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Effects of Buoyancy Flux on
Continental Shelf Circulation

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Abstract

The effects of buoyancy flux at the surface of a two-dimensional, steady state, continuously stratified continental shelf have been considered. Similarity solutions have been generated using an interactive technique. The solutions incorporate the influences of mechanical wind forcing, eddy diffusion, surface buoyancy flux and bottom topography. The traverse circulation can be decomposed into modes associated with the surface wind stress and the surface buoyancy flux and modified by stratification, mixing and bottom topography.

For the special cases of a wind stress with zero curl or for zero wind stress, upwelling circulations can be induced by both the sloping bottom and by gravitational convective effects. Alongshore interior currents counter to the direction of the wind are observed and are attributed to baroclinic pressure gradients.

The nature of the transverse and alongshore flows are characterized in terms of dimensionless ratios of: Coriolis to frictional forces, buoyancy to frictional forces; convective to diffusive forces; and horizontal to vertical mixing time scales and length scales.

Introduction

The relative importance of the various ocean surface processes which influence the dynamics and mass field distribution on a continental shelf have not yet been fully established either by theory or field observations. It is generally acknowledged that the effects of mechanical surface wind forcing are dominant in the relatively shallow continental margin regions of the earth. Consequently, most field programs and theoretical attempts aimed at deducing the physics of the continental shelves tend to overlook or simply ignore the fundamental influence of non-zero buoyancy flux at the free surface of the coastal zone. Buoyancy flux constraints or impositions may most generally be derived by mass or heat exchange with the atmosphere or by assuming that lateral coastal freshwater input, such as river and or estuarine outwelling and pluming occurs as wither a coastal surface feature or at least as a surface excess in the departure of a vertical integral density or volume constraint from a zero vertical plane, cross-sectional balance. There are several noteworthy exceptions to the apparent neglect of nonmechanical surface exchange forcing: Hansen and Rattray (1965) presented an estuarine flow model which contained modes of motion associated with wind stress, density gradients and mean freshwater flow; Phillips (1966) described a suggested buoyancy-driven circulation in the Red Sea;

Stommel and Leetma (1972) attempted to discriminate between wind-driven and river runoff modes of shelf circulation; Pietrafesa (1974) presented theory which differentiated between buoyancy and wind driven modes of circulation in the Red Sea; and Pietrafesa and Janowitz (1976) presented theoretical insights into modes of circulation associated with mass and heat exchange at the surface of a two-dimensional continental shelf.

Though there have been many attempts to actually measure the physics of continental margins, it was not until 1970 that a concerted, co-ordinated effort to understand the fundamental oceanography of the upwelling phenomenon was begun. This project, known as the Coastal Upwelling Ecosystems Analysis program is ongoing and includes data from the continental margins of Oregon-Washington, West Africa and Peru. Additionally, programs sponsored by ERDA, NOAA, BLM, NOS and NSF have increased the physical/dynamical data base of continental margins. Smith (1968) and O'Brien (1975) have presented excellent reviews of the state-of-the-art knowledge of coastal wind-driven upwelling. It is suggested herein that a buoyancy flux induced mode of upwelling (downwelling) may also exist.

In essence, the solutions demonstrate the interacting, mutual control of both stratification and circulation by both momentum and buoyancy sources at the surface.

Theoretical Formulation

The steady-state circulation of a stratified, incompressible Boussinesq fluid rotating with constant angular velocity $f/2$ about the vertical is assumed to be independent of the alongshore spatial variable except for the possible inclusion of longshore pressure gradients to partially balance the cross-shelf interior flow. Non-linear momentum effects and lateral momentum exchange are neglected though Garvine (1971) and Hidaka (1954) suggest that a lateral frictional boundary layer must be included in continental shelf circulation considerations. The effects of the neglected non-linear terms in the momentum equations will be considered after solutions have been obtained. Lateral mixing is included in the density equation, since Bowden (1965) showed that vertical turbulent mixing can, in the presence of a vertical velocity gradient, yield an effective lateral diffusion of density. Additionally, the Froude number is assumed small and the system is assumed to be in hydrostatic balance. The system of equations is thus not unlike those used by Hsueh and Kenney (1972), Pedlosky (1974¹ a, b), Pietrafesa (1973), and a host of others. X

The equations governing the coastal circulation in a right-handed Cartesian co-ordinate frame with x positive west, y positive north and z positive down (cf. figure 1) are X

$$fv = -\frac{1}{\rho_0} Px + (Au_z)_z, \quad (1)$$

$$-fu = -\frac{S}{\rho_0} + (Av_z)_z, \quad (2)$$

$$P_z = g\rho, \quad (3)$$

$$u_x + w_z = 0, \quad \text{and} \quad (4)$$

$$u\rho_x + w\rho_z = (D\rho_x)_x + (K\rho_z)_z, \quad (5)$$

where subscripts denote differentiation and u , v and w are x , y and z velocity components, respectively, P is pressure, ρ is the fluid density, g is gravitational acceleration, S denotes the x , z independent longshore pressure gradient, A and K denote the vertical eddy viscosity and eddy diffusion phenomenological transfer coefficients and D is the horizontal eddy diffusion exchange coefficient.

To obtain governing equations for the velocity and density fields*, we first introduce a stream function, ψ , such that

$$u = -\psi_z \text{ and } w = \psi_x, \quad (6)$$

thus identically satisfying (4).

The boundary conditions on the velocity field require that the exchanges of momentum be continuous

*Refer to Appendix A for another approach to obtaining the governing equations and boundary conditions.

across the surface, that there be a rigid lid on the system and that the velocity vanish at the bottom.

The conditions on the velocity field are

$$(a) \quad \psi(0, x) = 0, \quad (7)$$

$$(b) \quad A\psi_{zz}(0, x) = T_o(x),$$

$$(c) \quad Av_z(0, x) = -T_a(x),$$

$$(d) \quad \psi(h, x) = 0,$$

$$(e) \quad \psi_z(h, x) = 0,$$

and

$$(f) \quad v(h, x) = 0,$$

where

$T_o(x)$ and $T_a(x)$

are the x and y components, respectively, of the wind stress vector. If we now introduce the stream-function into (2) and integrate (2) with respect to z, we obtain

$$Av_z = f\psi + \frac{sz}{f} + g(x). \quad (8)$$

Using the boundary condition (7c) we find that

$$v_z(x, z) = \frac{f\psi}{A}(x, z) + \frac{sz}{f_o A} - \frac{T_a}{A}(x). \quad (9)$$

We next assume, A, K and D to be z-independent and integrate (9) in z using condition (7f) to obtain

$$v = \frac{T_a}{A} (x) (h(x)-z) + \frac{s}{2f_o A} (z^2-h^2)x) - \frac{f}{A} \int_f^h (x) \psi(x, z') dz' \quad (10)$$

Hence, v has been obtained in terms of ψ with all boundary constraints on v satisfied.

To obtain an equation for ψ , we first differentiate (1) with respect to z , use (3) to eliminate P and (9) to eliminate, v , to obtain

$$\psi_{zzzz} + \frac{f^2}{A^2} \psi = \frac{-g\rho_x}{\rho o A} + \frac{f}{A^2} T_a - \frac{fsz}{\rho o A^2} \quad (11)$$

The equation governing ρ is

$$\psi_x \rho_z - \psi_z \rho_x = (D\rho_x)_x + K\rho_{zz} \quad (12)$$

The boundary conditions on the non-linearly coupled set (11) and (12) include not only (7(a), (b), (d) and (3)) but also those of continuous exchange of buoyancy flux across the surface and of a non-conducting, impermeable bottom, i.e.

$$(a) \rho_z(x, 0) = Q(x)/K$$

and

$$(b) K\rho_z(x, h) - h_x D\rho_x(x, h) = 0 \quad ,$$

where Q represents both the flux of heat and the difference between evaporation and precipitation or rather the net freshwater addition (or depletion) at the surface.

The system of equations and boundary conditions now include (11) and (12) which with (7(a), (b), (d) and (e) and (13(a) and (b)) can be used to ψ and ζ and then, with (7(c)), to evaluate (10) for v . To solve this system we will attempt a separation transformation approach.

We now attempt a separation transformation of equations (11) and (12) with the introduction of a new set of variables such that

$$(a) \quad \zeta = \frac{x}{L} \quad , \quad (14)$$

$$(b) \quad \eta = \frac{z}{h_L} \zeta^\alpha \quad ,$$

$$(c) \quad h = h_L \zeta^{-\alpha} \quad ,$$

$$(d) \quad \phi = \frac{\psi}{\psi_L} \zeta^{-\alpha_1} \quad ,$$

$$(e) \quad \theta = (\theta(0) + \frac{(\rho(x,z) - \rho(L,0))}{\hat{\rho}}) \zeta^{-\alpha_2} \quad ,$$

$$\text{where } \hat{\rho} = \rho(L, h_L) - \rho(L, 0) \equiv \frac{|q|h_L}{K_L} \quad ,$$

$$(f) \quad A = A_L \zeta^{\alpha_3}$$

$$(g) \quad D = D_L \zeta^{\alpha_4}$$

$$(h) \quad K = K_L \zeta^{\alpha_5}$$

$$(i) \quad v = V_L \zeta^{\alpha_7}$$

$$(j) \quad \tau_o(x,0) = \tau_o \zeta^{\alpha_8}$$

$$(k) \quad \tau_a(x,0) = \tau_a \zeta^{\alpha_9}$$

and

$$(l) \quad Q(x,0) = q \zeta^{\alpha_{10}}$$

where L denotes the shelf zone width.

Substituting these transformed variables into equations (11), (12) and (10) and separating out the explicit x-dependences then yields the following set of ordinary differential equations,

$$\phi_{\eta\eta\eta\eta} + E^{-2}\phi, \quad (15)$$

$$= -Ra(\alpha_2\theta + \alpha\eta\theta_\eta) + Sn$$

$$(\Gamma + \alpha^2 \eta^2) \theta_{\eta\eta} + (\alpha\eta(3\alpha_2 - 1) - \gamma(\alpha_2 - 2\alpha - 1)\phi)\theta_\eta, \quad (16)$$

$$+ (\alpha_2(\alpha_2 - \alpha - 1) + \gamma\alpha_2\phi_\eta)\theta = 0$$

This relation could also be written as

$$(1 + I\alpha^2 \eta^2)\theta_{\eta\eta} + (I\alpha\eta(3\alpha_2 - 1) - Li(\alpha_z - 2\alpha - 1)\phi)\theta_\eta +$$

$$(I\alpha_2(\alpha_z - \alpha - 1) + Li\alpha_2\phi_\eta)\theta = 0,$$

$$\text{with } Ra = h_L^4 g \hat{\rho} / (A_L \psi_L L), E = A_L / f h_L^2, \Gamma = \frac{K_L L^2}{D_L h_L^2}$$

$$\text{and } \psi_L = (\tau_a^2 + \tau_o^2)^{1/2} \text{ or } \frac{g|q|A_L}{L f^2 K_L}, \text{ which ever}$$

$$\text{is larger, } \Gamma = \frac{K_L L^2}{D_L h_L^2}, Li = \frac{g|q|h_L A_L}{L^2 f^2 K_L} \quad (17)$$

$$\text{and } \gamma = \frac{\psi_L L}{K_L h_L}$$

The similarity conditions associated with the separation transformation are

$$\alpha_3 + 4\alpha = \alpha_2 - 1 - \alpha_1, \quad (18)$$

$$\alpha_3 + \alpha_2 - 1 = \alpha_1,$$

$$\alpha_1 - 1 - \alpha = \alpha_5,$$

$$\alpha_4 - 2 - 2\alpha - \alpha_5 = 0,$$

$$\alpha_3 + 2\alpha = 0,$$

and

$$\alpha_7 - \alpha_3 = \alpha_1 + 3\alpha.$$

It is of note that if $S \neq 0$, then we must also require that $\alpha_2 - \alpha = 1$.

The behavior of the system is characterized by dimensionless coefficients; E is the vertical Ekman No., Li is the Lineykin No., δ is an aspect ratio, γ

is a relative measure of the strength of circulation to horizontal mixing; the quantity $Ra\gamma$ is the density i.e., $\Delta S, \Delta T$ convective Raleigh No.; RaE^2 is a measure of the effect of stratification on the circulation; $\delta = h_L/L$ is an aspect ratio; and $\Gamma = \frac{K_L L^2}{h_L^2 D_L}$ is a ratio of vertical to horizontal mixing time scales.

The boundary conditions, relation (9), now become,

$$(a) \quad \phi_\eta(0) = 0, \quad (19)$$

$$(b) \quad \phi_{\eta\eta}(0) = (E^{-1}/f\psi_L)\tau_a,$$

$$(c) \quad \theta_\eta(0) = \frac{q}{K_L},$$

$$(d) \quad \phi_\eta(1) = 0,$$

$$(e) \quad \phi(1) = 0,$$

and

$$(f) \quad (\Gamma + \alpha^2)\theta_\eta(1) + \alpha\alpha_2\theta(1) = 0,$$

with constant conformal invariance constraints

$$(a) \quad \alpha_8 - 2\alpha = \alpha_1, \quad (20)$$

$$(b) \quad \alpha_{10} - \alpha = \alpha_2 + \alpha_5,$$

$$(c) \quad \alpha_6 = \alpha_1,$$

and

$$(d) \quad \alpha_9 - \alpha = \alpha_2 + \alpha_5.$$

Equations (15) and (16) are separately linear in either dependent variable but are non-linearly coupled in the variables θ and ϕ when either is expressed in terms of the other. The nature of this coupling sug-

gests that a straightforward solution may not be possible. It is of note that, in general, $E^{-2} \gg 1$ and typically, either $RaE^2 < 1$ so that either a perturbation* expansion or an interative* scheme could be used in an attempt to obtain solutions to the system.

We will now solve the coupled set (15) and (16), using a simultaneous iterative technique. We first rewrite equations (15) and (16), operationally as

$$L_1 \phi = RaE^2 L_2 \theta \quad (15a)$$

and

$$L_3(\phi, \theta, \gamma) = 0, \quad (16a)$$

Where L_1 , L_2 and L_3 denote appropriate differential operators. Then for the iterative approach, we rewrite these equations as

$$L_1 \phi_m = RaE^2 L_2 \theta_{m-1} \quad (21)$$

and

$$L_3(\phi_m, \theta_m) = 0, \quad (22)$$

where $m = 1, 2, 3, \dots$

Next, we let $\phi_0 = 0$ and solve for ϕ_1 from (21), subject to conditions (19 a, b, d and e). We then solve for θ_1 from (22), which becomes a linear equation with non-constant coefficients, subject to conditions (19 c and f). With θ , thus determined, ϕ_2

* refer to Appendix B for alternative solution approaches.

is found from (21) and then ϕ_2 from (22). We continue iterating and superposing partial solutions, in the manner described, until two consecutive solutions for θ and ϕ agree to, at least, four significant figures. For wind-driven cases, with $\gamma = 1.0$ and $RaE^2 \leq 0.5$, ten to fifteen iterations were required for solution convergence in both ϕ and θ . For strictly buoyancy driven cases, i.e., wind forcing set to zero, with $RaE^2 = 1.0$ but $\gamma < 0.1$, approximately five iterations were necessary for solution convergence. We note that since the homogeneous solutions of (21) are known, the particular solution for ϕ_m was explicitly determined by the variation of parameters technique; a Runge-Kutta scheme was used for (22).

Table 1

Case No.	Ra/Ta	(Ta) ^{1/4}	M	α	α_2	α_{10}	I	(x)	(y)	H	$\rho(L, h_L)$	$\rho(L, 0)$
1	0	20	1	-0.4	0.2		1	0	1	+1		
2	0	20	1	-0.4	0.2		1	0	-1	-1		
3	0	20	1	-0.4	2.2		1	0	1	+1		
4	0	20	1	-1.5	-2		1	0	1	+1		
5	0	20	1	-1.5	-2		1	0	-1	+1		
6	0	20	1	-1.5	0		1	0	-1	+1		
7	0	29	1	-1.5	0		1	0	1	+1		
8	0	20	1	-1.5	1		1	0	1	+1		
9	0	20	1	-1.5	1		1	0	-1	+1.0		
10	1	20	0	-1.5	-0.5		1	0	0	+1		
11	0.5	20	1	-1.5	1		1	1	1	+1		
12	0	20	1	-0.4	0.2		1	0	0	-1		
13	0.5	20	1	-0.4	0.2		1	0	1	-1		
14	0	20	1	-0.4	0.2		1	0	-1	1		
15	0.5	20	1	-0.4	0.2		1	0	-1	1		
16	0.5	20	1	-0.4	2.2		1	0	-1	1		
17	0.5	20	1	-0.4	2.2		1	0	1	1		
18	0.5	20	1	-1.5	-2		1	0	1	1		
19	0	20	1	-1.5	-2		1	0	-1	-1		
20	0.2	20	1	-0.4	2.2		1	0	-1	1		
21	0	20	1	-0.4	2.2		1	0	1	-1		
22	0.1	20	1	-0.4	2.2		1	0	-1	+1		
23	0	20	1	-0.4	1		1	1	0	1		
24	0	20	1	-0.4	1		1	-1	0	1		
25	1	20	0.2	-0.4	0.6		1	0	0	+1		
26	0.5	20	1	-1.5	-2		1	0	-1	-1		
27	0	20	1	-1.5	0.2		1	0	-1	+1		
28	0	20	1	-1.5	0.2		1	0	1	+1		
29	0.5	20	1	-1.5	1		1	0	1	+1		
30	0.1	20	1	-0.4	2.2		1	0	1	-1		
31	0.2	20	1	-0.4	2.2		1	0	1	-1		
32	0.1	20	1	-0.4	2.2		1	0	-1	+1		
33	0.3	20	1	-0.4	2.2		1	0	-1	+1		
34	0.5	20	1	-0.4	1		1	1	0	+1		
35	0	20	1	-0.4	2.2		1	0.0	1	+1		
36	0.2	20	1	-0.4	2.2		1	0	1	+1		

Table 1 (Cont'd)

Case No.	Ra/Ta	$(Ta)^{1/4}$	M	α	α_2	α_{10}	I	(x)	(y)	H	$\rho(L, h_L)$	$\rho(L, 0)$
37	0.5	20	1	-0.4	2.2		1	0	1	+1		
38	0.2	20	1	-0.4	2.2		1	0	-1	+1		
39	0.5	20	1	-0.4	1		1	-1	0	+1		
40	0.5	20	1	-1.5	1		1	-1	0	+1		
41	1	20	0.2	-1.5	-0.5		1	0	0	+1		
42	0	20	0	-1.5	0.2		1	0	-1	+1		
43	0	20	0.5	-1.5	0.1		1	0	-1	+1		
44	0	20	1	-1.5	0.2		1	0	-1	+1		
45	0.25	20	0	-1.5	0.2		1	0	-1	+1		
46	0.25	20	0.5	-1.5	0.2		1	0	-1	+1		
47	0.25	20	1	-1.5	0.2		1	0	-1	+1		
48	0.5	20	0	-1.5	0.2		1	0	-1	+1		
49	0.5	20	0.5	-1.5	0.2		1	0	-1	+1		
50	0.5	20	1	-1.5	0.2		1	0	-1	+1		
51	0.5	20	1	-0.5	1		1	1	0	+1		
52	0.5	20	1	-0.4	1		1	-1	0	+1		
53	1	20	0	-0.4	0.6		1	0	0	+1		
54	1	20	0.2	-0.4	0.6		1	0	0	+1		
55	0	20	1	-0.4	0.6		1	0	-1	+1		
56	0.5	20	1	-0.4	0.6		1	0	-1	+1		
57	0.5	20	1	-0.4	2.2		1	0	1	-1		
58	0	20	1	-0.4	2.2		1	0	-1	+1		

RESULTS AND CONCLUSIONS TO FOLLOW

Appendix A

The equations governing the coastal circulation in a right-handed Cartesian co-ordinate frame with x positive west, y positive north and z positive down (refer to Figure 1) are

$$fv = -\frac{1}{\rho_0} P_x + (Au_z)_z \quad (1)$$

$$-fu = -\frac{S}{\rho_0} + (Av_z)_z \quad (2)$$

$$P_z = g_\rho \quad (3)$$

$$u_x + w_z = 0 \quad (4)$$

$$u\rho_x + w\rho_z = (D\rho_x)_x + (K\rho_z)_z \quad (5)$$

where subscripts denote differentiation and u, v and w are x, y and z velocity components, P is pressure, ρ is the fluid density, g is gravitational acceleration, S denotes the x, z in dependent longshore pressure gradient, A and K denote the vertical eddy viscosity and eddy diffusion phenomenological transfer coefficients and D is the horizontal eddy diffusion exchange coefficient.

We first differentiate equation (1) twice in z and use (3) to eliminate P. Then taking A to be z-independent, we combine this resultant relation with equation (2), thereby eliminating v. With the introduction of the stream function ψ , such that

$$u = -\psi_z \text{ and } w = \psi_x \quad (6)$$

The following set of nonlinearly coupled equations in ψ and ρ , is obtained

$$(A(A\psi_{zz})_{zz})_z + f\psi_z = -\left(\frac{gA\rho_x}{f\rho_0}\right)_z + gS \quad (7)$$

and

$$\psi_x \rho_z - \psi_z \rho_x = (D\rho_x)_x + (K\rho_z)_z. \quad (8)$$

The boundary conditions at the free surface are: the exchanges of momentum and buoyancy flux across the interface are continuous and the vertical velocity is zero. The boundary conditions at the bottom are that there is no slip or flux across the non-conducting, impermeable wall. The conditions at the surface can be expressed (refer to Figure 1) as

$$(a) \psi_x(x,0) = 0, \quad (9)$$

$$(b) \frac{1}{f}(A_{zz}(x,0))_{zz} + \frac{g}{f\rho_0} \rho_x(x,0) = \tau_{yz}(x,0),$$

$$(c) A\psi_{zz}(x,0) = -\tau_{xz}(x,0),$$

and

$$(d) KP_z(x,0) = Q(x,0),$$

where Q represents both the flux of heat and the difference between evaporation and precipitation or rather the net freshwater addition (or depletion) at the surface. Eq. (9b) follows from the z derivative of equation (1). The conditions at the bottom are

$$(e) \quad \psi_z(x, h) = 0 \quad , \quad (9)$$

$$(f) \quad \psi_x(x, h) = 0 \quad ,$$

$$(g) \quad K \rho_z(x, h) - D h_x(x) \rho_x(x, h) = 0 \quad ,$$

where h is the bottom, and finally

$$(h) \quad v(x, h) = 0 \quad .$$

The longshore velocity component is an auxiliary calculation once ψ and P are known using the following relation

$$v_z = - \frac{g}{\rho_0 f} \rho_x - \frac{1}{f} (A(\psi_{zz}))_{zz} \quad (10)$$

and condition (9h)

We now attempt a separation transformation of equations (96) and (7) with the introduction of a new set of variables such that

$$(a) \quad \zeta = \frac{x}{L} \quad , \quad (11)$$

$$(b) \quad \eta = \frac{z}{h_L} \zeta^\alpha \quad ,$$

$$(c) \quad h = h_L \zeta^{-\alpha} \quad ,$$

$$(e) \quad \theta = (\theta(0) + \frac{(\rho(x, z) - \rho(L, 0))}{\hat{\rho}}) \zeta^{-\alpha/2} \quad ,$$

where

$$\hat{\rho} = \rho(L, h_L) - \rho(L, 0) \equiv \frac{|q| h_L}{K_L} \quad ,$$

$$(f) \quad A = A_L \zeta^{\alpha_3}$$

$$(g) \quad D = D_L \zeta^{\alpha_4}$$

$$(h) \quad K = K_L \zeta^{\alpha_5}$$

$$(i) \quad v = V_L \zeta^{\alpha_7}$$

$$(j) \quad \gamma_o(x,0) = \tau_o \zeta^{\alpha_8}$$

$$(k) \quad \gamma_a(x,0) = \tau_a \zeta^{\alpha_9}$$

and

$$(l) \quad Q(x,0) = q \zeta^{\alpha_{10}}$$

where L denotes the shelf zone width.

Substituting these transformed variables into equations (7), (8), (9) and (10) and separating out the explicit x -dependences then yields the following system of ordinary differential equations,

$$\phi_{\eta\eta\eta\eta\eta} + E^{-2} \phi_{\eta} \quad , \quad (12)$$

$$= -Ra((\alpha + \alpha_2)\theta_{\eta} + \alpha\eta\theta_{\eta\eta}) + S$$

$$(\Gamma + \alpha^2\eta^2)\theta_{\eta\eta} + (\alpha\eta(3\alpha_2 - 1) - \gamma(\alpha_2 - 2\alpha - 1)\phi)\theta_{\eta} \quad , \quad (13)$$

$$+ (\alpha_2(2\alpha_2 - \alpha - 1) + \gamma\alpha_2\phi_{\eta})\theta = 0$$

and

$$V_{\eta} = Ra(\alpha_2\theta + \alpha\eta\theta_{\eta}) + \phi_{\eta\eta\eta\eta} \quad , \quad (14)$$

with $V_L = (A_v \psi_L / fh_L^3)$, $E = A / fh_L$, $\psi_L = (\tau_a^2 + \tau_o^2)^{\frac{1}{2}}$

or $\frac{g|q|A_L}{Lf^2K_L}$, which ever is larger,

$$\Gamma = K_L L^2 / D_L h_L^2, \quad \gamma = \psi_L L / K_L h_L \quad \text{and} \quad Ra = h_L^4 \hat{g} \rho / A_L \psi_L L.$$

The similarity conditions associated with the separation transformation are

$$\alpha_3 + 4\alpha = \alpha_2 - 1 - \alpha_1, \quad (15)$$

$$\alpha_3 + \alpha_2 - 1 - \alpha_1 = 0,$$

$$\alpha_1 - 1 - \alpha - \alpha_5 = 0,$$

$$\alpha_4 - 2 - 2\alpha - \alpha_5 = 0,$$

$$\alpha_3 + 2\alpha = 0,$$

and

$$\alpha_7 - \alpha_3 = \alpha_1 + 3\alpha$$

The boundary conditions, relation (9), now become,

$$(a) \quad \phi_\eta(0) = 0, \quad (16)$$

$$(b) \quad \phi_{\eta\eta\eta\eta}(0) + \alpha_2 Ra \theta(0) = E^{-2} \frac{A}{f\psi_L} \tau_o,$$

$$(c) \quad \phi_{\eta\eta}(0) = (E^{-1}/f\psi_L) \tau_a,$$

$$(d) \quad \theta_\eta(0) = \frac{q}{K_L},$$

$$(e) \quad \phi_\eta(1) = 0,$$

$$(f) \quad \phi(1) = 0,$$

$$(g) \quad (\Gamma + \alpha^2)_{\theta} (1) + \alpha \alpha_2^{\theta} (1) = 0 \quad ,$$

and

$$(h) \quad v(1) = 0 \quad ,$$

with constant conformal invariance constraints

$$(a) \quad \alpha_8 - 2\alpha = \alpha_1 \quad , \quad \bullet$$

$$(b) \quad \alpha_{10} - \alpha = \alpha_2 + \alpha_5 \quad ,$$

$$(c) \quad \alpha_6 = \alpha_1 \quad ,$$

and

$$(d) \quad \alpha_9 - \alpha = \alpha_2 + \alpha_5$$

Appendix B

Equations (12) and (13) are separately linear in either dependent variable but are non-linearly coupled in the variables θ and ϕ when either is expressed in terms of the other. The nature of this coupling suggests that a straightforward solution may not be possible. It is of note that, in general, $E^{-2} \gg 1$ and typically, either $RaE^2 < 1$ or $RaME^2 < 1$ so that either a perturbation expansion or an iterative scheme could be used in an attempt to obtain solutions to the system.

Consider relation (12); if $RaE^2 \ll 1$ or $Ra \rightarrow 0$, then the remaining terms yield Ekman layer solutions, with scale $E^{\frac{1}{2}}$, near the boundaries $\eta=0, 1$. For finite Ra , it follows from relation (12) that in the region outside of the boundary frictional layers, i.e., the interior,

$$\phi_{\eta} \approx -RaE^2 (\alpha_2 \theta + \alpha \eta \theta_{\eta})_{\eta} + S \quad (17)$$

The θ_{η} influence will be determined, within the Ekman layers, where

$$\phi_{\eta} \sim O(E^{-\frac{1}{2}}) \quad , \quad (18)$$

by the solution of (13), using conditions (16 b, d and g). Specifically, condition (16d) will be the measure of the θ_{η} influence within the surface Ekman layer.

ϕ can be represented approximately by

$$\phi = \phi^A + \phi^B + \phi^C \quad (18)$$

where

$$\phi^A = O(RaE^2) \quad (19)$$

is obtained from (17) and

where

$$\phi^B = \text{Maximum } O(1) \quad (20)$$

and

$$\phi^C = \text{Maximum } O(RaE^2) \quad (21)$$

are the Ekman layer solutions

to

$$\phi_{\eta\eta\eta\eta\eta} + E^{-2} \phi_{\eta} = 0 \quad (22)$$

at the bottom and surface boundaries, respectively.

It follows from (19) - (21) that

$$RaE^2 \ll 1 \quad (23)$$

if boundary conditions are to be separately applied to (20) and (21). If (23) is not true then it could be possible that

$$\gamma RaE^2 \ll 1 \quad (24)$$

If (24) were true then the interaction of ϕ and θ in relation (13) could be neglected to lowest order

and either a perturbative or iterative series solution to the coupled set (12) and (13) could be tried. The ϕ , θ interaction terms in (13) are

$$-\gamma(\alpha_2 - 2\alpha - 1) \phi \theta_\eta \quad (25)$$

and

$$\gamma \alpha_2 \phi_\eta \theta \quad (26)$$

representing horizontal and vertical advection of density, respectively.

If we rewrite ϕ such that

$$\phi = \phi^* \text{RaE}^2 \quad (27)$$

then terms (25) and (26) would contain the multiplier $\text{RaE}^2 \gamma$ (28)

which, if small in coupled association with both ϕ^* and ϕ_η^* , would eliminate the ϕ , θ coupling to lowest order. Another way to consider this is to consider (18) using conditions (19), (20) and (21):

From these estimates, we obtain

$$\gamma \phi_\eta^\theta \sim \gamma \text{RaE}^2 \theta_\eta^0 + \gamma \phi_\eta^C \theta + \gamma \phi_\eta^B \theta. \quad (29)$$

The contribution from ϕ^A is negligible only if

$$\gamma \text{RaE}^2 \ll 1 \quad (30)$$

If (30) is not true, i.e., if $\gamma \text{RaE}^2 \geq 0(1)$ then equation (13) must be addressed in its full non-linearity.

It is of note that

$$\phi_{\eta}^B, \phi_{\eta}^C \sim O(E^{-\frac{1}{2}} \phi^B, \phi^C) \quad (31)$$

and exist, separately, on the scales of order $E^{\frac{1}{2}}$. The Ekman layer inputs are then of lesser importance, to lowest order, in (13) if

$$\gamma E^{\frac{1}{2}} \cdot \text{Maximum}(1, RaE^2) \ll 1 \quad (32)$$

If (31) and (32) are both true then terms (29) and (30) are both of negligible significance, to lowest order, in the solution of (13).

In general then the order of solution is such that equation can be solved first, in the form of (22) if $RaE^2 < O(1)$ and equation (13), without the non-linear terms $\phi_{\eta}\theta$ and ϕ, θ_{η} , may be solved first, if $Ra\gamma E^2 \sim O(1)$.

The order of solution and justifications thereof will be noted for the cases considered.

It is of note that a system of equations similar to the system being addressed herein, were treated by Pietrafesa and Rattray (1974) but they used a bilinear form of an expansion parameter based on the Shohat Renormalization Technique (Shohat, 1944). The range of validity of their results has been established by Struble and Pietrafesa (1975). When possible, inter-comparisons between valid results derived from the Pietrafesa - Rattray paper will be made with the results generated herein.

The system of ordinary differential equations (12) -

(14) can be rewritten in series, expansion approximation form as

$$\phi = \phi_0 + (RaE^2)\phi_1 + (RaE^2)^2 \phi_2 + \dots \quad (33)$$

$$\theta = \theta_0 + (RaE^2)\theta_1 + (RaE^2)^2 \theta_2 + \dots ,$$

$$v = v_0 + (RaE^2)v_1 + (RaE^2)^2 v_2 + \dots ,$$

$$\text{for } RaE^2 \ll 1 ,$$

or

$$\phi = \phi_0 + (RaE^2_\gamma)\phi_1 + (RaE^2_\gamma)^2 \phi_2 + \dots , \quad (34)$$

$$\theta = \theta_0 + (RaE^2_\gamma)\theta_1 + (RaE^2_\gamma)^2 \theta_2 + \dots ,$$

$$v = v_0 + (RaE^2_\gamma)v_1 + (RaE^2_\gamma)v_2 + \dots ,$$

$$\text{for } RaE^2_\gamma \ll 1 .$$

The boundary conditions, 16 a, e, f, g and h become, respectively,

$$(a) \quad \phi_\eta(0) = 0 , \quad (35)$$

$$(b) \quad \phi_\eta(1) = 0 ,$$

$$(c) \quad \phi(1) = 0 ,$$

$$(d) \quad (\Gamma + \alpha^2)\theta_\eta(1) + \alpha\alpha_2\theta(1) = 0 ,$$

and

$$(e) \quad v(1) = 0 .$$

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