d^3 u \quad (\text{IV-2})$$

It is convenient to introduce a collision invariant vector d_k^γ , where d_k^γ is equal to the velocity of the center of mass of the two interacting particles minus the mean flow velocity for particles of type γ . It can be shown that:

$$\text{a. } d_k^\gamma = u_k^\gamma - \frac{m^\nu}{m^\gamma + m^\nu} v_{r_k}^{\gamma, \nu} \quad (\text{IV-3})$$

$$\text{b. } d_k^{\gamma'} = u_k^{\gamma'} - \frac{m^\nu}{m^\gamma + m^\nu} v_{r_k}^{\gamma, \nu'}$$

For

$\psi^\gamma = m^\gamma u_k^\gamma$, $m^\gamma u_k^\gamma u_1^\gamma$ and $\frac{m^\gamma}{2} u_k^2 u_1^\gamma$, $(\psi^\gamma - \psi^{\gamma'})$ becomes, respectively:

$$a. \quad m^\gamma (u_k^\gamma - u_k^{\gamma'}) = \frac{m^\gamma m^\nu}{m^\gamma + m^\nu} \left(v_{r_k}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} \right)$$

$$b. \quad m^\gamma \left(u_k^\gamma u_1^\gamma - u_k^{\gamma'} u_1^{\gamma'} \right) = 2 \frac{m^\gamma m^\nu}{m^\gamma + m^\nu} d_k^\gamma \left(v_{r_1}^{\gamma, \nu} - v_{r_1}^{\gamma, \nu'} \right) \\ + \frac{m^\gamma m^{\nu 2}}{(m^\gamma + m^\nu)^2} \left(v_{r_k}^{\gamma, \nu} v_{r_1}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} v_{r_1}^{\gamma, \nu'} \right)$$

$$c. \quad \frac{m^\gamma}{2} \left(u_k^2 u_1^\gamma - u_k^2 u_1^{\gamma'} \right) = \frac{1}{2} \frac{m^\gamma m^\nu}{(m^\gamma + m^\nu)} d_k^{\gamma 2} \left(v_{r_k}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} \right) + \frac{m^\gamma m^{\nu 3}}{2(m^\gamma + m^\nu)^3} v_r^{\gamma, \nu 2} \left(v_{r_k}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} \right) \\ + \frac{m^\gamma m^\nu}{(m^\gamma + m^\nu)} d_k^\gamma d_l^\gamma \left(v_{r_1}^{\gamma, \nu} - v_{r_1}^{\gamma, \nu'} \right) \\ + \frac{m^\gamma m^{\nu 2}}{(m^\gamma + m^\nu)^2} d_l^\gamma \left(v_{r_k}^{\gamma, \nu} v_{r_1}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} v_{r_1}^{\gamma, \nu'} \right). \quad (IV-4)$$

At this point two assumptions are made. It is assumed that all collisions are elastic, and that the force between interacting particles can be expressed in the form

$$F = \frac{K_{12}}{r^n},$$

where K_{12} is a constant.

The collision integrals may be separated into a velocity integration part and an integration over angle of scatter. Considering the integral over angle of scatter, it is seen that insofar as velocity dependence is concerned, only two forms appear -- integrals containing

$$\left(v_{r_k}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} \right)$$

and those containing

$$\left(v_{r_k}^{\gamma, \nu} v_{r_1}^{\gamma, \nu} - v_{r_k}^{\gamma, \nu'} v_{r_1}^{\gamma, \nu'} \right).$$

Now if a spherical coordinate system in velocity space is chosen in which $v_{r_k}^{Y,\nu}$ is in the z direction, the vector $\left(v_{r_k}^{Y,\nu} - v_{r_k}^{Y,\nu'} \right) = (-v_r \sin X \cos \phi, -v_r \sin X \sin \phi, v_r (1 - \cos X))$ where X is the angle through which $\bar{v}_r$ is deflected. Upon integration over ϕ from 0 to 2π , this becomes:

$$2\pi v_{r_k}^{Y,\nu} (1 - \cos X). \quad (\text{IV-5})$$

If the tensor, $\left(v_{r_k}^{Y,\nu} v_{r_l}^{Y,\nu} - v_{r_k}^{Y,\nu'} v_{r_l}^{Y,\nu'} \right)$, is expressed in the same coordinate system and the integration over ϕ performed, the result is:

$$3\pi (1 - \cos^2 X) \left\{ v_{r_k}^{Y,\nu} v_{r_l}^{Y,\nu} - \frac{\delta_{kl}}{3} v_r^2 \right\}. \quad (\text{IV-6})$$

Directing attention to the integration over the impact parameter, b , the integrals of interest are seen from Equations IV-5, and IV-6 to be of the form:

$$\int_0^{b_{\max}} (1 - \cos^m X) v_r^2 b db, \quad (\text{IV-7})$$

where $m = 1$ or 2 . Following Chapman and Cowling,⁽³⁾ the equations of motion for two particles interacting with an arbitrary force,

$$F = \frac{K_{12}}{r^n},$$

are solved. The integrals of interest become:

$$\left\{ \frac{m^Y + m^\nu}{m^Y m^\nu} K_{12} \right\}^{\frac{2}{n-1}} v_r^{\frac{n-5}{n-1}} \int_0^{\zeta_{\max}} (1 - \cos^m X) \zeta_0 d\zeta_0, \quad (\text{IV-8})$$

where

$$X = \pi - 2 \int_0^{\zeta_0} \left\{ 1 - \zeta^2 - \left(\frac{2}{n-1} \right) \left(\frac{\zeta}{\zeta_0} \right)^{n-1} \right\}^{-\frac{1}{2}} d\zeta$$

and

$$\zeta = \frac{b}{r}$$

$$\zeta_o = b \left\{ \frac{m^\gamma m^\nu v_r^2}{(m^\gamma + m^\nu) K_{12}} \right\}^{\frac{1}{n-1}}$$

ζ_{oo} is the positive real root of the equation:

$$1 - \zeta^2 - \left(\frac{2}{n-1} \right) \left(\frac{\zeta}{\zeta_o} \right) = 0$$

$b_{\max}$ is the upper limit on the impact parameter; the appropriate values for $b_{\max}$ for each type of interaction will be discussed later.

The integration over angles of scatter has thus been reduced to solving the integral,

$$\int_0^{\zeta_{\max}} (1 - \cos^m \chi) \zeta_o d\zeta_o = A_m(n) \quad (\text{IV-9})$$

$A_m(n)$ is a function of n and its value will depend on the particular force law involved.

The collision transfer integrals may now be written:

$$2\pi (K_{12})^{\frac{2}{n-1}} A_1(n) \left\{ \frac{m^\gamma + m^\nu}{m^\gamma m^\nu} \right\}^{\left(\frac{2}{n-1} - 1 \right)} \int f^\gamma f^\nu v_r^{\frac{n-5}{n-1}} \Psi d^3 u^\gamma d^3 u^\nu \quad (\text{IV-10})$$

where Ψ , for momentum transfer, stress transfer, and heat flux transfer, respectively, is given by:

$$\begin{aligned} \text{a. } & v_{r_k}^{\gamma, \nu} \\ \text{b. } & 2 d_k^\gamma v_{r_1}^{\gamma, \nu} + \frac{3}{2} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_2(n)}{A_1(n)} v_{r_k}^{\gamma, \nu} \circ v_{r_1}^{\gamma, \nu} \\ \text{c. } & \frac{1}{2} d_k^{\gamma^2} v_{r_k}^{\gamma, \nu} + \frac{1}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 v_r^{\gamma, \nu} v_{r_k}^{\gamma, \nu} + d_k^\gamma d_1^\gamma v_{r_1}^{\gamma, \nu} + \frac{3}{2} \frac{m^\nu}{m^\gamma + m^\nu} d_1^\gamma \frac{A_2(n)}{A_1(n)} v_{r_k}^{\gamma, \nu} \circ v_{r_1}^{\gamma, \nu} \end{aligned} \quad (\text{IV-11})$$

The velocity space integrals become simpler in form if dimensionless variables, g_k and G_k , are introduced. These variables are defined by the relations:

$$\text{a. } \beta^{\gamma \frac{1}{2}} u_k^\gamma = \left(\frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right)^{1/2} \left\{ \left(\frac{\beta^\gamma}{\beta^\nu} \right)^{\frac{1}{2}} G_k + g_k - \epsilon_k^{\gamma, \nu} \right\}$$

$$b. \beta^{\nu \frac{1}{2}} u_k^{\nu} = \left(\frac{\beta^{\gamma}}{\beta^{\gamma} + \beta^{\nu}} \right)^{\frac{1}{2}} \left\{ - \left(\frac{\beta^{\nu}}{\beta^{\gamma}} \right)^{\frac{1}{2}} G_k - g_k - \epsilon_k^{\gamma, \nu} \right\} \quad (IV-12)$$

where

$$\epsilon_k^{\gamma, \nu} \equiv \left(\frac{\beta^{\gamma} \beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^{\frac{1}{2}} (w_k^{\gamma} - w_k^{\nu})$$

Using these new variables $v_{r_k}^{\gamma, \nu}$ and d_k^{γ} become:

$$a. v_{r_k}^{\gamma, \nu} = \left(\frac{\beta^{\gamma} + \beta^{\nu}}{\beta^{\gamma} \beta^{\nu}} \right)^{\frac{1}{2}} g_k \quad (IV-13)$$

$$b. d_k^{\gamma} = \left(\frac{\beta^{\gamma} + \beta^{\nu}}{\beta^{\gamma} \beta^{\nu}} \right)^{\frac{1}{2}} \left\{ \frac{(\beta^{\gamma} \beta^{\nu})^{\frac{1}{2}}}{\beta^{\gamma} + \beta^{\nu}} G_k + \kappa^{\gamma, \nu} g_k - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \epsilon_k^{\gamma, \nu} \right\}$$

where

$$\kappa^{\gamma, \nu} \equiv \left(\frac{\beta^{\gamma} \beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right) \frac{k}{m^{\gamma} + m^{\nu}} (T^{\gamma} - T^{\nu})$$

The expressions for Ψ^{γ} become:

$$a. \left(\frac{\beta^{\gamma} + \beta^{\nu}}{\beta^{\gamma} \beta^{\nu}} \right)^{\frac{1}{2}} g_k$$

$$b. \left(\frac{\beta^{\gamma} + \beta^{\nu}}{\beta^{\gamma} \beta^{\nu}} \right)^{\frac{1}{2}} \left\{ 2 \frac{(\beta^{\gamma} \beta^{\nu})^{\frac{1}{2}}}{\beta^{\gamma} + \beta^{\nu}} \overline{G_k g_1} + 2 \kappa^{\gamma, \nu} g_k g_1 - 2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \overline{\epsilon_k^{\gamma, \nu} g_1} + \frac{3}{2} \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{A_{2(n)}}{A_{1(n)}} g_k^o g_1 \right\}$$

$$c. \left(\frac{\beta^{\gamma} + \beta^{\nu}}{\beta^{\gamma} \beta^{\nu}} \right)^{\frac{3}{2}} \left\{ \frac{1}{2} \frac{\beta^{\gamma} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} G_k^2 g_k + \left[\frac{2 (\beta^{\gamma} \beta^{\nu})^{\frac{1}{2}}}{\beta^{\gamma} + \beta^{\nu}} \kappa^{\gamma, \nu} + \frac{(\beta^{\gamma} \beta^{\nu})^{\frac{1}{2}}}{\beta^{\gamma} + \beta^{\nu}} \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{3}{2} \frac{A_{2(n)}}{A_{1(n)}} \right] G_s g_s g_k - \frac{(\beta^{\gamma} \beta^{\nu})^{\frac{1}{2}}}{\beta^{\gamma} + \beta^{\nu}} \beta^{\nu} \right\}$$

$$\begin{aligned}
& \epsilon_s^{\gamma, \nu} G_s g_k + \left[\frac{3}{2} \kappa^{\gamma, \nu 2} + \frac{m^{\nu 2}}{2(m^\gamma + m^\nu)^2} + \kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_2(n)}{A_1(n)} \right] g_s^2 g_k \\
& + \left[-2\kappa^{\gamma, \nu} \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} - \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{3}{2} \frac{A_2(n)}{A_1(n)} \right] \epsilon_s^{\gamma, \nu} g_s g_k \\
& + \left[\frac{1}{2} \left(\frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right)^2 \right] \epsilon_s^{\gamma, \nu 2} g_k + \left[+ \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{1}{2} \frac{A_2(n)}{A_1(n)} - \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \kappa^{\gamma, \nu} \right] \epsilon_k^{\gamma, \nu} g_s^2 \\
& + \left[- \frac{m^\nu}{m^\gamma + m^\nu} \frac{1}{2} \frac{A_2(n)}{A_1(n)} \frac{(\beta^\gamma \beta^\nu)^{\frac{1}{2}}}{\beta^\gamma + \beta^\nu} - \frac{(\beta^\gamma \beta^\nu)^{\frac{1}{2}}}{\beta^\gamma + \beta^\nu} \kappa^{\gamma, \nu} \right] G_k g^2 \\
& + \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} G_j g_j g_k + \left[- \frac{\beta^\nu (\beta^\gamma \beta^\nu)^{\frac{1}{2}}}{(\beta^\gamma + \beta^\nu)^2} \right] \\
& \epsilon_s^{\gamma, \nu} g_s G_k + \left[- \frac{\beta^\nu (\beta^\gamma \beta^\nu)^{\frac{1}{2}}}{(\beta^\gamma + \beta^\nu)^2} \right] G_s g_s \epsilon_k^{\gamma, \nu} + \left(\frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right)^2 \epsilon_s^{\gamma, \nu} g_s \epsilon_k^{\gamma, \nu} \Bigg\} \quad (IV-14)
\end{aligned}$$

If this change of variable is made in the expression $f_f^{\gamma \nu} d^3 u d^3 u^\nu$, and the cross products of pressure tensors and heat flux vectors are neglected (the validity of dropping cross product terms is well established and has been discussed by Grad⁽¹⁾), then this may be written:⁽⁵⁾

$$\begin{aligned}
f_f^{\gamma \nu} d^3 u d^3 u^\nu &= \frac{n^\gamma n^\nu}{(2\pi)^3} e^{-\left[\frac{G^2}{2} + \frac{(\bar{g} - \bar{\epsilon}^{\gamma, \nu})^2}{2} \right]} \\
& \left\{ 1 + \sum_\beta P_{ij}^\beta (A_{ij}^\beta) + \sum_\beta Q_i^\beta (B_i^\beta) \right\} d^3 g d^3 G \quad (IV-15)
\end{aligned}$$

where:

$$\begin{aligned}
a. \quad A_{ij}^\beta &= a^{\beta 2} G_i^o G_j + g_i^o g_j + \epsilon_i^{\gamma, \nu o} \epsilon_j^{\gamma, \nu} - 2g_i^o \epsilon_j^{\gamma, \nu} + 2a^\beta (G_i g_j - G_i \epsilon_j^{\gamma, \nu}) \\
b. \quad C_i^\beta &= a^{\beta 2} \left[G^2 g_i + G_i G_j g_j - G_i^2 \epsilon_j^{\gamma, \nu} - 2G_i G_j \epsilon_j^{\gamma, \nu} - 5g_i + 5\epsilon_i^{\gamma, \nu} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[g^2 g_i - g^2 \epsilon_i^{\gamma, \nu} + \epsilon^{\gamma, \nu} g_i^2 - \epsilon^{\gamma, \nu} \epsilon_i^2 \epsilon^{\gamma, \nu} - 2 \epsilon_j^{\gamma, \nu} g_j g_i + 2 g_j \epsilon_j^{\gamma, \nu} \epsilon_i^{\gamma, \nu} - 5 g_i + 5 \epsilon_i^{\gamma, \nu} \right] \\
& + a \beta^3 \left[G^2 G_i - 5 G_i \right] + a \beta \left[g^2 G_i + 2 G_j g_j g_i - 2 g_j G_i \epsilon_j^{\gamma, \nu} - 2 G_j g_j \epsilon_i^{\gamma, \nu} - 2 G_j g_i \epsilon_j^{\gamma, \nu} \right. \\
& \left. + \epsilon^{\gamma, \nu} G_i + 2 G_j \epsilon_j^{\gamma, \nu} \epsilon_i^{\gamma, \nu} - 5 G_i \right] \tag{IV-16}
\end{aligned}$$

where:

$$a. \quad P_{ij}^{\beta} = \frac{P_{ij}^{\alpha\beta} \beta^2}{2p^{\beta}}$$

$$b. \quad Q_i^{\beta} = \frac{\beta^{\frac{3}{2}} Q_i^{\beta} \beta^3}{5\rho^{\beta}}$$

and

$$a. \quad a^{\gamma} = \left(\frac{\beta^{\gamma}}{\beta^{\nu}} \right)^{\frac{1}{2}} = -\frac{1}{a^{\nu}}$$

$$b. \quad b^{\gamma} = \left(\frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^{\frac{1}{2}}$$

$$c. \quad b^{\nu} = -\left(\frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^{\frac{1}{2}}$$

As may be seen, all the velocity space integrals involved are of the form:

$$\frac{1}{4\pi (2\pi)^{3/2}} \int e^{-\left[\frac{G^2}{2} + \frac{(\bar{g} - \bar{\epsilon}^{\gamma, \nu})^2}{2} \right]} G_{i_1} \dots G_{i_n} g^{\frac{n-5}{n-1}} g_{j_1} \dots g_{j_m} d^3 g d^3 G \tag{IV-17}$$

To facilitate evaluating these integrals, we make use of a special notation originated by Grad. ⁽²⁾

Define:

- a. $\{\delta\epsilon\}_{ijk} \equiv \delta_{ij}\epsilon_k + \delta_{ik}\epsilon_j + \delta_{jk}\epsilon_i$
- b. $\{\delta\epsilon\epsilon\}_{ijkl} \equiv \delta_{ij}\epsilon_k\epsilon_l + \delta_{ik}\epsilon_j\epsilon_l + \delta_{il}\epsilon_j\epsilon_k + \delta_{jk}\epsilon_i\epsilon_l + \delta_{jl}\epsilon_i\epsilon_k + \delta_{kl}\epsilon_i\epsilon_j$ (IV-18)
- c. $\{\delta\delta\}_{ijkl}, \{\delta\epsilon\epsilon\epsilon\}$ are defined similarly.

The $\bar{G}$ space integrals are straightforward, and have been tabulated by Grad.⁽¹⁾ These are:

- a. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} d^3G = 1$
- b. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} G_i G_j d^3G = \delta_{ij}$ (IV-19)
- c. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} G^2 d^3G = 3$
- d. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} G_i G_j G_k G_l d^3G = \{\delta\delta\}_{ijkl}$
- e. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} G^2 G_i G_j d^3G = 5\delta_{ij}$
- f. $\frac{1}{(2\pi)^{3/2}} \int e^{-\frac{G^2}{2}} G^4 d^3G = 15$

The $\bar{g}$ space integrals are more difficult and will be handled using a procedure devised by Everett.⁽⁵⁾

Let

$$\langle g_{i_1} \dots g_{i_n} \rangle \equiv \frac{1}{4\pi} \int e^{-\frac{(\bar{g} - \epsilon^{-\gamma, \nu})^2}{2}} g_{i_1} \dots g_{i_n} g^{\frac{n-5}{n-1}} d^3g \quad (IV-20)$$

and

$$I_k \equiv \frac{1}{4\pi} \int e^{-\frac{g^2}{2} + \bar{g} \cdot \epsilon^{-\gamma, \nu}} g_k g^{\frac{n-5}{n-1}} d^3g \quad (IV-21)$$

It is seen that

$$\langle g_{i_1} \dots g_{i_n} \rangle = e^{-\frac{\epsilon^{\gamma, \nu^2}}{2}} \frac{\partial^{m-1} I_{i_m}}{\partial \epsilon_{i_1}^{\gamma, \nu} \dots \partial \epsilon_{i_{m-1}}^{\gamma, \nu}} \quad (\text{IV-22})$$

Therefore, all the $\bar{g}$ space integrals are reduced to solving one integral, I_k . Now, let $x_k = g_k - \epsilon_k^{\gamma, \nu}$ in I_k . I_k becomes:

$$I_k = \frac{1}{4\pi} e^{\frac{\epsilon^{\gamma, \nu^2}}{2}} \int e^{-\frac{x^2}{2}} \left(x_k + \epsilon_k^{\gamma, \nu} \right) \left| \frac{x}{x + \epsilon^{\gamma, \nu}} \right|^{\frac{n-5}{n-1}} d^3 x \quad (\text{IV-23})$$

If a coordinate system is chosen in which $\bar{\epsilon}^{\gamma, \nu}$ is in the z direction, I_k can be written:

$$I_k = \frac{1}{4\pi} e^{\frac{\epsilon^{\gamma, \nu^2}}{2}} \frac{\epsilon_k^{\gamma, \nu}}{\epsilon^{\gamma, \nu}} \int e^{-\frac{x^2}{2}} \left(x \cos \theta + \epsilon^{\gamma, \nu} \right) x^2 \cdot \left(x^2 + 2x\epsilon^{\gamma, \nu} \cos \theta + \epsilon^{\gamma, \nu^2} \right)^{\frac{n-5}{2(n-1)}} \sin \theta d\theta d\phi dx \quad (\text{IV-24})$$

where (x, θ, ϕ) are spherical coordinates. Let $\mu = -\cos \theta$, then

$$I_k = \frac{1}{4\pi} e^{\frac{\epsilon^{\gamma, \nu^2}}{2}} \frac{\epsilon_k^{\gamma, \nu}}{\epsilon^{\gamma, \nu}} \int e^{-\frac{x^2}{2}} x^2 \left(-x\mu + \epsilon^{\gamma, \nu} \right) \cdot \left(x^2 - 2x\epsilon^{\gamma, \nu}\mu + \epsilon^{\gamma, \nu^2} \right)^{\frac{n-5}{2(n-1)}} d\mu dx d\phi \quad (\text{IV-25})$$

Integrating over ϕ from 0 to 2π , yields:

$$I_k = \frac{1}{2} \frac{\epsilon_k^{\gamma, \nu}}{\epsilon^{\gamma, \nu}} e^{\frac{\epsilon^{\gamma, \nu^2}}{2}} \int_{-1}^{+1} \int_0^\infty e^{-\frac{x^2}{2}} x^2 \left(-x\mu + \epsilon^{\gamma, \nu} \right) \cdot \left(x^2 - 2x\epsilon^{\gamma, \nu}\mu + \epsilon^{\gamma, \nu^2} \right)^{\frac{n-5}{2(n-1)}} d\mu dx \quad (\text{IV-26})$$

Before I_k can be further evaluated, the force function must be explicitly determined, that is, a value must be assigned to n .

Charged particles interact according to an inverse square force, $F = \frac{e^\gamma e^\nu}{r^2}$, and for this force law the integration over angle of scatter can be performed exactly provided a suitable $b_{\max}$ is chosen to prevent the well-known logarithmic divergence which arises due to the long range of the coulomb force. $b_{\max}$ is set equal to the Debye length, L_D , where

$$L_D = \left\{ \sum_i \frac{kT}{4\pi n_i e_i^2} \right\}^{1/2}$$

See Appendix A for a discussion of this. The results of this integration are:

$$a. A_1(2) = 2 \ell n \Lambda^{\gamma, \nu} = 2 \Omega^{(1)}_{\gamma, \nu}$$

$$b. A_2(2) = 4 \left(\ell n \Lambda^{\gamma, \nu} - \frac{1}{2} \right) = 4 \Omega^{(2)}_{\gamma, \nu} \quad (IV-27)$$

where

$$\Lambda^{\gamma, \nu} = \left\{ 1 + \left(\frac{m^\gamma m^\nu}{m^\gamma + m^\nu} \frac{L_D}{e^\gamma e^\nu} \right)^2 v_r^{\gamma, \nu^4} \right\}^{1/2} \approx \left(3 + \epsilon^{\gamma, \nu^2} \right) \left\{ \frac{m^\gamma m^\nu}{m^\gamma + m^\nu} \frac{\beta^\gamma + \beta^\nu}{\beta^\gamma \beta^\nu} \frac{L_D}{e^\gamma e^\nu} \right\}$$

γ and ν can refer to either ions or electrons.

For the inverse square force, the integrations over velocity space have been carried out by Everett.⁽⁵⁾ The results are summarized here.

It is found that

$$I_k(n=2) = \frac{\epsilon^{\gamma, \nu}}{\epsilon^{\gamma, \nu^3}} e^{\frac{\epsilon^{\gamma, \nu^2}}{2}} \int_0^{\epsilon^{\gamma, \nu}} \frac{x^2}{x^2} e^{-\frac{x^2}{2}} dx$$

and from this value of I_k the various $\langle g_{i_1}, g_{i_2}, \dots, g_{i_n} \rangle$ are determined. When the angular, $\bar{G}$ space, and $\bar{g}$ space integrations are combined the final results are:

$$a. \quad S^{\gamma, \nu} = \mathcal{J}_{(2)}^{\gamma, \nu} \left\{ \epsilon_k^{\gamma, \nu} F + \left(\frac{2}{\epsilon^{\gamma, \nu 2}} J - \frac{6}{\epsilon^{\gamma, \nu 2}} F \right) \right\}.$$

$$\sum_{\beta} P_{kj}^{\beta} \epsilon_j^{\gamma, \nu} + \left(-\frac{1}{\epsilon^{\gamma, \nu 2}} J - \frac{5}{\epsilon^{\gamma, \nu 4}} J + \frac{15}{\epsilon^{\gamma, \nu 4}} F \right).$$

$$\sum_{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} - J \sum_{\beta} Q_k^{\beta} + J.$$

$$\sum_{\beta} Q_i^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \left\{ \right.$$

(IV-28)

$$b. \quad S_k^{\gamma, \gamma} = 0$$

$$c. \quad R^{\gamma, \nu} = \mathcal{R}_{(2)}^{\gamma, \nu} \left\{ -2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(\frac{1}{3} \epsilon^{\gamma, \nu 2} F \right) + \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2 \kappa^{\gamma, \nu} \right) \right.$$

$$\left(\frac{1}{3} J + \frac{1}{3} \epsilon^{\gamma, \nu 2} F \right) + \left(-\frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right).$$

$$\begin{aligned} & \left(J + \epsilon^{\gamma, \nu 2} F \right) + \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(-\frac{1}{3} J - \frac{1}{\epsilon^{\gamma, \nu 2}} J + \frac{3}{\epsilon^{\gamma, \nu 2}} F \right) \right. \\ & \quad \left. + \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2 \kappa^{\gamma, \nu} \right) \left(-\frac{1}{3 \epsilon^{\gamma, \nu 2}} J + \frac{1}{\epsilon^{\gamma, \nu 2}} F \right) \right. \\ & \quad \left. + \left(\frac{-m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(-\frac{1}{\epsilon^{\gamma, \nu 2}} J + \frac{3}{\epsilon^{\gamma, \nu 2}} F \right) \right]. \end{aligned}$$

$$\sum_{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} + \left[2 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(\frac{2}{3(\epsilon^{\gamma, \nu})^2} J - \frac{2}{\epsilon^{\gamma, \nu 2}} F \right) \right].$$

$$\begin{aligned}
& \sum_{\beta} a^{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} + \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(-\frac{1}{3} J + \frac{\epsilon^{\gamma, \nu^2}}{3} J \right) \right. \\
& + \left. \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \frac{1}{3} J - \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} J \right] \sum_{\beta} Q_i^{\beta} \epsilon_i^{\gamma, \nu} \\
& + \left[-2 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} J \right] \sum_{\beta} a^{\beta} Q_i^{\beta} \epsilon_i^{\gamma, \nu} \Big\}
\end{aligned}$$

d. $R^{\gamma, \gamma} = 0$

$$\begin{aligned}
\text{e. } L_{kl}^{\circ \gamma, \nu} &= \mathcal{D}_{(2)}^{\gamma, \nu} \left\{ \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} F + \left(\frac{3m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \cdot \left(\frac{1}{\epsilon^{\gamma, \nu^2}} J + F - \frac{3}{\epsilon^{\gamma, \nu^2}} F \right) \right] \right. \\
& \epsilon_k^{\gamma, \nu} \epsilon_l^{\gamma, \nu} + \left[\left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \left(\frac{2}{\epsilon^{\gamma, \nu^2}} J + 2F - \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \right] \\
& \sum_{\beta} P_{kl}^{\beta} + \left[4 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} F \right] \sum_{\beta} a^{\beta} P_{kl}^{\beta} + \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(\frac{2}{\epsilon^{\gamma, \nu^2}} J - \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \right. \\
& + \left. \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \left(-\frac{20}{\epsilon^{\gamma, \nu^4}} J - \frac{12}{\epsilon^{\gamma, \nu^2}} F + \frac{60}{\epsilon^{\gamma, \nu^4}} F \right) \right] \\
& \sum_{\beta} \overline{P_{ks}^{\gamma, \nu} \epsilon_s^{\gamma, \nu} \epsilon_l^{\gamma, \nu}} + \left[2 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(\frac{2}{\epsilon^{\gamma, \nu^2}} J - \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \right] \\
& \sum_{\beta} a^{\beta} \overline{P_{ks}^{\gamma, \nu} \epsilon_s^{\gamma, \nu} \epsilon_l^{\gamma, \nu}} + \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(-\frac{1}{\epsilon^{\gamma, \nu^2}} J - \frac{5}{\epsilon^{\gamma, \nu^4}} J + \frac{15}{\epsilon^{\gamma, \nu^4}} F \right) \right. \\
& + \left. \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \left(\frac{2}{\epsilon^{\gamma, \nu^4}} J + \frac{35}{\epsilon^{\gamma, \nu^6}} J + \frac{15}{\epsilon^{\gamma, \nu^4}} F - \frac{105}{\epsilon^{\gamma, \nu^6}} F \right) \right]
\end{aligned}$$

$$\sum_{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \epsilon_l^{\gamma, \nu} + \left[2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} J + \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \right. \\ \left. \left(\frac{4}{\epsilon^{\gamma, \nu^2}} J - \frac{12}{\epsilon^{\gamma, \nu^2}} F \right) \right].$$

$$\sum_{\beta} \overline{Q_k^{\beta} \epsilon_l^{\gamma, \nu}} + \left[2 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(-J + \frac{4}{\epsilon^{\gamma, \nu^2}} J - \frac{12}{\epsilon^{\gamma, \nu^2}} F \right) \right].$$

$$\sum_{\beta} a^{\beta} \overline{Q_k^{\beta} \epsilon_l^{\gamma, \nu}} + \left[-2 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} J + \left(3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} + 2\kappa^{\gamma, \nu} \right) \right.$$

$$\left. \left(-\frac{2}{\epsilon^{\gamma, \nu^2}} J - \frac{10}{\epsilon^{\gamma, \nu^4}} J + \frac{30}{\epsilon^{\gamma, \nu^4}} F \right) \right] \sum_{\beta} Q_s^{\beta} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \epsilon_l^{\gamma, \nu} \\ + \left[2 \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(-\frac{2}{\epsilon^{\gamma, \nu^2}} J - \frac{10}{\epsilon^{\gamma, \nu^4}} J + \frac{30}{\epsilon^{\gamma, \nu^4}} F \right) \right].$$

$$\sum_{\beta} a^{\beta} Q_s^{\beta} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \epsilon_l^{\gamma, \nu} \left\{ \right.$$

$$f. L_{kl}^{\gamma, \gamma} = \mathcal{L}_{(2)}^{\gamma, \gamma} \left\{ \frac{4}{5} \frac{\Omega_{\gamma, \gamma}^{(2)}}{\Omega_{\gamma, \gamma}^{(1)}} P_{kl}^{\gamma} \right\}$$

$$g. M_k^{\gamma, \nu} = \mathcal{M}_{(2)}^{\gamma, \nu} \left\{ \left[\left(\frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^2 \frac{3 \epsilon^{\gamma, \nu^2}}{2} F - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \cdot \left(2\kappa^{\gamma, \nu} + 3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right. \right. \\ \left. \left(J - 2F + \epsilon^{\gamma, \nu^2} F \right) - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(\kappa^{\gamma, \nu} - \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(J + \epsilon^{\gamma, \nu^2} F \right) \right. \\ \left. \left. + \left(\frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \right)^2 + 2\kappa^{\gamma, \nu} \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(J - F + \epsilon^{\gamma, \nu^2} F \right) \right\} \right.$$

$$+ \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \frac{5}{2} F \left] \epsilon_k^{\gamma, \nu} + \left[\left(\frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right)^2 (J - 3F) + \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \left(\frac{5}{\epsilon^{\gamma, \nu^2}} J - \frac{15}{\epsilon^{\gamma, \nu^2}} F \right) \right.$$

$$- \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(- \frac{8}{\epsilon^{\gamma, \nu^2}} J - 4F + \frac{24}{\epsilon^{\gamma, \nu^2}} F \right)$$

$$+ \left(\frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 + 2\kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(- \frac{2}{\epsilon^{\gamma, \nu^2}} J - 2F + \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \left. \right]$$

$$\sum_\beta P_{kj}^{\prime\beta} \epsilon_j^{\gamma, \nu} + \left[- \frac{(\beta^\gamma \beta^\nu)^{1/2} \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \cdot (2J - 2F) + \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right.$$

$$\left. \left(\frac{4}{\epsilon^{\gamma, \nu^2}} J + 2F - \frac{12}{\epsilon^{\gamma, \nu^2}} F \right) + \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \left(\kappa^{\gamma, \nu} - \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) - 2F \right].$$

$$\sum_\beta a^\beta P_{ks}^{\prime\beta} \epsilon_s^{\gamma, \nu} + \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} 2F \sum_\beta a^{\beta^2} P_{ks}^{\prime\beta} \epsilon_s^{\gamma, \nu}$$

$$+ \left[\left(\frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right)^2 \left(- \frac{3}{2} J - \frac{11}{2\epsilon^{\gamma, \nu^2}} J + \frac{33}{2\epsilon^{\gamma, \nu^2}} F \right) + \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \cdot \right.$$

$$\left. \left(- \frac{5}{2\epsilon^{\gamma, \nu^2}} J - \frac{25}{2\epsilon^{\gamma, \nu^4}} J + \frac{75}{2\epsilon^{\gamma, \nu^4}} F \right) - \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \left(2\kappa^{\gamma, \nu} + \frac{3m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right.$$

$$\left. \left(\frac{1}{\epsilon^{\gamma, \nu^2}} J + \frac{20}{\epsilon^{\gamma, \nu^4}} J + \frac{9}{\epsilon^{\gamma, \nu^2}} F - \frac{60}{\epsilon^{\gamma, \nu^4}} F \right) - \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \left(\kappa^{\gamma, \nu} - \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right.$$

$$\left. \left(- \frac{1}{\epsilon^{\gamma, \nu^2}} J + \frac{3}{\epsilon^{\gamma, \nu^2}} F \right) + \left(- \frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right) + 2\kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right]$$

$$\begin{aligned}
& \left(\frac{5}{\epsilon^{\gamma, \nu^4}} J + \frac{3}{\epsilon^{\gamma, \nu^2}} F - \frac{15}{\epsilon^{\gamma, \nu^4}} F \right) \sum_{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \\
& + \left[- \frac{(\beta^{\gamma} \beta^{\nu})^{1/2} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} \left(\frac{4}{\epsilon^{\gamma, \nu^2}} J - \frac{12}{\epsilon^{\gamma, \nu^2}} F \right) + \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right. \\
& \left. \left(- \frac{10}{\epsilon^{\gamma, \nu^4}} J - \frac{6}{\epsilon^{\gamma, \nu^2}} F + \frac{30}{\epsilon^{\gamma, \nu^4}} F \right) \sum_{\beta} a^{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \right. \\
& \left. + \left[- \left(\frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^2 \frac{\epsilon^{\gamma, \nu^2}}{2} J - \frac{\beta^{\gamma} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} \frac{5}{2} J - \frac{\beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right. \right. \\
& \left. \left. (2J - 6F) + \left(\frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \right)^2 + 2\kappa^{\gamma, \nu} \cdot \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) (-2F) \right] \right. \\
& \left. \sum_{\beta} Q_k^{\beta} + \left[- \frac{(\beta^{\gamma} \beta^{\nu})^{1/2} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} \left(-\epsilon^{\gamma, \nu^2} J + 4J - 12F \right) + \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \right. \right. \\
& \left. \left(2\kappa^{\gamma, \nu} + 3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) (-J + 4F) + \frac{(\beta^{\gamma} \beta^{\nu})^{1/2}}{\beta^{\gamma} + \beta^{\nu}} \left(\kappa^{\gamma, \nu} - \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \right. \\
& \left. (-J - 2F) \right] \cdot \sum_{\beta} a^{\beta} Q_k^{\beta} + \frac{\beta^{\gamma} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} (2J + 9F) \\
& \sum_{\beta} a^{\beta^2} Q_k^{\beta} + \left[\left(\frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right)^2 \cdot \left(-J + \frac{3}{2} \epsilon^{\gamma, \nu^2} J \right) + \frac{\beta^{\gamma} \beta^{\nu}}{(\beta^{\gamma} + \beta^{\nu})^2} \frac{5}{2} J \right. \\
& \left. - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) \left(-J - \frac{6}{\epsilon^{\gamma, \nu^2}} J + \frac{18}{\epsilon^{\gamma, \nu^2}} F \right) - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \right. \\
& \left. \left(\kappa^{\gamma, \nu} - \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{\Omega_{\gamma, \nu}^{(2)}}{\Omega_{\gamma, \nu}^{(1)}} \right) J + \left(\frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \right)^2 \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + 2\kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega^{(2)}_{\gamma, \nu}}{\Omega^{(1)}_{\gamma, \nu}} \left(-\frac{2}{\epsilon^{\gamma, \nu^2}} J + \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \left] \sum_{\beta} Q_s^{\beta} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \right. \\
& + \left[-\frac{(\beta^\gamma \beta^\nu)^{1/2}}{(\beta^\gamma + \beta^\nu)^2} \beta^\nu \left(-5J - \frac{12}{\epsilon^{\gamma, \nu^2}} J + \frac{36}{\epsilon^{\gamma, \nu^2}} F \right) + \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \left(2\kappa^{\gamma, \nu} + 3 \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega^{(2)}_{\gamma, \nu}}{\Omega^{(1)}_{\gamma, \nu}} \right) \right. \\
& \left. \left(\frac{4}{\epsilon^{\gamma, \nu^2}} J - \frac{12}{\epsilon^{\gamma, \nu^2}} F \right) + \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \left(\kappa^{\gamma, \nu} - \frac{m^\nu}{m^\gamma + m^\nu} \frac{\Omega^{(2)}_{\gamma, \nu}}{\Omega^{(1)}_{\gamma, \nu}} \right) \left(-\frac{2}{\epsilon^{\gamma, \nu^2}} J + \frac{6}{\epsilon^{\gamma, \nu^2}} F \right) \right] \\
& \left. \sum_{\beta} a^{\beta} Q_s^{\beta} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu} + \left[\frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \left(\frac{9}{\epsilon^{\gamma, \nu^2}} J - \frac{27}{\epsilon^{\gamma, \nu^2}} F \right) \right] \sum_{\beta} a^{\beta^2} Q_s^{\beta} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \right\}
\end{aligned}$$

$$h. \quad M_k^{\gamma, \gamma} = \mathcal{M}_{(2)}^{\gamma, \gamma} \left\{ \frac{4}{3} \frac{\Omega^{(2)}_{\gamma, \gamma}}{\Omega^{(1)}_{\gamma, \gamma}} Q_k^{\gamma} \right\}$$

where:

$$a. \quad \mathcal{P}_{(2)}^{\gamma, \nu} = -\frac{\beta^\gamma \beta^\nu}{\beta^\gamma + \beta^\nu} 4(2\pi)^{1/2} n^\gamma n^\nu \frac{m^\gamma + m^\nu}{m^\gamma m^\nu} e^4 \Omega_{\gamma, \nu}^{(1)}$$

$$b. \quad \mathcal{R}_{(2)}^{\gamma, \nu} = -\left(\frac{\beta^\gamma \beta^\nu}{\beta^\gamma + \beta^\nu} \right)^{1/2} 6(2\pi)^{1/2} n^\gamma n^\nu \frac{m^\gamma + m^\nu}{m^\gamma m^\nu} e^4 \Omega_{\gamma, \nu}^{(1)}$$

$$c. \quad \mathcal{L}_{(2)}^{\gamma, \nu} = -\left(\frac{\beta^\gamma \beta^\nu}{\beta^\gamma + \beta^\nu} \right) 4(2\pi)^{1/2} n^\gamma n^\nu \frac{m^\gamma + m^\nu}{m^\gamma m^\nu} e^4 \Omega_{\gamma, \nu}^{(1)}$$

$$d. \quad \mathcal{M}_{(2)}^{\gamma, \nu} = -4(2\pi)^{1/2} n^\gamma n^\nu \frac{m^\gamma + m^\nu}{m^\gamma m^\nu} \Omega_{\gamma, \nu}^{(1)}$$

$$e. \quad J = e^{-\frac{\epsilon^{\gamma, \nu^2}}{2}}$$

$$f. \quad F = \frac{1}{\epsilon^{\gamma, \nu}_3} \int_0^{\epsilon^{\gamma, \nu}} x^2 e^{-\frac{x^2}{2}} dx \quad (IV-29)$$

In the above notation, the interpretation of the superscripts is specific; that is, if γ corresponds to electrons, then ν refers to ions and vice versa. Also, the assumption that collisions are elastic ensures that $N^\nu = 0$.

For all other interactions, it will be assumed that the particles obey a $\frac{K_{12}}{r^5}$ force law. For this case, the results of the angular integrations have been determined analytically by Chapman and Cowling.⁽³⁾ The results are:

$$\begin{aligned} a. \quad A_1(5) &= 0.422 \\ b. \quad A_2(5) &= 0.436 \end{aligned} \quad (IV-30)$$

In this case, the problem of logarithmic diversion does not arise and $b_{\max}$ is set equal to ∞ .

For $n = 5$, I_k becomes:

$$I_k = \frac{1}{2} \frac{\epsilon^{\gamma, \nu}_k}{\epsilon^{\gamma, \nu}} e^{-\frac{\epsilon^{\gamma, \nu}_k}{3}} \int_1^1 \int_0^\infty e^{-\frac{x^2}{2}} x^2 (-x\mu + \epsilon^{\gamma, \nu}) dx d\mu = \left(\frac{\pi}{2}\right)^{1/2} \epsilon^{\gamma, \nu} e^{-\frac{\epsilon^{\gamma, \nu}_k}{2}} \quad (IV-31)$$

For this value of I_k the following $\bar{g}$ space integrals were obtained:

$$\begin{aligned} a. \quad \langle g_k \rangle &= \left(\frac{\pi}{2}\right)^{1/2} \epsilon^{\gamma, \nu}_k \\ b. \quad \langle g_i g_j \rangle &= \left(\frac{\pi}{2}\right)^{1/2} \left[\delta_{ij} + \epsilon^{\gamma, \nu}_i \epsilon^{\gamma, \nu}_j \right] \\ c. \quad \langle g_i g_j g_k \rangle &= \left(\frac{\pi}{2}\right)^{1/2} \left[\left\{ \delta \epsilon^{\gamma, \nu} \right\}_{ijk} + \epsilon^{\gamma, \nu}_i \epsilon^{\gamma, \nu}_j \epsilon^{\gamma, \nu}_k \right] \\ d. \quad \langle g^2 g_i \rangle &= \left(\frac{\pi}{2}\right)^{1/2} \left[5 \epsilon^{\gamma, \nu}_i + \epsilon^{\gamma, \nu} \epsilon^{\gamma, \nu}_i \right] \\ e. \quad \langle g_i g_j g_k g_l \rangle &= \left(\frac{\pi}{2}\right)^{1/2} \left[\left\{ \delta \epsilon^{\gamma, \nu} \epsilon^{\gamma, \nu} \right\}_{ijkl} + \left\{ \delta \delta \right\}_{ijkl} + \epsilon^{\gamma, \nu}_i \epsilon^{\gamma, \nu}_j \epsilon^{\gamma, \nu}_k \epsilon^{\gamma, \nu}_l \right] \end{aligned} \quad (IV-32)$$

$$f. \langle g^2 g_i g_j \rangle = \left(\frac{\pi}{2}\right)^{1/2} \left[\left(5 + \epsilon^{\gamma, \nu^2}\right) \delta_{ij} + \left(7 + \epsilon^{\gamma, \nu^2}\right) \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \right]$$

$$g. \langle g_i g_j g_k g_\ell g_m \rangle = \left(\frac{\pi}{2}\right)^{1/2} \left[\left\{ \delta \epsilon^{\gamma, \nu} \epsilon^{\gamma, \nu} \epsilon^{\gamma, \nu} \epsilon^{\gamma, \nu} \right\}_{ijklm} + \left\{ \delta \delta \epsilon^{\gamma, \nu} \right\}_{ijklm} \right. \\ \left. + \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \epsilon_\ell^{\gamma, \nu} \epsilon_m^{\gamma, \nu} \right]$$

$$h. \langle g^2 g_i g_j g_k \rangle = \left(\frac{\pi}{2}\right)^{1/2} \left[\left(7 + \epsilon^{\gamma, \nu^2}\right) \left\{ \delta \epsilon^{\gamma, \nu} \right\}_{ijk} + \left(9 + \epsilon^{\gamma, \nu^2}\right) \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \right]$$

$$i. \langle g^4 g_i \rangle = \left(\frac{\pi}{2}\right)^{1/2} \left[\left(35 + 14 \epsilon^{\gamma, \nu^2} + \epsilon^{\gamma, \nu^4}\right) \epsilon_i^{\gamma, \nu} \right]$$

$$j. \langle g^4 g_i g_j \rangle = \left(\frac{\pi}{2}\right)^{1/2} \left[\left(35 + 14 \epsilon^{\gamma, \nu^2} + \epsilon^{\gamma, \nu^4}\right) \delta_{ij} + \left(63 + 18 \epsilon^{\gamma, \nu^2} + \epsilon^{\gamma, \nu^4}\right) \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \right]$$

The results for the collision transfer terms are:

$$a. S_k^{\gamma, \nu} = \mathcal{P}_{(5)}^{\gamma, \nu} \left\{ \epsilon_k^{\gamma, \nu} - \sum_{\beta} P_{ij}^{\beta} \epsilon_i^{\gamma, \nu} \epsilon_j^{\gamma, \nu} \epsilon_k^{\gamma, \nu} \right\}$$

$$b. S_k^{\gamma, \gamma} = 0$$

$$c. R^{\gamma, \nu} = \mathcal{R}_{(5)}^{\gamma, \nu} \left\{ \epsilon^{\gamma, \nu} \left(3 + \epsilon^{\gamma, \nu^2}\right) - \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \epsilon^{\gamma, \nu^2} \right\} \quad (1V-33)$$

$$d. R^{\gamma, \gamma} = 0$$

$$e. L_{kl}^{\gamma, \nu} = \mathcal{L}_{(5)}^{\gamma, \nu} \left\{ - \frac{2\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \epsilon_k^{\gamma, \nu} \epsilon_\ell^{\gamma, \nu} + \left[\frac{3}{2} \frac{m^{\nu}}{m^{\gamma} + m^{\nu}} \frac{A_{2(5)}}{A_{1(5)}} + 2\kappa^{\gamma, \nu} \right] \right\}$$

$$\epsilon_k^{\gamma, \nu} \epsilon_\ell^{\gamma, \nu} - 4 \frac{\beta^{\nu}}{\beta^{\gamma} + \beta^{\nu}} \sum_{\beta} \overline{P_{ls}^{\beta}} \epsilon_s^{\gamma, \nu} \epsilon_k^{\gamma, \nu}$$

$$+ \left[3 \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} + 4\kappa^{\gamma, \nu} \right] \sum_{\beta} P_{kl}^{\prime\beta} + 4 \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \sum_{\beta} a^\beta P_{kl}^{\prime\beta} \left\{ \right.$$

$$f. \quad L_{kl}^{\circ \gamma, \gamma} = \mathcal{L}_{(5)}^{\gamma, \gamma} \left\{ 3 \frac{A_{2(5)}}{A_{1(5)}} \overline{P_{kl}^{\prime\gamma}} \right\}$$

$$g. \quad M_k^{\gamma, \nu} = \mathcal{M}_{(5)}^{\gamma, \nu} \left[\frac{3}{2} \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} + 15\kappa^{\gamma, \nu^2} + \frac{5}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 + 5\kappa^{\gamma, \nu} \cdot \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right.$$

$$\left. - 5\kappa^{\gamma, \nu} \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right] \epsilon_k^{\gamma, \nu} + \left[\frac{3}{2} \kappa^{\gamma, \nu^2} + \frac{1}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 + \kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right.$$

$$\left. - 3\kappa^{\gamma, \nu} \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} - \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} + \frac{3}{2} \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \right] \cdot \epsilon_k^{\gamma, \nu^2} \epsilon_k^{\gamma, \nu}$$

$$+ 2 \frac{\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \sum_{\beta} a^{\beta^2} P_{ks}^{\prime\beta} \epsilon_s^{\gamma, \nu} + \left[4 \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \kappa^{\gamma, \nu} + 2 \frac{m^\nu}{m^\gamma + m^\nu} \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right.$$

$$\left. + 2\kappa^{\gamma, \nu} \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} - 2 \frac{(\beta^\gamma \beta^\nu)^{1/2} \beta^\nu}{(\beta^\gamma + \beta^\nu)^2} \right] \sum_{\beta} a^\beta P_{ks}^{\prime\beta} \epsilon_s^{\gamma, \nu} - 2 \frac{(\beta^\gamma \beta^\nu)^{1/2} \beta^\nu}{\beta^\gamma + \beta^\nu} \cdot$$

$$\sum_{\beta} a^\beta P_{sk}^{\prime\beta} \epsilon_s^{\gamma, \nu} + \left[-\frac{9}{2} \kappa^{\gamma, \nu^2} - \frac{3}{2} \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 - 3\kappa^{\gamma, \nu} \cdot \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right.$$

$$\left. - 4\kappa^{\gamma, \nu} \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} - 3 \frac{\beta^\nu}{\beta^\gamma + \beta^\nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right] \cdot \sum_{\beta} P_{kj}^{\prime\beta} \epsilon_j^{\gamma, \nu} + \frac{15\beta^\gamma \beta^\nu}{(\beta^\gamma + \beta^\nu)^2}$$

$$\sum_{\beta} a^{\beta^2} Q_k^{\prime\beta} + \left[15\kappa^{\gamma, \nu^2} + 5 \left(\frac{m^\nu}{m^\gamma + m^\nu} \right)^2 + 10\kappa^{\gamma, \nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right]$$

$$\sum_{\beta} Q_k^{\prime\beta} + \left[30 \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \kappa^{\gamma, \nu} + 10 \frac{(\beta^\gamma \beta^\nu)^{1/2}}{\beta^\gamma + \beta^\nu} \frac{m^\nu}{m^\gamma + m^\nu} \frac{A_{2(5)}}{A_{1(5)}} \right] \cdot \sum_{\beta} a^\beta Q_k^{\prime\beta}$$

$$h. \quad M_k^{\gamma, \gamma} = \mathcal{M}_{(5)}^{\gamma, \gamma} \left\{ 5 \frac{A_2(5)}{A_1(5)} Q_k^{\gamma} \right\}$$

where

$$a. \quad \mathcal{P}_{(5)}^{\gamma, \nu} = - \left(\frac{\beta^\gamma + \beta^\nu}{\beta^\gamma \beta^\nu} \right)^{1/2} 2\pi n^\gamma n^\nu \left(\frac{m^\gamma m^\nu}{m^\gamma + m^\nu} K_{12} \right)^{1/2} A_1(5)$$

$$b. \quad \mathcal{R}_{(5)}^{\gamma, \nu} = - \frac{\beta^\gamma + \beta^\nu}{\beta^\gamma \beta^\nu} 2\pi n^\gamma n^\nu \left(\frac{m^\gamma m^\nu}{m^\gamma + m^\nu} K_{12} \right)^{1/2} A_1(5)$$

$$c. \quad \mathcal{L}_{(5)}^{\gamma, \nu} = - \frac{\beta^\gamma + \beta^\nu}{\beta^\gamma \beta^\nu} 2\pi n^\gamma n^\nu \left(\frac{m^\gamma m^\nu}{m^\gamma + m^\nu} K_{12} \right)^{1/2} A_1(5)$$

$$d. \quad \mathcal{M}_{(5)}^{\gamma, \nu} = - \left(\frac{\beta^\gamma + \beta^\nu}{\beta^\gamma \beta^\nu} \right)^{3/2} 2\pi n^\gamma n^\nu \left(\frac{m^\gamma m^\nu}{m^\gamma + m^\nu} K_{12} \right)^{1/2} A_1(5)$$

Here as before, γ and ν must refer to different types of particles.

V. REDUCED COLLISION TERMS

At this point, it is convenient to reduce the collision transfer terms to a more manageable form. For a system which at most approaches full ionization, it is expected that $\epsilon^{\gamma, \nu}$ will be less than 1. Therefore, the transfer terms will be written keeping only terms of order less than ϵ^{γ, ν^4} , except for ion and neutral heat flux where it is sufficient to keep terms of first order in $\epsilon^{\gamma, \nu}$ only. Also, for simplicity assume that only one species of neutral particles exists and that at most single ionization occurs. Use will be made of the large mass difference which exists in most systems of interest. That is:

$$\begin{aligned} \text{a. } m^i &\approx m^N \gg m^e \\ \text{b. } \beta^i &\gg \beta^e \\ \text{c. } \beta^N &\gg \beta^e \end{aligned} \tag{V-1}$$

Also, it is assumed that the difference in temperature between ions and neutrals is small compared to the sum of the absolute temperature of both. This assumption, combined with the assumption on the mass difference, ensures that:

$$\begin{aligned} \text{a. } \kappa^{e, i} &\approx \kappa^{e, N} \ll 1 \\ \text{b. } \kappa^{i, N} &\ll 1 \end{aligned} \tag{V-2}$$

It is further assumed that either $w_k^e \gg w_k^i, w_k^N$ or that $w_k^i \approx w_k^N$ so that $\epsilon^{e, i} \approx \epsilon^{e, N} \equiv \epsilon$. That is, it is postulated that the difference between the flow velocities of electrons and neutrals divided by the square root of the sum of the squares of the thermal velocities of electrons and neutrals approximately equals the difference between the flow velocities of electrons and ions divided by the square root of the sum of the squares of the thermal velocities of electrons and ions.

Further, for the purpose of reducing the collision terms, order of magnitude sizes are postulated for $Q_k^{e, i}$ and $P_{ij}^{e, i}$. That is, it is postulated that:

$$\begin{aligned} \text{a. } Q_k^{e, i} &\approx \epsilon_k \\ \text{b. } P_{ij}^{e, i} &\approx \epsilon_i \epsilon_j \end{aligned} \tag{V-3}$$

The validity of this postulate is verified by inspecting the final expressions which are derived for $Q_k^{'e}$ and $P_{ij}^{'e}$, where it is seen that to lowest order, $Q_k^{'e} \approx \epsilon_k$ and $P_{ij}^{'e} \approx \epsilon_i \epsilon_j$.

Using these assumptions and postulates the reduced collision terms may be written:

A. Momentum transfer

$$1. S_k^{e,i} = - \frac{\rho^e}{\beta^e \lambda^e} \left\{ \left(\frac{1}{3} - \frac{\epsilon^2}{10} \right) \epsilon_k - \frac{1}{5} \frac{P_{kj}^{oe}}{p^e} \epsilon_j + \left(-1 + \frac{\epsilon^2}{2} \right) \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} + \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} \epsilon_k \epsilon_j \right\}$$

$$2. S_k^{e,e} = 0$$

$$3. S_k^{e,N} = \mathcal{P}_{(5)}^{e,N} \epsilon_k \quad (V-4)$$

$$4. S_k^{i,e} = \frac{\rho^e}{\beta^e \lambda^e} \left\{ \left(\frac{1}{3} - \frac{\epsilon^2}{10} \right) \epsilon_k - \frac{1}{5} \frac{P_{kj}^{oe}}{p^e} \epsilon_j + \left(-1 + \frac{\epsilon^2}{2} \right) \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} + \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} \epsilon_k \epsilon_j \right\}$$

$$5. S_k^{i,i} = 0$$

$$6. S_k^{i,N} = \mathcal{P}_{(5)}^{i,N} \epsilon_k$$

$$7. S_k^{N,e} = - \mathcal{P}_{(5)}^{e,N} \epsilon_k$$

$$8. S_k^{N,i} = - \mathcal{P}_{(5)}^{i,N} \epsilon_k$$

$$9. S_k^{N,N} = 0$$

B. Energy transfer

$$1. R^{e,i} = - \frac{\rho^e}{\beta^e{}^{3/2} \lambda^e} \left\{ \kappa^{e,i} \left(1 - \frac{\epsilon^2}{6} + \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} \epsilon_k \right) - \frac{\epsilon^2}{3} + \frac{\beta^e{}^{3/2} Q_k^e}{5\rho^e} \epsilon_k \right\}$$

$$2. R^{e,e} = 0$$

$$3. R^{e, N} = \mathcal{R}_{(5)}^{e, N} \left\{ \kappa^{e, N} (3 + \epsilon^2) - \epsilon^2 \right\}$$

$$4. R^{i, e} = \frac{\rho^e}{\beta^e \lambda^e} \left\{ \kappa^{e, i} \left(1 - \frac{\epsilon^2}{6} + \frac{\beta^e \epsilon_k^e}{5 \rho^e} \right) + \frac{\beta^e}{\beta^N} \frac{\epsilon^2}{3} + 4 \frac{\beta^e \epsilon_k^e}{5 \beta^i \rho^e} \epsilon_k \right\}$$

$$5. R^{ii} = 0$$

$$6. R^{i, N} = \mathcal{R}_{(5)}^{i, N} \left\{ \kappa^{i, N} (3 + \epsilon^{i, N^2}) - \frac{\beta^N}{\beta^i + \beta^N} \epsilon^{i, N^2} \right\}$$

$$7. R^{N, e} = -\mathcal{R}_{(5)}^{e, N} \left\{ \kappa^{e, N} (3 + \epsilon^2) + \frac{\beta^e}{\beta^N} \epsilon^2 \right\}$$

$$8. R^{n, i} = -\mathcal{R}_{(5)}^{N, i} \left\{ \kappa^{i, N} (3 + \epsilon^{i, N^2}) + \frac{\beta^i}{\beta^i + \beta^N} \epsilon^{i, N^2} \right\}$$

$$9. R^{N, N} = 0$$

C. Traceless Stress Transfer

$$1. L_{kl}^{oc, i} = - \frac{\rho^e}{\beta^e \lambda^e} \left\{ \left(-\frac{2}{3} + \frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \epsilon_k^o \epsilon_l + \frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \frac{P_{kl}^{oe}}{p^e} + \left(2 - \frac{12}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \frac{\beta^e \epsilon_k^e}{5 \rho^e} \epsilon_l \right\}$$

$$2. L_{kl}^{oe, e} = - \frac{\rho^e}{\beta^e \lambda^e} \left\{ \frac{(2)^{1/2}}{5} \frac{n^e}{n^i} \frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}} \frac{P_{kl}^{oe}}{p^e} \right\}$$

$$3. L_{kl}^{oe, N} = \mathcal{L}_{(5)}^{e, N} \left\{ \left(-2 + \frac{3}{2} \frac{A_2(5)}{A_1(5)} \right) \epsilon_k^o \epsilon_l + \frac{3}{2} \frac{A_2(5)}{A_1(5)} \frac{P_{kl}^{oe}}{p^e} \right\}$$

$$\begin{aligned}
4. \quad L_{kl}^{oi,e} &= - \frac{\rho^e}{\beta^e \lambda^e} \left\{ \left(- \frac{2}{3} \frac{\beta^e}{\beta^i} + \frac{2}{5} \frac{m^e}{m^i} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} - \frac{4}{5} \kappa^{e,i} \right) \right. \\
&\quad \left. \epsilon_k^o \epsilon_\ell + \frac{2}{15} \left(3 \frac{m^e}{m^i} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} - 2 \kappa^{e,i} - 5 \frac{\beta^e}{\beta^i} \right) \frac{P_{kl}^{oe}}{p^e} \right. \\
&\quad \left. + \frac{4}{25} \left(3 \frac{m^e}{m^i} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} - 2 \kappa^{e,i} - 7 \frac{\beta^e}{\beta^i} \frac{\beta^{e \frac{3}{2}} Q_k^e}{\rho^e} \epsilon_\ell \right) \right\} \\
5. \quad L_{kl}^{oi,i} &= - \frac{\rho^e}{\beta^e \lambda^e} \left\{ \left(\frac{\beta^e}{\beta^i} \right)^{1/2} \frac{(2)^{1/2}}{5} \frac{n^i}{n^e} \frac{m^e}{m^i} \frac{\Omega_{ii}^{(2)}}{\Omega_{ei}^{(1)}} \frac{P_{kl}^{oi}}{p^i} \right\} \\
6. \quad L_{kl}^{oi,N} &= \mathcal{L}_{(5)}^{i,N} \left\{ - 2 \frac{\beta^N}{\beta^i + \beta^N} \epsilon_k^{i,No} \epsilon_\ell^{i,N} + \frac{3}{4} \frac{A_2(5)}{A_1(5)} \epsilon_k^{i,No} \epsilon_\ell^{i,N} \right. \\
&\quad \left. + \frac{3}{4} \frac{A_2(5)}{A_1(5)} \left[\frac{\beta^N}{\beta^i + \beta^N} \frac{P_{kl}^{oi}}{p^i} + \frac{\beta^i}{\beta^i + \beta^N} \frac{P_{kl}^{oN}}{p^N} \right] \right\} \\
7. \quad L_{kl}^{oN,e} &= \mathcal{L}_{(5)}^{e,N} \left\{ - 2 \frac{\beta^e}{\beta^N} \epsilon_k^o \epsilon_\ell + \frac{3}{2} \frac{A_2(5)}{A_1(5)} \frac{m^e}{m^i} \epsilon_k^o \epsilon_\ell + \frac{3}{2} \frac{m^e}{m^i} \frac{A_2(5)}{A_1(5)} \frac{P_{kl}^{oe}}{p^e} \right\} \\
8. \quad L_{kl}^{oN,i} &= \mathcal{L}_{(5)}^{i,N} \left\{ - 2 \frac{\beta^i}{\beta^i + \beta^N} \epsilon_k^{i,No} \epsilon_\ell^{i,N} + \frac{3}{4} \frac{A_2(5)}{A_1(5)} \epsilon_k^{i,No} \epsilon_\ell^{i,N} \right. \\
&\quad \left. + \frac{3}{4} \frac{A_2(5)}{A_1(5)} \left[\frac{\beta^i}{\beta^i + \beta^N} \frac{P_{kl}^{oi}}{p^i} + \frac{\beta^N}{\beta^i + \beta^N} \frac{P_{kl}^{oN}}{p^N} \right] \right\} \\
9. \quad L_{kl}^{oN,N} &= \mathcal{L}_{(5)}^{N,N} \left\{ \frac{3}{4} \frac{A_2(5)}{A_1(5)} \frac{P_{kl}^{oN}}{p^N} \right\}
\end{aligned}$$

D. Heat Flux Transfer

$$1. \quad M_k^{e,i} = \frac{-\rho^e}{\beta^e \lambda^e} \left\{ \left[\frac{1}{3} + \left(\frac{13}{30} - \frac{4}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \epsilon^2 \right] \epsilon_k + \left(-\frac{4}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} - \frac{2}{15} \right) \frac{p_{kj}^{oe}}{2\rho^e} \epsilon_j \right. \\ \left. + \left[-\frac{1}{3} + \left(-\frac{2}{5} + \frac{6}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \epsilon^2 \right] \frac{\rho^e \beta^e Q_k^e}{5\rho^e} + \left(-\frac{4}{5} + \frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \frac{\beta^e \beta^e Q_s^e}{5\rho^e} \epsilon_s \epsilon_k \right\}$$

$$2. \quad M_k^{e,e} = -\frac{\rho^e}{\beta^e \lambda^e} \left\{ \frac{2(2)^{1/2}}{15} \frac{n^e}{n^i} \frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}} \frac{\beta^e \beta^e Q_k^e}{\rho^e} \right\}$$

$$3. \quad M_k^{e,N} = \mathcal{M}_{(5)}^{e,N} \left\{ \frac{5}{2} \epsilon_k + \left(2 - \frac{A_2(5)}{A_1(5)} \right) \epsilon^2 \epsilon_k + \left(-\frac{3}{2} - 3 \frac{A_2(5)}{A_1(5)} \right) \frac{p_{kj}^{oe}}{2\rho^e} + \frac{\beta^e \beta^e Q_k^e}{\rho^e} \right\}$$

$$4. \quad M_k^{i,e} = -\frac{\rho^e}{\beta^e \lambda^e} \left\{ \frac{\beta^e \beta^e Q_k^i}{\rho^i} \right\}$$

$$5. \quad M_k^{i,i} = -\frac{\rho^e}{\beta^e \lambda^e} \left\{ \frac{2(2)^{1/2}}{15} \frac{n^i}{n^e} \frac{m^e}{m^i} \frac{\Omega_{ii}^{(2)}}{\Omega_{ie}^{(1)}} \frac{\beta^i \beta^i Q_k^i}{\rho^i} \right\}$$

$$6. \quad M_k^{i,N} = \mathcal{M}_{(5)}^{i,N} \left\{ \left(\frac{3}{2} \frac{\beta^i \beta^N}{(\beta^i + \beta^N)^2} + \frac{5}{8} \right) \epsilon_k^{i,N} \right. \\ \left. + \frac{A_2(5)}{A_1(5)} \frac{(\beta^i \beta^N)^{1/2}}{\beta^i + \beta^N} \left[\left(\frac{\beta^i}{\beta^N} \right)^{1/2} \left(\frac{\beta^N}{\beta^i + \beta^N} \right)^{3/2} \frac{Q_k^i \beta^i^{3/2}}{5\rho^i} \right. \right. \\ \left. \left. + \left(\frac{\beta^N}{\beta^i} \right)^{1/2} \left(\frac{\beta^i}{\beta^i + \beta^N} \right)^{3/2} \frac{Q_k^N \beta^N^{3/2}}{5\rho^N} \right] \right\}$$

$$7. \quad M_k^{N, e} = \mathcal{M}_{(5)}^{e, N} \left\{ 3 \frac{\beta^e{}^{3/2} Q_k^N}{\rho^N} \right\}$$

$$8. \quad M_k^{N, i} = \mathcal{M}_{(5)}^{i, N} \left\{ - \left(\frac{3}{2} \frac{\beta^i \beta^N}{(\beta^i + \beta^N)^2} + \frac{5}{8} \right) \epsilon_k^{i, N} \right.$$

$$+ \frac{A_2(5)}{A_1(5)} \frac{(\beta^i \beta^N)^{1/2}}{\beta^i + \beta^N} \left[\left(\frac{\beta^N}{\beta^i} \right)^{1/2} \left(\frac{\beta^i}{\beta^i + \beta^N} \right)^{3/2} \frac{\beta^{N^{3/2}} Q_k^N}{5 \rho^N} \right. \\ \left. + \left(\frac{\beta^i}{\beta^N} \right)^{1/2} \left(\frac{\beta^N}{\beta^i + \beta^N} \right)^{3/2} \frac{\beta^{i^{3/2}} Q_k^i}{5 \rho^i} \right] \left. \right\}$$

$$9. \quad M_k^{N, N} = \mathcal{M}_{(5)}^{N, N} \left\{ \frac{1}{2(2)^{1/2}} \frac{A_2(5)}{A_1(5)} \frac{\beta^{N^{3/2}} Q_k^N}{\rho^N} \right\}$$

In the above relations, the electron mean free path with respect to ions, λ^e , has been introduced. λ^e is defined:

$$\lambda^e = \frac{(kT^e)^2}{4(2\pi)^{1/2} n_i^4 \Omega_{ei}^{(1)}} \quad (V-5)$$

VI. TRANSFER EQUATIONS

To complete the transfer equations it is necessary to solve for $\langle u_i u_r u_s \rangle$ and $\langle u^2 u_r u_s \rangle$ in addition to having expressions for the collision terms. The values for these expressions are obtained using the approximate form of the distribution function.

$$a. \langle u_i u_r u_s \rangle = \frac{2}{5} \frac{Q_k}{\rho} \left\{ \delta_{kr} \delta_{is} + \delta_{ki} \delta_{rs} + \delta_{ks} \delta_{ri} \right\} \quad (VI-1)$$

$$b. \langle u^2 u_r u_s \rangle = \frac{5}{\beta^2} \delta_{rs} + \frac{7}{\beta \rho} P_{rs}^0$$

Combining the above expressions, the reduced collision transfer expressions and Equations II-6 and II-7, gives the desired equations. The transfer equations for charged particles are:

$$a. \frac{D_\gamma n^\gamma}{Dt} + n^\gamma \frac{\partial w_i^\gamma}{\partial r_i} = 0 \quad (VI-2)$$

$$b. \frac{D_\gamma w_i^\gamma}{Dt} = \frac{e^\gamma}{m^\gamma} \left(E_i + \frac{1}{c} \epsilon_{ijk} w_j^\gamma H_k \right) - \frac{1}{\rho^\gamma} \frac{\partial P_{ij}^\gamma}{\partial r_j} + \frac{1}{\rho^\gamma} \left[S_i^{\gamma,e} + S_i^{\gamma,i} + S_i^{\gamma,N} \right]$$

$$c. \frac{D_\gamma T^\gamma}{Dt} = \frac{2}{3n^\gamma} \left\{ - \frac{\partial Q_i^\gamma}{\partial r_i} - P_{ij}^\gamma \frac{\partial w_j^\gamma}{\partial r_i} + R^{\gamma,e} + R^{\gamma,i} + R^{\gamma,N} \right\}$$

$$d. \frac{D_\gamma P_{rs}^{\circ\gamma}}{Dt} = - \frac{4}{5} \frac{\partial Q_s^\gamma}{\partial r_r} + 2 \frac{e^\gamma}{m^\gamma} \epsilon_{rjk} P_{js}^{\circ\gamma} H_k - P_{rs}^{\circ\gamma} \frac{\partial w_i^\gamma}{\partial r_i} - 2 P_{jr}^{\circ\gamma} \frac{\partial w_s^\gamma}{\partial r_j} \\ + L_{rs}^{\circ\gamma,e} + L_{rs}^{\circ\gamma,i} + L_{rs}^{\circ\gamma,N}$$

$$e. \frac{D_\gamma Q_r^\gamma}{Dt} = - \frac{7}{5} Q_r^\gamma \frac{\partial w_i^\gamma}{\partial r_i} - Q_s^\gamma \frac{\partial w_r^\gamma}{\partial r_s} - \frac{4}{5} Q_s^\gamma \frac{\partial w_s^\gamma}{\partial r_r} + \frac{P_{ir}^{\circ\gamma}}{\rho^\gamma} \frac{\partial P_{ij}^\gamma}{\partial r_j}$$

$$\begin{aligned}
& - \frac{kT^\gamma}{m^\gamma} \frac{\partial P_{rs}^{o\gamma}}{\partial r_s} - \frac{7}{2} \frac{k}{m^\gamma} P_{rs}^{o\gamma} \frac{\partial T^\gamma}{\partial r_s} + \frac{e^\gamma}{m^\gamma c} \epsilon_{rjk} Q_j^\gamma H_k - \frac{5}{2} \frac{kp^\gamma}{m^\gamma} \frac{\partial T^\gamma}{\partial r_r} \\
& - \left(\frac{5}{2} p^\gamma \delta_{ir} + P_{ir}^{o\gamma} \right) \cdot \frac{1}{\rho^\gamma} \left(S_i^{\gamma,e} + S_i^{\gamma,i} + S_i^{\gamma,N} \right) + \left(M_r^{\gamma,e} + M_r^{\gamma,i} + M_r^{\gamma,N} \right)
\end{aligned}$$

In the above equations γ can refer to either ions or electrons.

For the neutral specie, the transfer equations become:

$$a. \quad \frac{D_N n^N}{Dt} + n^N \frac{\partial w_i^N}{\partial r_i} = 0 \quad (VI-3)$$

$$b. \quad \frac{D_N w_i^N}{Dt} = - \frac{1}{\rho^N} \frac{\partial P_{ij}^N}{\partial r_j} + a_i + \frac{1}{\rho^N} \left[S_i^{N,e} + S_i^{N,i} + S_i^{N,N} \right]$$

$$c. \quad \frac{D_N T^N}{Dt} = \frac{2}{3n^N} \left\{ - \frac{\partial Q_i^N}{\partial r_i} - P_{ij}^N \frac{\partial w_j^N}{\partial r_i} + R^{N,e} + R^{N,i} + R^{N,N} \right\}$$

$$d. \quad \frac{D_N P_{rs}^{oN}}{Dt} = - \frac{4}{5} \frac{\partial Q_s^N}{\partial r_r} - P_{rs}^{oN} \frac{\partial w_i^N}{\partial r_i} - 2P_{jr}^N \frac{\partial w_s^N}{\partial r_j} + L_{rs}^{oN,e} + L_{rs}^{oN,i} + L_{rs}^{oN,N}$$

$$e. \quad \frac{D_N Q_r^N}{Dt} = - \frac{7}{5} Q_r^N \frac{\partial w_i^N}{\partial r_i} - Q_j^N \frac{\partial w_r^N}{\partial r_j} - \frac{4}{5} Q_s^N \frac{\partial w_s^N}{\partial r_r} + \frac{P_{ir}^{oN}}{\rho^N} \frac{\partial P_{ij}^N}{\partial r_j} - \frac{kT^N}{m^N} \frac{\partial P_{rs}^{oN}}{\partial r_s}$$

$$- \frac{7}{2} \frac{k}{m^N} P_{rs}^{oN} \frac{\partial T^N}{\partial r_s} - \frac{5}{2} \frac{kP^N}{m^N} \frac{\partial T^N}{\partial r_r} - \left(\frac{5}{2} p^N \delta_{ir} + P_{ir}^{oN} \right)$$

$$\frac{1}{\rho^N} \left(S_i^{N,e} + S_i^{N,i} + S_i^{N,N} \right) + \left(M_r^{N,e} + M_r^{N,i} + M_r^{N,N} \right)$$

Equations VI-2, VI-3 together with expressions (V-4) for the collision transfer terms constitute a closed set of transfer equations applicable to gaseous systems of all degrees of ionization. The magnetic and electric fields which appear are governed by Maxwell's equations. For the case of fully ionized systems only Equations VI-2 are of interest; in this limit the set of equations reduces to those of Everett.⁽⁵⁾ In the limit of no ionization, Equations VI-3 are the applicable equations; here the equations reduce to the

moment equations for neutral gases determined by Grad,⁽¹⁾ except that an acceleration term that accounts for the possible effect of external forces is kept for the sake of generality.

Also, it should be noted that the more familiar equations reflecting changes in the properties of the mixture as a whole rather than changes in the properties of the individual constituents are easily attained by combining the appropriate equations for each constituent.

For example, if Equation VI-2a is multiplied by m^{γ} for electrons and for ions and these two equations are added to Equation V-3a, multiplied by m^N and with the acceleration term set equal to zero the total mass conservative equation is obtained.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho w_i^T}{\partial r_i} = 0 \quad (\text{VI-4})$$

where

$$\text{a.} \quad \rho = \rho^i + \rho^e + \rho^N$$

$$\text{b.} \quad \rho w_i^T = \rho^i w_k^i + \rho^e w_k^e + \rho^N w_k^N \quad (\text{VI-5})$$

Equations for total charge conservation, and total momentum conservation are developed similarly. A generalized Ohm's law is developed in Section VII, from these equations, for use in the study of transport coefficients of the system.

VII. TRANSPORT RELATIONS

To obtain transport properties of a plasma system, it is necessary to derive transport relations for the properties of interest. That is, expressions which depict such things as the heat flow due to each constituent must be isolated from the complete set of transfer equations. These relations are obtained for systems with slowly varying flows; that is, it is assumed that the change in macroscopic parameters of the plasma is slight over a mean free path and during times of the order of collision times.

First look at the charged particle equations. Even with the restriction of slowly varying flows, the equations for the traceless stress and heat flux are still coupled equations. The procedure followed to decouple the equations and obtain explicit expressions for the heat flux and the traceless stress is to solve for the heat flux to first order in ϵ , use this expression to obtain the traceless stress to second order in ϵ and then finally to obtain an expression for the heat flux to third order in ϵ .

Using Equation VI-2e, the electron heat flux, to first order in ϵ , is found to be:

$$Q_r^e = \frac{\rho^e}{\beta^e^{3/2}} B \alpha_{ri}^e \epsilon_i \quad (\text{VII-1})$$

where:

$$\begin{aligned} \text{a. } B &= \frac{1}{2} \left[\frac{13}{30} + \frac{2(2)^{1/2}}{15} \frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}} + A \right] \\ \text{b. } A &= \frac{1}{4} \frac{n^N (2\pi)^{1/2} K_{12}^{1/2} (kT^e)^{3/2} A_1^{(5)}}{n^i e^4 \Omega_{ei}^{(1)}} \end{aligned} \quad (\text{VII-2})$$

If a local coordinate system, in which the magnetic field is in the z direction, is used, then:

$$\text{a. } \alpha_{ri}^e = \begin{pmatrix} \chi_{ee}^{(1)} & \chi_{ee}^{(2)} & 0 \\ -\chi_{ee}^{(2)} & \chi_{ee}^{(1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{b. } \chi^{e(1)} = \frac{1}{1 + 4\omega^e B^2 \lambda^e \beta^e} \quad (\text{VII-3})$$

$$\text{c. } \chi^{e(2)} = \frac{2B\lambda^e \omega^e \beta^{e1/2}}{1 + 4\omega^e B^2 \lambda^e \beta^e}$$

$|\omega^e|$ is the cyclotron frequency for electrons.

$$\omega^e = - \frac{eH}{m^e c} \quad (\text{VII-4})$$

It should be noted that the electron heat flux is dependent on the number density and hence, on the degree of ionization in other than a linear manner. That is, the heat flux does depend directly on the number density of electrons through the coefficient ρ^e ; however, the term, B, is also number density dependent and decreases as the level of ionization decreases. As can be seen the decrease in the electron heat flux due to B is a nonlinear effect. This effect is of importance in analyzing the expression for the resistivity of a plasma and will be discussed at that time.

The traceless stress, to second order in ϵ , is obtained from Equation VI-2d.

$$P_{rs}^{oe} - 2\lambda^e \omega^e \epsilon_{rjk} P_{js}^{oe} \hat{H}_k \beta^{e1/2} E = -p^e E \cdot \left\{ \left[-\frac{2}{3} + \frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} + \left(-2 + \frac{3}{2} \frac{A_2^{(5)}}{A_1^{(5)}} \right) A \right] \right. \\ \left. \epsilon_r^o \epsilon_s + \frac{1}{5} \left(2 - \frac{12}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) B \alpha_{ri}^e \epsilon_i^o \epsilon_s \right\} \quad (\text{VII-5})$$

where:

$$E = \frac{1}{\frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} + \frac{(2)^{1/2}}{5} \frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}} + \frac{3}{2} \frac{A_2^{(5)}}{A_1^{(5)}} A} \quad (\text{VII-6})$$

$\hat{H}_k$ is a unit vector in the direction of the magnetic field.

The electron heat flux to third order in ϵ is:

$$Q_r^e = \frac{\rho^e}{\beta^e} \alpha_{ri}^e \left\{ B \epsilon_i - 5B \frac{\lambda^e}{T} \frac{\partial T}{\partial r} + 2B \zeta \epsilon_r + 2B \Gamma_{is} \epsilon_s \right\} \quad (\text{VII-7})$$

where:

$$a. \quad \zeta = \left\{ \left[-\frac{41}{60} + \frac{4}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} - \left(2 - \frac{A_2^{(5)}}{A_1^{(5)}} \right) A \right] \epsilon^2 + \left(\frac{33}{50} - \frac{2}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) B \alpha_{si}^e \epsilon_i \epsilon_s \right\} \quad (VII-8)$$

$$b. \quad \Gamma_{is} = \left\{ \left(-\frac{1}{10} + \frac{4}{10} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \frac{P_{is}^{oe}}{p^e} + \left(\frac{33}{100} - \frac{6}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) B \alpha_{is}^e \epsilon^2 - B \alpha_{ks}^e \frac{P_{ik}^{oe}}{5p^e} \right. \\ \left. + A \left(\frac{3}{2} + 3 \frac{A_2^{(5)}}{A_1^{(5)}} \right) \frac{P_{is}^{oe}}{2p^e} \right\}$$

The ion heat flux is solved for in the same manner; however, due to the mass difference between electrons and ions, the ion heat flux is negligible compared to the electron heat flux except in the determination of the thermal conductivity perpendicular to a magnetic field. As will be seen later, because of the size of the ion mass and correspondingly larger radius of gyration, the perpendicular thermal conductivity is primarily due to ion motion.

The ion heat flux is calculated using Equation VI-2e, and to the order necessary for the determination of transport properties it is found to be:

$$Q_r^i = -\frac{\rho^i}{\beta^i^{3/2}} G \alpha_{ri}^i \left(\frac{5}{2} \frac{k p^i}{m^i} \frac{\beta^e{}^2 \lambda^e m^i}{\rho^e m^e} \right) \frac{\partial T}{\partial r_r} \quad (VII-9)$$

where:

$$G = \frac{1}{\left[\left(\frac{2(2)^{1/2}}{15} \frac{n^i}{n^e} \frac{\Omega_{ii}^{(2)}}{\Omega_{ei}^{(1)}} \right) - \left(\frac{\beta^e{}^2 \lambda^e m^i}{\rho^e m^e} \frac{T^i{}^{3/2} T^N}{(T^i + T^N)^{5/2}} \frac{A_2^{(5)}}{A_1^{(5)}} \mathcal{M}_{(5)}^{i,N} \right) \right]} \quad (VII-10)$$

The ion tensor, α_{ri}^i , in a local coordinate system in which the magnetic field is in the z direction is given by:

$$\alpha_{ri}^i = \begin{pmatrix} \chi^{i(1)} & \chi^{i(2)} & 0 \\ -\chi^{i(2)} & \chi^{i(1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (VII-11)$$

where:

$$\begin{aligned}
 \text{a. } X^{i(1)} &= \frac{1}{1 + \left(\frac{\rho^i}{\beta^{i3/2}} \right)^2 \left(\frac{\beta^{e2} \lambda^e m^i}{\rho^e m^e} \right)^2 \omega^i{}^2 G^2} \\
 \text{b. } X^{i(2)} &= \frac{\frac{\rho^i}{\beta^{i3/2}} \frac{\beta^{e2} \lambda^e m^i}{\rho^e m^e} \omega^i G}{1 + \left(\frac{\rho^i}{\beta^{i3/2}} \right)^2 \left(\frac{\beta^{e2} \lambda^e m^i}{\rho^e m^e} \right)^2 \omega^i{}^2 G^2} \\
 \text{c. } \omega^i &= \frac{eH}{m^i c}
 \end{aligned} \tag{VII-12}$$

The solution for the neutral heat flux is straightforward and is found using Equation VI-3e.

$$\begin{aligned}
 Q_r^N &= \frac{1}{-3 \frac{\beta^{e3/2}}{\rho^N} \mathcal{M}_{(5)}^{e,N} - \frac{1}{2(2)^{1/2}} \frac{A_{2(5)}}{A_{1(5)}} \mathcal{M}_{(5)}^{N,N} \frac{\beta^{N3/2}}{\rho^N}} \\
 &\quad \left\{ -\frac{5}{2} \frac{k p^N}{m^N} \frac{\partial T^N}{\partial r_r} - \frac{5}{2} \frac{p^N}{\rho^N} \mathcal{J}_{(5)}^{e,N} c_r - \frac{5}{2} \frac{p^N}{\rho^N} \mathcal{J}_{(5)}^{i,N} \epsilon_r^{i,N} \right\}
 \end{aligned} \tag{VII-13}$$

In addition to the transport relations for heat flux given above, an equation specifying the flow of current in the plasma is needed. That is, a generalized Ohm's law must be derived from the transfer equations. This is obtained by multiplying Equation VI-2b by e^γ and adding the equation for ions to that of electrons. The generalized Ohm's law is:

$$\begin{aligned}
 \frac{m^e}{n^e e^2} \left\{ \frac{D q_j^j}{Dt} + j_j \frac{\partial w_i^q}{\partial r_i} + j_i \frac{\partial w_j^q}{\partial r_i} \right\} &= \frac{1}{n^e e} \frac{\partial}{\partial r_i} \left(F_{ij}^e + \frac{m^e}{n^e e^2} j_i^j j_j \right) \\
 &+ \left(E_j + \frac{1}{c} \epsilon_{jkl} w_k^q H_l \right) - \frac{1}{n^e e c} \epsilon_{jkl} j_k H_l - \frac{1}{n^e e} S_j^e
 \end{aligned} \tag{VII-14}$$

where:

$$a. \frac{Dq}{Dt} = \frac{\partial}{\partial t} + w_k^q \frac{\partial}{\partial r_k}$$

$$b. w_k^q = \frac{\rho^i w_k^i + \rho^e w_k^e}{\rho^q}$$

$$c. \rho^q = \rho^e + \rho^i$$

$$d. j_k = J_k - q w_k^q \quad (VII-15)$$

$$e. q = q^i + q^e = n^i e - n^e e$$

Under the restrictions of slowly varying flows, the generalized Ohm's law may be written:

$$\frac{1}{n^e e} \frac{\partial p^e}{\partial r_k} + \left(E_k + \frac{1}{c} \epsilon_{k\ell m} w_\ell^q H_m \right) - \frac{1}{n^e e c} \epsilon_{k\ell m} j_\ell H_m - \frac{1}{n^e e} \left(S_k^{e,e} + S_k^{e,i} + S_k^{e,N} \right) = 0 \quad (VII-16)$$

At this point some further plausible assumptions about the system are made. It is assumed that the system is quasi-neutral; that is, $n^i \approx n^e$. Also, if κ is the fraction ionized, and $\mathcal{N}$ is the number of neutrals per unit volume in the unionized state, then $n^i \approx n^e = \kappa \mathcal{N}$ and $n^N = (1 - \kappa) \mathcal{N}$. Also, it should be noted that within the limits of the assumptions made,

$$\epsilon_k = - \frac{\beta^e 1/2}{n^e e} j_k$$

The generalized Ohm's law, to third order in ϵ , may now be written:

$$\begin{aligned} & \frac{1}{n^e e} \frac{\partial p^e}{\partial r_k} + \left(E_k + \frac{1}{c} \epsilon_{k\ell m} w_\ell^q H_m \right) - \frac{1}{n^e e c} \epsilon_{k\ell m} j_\ell H_m + \frac{Bk}{e} \alpha_{ki}^e \frac{\partial T^e}{\partial r_i} \\ & = \eta^0 \left\{ \delta_{k\ell} + \frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1(5)}{\eta_e^0 2} \delta_{k\ell} - \frac{3}{5} B \alpha_{k\ell}^e \right\} j_\ell \\ & + \eta^0 \left\{ \left(-\frac{3}{10} \epsilon^2 + \frac{3}{5} B \alpha_{sm}^e \epsilon_m \epsilon_s \right) \delta_{k\ell} \right\} \end{aligned}$$

$$+ \left(-\frac{6}{5} B \zeta \alpha_{k\ell}^e + \frac{3}{10} \epsilon^2 B \alpha_{k\ell}^e - \frac{3}{5} \frac{P_{k\ell}^{oe}}{p^e} - \frac{6}{5} B \alpha_{ki}^e \Gamma_{il} \right) \Big|_{j\ell} \quad (\text{VII-17})$$

where:

$$\eta^o = \frac{m^e}{3 \lambda^e \beta^e^{1/2} n^e e^2} = \frac{4(2\pi)^{1/2} (m^e)^{1/2} e^2 \Omega_{ei}^{(1)}}{3(kT^e)^{3/2}} \quad (\text{VII-18})$$

VIII. TRANSPORT COEFFICIENTS

The transport coefficients of a plasma system will be determined from the transport relations derived in Section VII.

A. Resistivity

The resistivity, η , is defined as the coefficient of $\mathbf{j}_\ell$ in the generalized Ohm's law, VII-17. Obviously this coefficient is unwieldy in its present form; it is instructive to look at the resistivity for certain specialized cases.

I. Resistivity Parallel to a Magnetic Field

In the case of the resistivity parallel to, or in the absence of, a magnetic field, the tensorial relations simplify. That is, $\alpha_{ri}^e = \delta_{ri}$, and $\eta_{||}$ may be written:

$$\begin{aligned} \eta_{||} = \eta^0 & \left\{ 1 - \frac{3}{5} B + \frac{1-\kappa}{\kappa} \frac{(2\pi)(m^e K_{12})^{1/2} A_1(5)}{e^2 \eta_0} \right\} \\ & + \eta^0 \left\{ \left[-\frac{3}{10} + \frac{9}{10} B + \frac{6}{5} B \left(\frac{41}{60} - \frac{4}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) + \frac{6}{5} B^2 \right. \right. \\ & \left. \left(-\frac{33}{50} + \frac{2}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) + \frac{6}{5} BA \left(2 - \frac{A_2(5)}{A_1(5)} \right) + \frac{2}{5} EF + \frac{4}{5} BEF \left(-\frac{1}{10} + \frac{4}{10} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \right. \\ & \left. \left. - \frac{6}{5} B^2 \left(\frac{33}{100} - \frac{6}{25} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) - \frac{12}{75} B^2 EF + \frac{2}{5} BAEF \left(\frac{3}{2} + 3 \frac{A_2(5)}{A_1(5)} \right) \right] \epsilon^2 \right\} \quad (\text{VIII-1}) \end{aligned}$$

where:

$$F = \left\{ -\frac{2}{3} + \frac{2}{5} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} + A \left(-3 + \frac{3}{2} \frac{A_2(5)}{A_1(5)} \right) + B \left(\frac{2}{5} - \frac{12}{15} \frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \right) \right\} \quad (\text{VIII-2})$$

The major contribution to the resistivity is the first or scalar term of VIII-1, the second term constitutes a corrective factor of the order ϵ^2 . It is instructive to examine the terms that constitute the scalar resistivity. The first part of the expression, η^0 , represents the resistivity due to charged particle interactions in the absence of any heat flux coupling. The second part, $-\frac{3}{5} \eta^0 B$, shows the influence of the electron heat flux coupling which acts to reduce the resistivity of the system. The

last expression, $\frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1}{e^2} (5)$ represents the effect of electron-neutral collisions. As would be expected this term is zero for full ionization and becomes very large for very small degrees of ionization; it approaches infinity as the degree of ionization approaches zero. As was mentioned in Section VII, the size of the electron heat flux depends, in a non-linear manner, on the degree of ionization. This shows that it is incorrect to consider the resistivity of a partially ionized gas to be simply a linear combination of an expression representing the resistivity of a fully ionized gas plus an expression which accounts for electron-neutral collisions.

The higher order terms, order of ϵ^2 , show the influence of the higher order heat flux coupling, the pressure tensor coupling, and the higher order effects of electron-neutral and charged-charged collisions.

For the fully ionized plasma, the expression for resistivity reduces to that of Everett.⁽⁵⁾

$$\eta_{||} = \eta^0 \left[0.52 + 0.1 \epsilon^2 \right] \quad (\text{VIII-3})$$

The fact that for a fully ionized plasma

$$\frac{\Omega_{ei}^{(2)}}{\Omega_{ei}^{(1)}} \approx \frac{\Omega_{ee}^{(2)}}{\Omega_{ei}^{(1)}} \approx 1$$

has been used in obtaining the above result.

Table I shows this value for the fully ionized scalar resistivity compared with the values obtained by Ahtye⁽⁶⁾ and Devoto⁽⁷⁾ who used the Chapman-Enskog formulation in their work. All results are divided by the familiar Spitzer-Harm⁽⁸⁾ value.

TABLE I
Comparison of Results

<u>Results Herein</u>	<u>Ahtye</u>	<u>Devoto</u>
1.023	0.939-0.994	1.006

Figure 1 shows a plot of the partially ionized conductivity for argon determined using the expressions developed in this paper and compared with the experimental results obtained by Lin, Resler, and Kantrowitz⁽⁹⁾. The cross section for electron-argon atom collisions was obtained by extrapolation from data on electron-atom elastic collisions presented by Moiseiwitsch⁽¹⁰⁾. The Spitzer-Harm⁽⁸⁾ value for the fully ionized conductivity is also plotted for comparison purposes.

Figure 1. Comparison of Theoretical and Experimentally Determined Electrical Conductivity of Partially Ionized Argon

In the other limit, no ionization, the resistivity has an infinite value, which is the same as saying that the conductivity of a gas consisting of only neutral particles is zero.

II. Resistivity Perpendicular to a Magnetic Field.

For the case of the resistivity perpendicular to a magnetic field, the tensorial relations do not simplify and the expression for the resistivity becomes slightly more complicated. It is sufficient to consider the resistivity to first order in ϵ . The resistivity may then be written:

$$\eta_{\perp} = \eta^0 \left\{ 1 - \frac{3}{5} B \chi^{e(1)} + \frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1(5)}{\eta^0 e^2} \right\} \quad (\text{VIII-4})$$

Or expressing $\chi^{e(1)}$ explicitly:

$$\eta_{\perp} = \eta^0 \left\{ 1 - \frac{3}{5} B \frac{1}{1 + 4B^2 \omega e^2 \lambda e^2 \beta^e} + \frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1(5)}{e^2 \eta^0} \right\} \quad (\text{VIII-5})$$

The presence of the magnetic field has the effect of reducing the amount of heat flux coupling, thus increasing the resistivity. If the magnetic field becomes large, the effect of the heat flux coupling becomes negligible and the resistivity may be written:

$$\eta_{\perp} = \eta^0 \left\{ 1 + \frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1(5)}{e^2 \eta^0} \right\} \quad (\text{VIII-6})$$

For the fully ionized case this reduces to that obtained by Spitzer. ⁽¹¹⁾

$$\eta_{\perp} = \eta^0 \quad (\text{VIII-7})$$

In the limit of no ionization the resistivity again has an infinite value.

B. Thermal Conductivity

The thermal conductivity, K^{γ} , of a specie is defined as the coefficient of the temperature gradient in the expression for the heat flow of that specie. Again it is instructive to look at the thermal conductivity for each specie in certain specialized cases.

I. Electron Thermal Conductivity Parallel to a Magnetic Field

As in the case of resistivity parallel to a magnetic field, the magnetic field tensor simplifies and the expression for the electron heat flow becomes relatively simple. The parallel electron heat flux, neglecting terms of second order in ϵ , may be written:

$$Q_{||}^e = \frac{\rho^e}{\beta^e{}^{3/2}} \left\{ B\epsilon_{||} - 5 \frac{B\lambda^e}{T^e} \frac{\partial T^e}{\partial r_{||}} \right\} \quad (\text{VIII-8})$$

The value of $K_{||}^e$ is taken from this expression, and is seen to be:

$$K_{||}^e = \frac{5\rho^e B\lambda^e}{\beta^e{}^{3/2} T^e} \quad (\text{VIII-9})$$

This may be simplified to:

$$K_{||}^e = \frac{5B(kT^e)^{5/2} k}{4(2\pi m^e)^{1/2} e^4 \Omega_{ei}^{(1)}} \quad (\text{VIII-10})$$

The value of $\eta_{||}$ to first order in ϵ has been used in evaluating the above. For a fully ionized gas, the electron thermal conductivity becomes:

$$K_{||}^e = \frac{0.402 (kT^e)^{5/2} k}{(m^e)^{1/2} e^4 \Omega_{ei}^{(1)}} \quad (\text{VIII-11})$$

This result is equivalent to the second order, second approximation of the familiar Chapman-Enskog theory, and is in agreement with the value obtained by Herdan and Liley.³⁰ To reproduce results equivalent with the Chapman-Enskog third or fourth order, second approximation would require a different ordering technique.

Figure 2 shows a plot of the partially ionized thermal conductivity of argon based upon the physical situation present in the experiment performed by Lin, et al.

In the limit of no ionization the electron thermal conductivity goes to zero.

II. Electron Thermal Conductivity Perpendicular to a Magnetic Field

In solving for the thermal conductivity perpendicular to a magnetic field, it is assumed that the temperature gradient and the electric field parallel to the magnetic field are zero. In this case $Q_{\perp}^e$ becomes:

$$Q_{\perp}^e = \frac{\rho^e}{\beta^e{}^{3/2}} B X^{e(1)} \epsilon_{\perp} - \frac{\rho^e}{\beta^e{}^{3/2}} X^{e(1)} \frac{5B\lambda^e}{T^e} \frac{\partial T^e}{\partial r_{\perp}} \quad (\text{VIII-12})$$

Figure 2. Plot of Theoretical Electron Thermal Conductivity of Partially Ionized Argon

As before, it is necessary to account for the coupling which exists between the expression for heat flow and the generalized Ohm's law. Proceeding as in the determination of the parallel thermal conductivity, $Q_{\perp}^e$ may be written:

$$Q_{\perp}^e = -\frac{kT^e B}{e\eta_{\perp}} \chi^{e(1)} E'_{\perp} - \frac{B(kT^e)^2 \chi^{e(1)}}{n^e e^2 \eta_{\perp}} \frac{\partial n^e}{\partial r_{\perp}} - \left[\frac{Bk^2 T^e \chi^{e(1)}}{e^2 \eta_{\perp}} + \frac{B^2 k^2 T^e \chi^{e(1)}}{e^2 \eta_{\perp}} + \frac{5B\lambda^e \rho^e \chi^{e(1)}}{\beta^e e^{3/2} T^e} \right] \frac{\partial T^e}{\partial r_{\perp}} \quad (\text{VIII-13})$$

where:

$$E'_{\perp} = E_{\perp} + \frac{1}{c} W_{\parallel}^q H \quad (\text{VIII-14})$$

The thermal conductivity, $K_{\perp}^e$, is seen to be:

$$K_{\perp}^e = \frac{Bk^2 T^e \chi^{e(1)}}{e^2 \eta_{\perp}} + \frac{B^2 k^2 T^e \chi^{e(1)}}{e^2 \eta_{\perp}} + \frac{5B\lambda^e \rho^e \chi^{e(1)}}{\beta^e e^{3/2} T^e} \quad (\text{VIII-15})$$

However, it is usually sufficient to express the thermal conductivity perpendicular to a magnetic field to the lowest order in the inverse power of H . Therefore $K_{\perp}^e$ may be written:

$$K_{\perp}^e = \frac{Bk^2 T^e \chi^{e(1)}}{e^2 \eta_{\perp}} + \frac{5B\lambda^e \rho^e \chi^{e(1)}}{\beta^e e^{3/2} T^e} \quad (\text{VIII-16})$$

which becomes:

$$K_{\perp}^e = \frac{(kT^e)^{5/2} k}{e^4 (m^e)^{1/2} \Omega_{ei}^{(1)}} \frac{B\chi^{e(1)}}{4(2\pi)^{1/2}} \left\{ \frac{3}{\left[1 + \frac{1-\kappa}{\kappa} \frac{2\pi (m^e K_{12})^{1/2} A_1(5)}{e^2 \eta^o} \right]} + 5 \right\} \quad (\text{VIII-17})$$

If H is large, a further simplification results. The tensor element $\chi^{e(1)}$ may be written:

$$\chi^{e(1)} \approx \frac{1}{4\omega^e B^2 \lambda^e \beta^e} = \frac{8\pi m^e c^2 (\kappa/\mu)^2 e^6 \Omega_{ei}^{(1)}}{B^2 (kT^e)^3 H^2} \quad (\text{VIII-18})$$

and $K_{\perp}^e$ becomes:

$$K_{\perp}^e = \frac{(2\pi)^{1/2}}{BH^2} \left(\frac{m_e k}{T^e} \right)^{1/2} e^2 c^2 (\kappa \mathcal{N})^2 \Omega_{ei}^{(1)} \left\{ \frac{3}{1 + \frac{1-\kappa}{\kappa} \frac{2\pi (m_e K_{12}^e)^{1/2} A_1(5)}{e^2 \eta^o}} + 5 \right\} \quad (\text{VIII-19})$$

The value of $\eta_{\perp}$ to first order in ϵ , in the presence of a strong magnetic field, has been used to obtain the final expression for $K_{\perp}^e$. In the fully ionized limit the perpendicular thermal conductivity reduces to:

$$K_{\perp}^e = 25.1 \left(\frac{m_e k}{T^e} \right)^{1/2} \frac{e^2 c^2 \mathcal{N}^2 \Omega_{ei}^{(1)}}{H^2} \quad (\text{VIII-20})$$

In the limit of no ionization, $K_{\perp}^e$ becomes zero.

III. Ion Thermal Conductivity Perpendicular to a Magnetic Field

The same assumptions are made in solving for $K_{\perp}^i$ as were made in solving for $K_{\perp}^e$; that is, it is assumed that the temperature gradient and the electric field parallel to the magnetic field are zero. The equation for the ion heat flux is used, and in this case $Q_{\perp}^i$ becomes:

$$Q_{\perp}^i = - \frac{\rho^i}{\beta^i^{3/2}} G \frac{\beta^e \lambda^e m^i}{\rho_e m^e} \frac{5}{2} \frac{k p^i}{m^i} \chi^{i(1)} \frac{\partial T^i}{\partial r_{\perp}} \quad (\text{VIII-21})$$

The ion thermal conductivity may be written:

$$K_{\perp}^i = \frac{G 5 (k T^i)^{5/2} k}{8 (2\pi m^i)^{1/2} e^4 \Omega_{ei}^{(1)}} \quad (\text{VIII-22})$$

For the case of a strong magnetic field a further simplification results. The tensor element $\chi^{i(1)}$ may be written:

$$\chi^{i(1)} \approx \frac{1}{\frac{\rho^i}{\beta^i^{3/2}} \frac{\beta^e \lambda^e m^i}{\rho_e m^e} \omega^i G^2} = \frac{32\pi m^i c^2 (\kappa \mathcal{N})^2 e^6 \Omega_{ei}^{(1)^2}}{(k T^i)^{3/2} H^2 G^2} \quad (\text{VIII-23})$$

and $K_{\perp}^i$ becomes:

$$K_{\perp}^i = \frac{10(2\pi)^{1/2}}{GH^2} \left(\frac{m_k^i}{T^i} \right)^{1/2} e^2 c^2 (\kappa_N)^2 \Omega_{ei}^{(1)} \quad (\text{VIII-24})$$

For the fully ionized plasma, in the presence of a strong magnetic field, this reduces to:

$$K_{\perp}^i = \frac{8(\pi)^{1/2}}{3} \left(\frac{m_k^i}{T^i} \right)^{1/2} \frac{\mathcal{N}^2 e^2 c^2 \Omega_{ii}^{(2)}}{H^2} \quad (\text{VIII-25})$$

This is the same result as that derived by Rosenbluth and Kaufman⁽¹³⁾. In the limit of no ionization, $K_{\perp}^i$ becomes zero.

It was mentioned previously that the perpendicular thermal conductivity was due primarily to the ion motion. This can be seen by comparing Equations VIII-25 and VIII-21.

IV. Neutral Particle Thermal Conductivity

The neutral particle thermal conductivity is obtained from the expression for the neutral heat flux and is considered to be the coefficient of the neutral temperature gradient. Using Equation VII-13, K^N is seen to be:

$$K^N = \frac{1}{\left[-3 \frac{\beta e^{3/2}}{\rho^N} \mathcal{M}_{(5)}^{e,N} - \frac{1}{2(2)^{1/2}} \frac{\beta^{N3/2}}{\rho^N} \frac{A_2^{(5)}}{A_1^{(5)}} \mathcal{M}_{(5)}^{N,N} \right]} \frac{5}{2} \frac{kp^N}{m^N} \quad (\text{VIII-26})$$

In the fully ionized limit, K^N becomes zero. For zero ionization K^N reduces to:

$$K^N = \frac{5(2)^{1/2}}{4\pi} \frac{k^2 T^N}{(m_{K_{12}}^N)^{1/2} A_2^{(5)}} \quad (\text{VIII-27})$$

This is identical to the result derived by Chapman and Cowling.⁽³⁾ This fact points out an interesting situation that was first stated by Grad. The 13-moment approximation is not an approximation for Maxwell molecules and, in addition, the cross product terms that were neglected in the collision integrals are identically zero. Therefore, the results of the collision integrals are exact and coefficients such as the neutral particle conductivity, in the absence of any ionization, would be expected to have the same value as those obtained by Chapman and Cowling for Maxwell molecules.

APPENDIX A

A REVIEW OF TRANSPORT EQUATIONS FOR A PLASMA

A-I. INTRODUCTION

An attempt is made to evaluate the justification for the use of the various forms of the transport equations which have been assumed to provide an adequate mathematical description of the behavior of the constituents of a fully ionized plasma system. These transport equations have been critically compared with careful attention given to the approximations and assumptions underlying their development. Since considerable confusion exists regarding the appropriateness of the Boltzmann form of the collision integral, considerable attention is given to the development of the arguments supporting its use. In particular, the erroneous belief that the effects of correlations have been neglected has been expelled. In addition, arguments are given which lead to the conclusion that the importance of multiparticle collisions may have been overemphasized.

In Section A-II the BBGKY method is reviewed with a discussion on its relation to the Boltzmann equation. The BGK method is briefly discussed in Section A-III. In Section A-IV, in view of the confusion with regard to the similarities and differences between the Fokker-Planck collision integral and the Boltzmann collision integral, careful attention is given to the particle behavior within a Debye sphere for the cases of sequential two-particle collision and for simultaneous multiparticle collisions, where two-particle space time paths are assumed. The collision effect is shown to be precisely the same for these two cases. Section A-V is devoted to a discussion of the Fokker-Planck equation and its relation to the Boltzmann equation. Section A-VI contains a brief presentation of the Vlasov equation. Finally, in Section A-VII the Boltzmann equation is discussed and arguments presented to justify its use.

A-II. BBGKY KINETIC THEORY

The BBGKY Kinetic Theory is the result of work by Bogolyubov,⁽¹⁴⁾ Born,⁽¹⁵⁾ Green,^(15, 16) Kirkwood,⁽¹⁷⁾ and Yvon.⁽¹⁸⁾ The kinetic equations are obtained directly from Liouville's equation and are sufficiently general that terms accounting for the influence of multiple particle interactions are in evidence. Unfortunately, only limited practical application is possible at this time because of the mathematical complexity involved.

The introductory development of the theory presented here follows that given by Montgomery and Tidman⁽¹⁹⁾ and for simplicity considers an electron plasma moving in a uniform background of fixed positive charge with no magnetic field interactions.

The probability distribution of the electrons is $D_N(\bar{X}_1, \bar{X}_2 \dots \bar{X}_N)$, where $\bar{X}_N = (r_N, p_N)$; this distribution gives the probability that electron 1 is at $\bar{X}_1$, electron 2 at $\bar{X}_2 \dots$ and electron N at $\bar{X}_N$ at time t. $D(\bar{X}_1 \dots, t)$ develops in time according to Liouville's Theorem as:

$$\frac{\partial D_N}{\partial t} + \left\{ D_N, H_N \right\} = 0$$

where

$$\left\{ D_N, H_N \right\}$$

represents the Poisson bracket of D_N and H_N ; H_N is the Hamiltonian of the system. Written explicitly:

$$\bar{H}_N = \sum_{i=1}^N \frac{\bar{p}_i^2}{2m} + \sum_{i < j=1}^N \phi_{ij}$$

where ϕ_{ij} is the scalar electrical potential at position $\bar{r}_i$, due to an electron at $\bar{r}_j$.

Now the unrestricted solution of Liouville's equation is equivalent to the solution of N equations representing the equations of motion for the N electrons of the system. Obtaining such a solution would, of course, be an impossible task.

The approach used in developing the BBGKY theory is to reduce the Liouville equation to a tractable set of equations. The assumptions made and procedures followed to bring about such a reduction are such that, although information is lost, the information pertaining to macroscopic behavior is retained.

It is assumed that we have an isolated system of N electrons and that like particles are indistinguishable; i. e., D_N is symmetric under the interchange of like particle coordinates. Then reduced probability distributions, f_s , are defined.

$$\frac{f_s}{V^s} \equiv \int D_N d\bar{X}_{s+1} \dots d\bar{X}_N$$

For a given s , f_s is the probability distribution of s particles and V is the configuration space volume.

The reduction is obtained by integrating each term of the Liouville equation over the six-dimensional phase space of each of the electrons $\bar{X}_{s+1}, \bar{X}_{s+2}, \dots, \bar{X}_N$. From this procedure an equation for f_s in terms of f_{s+1} is developed. That is:

$$\frac{\partial f_s}{\partial t} + \left\{ f_s; H_s \right\} = N_0 \int \left\{ \phi_{i,s+1}; f_{s+1} \right\} d\bar{X}_{s+1}$$

where N_0 is the average number of electrons/unit volume. Setting $s = 1$ the equation is seen to be the equation for the one particle distribution function; f_1 is coupled through the interaction term on the right with f_2 , the two particle distribution. Similarly setting $s = 2$, the equation is seen to be the equation for the two-particle distribution coupled with the three particle-distribution f_3 .

The solution of the above set of equations ($s = 1, 2, 3, \dots, N$) is equivalent to the solution of Liouville's equation. An untractable set of equations still exists. However, it is assumed that if the first two equations in the chain are used to solve for f_1 then a sufficiently accurate solution for f_1 will be obtained. That is, it is assumed that the higher order equations ($s = 3, 4, \dots, N$) change the solution for f_1 , from that obtained using only the equations for $s = 1$ and 2 , by a negligible amount. Thus the chain of equations is broken by a plausible, but unproved assertion. The first two equations in the chain are written:

$$\begin{aligned} 1. \quad \frac{\partial f_1}{\partial t} + \frac{\bar{P}_1}{m} \cdot \frac{\partial f_1}{\partial \bar{r}_1} &= N_0 \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} \cdot \frac{\partial f_2}{\partial \bar{r}_2} d\bar{X}_2 \\ 2. \quad \frac{\partial f_2}{\partial t} + \left(\frac{\bar{P}_1}{m} \cdot \frac{\partial}{\partial \bar{r}_1} + \frac{\bar{P}_2}{m} \cdot \frac{\partial}{\partial \bar{r}_2} \right) f_2 &- \left(\frac{\partial \phi_{12}}{\partial \bar{r}_1} \cdot \frac{\partial}{\partial \bar{P}_1} + \frac{\partial \phi_{12}}{\partial \bar{r}_2} \cdot \frac{\partial}{\partial \bar{P}_2} \right) f_2 \\ &= N_0 \int \left(\frac{\partial \phi_{13}}{\partial \bar{r}_1} \cdot \frac{\partial f_3}{\partial \bar{P}_1} + \frac{\partial \phi_{23}}{\partial \bar{r}_2} \cdot \frac{\partial f_3}{\partial \bar{P}_2} \right) d\bar{X}_3 \end{aligned} \quad (A)$$

This set of two equations is still not tractable; a method must be found to close out the set. That is, the number of unknown functions must be reduced to two. Hopefully, then, f_1 could be solved for and, from the probability distribution of one particle, knowledge of the behavior of the macroscopic properties of the plasma system could be obtained.

The method used to accomplish this is to write out f_2 and f_3 in terms of f_1 plus terms representing the correlation between either two or three particles. That is:

$$\begin{aligned}
f_1(\bar{X}_1) &= f_1(\bar{X}_1) \\
f_2(\bar{X}_1 \bar{X}_2) &= f_1(\bar{X}_1) f_1(\bar{X}_2) + P(\bar{X}_1 \bar{X}_2) \\
f_3(\bar{X}_1 \bar{X}_2 \bar{X}_3) &= f_1(\bar{X}_1) f_1(\bar{X}_2) f_1(\bar{X}_3) + f_1(\bar{X}_1) P(\bar{X}_2 \bar{X}_3) + f_1(\bar{X}_2) P(\bar{X}_3 \bar{X}_1) \\
&\quad + f_1(\bar{X}_3) P(\bar{X}_2 \bar{X}_1) + T(\bar{X}_1 \bar{X}_2 \bar{X}_3)
\end{aligned}$$

For noninteracting particles $f_2 = f_1(\bar{X}_1) f_1(\bar{X}_2)$. For interacting particles $f_2 = f_1(\bar{X}_1) f_1(\bar{X}_2) + P(\bar{X}_1 \bar{X}_2)$, where $P(\bar{X}_1 \bar{X}_2)$ is the correlation function and is a measure of the change in the two-particle distribution due to interactions from the non-interacting form. $P(\bar{X}_1 \bar{X}_2)$ will, of course, be small at large $|\bar{r}_2 - \bar{r}_1|$. An expansion of the form above is called a "Mayer Cluster Expansion"⁽²⁰⁾ and was developed in the study of equilibrium statistical mechanics.

This expansion will be useful if P and T are small in comparison to the factorable parts of f_2 and f_3 . The original work with this approach was concerned with molecular gases, where the arguments for the "smallness" of P and T are reasonably straightforward.

For a plasma system, even with the long range coulomb forces, it is still argued⁽⁶⁾ that $\frac{P}{f_1 f_1} \ll 1$.

The idea, then, is to make a non-equilibrium Mayer Cluster Expansion in the two equations (A), with the hope that the arguments on the size of P will still be valid, even for the non-equilibrium state. We also make the plausible, but unproved, assumption that $T(X_1 X_2 X_3)$ is negligible compared to $P f_1$.

Following this procedure we obtain two equations relating f_1 to P . In the more familiar velocity space notation the equations are written:

$$\begin{aligned}
1. \quad \frac{\partial f_1}{\partial t} + \bar{v}_1 \cdot \frac{\partial f_1}{\partial \bar{r}_1} - \frac{N_0}{m} \left\{ \int d\bar{r}_2 d\bar{v}_2 \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(\bar{r}_2 \bar{v}_2) \right\} \cdot \frac{\partial f(\bar{r}_1 \bar{v}_1)}{\partial \bar{v}_1} \\
= \frac{N_0}{m} \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} \cdot \frac{\partial P(\bar{r}_1 \bar{v}_1 \bar{r}_2 \bar{v}_2)}{\partial \bar{v}_1} d\bar{r}_2 d\bar{v}_2 \\
2. \quad \frac{\partial P}{\partial t} + \left(\bar{v}_1 \cdot \frac{\partial}{\partial \bar{r}_1} + \bar{v}_2 \cdot \frac{\partial}{\partial \bar{r}_2} \right) P = \frac{1}{m} \left[\frac{\partial \phi_{12}}{\partial \bar{r}_1} \cdot \frac{\partial}{\partial \bar{v}_1} + \frac{\partial \phi_{12}}{\partial \bar{r}_2} \cdot \frac{\partial}{\partial \bar{v}_2} \right] \cdot f(\bar{r}_1 \bar{v}_1) f(\bar{r}_2 \bar{v}_2) \\
+ \frac{N_0}{m} \left[\int d\bar{r}_3 d\bar{v}_3 f(\bar{r}_3 \bar{v}_3) \left(\frac{\partial \phi_{13}}{\partial \bar{r}_1} \cdot \frac{\partial}{\partial \bar{v}_1} + \frac{\partial \phi_{23}}{\partial \bar{r}_2} \cdot \frac{\partial}{\partial \bar{v}_2} \right) \right] P
\end{aligned} \tag{B}$$

$$+ \frac{N_o}{m} \left[\frac{\partial f(\bar{r}_1 \bar{v}_1)}{\partial \bar{v}_1} \left\{ \int \frac{\partial \phi_{13}}{\partial \bar{r}_1} P(\bar{r}_2 \bar{v}_2 \bar{r}_3 \bar{v}_3) \cdot d\bar{r}_3 d\bar{v}_3 + \int \frac{\partial \phi_{23}}{\partial \bar{r}_2} P(\bar{r}_1 \bar{v}_1 \bar{r}_3 \bar{v}_3) d\bar{r}_3 d\bar{v}_3 \right\} \right]$$

Equation 1 of (B) is an integro-differential equation that gives the development of f_1 with time in terms of the correlation function P . Equation 2 is another integro-differential equation which gives the development of the correlation function with time.

At this time no exact solutions of these two equations are available. Attempts have been made to obtain approximate solutions to Equation 2, and then to use these solutions for P in Equation 1 to obtain a useable kinetic equation for the one particle distribution.

It is instructive to look at Equation 1 of (B) and to recast it in a slightly different form so as to show the similarity between it and the Boltzmann equation. That is, Equation 1 is written:

$$\begin{aligned} \frac{\partial f}{\partial t} + \bar{v}_1 \cdot \frac{\partial f}{\partial \bar{r}_1} + \frac{N_o}{m} \left\{ \int_{L_D}^{\infty} f(\bar{r}_2 \bar{v}_2) \frac{\partial \phi_{12}}{\partial \bar{r}_1} d\bar{r}_2 d\bar{v}_2 \right\} \cdot \frac{\partial f(\bar{r}_1 \bar{v}_1)}{\partial \bar{v}_1} \\ + \frac{N_o}{m} \int_{L_D}^{\infty} \frac{\partial \phi_{12}}{\partial \bar{r}_1} \frac{\partial P(\bar{r}_1 \bar{v}_1 \bar{r}_2 \bar{v}_2)}{\partial \bar{v}_1} d\bar{r}_2 d\bar{v}_2 = \left\{ \int_0^{L_D} f(\bar{r}_2 \bar{v}_2) \cdot \frac{\partial \phi_{12}}{\partial \bar{r}_1} d\bar{r}_2 d\bar{v}_2 \cdot \frac{\partial f(\bar{r}_1 \bar{v}_1)}{\partial \bar{v}_1} \right. \\ \left. + \int_0^{L_D} \frac{\partial \phi_{12}}{\partial \bar{r}_1} \frac{\partial P(\bar{r}_1 \bar{v}_1 \bar{r}_2 \bar{v}_2)}{\partial \bar{v}_1} d\bar{r}_2 d\bar{v}_2 \right\} \end{aligned}$$

where L_D is the Debye shielding distance. The first integral on the left-hand side represents the electric field generated by particles outside the Debye sphere and is identical to the force term in the Boltzmann equation provided the concept of shielding is introduced and that the particles outside the shielding distance are considered to produce a macroscopic electric field. The second integral on the left-hand side represents the effect of $P(\bar{r}_1 \bar{v}_1 \bar{r}_2 \bar{v}_2)$ outside of the Debye sphere and is expected to be negligible. (Calculations on the size of P show that P becomes very small outside L_D ; which is another way of stating that L_D is the correct scale length to use as a shielding distance.⁽¹⁹⁾) In the Boltzmann equation this term does not appear. The right-hand side of the equation may now be roughly identified with the right-hand side of the Boltzmann equation; in both equations these terms represent "collision" effects, the interaction of particles within a Debye sphere. In the Boltzmann equation the effect of collisions is determined without the need to have a knowledge of the mathematical form of P . (See Section IV.) In the BBGKY theory, an attempt is made to determine the mathematical form of the correlation function and this is used to determine collision effects. The identification between the two equations becomes easier to make when the first integral on the right-hand side of Equation 1 is identified with the electric field due to particles inside the Debye sphere. As long as the gradient of number density inside the Debye sphere is small (a condition which must exist before the Boltzmann equation can be used) this term will be small compared

to the second integral. The second integral on the right-hand side is the integral that represents the effects of actual collisions between particles, that integral which corresponds to the classical Boltzmann collision integral.

The advantage claimed for the BBGKY theory is that since P is determined mathematically the possibility exists of calculating collision effects with a high degree of accuracy. Also, it is not necessary to introduce a shielding distance, in that the exact shielding distance, to which the Debye length is an approximation based on a static analysis, is introduced naturally by the mathematics of the equations. On the other hand, it is not unanimously felt that the BBGKY theory is completely correct. It is argued that the collision effects are overestimated, that certain types of collisions are "counted" twice.⁽²¹⁾

As has been mentioned, a major drawback of this procedure is the extreme mathematical complexity of the equations. Much work has been done in developing approximate forms for these equations. However, these approximate forms, even under the broadest assumptions, are still exceedingly complex and, by the nature of the assumptions, limited in scope.

A-III. BGK KINETIC THEORY

The BGK type kinetic equation is an attempt to generate a kinetic equation that is similar to the Boltzmann equation, but one that avoids some of the mathematical complexities inherent in the Boltzmann equation. Bhatnager, Gross and Krook did the original work on this approach. (22, 23)

The idea is to replace the collision integral in the classical Boltzmann equation with an approximate form which is mathematically simpler, but one which hopefully reproduces the effect of the Boltzmann collision integral. This approximate form must, of course, conserve such quantities as number density, total momentum and total energy.

Perhaps the simplest approach is to argue that the collision process causes the distribution function, f , to reduce to the Maxwellian distribution, f_0 , in some average τ . Therefore, the change in f with time due to collisions is written

$$\frac{f_0 - f(\vec{r}, \vec{v}, t)}{\tau}$$

One obvious difficulty that presents itself is the choice of a suitable τ . Also, since we represented the change in f due to collisions in a rough, average way, knowledge of the instantaneous conservation of quantities of interest has been lost.

More sophisticated approximations to the Boltzmann collision integral which provide for the instantaneous conservation of quantities of interest or which more closely approximate the Boltzmann collision integral become increasingly complex.

For example, Frisch and Morrison⁽²⁴⁾ use a BGK type kinetic equation where the collision terms are derived from an analysis of the solutions of the Boltzmann collision integral. Here

$$\left. \frac{\partial f^a}{\partial t} \right)_c$$

for a two-component neutral plasma consisting of α and β particles is written:

$$\begin{aligned} \frac{\partial f^a}{\partial t} = & \Gamma (f_0^a - f^a) - \frac{\mu f_0^a}{N^a k T^a} \left[\langle v_i^a \rangle - \langle v_i^\beta \rangle \right] \cdot \left[v_i - \langle v_i^a \rangle \right] \\ & + \frac{f_0^a}{N^a k T^a} \left[\mu_M \left(\langle v_i^a \rangle - \langle v_i^\beta \rangle \right) - \frac{3}{2} (T^a - T^\beta) \right] \cdot \left[\frac{(v_i - \langle v_i^a \rangle)^2}{3kT} - 1 \right] \end{aligned}$$

where:

$$\begin{aligned} \text{a.} \quad N^\nu &= \int f^\nu d^3v \\ \text{b.} \quad \langle v_i^\nu \rangle &= \frac{1}{N^\nu} \int v_i f^\nu d^3v \\ \text{c.} \quad T^\nu &= \frac{m^\nu}{3N^\nu k} \int (v - \langle v^\nu \rangle)^2 f^\nu d^3v \end{aligned}$$

The inverse relaxation times Γ, μ, μ_M and ζ are taken to be characteristic coefficients of the plasma.

The fundamental objection to the use of a BGK type equation for the study of transport properties of a plasma system lies in the inverse relaxation times that occur explicitly in the collision terms. These quantities are directly related to the transport properties of the plasma. Thus, the use of the BGK kinetic equation implies a priori knowledge of the transport coefficients.

An equation of this type is principally useful for the study of a plasma system in which the transport coefficients are assumed to be known; values for these coefficients may then be used, in a mathematical form which is somewhat simpler than the Boltzmann collision integral, to obtain other information about the system. For instance, in Ref. 24 shock waves in a steady-state model plasma are studied.

A-IV. BEHAVIOR OF PARTICLES WITHIN THE DEBYE SPHERE

In an attempt to obtain a better understanding of the behavior of particles during the collision process and an appreciation for the significance of the approximations stated or implied in the course of developing models for collisions, we will focus attention on a test particle located at the center of its Debye sphere and field particles which pass through the sphere. Several cases of interest will be considered.

A. Tenuous Spatially Homogeneous Gas

The tenuous gas (i. e. , system within which the likelihood of other than two-particle collisions is extremely small, e. g. , a neutral particle system) has been treated in what appears to be an adequate manner by the Boltzmann two-particle collision model and is considered here only because of the misunderstanding which prevails regarding the approximations inherent to the method.

For the sake of simplicity and ease of understanding, we will consider the average momentum transfer rate to the test particle, which may be written,

$$\left\langle \frac{\Delta \bar{p}}{\Delta \tau} \right\rangle = \frac{1}{\Delta \tau} \int_0^{\Delta \tau} \bar{F}(t) dt$$

where $\Delta \tau$ is chosen large enough that sufficient collisions take place within that time to make the average meaningful.

Assume that at time t_1 within $\Delta \tau$ a field particle enters the sphere and interacts for time $\Delta \tau_1$ ($\Delta \tau_1 \ll \Delta \tau$); subsequently, a second field particle enters the sphere at time t_2 interacting for time $\Delta \tau_2$, etc.

$$\text{Then: } \left\langle \frac{\Delta \bar{p}}{\Delta \tau} \right\rangle = \frac{1}{\Delta \tau} \left\{ \int_{t_1}^{t_1 + \Delta \tau_1} \bar{F}_1(t) dt + \int_{t_2}^{t_2 + \Delta \tau_2} \bar{F}_2(t) dt + \dots \right\} = \frac{1}{\Delta \tau} \left\{ \Delta \bar{p}_1 + \Delta \bar{p}_2 + \dots \right\}$$

Figure 3. Sequential Binary Collisions in a Debye Sphere

For each pair of interacting particles the force function is well known, so given initial conditions (i. e., velocity and position on Debye sphere of field particle at time of entry), the change of momentum is easily calculated.

Furthermore, if the two-particle distribution function $f_2(\bar{r}_1 \bar{v}_1 \bar{r}_2 \bar{v}_2)$ is known where $|\bar{r}_2 - \bar{r}_1| = L_D$ (i. e., when field particle 2 is on the Debye sphere of test particle 1), the probability of occurrence of a particular interaction is known and by integrating over all velocity space of the field particles, one obtains an explicit expression for $\langle \frac{\Delta p}{\Delta t} \rangle$. It should be noted that if, as assumed, only two-particle collisions occur and if the force function is such that the kinematic behavior is known, the only inaccuracy is that which results from an inability to express f_2 accurately.

With reference to the Boltzmann collision integral in Section VI, we note that our concern is with $(f'_1 f'_1 - f_1 f_1)$. We now write

$$f_2 = f_1(\bar{r}_1 \bar{v}_1) f_1(\bar{r}_2 \bar{v}_2) + P(\bar{r}_1 \bar{r}_2 \bar{v}_1 \bar{v}_2)$$

and

$$f'_2 = f'_1(\bar{r}_1 \bar{v}_1) f'_1(\bar{r}_2 \bar{v}_2) + P'(\bar{r}_1 \bar{r}_2 \bar{v}_1 \bar{v}_2)$$

and obtain:

$$f'_2 - f_2 = (f'_1 f'_1 - f_1 f_1) + (P' - P)$$

Fortunately, for spatially homogeneous systems,

$$P(\bar{r}_1 \bar{r}_2 \bar{v}_1 \bar{v}_2) = P(|\bar{r}_1 - \bar{r}_2| \bar{v}_1 \bar{v}_2)$$

(i. e., the correlation function is a function of the distance of separation only). Further, as pointed out previously, the particular values of P' and P required are those where $|\bar{r}_2 - \bar{r}_1| = L_D$ (i. e., $P'(L_D \bar{v}_1 \bar{v}_2)$ and $P(L_D \bar{v}_1 \bar{v}_2)$). However, $P'(L_D \bar{v}_1 \bar{v}_2) = P(L_D \bar{v}_1 \bar{v}_2)$ so that $f'_2 - f_2 = f'_1 f'_1 - f_1 f_1$ which is the representation first proposed by Boltzmann.*

B. Dense Spatially Homogeneous Gas

The class of systems of interest here are those in which the multiparticle collisions are responsible for the changes of interest, for example, plasma systems. The actual path of a field particle in the Debye sphere will be other than that predicted for the two-particle collision model. However, in an attempt to understand the significance of the work that has been done by Rosenbluth, MacDonald, and Judd,⁽²⁵⁾ where a two-particle model has been used to obtain the so called Fokker-Planck coefficients, we will assume that the interactions within time Δt (time for enough collisions to take place so that the

*Also since the Debye distance corresponds physically to the distance beyond which the test particle cannot feel the influence of the field particle, $P(L_D \bar{v}_1 \bar{v}_2)$ is very small.

average is statistically significant) are occurring simultaneously but that the field particles follow the same paths in space and time as they would follow if the interactions were two particle interactions.

Figure 4. Simultaneous Binary Collisions in a Debye Sphere

One now obtains for the rate of change of momentum of the test particle

$$\langle \frac{\Delta \bar{p}}{\Delta \tau} \rangle = \frac{1}{\Delta \tau} \int_0^{\Delta \tau} (\bar{F}_1(t) + \bar{F}_2(t) + \dots) dt$$

which can be written:

$$\langle \frac{\Delta \bar{p}}{\Delta \tau} \rangle = \frac{1}{\Delta \tau} \int_0^{\Delta \tau} \bar{F}_1(t) dt + \frac{1}{\Delta \tau} \int_0^{\Delta \tau} \bar{F}_2(t) dt + \dots$$

or:

$$\langle \frac{\Delta \bar{p}}{\Delta \tau} \rangle = \frac{1}{\Delta \tau} \left\{ \Delta \bar{p}_1 + \Delta \bar{p}_2 + \dots \right\}$$

This is the same result as that of part A since the force functions and particle paths in space and time are assumed to be precisely the same. Hence, we conclude that for given paths the computed value of $\langle \frac{\Delta \bar{p}}{\Delta \tau} \rangle$ is the same whether collisions are sequential, simultaneous, or anything in between. This observation has been alluded to, if not explicitly presented, by Grad⁽²⁶⁾ and others.

Since for plasma systems one is concerned with multiple particle collisions and since no one has included the perturbation of the space time path of the particles due to the presence of other particles, it would be highly desirable to determine the magnitude of the effect of this perturbation. Crude calculations were made to determine the perturbation to the path with impact parameter of a magnitude corresponding to those collisions which contribute more significantly to the changes of interest. The results indicate that the perturbation is extremely small.

A-V. THE FOKKER-PLANCK EQUATION

When a dense system of particles is considered, the physical interpretation implied by the Boltzmann collision integral (that of successive binary encounters) is obviously false. A mathematical form which would depict many simultaneous encounters is desired. The Fokker-Planck equation is one which attempts to mathematically portray the physically correct situation.

For example, look at some characteristic distances in a plasma:

Let: b = distance of closest approach of two particles in absence of any mutual force

d_0 = the mean interparticle distance

b_0 = the value of the collision parameter producing a 90° deflection

L_D = the Debye shielding distance.

Now for almost all situations of interest $b_0 \ll d_0 \ll L_D$. In region I, where $b < b_0$, close encounters (encounters producing large deflections) occur. These certainly are well described by the Boltzmann collision integral. In region II, $b_0 < b < d_0$, encounters lead to relatively small deflections. Here again the Boltzmann collision integral provides an adequate description, since it is argued that a collision with one particle will be over before another particle approaches to a distance less than d .

In region III, $d_0 < b < L_D$, it would appear as though binary collision analysis were invalid and that an approach considering, say, charge density fluctuations would have to be used.

Chandrasekhar, ⁽²⁷⁾ using the theory of stochastic processes, developed the F-P equation in a general form for the problem of Brownian movement. This type of analysis is also applicable to region III of a plasma. The collision term may be written:

$$\Delta t \frac{\partial f}{\partial t} \bigg|_c = - \frac{\partial}{\partial v_i} \left[f \langle \Delta v_i \rangle \right] + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left[f \langle \Delta v_i \Delta v_j \rangle \right] + O \left[\langle \Delta v_i \Delta v_j \Delta v_k \rangle \right]$$

where

$$\langle \Delta v_i \rangle = \int \psi(\vec{v}, \Delta \vec{v}) d\Delta \vec{v} \Delta v_i$$

and

$$\langle \Delta v_i \Delta v_j \rangle = \int \psi(\vec{v}, \Delta \vec{v}) d\Delta \vec{v} \Delta v_i \Delta v_j$$

$\psi(\bar{v}, \Delta\bar{v})$ is the probability that in time Δt a particle's velocity changes from $\bar{v}$ to $\bar{v} + \Delta\bar{v}$. It should be noted that in the Fokker-Planck formulation only collisions producing small changes in $\bar{v}$ are considered. Spitzer⁽²⁸⁾ has shown that the cumulative effect of many distant encounters (those producing small changes in $\bar{v}$) far outweighs the effect of the relatively infrequent close encounters.

The interesting thing about the Fokker-Planck equation when applied to a plasma system is that it only exists to dominant order; terms containing $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ are terms of order $\ln \Lambda$, where Λ is the ratio of the Debye shielding distance to the classical distance of closest approach. Now as Grad⁽²⁵⁾ has pointed out, the dominant order terms of any theory of the collision process will be identical to those obtained from the use of a binary collision analysis, provided the same shielding cutoff is used. This fact is evidenced by the work of Kaufman⁽²⁹⁾ who derived the Fokker-Planck equation from the Liouville equation assuming binary collisions in region II and charge density fluctuations in region III. Kaufman's results are identical with those of Rosenbluth, MacDonald and Judd⁽²⁵⁾ who derived the Fokker-Planck collision integral for inverse square forces under the assumption of binary collisions in both regions II and III.

Rosenbluth, MacDonald and Judd obtain the following values for $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$.

$$\langle \Delta v_i \rangle = - \frac{4\pi e^4 \ln \Lambda}{m^\gamma m^\beta} \frac{|\bar{g}|}{m^\gamma + m^\beta}$$

$$\langle \Delta v_i \Delta v_j \rangle = \frac{4\pi e^4 \ln \Lambda}{m^\gamma m^\beta} \frac{[\delta_{ij} - g_i g_j]}{g^2}$$

where $\bar{g}$ is the relative velocity of two particles. Therefore, $\left. \frac{\partial f^\gamma}{\partial t} \right|_c$ is written:

$$\left. \frac{\partial f^\gamma}{\partial t} \right|_c = \Gamma^\gamma \left[- \frac{\partial}{\partial v_i} \left(f^\gamma \frac{\partial H^\gamma}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f^\gamma \frac{\partial^2 G}{\partial v_i \partial v_j} \right) \right]$$

where:

$$a. \quad \Gamma^\gamma = \frac{4\pi e^4 \ln \Lambda}{m^\gamma m^2}$$

$$b. \quad H^\gamma(v) = \sum_\beta \frac{m^\gamma + m^\beta}{m^\beta} \int d\bar{v}' \frac{f^\beta(\bar{v}')}{|\bar{g}|}$$

$$c. \quad G^\gamma(v) = \sum_\beta \int d\bar{v}' f^\beta(\bar{v}') |\bar{g}|$$

If we can use binary analysis to compute the Fokker-Planck collision terms (that is, we consider each field particle to exert its influence on a test particle independently of the other field particles), then we may conclude that the Boltzmann equation is more accurate than the Fokker-Planck equation since the Boltzmann equation includes everything in the Fokker-Planck equation plus the effects of close encounters. To see that this is so, it must be realized that even though the physical interpretation of the two forms is different, they are mathematically equivalent within the restrictions inherent to the Fokker-Planck equation. This fact is discussed more fully in Section A-IV and can be verified by expanding the Boltzmann collision integral in small changes in $\Delta \bar{v}$ and retaining terms to second order in $\Delta \bar{v}$. A collision integral identical to that of Rosenbluth, MacDonald and Judd is obtained.

It has been frequently stated that the use of the Boltzmann equation to describe a plasma is invalid and that the Fokker-Planck equation should be used instead. We can see that this is incorrect, and for a given problem the choice of which equation to use depends upon which equation lends itself more readily to solution. All things being equal the Boltzmann equation should be chosen since it should produce the more accurate results.

Finally it must be noted that there are restrictions on the use of the Fokker-Planck equation. These are the same as those of the Boltzmann equation and will be more fully covered in the discussion of that equation. Briefly the Fokker-Planck equation is valid only if the spatial variation of the plasma is small compared to the Debye length and if the time variation is small compared to the plasma frequency.

A - VI. VLASOV EQUATION

The Vlasov, or collisionless Boltzmann equation, is most easily obtained by letting the collision term of the Boltzmann equation go to zero. The force term is now interpreted in a slightly different way; the macroscopic electric field is obtained by considering all of the plasma rather than just the particles outside the Debye sphere. The identical equation may be obtained from the BBGKY theory by allowing P to approach zero in equation 1(D) of Section A-II (letting P approach zero is just another way of saying that collision effects are negligible). The Vlasov equation is thus written:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + a_i \frac{\partial f}{\partial v_i} = 0$$

where

$$\bar{a} = \frac{|q| |\bar{E}|}{m} = \frac{N_0}{m} \int_0^\infty d\bar{r}_2 d\bar{v}_2 \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(\bar{r}_2, \bar{v}_2, t)$$

in the notation of Section II. The Vlasov equation is simply a statement that the total time derivative of f , taken along a particle trajectory, is zero.

A "derivation" such as the above may be misleading; the Vlasov equation is not just a special case of the Boltzmann equation. It is a formulation that has its own restricted range of validity. Certainly the Vlasov equation could be used to describe the uninteresting case of a plasma so tenuous that collision effects are vanishingly small. Its most important use lies in providing a description of a plasma system undergoing small wavelength oscillations. The Vlasov equation applies to situations where the Boltzmann equation, because of the restriction that f must be slowly varying, is inapplicable. Also, since it is used to describe plasma systems in which collision effects are not of importance, it is, in general, not suitable for the study of transport properties.

A-VII. THE BOLTZMANN EQUATION

The Boltzmann equation is a deceptively simple integrodifferential equation which depicts the time development of the one-particle distribution. Essentially it is the statement that total time derivative of f computed along a particle trajectory in phase space is equal to the change in f with time due to collisions. That is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + a_i \frac{\partial f}{\partial v_i} = \frac{\partial f}{\partial t}_c$$

The development of the collision integral has the interesting characteristic that it has as its basis a mixture of a microscopic and a macroscopic model. The probability of particle 1 being located in the phase space element $d^3r_1 d^3v_1$ is $f(\bar{r}_1, \bar{v}_1) d^3r_1 d^3v_1$ and the probability of particle 2 being located in the phase space element $d^3r_2 d^3v_2$ is $f(\bar{r}_2, \bar{v}_2) d^3r_2 d^3v_2$, which is a macroscopic approach. (The phase space elements $d^3r_1 d^3v_1$ and $d^3r_2 d^3v_2$ of interest here are six-dimensional volume elements far enough apart and of such velocities that they represent pre-collision elements.) However, the probability of particle 1 ending up, after a collision, in $d^3r'_1 d^3v'_1$ and particle 2 ending up in $d^3r'_2 d^3v'_2$ is determined by the kinematics of the interacting particles, which is a microscopic view.

With this type of approach, the net gain of particles into an element $d^3r_1 d^3v_1$ is shown to be

$$\left\{ \int |\bar{g}| (f'_1 f'_1 - f_1 f_1) d\Omega d^3v_2 \right\} d^3r_1 d^3r_2$$

where $\bar{g}$ is the relative velocity involved. As pointed out in Section A-IV, the required distribution is f_2 which before interaction equals $f_1 f_1$ (i.e., the pre-collision two-particle distribution function). We have no need to know the deviation of f_2 from the product, $f_1 f_1$, which f_2 experiences during the course of a collision.

An heuristic derivation of the Boltzmann equation, following the original arguments of Boltzmann, may be found in Chapman and Cowling.⁽³⁾ A more mathematical approach starting with the Liouville equation may be found in the work of Bogolyubov,⁽¹⁴⁾ Grad,⁽¹⁾ or Kirkwood.⁽¹⁷⁾

Regardless of the method used to derive the Boltzmann equation, certain assumptions are implicitly made. These are:

a. Binary Paths -- Whether events are sequential or simultaneous, the paths of interacting particles do not deviate, in space and time, from the binary collision paths.

b. Slowly Varying f -- It is assumed that the spatial variation of f is small compared to the inter-particle force range and that the time variation of f is slow compared to the time involved in a collision

process. In a plasma system it is required that the spatial variation be small compared to the Debye length and the time variation be small compared to the plasma frequency.

c. Molecular Chaos -- It is assumed that if two particles are not in each other's field no correlation between their velocities exists.

These assumptions are examined in more detail by Kirkwood⁽¹⁷⁾ and Grad⁽¹⁾. However, it is worthwhile to comment further on assumption c. Boltzmann applies the molecular chaos hypothesis to non-interacting particles. It has often been stated or implied that this hypothesis applies to interacting particles and that correlations between interacting particles are therefore neglected. This is an incorrect interpretation of the Boltzmann collision integral.

One further comment on some criticisms of the Boltzmann equation will perhaps be appropriate. It has been stated that the Boltzmann equation does not show the effect the inertial terms (those on the left-hand side of the equation) have on the collision integral. This criticism does have some validity; Bogolyubov⁽¹⁴⁾ showed (while deriving kinetic equations for a molecular gas) that in the spatially homogeneous case these terms are, indeed, independent, but that if spatial gradients exist the inertial terms influence the collision process. However, if the variation of f is small this connection between the two is negligible for both molecular gases and for plasmas.

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