PATHFINDER ATOMIC POWER PLANT
STRESS AND DEFLECTION OF BOILER
FUEL ELEMENT BOXES

Submitted to
U. S. ATOMIC ENERGY COMMISSION
NORTHERN STATES POWER COMPANY
and
CENTRAL UTILITIES ATOMIC POWER ASSOCIATES
by

ALLIS-CHALMERS MANUFACTURING COMPANY
ATOMIC ENERGY DIVISION
Milwaukee 1, Wisconsin

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor Allis-Chalmers Manufacturing Company, nor any person acting on behalf of the Commission or Allis-Chalmers Manufacturing Company :

A. Makes any warranty or representation to others, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities to others with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, 'person acting on behalf of the Commission or Allis-Chalmers Manufacturing Company' includes any employee or contractor of the Commission, or Allis-Chalmers Manufacturing Company or employee of such contractor, to the extent that such employee or contractor of the Commission, or Allis-Chalmers Manufacturing Company or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission or Allis-Chalmers Manufacturing Company or his employment with such contractor.

**PATHFINDER ATOMIC POWER PLANT
STRESS AND DEFLECTION OF BOILER
FUEL ELEMENT BOXES**

by
Lloyd E. Vlies

Submitted to

**U. S. ATOMIC ENERGY COMMISSION
NORTHERN STATES POWER COMPANY
and
CENTRAL UTILITIES ATOMIC POWER ASSOCIATES**

by
ALLIS-CHALMERS MANUFACTURING COMPANY

Under
Agreement dated 2nd Day of May 1957, as Amended
between

Allis-Chalmers Mfg. Co. & Northern States Power Co.
under

AEC Contract No. AT(11-1)-589

April 31, 1962

Classification - UNCLASSIFIED

Reviewed by *R. G. Michel*

Authorized Classifying Official

Approved: *C. B. Graham*

C. B. Graham
Manager
Nuclear Power Dept.-Greendale

**ALLIS-CHALMERS MANUFACTURING COMPANY
ATOMIC ENERGY DIVISION
MILWAUKEE 1, WISCONSIN**

Approved: *Hibbert Hill*

Hibbert Hill
Chief Engineer

**NORTHERN STATES POWER COMPANY
15 SOUTH FIFTH STREET
MINNEAPOLIS 2, MINNESOTA**

ACNP-62010

PATHFINDER ATOMIC POWER PLANT
STRESS AND DEFLECTION OF BOILER
FUEL ELEMENT BOXES

DISTRIBUTION

USAEC, Chicago Operations Office-- 9800 South Cass Avenue, Argonne, Illinois	8
USAEC, Division of Reactor Development-- Washington 25, D. C.	8
USAEC, OTIE-- Oak Ridge, Tennessee	20
OFFSET MASTERS PLUS	
Northern States Power Company and CUAPA	35
Allis-Chalmers Manufacturing Company	39
TOTAL	<hr/> 110

FOREWORD

As part of the research and development program for the Pathfinder Atomic Power Plant, calculational methods were developed for determining the stress and deflection in the boiler fuel element boxes. The derivation of equations and a test program to check the accuracy of the equations is described in this report.

The Pathfinder Atomic Power Plant is a 66-mwe plant that will be owned and operated by Northern States Power Company of Minneapolis. The plant will be built near Sioux Falls, South Dakota and is scheduled to become critical in mid-1962.

Contributing toward the research and development program are the U. S. Atomic Energy Commission and Central Utilities Atomic Power Associates, a group of ten midwestern private utility companies. CUAPA members include the following: Central Electric and Gas Company, Interstate Power Company, Iowa Power and Light Company, Iowa Southern Utilities Company, Madison Gas and Electric Company, Northern States Power Company, Northwestern Public Service Company, Otter-Tail Power Company, St. Joseph Light and Power Company, and Wisconsin Public Service Corporation.

Allis-Chalmers Manufacturing Company is prime contractor for design and construction of the plant. The plant will incorporate the Controlled Recirculation Boiling Reactor with Integral Nuclear Superheater.

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

CONTENTS

Foreword	5
ABSTRACT	9
1. Introduction	11
2. Object	11
3. Conclusion	11
4. Description of Fuel Element Boxes	12
5. Theory	
5.1 Equations for stress in single-element box	12
5.2 Deflection equation for single-element box	15
5.3 Equations for stress and deflection of four-element box	16
6. Experimental Analysis of the Fuel Element boxes	18
7. Discussion	20
References	22
Appendix A - Nomenclature	
Appendix B - Derivation of Equations	

TABLES

- I Measured strains in single-element box.
- II Experimental and theoretical stress and deflection values for single-element box.
- III Measured strains in the four-element box.
- IV Measured deflections of four-element box.
- V Experimental and theoretical stress in four-element box.

FIGURES

- 1. Boiler fuel element boxes
- 2. One quadrant of a single-element box.
- 3. Section of four-element box.
- 4. Test of single-element box
- 5. Test of four-element box.

FIGURES (Cont'd.)

6. Stress in single-element boiler fuel element box as a function of internal pressure.
7. Deflection of single-element box as a function of internal pressure.
8. Stress in sides of four-element box as a function of internal pressure. Pressure in the control rod channel is equal to the pressure in the compartments.
9. Maximum deflection of box sides as a function of maximum stress and strain at inside corner.
10. Stress in corners of four-element box as a function of compartment internal pressure with reduced pressure in control rod channels.
11. Stress in corners of four-element box as a function of compartment internal pressure with no pressure in the control rod channel.
12. Stress in corner of four-element box as a function of compartment internal pressure and control rod channel pressure.
13. Curved beam factor.
14. Determination of modulus of elasticity for plexiglas used in model of four-element box.

ABSTRACT

Equations were derived by strain-energy methods for analysing the stress and deflection in the boiler fuel element boxes used in the Pathfinder nuclear reactor. These boxes are of two types, both constructed of Zircaloy 11. One box is approximately 5 in. sq. The second is approximately $10\frac{1}{2}$ in. sq. divided internally into four compartments (5 in. sq.) by a cruciform control-rod guide structure. Stress and deflection in the box walls is caused by pressure differences across the walls. The equations were verified by testing an aluminum model of the single-element box and a Plexiglas model of the four-element box. There was less than 10 per cent difference between theoretical and measured values for the maximum stress and deflection, and therefore, use of the equations for designing the Pathfinder fuel element boxes is justified.

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

1. INTRODUCTION

The boiler fuel elements for the Pathfinder nuclear reactor are composed of fuel-rod bundles enclosed by square Zircaloy-II boxes. In order to reduce cost of materials and to minimize the amount of absorbing material in the core, it is necessary to design the boxes with a minimum wall thickness.

The only significant load placed on the box walls is the pressure drop from the inside to the outside of the box due to the pressure drop through the reactor core. Therefore, in designing for a minimum box-wall thickness, an accurate calculational method for determining the stresses and deflections in the box walls due to the pressure difference is necessary.

2. OBJECT

A program was initiated to develop calculational methods for determining stress and deflection in Pathfinder boiler fuel element boxes and to determine accuracy of these methods by a suitable test program.

3. CONCLUSION

The calculated values for stress and deflection of the fuel element boxes agree very well with the measured values. For the single-element box with internal pressure up to 20 psi, there is no difference between the theoretical and measured maximum stress values, and there is a maximum difference of 4.2 percent between the theoretical and measured deflection. There is a 6 percent difference between the theoretical and measured maximum stress in the four element box, and there is a 4 percent difference between the

theoretical and measured maximum deflection of the box side. Using the theoretical stress and deflection equations, it was determined that the minimum wall thickness of the Zircaloy-II fuel element boxes for the Pathfinder reactor should be 0.150 in.

4. DESCRIPTION OF FUEL ELEMENT BOXES

Two types of boxes (Figure 1) are used in the Pathfinder reactor. One is a single-element box approximately 5 in. sq.; the other is a four-element box where the four compartments are separated by a 17/32-in wide cruciform control rod channel.

During normal operation with the control rod withdrawn, the maximum pressure drop across the outside box walls is about 13 psi. The maximum pressure drop across the control-rod-channel walls is about 3.5 psi. When the control rods are scrammed, the pressure in the channel may be equal or greater than the pressure in the fuel compartments.

5. THEORY

Strain energy methods were used to derive equations for analysing the stress and deflection of the box walls caused by these pressure differences. The accuracy of the equations was checked by testing model boxes.

5.1 Equations for Stress in the Single-Element Box. The bending moment at any section of the box wall must be determined in order to get the stress distribution. A unit length of the box is considered, and, because of symmetry, only one quadrant is analysed. This quadrant is assumed to be fixed at one end, as shown in Figure 2. Nomenclature is defined in Appendix A.

A summation of all moments acting on this quadrant gives the moment equation for the flat side of the box as:

$$M_x = M_o - \frac{px^2}{2} \quad (1)$$

and the moment equation for the curved portion of the corner is:

$$M_\phi = M_o - \frac{pL^2}{2} - pLR (\sin \phi + \cos \phi - 1) \quad (2)$$

The deflections are assumed to be small. Therefore, the change in the moment due to deflection can be neglected. Details of all derivations are given in Appendix B.

The moment, M_o , in eq. (1), (2) cannot be determined from the conditions of static equilibrium. This moment is determined by strain energy methods with the condition that there is no rotation of the section at $x = 0$. Castigliano's theorem for determining rotation at any section states that,

$$\theta_o = \frac{\partial U}{\partial M_o} \quad (3)$$

where, θ_o is the angle of rotation of the section, M_o is the moment at the section, and U is the total strain energy of the member. For thin straight members, the total strain energy due to bending is,

$$U = \int \frac{M^2}{2EI} dx \quad (4)$$

where, M is the bending moment at any section. Eq. (4) also holds for curved members if $(R/c) > 4$, where, R is the radius of curvature, and c is 1/2 the wall thickness (Ref. 1). For a curved member, dx is replaced by $R d\phi$.

Since the slope at $x = 0$ remains zero, $\theta_0 = 0$, and solving for M_0 gives:

$$M_0 = \frac{2p}{(4L - \pi R)} \left[\frac{L^3}{3} + \frac{\pi L^2 R}{4} + L^2 R \frac{4 - \pi}{2} \right] \quad (5)$$

With eq. (5), (1), (2), the moment at any section of the box wall can be determined.

The stress in a transverse section of the box is a combination of the bending stress and the direct stress. For the flat side of the box from $x = 0$ to $x = L$, the stress is:

$$\sigma_x = \frac{P_x}{A_x} + \frac{M_x y}{I} \quad (6)$$

and at the corner, from $\varphi = 0$ to $\varphi = \pi/2$, the stress is,

$$\sigma_\varphi = \frac{P_n}{A_x} + \frac{M_\varphi}{A_x R} \left[1 - \frac{1}{Z} \left(\frac{y}{R+y} \right) \right] \quad (\text{Ref. 1}) \quad (7)$$

where, $Z = 1/3 (c/R)^2 + 1/5 (c/R)^4 + 1/7 (c/R)^6 + \dots$,

$$P_n = pL (\sin \varphi + \cos \varphi) + pR_i,$$

$$A_x = \text{area of the section } (A_x = t), \text{ and}$$

$$y = \text{distance from the centroidal axis of the section.}$$

Values for y are positive when measured toward the outside of the box wall, and negative when measured toward the inside. Thus values for tensile stress are positive.

The maximum stresses will occur at the inside and outside surfaces of the box wall, where, $y = \pm c$. Then, since $c = t/2$, $A_x = t$, and $I = t^3/12$, the following equations can be written for the transverse stress at the wall surfaces:

$$\sigma_x = \frac{p(L+R_i)}{t} \pm \frac{6M_x}{t^2} \quad (6a)$$

$$\sigma_\varphi = \frac{pL(\sin \varphi + \cos \varphi) + pR_i}{t} \pm K \frac{6M}{t^2} \quad (7a)$$

K is the correction factor for curved beams, and is plotted in Figure 13.

$$K = \frac{M_\varphi}{A_x R} \left[1 + \frac{1}{Z} \frac{c}{R+c} \frac{1}{M_\varphi c} \right]$$

In eqs. (6a), (7a), the positive sign is used for the stress at the outside surface, and the negative sign is used at the inside surface. The maximum stress occurs at the inside corner at $\varphi = 45^\circ$.

The bending of the transverse section of the box also causes stress in the longitudinal direction. The sides of the box are assumed to act as long flat plates being bent to a cylindrical surface by a uniform load. If the box has closed ends, the direct axial stress must be added to that due to bending. The following equation for the longitudinal stress at some distance from the ends of a closed box is derived in the appendix:

$$\sigma_z = \frac{P_z}{A_z} + \nu \left(\sigma_x - \frac{P_x}{A_x} \right) \quad (8)$$

P_z is the force exerted by the internal pressure, p , acting on the cross-sectional area inside the box; and A_z is the cross-sectional area of the box wall.

5.2 Deflection Equation for the Single-Element Box. The equation for the deflection at the center of the box side relative to the corner is derived by applying a dummy load, Q , at this point and writing the strain energy

equation for the section:

$$\delta Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial Q} dx + \int_0^{\pi/4} \frac{M}{EI} \frac{\partial M}{\partial Q} R d\phi \quad (9)$$

Solving this equation gives:

$$\delta Q = \frac{1}{EI} \left[\frac{pL^4}{8} + \frac{pL^3 R \pi}{8} + pL^2 R^2 (0.3611) + 0.1 pLR^3 \right. \\ \left. - M_0 \left(\frac{L^2}{2} + \frac{RL \pi}{4} + (0.293) R^2 \right) \right] \quad (10)$$

5.3 Equations for the Stress and Deflection of the Four-Element Box. A

free-body diagram of 1/8 of the box cross-section is shown in Figure 3-C.

The quarter section of the box is symmetric about the 45° line; therefore, 1/8 of the cross-section can be used to derive the moment equation. The curvature at the corners is neglected in the derivation of the equations for bending moments. It will be accounted for later.

There are three unknowns that must be determined, i.e., M_A , M_I , and P_{N_I} . When these are known, the other forces and moments can be determined. Using Castigliano's theorem for rotation and deflections:

$$\text{Rotation at } M_I, \theta_I = \frac{\partial U}{\partial M_I} = \int \frac{M}{EI} \frac{\partial M}{\partial M_I} dx = 0 \quad (11)$$

$$\text{Rotation at } M_A, \theta_A = \frac{\partial U}{\partial M_A} = \int \frac{M}{EI} \frac{\partial M}{\partial M_A} dx = 0 \quad (12)$$

Deflection at P_{N_I} in the direction of the force P_{N_I} ,

$$\delta_{P_{N_I}} = \frac{\partial U}{\partial P_{N_I}} = \int \frac{M}{EI} \frac{\partial M}{\partial P_{N_I}} dx = 0 \quad (13)$$

Solution of the three strain energy equations yields three equations containing the three unknowns which can then be solved for. Details of the solution are given in the Appendix. Solving for all the forces and moments with corrections for the curvature at the corners gives:

$$M_A = \frac{L^2}{96} (P_2 - P_1) - p^2 \left(\frac{1}{64} + \frac{1}{48L} \right) \quad (14)$$

$$M_1 = P_2 L^2 \left(\frac{41}{384} \right) - P_1 L^2 \left(\frac{9}{384} \right) + p^2 \left(\frac{7}{256} - \frac{1}{192L} \right) - (0.207) P_{N_1} R \quad (15)$$

$$P_{N_1} = \frac{3}{4\sqrt{2} L} \left[\frac{25}{48} P_2 L^2 + \frac{39}{48} P_1 L^2 + p^2 \left(\frac{7}{32} + \frac{1}{48L} \right) \right] \quad (16)$$

$$P_{2y} = \frac{\sqrt{2}}{2} P_{N_1} \quad (17)$$

$$P_{NB} = P_2 L \frac{2}{\sqrt{2}} + \frac{pL}{\sqrt{2}} - P_{N_1} \quad (18)$$

$$M_2 = M_1 + \frac{P_1 L^2}{2} - P_{N_1} \frac{\sqrt{2} L}{2} + 0.207 P_{N_1} R \quad (19)$$

$$M_B = M_A + M_2 + \frac{pL^2}{8} + \frac{P_2 L^2}{2} - \frac{\sqrt{2}}{2} P_{N_1} L - (0.207) P_{NB} R \quad (20)$$

The above equations can be used for any ratio of p -to- P_2 . For particular cases, the general equations can be simplified.

After the maximum bending moments and normal forces are determined, the stress at the inside corners, which is the maximum stress, can be determined

from the equation:

$$\sigma = \frac{P_N}{t} + K_i \frac{Mc}{I} \quad (21)$$

Since the bending moments and normal forces are per unit length of the box, $A = t$ and $I = t^3/12$. Eq. (21) then reduces to:

$$\sigma = \frac{P_N}{t} + K_i \frac{(6M)}{t^2} \quad (21a)$$

K_i is the curved beam factor to account for the curvature at the inside corner (Figure 13).

The equation for the deflection of the center of the box side is derived by applying a dummy load, Q , at this point and writing the strain energy equation for the section shown in Figure 3-d:

$$\delta Q = \frac{\partial U}{\partial Q} = \int \frac{M}{EI} \frac{\partial M}{\partial Q} dx \quad (22)$$

Solving this equation gives:

$$\begin{aligned} \delta Q = \frac{1}{EI} \left[\frac{M_A}{2} \left(\frac{2}{4} + L + L^2 \right) + \frac{P}{16} \left(\frac{2}{8} + L + L^2 \right) \right. \\ \left. + \frac{M_2}{2} (L + L^2) - P_{2y} \left(\frac{L^2}{4} - \frac{L^3}{3} \right) + \frac{P_2}{4} \left(\frac{L^2}{3} - \frac{L^4}{2} \right) \right] \quad (23) \end{aligned}$$

6. EXPERIMENTAL ANALYSIS OF THE FUEL ELEMENT BOXES

An aluminum single-element box was tested under internal pressure to determine stresses and deflections in the box walls, so that a comparison could be made with theoretical value. The box was 36 in. long and 4-15/16 in. sq.

(inside) with a 5/32-in radius at the corners and 1/8-in thick walls (Figure 4).

Strain gages were used to measure the biaxial strains at the inside and outside corners of the box and at the center of the box side. These are the points where the theoretical equations indicate that the stress is highest. Foil gages, 1/32-in gage length, were used at the corners, and 1/4-in wire gages were used on the flat sides. A dial indicator was used to measure the deflection of the box side. The strains and deflection were recorded as the pressure was increased in small steps from zero to 50 psi.

The maximum stress and deflection of the four-element box was determined by testing a Plexiglas model of the box (Figure 5). The box was 48 in. long with the four square fuel compartments each 5.00 ± 0.01 in. sq. inside with a center cruciform control rod channel $17/32 \pm 1/64$ in. wide between the compartments. The wall thickness was 0.150 ± 0.005 in. and the inside radius at the corners was $11/32 \pm 1/64$ in.

Eleven strain gages, 1/32 in. gage length, were used to measure the biaxial strains at the corners, and six dial indicators were used to measure the deflection of the box sides. The strains and deflections were recorded as the internal pressure was increased in steps from 0 to 35 in. water. Readings were taken with several different pressure ratios between the boxes and the center channel.

The experimental stress values were calculated using the equation:

$$\sigma_1 = \frac{E}{1 - \nu} (\epsilon_1 + \nu \epsilon_2) \quad (24)$$

Where, σ_1 is a principle stress, E is Young's modulus, ν is Poisson's ratio, ϵ_1 is the strain in the direction of the desired stress, and ϵ_2 is the strain perpendicular to ϵ_1 .

For the 61 S - T4 aluminum single-element box, $E = 10 \times 10^6$ psi and $\nu = 0.33$ (Ref. 4). Published values of ν and E for Plexiglas varied considerably; therefore, a sample of the material used for the Plexiglas four-element box was tested to determine these properties. This was done by applying a uniaxial load to a strip of the material (Plexiglas II UVA) and measuring the strain in the axial and transverse directions. The strip was 4 in. long by 0.740 in. wide by 0.1535 in. thick. Figure 14 shows the results of this test. For a uniaxial load acting along the y axis

$$\sigma_y = P/A = E \epsilon_y$$

$$\sigma_x = 0$$

Therefore, $E = \sigma_y/\epsilon_y$ and $\nu = \epsilon_x/\epsilon_y$. This gave values of approximately 6×10^5 psi for E and 0.35 for ν .

The measured strains and deflections and the experimental and theoretical stress and deflection values are given in Tables 1, 2, 3, 4, and 5. It was assumed in all the calculations that the yield strength of the material was not exceeded. Values for stress and deflection for both boxes obtained by theory and test are plotted in Figure 6 through 12.

7. DISCUSSION

The equations derived for the stress and deflection of the fuel element boxes give results that agree very well with the experimental values.

There is excellent agreement between the theoretical and measured stress values for the single-element box up to the point where the yield strength is reached (Figure 6). There is a 4.2-percent difference between the theoretical and measured deflection at an internal pressure of 20 psi (Figure 7).

The theoretical stress and deflections for the four-element box also agree quite well with the measured values. At the point of the maximum stress (σ_B), with $p = P_2$, there is a 6-percent difference between the experimental and theoretical values (Figure 8). There is a 4-percent difference between the measured and theoretical maximum deflection of the box side (Figure 9). The maximum stress and deflection occurs when the pressure inside the control rod channel is equal to or greater than the pressure in the fuel boxes. This condition may occur when the control rods are scrambled.

Using the theoretical stress and deflection equation, it was determined that the minimum wall thickness of the Zircaloy-III boiler fuel element boxes for the Pathfinder reactor should be 0.150 in. thick. This was with the maximum expected pressure difference of 13.4 psi across the box walls.

REFERENCES

1. Seely, F. B. and Smith, J. O., Advanced Mechanics of Materials, Second Edition, John Wiley & Sons Inc., New York, 1952.
2. Timoshenko, S., Strength of Materials - Part II, D. Van Nostrand Company, Inc., New York, 1941.
3. Singer, F. L., Strength of Materials, Harper & Brothers, New York, 1951.
4. Metals Handbook, American Society for Metals, Cleveland, 1948.

Appendix A

Nomenclature

A = Area

c = One-half of wall thickness

E = Young's Modulus

F = Force

I = Moment of inertia

K = Curved beam factor

$L = [L_i - R_i + (\sqrt{2}/2)R + t/2]$ for four-element box

$= 1/2(L_i - 2R_i)$ for single-element box

L_i = Inside dimension of box

l = Width of control rod channel plus (t)

M = Bending moment

p = Pressure in control rod channel for four-element box

$=$ Pressure in single-element box

P_2 = Pressure in fuel boxes of four-element box

$P_1 = P_2 - p$

P_N = Normal force, P_x = Force in "x" direction, P_y = Force in "y" direction

Q = Dummy load

P_z = Force in "z" (Axial) direction

R = Mean radius at the corner $= R_i + t/2$

R_i = Inside radius

t = Wall thickness

U = Total strain energy

x, y, z = Coordinates

Z = Curved section property

φ = Angular coordinate at corner

θ = Angular rotation of a section of the box wall

ϵ = Unit strain

σ = Stress

δ = Deflection

ν = Poisson ratio

Appendix B

Derivation of Equations for Single-element Box:

From Figure 2 the bending moment from $(X = 0)$ to $(X = L)$ is:

$$M_X = M_0 - (pX^2/2) \quad (1)$$

The moment from $(\varphi = 0)$ to $[\varphi = (\pi/2)]$ is:

$$\begin{aligned} M_\varphi = M_0 - pL[(L/2) + R \sin\varphi] + P_X R(1 - \cos\varphi) - F_X[R_i \cos(\varphi/2) - R \cos\varphi] \\ - F_Y[R \sin\varphi - R_i \sin(\varphi/2)] \end{aligned}$$

$$dF = p dA = p R_i d\varphi, \quad F_Y = \int_0^\varphi p R_i \cos\varphi d\varphi = p R_i \sin\varphi$$

$$F_X = \int_0^\varphi p R_i \sin\varphi d\varphi = p R_i (1 - \cos\varphi)$$

Therefore, $M_\varphi = M_0 - p(L^2/2) - pLR \sin\varphi + pRL - pLR \cos\varphi + pRR_i$

$$- pRR_i \cos\varphi - pR_i^2 \cos(\varphi/2) + pR_i R \cos\varphi + pR_i^2 \cos(\varphi/2) \cos\varphi$$

$$- pRR_i \cos^2\varphi - pR_i R \sin^2\varphi + pR_i^2 \sin(\varphi/2) \sin\varphi$$

This reduces to:

$$M_\varphi = M_0 - (pL^2/2) - pLR(\sin\varphi + \cos\varphi - 1) \quad (2)$$

(M_0) in Eq. (1), (2) will be determined from the condition that there is no rotation of the section at $(X = 0)$. That is, the slope at the end remains zero ($\theta_0 = 0$). From Castigliano's theorem:

$$\theta_0 = \partial U / \partial M_0 \quad (3)$$

where, for thin straight members and also for curved members if $(R/C) > 4$ (Ref. 1). The strain energy due to bending is:

$$U = \int (M^2/2EI) dx \quad (4)$$

Therefore,

$$\theta_0 = \int (M/EI) (\partial M / \partial M_0) dx \quad (3a)$$

From Eqs. (1), (2), $\partial M / \partial M_0 = 1$

And $\theta_0 = 0$

$$= 2 \int_0^L \frac{M_0 - (pX^2/2)}{EI} dx + 2 \int_0^{\pi/4} \left[\frac{M_0 - (pL^2/2) - pLR(\sin\phi + \cos\phi - 1)}{EI} \right] R d\phi \quad (3b)$$

Because of symmetry the integrals are written for one-half the quadrant and are multiplied by two.

Integrating Eq. (3b) gives:

$$0 = M_0 L - \frac{pL^3}{6} + M_0 R (\pi/4) - p \frac{L^2 R \pi}{8} - pLR^2(1 - \pi/4)$$

$$M_0 = \frac{2p}{(4L - \pi R)} \left[\frac{L^3}{3} + \frac{\pi L^2 R}{4} + LR^2 \left(\frac{4 - \pi}{2} \right) \right] \quad (5)$$

The conventional equation for combined bending and direct stress for a straight beam is:

$$\sigma_x = \frac{P_x}{A_x} + \frac{M_x y}{I} \quad (6)$$

The equation for a curved beam is derived in Ref. 1:

$$\sigma_\phi = \frac{P_n}{A_x} + \frac{M_\phi}{A_x R} \left[1 + \frac{L}{Z} \left(\frac{1}{R + Y} \right) \right] \quad (7)$$

From Figure 2:

$$\begin{aligned}
 P_n &= P_y \sin\varphi + P_x \cos\varphi + F_y \sin\varphi - F_x \cos\varphi \\
 &= pL \sin\varphi + p(l + R_i) \cos\varphi + pR_i \sin^2\varphi - pR_i \cos\varphi(1 - \cos\varphi) \\
 P_n &= pL(\sin\varphi + \cos\varphi) + pR_i
 \end{aligned}$$

To determine the stress in the longitudinal direction (σ_z) the side of the box is assumed to be a long flat plate (Figure A-1).

Figure A-1. Forces acting on flat side of box.

The unit elongations in the (X) and (Z) directions are:

$$\epsilon_X = \frac{\sigma_X - \nu \sigma_Z}{E}$$

$$\epsilon_Z = \frac{\sigma_Z - \nu \sigma_X}{E}$$

For a very long, narrow plate it can be assumed that there is no bending of the (Z) axis; therefore, the strain in the (Z) direction at any point (ϵ_Z) is equal to the strain at the neutral axis (ϵ_Z)_{N.A.}.

Therefore:

$$\epsilon_X = (\epsilon_X)_{N.A.}$$

$$(\epsilon_Z)_{N.A.} = \frac{P_Z}{A_Z E} - \frac{\nu P_X}{A_X E}$$

and,

$$\frac{\sigma_Z - \nu \sigma_X}{E} = \frac{P_Z}{A_Z E} - \frac{\nu P_X}{A_X E}$$

Solving for σ_Z gives:

$$\sigma_Z = \frac{P_Z}{A_Z} + \nu \left(\sigma_X - \frac{P_X}{A_X} \right) \quad (8)$$

The equation for the deflection of the center of the box side is divided by applying a dummy load (Q) at this point.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial Q} dx + \int_0^{\pi/4} \frac{M}{EI} \frac{\partial M}{\partial Q} R d\phi \quad (9)$$

The bending moment from $x = 0$ to $x = L$

$$M_X = M_0 - (Px^2/2) - QX \quad (10)$$

And the moment from $\varphi = 0$ to $\varphi = \pi/4$ is:

$$M_{\varphi} = M_0 - (pL^2/2) - pLR (\sin\varphi + \cos\varphi - 1) - Q(L + R \sin\varphi) \quad (11)$$

$$\frac{\partial M_x}{\partial Q} = -x$$

$$\frac{\partial M_{\varphi}}{\partial Q} = - (L + R \sin\varphi)$$

Therefore,

$$\begin{aligned} \delta_Q = \frac{1}{EI} & \left[\int_0^L (-M_0 x + p x^3/20) dx + \int_0^{\pi/4} R [-M_0 L - M_0 R \sin\varphi \right. \\ & + (pL^3/2) + (pL^2 R \sin\varphi)/2 + pL^2 R (\sin\varphi + \cos\varphi - 1) \\ & \left. + pLR^2(\sin^2\varphi + \sin\varphi \cos\varphi - \sin\varphi)] d\varphi \right] \end{aligned}$$

Integrating gives:

$$\begin{aligned} \delta_Q = \frac{1}{EI} & \left[\frac{pL^4}{8} + \frac{pL^3 R \pi}{8} + pL^2 R (0.3611) + (0.1) pLR^3 \right. \\ & \left. - M_0 [(L^2/2) + (RL\pi/4) + 0.293R^2] \right] \quad (12) \end{aligned}$$

Derivation of Equations for Four-Element Box:

From Figure 3-C the bending moment from $y = 0$ to $y = L$ is:

$$M_{y=0 \text{ to } L} = M_I - P_{NI} \cos 45^\circ y - (P_I y^2/2)$$

$$\frac{\partial M}{\partial M_I} = 1, \quad \frac{\partial M}{\partial M_A} = 0, \quad \frac{\partial M}{\partial P_{NI}} = -\frac{\sqrt{2}}{2} y$$

The moment from $X_A = 0$ to $X_A = l/2$ is:

$$M_{X_A=0 \text{ to } l/2} = M_A + p (X_A^2/2)$$

$$\frac{\partial M}{\partial M_I} = 0, \quad \frac{\partial M}{\partial M_A} = 1, \quad \frac{\partial M}{\partial P_{NI}} = 0$$

The moment from $X = 0$ to $X = L$ is:

$$\begin{aligned} M_{X=0 \text{ to } L} &= M_A + (p^2/8) + M_2 - (\sqrt{2}/2)P_{NI}X + P_2 (X^2/2) \\ &= M_A + (p^2/8) + M_I + (p_1 L^2/2) - P_{NI} (\sqrt{2}/2) + P_2 (X^2/2) \end{aligned}$$

$$\frac{\partial M}{\partial M_I} = 1, \quad \frac{\partial M}{\partial M_A} = 1, \quad \frac{\partial M}{\partial P_{NI}} = -\frac{\sqrt{2}}{2} (L + X)$$

From Castigliano's theorem:

$$\begin{aligned} \theta_I = \frac{\partial U}{\partial M_I} = 0 &= \int_0^L \left(M_I - P_{NI} \frac{\sqrt{2}}{2} y + P_I \frac{y^2}{2} \right) dy \\ &+ \int_0^L \left(M_A + \frac{p^2}{8} + M_I + \frac{p_1 L^2}{2} - P_{NI} \frac{\sqrt{2}}{2} (L+X) + P_2 \frac{X^2}{2} \right) dx \end{aligned}$$

Integrating gives:

$$M_A = P_{NI} \sqrt{2}L - 2M_I - \frac{2}{3}p_1 L^2 - \frac{P_2 L^2}{6} - \frac{p_1^2}{8} \quad (A)$$

$$\begin{aligned} \theta_A = \frac{\partial U}{\partial M_A} &= \int_0^{l/2} \left(M_A + \frac{p X_A^2}{2} \right) dX_A + \int_0^L \left[M_A + \frac{p_1 L^2}{8} + M_I + \frac{p_1 L^2}{2} \right. \\ &\quad \left. - P_{NI} \frac{\sqrt{2}}{2} (L + X) + \frac{P_2 X^2}{2} \right] dX = 0 \end{aligned}$$

Integrating gives:

$$M_I = P_{NI} \frac{\sqrt{2}}{4} L - M_A \left(2 + \frac{l}{2L} \right) - \frac{P_1 l^3}{8L} + 2 \left(-\frac{P_1 L^2}{2} - \frac{P_2 L^2}{6} \right) \quad (B)$$

$$\begin{aligned} \delta_{P_{NI}} = \frac{\partial U}{\partial P_{NI}} &= \frac{\sqrt{2}}{2} \int_0^L \left(M_I Y - P_{NI} \frac{\sqrt{2}}{2} y^2 + \frac{1}{2} P_1 Y^3 \right) dy \\ &- \frac{\sqrt{2}}{2} \int_0^L \left[M_A (L+x) + \frac{P_1 l^2}{8} (L+x) + M_I (L+x) + \frac{P_1 L^2}{2} (L+x) \right. \\ &\quad \left. - P_{NI} \frac{\sqrt{2}}{2} (L+x)^2 + \frac{P_2}{2} (x^2 L + x^3) \right] dx \end{aligned}$$

Integrating gives:

$$P_{NI} = \frac{3}{4\sqrt{2}L} \left[2M_I + \frac{7}{8} P_1 L^2 + \frac{3}{2} M_A + \frac{3}{16} P_1 l^2 + \frac{7}{24} P_2 L^2 \right] \quad (C)$$

Solving equations (A), (B) and (C) simultaneously gives:

$$M_A = \frac{L^2}{96} (P_2 - P_1) - P_1 l^2 \left(\frac{1}{64} + \frac{l}{48L} \right)$$

$$M_I = P_2 L^2 \left(\frac{41}{384} \right) - P_1 L^2 \left(\frac{9}{384} \right) + P_1 l^2 \left(\frac{7}{256} - \frac{l}{182L} \right)$$

$$P_{NI} = \frac{3}{4\sqrt{2}L} \left[\frac{25}{48} P_2 L^2 + \frac{39}{48} P_1 L^2 + P_1 l^2 \left(\frac{7}{32} + \frac{l}{24L} \right) \right]$$

Correcting the moments for the curvature at the corner and solving for all the required forces and moments gives eqs. (14) through (20).

The deflection at the center of the box side is obtained by applying a dummy load (Q) at this point as shown in Figure 3-d and writing the deflection equation.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int \frac{M}{EI} \frac{\partial M}{\partial Q} dx$$

$$M_{X_A=0 \text{ to } l/2} = M_A + QX + \frac{pX_A^2}{2}, \quad \frac{\partial M}{\partial Q} = X, \quad Q = 0$$

Therefore:

$$M \frac{\partial M}{\partial Q} = M_A X_A + \frac{pX_A^3}{2}$$

$$M_{X=0 \text{ to } L} = M_A + \frac{pl^2}{8} + Q \left(\frac{l}{2} + X \right) + M_2 + P_{2Y}X + \frac{P_2X^2}{2}$$

$$\frac{\partial M}{\partial Q} = \left(\frac{l}{2} + X \right)$$

$$\begin{aligned} M \frac{\partial M}{\partial Q} &= M_A \left(\frac{l}{2} + X \right) + \frac{pl^2}{8} \left(\frac{l}{2} + X \right) + M_2 \left(\frac{l}{2} + X \right) - P_{2Y} \left(\frac{Xl}{2} + X^2 \right) \\ &\quad + \frac{P_2}{2} \left(\frac{X^2l}{2} + X^3 \right) \end{aligned}$$

$$\begin{aligned}
 \delta_Q &= \int_0^{l/2} \frac{M}{EI} \cdot \frac{\partial M}{\partial Q} dx_A + \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial Q} dx \\
 &= \frac{1}{EI} \left[\frac{M_A}{2} \left(\frac{l^2}{4} + lL + L^2 \right) + \frac{Pl^2}{16} \left(\frac{2}{8} + lL + L^2 \right) \right. \\
 &\quad \left. + \frac{M_2}{2} (lL + L^2) - P_{2y} \left(\frac{L^2 l}{4} + \frac{L^3}{3} \right) + \frac{P_2}{4} \left(\frac{L^3 l}{3} + \frac{L^4}{2} \right) \right]
 \end{aligned}$$

TABLE I
MEASURED STRAINS IN SINGLE-ELEMENT BOX

Pressure, psi Strain, μ in/in	10	14	18	20	22	26	28	30	34	38	42	46	50
ϵ_B	-500	-700	-900	----	-1150	-1380	----	-1590	-1780	-2000	-2280	-2500	-2730
ϵ_E	-23	-31	-41	----	-50	-57	----	-68	-77	-85	-94	-103	-112
ϵ_A	850	1200	1520	----	1950	2300	----	2740	3150	3620	4220	4670	5200
ϵ_C	360	510	620	720	----	920	990	----	1200	1350	1490	1670	1785
ϵ_D	9	15	17	16	----	25	26	----	31	----	41	----	46

Strain gage locations are indicated in Figure 6.

TABLE II

EXPERIMENTAL & THEORETICAL STRESS AND DEFLECTION VALUES FOR SINGLE-ELEMENT BOX

Pressure, psi Stress, psi	10	14	18	20	22	26	28	30	34	38	42	46	50	Theoretical σ/p
$-\sigma_B$	5700	7970	10250	----	13080	15710	-----	18100	20250	22770	25920	28400	31050	6.11
$-\sigma_E$	2100	2940	3790	----	4825	5770	-----	6650	7450	8360	9490	10410	11370	20.1
σ_A	9540	13470	17050	----	21900	25800	-----	30770	35300	40700	47350	52400	58400	96.1
σ_C	4080	5780	-----	8140	-----	-----	11,210	-----	13590	-----	16900	-----	20150	41.3
σ_D	1433	2060	-----	2840	-----	-----	3970	-----	4290	-----	5980	-----	7100	139.6
Deflection, in.	δ_T/n													
δ side	0.0091	----	----	0.0181	-----	-----	-----	0.0266	-----	-----	-----	-----	0.0440	$0.966 (10)^{-3}$
$-\delta$ corner	0.0001	----	----	0.0005	-----	-----	-----	0.00095	-----	-----	-----	-----	0.0022	

$$\delta_{\text{Total, calculated}} = \left(\delta_{\text{side}} - \frac{\sqrt{2}}{2} \delta_{\text{corner}} \right)_{\text{measured}}$$

TABLE III
MEASURED STRAINS IN FOUR-ELEMENT BOX

PRESSURE, IN. H ₂ O		STRAIN, MICROINCHES PER INCH										
P	P ₂	ε ₁	ε ₂	ε ₃	ε ₄	ε ₅	ε ₆	ε ₇	ε ₈	ε ₉	ε ₁₀	ε ₁₁
0	5.3	0	105	0	120	120	140	0	140	0	140	0
2.85	5.8	0	210	0	0	0	190	-80	170	-80	250	0
5.3	5.3	10	300	0	-100	-100	158	-100	180	-100	370	40
0	10.2	0	300	0	200	220	310	0	310	0	380	0
4.9	9.9	40	490	10	0	0	330	-90	320	-90	500	80
10.0	10.0	80	600	30	-200	-200	380	-200	320	-200	690	100
0	20.3	0	530	0	400	490	630	0	620	0	700	0
5	20.3	20	720	5	200	260	670	-100	660	-100	880	60
14.9	20.3	120	1000	100	-180	-180	720	-300	700	-300	1200	150
14.8	20.2	100	1000	80	-200	-200	700	-300	670	-320	1120	120
20.75	20.75	130	1190	100	-440	-470	720	-430	700	-480	1340	190
5.3	0	8	200	50	-200	-200	80	-80	80	-100	210	100
5.3	2.6	0	200	0	-180	-180	100	-100	100	-100	230	30
15.1	0	80	420	70	-670	-720	30	-330	0	-380	420	100
15.1	7.75	60	600	10	-510	-550	270	-370	210	-400	690	100
25.0	29.9	170	1560	120	-460	-400	1010	-510	990	-550	1770	210
35.0	35.0	300	2160	280	-700	-730	1400	-700	1320	-740	2440	310
17.5	35.0	170	1530	100	0	50	1210	-350	1170	-370	1800	190
35.0	17.5	310	1725	360	-1180	-1300	800	-760	710	-810	1875	350

$$\sigma_B = [E/(1 - \nu^2)] [\epsilon_2 + \nu \epsilon_1] \text{ or } [E/(1 - \nu^2)] [\epsilon_{10} + \nu \epsilon_{11}]$$

$$\sigma_1 = [E/(1 - \nu^2)] [\epsilon_6 + \nu \epsilon_7] \text{ or } [E/(1 - \nu^2)] [\epsilon_8 + \nu \epsilon_9]$$

$$\sigma_2 = [E/(1 - \nu^2)] [\epsilon_4 \text{ or } \epsilon_5 + \nu \epsilon_3]$$

(See Figure 3-b for location of σ_B , σ_1 , and σ_2)

TABLE IV MEASURED DEFLECTIONS OF FOUR ELEMENT BOX

PRESSURE, IN. H ₂ O		DEFLECTION, .001 INCHES					
p	P ₂	#1	#2	#3	#4	#5	#6
0	5.3	.85	-.32	.95	2.0	-.5	2.5
2.85	5.8	4.0	4.9	4.2	6.0	5.0	6.5
5.3	5.3	6.2	9.8	6.5	7.5	10.0	9.0
0	10.2	2.7	-.2	2.55	2.7	-.6	4.1
4.9	9.9	7.5	8.4	7.4	8.6	8.5	9.0
10.0	10.0	12.1	17.3	12.1	13.8	18.0	15.0
0	20.3	5.8	-.3	5.5	8.1	-.3	9.5
5	20.3	10.5	8.9	10.8	13.1	8.8	15.8
14.9	20.3	19.8	16.4	20.0	21.9	27.0	25.5
14.8	20.2	19.5	26.3	19.6	21.5	26.0	25.0
20.75	20.75	25.3	36.7	25.4	27.0	37.0	32.0
5.3	0	4.8	9.6	5.0	5.3	9.9	6.0
5.3	2.6	5.8	9.0	5.8	5.5	9.2	6.7
15.1	0	13.8	27.0	14.0	13.9	28.0	16.2
15.1	7.75	16.2	27.3	16.4	17.1	27.5	20.0
25.0	29.9	31.9	45.0	32.3	34.5	46.0	40.0
35.0	35.0	----	61.8	----	----	62.9	----
17.5	35.0	----	31.0	----	----	32.0	----
35.0	17.5	----	64.7	----	----	67.4	----

Position of Dial Indicators

From Figure (9), for $p = P_2$ Experimental, $\delta_{\max} = .0265 \epsilon_{\max}$ Theoretical, $\delta_{\max} = .0276 \epsilon_{\max}$ $\delta_{\max}$ = max. deflection of box side
(indicated by dial 2 and 5) $\epsilon_{\max}$ = max. strain at corner B .

TABLE V EXPERIMENTAL & THEORETICAL STRESS IN FOUR-ELEMENT BOX

p		P ₂		σ_B/P_2 (psi/psi)		σ_1/P_2		σ_2/P_2	
In. H ₂ O	psi	In. H ₂ O	psi	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.
0	0	5.3	.1912	500	517	500	517	429	526
2.85	.103	5.8	.2095	818	999	530	620	0	-4.2
5.3	.1912	5.3	.1912	1372	1469	518	719	-357	-552
0	0	10.2	.3685	575	517	575	517	408	526
4.9	.177	9.9	.358	964	999	572	620	0	-4.2
10	.3613	10.0	.3613	1372	1469	587.5	719	-549	-552
0	0	20.3	.734	570	517	570	517	450	526
5	.1806	20.3	.734	840	---	592	---	244	---
14.9	.539	20.3	.734	1170	---	575	---	-135	---
14.8	.535	20.2	.730	1090	---	558	---	-161	---
20.75	.75	20.75	.750	1282	1469	520	719	-398	-552
5.3	.1912	0	0	876	----	161	---	-653	---
5.3	.1912	2.6	.094	860	1207	233	462	-644	-811
15.1	.546	0	0	570	----	-108	---	-872	---
15.1	.546	7.75	.28	909	1207	176	462	-685	-811
25.0	.903	29.9	1.08	1167	----	527	---	-265	---
35	1.265	35	1.265	1380	1469	624	719	-341	-552
17.5	.632	35	1.265	1010	999	588	620	+35	-4.2
35	1.265	17.5	.632	1180	1207	289	462	-635	-811

Location of Maximum Stresses

Results are plotted on Figures 8, 10, 11, & 12.

Figure 1. Boiler fuel element boxes. (AC Dwg 43-024-845)

Figure 2. One quadrant of a single-element box. (AC Dwg 43-024-848)

Figure 3. Section of four-element box. (AC Dwg 43-024-847)

Figure 4. Test of single-element box. (AC Photo 207009)

Figure 5. Test of four-element box. (AC Photo 212243A)

Figure 6. Stress in single-element boiler fuel element box as a function of internal pressure. Material = 61ST4 aluminum; $E = 10 \times 10^6$ psi; $\nu = 0.33$; yield str. = 21,000 psi. Stress calculation assumed yield strength was not exceeded. (AC Dwg 43-024-158)

Figure 7 Deflection of aluminum single-element box as a function of internal pressure. (AC Dwg 43-024-849)

Figure 8. Stress in four-element box as a function of internal pressure. Dashed curves denote calculated values. Points are experimental values. Pressure in the control rod channel is equal to pressure in the compartments. (A-C Dwg. 43-024-876)

Figure 9. Maximum deflection of sides of four-element box as a function maximum strain and stress at inside corner. (AC Dwg 43-024-873)

Figure 10. Stress in corners of four-element box as a function of compartment internal pressure with control-rod-channel pressure reduced to one-half. (AC Dwg 43-024-874)

Figure 11. Stress in corners of four-element box as a function of compartment internal pressure with no pressure in control rod channel.
(AC Dwg 43-024-871)

Figure 12. Stress in corner of four-element box as a function of compartment internal pressure with control-rod-channel pressure held two times higher than the compartment pressure (AC Dwg 43-024-875)

Figure 13. Curved beam factor. (AC Dwg 43-024-872)

Figure 14. Determination of modulus of elasticity of Plexiglas II UVA in four-element test at 75 F. (AC Dwg 43-024-850)