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Aerodynamic Separation of Gases and Isotopes

Stage Separation Theory - E. Von Halle
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I. INTRODUCTION

Classification of Isotope Separation Processes. Isotope separation processes can be classified in a number of ways. An extremely important classification when one is concerned with processes which produce only small enrichments per stage is whether from a thermodynamical point of view the separation process is

- a) reversible (R),
- b) irreversible (I), or
- c) semi-reversible (SR).

The importance of this classification arises from the fact that irreversible isotope separation processes consume large amounts of power. In fact, the gaseous diffusion process for uranium isotope separation at one time used about 10% of the electrical power generation of the United States. In comparison, reversible processes only require an amount of power proportional to the free energy change effected by the process, which is orders of magnitude smaller than that required by an irreversible process. A semi-reversible process is one, such as chemical exchange, in which the separation actually takes place reversibly, but the reflux involves irreversible processes.

Secondly, we can classify isotope separation processes as to whether they are stage-wise enrichment processes, such as gaseous diffusion or distillation in a bubble cap plate column, or continuous, countercurrent column processes, such as the countercurrent gas centrifuge process or

EF

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distillation in a packed column. It will be shown that even though the same mathematical design equations can be used for both types of processes, there are fundamental differences between them. This classification will be designated by

- a) stage-wise separation process (S-W) and
- b) continuous countercurrent column process (C-C).

We may also attempt to classify isotope separation processes on the basis of whether the fundamental separation mechanism depends on

- a) an equilibrium process (E) or
- b) a kinetic process (K),

and also by whether the basic separation mechanism depends on

- a) a chemical process (Ch) or
- b) a physical process (Ph).

These two classification schemes, however, depend to a great extent in some cases on the definitions one chooses to use rather than on any fundamental differences between processes.

Finally, we may attempt to classify the isotope separation processes on the basis of economic considerations with respect to one another as to whether the process is relatively economical for separating the isotopes

- a) of the light elements, M less than 40 (L),
- b) of the elements of medium weight, M between 40 and 150 (M), or
- c) of the heavy elements, M greater than 150 (H),

and also as to whether the process is relatively economical for separating isotopes

- a) on a small scale, mg per day (SS),
- b) on a medium scale, g per day (MS), or
- c) on a large scale, kg per day (LS).

Table 1 lists several frequently mentioned methods of isotope separation and classifies them according to the categories listed above.

II. FUNDAMENTAL CONCEPTS

1. The Stage Separation Factor. For each and every isotope separation process there exists a fundamental quantity, α , which is an index of the amount of separation that can be attained at steady-state in an element of the process equipment. The separation stage, or in a continuous countercurrent column process, the transfer unit or equivalent theoretical plate, may be considered as a device which separates a feed stream, or streams in the latter case, into two product streams (often called "heads" and "tails" or "product" and "waste") such that the concentrations of the components in the two effluent streams are related by the quantity α . For the case of the separation of a binary mixture this relationship is

$$\frac{y/(1-y)}{x/(1-x)} = \alpha \quad (\text{II.1.1})$$

where y is the mole fraction of the desired or specified component in the "upflowing" (heads or product) stream from the stage and x is the mole fraction of the same component in the "downflowing" (tails or waste) stream from the stage. The ratio of the mole fractions, $y/(1-y)$ or $x/(1-x)$, is frequently termed the abundance ratio of the specified isotope. The quantity, α , is usually called the stage separation factor.

For the case of separating a binary mixture the following conventions are used in assigning nomenclature. Usually the separation process is for the purpose of obtaining one component of the mixture in an enriched form; the concentrations of the streams are then specified by the mole fraction of the desired component. If both components are desired, then the choice of the desired component is arbitrary. In either case it will be said

TABLE 1
CLASSIFICATION OF ISOTOPE SEPARATION PROCESSES

<u>Separation Method</u>	<u>I,SR,R</u>	<u>S-W,C-C</u>	<u>E,K</u>	<u>Ch,Ph</u>	<u>Economic Consideration</u>	
					<u>L,M,H</u>	<u>SS,MS,LS</u>
1. Electromagnetic (calutron)*	I	S-W	K	Ph	L,M,H	SS
2. Thermal diffusion	I	C-C	E	Ph	L,M	SS
3. Gaseous diffusion	I	S-W	K	Ph	M,H	LS
4. Distillation (plate column)	R	S-W	E	Ch	L	SS,MS,LS
" (packed column)	R	C-C	E	Ch	L	SS,MS,LS
5. Chemical exchange	SR	C-C	E	Ch	L	SS,MS,LS
6. Dual temperature chem. exch.	SR	C-C	E	Ch	L	MS,LS
7. Gas centrifuge	R	C-C	E	Ph	M,H	LS,MS
8. Electrolysis	I	S-W	K	Ch	L	LS,MS
9. Mass and sweep diffusion	I	C-C	E	Ph	M	MS
10. Molecular distillation	I	S-W	K	Ch	L	LS
11. Separation nozzle	I	S-W	K	Ph	H	LS
12. Ion exchange	SR	C-C	E	Ch	L	
13. Electromigration	I	C-C	K	Ph	L	
14. Photochemical rx. (laser processes)	I	S-W	K	Ch	L,M,H	?

*Based on the availability of such a facility, or the requirement for a large number of enriched isotopes.

that the "upflowing" stream from the separation stage is the one in which the specified component is enriched, and, by virtue of this convention, α is defined as a quantity whose value is always greater than unity. However, for many of the processes which we will consider, α exceeds unity by only a very small fraction.

It can be seen from equation (II.1.1) that for a stage which yields no separation (i.e., $y = x$), the value of the stage separation factor is equal to unity. For this reason, a somewhat more useful parameter is the quantity, ψ , defined by

$$\psi \equiv \alpha - 1 \quad (\text{II.1.2})$$

which is equal to zero for the case of a non-separating stage. We will call ψ the stage enrichment factor. In chemical engineering terminology, the relationship between the concentrations of the streams leaving a stage is called the equilibrium equation. Combining equations (II.1.1) and (II.1.2), we obtain the "exact form" of the equilibrium relationship which can be written

$$y = x + \frac{\psi x(1 - x)}{1 + \psi x} \quad (\text{II.1.3a})$$

Since, for most isotope separation processes (the main exceptions are the electromagnetic processes and the photochemical and photophysical processes), α exceeds unity by only a very small fraction, it follows that ψ is a positive quantity very much less than unity. Although this results in practical disadvantages — large plants with many stages to effect a desired separation, it results also in analytical advantages as it permits us to use simplified equations without introducing appreciable error. Expanding the term $1/(1 + \psi x)$ in equation (II.1.3a) in a power series in x and neglecting terms of greater than the first power in ψ , we obtain the "approximate form" of the equilibrium relationship, valid for small

ψ -values

$$y = x + \psi x(1 - x)\{1 - \psi x + \psi^2 x^2 - \dots\} \approx x + \psi x(1 - x) \quad (\text{II.1.3b})$$

in which the fractional error introduced by the approximation in the concentration difference, $y - x$, is always less than ψx .

Other interesting forms of the equilibrium equation which will prove useful in special applications are the "dilute approximation" for the term $x(1 - x)$ which yields

$$y = x + \psi x \quad (\text{II.1.3c})$$

valid when x is small, the "linear approximation" for the term $x(1 - x)$ which yields

$$y = x + \psi(px + q) \quad (\text{II.1.3d})$$

valid, with the proper choice of p and q , over any range of concentrations which is not too great, and the "constant approximation" for the term $x(1 - x)$ which yields

$$y = x + \psi q \quad (\text{II.1.3e})$$

which is valid in the concentration range in the neighborhood of x equal to 0.5.

The form of the equilibrium relationship given by equation (II.1.1) was chosen because for most separation processes the stage separation factor α , so defined, is essentially independent of the concentration of the stage. This assertion is supported by the following arguments.

a) For a kinetic type process, such as gaseous diffusion, one has a current of each component through the barrier proportional to the concentration of that component on the high pressure side of the barrier and proportional to a rate constant specific for that component. We can therefore write:

$$J_1 = k_1 x' \quad \text{and} \quad J_2 = k_2 (1 - x')$$

$$\frac{y'}{1 - y'} = \frac{J_1}{J_2} = \frac{k_1}{k_2} \frac{x'}{1 - x'}$$

which is of the form of equation (II.1.1) with $\alpha = k_1/k_2$ and therefore independent of concentration.

b) For an equilibrium process, such as chemical exchange, one has the situation of equal and opposite currents of each component when equilibrium exists. We can therefore write

$$\begin{array}{l} k_1 y' = k_1 x' \\ k_2 (1 - y') = k_2 (1 - x') \\ \frac{y'}{1 - y'} = \frac{k_1 k_2}{k_2 k_1} \frac{x'}{1 - x'} \end{array} \quad \begin{array}{c} \uparrow \\ x' \quad \leftarrow \quad J_1 = k_1 x' \\ \downarrow \\ 1-x' \quad \leftarrow \quad J_1' = k_1' y' \quad y' \\ \downarrow \\ 1-x' \quad \leftarrow \quad J_2 = k_2 (1-x') \quad 1-y' \\ \downarrow \\ 1-x' \quad \leftarrow \quad J_2' = k_2' (1-y') \end{array}$$

which is again of the same form as equation (II.1.1) with $\alpha = k_1 k_2' / (k_2 k_1')$ independent of concentration.

2. The Value Function and the Separative Capacity of a Separation Stage.

In a stage of a separation process there is a certain amount of energy expended in order to effect the separation. By virtue of this investment of energy, the effluent streams from the stage should have more value than the feed to the stage. We shall now endeavor to find a "value function" which reflects the value of a unit of material of a given concentration, and which will enable us to evaluate the increase in the value of the streams effected by the stage. The function we seek should have the following properties:

a) If $v(x)$ represents the value of one unit of material of concentration x , then the value of M units of material of the same concentration is $Mv(x)$.

b) Since the energy expended in a stage on a given quantity of feed material is independent of its concentration, the increase in the

value of the streams effected by the stage should also be independent of the concentration of the material being processed.

A simple separation stage, shown schematically in Figure II.1, takes a feed stream and splits it into two fractions, one slightly enriched in the specified component and the other slightly depleted.

FIGURE II.1 A SIMPLE SEPARATION STAGE

In Figure II.1, G is the stage feed rate, L is the stage upflow rate, and L' is the stage downflow rate, all measured in consistent units of mass per unit time. The concentration of the specified component in the feed stream is z , in the stage upflow stream y , and in the stage downflow stream x . The quantity θ , termed the stage cut, is defined as the fraction of the stage feed which leaves the stage as the upflow stream: thus $\theta = L/G$.

Assuming the existence of a value function, $v(x)$, the increase in the value of the material processed per unit time, δU , effected by the separation stage is obtained by making a value balance around the stage as follows:

$$\delta U = \theta G v(y) + (1 - \theta) G v(x) - G v(z) \quad (\text{II.2.1})$$

Since the enrichment which can be achieved in a single stage of an isotope separation process is small, i.e., x , y , and z differ only slightly from one another, we can expand $v(y)$ and $v(z)$ in Taylor series about x and

retain only the first non-vanishing term in (II.2.1) without introducing appreciable error. Thus we write

$$v(y) = v(x) + (y - x)v'(x) + \frac{1}{2}(y - x)^2v''(x) + \dots \quad (\text{II.2.2a})$$

$$\text{and } v(z) = v(x) + (z - x)v'(x) + \frac{1}{2}(z - x)^2v''(x) + \dots \quad (\text{II.2.2b})$$

Furthermore, from a specified component material balance around the separation stage, we have the relationship

$$z = \theta y + (1 - \theta)x, \text{ or } z - x = \theta(y - x) \quad (\text{II.2.3})$$

Substituting the last three expressions into equation (II.2.1) and neglecting terms containing the very small concentration difference, $y - x$, raised to powers greater than two, we obtain

$$\begin{aligned} \delta U &= \theta Gv(x) + \theta G(y - x)v'(x) + \theta G\frac{1}{2}(y - x)^2v''(x) \\ &\quad + (1 - \theta)Gv(x) - Gv(x) - G\theta(y - x)v'(x) - G\frac{1}{2}\theta^2(y - x)^2v''(x) \\ \delta U &= \frac{1}{2} G\theta(1 - \theta)(y - x)^2v''(x). \end{aligned} \quad (\text{II.2.4})$$

The approximate stage equilibrium relationship, equation (II.1.3b), is now used to replace the concentration difference, $y - x$, in the preceding equation with the result

$$\delta U = \frac{1}{2} G\theta(1 - \theta)\psi^2x^2(1 - x)^2v''(x) \quad (\text{II.2.5})$$

Thus, we see that in order for δU to be independent of concentration, it is necessary to set the product of the terms in x in equation (II.2.5) equal to a constant: for convenience, let this constant be unity. We now have a grip on the value function we are after by its second derivative.

The change in the value of the material processed effected by the stage per unit time will be called the separative capacity of the stage. Other authors have called it the separative power. With the value function defined as indicated above, we see from equation (II.2.5) that the separative capacity of a stage is given by

$$\delta U = \frac{1}{8} G\psi^2 = \frac{1}{4} L\psi^2 . \quad (\text{II.2.7})$$

The value function itself can now be obtained from the second order differential equation

$$v''(x) = \frac{1}{x^2(1-x)^2} . \quad (\text{II.2.8})$$

This expression is readily integrated with the use of partial fractions.

We find that

$$v''(x) = \frac{2}{x} + \frac{2}{1-x} + \frac{1}{x^2} + \frac{1}{(1-x)^2}$$

which can be integrated once with respect to x to yield

$$v'(x) = 2\ln x - 2\ln(1-x) - \frac{1}{x} + \frac{1}{1-x} + C_1 .$$

A second integration with respect to x yields

$$\begin{aligned} v(x) &= 2x\ln x - 2x + 2(1-x)\ln(1-x) - 2(1-x) \\ &\quad - \ln x - \ln(1-x) + C_1x + C_2 . \end{aligned}$$

Therefore, collecting terms, we obtain the following expressions:

$$v'(x) = 2\ln \frac{x}{1-x} + \frac{2x-1}{x(1-x)} + C_1 \quad (\text{II.2.9})$$

$$v(x) = (2x-1)\ln \frac{x}{1-x} - 2 + C_1x + C_2 \quad (\text{II.2.10})$$

where C_1 and C_2 are constants of integration. The most convenient forms of these equations for use are obtained by letting

$$v'(0.5) = 0, \text{ whereby } C_1 = 0, \text{ and} \quad (\text{II.2.11})$$

$$v(0.5) = 0, \text{ whereby } C_2 = 2, \quad (\text{II.2.12})$$

so that we obtain as a final result:

$$v(x) = (2x-1)\ln \frac{x}{1-x} \quad (\text{II.2.13})$$

$$v'(x) = 2\ln \frac{x}{1-x} + \frac{2x-1}{x(1-x)} \quad (\text{II.2.14})$$

$$v''(x) = \frac{1}{x^2(1-x)^2} . \quad (\text{II.2.15})$$

It may be noted that the value function, $v(x)$, as here defined, is

symmetric about $x = 0.5$, that is, $v(1 - x) = v(x)$; but that $v'(1 - x) = -v'(x)$. Tables of the value function (Table 2) and of the derivative of the value function (Table 3) have been prepared for $0.001 \leq x \leq 0.5$.

Illustration II.1 Calculate the separative capacity required to produce 1 lb-mole uranium per year containing 90 mole % U-235 from normal uranium containing 0.72 mole % U-235. The waste or reject material may be assumed to contain 0.36 mole % U-235.

The separative capacity required is obtained by taking a value balance around the cascade, similar to that taken around the stage. The required separative capacity, U , is therefore given by

$$U = Pv(y_p) + Wv(x_w) - Fv(x_F)$$

where P is the product rate and y_p the product concentration, W is the waste rate and x_w its concentration, and F is the feed rate and x_F the feed concentration. From the total and specified component material balances around the cascade, we obtain W and F in terms of P and the concentrations.

$$P + W = F$$

$$Py_p + Wx_w = Fx_F$$

From these two material balance equations we obtain

$$P(y_p - x_w) = F(x_F - x_w)$$

$$F = P \frac{y_p - x_w}{x_F - x_w} = 1 \frac{0.9000 - 0.0036}{0.0072 - 0.0036} = 249.000 \text{ lb-mole/yr}$$

$$W = F - P = 248.000 \text{ lb-mole/yr}$$

Therefore, with the aid of the value unit tables, we find that

$$U = (1)(1.75778) + (248)(5.58273) - (249)(4.85551)$$

TABLE 2
VALUE FUNCTION

$$V(X) = (2X-1)\text{LN}(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.0010	6.89294	6.88286	6.87288	6.86300	6.85321	6.84351	6.83391	6.82439	6.81496	6.80562
.0011	6.79636	6.78718	6.77809	6.76907	6.76014	6.75128	6.74250	6.73379	6.72515	6.71659
.0012	6.70809	6.69967	6.69132	6.68303	6.67481	6.66665	6.65856	6.65053	6.64257	6.63466
.0013	6.62682	6.61903	6.61130	6.60363	6.59602	6.58846	6.58096	6.57351	6.56612	6.55878
.0014	6.55149	6.54425	6.53706	6.52992	6.52283	6.51579	6.50880	6.50185	6.49495	6.48810
.0015	6.48129	6.47452	6.46780	6.46112	6.45449	6.44790	6.44135	6.43484	6.42837	6.42194
.0016	6.41555	6.40921	6.40290	6.39662	6.39039	6.38419	6.37803	6.37191	6.36582	6.35977
.0017	6.35375	6.34777	6.34182	6.33590	6.33002	6.32418	6.31836	6.31258	6.30683	6.30111
.0018	6.29542	6.28977	6.28414	6.27854	6.27298	6.26744	6.26194	6.25646	6.25101	6.24559
.0019	6.24020	6.23483	6.22949	6.22418	6.21890	6.21365	6.20842	6.20321	6.19803	6.19288
.0020	6.18776	6.18265	6.17758	6.17252	6.16750	6.16249	6.15751	6.15256	6.14762	6.14271
.0021	6.13783	6.13296	6.12812	6.12330	6.11851	6.11373	6.10898	6.10425	6.09954	6.09485
.0022	6.09018	6.08553	6.08091	6.07630	6.07171	6.06715	6.06260	6.05807	6.05357	6.04908
.0023	6.04461	6.04016	6.03573	6.03132	6.02692	6.02255	6.01819	6.01385	6.00953	6.00523
.0024	6.00094	5.99667	5.99242	5.98819	5.98397	5.97977	5.97559	5.97142	5.96727	5.96313
.0025	5.95902	5.95491	5.95083	5.94676	5.94270	5.93867	5.93464	5.93064	5.92664	5.92266
.0026	5.91870	5.91475	5.91082	5.90690	5.90300	5.89911	5.89524	5.89137	5.88753	5.88369
.0027	5.87988	5.87607	5.87228	5.86850	5.86474	5.86099	5.85725	5.85352	5.84981	5.84612
.0028	5.84243	5.83876	5.83510	5.83145	5.82782	5.82419	5.82058	5.81699	5.81340	5.80983
.0029	5.80627	5.80272	5.79918	5.79566	5.79214	5.78864	5.78515	5.78167	5.77820	5.77475
.0030	5.77130	5.76787	5.76445	5.76103	5.75763	5.75424	5.75086	5.74750	5.74414	5.74079
.0031	5.73745	5.73413	5.73081	5.72751	5.72421	5.72093	5.71765	5.71439	5.71113	5.70789
.0032	5.70465	5.70143	5.69821	5.69501	5.69181	5.68863	5.68545	5.68228	5.67913	5.67598
.0033	5.67284	5.66971	5.66659	5.66348	5.66037	5.65728	5.65420	5.65112	5.64805	5.64500
.0034	5.64195	5.63891	5.63587	5.63285	5.62984	5.62683	5.62383	5.62084	5.61786	5.61489
.0035	5.61193	5.60897	5.60602	5.60308	5.60015	5.59723	5.59431	5.59140	5.58850	5.58561
.0036	5.58273	5.57985	5.57698	5.57412	5.57127	5.56842	5.56558	5.56275	5.55993	5.55712
.0037	5.55431	5.55151	5.54871	5.54593	5.54315	5.54038	5.53761	5.53485	5.53210	5.52936
.0038	5.52662	5.52389	5.52117	5.51846	5.51575	5.51304	5.51035	5.50766	5.50498	5.50230
.0039	5.49964	5.49698	5.49432	5.49167	5.48903	5.48639	5.48377	5.48114	5.47853	5.47592
.0040	5.47331	5.47072	5.46813	5.46554	5.46296	5.46039	5.45782	5.45526	5.45271	5.45016
.0041	5.44762	5.44508	5.44255	5.44003	5.43751	5.43500	5.43249	5.42999	5.42750	5.42501
.0042	5.42253	5.42005	5.41758	5.41511	5.41265	5.41020	5.40775	5.40530	5.40287	5.40043
.0043	5.39801	5.39558	5.39317	5.39076	5.38835	5.38595	5.38356	5.38117	5.37878	5.37640
.0044	5.37403	5.37166	5.36930	5.36694	5.36459	5.36224	5.35990	5.35756	5.35523	5.35290
.0045	5.35058	5.34826	5.34594	5.34364	5.34133	5.33904	5.33674	5.33445	5.33217	5.32989

TABLE 2 CONT'D.

VALUE FUNCTION

$$V(X) = (2X-1)\text{LN}(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.0046	5.32762	5.32535	5.32309	5.32083	5.31857	5.31632	5.31408	5.31183	5.30960	5.30737
.0047	5.30514	5.30292	5.30070	5.29849	5.29628	5.29407	5.29187	5.28968	5.28749	5.28530
.0048	5.28312	5.28094	5.27877	5.27660	5.27443	5.27227	5.27012	5.26796	5.26582	5.26367
.0049	5.26153	5.25940	5.25727	5.25514	5.25302	5.25090	5.24879	5.24668	5.24457	5.24247
.0050	5.24037	5.23828	5.23619	5.23410	5.23202	5.22994	5.22787	5.22580	5.22373	5.22167
.0051	5.21961	5.21756	5.21551	5.21346	5.21142	5.20938	5.20735	5.20531	5.20329	5.20126
.0052	5.19924	5.19723	5.19521	5.19321	5.19120	5.18920	5.18720	5.18521	5.18322	5.18123
.0053	5.17925	5.17727	5.17529	5.17332	5.17135	5.16938	5.16742	5.16546	5.16351	5.16156
.0054	5.15961	5.15767	5.15572	5.15379	5.15185	5.14992	5.14799	5.14607	5.14415	5.14223
.0055	5.14032	5.13841	5.13650	5.13460	5.13270	5.13080	5.12891	5.12702	5.12513	5.12324
.0056	5.12136	5.11949	5.11761	5.11574	5.11387	5.11201	5.11015	5.10829	5.10643	5.10458
.0057	5.10273	5.10088	5.09904	5.09720	5.09537	5.09353	5.09170	5.08987	5.08805	5.08623
.0058	5.08441	5.08259	5.08078	5.07897	5.07717	5.07536	5.07356	5.07176	5.06997	5.06818
.0059	5.06639	5.06460	5.06282	5.06104	5.05926	5.05749	5.05572	5.05395	5.05218	5.05042
.0060	5.04866	5.04690	5.04515	5.04339	5.04165	5.03990	5.03816	5.03641	5.03468	5.03294
.0061	5.03121	5.02948	5.02775	5.02603	5.02431	5.02259	5.02087	5.01916	5.01745	5.01574
.0062	5.01403	5.01233	5.01063	5.00893	5.00724	5.00554	5.00385	5.00217	5.00048	4.99880
.0063	4.99712	4.99544	4.99377	4.99209	4.99043	4.98876	4.98709	4.98543	4.98377	4.98211
.0064	4.98046	4.97881	4.97716	4.97551	4.97387	4.97222	4.97058	4.96895	4.96731	4.96568
.0065	4.96405	4.96242	4.96080	4.95917	4.95755	4.95593	4.95432	4.95270	4.95109	4.94948
.0066	4.94788	4.94627	4.94467	4.94307	4.94148	4.93988	4.93829	4.93670	4.93511	4.93352
.0067	4.93194	4.93036	4.92878	4.92720	4.92563	4.92406	4.92249	4.92092	4.91935	4.91779
.0068	4.91623	4.91467	4.91311	4.91156	4.91000	4.90845	4.90690	4.90536	4.90381	4.90227
.0069	4.90073	4.89920	4.89766	4.89613	4.89460	4.89307	4.89154	4.89002	4.88849	4.88697
.0070	4.88545	4.88394	4.88242	4.88091	4.87940	4.87789	4.87638	4.87488	4.87338	4.87188
.0071	4.87038	4.86888	4.86739	4.86590	4.86441	4.86292	4.86143	4.85995	4.85847	4.85699
.0072	4.85551	4.85403	4.85256	4.85108	4.84961	4.84814	4.84668	4.84521	4.84375	4.84229
.0073	4.84083	4.83937	4.83792	4.83647	4.83501	4.83356	4.83212	4.83067	4.82923	4.82778
.0074	4.82634	4.82491	4.82347	4.82204	4.82060	4.81917	4.81774	4.81632	4.81489	4.81347
.0075	4.81204	4.81062	4.80921	4.80779	4.80637	4.80496	4.80355	4.80214	4.80073	4.79933
.0076	4.79792	4.79652	4.79512	4.79372	4.79233	4.79093	4.78954	4.78814	4.78675	4.78537
.0077	4.78398	4.78259	4.78121	4.77983	4.77845	4.77707	4.77570	4.77432	4.77295	4.77158
.0078	4.77021	4.76884	4.76747	4.76611	4.76474	4.76338	4.76202	4.76066	4.75931	4.75795
.0079	4.75660	4.75525	4.75390	4.75255	4.75120	4.74986	4.74852	4.74717	4.74583	4.74449
.0080	4.74316	4.74182	4.74049	4.73916	4.73782	4.73650	4.73517	4.73384	4.73252	4.73119

TABLE 2 CONT'D.

VALUE FUNCTION

$$V(X) = (2X-1)\text{LN}(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.0081	4.72987	4.72855	4.72723	4.72592	4.72460	4.72329	4.72198	4.72067	4.71936	4.71805
.0082	4.71674	4.71544	4.71414	4.71283	4.71153	4.71024	4.70894	4.70764	4.70635	4.70506
.0083	4.70376	4.70247	4.70119	4.69990	4.69861	4.69733	4.69605	4.69477	4.69349	4.69221
.0084	4.69093	4.68966	4.68839	4.68711	4.68584	4.68457	4.68330	4.68204	4.68077	4.67951
.0085	4.67825	4.67699	4.67573	4.67447	4.67321	4.67196	4.67070	4.66945	4.66820	4.66695
.0086	4.66570	4.66445	4.66321	4.66196	4.66072	4.65948	4.65824	4.65700	4.65576	4.65453
.0087	4.65329	4.65206	4.65083	4.64960	4.64837	4.64714	4.64591	4.64469	4.64346	4.64224
.0088	4.64102	4.63980	4.63858	4.63736	4.63615	4.63493	4.63372	4.63251	4.63130	4.63009
.0089	4.62888	4.62767	4.62646	4.62526	4.62406	4.62285	4.62163	4.62045	4.61926	4.61806
.0090	4.61686	4.61567	4.61448	4.61328	4.61209	4.61090	4.60972	4.60853	4.60734	4.60616
.0091	4.60498	4.60379	4.60261	4.60143	4.60025	4.59908	4.59790	4.59673	4.59555	4.59438
.0092	4.59321	4.59204	4.59087	4.58970	4.58854	4.58737	4.58621	4.58505	4.58388	4.58272
.0093	4.58157	4.58041	4.57925	4.57809	4.57694	4.57579	4.57464	4.57348	4.57233	4.57119
.0094	4.57004	4.56889	4.56775	4.56660	4.56546	4.56432	4.56318	4.56204	4.56090	4.55976
.0095	4.55863	4.55749	4.55636	4.55523	4.55409	4.55296	4.55183	4.55071	4.54958	4.54845
.0096	4.54733	4.54620	4.54508	4.54396	4.54284	4.54172	4.54060	4.53948	4.53837	4.53725
.0097	4.53614	4.53503	4.53392	4.53280	4.53170	4.53059	4.52948	4.52837	4.52727	4.52616
.0098	4.52506	4.52396	4.52286	4.52176	4.52066	4.51956	4.51846	4.51737	4.51627	4.51518
.0099	4.51409	4.51300	4.51190	4.51082	4.50973	4.50864	4.50755	4.50647	4.50538	4.50430
.0100	4.50322	4.50214	4.50106	4.49998	4.49890	4.49782	4.49674	4.49567	4.49460	4.49352
.0101	4.49245	4.49138	4.49031	4.48924	4.48817	4.48710	4.48604	4.48497	4.48391	4.48285
.0102	4.48178	4.48072	4.47966	4.47860	4.47754	4.47649	4.47543	4.47437	4.47332	4.47227
.0103	4.47121	4.47016	4.46911	4.46806	4.46701	4.46597	4.46492	4.46387	4.46283	4.46178
.0104	4.46074	4.45970	4.45866	4.45762	4.45658	4.45554	4.45450	4.45347	4.45243	4.45140
.0105	4.45036	4.44933	4.44830	4.44727	4.44624	4.44521	4.44418	4.44315	4.44213	4.44110
.0106	4.44008	4.43905	4.43803	4.43701	4.43599	4.43497	4.43395	4.43293	4.43191	4.43090
.0107	4.42988	4.42887	4.42785	4.42684	4.42583	4.42482	4.42381	4.42280	4.42179	4.42078
.0108	4.41978	4.41877	4.41777	4.41676	4.41576	4.41476	4.41375	4.41275	4.41175	4.41076
.0109	4.40976	4.40876	4.40776	4.40677	4.40577	4.40478	4.40379	4.40280	4.40181	4.40081
.0110	4.39983	4.39884	4.39785	4.39686	4.39588	4.39489	4.39391	4.39292	4.39194	4.39096
.0111	4.38998	4.38900	4.38802	4.38704	4.38606	4.38509	4.38411	4.38313	4.38216	4.38119
.0112	4.38021	4.37924	4.37827	4.37730	4.37633	4.37536	4.37439	4.37343	4.37246	4.37150
.0113	4.37053	4.36957	4.36860	4.36764	4.36668	4.36572	4.36476	4.36380	4.36284	4.36188
.0114	4.36093	4.35997	4.35902	4.35806	4.35711	4.35616	4.35520	4.35425	4.35330	4.35235
.0115	4.35140	4.35046	4.34951	4.34856	4.34762	4.34667	4.34573	4.34478	4.34384	4.34290

TABLE 2 CONT'D.

VALUE FUNCTION

$$V(X) = (2X-1)\ln(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.0116	4.34196	4.34102	4.24008	4.33914	4.33820	4.33726	4.33632	4.33539	4.33445	4.33352
.0117	4.33259	4.33165	4.33072	4.32979	4.32886	4.32793	4.32700	4.32607	4.32514	4.32422
.0118	4.32329	4.32236	4.32144	4.32052	4.31959	4.31867	4.31775	4.31683	4.31591	4.31499
.0119	4.31407	4.31315	4.31223	4.31131	4.31040	4.30948	4.30857	4.30765	4.30674	4.30583
.0120	4.30492	4.30401	4.30310	4.30219	4.30128	4.30037	4.29946	4.29855	4.29765	4.29674
.0121	4.29584	4.29493	4.29403	4.29313	4.29223	4.29133	4.29042	4.28953	4.28863	4.28773
.0122	4.28683	4.28593	4.28504	4.28414	4.28325	4.28235	4.28146	4.28056	4.27967	4.27878
.0123	4.27789	4.27700	4.27611	4.27522	4.27433	4.27345	4.27256	4.27167	4.27079	4.26990
.0124	4.26902	4.26813	4.26725	4.26637	4.26549	4.26461	4.26373	4.26285	4.26197	4.26109
.0125	4.26021	4.25933	4.25846	4.25758	4.25671	4.25583	4.25496	4.25409	4.25321	4.25234
.0126	4.25147	4.25060	4.24973	4.24886	4.24799	4.24713	4.24626	4.24539	4.24453	4.24366
.0127	4.24280	4.24193	4.24107	4.24021	4.23934	4.23848	4.23762	4.23676	4.23590	4.23504
.0128	4.23418	4.23333	4.23247	4.23161	4.23076	4.22990	4.22905	4.22819	4.22734	4.22649
.0129	4.22564	4.22478	4.22393	4.22308	4.22223	4.22138	4.22054	4.21969	4.21884	4.21799
.0130	4.21715	4.21630	4.21546	4.21461	4.21377	4.21293	4.21208	4.21124	4.21040	4.20956
.0131	4.20872	4.20788	4.20704	4.20620	4.20537	4.20453	4.20369	4.20286	4.20202	4.20119
.0132	4.20035	4.19952	4.19869	4.19786	4.19702	4.19619	4.19536	4.19453	4.19370	4.19287
.0133	4.19205	4.19122	4.19039	4.18957	4.18874	4.18791	4.18709	4.18627	4.18544	4.18462
.0134	4.18380	4.18297	4.18215	4.18133	4.18051	4.17969	4.17887	4.17806	4.17724	4.17642
.0135	4.17560	4.17479	4.17397	4.17316	4.17234	4.17153	4.17072	4.16990	4.16909	4.16828
.0136	4.16747	4.16666	4.16585	4.16504	4.16423	4.16342	4.16261	4.16181	4.16100	4.16019
.0137	4.15939	4.15858	4.15778	4.15697	4.15617	4.15537	4.15456	4.15376	4.15296	4.15216
.0138	4.15136	4.15056	4.14976	4.14896	4.14817	4.14737	4.14657	4.14577	4.14498	4.14418
.0139	4.14339	4.14259	4.14180	4.14101	4.14022	4.13942	4.13863	4.13784	4.13705	4.13626
.0140	4.13547	4.13468	4.13389	4.13310	4.13232	4.13153	4.13074	4.12996	4.12917	4.12839
.0141	4.12760	4.12682	4.12604	4.12525	4.12447	4.12369	4.12291	4.12213	4.12135	4.12057
.0142	4.11979	4.11901	4.11823	4.11746	4.11668	4.11590	4.11513	4.11435	4.11357	4.11280
.0143	4.11203	4.11125	4.11048	4.10971	4.10893	4.10816	4.10739	4.10662	4.10585	4.10508
.0144	4.10431	4.10354	4.10278	4.10201	4.10124	4.10048	4.09971	4.09894	4.09818	4.09741
.0145	4.09665	4.09589	4.09512	4.09436	4.09360	4.09284	4.09207	4.09131	4.09055	4.08979
.0146	4.08903	4.08828	4.08752	4.08676	4.08600	4.08525	4.08449	4.08373	4.08298	4.08222
.0147	4.08147	4.08071	4.07996	4.07921	4.07846	4.07770	4.07695	4.07620	4.07545	4.07470
.0148	4.07395	4.07320	4.07245	4.07170	4.07096	4.07021	4.06946	4.06872	4.06797	4.06722
.0149	4.06648	4.06573	4.06499	4.06425	4.06350	4.06276	4.06202	4.06128	4.06054	4.05979
.0150	4.05905	4.05831	4.05757	4.05684	4.05610	4.05536	4.05462	4.05388	4.05315	4.05241

TABLE 2 CONT'D.

VALUE FUNCTION

$$V(X) = (2X-1)\text{LN}(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.01	4.50322	4.39983	4.30492	4.21715	4.13547	4.05905	3.98723	3.91944	3.85525	3.79426
.02	3.73615	3.68064	3.62751	3.57654	3.52755	3.48038	3.43490	3.39098	3.34851	3.30739
.03	3.26753	3.22886	3.19129	3.15476	3.11922	3.08461	3.05087	3.01796	2.98584	2.95447
.04	2.92381	2.89383	2.86450	2.83578	2.80765	2.78009	2.75308	2.72658	2.70058	2.67506
.05	2.65000	2.62538	2.60119	2.57741	2.55403	2.53103	2.50840	2.48613	2.46421	2.44262
.06	2.42135	2.40040	2.37976	2.35941	2.33934	2.31956	2.30005	2.28080	2.26180	2.24306
.07	2.22455	2.20629	2.18825	2.17044	2.15284	2.13546	2.11829	2.10131	2.08454	2.06796
.08	2.05157	2.03537	2.01934	2.00349	1.98781	1.97231	1.95696	1.94178	1.92676	1.91190
.09	1.89718	1.88262	1.86820	1.85392	1.83978	1.82579	1.81192	1.79820	1.78460	1.77113
.10	1.75778	1.74456	1.73146	1.71848	1.70561	1.69286	1.68023	1.66770	1.65529	1.64298
.11	1.63078	1.61868	1.60669	1.59479	1.58300	1.57130	1.55971	1.54820	1.53679	1.52547
.12	1.51425	1.50311	1.49206	1.48110	1.47022	1.45943	1.44873	1.43810	1.42756	1.41709
.13	1.40671	1.39640	1.38617	1.37602	1.36595	1.35594	1.34601	1.33615	1.32637	1.31665
.14	1.30701	1.29743	1.28792	1.27848	1.26911	1.25980	1.25056	1.24138	1.23226	1.22321
.15	1.21422	1.20529	1.19642	1.18762	1.17887	1.17018	1.16155	1.15297	1.14446	1.13600
.16	1.12760	1.11925	1.11095	1.10271	1.09453	1.08640	1.07832	1.07029	1.06231	1.05439
.17	1.04651	1.03869	1.03092	1.02319	1.01552	1.00789	1.00031	.99278	.98529	.97785
.18	.97046	.96312	.95582	.94856	.94135	.93418	.92706	.91998	.91295	.90596
.19	.89901	.89210	.88523	.87841	.87163	.86488	.85818	.85152	.84490	.83832
.20	.83178	.82527	.81881	.81238	.80600	.79965	.79333	.78706	.78082	.77462
.21	.76846	.76233	.75624	.75018	.74416	.73818	.73223	.72631	.72043	.71458
.22	.70877	.70300	.69725	.69154	.68586	.68022	.67461	.66903	.66348	.65797
.23	.65249	.64704	.64162	.63623	.63088	.62555	.62026	.61500	.60977	.60456
.24	.59939	.59425	.58914	.58406	.57901	.57398	.56899	.56403	.55909	.55418
.25	.54931	.54446	.53964	.53484	.53008	.52534	.52063	.51595	.51129	.50667
.26	.50206	.49749	.49295	.48843	.48393	.47947	.47503	.47061	.46622	.46186
.27	.45753	.45322	.44893	.44467	.44044	.43623	.43205	.42789	.42376	.41965
.28	.41556	.41150	.40747	.40346	.39947	.39551	.39157	.38766	.38377	.37990
.29	.37606	.37224	.36845	.36468	.36093	.35720	.35350	.34982	.34616	.34253
.30	.33892	.33533	.33177	.32822	.32470	.32120	.31773	.31427	.31084	.30743
.31	.30405	.30068	.29734	.29401	.29071	.28743	.28418	.28094	.27772	.27453
.32	.27136	.26821	.26508	.26197	.25888	.25581	.25276	.24974	.24673	.24375
.33	.24078	.23784	.23492	.23201	.22913	.22627	.22342	.22060	.21780	.21502
.34	.21225	.20951	.20679	.20409	.20140	.19874	.19609	.19347	.19086	.18828
.35	.18571	.18316	.18064	.17813	.17564	.17317	.17072	.16828	.16587	.16348

TABLE 2 CONT'D.

VALUE FUNCTION

$$V(X) = (2X-1)\text{LN}(X/(1-X))$$

X	0	1	2	3	4	5	6	7	8	9
.36	.16110	.15875	.15641	.15409	.15179	.14951	.14724	.14500	.14277	.14057
.37	.13838	.13621	.13405	.13192	.12980	.12771	.12563	.12357	.12152	.11950
.38	.11749	.11550	.11353	.11158	.10964	.10773	.10583	.10395	.10208	.10024
.39	.09841	.09660	.09481	.09303	.09127	.08953	.08781	.08610	.08442	.08275
.40	.08109	.07946	.07784	.07624	.07465	.07309	.07154	.07001	.06849	.06699
.41	.06551	.06405	.06260	.06117	.05976	.05837	.05699	.05563	.05428	.05295
.42	.05164	.05035	.04907	.04781	.04657	.04534	.04413	.04294	.04176	.04060
.43	.03946	.03833	.03722	.03613	.03505	.03399	.03295	.03192	.03091	.02992
.44	.02894	.02798	.02703	.02611	.02519	.02430	.02342	.02256	.02171	.02088
.45	.02007	.01927	.01849	.01772	.01698	.01624	.01553	.01483	.01415	.01348
.46	.01283	.01219	.01157	.01097	.01039	.00982	.00926	.00872	.00820	.00770
.47	.00721	.00674	.00628	.00584	.00541	.00500	.00461	.00423	.00387	.00353
.48	.00320	.00289	.00259	.00231	.00205	.00180	.00157	.00135	.00115	.00097
.49	.00080	.00065	.00051	.00039	.00029	.00020	.00013	.00007	.00003	.00001
.50	0.00000	.00001	.00003	.00007	.00013	.00020	.00029	.00039	.00051	.00065

TABLE 3
DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2LN[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.0010	1012.81	1002.89	993.165	983.627	974.272	965.096	956.092	947.256	938.584	930.071
.0011	921.712	913.504	905.443	897.523	889.743	882.098	874.584	867.199	859.939	852.800
.0012	845.781	838.877	832.086	825.406	818.833	812.365	806.000	799.735	793.568	787.496
.0013	781.518	775.630	769.832	764.121	758.495	752.952	747.491	742.109	736.805	731.577
.0014	726.424	721.344	716.335	711.397	706.520	701.723	696.986	692.313	687.703	683.154
.0015	678.667	674.238	669.868	665.555	661.298	657.096	652.947	648.851	644.807	640.814
.0016	636.871	632.976	629.130	625.330	621.577	617.870	614.207	610.587	607.011	603.477
.0017	599.984	596.533	593.121	589.749	586.415	583.120	579.861	576.640	573.455	570.305
.0018	567.190	564.110	561.063	558.040	555.069	552.120	549.203	546.317	543.462	540.637
.0019	537.842	535.076	532.338	529.629	526.948	524.295	521.668	519.068	516.494	513.946
.0020	511.423	508.926	506.453	504.004	501.580	499.179	496.801	494.446	492.114	489.804
.0021	487.516	485.249	483.004	480.780	478.577	476.394	474.232	472.089	469.966	467.862
.0022	465.777	463.712	461.664	459.635	457.624	455.631	453.656	451.698	449.757	447.833
.0023	445.925	444.035	442.160	440.301	438.459	436.632	434.820	433.024	431.242	429.476
.0024	427.724	425.987	424.264	422.555	420.860	419.179	417.512	415.858	414.217	412.590
.0025	410.975	409.374	407.785	406.208	404.644	403.093	401.553	400.025	398.509	397.005
.0026	395.512	394.031	392.561	391.102	389.654	388.217	386.791	385.375	383.970	382.576
.0027	381.191	379.817	378.453	377.099	375.755	374.420	373.096	371.780	370.474	369.178
.0028	357.891	356.613	355.343	354.083	352.832	351.589	350.356	349.130	347.913	346.705
.0029	355.505	354.313	353.129	351.954	350.786	349.626	348.474	347.330	346.193	345.064
.0030	343.943	342.828	341.722	340.622	339.530	338.445	337.367	336.296	335.232	334.174
.0031	333.124	332.080	331.043	330.013	328.989	327.972	326.961	325.956	324.958	323.966
.0032	322.980	322.000	321.026	320.058	319.097	318.141	317.191	316.246	315.308	314.375
.0033	313.448	312.526	311.610	310.700	309.795	308.895	308.001	307.111	306.228	305.349
.0034	304.475	303.607	302.744	301.885	301.032	300.183	299.340	298.501	297.667	296.838
.0035	296.014	295.194	294.379	293.568	292.762	291.961	291.164	290.372	289.584	288.800
.0036	288.021	287.246	286.475	285.708	284.946	284.188	283.434	282.684	281.938	281.196
.0037	280.458	279.724	278.994	278.268	277.546	276.827	276.113	275.402	274.695	273.992
.0038	273.292	272.596	271.904	271.215	270.530	269.848	269.170	268.495	267.824	267.156
.0039	266.492	265.831	265.174	264.519	263.868	263.221	262.576	261.935	261.297	260.662
.0040	260.031	259.402	258.777	258.155	257.536	256.919	256.306	255.696	255.089	254.485
.0041	253.884	253.285	252.690	252.097	251.508	250.921	250.337	249.755	249.177	248.601
.0042	248.028	247.458	246.890	246.325	245.763	245.203	244.646	244.091	243.540	242.990
.0043	242.443	241.899	241.357	240.818	240.281	239.747	239.215	238.686	238.159	237.634
.0044	237.112	236.592	236.074	235.559	235.046	234.535	234.027	233.521	233.017	232.515
.0045	232.016	231.519	231.024	230.531	230.040	229.552	229.065	228.581	228.099	227.619

TABLE 3 CONT'D.
DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2\text{LN}[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.0046	227.141	226.665	226.191	225.719	225.249	224.782	224.316	223.852	223.390	222.930
.0047	222.472	222.016	221.562	221.110	220.660	220.211	219.765	219.320	218.877	218.436
.0048	217.997	217.560	217.124	216.691	216.259	215.829	215.400	214.973	214.549	214.125
.0049	213.704	213.284	212.866	212.450	212.035	211.622	211.211	210.801	210.393	209.986
.0050	209.582	209.178	208.777	208.377	207.978	207.581	207.186	206.792	206.400	206.009
.0051	205.620	205.232	204.846	204.462	204.078	203.697	203.317	202.938	202.560	202.185
.0052	201.810	201.437	201.066	200.696	200.327	199.959	199.593	199.229	198.866	198.504
.0053	198.143	197.784	197.426	197.070	196.715	196.361	196.009	195.657	195.308	194.959
.0054	194.612	194.266	193.921	193.577	193.235	192.894	192.554	192.216	191.879	191.542
.0055	191.208	190.874	190.542	190.210	189.880	189.551	189.224	188.897	188.572	188.248
.0056	187.925	187.603	187.282	186.962	186.644	186.326	186.010	185.695	185.381	185.068
.0057	184.756	184.445	184.136	183.827	183.519	183.213	182.907	182.603	182.300	181.997
.0058	181.696	181.396	181.097	180.799	180.501	180.205	179.910	179.616	179.323	179.031
.0059	178.739	178.449	178.160	177.872	177.584	177.298	177.013	176.728	176.445	176.162
.0060	175.881	175.600	175.320	175.041	174.763	174.486	174.210	173.935	173.661	173.387
.0061	173.115	172.843	172.573	172.303	172.034	171.766	171.498	171.232	170.967	170.702
.0062	170.438	170.175	169.913	169.652	169.391	169.132	168.873	168.615	168.358	168.101
.0063	167.846	167.591	167.337	167.084	166.831	166.580	166.329	166.079	165.830	165.581
.0064	165.334	165.087	164.841	164.595	164.351	164.107	163.864	163.621	163.380	163.139
.0065	162.898	162.659	162.420	162.182	161.945	161.709	161.473	161.238	161.003	160.770
.0066	160.537	160.304	160.073	159.842	159.612	159.382	159.153	158.925	158.698	158.471
.0067	158.245	158.019	157.795	157.570	157.347	157.124	156.902	156.681	156.460	156.240
.0068	156.020	155.801	155.583	155.365	155.148	154.932	154.716	154.501	154.286	154.072
.0069	153.859	153.647	153.434	153.223	153.012	152.802	152.592	152.383	152.175	151.967
.0070	151.760	151.553	151.347	151.141	150.937	150.732	150.528	150.325	150.123	149.920
.0071	149.719	149.518	149.318	149.118	148.919	148.720	148.522	148.324	148.127	147.931
.0072	147.735	147.539	147.344	147.150	146.956	146.763	146.570	146.378	146.185	145.995
.0073	145.804	145.614	145.424	145.235	145.046	144.858	144.671	144.484	144.297	144.111
.0074	143.925	143.740	143.556	143.372	143.188	143.005	142.822	142.640	142.458	142.277
.0075	142.096	141.916	141.736	141.557	141.378	141.200	141.022	140.845	140.668	140.491
.0076	140.315	140.140	139.965	139.790	139.616	139.442	139.269	139.096	138.923	138.751
.0077	138.580	138.409	138.238	138.068	137.898	137.729	137.560	137.392	137.224	137.056
.0078	136.889	136.722	136.556	136.390	136.224	136.059	135.895	135.730	135.567	135.403
.0079	135.240	135.078	134.915	134.754	134.592	134.431	134.271	134.111	133.951	133.791
.0080	133.632	133.474	133.316	133.158	133.001	132.843	132.687	132.531	132.375	132.219

TABLE 3 CONT'D.
DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2\ln[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.0081	132.064	131.909	131.755	131.601	131.447	131.294	131.141	130.989	130.837	130.685
.0082	130.534	130.383	130.232	130.082	129.932	129.782	129.633	129.484	129.336	129.188
.0083	129.040	128.892	128.745	128.599	128.452	128.306	128.161	128.015	127.870	127.726
.0084	127.581	127.437	127.294	127.150	127.007	126.865	126.723	126.581	126.439	126.298
.0085	126.157	126.016	125.876	125.736	125.596	125.457	125.318	125.179	125.041	124.903
.0086	124.765	124.628	124.491	124.354	124.217	124.081	123.945	123.810	123.675	123.540
.0087	123.405	123.271	123.137	123.003	122.870	122.737	122.604	122.471	122.339	122.207
.0088	122.076	121.945	121.814	121.683	121.552	121.422	121.292	121.163	121.034	120.905
.0089	120.776	120.648	120.520	120.392	120.264	120.137	120.010	119.883	119.757	119.631
.0090	119.505	119.379	119.254	119.129	119.004	118.880	118.756	118.632	118.508	118.385
.0091	118.262	118.139	118.016	117.894	117.772	117.650	117.528	117.407	117.286	117.165
.0092	117.045	116.925	116.805	116.685	116.566	116.446	116.328	116.209	116.090	115.972
.0093	115.854	115.737	115.619	115.502	115.385	115.268	115.152	115.036	114.920	114.804
.0094	114.689	114.573	114.459	114.344	114.229	114.115	114.001	113.887	113.774	113.660
.0095	113.547	113.435	113.322	113.210	113.098	112.986	112.874	112.763	112.651	112.540
.0096	112.430	112.319	112.209	112.099	111.989	111.879	111.770	111.661	111.552	111.443
.0097	111.335	111.226	111.118	111.011	110.903	110.796	110.688	110.582	110.475	110.368
.0098	110.262	110.156	110.050	109.944	109.839	109.734	109.629	109.524	109.419	109.315
.0099	109.211	109.107	109.003	108.899	108.796	108.693	108.590	108.487	108.385	108.282
.0100	108.180	108.078	107.976	107.875	107.774	107.672	107.572	107.471	107.370	107.270
.0101	107.170	107.070	106.970	106.871	106.771	106.672	106.573	106.474	106.376	106.277
.0102	106.179	106.081	105.983	105.886	105.788	105.691	105.594	105.497	105.400	105.304
.0103	105.207	105.111	105.015	104.920	104.824	104.729	104.633	104.538	104.444	104.349
.0104	104.254	104.160	104.066	103.972	103.878	103.785	103.691	103.598	103.505	103.412
.0105	103.319	103.227	103.134	103.042	102.950	102.858	102.766	102.675	102.584	102.492
.0106	102.401	102.311	102.220	102.129	102.039	101.949	101.859	101.769	101.679	101.590
.0107	101.501	101.411	101.322	101.234	101.145	101.056	100.968	100.880	100.792	100.704
.0108	100.616	100.529	100.441	100.354	100.267	100.180	100.094	100.007	99.9205	99.8343
.0109	99.7482	99.6622	99.5764	99.4908	99.4053	99.3199	99.2348	99.1497	99.0649	98.9801
.0110	98.8956	98.8111	98.7269	98.6428	98.5588	98.4750	98.3913	98.3078	98.2244	98.1412
.0111	98.0582	97.9752	97.8925	97.8098	97.7274	97.6450	97.5628	97.4808	97.3989	97.3172
.0112	97.2355	97.1541	97.0728	96.9916	96.9105	96.8297	96.7489	96.6683	96.5878	96.5075
.0113	96.4273	96.3473	96.2674	96.1876	96.1080	96.0285	95.9491	95.8699	95.7909	95.7119
.0114	95.6331	95.5545	95.4759	95.3975	95.3193	95.2412	95.1632	95.0853	95.0076	94.9300
.0115	94.8526	94.7753	94.6981	94.6210	94.5441	94.4673	94.3906	94.3141	94.2377	94.1615

TABLE 3 CONT'D.

DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2LN[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.0116	94.0853	94.0093	93.9334	93.8577	93.7821	93.7066	93.6312	93.5560	93.4809	93.4059
.0117	93.3310	93.2563	93.1817	93.1072	93.0329	92.9587	92.8846	92.8106	92.7367	92.6630
.0118	92.5894	92.5159	92.4426	92.3693	92.2962	92.2232	92.1503	92.0776	92.0050	91.9324
.0119	91.8601	91.7878	91.7156	91.6436	91.5717	91.4999	91.4282	91.3567	91.2853	91.2139
.0120	91.1427	91.0717	91.0007	90.9298	90.8591	90.7885	90.7180	90.6476	90.5773	90.5072
.0121	90.4371	90.3672	90.2974	90.2277	90.1581	90.0886	90.0193	89.9500	89.8809	89.8119
.0122	89.7430	89.6742	89.6055	89.5369	89.4684	89.4001	89.3318	89.2637	89.1956	89.1277
.0123	89.0599	88.9922	88.9246	88.8571	88.7898	88.7225	88.6553	88.5883	88.5213	88.4545
.0124	88.3878	88.3211	88.2546	88.1882	88.1219	88.0557	87.9896	87.9236	87.8577	87.7919
.0125	87.7262	87.6607	87.5952	87.5298	87.4645	87.3994	87.3343	87.2693	87.2045	87.1397
.0126	87.0751	87.0105	86.9461	86.8817	86.8175	86.7533	86.6893	86.6253	86.5614	86.4977
.0127	86.4340	86.3705	86.3070	86.2437	86.1804	86.1172	86.0542	85.9912	85.9283	85.8656
.0128	85.8029	85.7403	85.6778	85.6154	85.5531	85.4910	85.4289	85.3668	85.3049	85.2431
.0129	85.1814	85.1198	85.0582	84.9968	84.9355	84.8742	84.8130	84.7520	84.6910	84.6301
.0130	84.5693	84.5087	84.4481	84.3875	84.3271	84.2668	84.2066	84.1464	84.0864	84.0264
.0131	83.9665	83.9067	83.8470	83.7874	83.7279	83.6685	83.6092	83.5499	83.4908	83.4317
.0132	83.3727	83.3138	83.2550	83.1963	83.1377	83.0791	83.0207	82.9623	82.9040	82.8458
.0133	82.7877	82.7297	82.6717	82.6139	82.5561	82.4984	82.4408	82.3833	82.3259	82.2686
.0134	82.2113	82.1541	82.0970	82.0400	81.9831	81.9263	81.8695	81.8129	81.7563	81.6998
.0135	81.6433	81.5870	81.5307	81.4746	81.4185	81.3625	81.3065	81.2507	81.1949	81.1392
.0136	81.0836	81.0281	80.9726	80.9173	80.8620	80.8068	80.7517	80.6966	80.6416	80.5867
.0137	80.5319	80.4772	80.4226	80.3680	80.3135	80.2591	80.2047	80.1505	80.0963	80.0422
.0138	79.9882	79.9342	79.8803	79.8265	79.7728	79.7192	79.6656	79.6121	79.5587	79.5054
.0139	79.4521	79.3989	79.3458	79.2927	79.2398	79.1869	79.1341	79.0813	79.0287	78.9761
.0140	78.9236	78.8711	78.8188	78.7665	78.7142	78.6621	78.6100	78.5580	78.5061	78.4542
.0141	78.4024	78.3507	78.2991	78.2475	78.1960	78.1446	78.0933	78.0420	77.9908	77.9396
.0142	77.8886	77.8376	77.7866	77.7358	77.6850	77.6343	77.5836	77.5331	77.4825	77.4321
.0143	77.3817	77.3315	77.2812	77.2311	77.1810	77.1310	77.0810	77.0311	76.9813	76.9316
.0144	76.8819	76.8323	76.7827	76.7332	76.6838	76.6345	76.5852	76.5360	76.4869	76.4378
.0145	76.3888	76.3399	76.2910	76.2422	76.1934	76.1448	76.0962	76.0476	75.9991	75.9507
.0146	75.9024	75.8541	75.8059	75.7577	75.7097	75.6616	75.6137	75.5658	75.5180	75.4702
.0147	75.4225	75.3749	75.3273	75.2798	75.2323	75.1849	75.1376	75.0904	75.0432	74.9960
.0148	74.9490	74.9020	74.8550	74.8082	74.7613	74.7146	74.6679	74.6213	74.5747	74.5282
.0149	74.4817	74.4353	74.3890	74.3428	74.2966	74.2504	74.2043	74.1583	74.1124	74.0665
.0150	74.0206	73.9748	73.9291	73.8835	73.8379	73.7923	73.7468	73.7014	73.6561	73.6108

TABLE 3 CONT'D.
DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2\ln[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.01	108.180	98.8956	91.1427	84.5693	78.9236	74.0206	69.7218	65.9210	62.5357	59.5005
.02	56.7632	54.2816	52.0210	49.9527	48.0529	46.3015	44.6815	43.1784	41.7798	40.4750
.03	39.2546	38.1106	37.0359	36.0243	35.0702	34.1687	33.3156	32.5069	31.7391	31.0093
.04	30.3144	29.6521	29.0200	28.4161	27.8334	27.2852	26.7550	26.2462	25.7576	25.2880
.05	24.8362	24.4013	23.9821	23.5780	23.1880	22.8113	22.4474	22.0954	21.7549	21.4253
.06	21.1059	20.7964	20.4962	20.2049	19.9221	19.6474	19.3805	19.1210	18.8686	18.6229
.07	18.3838	18.1509	17.9240	17.7029	17.4872	17.2769	17.0716	16.8712	16.6756	16.4845
.08	16.2977	16.1152	15.9368	15.7622	15.5915	15.4244	15.2608	15.1006	14.9438	14.7901
.09	14.6395	14.4919	14.3472	14.2052	14.0660	13.9295	13.7954	13.6639	13.5347	13.4079
.10	13.2833	13.1610	13.0407	12.9226	12.8064	12.6922	12.5799	12.4695	12.3609	12.2540
.11	12.1488	12.0453	11.9434	11.8431	11.7443	11.6470	11.5512	11.4568	11.3638	11.2722
.12	11.1818	11.0928	11.0050	10.9185	10.8331	10.7490	10.6659	10.5840	10.5032	10.4235
.13	10.3448	10.2671	10.1905	10.1148	10.0400	9.96625	9.89337	9.82139	9.75029	9.68005
.14	9.61065	9.54207	9.47431	9.40733	9.34113	9.27570	9.21100	9.14704	9.08380	9.02125
.15	8.95940	8.89622	8.83770	8.77784	8.71861	8.66000	8.60202	8.54463	8.48784	8.43162
.16	8.37598	8.32090	8.26637	8.21237	8.15891	8.10597	8.05355	8.00162	7.95019	7.89925
.17	7.84879	7.79879	7.74926	7.70018	7.65155	7.60336	7.55560	7.50826	7.46135	7.41484
.18	7.36874	7.32303	7.27772	7.23280	7.18825	7.14408	7.10027	7.05683	7.01374	6.97100
.19	6.92861	6.88656	6.84484	6.80346	6.76240	6.72165	6.68123	6.64111	6.60131	6.56180
.20	6.52259	6.48367	6.44504	6.40670	6.36864	6.33085	6.29334	6.25609	6.21911	6.18239
.21	6.14593	6.10973	6.07377	6.03806	6.00259	5.96737	5.93238	5.89763	5.86310	5.82881
.22	5.79474	5.76039	5.72726	5.69384	5.66064	5.62765	5.59486	5.56228	5.52991	5.49773
.23	5.46575	5.43396	5.40237	5.37096	5.33974	5.30871	5.27786	5.24719	5.21670	5.18638
.24	5.15624	5.12626	5.09646	5.06683	5.03736	5.00805	4.97890	4.94992	4.92109	4.89241
.25	4.86389	4.83552	4.80730	4.77923	4.75131	4.72353	4.69589	4.66840	4.64104	4.61382
.26	4.58674	4.55979	4.53298	4.50630	4.47975	4.45332	4.42703	4.40086	4.37481	4.34889
.27	4.32309	4.29740	4.27184	4.24640	4.22107	4.19585	4.17075	4.14577	4.12089	4.09612
.28	4.07146	4.04691	4.02247	3.99813	3.97389	3.94976	3.92573	3.90179	3.87796	3.85423
.29	3.83059	3.80705	3.78361	3.76026	3.73700	3.71384	3.69076	3.66778	3.64488	3.62208
.30	3.59936	3.57672	3.55418	3.53172	3.50934	3.48704	3.46483	3.44269	3.42064	3.39867
.31	3.37677	3.35495	3.33321	3.31155	3.28996	3.26844	3.24700	3.22563	3.20433	3.18311
.32	3.16196	3.14087	3.11986	3.09891	3.07803	3.05722	3.03647	3.01579	2.99517	2.97462
.33	2.95414	2.93371	2.91335	2.89305	2.87281	2.85263	2.83251	2.81245	2.79245	2.77250
.34	2.75261	2.73278	2.71301	2.69329	2.67362	2.65401	2.63446	2.61496	2.59551	2.57611
.35	2.55676	2.53746	2.51822	2.49902	2.47988	2.46078	2.44173	2.42273	2.40377	2.38487

TABLE 3 CONT'D.
DERIVATIVE OF VALUE FUNCTION

$$-(D[V(X)]/DX) = -2\text{LN}[X/(1-X)] - (2X-1)/[X(1-X)]$$

X	0	1	2	3	4	5	6	7	8	9
.36	2.36601	2.34719	2.32842	2.30970	2.29102	2.27238	2.25378	2.23523	2.21673	2.19826
.37	2.17983	2.16145	2.14311	2.12481	2.10654	2.08832	2.07013	2.05199	2.03388	2.01581
.38	1.99777	1.97977	1.96181	1.94389	1.92600	1.90814	1.89032	1.87254	1.85479	1.83707
.39	1.81938	1.80173	1.78411	1.76652	1.74896	1.73144	1.71394	1.69648	1.67904	1.66164
.40	1.64426	1.62692	1.60960	1.59231	1.57505	1.55781	1.54060	1.52342	1.50627	1.48914
.41	1.47204	1.45496	1.43791	1.42088	1.40388	1.38690	1.36995	1.35302	1.33611	1.31922
.42	1.30236	1.28552	1.26870	1.25190	1.23513	1.21837	1.20164	1.18492	1.16823	1.15155
.43	1.13490	1.11826	1.10164	1.08504	1.06846	1.05190	1.03535	1.01882	1.00231	.98582
.44	.96934	.95287	.93643	.92000	.90358	.88718	.87079	.85442	.83806	.82171
.45	.80538	.78906	.77276	.75647	.74018	.72392	.70766	.69142	.67518	.65896
.46	.64275	.62654	.61035	.59417	.57800	.56184	.54568	.52954	.51340	.49728
.47	.48116	.46504	.44894	.43284	.41675	.40067	.38459	.36852	.35246	.33640
.48	.32034	.30429	.28825	.27221	.25617	.24014	.22412	.20809	.19207	.17606
.49	.16004	.14403	.12802	.11201	.09601	.08001	.06400	.04800	.03200	.01600
.50	0.00000	-0.0160	-0.0320	-0.0480	-0.0640	-0.0800	-0.0960	-0.1120	-0.1280	-0.1440

$U = 177.253 \text{ lb-mole U/yr of separative capacity .}$

Illustration II.2 Calculate the separative work required to produce 1 kg of uranium containing 90 wt % U-235 from normal uranium containing 0.7115 wt % U-235. The reject material may be assumed to contain 0.20 wt % U-235.

The procedure is the same as in illustration II.1, but in this case one uses amounts rather than rates in performing the computation. The separative work required is obtained from

$$U' = P'v(y_p) + W'v(x_w) - F'v(x_F)$$

where P' is the amount of product, F' is the amount of feed necessary to produce the product, and W' is the associated amount of waste. One obtains:

$$U' = (1)(1.75778) + (174.562)(6.18776) - (175.462)(4.86814)$$

$$U' = 227.245 \text{ kg U separative work .}$$

3. Rayleigh Distillation, Separative Work in. Firstly, let us consider the vapor-liquid equilibrium relationship in distillation. Consider a binary liquid isotopic mixture in equilibrium with its vapor. Let us designate the component enriched in the vapor phase as component 1, the specified component, the other component as component 2, and let the mole fraction of component 1 in the liquid phase be x' and in the vapor phase y' .

The concentrations x' and y' are related by an equilibrium equation which can be obtained as follows. If the liquid phase is ideal, the partial pressure of each component is given by Raoult's law,

$$p_1 = \pi_1 x' \quad \text{and} \quad p_2 = \pi_2 (1 - x') , \quad (\text{II.3.1})$$

where p_1 and p_2 are the partial pressures of the two components and

π_1 and π_2 are the vapor pressures of the pure components at the temperature of the system. If the vapor phase is ideal, Dalton's law is valid, and we can write

$$p_1 = \pi y' \quad \text{and} \quad p_2 = \pi(1 - y') , \quad (\text{II.3.2})$$

where π is the total pressure of the system. Since, at equilibrium, the partial pressure of each component must be the same in both phases, we obtain

$$\pi y' = \pi_1 x' \quad \text{and} \quad \pi(1 - y') = \pi_2(1 - x') \quad (\text{II.3.3})$$

and, hence, the equilibrium equation

$$\frac{y'}{1 - y'} = \frac{\pi_1}{\pi_2} \frac{x'}{1 - x'} . \quad (\text{II.3.4})$$

This equation has the same form as equation (II.1.1), so we see that in distillation the ratio, π_1/π_2 , called the relative volatility, plays a role similar to that of the stage separation factor. Let us designate the relative volatility by α' . Therefore, at equilibrium, the mole fraction of component 1 in the vapor is related to its mole fraction in the liquid by

$$y' = x' + \frac{\psi' x' (1 - x')}{1 + \psi' x'} , \quad (\text{II.3.5})$$

where $\psi' = \alpha' - 1$, and if ψ' is sufficiently small, by the approximate form of the equilibrium equation

$$y' = x' + \psi' x' (1 - x') . \quad (\text{II.3.6})$$

Now let us turn our attention to a consideration of the Rayleigh distillation process. Rayleigh distillation is a simple batch-wise differential distillation in which each increment of vapor evaporated from the liquid surface is in equilibrium with the liquid remaining in the still pot. The vapor is removed from the still so that no partial condensation and revaporization occurs. Figure II.2 shows a schematic representation of

the process and indicates the nomenclature used.

- M_0 = Moles of liquid feed charged to the still
- z = Mole fraction of more volatile component in the feed
- M = Moles of liquid remaining in still at time t
- x' = Mole fraction of more volatile component in the liquid at time t
- m = Vaporization rate
- y' = Mole fraction of more volatile component in the vapor at time t

FIGURE II.2 THE RAYLEIGH DISTILLATION PROCESS

An overall material balance around the material in the still pot yields

$$-\frac{d}{dt} M = m ; \quad dM = -m dt \quad (\text{II.3.7})$$

that is, the rate of decrease of the liquid in the still pot is equal to the vaporization rate. A similar material balance for the more volatile component yields

$$-\frac{d}{dt}(Mx') = my' ; \quad Mdx' + x'dM = -y'mdt \quad (\text{II.3.8})$$

Combining these two equations so as to eliminate $m dt$, we obtain

$$Mdx' = (y' - x')dM, \quad \text{or} \quad \int_{M_0}^{M_f} \frac{dM}{M} = \int_z^x \frac{dx'}{y' - x'} \quad (\text{II.3.9})$$

where M_f is the amount of liquid left in the still pot at the end of the distillation and x is the final mole fraction of the more volatile component in the remaining liquid. The quantity, y' , appearing in equation (II.3.9) may now be expressed in terms of x' by means of the equilibrium relationship and the indicated integration carried out.

a) Exact solution. If one substitutes the exact form of the equilibrium equation, (II.3.5), into the Rayleigh distillation equation,

(II.3.9), one obtains

$$\int_{M_0}^{M_F} \frac{dM}{M} = \int_z^x \frac{1 + \psi' x'}{\psi' x' (1 - x')} dx' \quad (\text{II.3.10})$$

which is readily integrated to yield

$$\psi' \ln \frac{M_F}{M_0} = \ln \frac{x}{1-x} \frac{1-z}{z} - \psi' \ln \frac{1-x}{1-z} \quad (\text{II.3.11})$$

which gives the exact expression for the relationship between x and z in the Rayleigh distillation process.

b) Approximate solution. If one substitutes the approximate form of the equilibrium equation, (II.3.6), into the Rayleigh distillation equation, one obtains in place of (II.3.10) the relationship

$$\int_{M_0}^{M_F} \frac{dM}{M} = \int_z^x \frac{1}{\psi' x' (1 - x')} dx' \quad (\text{II.3.12})$$

which is readily integrated to yield

$$\psi' \ln \frac{M_F}{M_0} = \ln \frac{x}{1-x} \frac{1-z}{z} \quad (\text{II.3.13})$$

which is an approximate form of (II.3.11), valid when ψ' is sufficiently small.

c) A more approximate solution. If in addition to ψ' being small, the value of $\psi' \ln (M_0/M_F)$ is also small, that is, if the fraction evaporated is not too large, we can make the additional approximation

$$\frac{z}{1-z} = \frac{x}{1-x} e^{\psi' \ln \frac{M_0}{M_F}} = \frac{x}{1-x} (1 + \psi' \ln \frac{M_0}{M_F})$$

which is a familiar form which we have already seen can be written

$$z = x + (\psi' \ln \frac{M_0}{M_F}) x (1-x) \quad (\text{II.3.14})$$

If we now compare the Rayleigh distillation with a separation stage, we note that the stage upflow, L' , is equivalent to the amount evaporated,

$M_0 - M_f$, and therefore that the cut, θ , in Rayleigh distillation is

$$\theta = 1 - \frac{M_f}{M_0} .$$

If we let y designate the average mole fraction of the more volatile component in the product material collected, we have the material balance

$$z = \theta y + (1 - \theta)x ; \quad z - y = \theta(y - x) .$$

With these last two relationships, we can write equation (II.3.14) in the form

$$y = x + \left(\frac{1}{\theta} \ln \frac{1}{1 - \theta} \right) \psi' x (1 - x) . \quad (\text{II.3.15})$$

Comparing this result with equation (II.1.3b) which defines the stage enrichment factor, we see that in the case of Rayleigh distillation

$$\psi = \left(\frac{1}{\theta} \ln \frac{1}{1 - \theta} \right) \psi' , \quad (\text{II.3.16})$$

that is, the stage enrichment factor is equal to the equilibrium enrichment factor multiplied by a function of the cut. This correction factor is commonly known as the cut correction.

The separative work accomplished by Rayleigh distillation can now be readily expressed with the help of the previously derived equation, (II.2.6). It takes the form

$$\delta U' = \frac{1}{2} M_0 \theta (1 - \theta) \left(\frac{1}{\theta} \ln \frac{1}{1 - \theta} \right)^2 \psi'^2 . \quad (\text{II.3.17})$$

Since the energy input to the process is proportional to the upflow, another interesting quantity is the separative work produced per unit upflow

$$\frac{\delta U'}{\theta M_0} = \frac{1}{2} (1 - \theta) \left(\frac{1}{\theta} \ln \frac{1}{1 - \theta} \right)^2 \psi'^2 \quad (\text{II.3.18})$$

The separative work and the separative work per unit upflow are shown plotted as a function of the cut in Figure II.3. It can be seen that the separative work is a maximum at a cut of about 0.8, while the separative

FIGURE II.3 RAYLEIGH DISTILLATION: VARIATION OF THE SEPARATIVE WORK AND OF THE SEPARATIVE WORK PER UNIT UPFLOW WITH THE CUT, θ

TABLE 4
RAYLEIGH DISTILLATION

θ	$\frac{1}{\theta} \ln \frac{1}{1-\theta}$	$\frac{2\delta U'}{M_0 \psi'^2}$	$\frac{2\delta U'}{\theta M_0 \psi'^2}$
.01	1.00503	9.99992E-3	.999992
.02	1.01014	1.99993E-2	.999966
.03	1.01531	2.99977E-2	.999923
.04	1.02055	3.99944E-2	.999861
.05	1.02587	.049989	.999781
.06	1.03126	5.99809E-2	.999681
.07	1.03672	6.99693E-2	.999561
.08	1.04227	7.99537E-2	.999421
.09	1.0479	8.99333E-2	.999259
.1	1.05361	9.99075E-2	.999075
.11	1.0594	.109876	.998869
.12	1.06528	.119837	.998639
.13	1.07125	.12979	.998385
.14	1.07731	.139735	.998107
.15	1.08346	.14967	.997802
.16	1.08971	.159595	.997471
.17	1.09606	.169509	.997112
.18	1.10251	.17941	.996725
.19	1.10906	.189299	.996308
.2	1.11572	.199172	.995861
.21	1.12249	.20903	.995382
.22	1.12937	.218872	.994871
.23	1.13637	.228695	.994327
.24	1.14349	.238499	.993747
.25	1.15073	.248283	.993132
.26	1.1581	.258044	.992479
.27	1.1656	.267783	.991787
.28	1.17323	.277496	.991055
.29	1.181	.287182	.990282
.3	1.18892	.29684	.989466
.31	1.19698	.306467	.988605
.32	1.2052	.316063	.987697
.33	1.21357	.325625	.986741
.34	1.2221	.33515	.985736
.35	1.23081	.344637	.984678
.36	1.23969	.354084	.983566
.37	1.24874	.363487	.982399
.38	1.25799	.372846	.981172
.39	1.26743	.382155	.979886
.4	1.27706	.391414	.978536
.41	1.28691	.400619	.97712
.42	1.29697	.409767	.975635
.43	1.30725	.418854	.974079
.44	1.31777	.427878	.972449
.45	1.32853	.436833	.970741
.46	1.33954	.445718	.968951
.47	1.3508	.454526	.967077
.48	1.36235	.463255	.965114
.49	1.37417	.471899	.963059
.5	1.38629	.480453	.960906

TABLE 4 CONT'D.
RAYLEIGH DISTILLATION

θ	$\frac{1}{\theta} \ln \frac{1}{1-\theta}$	$\frac{2\delta U'}{M_0 \psi'^2}$	$\frac{2\delta U'}{\theta M_0 \psi'^2}$
.51	1.39873	.438912	.953652
.52	1.41148	.497271	.956291
.53	1.42457	.505524	.953319
.54	1.43802	.513664	.95123
.55	1.45183	.521685	.948517
.56	1.46604	.529579	.945676
.57	1.48065	.537338	.942699
.58	1.49569	.544955	.939578
.59	1.51118	.552421	.936307
.6	1.52715	.559726	.932876
.61	1.54362	.56686	.929278
.62	1.56062	.573812	.925503
.63	1.57818	.58057	.921539
.64	1.59633	.587121	.917377
.65	1.61511	.593453	.913004
.66	1.63456	.599549	.908407
.67	1.65472	.605394	.903573
.68	1.67564	.61097	.898485
.69	1.69737	.616257	.893127
.7	1.71996	.621236	.88748
.71	1.74349	.625882	.881525
.72	1.76801	.630172	.875239
.73	1.79361	.634076	.868597
.74	1.82037	.637565	.861574
.75	1.84839	.640604	.854139
.76	1.87778	.643156	.846258
.77	1.90867	.645179	.837895
.78	1.94119	.646626	.829008
.79	1.9755	.647444	.819549
.8	2.0118	.647573	.809466
.81	2.05029	.646945	.798697
.82	2.09122	.645433	.787174
.83	2.13489	.643093	.774817
.84	2.18164	.639687	.761532
.85	2.23191	.635129	.747211
.86	2.28618	.629284	.731725
.87	2.34508	.621983	.714923
.88	2.40939	.613025	.696619
.89	2.48003	.602165	.67659
.9	2.55843	.5891	.654555
.91	2.64609	.573449	.630163
.92	2.74536	.554722	.602959
.93	2.85942	.532276	.57234
.94	2.99299	.505231	.537479
.95	3.1534	.472337	.497197
.96	3.353	.431715	.449703
.97	3.61501	.380287	.392049
.98	3.99186	.312325	.318699
.99	4.65169	.214218	.216382

work per unit upflow is a maximum when the cut is equal to zero. For a cut of $1/2$, the separative work per unit upflow has dropped only to 96% of its maximum value, indicating that only 4% of the separative work produced is lost when the vapor corresponding to $1/2$ of the material charged is mixed together in the product collecting vessel.

4. The Separative Capacity of an Element of Column Length in a Continuous Countercurrent Column Process. Let us now attempt to compute the separative capacity of a length, dz , of column in a simple, continuous countercurrent process. For a process of this type (e.g., distillation in a packed column) there exists an equilibrium separation factor, α' , which relates the concentrations of the specified component in the two phases (e.g., vapor and liquid) when equilibrium between them exists. This relationship is, of course, of the form of equation (II.1.1), and for the case of isotope separation can be written in the approximate form

$$y' = x' + \psi' x' (1 - x') , \quad (\psi' \equiv \alpha' - 1) , \quad (\text{II.4.1})$$

it having previously been shown that the approximation introduces a fractional error less than $\psi' x'$ in the concentration difference, $y' - x'$.

The column section under consideration is shown schematically in Figure II.4. In an actual column the two phases are normally quite dispersed and intermixed in order to provide a large interfacial area for mass transfer between them. The current of the specified component, J_1' , measured in moles per unit time per unit area of interface, from the downflowing stream (the liquid stream in the case of distillation) to

FIGURE II.4 SECTION OF A CONTINUOUS COUNTERCURRENT COLUMN PROCESS

the upflowing stream (vapor stream) is taken proportional to the departure of the concentrations of the two phases from the equilibrium condition.

Thus we write the expression for current of specified component from the downflowing stream, in which the concentration of specified component is designated by x , to the upflowing stream of concentration y in the following form, familiar to chemical engineers

$$J_1' = k(y^* - y) , \quad (\text{II.4.2})$$

where k is a coefficient of mass transfer for the system under consideration. Its value depends on the physical properties of the phases and on the flow rates and geometry of the system; it is inversely proportional to the resistance to mass transfer between the phases. The quantity y^* represents the mole fraction in the upflowing stream which would be in equilibrium with the downflowing stream of concentration x . If we let a represent the interfacial area per unit volume of column, and let A represent the cross section area of the column, we can rewrite equation (II.4.2) in the form

$$J_1 = ka A [\psi'x(1-x) - (y-x)] , \quad (\text{II.4.3})$$

in which the approximate form of the equilibrium relationship has been used to express y^* in terms of x . Equation (II.4.3) gives the rate of mass transfer between the phases, J_1 , in moles of specified component per unit time per unit length of column. A material balance around each phase yields for the column section

$$\begin{aligned} L'(x+dx) &= L'x + J_1dz ; & Ly + J_1dz &= L(y+dy) \\ L'dx &= J_1dz = Ldy . \end{aligned} \quad (\text{II.4.4})$$

The separative capacity of the column section is calculated, as in the case of the stage, by making a value balance around the section and setting the separative capacity equal to the increase in the value of the streams effected by the column section per unit time.

$$\delta U = L'v(x) + Lv(y+dy) - Lv(y) - L'v(x+dx) \quad (\text{II.4.5})$$

Since the concentrations in equation (II.4.5) differ only slightly from each other when we consider isotope separation processes, we may expand the value functions in Taylor series about x and neglect terms containing concentration differences raised to powers greater than two. So doing, we obtain

$$\begin{aligned} \delta U = & L'v(x) \\ & + Lv(x) + L(y+dy-x)v'(x) + \frac{L}{2}(y+dy-x)^2v''(x) \\ & - Lv(x) - L(y-x)v'(x) - \frac{L}{2}(y-x)^2v''(x) \\ & - L'v(x) - L'dxv'(x) - \frac{L'}{2}dx^2v''(x) . \end{aligned}$$

Combining terms and using the material balance relationship, (II.4.4), to express dx and dy in terms of J_1dz , the above equation reduces to

$$\delta U = Ldy(y-x)v''(x) = J_1dz(y-x)v''(x) . \quad (\text{II.4.6})$$

The term, $(Ldy^2 - L'dx^2)v''(x)/2$, was neglected in writing (II.4.6) since, as will be seen in the following section, L and L' are almost equal for the case of isotope separation so that this term which can be written $Ldy^2(1-L/L')v''(x)/2$ is indeed negligible. Introducing the expression J_1 from equation (II.4.3), we find that

$$\delta U = kaDz[\psi'x(1-x) - (y-x)]\frac{y-x}{x^2(1-x)^2} . \quad (\text{II.4.7})$$

Thus we see that the separative capacity of the section depends on the concentration difference, $y-x$, and is equal to zero both when x and y are in equilibrium and when x is equal to y . This should have been anticipated, since when x and y are in equilibrium, there is no transport between the phases, and when x is equal to y , the streams flowing into and out of the section are equivalent, and hence there is no change in value brought about by the section. We now inquire for what value of the concentration difference, $y-x$, will the separative capacity be a maximum.

Differentiating equation (II.4.7) with respect to $(y-x)$, and equating the result to zero, we find that the condition necessary for δU to attain its maximum value is

$$y - x = \frac{\psi'x(1-x)}{2} \quad (\text{II.4.8})$$

and that

$$\delta U_{\max} = kaA \frac{\psi'^2}{4} dz . \quad (\text{II.4.9})$$

We have seen that in a stage-wise process the two streams leaving a stage are related by the stage separation factor: let us consider, in the case of a continuous countercurrent process, what length of column is necessary in order that the streams leaving the section be similarly related by the process equilibrium separation factor. In other words, what length of column is required in order for the concentrations of the two streams leaving to be in equilibrium. Taking this column length to be Δz , and the concentrations of the streams flowing out of this element to be $y + \Delta y$ and x , we wish to determine Δz such that

$$y + \Delta y = x + \psi'x(1-x) ; \quad \Delta y = \psi'x(1-x) - (y-x) .$$

From equations (II.4.4) and (II.4.3), we obtain

$$\Delta y = \frac{J_1}{L} \Delta z = \frac{kaA}{L} [\psi'x(1-x) - (y-x)] \Delta z .$$

Equating the two preceding expressions for Δy , we find that the required length is given by

$$\Delta z = \frac{L}{kaA} . \quad (\text{II.4.10})$$

In analogy with the stage-wise process, we call this length a stage or stage length or HTU (height of a transfer unit) or HETP (height equivalent to a theoretical plate). From equations (II.4.9) and (II.4.10), we see that the maximum separative capacity of a stage in a continuous countercurrent column process is given by

$$\delta U_{\max} = \frac{L\psi'^2}{4} \quad . \quad (\text{II.4.11})$$

In conclusion we may make the following observations regarding the separative capacity of a stage in a continuous countercurrent process in comparison with the separative capacity of a stage in a stage-wise process:

- 1) The separative capacity of a S-W stage with a cut equal to $\frac{1}{2}$ is equal to $L\psi'^2/4$.
- 2) The maximum separative capacity of a C-C stage is equal to $L\psi'^2/4$.
- 3) In the expression for the separative capacity of the S-W stage the stage enrichment factor appears. As was shown in the discussion of Rayleigh distillation, the stage enrichment factor is equal to the equilibrium enrichment factor multiplied by a correction factor (the cut correction).
- 4) In the expression for the separative capacity of the C-C stage the equilibrium enrichment factor appears: there is no correction factor equivalent to the cut correction.
- 5) In the case of a C-C stage in which the concentration difference, $y - x$, is such that the separative capacity is maximized, it can be shown that the concentrations in the two streams entering the stage are identical. In this case the analogy between the C-C stage and the S-W stage is accentuated: they may both be considered to be devices which divide a feed stream into two effluent streams, the concentrations of which are related by the "stage" separation factor.

The properties of a stage in a continuous countercurrent process will be considered further in subsequent sections dealing with the derivation of

the gradient equations for continuous processes.

5. The Reversible, Isothermal Work of Separation. In the preceding sections we have been talking about a mathematically defined quantity, important in isotope separation, called separative work. Those who are not familiar with the theory of isotope separation often confuse separative work with the reversible thermodynamic work of separation. It is therefore appropriate to take a moment to differentiate between these two concepts.

You will recall from thermodynamics that the work done on a gaseous system during a thermodynamically reversible process is given by

$$w = - \int p dV , \quad (\text{II.5.1})$$

where p is the pressure on the system and V the volume of the gas. For the case of an ideal gas and an isothermal process, one has the equation of state

$$pV = nRT , \quad \text{whence} \quad p dV + V dp = 0 , \quad (\text{II.5.2})$$

so that the work done on the system is also given by

$$w = \int V dp . \quad (\text{II.5.3})$$

Furthermore, you will recall from thermodynamics that the change in the Gibbs free energy of the system is given by

$$\Delta G = \Delta H - \Delta(TS) = - \int S dT + \int V dp \quad (\text{II.5.4})$$

so that the work required for a reversible, isothermal separation is equal to the Gibbs free energy change of the system. Since, for the isothermal process, $\Delta G = -T\Delta S$, it follows that a separation process effects a decrease in the entropy of the system (i.e., brings about a more ordered, less random, arrangement). For an isothermal process involving an ideal gas, equation (II.5.4) yields

$$G = \int_{p_i}^{p_f} \frac{nRT}{p} dp = nRT \ln \frac{p_f}{p_i} \quad , \quad (\text{II.5.5})$$

where p_f and p_i represent the final and initial pressures, respectively. That the work required to effect a reversible, isothermal separation of a gas mixture is given by equation (II.5.5) is frequently illustrated with the use of the concept of ideal semi-permeable membranes. Consider a reservoir containing a binary gas mixture at pressure π in which the concentration of specified component is z . For the separation of n moles of this mixture into the two pure components, we permit $n z$ moles of the specified component to pass at its partial pressure through a semi-permeable membrane which is impermeable to the other component and then compress this gas to the original system pressure π . Similarly, we permit $n (1 - z)$ moles of the other component to pass at its partial pressure through another semi-permeable membrane which is impermeable to the specified component and then compress this fraction to pressure π . Thus, the work required to separate n moles of the mixture into pure components by a reversible, isothermal process is just this work of compression and is given by

$$w = \Delta G = nzRT \ln \frac{\pi}{\pi z} + n(1 - z)RT \ln \frac{\pi}{\pi(1 - z)}$$

$$w = \Delta G = -nRT[z \ln z + (1 - z) \ln(1 - z)] \quad . \quad (\text{II.5.6})$$

The reversible, isothermal work of separation accomplished by a separation stage per unit time can now be computed by making a free energy balance around the stage, similar to the procedure followed in the determination of the separative capacity. We write that the free energy production is equal to that required to separate the feed stream into its pure components less that to reconstitute the effluent streams from the pure components:

$$\begin{aligned} \frac{\Delta \bar{G}}{RT} = & -G[z \ln z + (1-z) \ln(1-z)] + \theta G[y \ln y + (1-y) \ln(1-y)] \\ & + (1-\theta) G[x \ln x + (1-x) \ln(1-x)] , \end{aligned} \quad (\text{II.5.7})$$

where we have used $\bar{G}$ to designate the Gibbs free energy so as to differentiate it from the stage feed rate G .

The functions appearing in (II.5.7) are then expanded in Taylor's series about x and terms containing the small concentration difference, $y-x$, or $z-x$, raised to a power greater than two are neglected with the result

$$\begin{aligned} \frac{\Delta \bar{G}}{RT} = & -G[f_1(x) + (z-x)f_1'(x) + \frac{(z-x)^2}{2} f_1''(x) \\ & + f_2(x) + (z-x)f_2'(x) + \frac{(z-x)^2}{2} f_2''(x)] \\ & + \theta G[f_1(x) + (y-x)f_1'(x) + \frac{(y-x)^2}{2} f_1''(x) \\ & + f_2(x) + (y-x)f_2'(x) + \frac{(y-x)^2}{2} f_2''(x)] \\ & + (1-\theta)G[f_1(x) + f_2(x)] \end{aligned}$$

where

$$\begin{aligned} f_1(x) &= x \ln x \quad \text{and} \\ f_2(x) &= (1-x) \ln(1-x) . \end{aligned}$$

Making use of the stage material balance relationship, $z-x = \theta(y-x)$, the above expression for the free energy change produced by the stage reduces to

$$\begin{aligned} \frac{\Delta \bar{G}}{RT} &= \frac{1}{2} G \theta (1-\theta) (y-x)^2 [f_1''(x) + f_2''(x)] \\ \frac{\Delta \bar{G}}{RT} &= \frac{1}{2} G \theta (1-\theta) \frac{(y-x)^2}{x(1-x)} . \end{aligned} \quad (\text{II.5.8})$$

This expression may be compared with equation (II.2.4) for the separative capacity of a stage. More explicitly, the thermodynamic efficiency of a

stage may be defined as the ratio of the free energy produced to the actual energy consumption of the stage. Thus, the thermodynamic efficiency should be directly proportional to the ratio

$$\frac{\overline{\Delta G}}{\delta U} = RTx(1-x) \quad (\text{II.5.9})$$

so that it is definitely concentration dependent and will be very small indeed when one is working with extremely dilute mixtures. We will find the separative capacity to be a much more useful quantity than the free energy production in isotope separation work.

III. ELEMENTS OF CASCADE DESIGN: THE IDEAL CASCADE

1. The Separation Cascade. A cascade is the name given to an assembly of separating elements, or stages, connected together for the purpose of accomplishing some specific separation task. The scheme used to form a cascade from the individual separating elements depends to some extent on the type of equipment under consideration.

a) In the simplest case we may consider a cascade utilizing a stage-wise separation process for which stages with any desired throughput are readily available. In such a case, a simple series connection of the stages (see Figure III.1.a) would normally be employed.

b) It is a useful concept, and a situation which may sometimes occur in practice, to consider that the cascade is composed of many identical separating elements with throughputs which are small with respect to the desired cascade interstage flows. In such a case, a number of these elements are connected in parallel to form a stage with the desired upflow, and these stages are connected in series as in (a) to form the cascade (see Figure III.1b).

c) In the case that a continuous countercurrent column process

FIGURE III.1a A CASCADE SECTION
COMPOSED OF INDIVIDUAL SEPARATION
STAGES CONNECTED IN SERIES

FIGURE III.1b A CASCADE SECTION COMPOSED OF SMALL
SEPARATING ELEMENTS (STAGES OR COLUMNS) CONNECTED
IN A SERIES-PARALLEL ARRANGEMENT

is being utilized, it will still be necessary, in general, to interconnect a large number of columns to achieve the desired separation. As was indicated in section II.4, each column can be considered to be composed of a number of theoretical stages connected in series, and is itself a cascade. Thus a cascade composed of such columns could be considered to be a cascade of cascades. When possible, columns are normally operated in a cascade as 4-stream separation devices, that is each column has two feed streams and two effluent streams, one feed stream and one withdrawal stream at each end (see Figure III.1c).

d) In some cases, this type of connection is not practical, and the columns are operated as 3-stream separating elements with a single feed stream, near the middle of the column, and two withdrawal streams, one at each end. A schematic diagram of this type of cascade would look similar to Figure III.1b, but with the individual elements in this case being columns rather than single separating stages.

Figure III.2 is a schematic representation of a simple separation cascade. In a simple cascade, a feed stream is divided into a stream enriched in the specified component and a stream depleted in the specified component, termed product and waste, respectively; however, in a more general case there may be both side withdrawal streams and additional feed streams.

For the simple cascade one has the following two material balance relationships: the total material balance,

$$F = P + W, \quad (\text{III.1.1})$$

and the specified component material balance,

$$F x_F = P y_P + W x_W. \quad (\text{III.1.2})$$

FIGURE III.1c A CASCADE SECTION COMPOSED OF 4-STREAM SEPARATING DEVICES (COUNTER-CURRENT COLUMNS). FREQUENTLY THE UPFLOWS AND DOWNFLOWS ARE MIXED AND REDISTRIBUTED BETWEEN EACH BANK OF COLUMNS.

FIGURE III.2 SECTION OF THE ENRICHER IN A SIMPLE CASCADE COMPOSED OF SEPARATION STAGES CONNECTED IN SERIES

FIGURE III.2 A SIMPLE SEPARATION CASCADE

2. Derivation of the Gradient Equation for Stage-Wise Processes. The gradient equation for a section of a cascade gives the change in the concentration of the specified component per stage in a stage-wise process, or per unit length in a continuous countercurrent column process. The gradient equations for the cascade are obtained from a combination of the stage separation factor relationship, usually called the equilibrium-line equation, and the cascade material balance equations, frequently called the operating-line equations.

a) The equilibrium-line equation is the relationship between the concentrations of the specified component in the two streams leaving a stage in the cascade: it is the same relationship that was discussed in some detail in Section II.1. As we have seen, there are several forms that this relationship may take, one of which is the "exact form"

$$y_n - x_n = \frac{\psi_n x_n (1 - x_n)}{1 + \psi_n x_n} \quad (\text{III.2.1})$$

b) The operating-line equation for the enriching section of a cascade is obtained from material balances taken around the top of the cascade above the n^{th} stage (see Figure III.2 for reference). The total material balance can be written

$$L_n = L_{n+1}' + P \quad (\text{III.2.2})$$

and the specified component material balance

$$L_n y_n = L_{n+1}' x_{n+1} + P y_p \quad (\text{III.2.3})$$

Combining these two material balance equations, we obtain

$$L_n y_n = (L_n - P) x_{n+1} + P y_p \quad (\text{III.2.4})$$

which is the operating-line equation for the enricher in a simple cascade.

The gradient equation for the enriching section is now obtained by combining equations (III.2.1) and (III.2.4). First, subtract the quantity,

$(L_n - P)x_n$, from both sides of (III.2.4) and rewrite it in the form

$$(L_n - P)(x_{n+1} - x_n) = L_n(y_n - x_n) - P(y_P - x_n)$$

and then replace the concentration difference, $y_n - x_n$, by means of the equilibrium-line equation. The resulting equation can be written

$$x_{n+1} - x_n = \frac{L_n}{L_n - P} \left[\frac{\psi_n x_n (1 - x_n)}{1 + \psi_n x_n} - \frac{P}{L_n} (y_P - x_n) \right] \quad (\text{III.2.5})$$

and may be called the exact, finite difference form of the gradient equation for the enriching section. Proceeding in the same manner, but taking the material balances below the feed point around the bottom of the cascade, one obtains the operating-line equation for the stripping section and the gradient equation for the stripper:

$$x_{m+1} - x_m = \frac{L_m}{L_m + W} \left[\frac{\psi_m x_m (1 - x_m)}{1 + \psi_m x_m} - \frac{W}{L_m} (x_m - x_W) \right] \quad (\text{III.2.6})$$

where m is the index used to count stages in the stripper.

In the case of isotope separation where the enrichment per stage is quite small, it is possible to make some simplifying assumptions in the gradient equations without introducing appreciable error. Firstly, the enrichment per stage, $x_{n+1} - x_n$, can be approximated by the differential enrichment, dx/dn . Making this substitution in the preceding pair of equations, we obtain a set of gradient equations which we will call the exact, differential forms:

$$\frac{dx}{dn} = \frac{L}{L - P} \left[\frac{\psi x (1 - x)}{1 + \psi x} - \frac{P}{L} (y_P - x) \right], \quad (\text{III.2.7})$$

$$\frac{dx}{dn} = \frac{L}{L + W} \left[\frac{\psi x (1 - x)}{1 + \psi x} - \frac{W}{L} (x - x_W) \right]. \quad (\text{III.2.8})$$

Secondly, in the case of isotope separation where the stage enrichment factor is small with respect to unity, it follows that both P and W must also be small with respect to L everywhere in the cascade with the

exception of the immediate neighborhood of the product and waste withdrawal points. This fact will be justified in the next section. When we neglect P and W with respect to L in the gradient equations, we obtain a pair of equations which we will call the design equations:

$$\frac{dx}{dn} = \frac{\psi x(1-x)}{1+\psi x} - \frac{P}{L} (y_P - x) , \quad (\text{III.2.9})$$

$$\frac{dx}{dm} = \frac{\psi x(1-x)}{1+\psi x} - \frac{W}{L} (x - x_W) . \quad (\text{III.2.10})$$

Thus, for any form of the equilibrium-line equation that we choose to use, we can obtain three pairs of gradient equations, depending on the number of assumptions that are justified. We have previously discussed five different forms of the equilibrium relationship, so we are now in a position to write fifteen pairs of gradient equations. These equations will be examined in more detail in Section IV dealing with square cascade sections.

The pair of equations which we will use most frequently in isotope separation cascade design work is the pair of design equations which result when one uses the approximate equilibrium-line equation:

$$\frac{dx}{dn} = \psi x(1-x) - \frac{P}{L} (y_P - x) , \quad (\text{III.2.11})$$

$$\frac{dx}{dm} = \psi x(1-x) - \frac{W}{L} (x - x_W) . \quad (\text{III.2.12})$$

3. The McCabe-Thiele Diagram. Some Limiting Relationships. Some insight into the separation process taking place in a cascade can be gained by reference to a McCabe-Thiele, or x - y operating diagram. To construct a McCabe-Thiele diagram, we first plot on an x - y coordinate system the 45-degree line ($x = y$), the equilibrium line, and the operating line (see Fig. III.3). Let us assume for this illustration the approximate form of the equilibrium-line equation (II.1.3b) and an operating line equation for the top section in the enricher in which L is constant (a square enriching section).

The product concentration, y_p , occurs at the intersection of the operating line and the 45-degree line. A horizontal line on the diagram from this point will intersect the equilibrium line at (x_N, y_p) , giving the concentration in the downflowing stream from the top stage, in equilibrium with y_p . A vertical line from this point will intersect the operating line at (x_N, y_{N-1}) , giving the concentration in the upflowing stream from the $N-1^{\text{st}}$ stage. Proceeding in this manner, one can determine the concentration in the cascade after any desired number of stages.

a) The minimum number of stages. It should be evident from an inspection of the McCabe-Thiele diagram that the minimum number of stages to span a specified concentration range, $y_p - x_B$, results from this stepping off procedure when the operating line has its maximum permissible slope. Since the slope of the operating line is $(1 - P/L)$, it will be a maximum when P is equal to zero. This case, no product withdrawal, is often called the total reflux case, or total reflux operation. Using the design gradient equation as an example, (III.2.11), it is clear that for total reflux

$$\frac{dx}{dn} = \psi x(1 - x) , \quad (\text{III.3.1a})$$

and that the minimum number of stages to span the concentration difference, $y_p - x_B$, is given by

$$N_{\min} = \frac{1}{\psi} \ln \frac{y_p(1 - x_B)}{x_B(1 - y_p)} . \quad (\text{III.3.1b})$$

b) The minimum interstage flow. Inspection of the McCabe-Thiele diagram also reveals another interesting feature: for a given value of y_p and a given slope to the operating line, there is a minimum value of the concentration which can be reached, even with an infinite number of separation stages. This limiting concentration occurs at the intersection of

the equilibrium line and operating line, and is called a pinch point in distillation practice. Therefore, to be able to attain a specified value of x in a cascade, there is some minimum value that the slope of the operating line may have. This may be expressed also as a maximum value of P/L , or a minimum value of L for a specified product withdrawal rate. Thus one may obtain the value of $L_{\min}$ by finding the value of L which gives the desired point of intersection of the equilibrium and operating lines. However, since the enrichment per stage goes to zero at such a pinch point, the value of $L_{\min}$ can be determined quite simply by setting $x_{n+1} - x_n$, or dx/dn , equal to zero in the gradient equation. Using the design gradient equation as an example, (III.2.11), one finds that in the enriching section

$$L_{\min} = \frac{P(y_P - x)}{\psi_X(1 - x)} \quad (\text{III.3.2})$$

and in the stripping that

$$L_{\min} = \frac{W(x - x_W)}{\psi_X(1 - x)} \quad (\text{III.3.3})$$

It can be seen from the two equations for $L_{\min}$ given above that, except in the neighborhood of the product and waste withdrawal points, P and W are everywhere in a long cascade small with respect to the interstage flow. This fact provides the justification for neglecting P/L and W/L with respect to unity when writing the design equations.

The x -coordinates of the points of intersection of the equilibrium and operating lines play an important role in the solutions of the gradient equation for square sections. They are designated in the diagram by X_0 and X_1 , and will be examined in Section IV which deals with real cascades.

FIGURE III.3 MCCABE-THIELE DIAGRAM FOR AN ENRICHING SECTION

4. Separative Capacity of a Cascade Section. We shall next derive an expression for the separative capacity of a section of a cascade consisting of a group of adjacent stages. The approach used is similar to that of Section II.4, but here we are considering a stage-wise process. It will also be convenient at this time to define some new terms which permit writing the gradient equation in a short-hand form. The design gradient equation for the enriching section has the form

$$\frac{dx}{dn} = \psi x(1-x) - \frac{P}{L} (y_P - x) \quad . \quad (\text{III.4.1})$$

If we divide each term by the product, $x(1-x)$, and define the stage enrichment, ϵ , by

$$\epsilon \equiv \frac{1}{x(1-x)} \frac{dx}{dn} \quad , \quad (\text{III.4.2})$$

and the parameter, K , by

$$K \equiv \frac{P(y_P - x)}{x(1-x)} = \psi I_{\min} \quad , \quad (\text{III.4.3})$$

we can write the gradient equation (III.4.1) in the abbreviated form

$$\frac{\epsilon}{\psi} = 1 - \frac{K}{L\psi} = 1 - \frac{I_{\min}}{L} \quad . \quad (\text{III.4.4})$$

Now let us consider a cascade section such as is shown schematically in Figure III.4. For the separative capacity of this cascade section we can use the expression (II.4.6), derived for a continuous countercurrent column process:

$$\delta U = L dy (y - x) v''(x) \quad . \quad (\text{III.4.5})$$

Making use of the fact that $L dy = L' dx = L dx$, and the material balance around the top of the cascade given in Figure III.4, we can write

$$\delta U = L dx \frac{P(y_P - x)}{L} v''(x) \quad . \quad (\text{III.4.6})$$

Substituting for dx in equation (III.4.6) by means of the gradient equation, we can write

Material Balance Relationships:

$$L = L' + P$$

$$Ly = L'x + Py_P$$

$$Ly = (L - P)x + Py_P$$

$$y - x = \frac{P}{L}(y_P - x)$$

FIGURE III.4 A CASCADE SECTION IN A CASCADE CONSISTING OF SEPARATION STAGES

$$\delta U = L \left(\psi - \frac{P(y_p - x)}{Lx(1-x)} \right) \frac{P(y_p - x)}{Lx(1-x)} dn \quad (\text{III.4.7})$$

or using our short-hand notation, that

$$\delta U = L\psi^2 \left(1 - \frac{\varepsilon}{\psi} \right) \frac{\varepsilon}{\psi} dn = L\psi^2 \frac{L_{\min}}{L} \left(1 - \frac{L_{\min}}{L} \right) dn . \quad (\text{III.4.8})$$

It may be noted at this point that the maximum separative capacity of a cascade section of one stage ($dn=1$), according to the above equation is $L\psi^2/4$, obtained when $\varepsilon/\psi = 1/2$. Values less than this must be attributed to mixing losses between stages.

5. The Ideal Cascade. An ideal cascade can be defined in any one of several different, but equivalent ways.

Definition 1. For a cascade composed of separation stages, an ideal cascade is one in which there is no mixing of streams of differing concentrations at the inlet to a separation stage to form the feed to that stage. Referring to Figure III.2, it can be seen that for the simple series connected cascade this condition is met at the entrance to the $n+1^{\text{st}}$ stage if

$$y_n = x_{n+2} . \quad (\text{III.5.1})$$

Assuming that the enrichment per stage, dx/dn , and the product, $x(1-x)$, do not vary appreciably over the range of a few adjacent separation stages, we replace x_{n+2} by $x_n + 2 \frac{dx}{dn}$ and y_n by the approximate equilibrium line relationship with the result

$$x_n + \psi x(1-x) = x_n + 2 \frac{dx}{dn}$$

and using our shorthand notation of the preceding section, this can be written in the form

$$\frac{\varepsilon}{\psi} = \frac{1}{2} , \quad (\text{i.e., } \frac{dx}{dn} = \frac{\psi x(1-x)}{2}) . \quad (\text{III.5.2})$$

Integration of this equation shows that the number of stages required to effect a given separation in an ideal plant is precisely twice the minimum number of stages:

$$N_I = \frac{2}{\psi} \ln \frac{y_P(1-x_B)}{x_B(1-y_P)} = 2N_{\min} , \quad (\text{III.5.3})$$

and from equation (III.4.4), it can be seen that the interstage flow in an ideal cascade is precisely twice the minimum flow:

$$L_I = \frac{2P(y_P - x)}{\psi x(1-x)} = 2L_{\min} . \quad (\text{III.5.4})$$

Definition 2. An ideal cascade is a cascade in which the separative capacity per unit upflow in any section is at its maximum value. It can be seen from equation (III.4.8), derived in the preceding section, that $\delta U/L$ will be a maximum when

$$\frac{\varepsilon}{\psi} = \frac{L_{\min}}{L} = \frac{1}{2} \quad (\text{III.5.5})$$

in agreement with definition 1. Furthermore, it follows from (III.4.8) that when the separative capacity per unit upflow is maximized, the separative capacity of a stage is given by

$$U = \frac{L\psi^2}{4} \quad (\text{III.5.6})$$

which is the same expression as that given in (II.2.7) for the separative capacity of an isolated stage. This result should have been anticipated since in a cascade in which there is no mixing of different concentrations between stages, the separative capacity of a group of stages should be simply equal to the sum of the separative capacities of the individual stages.

Definition 3. An ideal cascade is a cascade which performs a given separation task with the minimum summation of stage upflows, that is

$$\sum L_n = \int_{x_F}^{y_P} \frac{L}{\psi x(1-x) - \frac{P}{L}(y_P - x)} dx \quad (III.5.7)$$

This integral will be a minimum when the integrand has its minimum value at every point over the range of integration. To find the condition which minimizes the integrand, we differentiate it with respect to L and set the derivative equal to zero with the result

$$\psi x(1-x) - \frac{P}{L}(y_P - x) = \frac{P}{L}(y_P - x) ,$$

or

$$L = \frac{2P(y_P - x)}{\psi x(1-x)} = 2L_{\min} \quad (III.5.8)$$

so that it can be seen that all three definitions of the ideal cascade are equivalent in that they lead to the identical two equations for the number of stages and for the interstage flow in an ideal cascade, initially called equations (III.5.3) and (III.5.4). Knowing the ψ -value, an ideal cascade can be designed for a given separation job by using the concentration, x , as a parameter and solving for N_I and L_I at various values of x in the cascade. A plot of N_I vs L_I gives the shape of the ideal cascade. Equations corresponding to (III.5.3) and (III.5.4) for use in designing the stripping section of the ideal cascade are similarly obtained with the use of the gradient equation for the stripper; they are:

$$M_I = \frac{2}{\psi} \ln \frac{x_T(1-x_W)}{x_W(1-x_T)} \quad (III.5.9)$$

and

$$L_I = \frac{2W(x - x_W)}{\psi x(1-x)} \quad (III.5.10)$$

Summation of stage upflows. It is now possible, and will prove very informative, to compute the sum of the stage upflows in an ideal cascade. Let us first form the sum of the stage upflows for the enriching

section of the cascade. We can write

$$\sum_{n=0}^N L_n = \int_{x_F}^{y_P} L \frac{dn}{dx} dx = \int_{x_F}^{y_P} \frac{2P(y_P - x)}{\psi x(1-x)} \frac{2}{\psi x(1-x)} dx \quad (\text{III.5.11})$$

where we have used (III.5.2) to express dx/dn in an ideal cascade and (III.5.4) to express L as a function of the concentration in an ideal cascade. Recalling equation (II.2.15) for the second derivative of the value function, we can write the preceding equation in the form

$$\sum_{n=0}^N L_n = \frac{4P}{\psi^2} \int_{x_F}^{y_P} (y_P - x) v''(x) dx \quad (\text{III.5.12})$$

which is readily integrated, using integration by parts, to yield

$$\begin{aligned} \sum L(\text{enr}) &= \frac{4P}{\psi^2} \{ y_P [v'(y_P) - v'(x_F)] - x v'(x) \Big|_{x_F}^{y_P} + \int_{x_F}^{y_P} v'(x) dx \} ; \\ \sum L(\text{enr}) &= \frac{4P}{\psi^2} [v(y_P) - v(x_F) - (y_P - x_F) v'(x_F)] . \end{aligned} \quad (\text{III.5.13})$$

We obtain the sum of the stage upflows for the stripping section of the cascade in a similar manner; we write

$$\sum_{m=0}^M L_m = \int_{x_W}^{x_F} L \frac{dm}{dx} dx = \int_{x_W}^{x_F} \frac{2W(x - x_W)}{\psi x(1-x)} \frac{2}{\psi x(1-x)} dx , \quad (\text{III.5.14})$$

where we have used (III.5.10) to express the flow in the stripper of an ideal cascade as a function of the concentration. This equation can be written in the form

$$\sum_{m=0}^M L_m = \frac{4W}{\psi^2} \int_{x_W}^{x_F} (x - x_W) v''(x) dx \quad (\text{III.5.15})$$

which is integrated, using integration by parts, to yield

$$\sum L(\text{str}) = \frac{4W}{\psi^2} [v(x_W) - v(x_F) - (x_W - x_F) v'(x_F)] . \quad (\text{III.5.16})$$

The summation of stage upflows for the entire cascade will be given by the

sum of the stage upflows in the enriching and stripping sections. Adding equations (III.5.13) and (III.5.16), we obtain

$$\begin{aligned} \sum L(\text{tot}) = \frac{4}{\psi^2} [Pv(y_P) - Pv(x_F) - P(y_P - x_F)v'(x_F) \\ + Wv(x_W) - Wv(x_F) - Wv(x_W - x_F)v'(x_F)] \quad . \quad (\text{III.5.17}) \end{aligned}$$

Making use of the cascade material balances: $F = P + W$, and also

$P(y_P - x_F) = W(x_F - x_W)$, equation (III.5.17) can be written in the form

$$\sum L(\text{tot}) = \frac{4}{\psi^2} [Pv(y_P) + Wv(x_W) - Fv(x_F)] \quad . \quad (\text{III.5.18})$$

The quantity in the square brackets in equation (III.5.18) is the result of a value balance around the cascade and is the *separative capacity* of the cascade. Thus we see that in the case of an ideal cascade, the separative capacity is directly proportional to the summation of the interstage flows:

$$U = \frac{\psi^2}{4} \sum L(\text{tot}) \quad . \quad (\text{III.5.19})$$

For isotope separation plants, using equipment of a particular type, we would expect that both the amount of equipment necessary to perform the separation and the amount of power required would be proportional to the summation of interstage flows. Although the latter is not necessarily the case when reversible separation methods are under consideration, it is the case for irreversible separation processes where power consumption is of prime importance. Since, as has been shown, the ideal cascade minimizes the summation of the stage upflows, we may make the following statement: The separative capacity required to effect a specified separation job is an index of the amount of equipment required in the separation plant, and in the case of irreversible separation processes an index of the power requirement. Clearly, from equation (III.5.19), given

the separative capacity required and the separation factor for the process, one can compute the minimum summation of stage upflows in the cascade to do the separation job. Then to estimate the total equipment and power requirement, one must know the equipment and power associated with one unit of interstage flow for the separation process under consideration.

It is worth pointing out at this time that the theory of ideal cascades is of more than just academic interest. Since, in an ideal cascade, no two stages are of the same size, it would be quite expensive to construct one. In practice, a separation cascade consists of sections in which the stages are identical. Such cascades are called real cascades or squared-off cascades. Since the separative capacity per unit upflow in a cascade section is given by equation (III.4.8)

$$\frac{\delta U}{L} = \psi^2 \frac{L_{\min}}{L} \left(1 - \frac{L_{\min}}{L}\right) dn$$

it can be seen that for an ideal cascade, $\delta U/L = \psi^2 dn/4$, and that for values of L that differ by as much as 20% from the ideal cascade value, the separative capacity per unit upflow decreases by only 4% from that in an ideal section. Thus, it follows that reasonably well designed real cascades approximate very closely the ideal cascade and their behavior can be calculated from ideal cascade theory.

Equilibrium time estimates for ideal cascades. An estimate of the equilibrium time of a cascade can be obtained by dividing the net enriched inventory of the cascade at steady state by the average rate that this material is supplied during the transient operating period. By the equilibrium time we mean the time that the cascade must operate from the time that it is filled with feed material to the time that

product material can be withdrawn at the rate P and concentration y_p . The initial rate at which enriched inventory is supplied to the enricher is

$$L_F(y - x)_F = L_F y_F (1 - x_F)$$

where F indicates the feed point of the cascade. When this maximum transport is used, one obtains a minimum estimate of the equilibrium time. As the concentration gradient in the enricher builds up, the transport of enriched material past the feed point decreases until at steady state it is equal to

$$P(y_p - x_F) .$$

When this minimum transport is used to calculate the equilibrium time, one obtains a maximum estimate of the time required.

The net enriched inventory, abbreviated N.E.I., contained in the enriching section of a cascade at steady state is given by

$$\text{N.E.I.} = \sum_{n=0}^N h_n (x_n - x_F) = \int_0^N h(x - x_F) dn , \quad (\text{III.5.20})$$

where h_n represents the hold-up of process gas, or process gas inventory, of the n^{th} stage. Since, for most processes, the size of a stage is directly proportional to the stage upflow, we assume that the quantity, h_n/L_n , is a constant for a given separation process. The quantity, h/L , has the units of time and is frequently termed the turn-over time or the hold-up time of the process. The expression for the net enriched inventory can therefore be written in the form

$$\text{N.E.I.} = \frac{h}{L} \int_0^N L(x - x_F) dn = \frac{h}{L} \int_{x_F}^{y_p} (x - x_F) L \frac{dn}{dx} dx . \quad (\text{III.5.21})$$

Inserting into the integral the expressions for L and for dn/dx for an ideal enricher as a function of concentration, we obtain

$$\text{N.E.I.} = \frac{h}{L} \frac{4P}{\psi^2} \int_{x_F}^{y_P} (x - x_F) (y_P - x) v''(x) dx . \quad (\text{III.5.22})$$

Integration of the expression for the net enriched inventory may be carried out as follows.

$$\frac{\text{N.E.I.}}{\frac{h}{L} \frac{4P}{\psi^2}} = \int_{x_F}^{y_P} (y_P x - x^2) v''(x) dx - \int_{x_F}^{y_P} x_F (y_P - x) v''(x) dx$$

The second integral has been evaluated in the calculation of the sum of the interstage flows. For the above expression one obtains

$$\begin{aligned} \frac{\text{N.E.I.}}{\frac{h}{L} \frac{4P}{\psi^2}} &= y_P^2 v'(y_P) - y_P x_F v'(x_F) - y_P v(y_P) + y_P v(x_F) \\ &\quad - \frac{1}{1 - y_P} + \frac{1}{1 - x_F} \\ &\quad - x_F [v(y_P) - v(x_F) - (y_P - x_F) v'(x_F)] . \end{aligned} \quad (\text{III.5.23})$$

Substituting into equation (III.5.23) the definitions of the value function and the first derivative of the value function given by (II.2.13) and (II.2.14), and algebraically simplifying the resulting expression, we obtain

$$\text{N.E.I.} = \frac{h}{L} \frac{4P}{\psi^2} [(y_P - 2y_P x_F + x_F) \ln \frac{y_P (1 - x_F)}{(1 - y_P) x_F} - 2(y_P - x_F)] \quad (\text{III.5.24})$$

and the estimates of the equilibrium time are:

$$\text{Eq.T. (max)} = \frac{\text{N.E.I.}}{P (y_P - x_F)} , \quad (\text{III.5.25})$$

$$\text{Eq.T. (min)} = \frac{\text{N.E.I.}}{L_F \psi x_F (1 - x_F)} . \quad (\text{III.5.26})$$

It may be noted that for the case of an ideal enricher, the maximum equilibrium time estimate is always precisely twice the minimum equilibrium time estimate.

6. Application of the Ideal Plant Concept: Optimum Tails Concentration

and the Cost of Enriched Isotopes. When dealing with the separation of isotopes on a large scale, it is usually possible to establish with some accuracy a reliable number for the cost of a unit of separative work, taking into account both the capital and the operating costs of the plant. The cost of product can then be written:

$$\bar{P}C_P = \bar{F}C_F + k[\bar{P}v(y_P) + \bar{W}v(x_W) - \bar{F}v(x_F)] , \quad (\text{III.6.1})$$

where $\bar{P}$, $\bar{F}$, and $\bar{W}$ are amounts of product, feed, and waste (not rates), C_P is the cost per unit of product, C_F is the cost per unit of feed material, and k is the cost per unit of separative work. The term in brackets is, of course, the amount of separative work required to produce $\bar{P}$ units of product from feed material of concentration x_F with the waste concentration set at x_W . The first question to be answered is: "What is the optimum value of the waste concentration, x_W ?" The longer the stripper the less is the amount of feed required, but the larger the amount of separative work required. Thus the optimum value of x_W implies a balance between the feed cost and the stripper cost and will depend on the ratio of the given costs, C_F and k .

The optimum value of x_W is, of course, the value which minimizes the cost of the product material. To determine the optimum value, we differentiate the cost of product, $\bar{P}C_P$, with respect to x_W , realizing that $\bar{P}$, y_P , and x_F are constant, and set the derivative equal to zero.

$$\frac{d(\bar{P}C_P)}{dx_W} = \frac{d\bar{F}}{dx_W} C_F + k[\bar{W}'v(x_W) + \frac{d\bar{W}}{dx_W} v(x_W) - \frac{d\bar{F}}{dx_W} v(x_F)] . \quad (\text{III.6.2})$$

From the cascade material balances we obtain the relationships

$$\bar{P} + \bar{W} = \bar{F} ; \quad \frac{d\bar{W}}{dx_W} = \frac{d\bar{F}}{dx_W} , \quad \text{and} \quad (\text{III.6.3})$$

$$\bar{P}y_P + \bar{W}x_W = \bar{F}x_F ; \quad \bar{W} + \frac{d\bar{W}}{dx_W} x_W = \frac{d\bar{F}}{dx_W} x_F . \quad (\text{III.6.4})$$

Eliminating $\bar{W}$ and $d\bar{W}/dx_W$ from equation (III.6.2) by means of the two material balance relationships, we obtain

$$\frac{d\bar{F}}{dx_W} C_F + k \frac{d\bar{F}}{dx_W} [(x_F - x_W)v'(x_W) + v(x_W) - v(x_F)] = 0 \quad (\text{III.6.5})$$

or

$$v(x_F) - v(x_W) - (x_F - x_W)v'(x_W) = \frac{C_F}{k} . \quad (\text{III.6.6})$$

Thus, with x_F given, and the value of the ratio C_F/k known, the above equation can be solved (numerically) to yield the value of x_W (optimum) which minimizes the cost of product.

If equation (III.6.6) is solved for C_F and the result inserted into equation (III.6.1), one obtains

$$\begin{aligned} \bar{P}C_P = k [&\bar{F}v(x_F) - \bar{F}v(x_W) - \bar{F}(x_F - x_W)v'(x_W) + \bar{P}v(y_P) \\ &+ \bar{W}v(x_W) - \bar{F}v(x_F)] , \end{aligned} \quad (\text{III.6.7})$$

and, making use of the material balance expressions, (III.6.3) and (III.6.4), the above equation can be written in the form

$$\bar{P}C_P = k [\bar{P}v(y_P) - \bar{P}v(x_W) - \bar{P}(y_P - x_W)v'(x_W)] \quad (\text{III.6.8})$$

or

$$C_P = k [v(y_P) - v(x_W) - (y_P - x_W)v'(x_W)] . \quad (\text{III.6.9})$$

It is clear that C_P , the cost of a unit mass of enriched isotope at concentration y_P , is a function both of y_P and of x_W which in the above equations is the tails concentration which minimizes the cost of the product. From equation (III.6.9), it can be seen that the cost of material at concentration x_W is exactly zero. Furthermore, the expression in brackets in equation (III.6.9) is recognizable as the separative work performed by an enricher producing one unit of product at concentration

y_p from feed material of concentration x_w . Thus the cost curve can be obtained from the costs of enrichers to produce the desired product from material of zero value.

7. Application of the Ideal Plant Concept: Differential Constants. In order to evaluate small changes in the production rate of an ideal cascade due to certain kinds of operational perturbations, it is often convenient to employ differential constants.

a) The differential constant, $dP/\delta U$. The differential constant, $dP/\delta U$, gives the change in the production rate of an ideal cascade which is associated with the change, δU , in the separative capacity of the cascade, the product concentration remaining constant. As an example of the use of this differential constant, consider the question of determining the change in the product rate of an ideal cascade due to the addition or removal of one or more stages. The answer can be written

$$dP = \frac{dP}{\delta U} \delta U = \frac{dP}{\delta U} (L\psi^2 N/4) , \quad (\text{III.7.1})$$

where $L\psi^2 N/4$ represents the increase in separative capacity of an ideal cascade due to the addition of N stages of size L .

The differential constant, $dP/\delta U$, is evaluated as follows. One starts with the three equations representing the overall material balance, the desired component material balance, and the value balance around the cascade and forms the derivatives with respect to P , realizing that y_p , F , and x_F are to remain invariant. Thus one obtains

$$F = P + W ; \quad 1 + \frac{dW}{dP} = 0 \quad (\text{III.7.2})$$

$$F x_F = P y_p + W x_w ; \quad y_p + W \frac{dx_w}{dP} + x_w \frac{dW}{dP} = 0 \quad (\text{III.7.3})$$

$$U = Pv(y_P) + Wv(x_W) - Fv(x_F) ; \quad (\text{III.7.4a})$$

$$\frac{\delta U}{dP} = v(y_P) + Wv'(x_W) \frac{dx_W}{dP} + \frac{dW}{dP} v(x_W) . \quad (\text{III.7.4b})$$

From equations (III.7.2) and (III.7.3) we have the relationships

$$\frac{dW}{dP} = -1 , \quad \text{and} \quad W \frac{dx_W}{dP} = x_W - y_P ,$$

which when substituted into equation (III.7.4b) yields

$$\frac{\delta U}{dP} = v(y_P) - v(x_W) - (y_P - x_W) v'(x_W) , \quad (\text{III.7.5})$$

or

$$\frac{dP}{\delta U} = \frac{1}{v(y_P) - v(x_W) - (y_P - x_W) v'(x_W)} . \quad (\text{III.7.6})$$

b) The effect of a restricted interstage flow. An interesting application of the differential constant, $dP/\delta U$, is to the evaluation of the loss of production due to a faulty stage in which the stage upflow is appreciably less than the design value. One must first evaluate the loss in separative capacity associated with the reduced value of L ; this can be done with the results of Section (III.4).

Let f be the ratio of the actual interstage flow to the ideal interstage flow: $L = fL_I$. From equation (III.4.4), we see that the enrichment effected by an ideal stage is equal to $\psi/2$, while for a stage with the flow fL_I , the enrichment is given by

$$\epsilon = \psi \left(1 - \frac{1}{2f}\right) .$$

Thus the change in the enrichment caused by throttling the upflow from an ideal stage is given by

$$\Delta \epsilon = \frac{\psi}{2} \left(1 - \frac{1}{f}\right) . \quad (\text{III.7.7})$$

Thus it can be seen that when f is small, the loss in enrichment can be equal to that of several ideal stages: for example, when $f = 0.1$, the

loss in enrichment, compared with the ideal cascade, is equal to that of nine ideal stages. Similarly, from equation (III.4.8), we see that the loss in separative capacity due to restricting the stage upflow is given by

$$\Delta U = \left(1 - \frac{1}{f}\right) \frac{\psi^2}{4} L_I = \left(1 - \frac{1}{f}\right) \psi \frac{P(y_p - x)}{2x(1-x)} \quad , \quad (\text{III.7.8})$$

and it can be seen that the loss in separative capacity which results when the upflow from a stage is reduced from its ideal value to 0.1 of the ideal flow is similarly equal to the separative capacity of nine ideal stages of the same size. When a reduction in the flow of a stage results in the loss of more than one equivalent stage, a reversal of the gradient (called a negative gradient) occurs at the perturbed stage.

The loss in production can be calculated from equation (III.7.8) and the differential constant, $dP/\delta U$:

$$\Delta P = \frac{dP}{\delta U} \Delta U \quad . \quad (\text{III.7.9})$$

c) Another differential constant, dP/dE . The differential constant, dP/dE , gives the change in the production rate of an ideal cascade which occurs when a small sidestream, dE at concentration x_E , is withdrawn from the cascade. The procedure for evaluating dP/dE is the same as that for obtaining $dP/\delta U$: one writes the overall material balance, the specified component material balance, and the value balance, and forms the derivatives with respect to E , realizing that y_p , x_E , x_F , F , and U are invariant. Thus one obtains:

$$F = P + E + W ; \quad \frac{dP}{dE} + 1 + \frac{dW}{dE} = 0 \quad (\text{III.7.10})$$

$$F x_F = P y_p + E x_E + W x_W ; \quad y_p \frac{dP}{dE} + x_E + W \frac{d x_W}{dE} + x_W \frac{dW}{dE} = 0 \quad (\text{III.7.11})$$

$$U = Pv(y_P) + Ev(x_E) + Wv(x_W) - Fv(x_F) ; \quad (\text{III.7.12a})$$

$$v(y_P)\frac{dP}{dE} + v(x_E) + Wv'(x_W)\frac{dx_W}{dE} + \frac{dW}{dE}v(x_W) = 0 . \quad (\text{III.7.12b})$$

From equations (III.7.10) and (III.7.11), we have the relationships

$$\frac{dW}{dE} = -1 - \frac{dP}{dE} , \quad \text{and} \quad W\frac{dx_W}{dE} = -x_E - y_P\frac{dP}{dE} + x_W + x_W\frac{dP}{dE} ,$$

which when substituted into equation (III.7.12b) yields

$$v(y_P)\frac{dP}{dE} + v(x_E) + v'(x_W)(-x_E - y_P\frac{dP}{dE} + x_W + x_W\frac{dP}{dE}) - v(x_W) - v(x_W)\frac{dP}{dE} = 0 .$$

Solving the above for dP/dE , we obtain

$$\frac{dP}{dE} = - \frac{v(x_E) - v(x_W) - (x_E - x_W)v'(x_W)}{v(y_P) - v(x_W) - (y_P - x_W)v'(x_W)} . \quad (\text{III.7.13})$$

The change in the product rate due to the withdrawal of the side stream is then obtained from

$$\Delta P = \frac{dP}{dE} \Delta E . \quad (\text{III.7.14})$$

The expression given by equation (III.7.13) for the differential constant dP/dE could have been obtained directly from a previous result by realizing that the dollar value of the streams produced by an ideal cascade of constant separative capacity is a fixed number and writing

$$PC_P + EC_E = \$, \quad \text{a constant,}$$

where C_P represents the cost of a unit of product at concentration y_P and C_E represents the cost of a unit of side stream material at concentration x_E . Differentiating with respect to E , one obtains

$$\frac{dP}{dE} = - \frac{C_E}{C_P} ,$$

which reference to equation (III.6.9) shows to be equivalent to equation (III.7.13) above.

8. Ideal Cascades Composed of Stages with Large Separation Factors:

The General Case. Consider now the case where one is not justified in making the approximations associated with stage separation factors close to unity; that is, the approximate equilibrium line equation, the differential equation in place of the finite difference equation, and the replacement of $(1 - P/L)$ by unity.

In this analysis use will be made of the quantity R , termed the abundance ratio, as a measure of the concentration of the desired component; it is related to the mole fraction z by the expressions

$$R \equiv \frac{z}{1-z} \quad \text{and} \quad z = \frac{R}{1+R} . \quad (\text{III.8.1})$$

One may now define, for convenience, the following stage separation factors relating the concentrations of the streams entering and leaving a stage

$$\alpha \equiv \frac{R'}{R''} = \frac{y}{1-y} \frac{1-x}{x} ; \quad (\text{III.8.2})$$

$$\beta \equiv \frac{R'}{R} = \frac{y}{1-y} \frac{1-z}{z} ; \quad y = \frac{\beta R}{1+\beta R} = \frac{\beta z}{1+(\beta-1)z} ; \quad (\text{III.8.3})$$

$$\gamma \equiv \frac{R}{R''} = \frac{z}{1-z} \frac{1-x}{x} ; \quad x = \frac{R}{\gamma+R} = \frac{z}{\gamma-(\gamma-1)z} . \quad (\text{III.8.4})$$

The nomenclature is similar to that used previously and is shown in the figure depicting two consecutive stages on the following page. It follows from the above definitions that $\alpha = \beta\gamma$. If the stages which make up the cascade are similar, that is, they all have the same values of α , β , and γ , but differ in size (i.e., throughput), the criterion for ideality (that is, no mixing of streams of unequal concentrations) can be expressed

$$R'_n = R_{n+1} \quad \text{and} \quad R''_{n+1} = R_n ,$$

$$\frac{R'_n}{R_n} = \frac{R_{n+1}}{R''_{n+1}} , \text{ or}$$

$$\beta = \gamma = \sqrt{\alpha} . \quad (\text{III.8.5})$$

The gradient is determined from the relationship

$$R_{n+1} = \beta R_n . \quad (\text{III.8.6})$$

The solution to this simple difference equation is

$$R_n = c\beta^n . \quad (\text{III.8.7})$$

Let the enricher consist of N stages excluding the feed stage, and let the feed stage be designated as the zeroth stage. The abundance ratio in the product material is given by $R_p = \beta R_N$, and one can write the gradient in the following forms

$$R_n = R_0\beta^n = R_F\beta^n ; \quad z_n = \frac{R_F\beta^n}{1 + R_F\beta^n} , \quad (\text{III.8.8a})$$

$$R_n = R_p\beta^{n-N-1} ; \quad z_n = \frac{R_p\beta^{n-N-1}}{1 + R_p\beta^{n-N-1}} \quad (\text{III.8.8b})$$

An expression for the cut θ of a stage in an ideal cascade is obtained from the material balance around the stage

$$\theta y + (1 - \theta)x = z ; \quad \theta = \frac{z - x}{y - x} . \quad (\text{III.8.9})$$

Substituting the expressions for y and x from equations (III.8.3) and (III.8.4) into the above equation, one obtains

$$\theta = \frac{(\gamma - 1)(1 - z)\{1 + (\beta - 1)z\}}{\beta\gamma - 1 - \beta(\gamma - 1)z - (\beta - 1)z} \quad (\text{III.8.10})$$

which, when the ideality condition ($\gamma = \beta$) is introduced, reduces to

$$\theta = \frac{1 + (\beta - 1)z}{\beta + 1} . \quad (\text{III.8.11})$$

Thus, for this general case, the value of the stage cut is seen to depend on the concentration of the material fed to the stage as well as on the magnitude of the stage separation factor.

An expression for the feed rate G to a stage in an ideal cascade is obtained from the material balance around the top of the cascade above the stage under consideration

$$\theta Gy = (\theta G - P)z + Py_P ; \quad G = \frac{P(y_P - z)}{\theta(y - z)} . \quad (\text{III.8.12})$$

Substituting for y by means of equation (III.8.3) and for θ by means of equation (III.8.11), one obtains

$$G = \frac{P(y_P - z)\{1 + (\beta - 1)z\}}{\theta(\beta - 1)z(1 - z)} = P \frac{\beta + 1}{\beta - 1} \frac{y_P - z}{z(1 - z)} . \quad (\text{III.8.13})$$

The sum of the stage feed flows in the enriching section, including that to the feed stages, is given by

$$\sum_{n=0}^N G_n = P \frac{\beta + 1}{\beta - 1} \sum_{n=0}^N \frac{y_P - z_n}{z_n(1 - z_n)} = P \frac{\beta + 1}{\beta - 1} \sum_{n=0}^N \left\{ y_P \frac{(1 + R_n)^2}{R_n} - (1 + R_n) \right\} . \quad (\text{III.8.14})$$

Substituting for R_n by means of the gradient equation (III.8.8a), one obtains

$$\sum_{n=0}^N G_n = P \frac{\beta + 1}{\beta - 1} \sum_{n=0}^N \left\{ y_P \left(\frac{1}{R_F \beta^n} + 2 + R_F \beta^n \right) - (1 + R_F \beta^n) \right\} . \quad (\text{III.8.15})$$

The sum can be obtained by finite integration of the right hand side of the above expression, the limits of integration being 0 and $N+1$

$$\begin{aligned} \sum_{n=0}^N G_n &= P \frac{\beta + 1}{\beta - 1} \left\{ (2y_P - 1)n + (y_P - 1)R_F \frac{\beta^n}{\beta - 1} + \frac{y_P}{R_F} \frac{1}{\beta^n} \frac{\beta}{1 - \beta} \right\}_0^{N+1} \\ &= P \frac{\beta + 1}{\beta - 1} \left\{ (2y_P - 1)(N+1) + \frac{(y_P - 1)R_F(\beta^{N+1} - 1)}{\beta - 1} + \frac{y_P}{R_F} \frac{\beta}{1 - \beta} \left(\frac{1}{\beta^{N+1}} - 1 \right) \right\} . \end{aligned} \quad (\text{III.8.16})$$

Making use of the fact that $R_P = R_F \beta^{N+1}$, the total feed flow in the enriching section (including the feed stage) can be written

$$\sum G(\text{enr}) = P \frac{\beta+1}{\beta-1} \left\{ (2y_P - 1) \frac{\ln(R_P/R_F)}{\ln \beta} + \frac{\beta - (\beta+1)x_F}{\beta-1} \frac{y_P - x_F}{x_F(1-x_F)} \right\}. \quad (\text{III.8.17})$$

An expression for the feed rate G to a stage in the stripping section of an ideal cascade is obtained from the material balance around the bottom of the cascade including the stage under consideration

$$\theta G y + W x_w = (\theta G + W) z; \quad G = \frac{W(z - x_w)}{\theta(y - z)}. \quad (\text{III.8.18})$$

Substituting for y by means of equation (III.8.3) and for θ by means of equation (III.8.11), one obtains

$$G = \frac{W(z - x_w) \{1 + (\beta - 1)z\}}{\theta(\beta - 1)z(1 - z)} = W \frac{\beta + 1}{\beta - 1} \frac{z - x_w}{z(1 - z)}. \quad (\text{III.8.19})$$

Let the stripper consist of M stages excluding the feed stage, and let the bottom stage be designated as the first stage. The gradient in the stripper can be expressed

$$R_m = R_w \beta^m; \quad z_m = \frac{R_w \beta^m}{1 + R_w \beta^m}. \quad (\text{III.8.20})$$

The sum of the stage feed flows in the stripping section is given by

$$\begin{aligned} \sum_{m=1}^M G_m &= W \frac{\beta + 1}{\beta - 1} \sum_{m=1}^M \frac{z_m - x_w}{z_m(1 - z_m)} \\ &= W \frac{\beta + 1}{\beta - 1} \sum_{m=1}^M \left\{ 1 + R_w \beta^m - x_w \left(\frac{1}{R_w \beta^m} + 2 + R_w \beta^m \right) \right\}. \quad (\text{III.8.21}) \end{aligned}$$

The sum can be obtained by finite integration of the right hand side of the above expression, the limits of integration being 1 and $M+1$.

$$\begin{aligned} \sum G(\text{str}) &= W \frac{\beta + 1}{\beta - 1} \left\{ (1 - 2x_w)m + (1 - x_w)R_w \frac{\beta^m}{\beta - 1} - \frac{x_w}{R_w} \frac{1}{\beta^m} \frac{\beta}{1 - \beta} \right\}_1^{M+1} \\ &= W \frac{\beta + 1}{\beta - 1} \left\{ (1 - 2x_w)M + \frac{(1 - x_w)R_w(\beta^{M+1} - \beta)}{\beta - 1} \right. \\ &\quad \left. - \frac{x_w}{R_w} \frac{\beta}{1 - \beta} \left(\frac{1}{\beta^{M+1}} - \frac{1}{\beta} \right) \right\}. \quad (\text{III.8.22}) \end{aligned}$$

Making use of the fact that $R_F = R_W \beta^{M+1}$, the total feed flow in the stripping section can be written

$$\sum G(\text{str}) = W \frac{\beta+1}{\beta-1} \left\{ (2x_W - 1) \frac{\ln(R_W/R_F)}{\ln \beta} - \frac{\beta - (\beta+1)x_F}{\beta-1} \frac{x_F - x_W}{x_F(1-x_F)} \right\}. \quad (\text{III.8.23})$$

The total stage feed flow in the ideal cascade is given by the sum of (III.8.17) and (III.8.23)

$$\begin{aligned} \sum G(\text{casc}) &= \frac{\beta+1}{\beta-1} \frac{1}{\ln \beta} \left\{ P(2y_P - 1) \ln \frac{R_P}{R_F} + W(2x_W - 1) \ln \frac{R_W}{R_F} \right\} \\ &= \frac{\beta+1}{\beta-1} \frac{1}{\ln \beta} \left\{ P(2y_P - 1) \ln R_P + W(2x_W - 1) \ln R_W - F(2x_F - 1) \ln R_F \right\} \end{aligned} \quad (\text{III.8.24a})$$

$$= \frac{\beta+1}{\beta-1} \frac{1}{\ln \beta} U. \quad (\text{III.8.24b})$$

At times one is more interested in the summation of stage upflows than in the summation of stage feeds. An expression for $\sum L$ in an ideal cascade is obtained as follows. The sum of the stage upflows in the enriching section, including that of the feed stage, is given by

$$\begin{aligned} \sum L(\text{enr}) &= \sum_{n=0}^N \theta_n G_n = \frac{P}{\beta-1} \sum_{n=0}^N \left\{ y_P \frac{(1+R_n)(1+\beta R_n)}{R_n} - (1+\beta R_n) \right\} \\ &= \frac{P}{\beta-1} \sum_{n=0}^N \left\{ y_P \left(\frac{1}{R_F \beta^n} + \beta + 1 + R_F \beta^{n+1} \right) - (1 + R_F \beta^{n+1}) \right\}. \end{aligned} \quad (\text{III.8.25})$$

The sum is obtained by finite integration, the limits of integration being 0 and $N+1$:

$$\begin{aligned} \sum L(\text{enr}) &= \frac{P}{\beta-1} \left\{ (\beta y_P + y_P - 1)n + (y_P - 1) R_F \frac{\beta^{n+1}}{\beta-1} + \frac{y_P}{R_F} \frac{1}{\beta^n} \frac{\beta}{1-\beta} \right\}_0^{N+1} \\ &= \frac{P}{\beta-1} (\beta y_P + y_P - 1)(N+1) + \frac{\beta}{\beta-1} (y_P - 1) R_F (\beta^{N+1} - 1) \\ &\quad - \frac{\beta}{\beta-1} \frac{y_P}{R_F} \left(\frac{1}{\beta^{N+1}} - 1 \right). \end{aligned} \quad (\text{III.8.26})$$

Making use of the fact that $R_P = R_F \beta^{N+1}$, the total upflow in the enriching section (including the feed stage) can be written

$$\sum L(\text{enr}) = \frac{P}{\beta - 1} \{ (\beta y_P + y_P - 1) \frac{\ln(R_P/R_F)}{\ln \beta} + \frac{\beta}{\beta - 1} \frac{(y_P - x_F)(1 - 2x_F)}{x_F(1 - x_F)} \}. \quad (\text{III.8.27})$$

The sum of the stage upflows in the stripping section is given by

$$\begin{aligned} \sum L(\text{str}) &= \sum_{m=1}^M \theta_m G_m = \frac{W}{\beta - 1} \sum_{m=1}^M \left\{ 1 + \beta R_m - x_W \frac{(1 + R_m)(1 + \beta R_m)}{R_m} \right\} \\ &= \frac{W}{\beta - 1} \sum_{m=1}^M \left\{ 1 + R_m \beta^{m+1} - x_W \left(\frac{1}{R_m \beta^m} + \beta + 1 + R_m \beta^{m+1} \right) \right\}. \end{aligned} \quad (\text{III.9.28})$$

The sum is obtained by finite integration, the limits of integration being 1 and $M+1$:

$$\begin{aligned} \sum L(\text{str}) &= \frac{W}{\beta - 1} \left\{ (1 - \beta x_W - x_W) M + (1 - x_W) R_W \frac{\beta^{M+1}}{\beta - 1} - \frac{x_W}{R_W} \frac{1}{\beta^M} \frac{\beta}{1 - \beta} \right\}_1^{M+1} \\ &= \frac{W}{\beta - 1} \left\{ (1 - \beta x_W - x_W) (M) + \frac{\beta}{\beta - 1} (1 - x_W) R_W (\beta^{M+1} - \beta) \right. \\ &\quad \left. + \frac{\beta}{\beta - 1} \frac{x_W}{R_W} \left(\frac{1}{\beta^{M+1}} - \frac{1}{\beta} \right) \right\}. \end{aligned} \quad (\text{III.8.29})$$

Making use of the fact that $R_F = R_W \beta^{M+1}$, the total upflow in the stripping section can be written

$$\sum L(\text{str}) = \frac{W}{\beta - 1} \left\{ (\beta x_W + x_W - 1) \frac{\ln(R_W/R_F)}{\ln \beta} - \frac{\beta}{\beta - 1} \frac{(x_F - x_W)(1 - 2x)}{x_F(1 - x_F)} \right\}. \quad (\text{III.8.30})$$

The total upflow in the ideal cascade is given by the sum of (III.8.27) and (III.8.30):

$$\begin{aligned} \sum L(\text{casc}) &= \frac{1}{\beta - 1} \frac{1}{\ln \beta} \left\{ P(\beta y_P + y_P - 1) \ln \frac{R_P}{R_F} + W(\beta x_W + x_W - 1) \ln \frac{R_W}{R_F} \right\} \\ &= \frac{1}{\beta - 1} \frac{1}{\ln \beta} \left\{ P(\beta y_P + y_P - 1) \ln R_P + W(\beta x_W + x_W - 1) \ln R_W \right. \\ &\quad \left. - F(\beta x_F + x_F - 1) \ln R_F \right\} \end{aligned} \quad (\text{III.8.31})$$

$$\sum L(\text{casc}) = \frac{1}{\beta - 1} \frac{1}{\ln \beta} U + \frac{1}{\ln \beta} \{P y_P \ln R_P + W x_W \ln R_W - F x_F \ln R_F\} . \quad (\text{III.8.32})$$

9. Ideal Cascades Composed of Stages with Large Separation Factors:

The Dilute Case. The dilute case is characterized by the dilute approximation: $(1 - z) \approx 1$, where z represents any concentration occurring within the cascade. For this case the equations of the preceding section take on somewhat simpler forms. Here the stage separation factors are defined by

$$\alpha \equiv \frac{y}{x} \quad (\text{III.9.1})$$

$$\beta \equiv \frac{y}{z} ; \quad y = \beta z \quad (\text{III.9.2})$$

$$\gamma \equiv \frac{z}{x} ; \quad x = \frac{z}{\gamma} . \quad (\text{III.9.3})$$

The condition for ideality remains as before

$$\beta = \gamma = \sqrt{\alpha} . \quad (\text{III.9.4})$$

However, the gradient in the dilute case can be written in the form

$$z_n = z_0 \beta^n = x_F \beta^n \quad \text{or} \quad z_n = y_P \beta^{n-N-1} \quad (\text{III.9.5})$$

in the enricher, and

$$z_m = x_W \beta^m \quad (\text{III.9.6})$$

in the stripper, and the expression for the cut in the ideal cascade reduces to

$$\theta = \frac{1}{\beta + 1} \quad (\text{III.9.7})$$

and is constant throughout the cascade.

For the feed rate to a stage in the enriching section one now obtains

$$G = P \frac{\beta + 1}{\beta - 1} \frac{y_P - z}{z} \quad \text{and} \quad L = \frac{P}{\beta - 1} \frac{y_P - z}{z} . \quad (\text{III.9.8})$$

The sum of the stage flows in the enriching section, including that to

the feed stage, is given by

$$\begin{aligned}
 \sum G(\text{enr}) &= P \frac{\beta+1}{\beta-1} \sum_{n=0}^N \frac{y_P - z_n}{z_n} = P \frac{\beta+1}{\beta-1} \sum_{n=0}^N \left(\frac{y_P}{x_F} \frac{1}{\beta^n} - 1 \right) \\
 &= P \frac{\beta+1}{\beta-1} \left\{ \frac{y_P}{x_F} \frac{1}{\beta^n} \frac{\beta}{1-\beta} - n \right\}_0^{N+1} = P \frac{\beta+1}{\beta-1} \left\{ - (N+1) - \frac{\beta}{\beta-1} \frac{y_P}{x_F} \left(\frac{1}{\beta^{N+1}} - 1 \right) \right\} \\
 \sum G(\text{enr}) &= P \frac{\beta+1}{\beta-1} \left\{ \frac{\ln(x_F/y_P)}{\ln \beta} + \frac{\beta}{\beta-1} \frac{y_P - x_F}{x_F} \right\} . \quad (\text{III.9.9})
 \end{aligned}$$

For the feed rate to a stage in the stripping section one now obtains

$$G = W \frac{\beta+1}{\beta-1} \frac{z - x_W}{z} \quad \text{and} \quad L = \frac{W}{\beta-1} \frac{z - x_W}{z} . \quad (\text{III.9.10})$$

The sum of the stage feed flows in the stripping section is given by

$$\begin{aligned}
 \sum G(\text{str}) &= W \frac{\beta+1}{\beta-1} \sum_{m=1}^M \frac{z_m - x_W}{z_m} = W \frac{\beta+1}{\beta-1} \sum_{m=1}^M \left(1 - \frac{1}{\beta^m} \right) \\
 &= W \frac{\beta+1}{\beta-1} \left\{ m - \frac{\beta}{1-\beta} \frac{1}{\beta^m} \right\}_1^{M+1} = W \frac{\beta+1}{\beta-1} \left\{ M + \frac{\beta}{\beta-1} \left(\frac{1}{\beta^{M+1}} - \frac{1}{\beta} \right) \right\} \\
 \sum G(\text{str}) &= W \frac{\beta+1}{\beta-1} \left\{ \frac{\ln(x_F/x_W)}{\ln \beta} - \frac{\beta}{\beta-1} \frac{x_F - x_W}{x_F} \right\} . \quad (\text{III.9.11})
 \end{aligned}$$

The total stage feed flow in the ideal cascade is given by the sum of (III.9.9) and (III.9.11)

$$\sum G(\text{casc}) = \frac{\beta+1}{\beta-1} \frac{1}{\ln \beta} \left\{ P \ln \frac{1}{y_P} + W \ln \frac{1}{x_W} - F \ln \frac{1}{x_F} \right\} \quad (\text{III.9.12})$$

$$= \frac{\beta+1}{\beta-1} \frac{1}{\ln \beta} U . \quad (\text{III.9.13})$$

The total stage upflow in the ideal cascade, since θ is constant, is given by

$$\sum L(\text{casc}) = \theta \sum G(\text{casc}) = \frac{U}{(\beta-1) \ln \beta} . \quad (\text{III.9.14})$$

IV. ELEMENTS OF CASCADE DESIGN: SQUARED-OFF CASCADES

The ideal cascades discussed in the previous chapter are tapered cascades in which no two stages in the enriching section or in the stripping section are the same size. Although, as has been shown, the ideal cascade has the property of minimizing the total interstage flow and, hence, the power and volume of equipment necessary to carry out a given separation task, the ideal cascade *is not the minimum cost cascade*. It is the nature of the manufacturing process that the cost per unit decreases as the number of units produced increases. Thus, appreciable cost savings are realized when reasonably large numbers of identical stages are used in the construction of a separation cascade.

A cascade section composed of identical separation stages is called a square section. In order either to design a separation cascade for a given task or to compute the separation performance of a given cascade it is first necessary to obtain solutions to the gradient equations for a square cascade section. The form of the gradient equation depends on the form of the equilibrium relationship used, on the number of approximations made in the derivation, and whether one is concerned with the enriching section, stripping section, or the region between side-streams. In section II.1 we discussed five forms of the equilibrium relationship, and in section III.2 we discussed three degrees of approximation (the difference equation, the differential equation, and the design equation): thus, we are confronted with 45 forms of the gradient equation to solve for a square section.

Although we will now discuss some analytic, explicit solutions to the various forms of the gradient equation, it should be pointed out that the advent and ubiquity of high speed electronic computing equipment tends to

make this material superfluous: in many instances it is more efficient to obtain solutions by simple numerical means, using alternately the equilibrium relationship and the operating-line relationship for the cascade.

1. The Exact, Finite-Difference Form of the Gradient Equation. This form of the gradient equation, previously given in equations (III.2.5) and (III.2.6), may be regarded as the exact form of the gradient equation for stage-wise isotope separation processes. These equations would be valid also for the separation of non-isotopic mixtures, provided that ψ and L were constant. For the case of distillation, for example, these restrictions imply an ideal mixture (ψ independent of concentration) and that the components of the mixture had equal latent heats of vaporization (constant L).

a) The enriching section. The exact, finite-difference form of the gradient equation for the enriching section is

$$x_{n+1} - x_n = \frac{L}{L - P} \left[\frac{\psi x_n (1 - x_n)}{1 + \psi x_n} - \frac{P}{L} (y_p - x_n) \right] . \quad (\text{IV.1.1})$$

Before attempting a solution to this equation, let us first examine the two limiting conditions which give rise to minimum N and minimum L .

$N_{\min}$ is defined as the minimum number of stages required to span a given concentration difference, or the number of stages required to span the concentration difference at total reflux ($P = 0$). From equation (IV.1.1) one obtains at total reflux

$$x_{n+1} = x_n + \frac{\psi x_n (1 - x_n)}{1 + \psi x_n} = \frac{(1 + \psi) x_n}{1 + \psi x_n} . \quad (\text{IV.1.2})$$

This difference equation is most readily solved by making the substitution $x = R/(1 + R)$. Equation (IV.1.2) then takes the form

$$\frac{R_{n+1}}{1 + R_{n+1}} = \frac{(1 + \psi) R_n}{1 + (1 + \psi) R_n} ; \quad R_{n+1} = (1 + \psi) R_n . \quad (\text{IV.1.3})$$

It can be readily seen by induction that the solution to this difference equation can be written

$$R_T = (1 + \psi)^N R_B, \text{ where } R = \frac{x}{1 - x}. \quad (\text{IV.1.4})$$

and

$$N_{\min} = \frac{\ln(R_T/R_B)}{\ln(1 + \psi)} = \frac{\ln(R_D/R_F)}{\ln(1 + \psi)}. \quad (\text{IV.1.5})$$

$L_{\min}$ is defined as the value of the interstage flow at which the stage enrichment vanishes. Setting $x_{n+1} - x_n$ equal to zero in equation (IV.1.1), we find that

$$L_{\min} = \frac{P(y_P - x)(1 + \psi x)}{\psi x(1 - x)}. \quad (\text{IV.1.6})$$

2. The Exact, Finite-Difference Gradient Equation. It follows from equation (III.2.5) derived in the preceding section, that the exact, finite-difference gradient equation applied to a square section in the enricher of a cascade has the form

$$x_{n+1} - x_n = \frac{L}{L - P} \left\{ \frac{\psi x_n (1 - x_n)}{1 + \psi x_n} - \frac{P}{L} (y_P - x_n) \right\}. \quad (\text{IV.2.1})$$

Multiplying each term by the quantity $(1 + \psi x_n)/\psi$, one obtains

$$\begin{aligned} \frac{1}{\psi} x_{n+1} + x_{n+1} x_n - \frac{1}{\psi} x_n - x_n^2 &= \frac{L}{L - P} (x_n - x_n^2) \\ &\quad - \frac{P}{L - P} \left(\frac{1}{\psi} y_P + y_P x_n - \frac{1}{\psi} x_n - x_n^2 \right) \end{aligned}$$

which can be written in the form

$$x_{n+1} x_n + A x_{n+1} + B x_n + C = 0, \quad (\text{IV.2.2})$$

where $A = \frac{1}{\psi}$, $B = \frac{P\psi y_P - L - L\psi}{\psi(L - P)}$, and $C = \frac{P y_P}{\psi(L - P)}$.

This is a standard form, called a Riccati difference equation. It may be solved by making the substitution $x_n = \frac{Z_{n+1}}{Z_n} - A$ which leads to a homogeneous, linear difference equation of 2nd order with constant coefficients

$$\left(\frac{Z_{n+2}}{Z_{n+1}} - A\right)\left(\frac{Z_{n+1}}{Z_n} - A\right) + A\left(\frac{Z_{n+2}}{Z_{n+1}} - A\right) + B\left(\frac{Z_{n+1}}{Z_n} - A\right) + C = 0$$

or

$$Z_{n+2} + (B - A)Z_{n+1} + (C - AB)Z_n = 0. \quad (\text{IV.2.3})$$

A difference equation of this type is solved in a manner rather similar to that employed in the case of the analogous differential equation. One determines, first of all, the roots of the auxiliary equation:

$$m^2 + (B - A)m + (C - AB) = 0.$$

The roots of the auxiliary equation are

$$m = \frac{A - B \pm \sqrt{(A - B)^2 - 4(C - AB)}}{2}$$

which, for convenience, can be written

$$m = A + \frac{-A - B \pm \sqrt{(A - B)^2 - 4(C - AB)}}{2}$$

or

$$m = \frac{1}{\psi} + x_1 \quad \text{and} \quad m = \frac{1}{\psi} + x_0. \quad (\text{IV.2.4})$$

Substitution of the values of A, B, and C into the above expression yields the following definitions for x_1 and x_0

$$\begin{aligned} x_1 &= \frac{-(A + B) + \sqrt{(A + B)^2 - 4C}}{2} \\ &= \frac{L\psi + P - P\psi y_P + \sqrt{(L\psi + P - P\psi y_P)^2 - 4P(L - P)\psi y_P}}{2(L - P)\psi}, \quad (\text{IV.2.5a}) \end{aligned}$$

$$\begin{aligned} x_0 &= \frac{-(A + B) - \sqrt{(A + B)^2 - 4C}}{2} \\ &= \frac{L\psi + P - P\psi y_P - \sqrt{(L\psi + P - P\psi y_P)^2 - 4P(L - P)\psi y_P}}{2(L - P)\psi}. \quad (\text{IV.2.5b}) \end{aligned}$$

The solution to the difference equation (IV.2.3) can be written in terms of the roots of the auxiliary equation in the form

$$Z_n = C_1\left(\frac{1}{\psi} + x_1\right)^n + C_2\left(\frac{1}{\psi} + x_0\right)^n \quad (\text{IV.2.6})$$

and the solution of the original difference equation for x_n can be written

$$x_n = \frac{C_1 \left(\frac{1}{\psi} + x_1\right)^{n+1} + C_2 \left(\frac{1}{\psi} + x_0\right)^{n+1}}{C_1 \left(\frac{1}{\psi} + x_1\right)^n + C_2 \left(\frac{1}{\psi} + x_0\right)^n} - \frac{1}{\psi}, \quad (\text{IV.2.7})$$

or

$$x_n = \frac{1}{\psi} \left\{ \frac{(1 + \psi x_1)^{n+1} + C(1 + \psi x_0)^{n+1}}{(1 + \psi x_1)^n + C(1 + \psi x_0)^n} - 1 \right\}. \quad (\text{IV.2.8})$$

The constant of integration, C , appearing in (IV.2.8) is evaluated from a boundary condition. Let the square section under consideration consist of N stages, but, for convenience, let them be numbered from $n = 0$ to $n = N-1$. The boundary conditions can then be written $x_0 = x_B$ and $x_N = x_T$. It follows from the first of these conditions that

$$x_B = \frac{1}{\psi} \left\{ \frac{1 + \psi x_1 + C(1 + \psi x_0)}{1 + C} - 1 \right\}; \quad C = \frac{x_1 - x_B}{x_B - x_0}. \quad (\text{IV.2.9})$$

and from the second condition it follows that

$$x_T = \frac{1}{\psi} \left\{ \frac{(x_B - x_0)(1 + \psi x_1)^{N+1} + (x_1 - x_B)(1 + \psi x_0)^{N+1}}{(x_B - x_0)(1 + \psi x_1)^N + (x_1 - x_B)(1 + \psi x_0)^N} - 1 \right\}. \quad (\text{IV.2.10})$$

This equation may be rearranged to yield

$$\frac{(x_T - x_0)(x_1 - x_B)}{(x_B - x_0)(x_1 - x_T)} = \frac{(1 + \psi x_1)^N}{(1 + \psi x_0)^N}; \quad N = \frac{\ln \frac{(x_T - x_0)(x_1 - x_B)}{(x_B - x_0)(x_1 - x_T)}}{\ln \frac{1 + \psi x_1}{1 + \psi x_0}}. \quad (\text{IV.2.11})$$

3. The Approximate Design Differential Gradient Equation. The approximate design differential gradient equation differs from the preceding form in that the approximate equilibrium line equation is used in place of the exact equilibrium relationship. Thus the approximate design differential gradient equation applied to a square section in the enriching section of a cascade has the form

$$\frac{dx}{dn} = \psi x(1 - x) - \frac{P}{L}(y_P - x). \quad (\text{IV.3.1})$$

It is convenient to solve this equation by first factoring the right hand sides and rewriting it in the form

$$\frac{dx}{dn} = \psi(x_1 - x)(x - x_0) . \quad (\text{IV.3.2})$$

A comparison of these two forms of the gradient equation (expanding each in a quadratic expression in x and equating the coefficients of like powers of x) shows that the constants x_1 and x_0 are related to the cascade parameters by

$$x_1 + x_0 = 1 + \frac{P}{L\psi} \quad \text{and} \quad x_1 x_0 = \frac{P\psi}{L\psi} . \quad (\text{IV.3.3})$$

These two equations can then be used to solve for x_1 and x_0 with the result

$$x_1, x_0 = \frac{L\psi + P \pm \sqrt{(L\psi + P)^2 - 4L\psi P\psi}}{2L\psi} . \quad (\text{IV.3.4})$$

By convention x_1 is taken to be the larger of the two terms; that is, the plus sign is used with x_1 , the minus sign with x_0 . It may be demonstrated that x_1 and x_0 are the x -coordinates of the points of intersection of the operating line and the approximate equilibrium line.

The gradient equation (IV.3.2) can now be written in the form

$$\frac{dx}{(x_1 - x)(x - x_0)} = \psi dn \quad (\text{IV.3.5})$$

and integrated from $n = 0$, $x = x_B$, to $n = N$, $x = x_T$. The integration is readily carried out with the use of the method of partial fractions. One writes

$$\frac{1}{(x_1 - x)(x - x_0)} = \frac{a}{x_1 - x} + \frac{b}{x - x_0} ; \quad 1 = a(x - x_0) + b(x_1 - x) ;$$

$$a = b = \frac{1}{x_1 - x_0} ;$$

and equation (IV.3.5) can be expressed

$$\int_{x_B}^{x_T} \left(\frac{1}{x_1 - x} + \frac{1}{x - x_0} \right) dx = \int_0^N (x_1 - x_0) \psi dn . \quad (\text{IV.3.6})$$

The result of the integration is

$$N = \frac{1}{\psi(x_1 - x_0)} \ln \frac{x_1 - x_B}{x_1 - x_T} \frac{x_T - x_0}{x_B - x_0} \quad (\text{IV.3.7})$$

This equation may be rearranged to give a solution for x_T when N and x_B are known:

$$x_T = \frac{x_0(x_1 - x_B) + x_1(x_B - x_0)e^{(x_1 - x_0)\psi N}}{x_1 - x_B + (x_B - x_0)e^{(x_1 - x_0)\psi N}} \quad (\text{IV.3.8})$$

An equation for x_B in terms of x_T and N is

$$x_B = \frac{x_1(x_T - x_0) + x_0(x_1 - x_T)e^{(x_1 - x_0)\psi N}}{x_T - x_0 + (x_1 - x_T)e^{(x_1 - x_0)\psi N}} \quad (\text{IV.3.9})$$

If one takes the upper limit of integration in (IV.3.6) to be an interior stage number n and the corresponding internal concentration x , one can obtain an equation of the form of (IV.3.8) from which one can calculate the concentration gradient within the section

$$x = \frac{x_0(x_1 - x_B) + x_1(x_B - x_0)e^{(x_1 - x_0)\psi n}}{x_1 - x_B + (x_B - x_0)e^{(x_1 - x_0)\psi n}} \quad (\text{IV.3.10})$$

This equation can also be used to calculate the average concentration in the section. Since each stage in the square section is presumed to have the same holdup, the average concentration in the section is given by

$$\bar{x} = \frac{1}{N} \int_0^N x dn \quad (\text{IV.3.11})$$

After carrying out the indicated integration one obtains

$$\bar{x} = x_0 + \frac{1}{N\psi} \ln \frac{x_1 - x_B}{x_1 - x_T} \quad (\text{IV.3.12})$$

The approximate design differential gradient equation applied to a square section in the stripping section of a cascade has the form

$$\frac{dx}{dm} = \psi x(1 - x) - \frac{W}{L}(x - x_w) \quad (\text{IV.3.13})$$

The solution for a section in the stripper can be obtained in the same

form as that for a section in the enricher by first rewriting the gradient equation in the form

$$\frac{dx}{dm} = \psi(x_1 - x)(x - x_0) \quad . \quad (\text{IV.3.14})$$

A comparison of these two forms of the gradient equation shows that the constants x_1 and x_0 for a section in the stripper are related to the cascade parameters by

$$x_1 + x_0 = 1 - \frac{W}{L\psi} \quad \text{and} \quad x_1 x_0 = -\frac{Wx_W}{L\psi} \quad . \quad (\text{IV.3.15})$$

It follows from these two equations that x_1 and x_0 in the stripper are given by

$$x_1, x_0 = \frac{L\psi - W \pm \sqrt{(L\psi - W)^2 + 4L\psi Wx_W}}{2L\psi} \quad . \quad (\text{IV.3.16})$$

Again, it can be demonstrated that x_1 and x_0 are the x -coordinates of the points of intersection of the operating line and the approximate equilibrium line. Integration of the gradient equation (IV.3.14) between the limits of $m = 0$, $x = x_B$, and $m = M$, $x = x_T$ yields

$$M = \frac{1}{\psi(x_1 - x_0)} \ell n \frac{x_1 - x_B}{x_1 - x_T} \frac{x_T - x_0}{x_B - x_0} \quad . \quad (\text{IV.3.17})$$

Since this equation is identical in form to equation (IV.3.7) for the enricher, but with M in place of N , it follows that equations similar to (IV.3.8), (IV.3.9), (IV.3.10), and (IV.3.12) are also valid for the stripper when N is replaced by M .

4. The Design of Real Cascades. The general problem is to design a cascade which will perform a specified separation job at a minimum cost. This is usually a complex and difficult task.

Consider therefore the following simpler problem. Suppose that you have at your disposal stages of types 1, 2, 3, etc. Type 1 stages have a stage

upflow L_1 , and enrichment factor ψ_1 , and an associated cost per stage C_1 . The problem is how to determine how many stages of each type to use in a cascade to do a given separation. An answer to this question may be obtained by adopting the philosophy of minimizing the cost per unit of separative work everywhere in the cascade by choosing the stage at each concentration which yields the lowest value of $C/\delta U$.

From equation (III.4.8) it may be seen that the separative work produced by a stage in a cascade is

$$\delta U_j = L_j \psi_j^2 \frac{L_{\min}}{L_j} \left(1 - \frac{L_{\min}}{L_j}\right) \quad (\text{IV.4.1})$$

One can therefore construct a curve of $C_i/\delta U_i$ vs $L_{\min}$ over the range of values of $L_{\min}$ pertinent to the separation job under consideration. This curve will have a minimum where $L_j = 2L_{\min}$. At the change-over point one has the relationship

$$\frac{C_j}{\delta U_j} = \frac{C_{j+1}}{\delta U_{j+1}} \quad (\text{IV.4.2})$$

At the change-over point between sizes 1 and 2, one can write

$$\frac{L_1 \psi_1^2 \frac{L_{\min}}{L_1} \left(1 - \frac{L_{\min}}{L_1}\right)}{C_1} = \frac{L_2 \psi_2^2 \frac{L_{\min}}{L_2} \left(1 - \frac{L_{\min}}{L_2}\right)}{C_2} \quad (\text{IV.4.3})$$

This equation can be used to solve for the value of $L_{\min}$, and hence for the concentration, at the change-over point.

Consider the special case, one of particular interest, in which the cost

of a stage is directly proportional to its size: $C_1 = CL_1$ and $C_2 = CL_2$.

Assume also that the ψ -values are the same for both equipment sizes.

Under these conditions equation (IV.4.3) reduces to

$$L_m = \frac{L_1 L_2}{L_1 + L_2} \quad (\text{IV.4.4})$$

Suppose further that the stages differ in size by the factor A: $L_1 = AL_2$.

It follows then that

$$L_m = \frac{1}{A + 1} L_1 = \frac{A}{A + 1} L_2 \quad (\text{IV.4.5})$$

Thus one changes from type 1 stages to type 2 stages when $L_2 = (1 + \frac{1}{A})L_{\min}$.

As would be expected, when A is close to 1, $L_2 \approx 2L_{\min}$ and the cascade is close to ideal (this implies many stage sizes available), and when A is large, $L_2 \approx L_{\min}$ and one suffers severe "pinching" (this implies few stage sizes available).

V. THE CONTINUOUS COUNTERCURRENT SEPARATION PROCESS

1. The Column Process. The continuous countercurrent isotope separation process discussed briefly in section II.4 will now be examined in more detail. The type of process which is usually considered in this category is a column process such as distillation in a packed column in which there is an upflowing vapor phase and a downflowing liquid phase. In order to provide good contacting between the two phases, one of them is frequently discontinuous and dispersed in the form of droplets or bubbles in the continuous phase. In the case of isotope separation, however, one frequently encounters processes in which there is a circulating gas or vapor flowing upwards in one part of the column and downwards in another. Such processes can also be considered as two-phase processes, consisting of one gas phase flowing upwards and a second gas phase flowing downwards.

Since there is no possibility of immiscible gases, each of the two gas phases will be continuous and they will be separated by an imaginary surface on which the gas velocity is zero.

2. A Simple Model (Cartesian Coordinates). Consider a continuous countercurrent isotope separation process taking place in a column defined by two flat vertical parallel plates a distance a apart. The coordinate system used is shown in the figure at the right:

r is the distance measured perpendicular to the plates, z is the distance measured parallel to the plates. Let $v(r)$ represent the local velocity of the process gas in the $+z$ -direction and let $c(r)$ represent the local value of the molar density of the process gas. Thus the molar velocity of the process gas in the $+z$ -direction is now given by cv . Only the case of non-decaying flow (that is, the case in which cv is independent of z) will be considered here.

Thus, in this model there is no net transport of process gas in the r -direction; there is however a transport of isotopic species in the r -direction. It is this isotopic transport that is the basis of the separation effect. In the absence of any horizontal (radial) bulk flow of gas, the transport of desired component in the $+r$ -direction by gas phase diffusion processes can be expressed by a current equation which has the form

$$J_{1r} = cD\left\{f(r) \cdot \xi(1 - \xi) - \frac{\partial \xi}{\partial r}\right\}, \quad (\text{V.2.1})$$

where J_{1r} is the current of desired component (component 1) in the

+r-direction in moles per unit area per unit time, D is the coefficient of self-diffusion of the process gas, $f(r)$ is a function of r which depends upon the separation process under consideration as well as on the physical properties of the components, and ξ is the mole fraction of the desired component in the process gas mixture.

3. Derivation of the Gradient Equation (after Onsager). The approach that will be used here to develop the gradient equation was first described by L. Onsager in 1939 in connection with the theory of thermal diffusion columns. For ease in following this derivation, the procedure will be described stepwise.

Step 1) Form an expression for the net transport of desired component in the +z-direction in a column operating at total reflux. There are two contributions to this transport, one due to convection the other due to ordinary molecular diffusion. The net transport is given by the sum of these two terms integrated over the column spacing.

$$T_1 = \int_0^a (cv\xi - cD \frac{\partial \xi}{\partial z}) dr . \quad (V.3.1)$$

Step 2) Define an auxiliary function $G(r)$ as follows:

$$G(r) \equiv \int_0^r cvdr . \quad (V.3.2)$$

$G(r)$ is therefore a measure of the net transport of process gas in the +z-direction in the column between the coordinates $r = 0$ and r . Therefore, at total reflux, $G(a) = 0$, and one can write the following relationships

$$G(0) = 0, \quad G(a) = 0, \quad \text{and} \quad \frac{dG}{dr} = cv . \quad (V.3.3)$$

Step 3) Making use of the properties of the auxiliary function given in

(V.3.3), integrate the first term in the transport equation (V.3.1) by parts. The transport equation can now be written

$$T_1 = \int_0^a \frac{dG}{dr} \xi dr - \int_0^a cD \frac{\partial \xi}{\partial z} dr . \quad (V.3.4)$$

After integration by parts of the first term, using the boundary values on G, the transport equation assumes the form

$$T_1 = - \int_0^a G \frac{\partial \xi}{\partial r} dr - \int_0^a cD \frac{\partial \xi}{\partial z} dr . \quad (V.3.5)$$

Step 4) Obtain an expression for $\partial \xi / \partial r$ from the equation of continuity in conjunction with the current equation (V.2.1). For the process under consideration the equation of continuity takes the form

$$\frac{\partial}{\partial r} (J_{1r}) + \frac{\partial}{\partial z} (cv\xi - cD \frac{\partial \xi}{\partial z}) = 0 . \quad (V.3.6)$$

The key step in the Onsager method is the approximation introduced at this point that the diffusion in the z-direction can be neglected in writing the equation of continuity. Thus equation (V.3.6) is approximated by

$$\frac{\partial}{\partial r} (J_{1r}) \approx - \frac{\partial}{\partial z} (cv\xi) = - cv \frac{\partial \xi}{\partial z} . \quad (V.3.7)$$

Noting that the current of desired component, J_{1r} , must vanish at the walls, $r = 0$ and $r = a$, integration of equation (V.3.7) with respect to r yields

$$J_{1r} = - \int_0^r cv \frac{\partial \xi}{\partial z} dr' . \quad (V.3.8)$$

Substitution of the expression given by the current equation (V.2.1) for J_{1r} and rearrangement of terms provides the desired result

$$\frac{\partial \xi}{\partial r} = f(r) \cdot \xi(1 - \xi) + \frac{1}{cD} \int_0^r cv \frac{\partial \xi}{\partial z} dr' . \quad (V.3.9)$$

Step 5) Substitute the above results for $\partial \xi / \partial r$ into the transport

equation (V.3.5). This substitution yields

$$T_1 = - \int_0^a Gf(r) \cdot \xi(1-\xi)dr - \int_0^a \frac{G}{cD} dr \int_0^r cv \frac{\partial \xi}{\partial z} dr' - \int_0^a cD \frac{\partial \xi}{\partial z} dr . \quad (V.3.10)$$

Step 6) Assume that the product $\xi(1 - \xi)$ and the derivative of ξ with respect to z are essentially independent of r and may be taken outside the integral signs. Letting x represent the appropriate average of ξ , the above equation can be rewritten in the form

$$T_1 = - \int_0^a Gf(r)dr \cdot \overline{\xi(1-\xi)} - \frac{d\bar{\xi}}{dz} \left\{ \int_0^a \frac{G^2}{cD} dr + \int_0^a cDdr \right\} ,$$

$$T_1 = - \int_0^a Gf(r)dr \cdot x(1-x) - \frac{dx}{dz} \left\{ \int_0^a \frac{G^2}{cD} dr + \int_0^a cDdr \right\} . \quad (V.3.11)$$

Step 7) Now consider the case in which the column is not at total reflux, but is operated with feed and product and waste withdrawal. The net transport of desired component will be increased by the amount Px in the enriching section and by the amount $-Wx$ in the stripping section. The quantity T_1 which at steady state is equal to zero for total reflux operation must equal $P y_p$ in the enriching section for the case of product withdrawal and must equal $-W x_w$ in the stripping section for the case of waste withdrawal. The gradient equation for the enriching section of a countercurrent isotope separation process can therefore be written in the form

$$\left\{ \int_0^a \frac{G^2}{cD} dr + \int_0^a cDdr \right\} \frac{dx}{dz} = - \int_0^a Gf(r)dr \cdot x(1-x) - P(y_p - x) . \quad (V.3.12)$$

Step 8) Equation (V.3.12) is a perfectly valid form of the gradient equation and indeed this form is often used for computation. However, because of our familiarity with stage-wise processes, it is convenient for us to divide each term in the above equation by the column upflow L

and compare the result with the approximate design differential gradient of section III. The column upflow is given by

$$L = \int_0^a cvdr$$

with v set equal to zero at all r -values at which v is negative.

The gradient equation now takes the form

$$\left\{ \frac{\int_0^a \frac{G^2}{cD} dr}{L} + \frac{\int_0^a cDdr}{L} \right\} \frac{dx}{dz} = \frac{-\int_0^a Gf(r)dr}{L} x(1-x) - \frac{P}{L} (y_P - x) \quad (V.3.13)$$

It can be seen by inspection that the term in brackets in equation (V.3.13) plays the role of a stage length and the first term on the right can be called the stage enrichment factor. The gradient equation can be expressed

$$(S_C + S_D) \frac{dx}{dz} = \frac{dx}{dn} = \psi x(1-x) - \frac{P}{L} (y_P - x) \quad (V.3.14)$$

where

$$S_C = \frac{\int_0^a \frac{G^2}{cD} dr}{L}$$

is the convective contribution to the stage length,

$$S_D = \frac{\int_0^a cDdr}{L}$$

is the contribution of back-diffusion to the stage length,

and

$$\psi = - \frac{\int_0^a Gf(r)dr}{L}$$

is the stage enrichment factor.

Step 8) The final step is a modification in the form of the gradient equation which was introduced by K. Cohen. Examination of the expression for the stage length yields the fact that S_C is directly proportional to L and S_D is inversely proportional to L . Thus one can write

$$S = S_C + S_D = s_C L + \frac{s_D}{L} \quad (V.3.15)$$

There is a value of L which minimizes the stage length; Cohen designates

this value by L_0 . One perceives from (V.3.15) that

$$L_0 = \sqrt{s_d/s_c} \quad (\text{V.3.16})$$

and that the minimum value of the stage length, designated by S_0 , obtained when $L = L_0$, is

$$S_0 = 2\sqrt{s_c s_d} \quad (\text{V.3.17})$$

If one now denotes the ratio of the actual value of L to L_0 by m , the gradient equation can be written in the form

$$\left(m + \frac{1}{m}\right) \frac{S_0}{2} \frac{dx}{dz} = \psi x(1-x) - \frac{P}{mL_0} (y_P - x) \quad (\text{V.3.18})$$

4. Sample Calculation. Assume that the current equation is given by

$$J_{1r} = cD\left\{k\xi(1-\xi) - \frac{\partial \xi}{\partial r}\right\}, \quad \text{that is } f(r) = k, \text{ a constant.}$$

Assume that the velocity profile consists of a δ -function at each wall.

The separation parameters can then be evaluated as follows

$$G = -L$$

$$\psi = ka$$

$$S_c = \frac{L}{cD} a; \quad s_c = \frac{a}{cD}$$

$$S_d = \frac{cDa}{L}; \quad s_d = cDa$$

$$L_0 = cD$$

$$S_0 = 2a$$

$$\therefore \left(m + \frac{1}{m}\right) a \frac{dx}{dz} = ka x(1-x) - \frac{P}{L} (y_P - x) \quad .$$

Assume that the velocity profile consists of a step function as shown below.

The column upflow is given by

$$L = B \frac{a}{2} \quad .$$

The G-function is given by

$$G = \int_0^r c v dr = -\frac{2L}{a} r \quad (0 \leq r \leq \frac{a}{2})$$

$$G = -L + \int_{a/2}^r c v dr = -L + \frac{2L}{a} (r - \frac{a}{2})$$

$$= -2L + \frac{2L}{a} r = 2L(\frac{r}{a} - 1) \quad (\frac{a}{2} \leq r \leq a) .$$

The separation parameters are obtained as follows

$$\begin{aligned} \psi &= -\frac{k}{L} \int_0^a G dr = -\frac{k}{L} \left\{ \int_0^{a/2} (-2L \frac{r}{a}) dr + \int_{a/2}^a 2L(\frac{r}{a} - 1) dr \right\} \\ &= -k \left\{ \left(-\frac{r^2}{a} \right)_0^{a/2} + 2 \left(\frac{r^2}{2a} - r \right)_{a/2}^a \right\} = -k \left\{ -\frac{a}{4} + 2 \left(\frac{a}{2} - a - \frac{a}{8} + \frac{a}{2} \right) \right\} \\ &= -ka \left(-\frac{1}{4} + 1 - 2 - \frac{1}{4} + 1 \right) = \frac{ka}{2} . \end{aligned}$$

$$\begin{aligned} s_c &= \frac{1}{L^2 c D} \int_0^a G^2 dr = \frac{1}{L^2 c D} \left\{ \int_0^{a/2} \frac{4L^2 r^2}{a^2} dr + \int_{a/2}^a 4L^2 \left(\frac{r^2}{a^2} - 2\frac{r}{a} + 1 \right) dr \right\} \\ &= \frac{1}{c D} \left\{ \left(\frac{4r^3}{3a^2} \right)_0^{a/2} + 4 \left(\frac{r^3}{3a^2} - \frac{r^2}{a} + r \right)_{a/2}^a \right\} \\ &= \frac{a}{c D} \left\{ \frac{4}{24} + 4 \left(\frac{1}{3} - 1 + 1 \right) - 4 \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{2} \right) \right\} \\ &= \frac{a}{c D} \left\{ \frac{4}{3} + 1 - 2 \right\} = \frac{1}{3} \frac{a}{c D} \end{aligned}$$

Illustrative Problem No. 1.

Design an ideal cascade composed of simple, series-connected stages to do the following separation job.

The feed is to be normal uranium containing 0.711 wt. % U-235. The waste stream is to be stripped to a U-235 content of 0.25 wt. %. Product material containing 4.0 wt. % U-235 is desired, and the cascade is to have a separative capacity of 8.75 million SWU/yr (kg U separative work/yr).

Assume that the stage enrichment factor, ψ , is equal to 0.00429 for all stages. Calculate P , F , W , N , M , L_F , $\Sigma L(\text{enr})$, $\Sigma L(\text{str})$, and plot L vs stage number for the cascade.

Illustration Problem No. 1 (continued)

We can construct the following table for an ideal cascade producing 1.0 kg U/yr containing 4.0 wt. % U-235.

	<u>Product</u>	<u>Feed</u>	<u>Waste</u>	
conc., x (wt. %)	0.04	0.00711	0.0025	
$\nu(x)$	2.923809	4.868883	5.959017	
rate	1.0	8.134490	7.134490	$\{F = P \frac{\eta_P - x_W}{x_F - x_W}\}$
rate $\times \nu(x)$	2.923809	39.605880	42.514547	

The separative capacity of this cascade is $U = P\nu(\eta_P) + W\nu(x_W) - F\nu(x_F)$:
 $U = 5.832471$ SWU/yr. Therefore, multiplying the rates in the above table by $8.75 \times 10^6 / 5.832471 = 1.500222 \times 10^6$ gives the rates corresponding to a 8.75 million SWU/yr cascade.

$$P = 1.500222 \times 10^6 \text{ kg U/yr}$$

$$F = 12.203539 \times 10^6 \text{ kg U/yr}$$

$$W = 10.703317 \times 10^6 \text{ kg U/yr}$$

$$N = \frac{2}{\psi} \ln \frac{R_P}{R_F} = 821.0$$

$$M = \frac{2}{\psi} \ln \frac{R_F}{R_W} = 489.4$$

$$L_F = \frac{2P(\eta_P - x_F)}{\psi x_F(1 - x_F)} = 3.258527 \times 10^9 \text{ kg U/yr}$$

$$\begin{aligned} \Sigma L(\text{ann}) &= \frac{4P}{\psi^2} \{ \nu(\eta_P) - \nu(x_F) - (\eta_P - x_F) \nu'(x_F) \} \\ &= 9.692452 \times 10^{11} \text{ kg U/yr} \end{aligned}$$

$$\begin{aligned} \Sigma L(\text{enr}) &= \frac{4W}{\psi^2} \{ \nu(x_W) - \nu(x_F) - (x_W - x_F) \nu'(x_F) \} \\ &= 9.325049 \times 10^{11} \text{ kg U/yr} \end{aligned}$$

Illustrative Problem No. 2

Assume that the uranium inventory of the cascade described in I.P. No. 1 is 0.1 kg per SWU per year. Estimate the equilibrium time of that ideal cascade.

Equation (III.5.25) can be written in the form

$$\text{Eq. T. (max)} = \frac{h}{L} \frac{4}{\psi^2} \left\{ \frac{(R_P/R_F) + 1}{(R_P/R_F - 1)} \ln \frac{R_P}{R_F} - 2 \right\} ,$$

$$\text{where } R_P = \frac{y_P}{1-y_P} \text{ and } R_F = \frac{x_F}{1-x_F} .$$

From the given information we obtain

$$\frac{h}{L} \frac{4}{\psi^2} = 0.1 \text{ yr} ,$$

or that with $\psi = 0.00429$, the turnover time of a stage ($\frac{h}{2L}$) is equal to $7\frac{1}{4}$ seconds. It follows that the estimated equilibrium time of an ideal cascade producing 4.0% U-235 from normal feed is 18.0 days.

Illustrative Problem No 2 (continued)

From the foregoing information one can construct the following instructive table showing the estimated equilibrium time of an ideal cascade as a function of its product concentration, under the assumption of normal feed and $4h/(L\psi^2)$ equal to 36.5 days.

N_F	Eg.T. (days)	N_F	Eg.T. (days)
.01	0.7	.10	40.9
.02	6.6	.20	64.3
.03	12.6	.30	81.4
.04	18.0	.40	96.1
.05	22.7	.50	109.9
.06	27.0	.60	123.9
.07	30.9	.70	139.5
.08	34.5	.80	158.7
.09	37.8	.90	187.9
		.95	215.0

Illustrative Problem No. 3

Suppose that cascades were designed to do the separation job described in I.P. No. 1 (a) with L for each stage equal to 0.8 times $L(\text{ideal})$ and (b) with L for each stage equal to 1.25 times $L(\text{ideal})$. For each of these two cases, indicate how the number of stages, the total stage upflow (ΣL), and the estimated equilibrium time would change with respect to the ideal cascade.

Let $L = Q \cdot L(\text{ideal})$. For the enriching section this relationship can be written

$$L = Q \frac{2F(\eta_p - x)}{\psi x(1-x)}.$$

Substituting this expression for L into the gradient equation for the enriching section yields

$$\frac{dx}{dn} = \psi x(1-x) - \frac{F}{L}(\eta_p - x) = (1 - \frac{1}{2Q})\psi x(1-x).$$

Integration of this equation gives the result

$$N = \frac{1}{2 - \frac{1}{Q}} \frac{2}{\psi} \ln \frac{R_T}{R_B} = \frac{1}{2 - (1/Q)} N(\text{ideal}).$$

Illustrative Problem No 3 (continued)

The total stage upflow in the enriching section is given by

$$\sum L = \int_0^N L dn = \int_{x_F}^{y_P} L \frac{dn}{dx} dx = \frac{Q}{2-(1/Q)} \int_{x_F}^{y_P} \frac{4P}{\psi^2} \frac{y_P - x}{x^2(1-x)^2} dx$$

$$\therefore \sum L = \frac{Q}{2-(1/Q)} \sum L(\text{ideal})$$

In the same manner, one can determine that for cascades of this Type ($L = Q \cdot L(\text{ideal})$), assuming that the turnover time, $t/2L$, is unchanged, the estimated equilibrium time is directly proportional to $\sum L$. Since one obtains the same results for the ratios $N/N(\text{ideal})$ and $\sum L/\sum L(\text{ideal})$ for the stripping section as obtained above for the enriching section, one can construct the following table and graph.

Q	$N/N(\text{ideal})$	$\sum L/\sum L(\text{ideal})$
0.8	1.3333	1.0667
1.25	0.8333	1.0417

Illustrative Problem No 4

Consider a cascade composed of stages, all of which are the same size, to do the separation job described in I.P. No. 1. Find the value of L for the stage which minimizes ΣL for the cascade. Compare ΣL , the number of stages required, and the estimated equilibrium time for the minimum ΣL case with the corresponding values for the ideal cascade.

For a square enriching section, the following equations are valid when ψ is small with respect to unity

$$N(\text{enr}) = \left(1 - \frac{P}{L}\right) \frac{1}{\psi(X_1 - X_0)} \ln \frac{(X_1 - X_F)(y_P - X_0)}{(X_1 - y_P)(X_F - X_0)}$$

$$X_1, X_0 = \frac{L\psi + P \pm \sqrt{(L\psi + P)^2 - 4L\psi F y_P}}{2L\psi}$$

$$\bar{X}(\text{enr}) = X_0 + \left(1 - \frac{P}{L}\right) \frac{1}{N\psi} \ln \frac{X_1 - X_F}{X_1 - y_P}$$

and for the stripping section

$$N(\text{str}) = \left(1 + \frac{W}{L}\right) \frac{1}{\psi(X_1 - X_0)} \ln \frac{(X_1 - X_W)(X_F - X_0)}{(X_1 - X_F)(X_W - X_0)}$$

Illustrative Problem No. 4 (continued)

$$X_1, X_0 = \frac{L\psi - W \pm \sqrt{(L\psi - W)^2 + 4L\psi W X_w}}{2L\psi}$$

The value of L which minimizes ΣL for a square cascade is found by "trial/err" (that is, numerically, using the above equations). We know that the optimum value of L lies between $L_F(\text{ideal})/2$ and $L_F(\text{ideal})$. We therefore let $L = Q \cdot L_F(\text{ideal})$ and construct the following table.

Q	$L \times 10^{-6}$	N	M	$\Sigma L \times 10^{-12}$	$\frac{\Sigma L(\text{ideal})}{\Sigma L(\text{square})}$	$\bar{\alpha}(\text{avr})$
0.55	1792	818	786	2.874	.6617	0.016343
0.60	1955	708	612	2.580	.7372	
0.65	2118	650	528	2.496	.7620	
0.6825	2224	624	493	2.485	.7653	
0.70	2281	613	477	2.487	.7645	
0.75	2444	587	443	2.516	.7559	
0.80	2607	567	418	2.566	.7412	
0.85	2770	551	398	2.629	.7234	
0.90	2933	538	383	2.701	.7040	
0.95	3096	528	370	2.780	.6840	
1.00	3259	519	360	2.864	.6641	

Illustrative Problem No. 4 (continued)

The estimated equilibrium time in this case is given by

$$\text{Eq. T.} = \frac{\text{Net Enriched Inventory}}{P(\eta_P - x_F)} = \frac{N h (\bar{x} - x_F)}{P(\eta_P - x_F)}$$

$$\text{Eq. T.} = \frac{h}{L} \frac{N L (\bar{x} - x_F)}{P(\eta_P - x_F)} = 259,700 \frac{h}{L}$$

Taking h/L equal to 14.5 seconds, as in I.P. No. 2, one obtains an estimated equilibrium time of 43.6 days, or 2.4 times greater than that for the ideal cascade.

Illustrative Problem No 5

Consider a cascade composed of stages of three different sizes, L_1 , L_2 , and L_3 , to do the separation job described in I.P. No. 1. Let L_1 be the largest stage size, and assume that $L_2/L_1 = 0.5432$ and that $L_3/L_2 = 0.5682$. Assume further that the total cost of a stage is directly proportional to the L of the stage. Design a reasonably optimum cascade using the three stage sizes as needed, and compare ΣL , the total number of stages, and the estimated equilibrium time with the corresponding values for the ideal cascade.

The cost per unit separative work produced by a stage as a function of its position in the cascade is given by

$$\frac{\$}{\text{SWU}} = \frac{C}{\psi^2 \frac{L(\text{ideal})}{2L} \left\{ 1 - \frac{L(\text{ideal})}{2L} \right\}}$$

We take as our reasonable design criterion that adjacent stages of two different sizes have the same unit cost of separative work. This leads to the following requirement at the interface between size 1 and size 2:

$$\frac{L(\text{ideal})}{2} = L_1 \frac{L_2/L_1}{1 + (L_2/L_1)}$$

Illustrative Problem No 5 (continued)

Thus, if we knew L_1 , we could solve for the concentration at the interface between size 1 and size 2. Let us adopt as a second reasonable design criterion that the unit cost of reparative work from the feed stage be equal to the unit cost of reparative work from the terminal stages of the size 1 equipment. This condition leads to

$$\frac{L_F(\text{ideal})}{2L_1} \left\{ 1 - \frac{L_F(\text{ideal})}{2L_1} \right\} = \frac{L_2/L_1}{1+(L_2/L_1)} \left\{ 1 - \frac{L_2/L_1}{1+(L_2/L_1)} \right\} = 0.2281$$

$$\therefore L_1 = 0.77 L_F(\text{ideal})$$

We can now construct the following cascade description:

Sect	Conc	$L \times 10^{-6}$	N	Total number of stages is 1214 vs. 1310 for the ideal cascade
3 Enr.	0.04000	774	310	
2 Enr.	0.01675	1363	184	The Total ΣL is 2.029×10^{12} vs 1.9017×10^{12} for the ideal cascade for a reparative efficiency of 93.7%
1 Enr.	0.00711	2509	226	
1 Str.	0.00386	2509	305	
2 Str.	0.00311	1363	106	
3 Str.	0.00250	774	83	