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MODERN CONTROL ASPECTS OF NUCLEAR REACTORS

Technical Report EE-157

by

J. Phelan
H. K. Knudsen
R. R. Mohler

July 1968

Final Report of research supported by
Sandia Corporation on Contract No. 53-6574

Buyer: M. S. Cotter - 4314
Requester: R. Banks - 5150

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SUMMARY

The purpose of this report is to review the research performed under Sandia Corporation Contract 53-6574 for the period of August 1, 1967 to May 1, 1968, when it was necessary to terminate the project.

Optimal control and the synthesis of practical suboptimal (nearly optimal) controls have received a major part of the research effort. Since physical reactor systems are nonlinear in nature and involve numerous constraints in state and control, it is convenient to utilize numerical analysis and digital computer solutions. Therefore, a quasi-linear programming algorithm was developed as that solves the problem by a sequence of linear programming solutions. The system is put into the realm of linear programming problems by a set of difference equations and a linearization procedure. Availability of canned linear programming routines (such as the Mathematical Programming System (MPS/360) with the IBM360) makes the technique most useful. The results of this work are published in the IEEE Transactions on Nuclear Science, Vol. NS-15, February 1968, pp. 64-73, and a copy of this paper is included in the appendix of this report.

Though the suboptimal control analysis was not complete at the termination of the project, past research indicates that practical suboptimal controllers can be designed from

a knowledge of the optimal trajectories. To accomplish this, so-called constant quality surfaces in state space are used, and the technique conveniently yields sub-optimal controls within some performance tolerance.

Closely related to the optimal control problem is system controllability, and the reachable states have been derived for a reactivity-constrained, two-dimensional, neutronics. Obviously, the region of controllability is strongly dependent on certain design parameters and system constraints. But, more effort is needed to relate controllability to practical reactor designs.

Reactor-type systems (involving a bilinear control mode) are generally more controllable than linear systems. Therefore, some preliminary results were obtained relative to the controllability of these systems.

Stability is another important consideration of reactor control systems. By means of a Popov stability analysis, a frequency response for a linearized reactor control system may be extended to include certain time-invariant nonlinearizations such as appear in control actuators.

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MODERN CONTROL ASPECTS OF NUCLEAR REACTORS

Supoptimal Control

The goal of this research was the realization of sub-optimal feedback controllers for a class of systems which includes nuclear reactors. The following progress was made toward this goal. A new approach to the design of suboptimal controllers--the constant quality surface--was developed; an algorithm to compute points on constant quality surfaces was developed and programmed, and methods of design for a feedback controller which utilizes the knowledge of constant quality surfaces were investigated.

It appears that suboptimal controllers superior to those designed using present techniques (References 1, 2) can be designed with knowledge of the system's constant quality surfaces.

Constant Quality Surfaces

The constant quality surface is defined as the locus of points in state space on which local sensitivity is constant. The local sensitivity $\delta S(\underline{x})$ at a point $\underline{x}$ in state space is given by

$$\delta S(\underline{x}) = \frac{\left[\frac{\partial J^*}{\partial \underline{x}}(\underline{x}) \right] \cdot f(\underline{x}, u_w)}{\left[\frac{\partial J^*}{\partial \underline{x}}(\underline{x}) \right] \cdot f(\underline{x}, u^*)}$$

where

$\frac{\partial J^*}{\partial \underline{x}}$ is the gradient of the performance index,
 $\dot{\underline{x}} = \underline{f}(\underline{x}, u)$ is the equation describing the dynamical
 system (reactor),
 u_w is the worst control at $\underline{x}$, and
 u^* is the optimal control at $\underline{x}$.

The local sensitivity is a measure of the importance of using the correct control at $\underline{x}$. It was found that (for performance indices of the form $J = \int_{t_0}^{t_1} f_0(\underline{x}) dt$) a sub-optimal controller which generates feedback controls that satisfy the condition $\delta S(\underline{x}) < q$ ($0 < q < 1$) will transfer any state to the target with a cost of less than $\frac{1}{q}$ of the optimal.

Constant quality surfaces are important in design, since they can be used to determine regions in the state space where the system is insensitive to the control. For example, consider the control policy shown in Figure 1, where the optimal control policy is $u=-1$ above the optimal switch curve, and $u=1$ below it. The local sensitivity in the shaded regions (between the optimal switch curve and the 0.8 constant quality curve) ranges from 0.8 to 1.0. A suboptimal controller designed to give a control, $u=-1$, for states above the approximate switch curve shown in Figure 1 and $u=+1$ below it will give a performance that is better than 80 percent of the optimal.

The Computation of Local Sensitivities

It is impossible, except for some simple problems, to obtain analytic solutions for local sensitivities or constant quality surfaces. The best that can be done is to compute local sensitivities point by point. If the performance-index gradient $\frac{\partial J^*}{\partial \underline{x}}$ can be evaluated at $\underline{x}$, the computation of the local sensitivity at that point is almost trivial for the class of problems being studied. $\frac{\partial J^*}{\partial \underline{x}}$ is, at regular points (Reference 3), equal to $-\underline{\lambda}^*$, where $\underline{\lambda}^*$ is the adjoint vector corresponding to the optimal trajectory and control. If unique extremal controls exist for the regulator type problem, $\underline{\lambda}^*(t_1)$, the final value of $\underline{\lambda}^*$ can be related to an optimal trajectory (Reference 4). That is, given $\underline{\lambda}^*(t_1)$, the adjoint differential equation can be integrated backward in time to give $\underline{\lambda}^*(t_1-t)$. The optimal control policy, $u^*(t_1-t)$, and the corresponding optimal trajectory, $\underline{x}^*(t_1-t)$, can then be computed from $\underline{\lambda}^*(t_1-t)$. A computer program was written, which, given $\underline{x}^*(t_1)$, t_1 , and t , generates the corresponding $\underline{\lambda}^*(t_1-t)$, $u^*(t_1-t)$, and $\underline{x}^*(t_1-t)$ for the time optimal control of a regulator. (Time optimal controls are generated for linear dynamical systems with bounded input and a hyperspherical target centered at the origin.) Other programs have been written to perform this task (Reference 5). A new program was written to incorporate a recursive algorithm for the generation of $\underline{\lambda}^*(t_1-t)$. This was done to reduce computation time. Using the program, it is possible to compute the

local sensitivity and the optimal control at a number of points in the state space.

Research on the utilization of local sensitivities in the realization of suboptimal feedback controllers was in progress when the project terminated. Preliminary results are discussed below.

The Realization of Suboptimal Feedback Controllers

Realizing a suboptimal feedback controller, given a set of points U in state space at which the optimal controls and local sensitivities are known, is a complicated problem. Questions which must be answered are:

1. What set of points U should be used?
2. What is a good structure for the feedback controller?
3. How should local sensitivities be used?

Question 1 is not answered.

The structure of the feedback controller is fixed somewhat by the form of the optimal control. The systems being studied have v -level optimal controls; that is, the optimal control at (almost) every instant of time has one of v discrete levels. For this class of system, the optimal controller can be considered to be a pattern classifier which classifies each point in state space into one of v categories. Preliminary research indicates that a pattern classifier with the structure shown in Figure 2 should be used as a feedback controller. This pattern classifier is a generalization of the Φ machine (Reference 6), a

pattern classifier commonly used as a learning machine.

This type of feedback controller is promising because:

1. the complexity of the controller can be regulated by the choice of $\underline{g}(\underline{x})$ and the number of threshold logic units (TLU's) (see Figure 2).
2. the controller can be optimized through parameter adjustment (the vectors $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_r$ must be set).

One method for using local sensitivities in the adjustment of the controller parameters is the following:

1. Calculate the optimal control and local sensitivity at each of a set of points U in the state space.
2. Assign a penalty (some function of the local sensitivity) for misclassification (by the controller) to each point belonging to U .
3. Develop a total penalty P which is some concave function of the misclassification penalties of the individual points in U .
4. Minimize P .

Minimizing P is a problem in linear or mathematical programming, depending on the total penalty function (Reference 7).

Figure 1. Constant Quality Surfaces;
Optimal and Suboptimal Control Policies

Figure 2. Structure of the Feedback Controller

Controllability of Reactors and Bilinear Systems

Reactable Zones at Neutron Kinetics

A simple two-dimensional bilinear model of a nuclear reactor is the classical one-energy-group, single-precursor neutron kinetic equations. This model will be shown to be controllable on a set whose boundaries are eigenvectors of the system matrix, but the system is not completely controllable. Here, complete controllability means that any state can be reached from any other state with an admissible control in infinite time, and controllability on a set refers to "complete" controllability for a particular set of states.

According to this model, (Reference 8), the average rate of change of the neutron density is equal to

$$\frac{dn}{dt} = \frac{k(1-\beta)-1}{\ell} n + \lambda c, \quad (1)$$

where

k is the effective multiplication factor,

β is the portion of neutrons in each generation that is delayed,

ℓ is the average time between neutron generations,

n is the neutron density,

c is the precursor density, and

λ is the decay constant of the precursors.

The rate of change of the precursor density is

$$\frac{dc}{dt} = \frac{k\beta n}{l} - \lambda c. \quad (2)$$

Since the neutron and precursor densities correspond to physical quantities, the regions of interest for these equations are where $n \geq 0$ and $c \geq 0$.

The effective multiplication constant k , which is the control for the system, is taken to be non-negative, and, to prevent the neutron density from increasing too rapidly, k is constrained to be less than or equal to $\gamma/(1-\beta)$, where γ is a specified constant, so $0 \leq \gamma < 1$.

For convenience, a state vector, $\underline{x}$, may be defined to have the components n and c . Also for convenience, a control variable, u , is substituted for k and the constants a_1 and a_2 are substituted for $1/l$ and λ . Then, Equations 1 and 2 can be written as

$$\dot{\underline{x}} = (A + uB)\underline{x} \quad (3)$$

where

$$A = \begin{bmatrix} -a_1 & a_2 \\ 0 & -a_2 \end{bmatrix} \quad (4)$$

and

$$B = \begin{bmatrix} a_1(1-\beta) & 0 \\ a_1\beta & 0 \end{bmatrix} \quad (5)$$

The state and control restraints can be written

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \frac{\gamma}{1-\beta} \geq u \geq 0, \quad (6)$$

The equilibrium set (the set of all $\underline{x}$'s for which $\dot{\underline{x}} = \underline{0}$ with an admissible u) for Equations 3, 4, and 5 can be written

$$x_2 = \frac{a_1\beta}{a_2} x_1 \quad (7)$$

with

$$u = 1, \quad (8)$$

In Reference 0, Lee and Markus derive a theorem that provides a test for local controllability of a nonlinear system about an equilibrium point. Here it is assumed that the process has M scalar controls and that the equation defining the dynamics can be written as

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u). \quad (9)$$

If $\underline{x}_0$ is an equilibrium point with u_0 the corresponding equilibrium control so that

$$\underline{f}(\underline{x}_0, u_0) = \underline{0}, \quad (10)$$

Equation 9 is locally controllable about the equilibrium point if the matrix $[\underline{e}, D\underline{e}, D^2\underline{e}, \dots, D^{n-1}\underline{e}]$ has rank n , where

$$D = \frac{\partial \underline{f}(\underline{x}_0, u_0)}{\partial \underline{x}} \quad (11)$$

and

$$\underline{e} = \frac{\partial \underline{f}(\underline{x}_0, u_0)}{\partial u}, \quad (12)$$

For Equation 3, the matrix D can be written as

$$D = A + u_0 B, \quad (13)$$

and the vector $\underline{e}$ can be written as

$$\underline{e} = \underline{Bx}_0. \quad (14)$$

Substituting Equations 4 and 5 into Equation 13, D can be written

$$D = \begin{bmatrix} -a_1\beta & a_2 \\ a_1\beta & -a_2 \end{bmatrix} \quad (15)$$

and substituting Equations 4 and 4 into Equation 14, $\underline{e}$ can be written

$$\underline{e} = \begin{bmatrix} a_1(1-\beta)x_{10} \\ a_1\beta x_{10} \end{bmatrix} \quad (16)$$

Equation 3 can be found to be locally controllable about a point $\underline{x}_0$ on the equilibrium line, provided that $x_{10} \neq 0$ and provided that

$$a_1(1-\beta) \neq a_2. \quad (17)$$

For $u = 1$, Equation 3 can be written

$$\dot{\underline{x}} = \begin{bmatrix} -a_1\beta & a_2 \\ a_1\beta & -a_2 \end{bmatrix} \underline{x} \quad (18)$$

The trajectories starting at points $x_1 \gg 0$ and $x_2 \gg 0$ generated by Equation 18 approach the equilibrium line asymptotically in time. Since the system is locally controllable about the equilibrium line, the equilibrium line is reachable from any point $x_1 \gg 0$ or $x_2 \gg 0$.

For $u = 0$, Equation 3 can be written

$$\dot{\underline{x}} = \begin{bmatrix} -a_1 & a_2 \\ 0 & -a_2 \end{bmatrix} \underline{x} \quad (19)$$

If $a_1 > a_2$, Equation 19 has an eigenvector,

$$\underline{x} = \begin{pmatrix} 1 \\ \frac{a_1 - a_2}{a_1} \end{pmatrix}, \quad (20)$$

that lies in the regions $x_1 \geq 0$ and $x_2 \geq 0$.

By using the control $u = 0$, any point between the eigenvector and the equilibrium line can be reached from the equilibrium line.

For $u = \gamma/(1-\beta)$, Equation 3 can be written

$$\dot{\underline{x}} = \begin{bmatrix} (\gamma-1)a_1 & a_2 \\ \frac{\gamma\beta a_1}{1-\beta} & -a_2 \end{bmatrix} \underline{x}. \quad (21)$$

For $\gamma \approx 1$ and $(a_1 - a_2)/a_1 \gg 1$, Equation 21 has an eigenvector,

$$\underline{x} = \begin{pmatrix} 1 \\ -1/2 + \left[1/4 + \frac{a\beta}{b(1-\beta)} \right]^{1/2} \end{pmatrix}, \quad (22)$$

that lies in the regions $x_1 \geq 0$ and $x_2 \geq 0$. By using the control $u = \gamma/(1-\beta)$ with $\gamma \approx 1$, any point between the equilibrium line and the eigenvector (Equation 22) for $u = \gamma/(1-\beta)$ can be reached.

The eigenvectors (Equations 20 and 22) that correspond to the controls $u = 0$ and $u = \gamma/(1-\beta)$ lie on opposite sides of the equilibrium line. Since any point within these two eigenvectors is reachable from the equilibrium line, and since the equilibrium line is reachable from any point in

the region $x_1 > 0$ and $x_2 > 0$, the interior of the region bounded by the eigenvectors for $u = 0$ and $u = \gamma/(1-\beta)$ is a set on which this model for a nuclear reactor is controllable. Figure 3 shows this region.

Since the model of the reactor contains a degenerate equilibrium point at $n = c = 0$, the model is not completely controllable and it cannot be controllable on any linear vector manifold.

Controllability of Bilinear Systems

Here, some simple generalizations of controllability are made for a class of reactor-like processes. More, precisely, these so-called bilinear time-invariant (BTI) systems can be represented by the equation,

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \sum_{i=1}^m u_i(t)B_i \underline{x}(t) + C \underline{u}(t),$$

where

$\underline{x}(t)$ is an n -vector (state vector),

$\underline{u}(t)$ is an m -vector (control vector),

A and all B_i are fixed $n \times n$ matrices, and

C is a fixed $n \times m$ matrix.

In mathematical terminology, a system of this form will be said to be completely controllable if, for every pair of points $\underline{x}_0$ and $\underline{x}_1 \in R^n$ there exists a bounded measurable control, $\underline{u}(t)$, so that $\underline{u}(t)$, steers the system from $\underline{x}(0) =$

O - B EQUILIBRIUM LINE

O - A EIGENVECTOR FOR $u = \frac{\gamma}{1-\beta}$

O - C EIGENVECTOR FOR $u = 0$

Figure 3. Controllable Region

$\underline{x}_0$ to $\underline{x}(t_1) = \underline{x}_1$ in some finite time, $t_1 \geq 0$.

A BTI system will be said to be controllable on a set, S , if for every pair of points $\underline{x}_0$ and $\underline{x}_1 \in S$, there exists a bounded measurable control, $\underline{u}(t)$, so that $\underline{u}(t)$ steers $\underline{x}(0) = \underline{x}_0$ to $\underline{x}(t_1) = \underline{x}_1$ in some finite time, $t_1 \geq 0$. If the system is completely controllable, it is controllable on any subset S of R^n . If the system is not completely controllable, it may be controllable on a set that is a linear vector manifold of dimension less than n .

An equilibrium point of the BTI system is a point $\underline{x}_e$, so

$$A \underline{x}_e + \sum_{i=1}^m \underline{u}_i B_i \underline{x}_e + C \underline{u} = \underline{0}$$

for some allowable $\underline{u}$. If $A \underline{x}_e + \sum_{i=1}^m \underline{u}_i B_i \underline{x}_e + C \underline{u} = \underline{0}$

for all $\underline{u}$, the equilibrium point $\underline{x}_e$ will be said to be degenerate. If a BTI system has a degenerate equilibrium point at the origin, the system will not be completely controllable or controllable on any linear vector manifold but there will be one or more disjoint sets on which the system is controllable.

If every point is an equilibrium point and the system is locally controllable about each equilibrium point, the system is completely controllable. Local controllability implies that every point in a neighborhood of the equilibrium point is reachable, using a bounded

measurable controller, and that the equilibrium point is reachable from any point in that same neighborhood.

If a BTI system contains a degenerate equilibrium point, the system is not completely controllable, since no point can be reached from the degenerate equilibrium point.

The simplest form of the BTI system is the one-dimensional scalar control system. This type of system can be shown not to be completely controllable.

Theorem I. Consider a system described by the equation

$$\dot{x} = ax + ubx + cu, \quad (23)$$

where a , b , and c are constants, and where u is bounded by $|u| \leq 1$. This system is completely controllable if, and only if, $a = b = 0$ and $c \neq 0$.

Proof (sufficiency). Assume $a = b = 0$ and $c \neq 0$. Then, Equation 23 becomes

$$\dot{x} = cu \quad (24)$$

and each x becomes an equilibrium point corresponding to the locally controllable about each equilibrium point, and hence is completely controllable. Therefore, a sufficient condition that the system described by Equation 23 be completely controllable is that $a = b = 0$ and $c \neq 0$.

Proof (necessity). Assume $c = 0$. Then, Equation 23 becomes

$$\dot{x} = (a + ub) x \quad (25)$$

and $x = 0$ becomes a degenerate equilibrium point. Hence, the system described by Equation 23 is not completely controllable if $c = 0$.

Now, assume $a = 0$, $b \neq 0$, and $c \neq 0$. Then, Equation 23 becomes

$$\dot{x} = u(bx + c) \quad (26)$$

and $x = -c/b$ becomes a degenerate equilibrium point. Hence, the system described by Equation 23 is not completely controllable if $a = 0$, $b \neq 0$, and $c \neq 0$.

Now assume that $a \neq 0$, $b = 0$, and $c \neq 0$. Then Equation 23 becomes

$$\dot{x} = ax + cu. \quad (27)$$

If $a > 0$, for $x > |c/a|$ and $\dot{x} > 0$, the origin cannot be reached from $x(0) > |c/a|$. If $a < 0$, $x < 0$ for $x > |c/a|$ and $x(t_1) > |c/a|$ cannot be reached from the origin. Hence, the system described by Equation 23 is not completely controllable if $a \neq 0$, $b = 0$, and $c \neq 0$.

Now, assume $a \neq 0$, $b \neq 0$, $c \neq 0$ and $a \geq b$. Then, at $x = -c/(a+b)$,

$$\dot{x} = c \left[-\frac{(a+ub)}{a+b} + u \right].$$

If $c > 0$, $\dot{x} \leq 0$ for $x(0)$ cannot be reached. If $c < 0$, $\dot{x} \geq 0$ for $x(0) = -c/(a+b)$ and $x(t_1) < x(0)$ cannot be reached. Hence, if $a \neq 0$, $b \neq 0$, $c \neq 0$, and $a \geq b$, the system described by Equation 23 is not completely controllable. Now assume $a \neq 0$, $b \neq 0$, $c \neq 0$, and $a < b$. Then, for $x = c/a-b$,

$$\dot{x} = c \left[\frac{a+ub}{a-b} + u \right].$$

If $ac > 0$, $\dot{x} \leq 0$ for $x(0) = c/a-b$ and $x(t_1) > x(0)$ cannot be reached. If $ac < 0$, $\dot{x} \geq 0$ for $x(0) = c/a-b$ and $x(t_1) < x(0)$ cannot be reached. Hence, the system described by Equation 23 is not completely controllable if $a \neq 0$, $b \neq 0$, $c \neq 0$, and $b > a$.

Therefore, a necessary condition that the system described by Equation 23 be completely controllable is that $a = 0$, $b = 0$, and $c \neq 0$.

If the single control of the previous theorem is replaced by two independent controls, it is possible for the one-dimensional BTI system to be completely controllable.

Theorem II. Consider a system described by the equation

$$\dot{x} = ax + u_1bx + cu_2, \quad (28)$$

where a , b , and c are constants and where u_1 and u_2 are bounded by $|u_1| \leq 1$ and $|u_2| \leq 1$. This system is completely controllable if, and only if, $|b| \geq |a|$ and $c \neq 0$.

Proof (sufficiency). Assume $|b| \geq |a|$ and $c \neq 0$. Then, by fixing u_1 so that $a + u_1b = 0$, Equation 28 becomes

$$\dot{x} = u_2c. \quad (29)$$

Since systems described by Equation 29 were shown to be completely controllable in Theorem I, a sufficient condition that a system described by Equation 28 be completely controllable is that $|b| \geq |a|$ and $c \neq 0$.

Proof (necessity). Assume $c = 0$. Then Equation 28 becomes

$$\dot{x} = (a + u_1b)x. \quad (30)$$

Any system described by Equation 30 was shown not to be completely controllable in Theorem I.

Assume $|a| > |b| > 0$ and $c \neq 0$. Then, for the point

$$x = \frac{c}{\min_{|u_1| \leq 1} (a + u_1 b)}$$

$$\dot{x} = c \left[\frac{a + u_1 b}{\min_{|u_1| \leq 1} (a + u_1 b)} + u_2 \right]$$

If $c > 0$, $\dot{x} \geq 0$ for $x(0) = \frac{c}{\min(a+u_1 b)}$, and any point $x(t_1) < x(0)$ cannot be reached. Hence, if $|a| > |b|$ and $c \neq 0$, a system described by Equation 28 will not be completely controllable. Therefore, a necessary condition that a system described by Equation 28 be completely controllable is that $|b| \geq |a|$ and $c \neq 0$.

It is obvious from these results that this is only the introduction to opening a fruitful research area that will be exploited.

Stability

It is the presence of an unstable mode that generally makes bilinear systems and the reactor neutronics so controllable. For these systems to be useful, however, the control function must stabilize the system at the proper time. In a sense, such processes involve a controlled use of instability.

In this section, stability of an equilibrium state such as that needed at the terminal end of most control problems is the primary concern, although it should be cautioned that the neutronic transfer function about criticality ($k=1$) has a pole at the origin (and the system is thereby said to be structurally unstable), this

condition proved to be useful in practice. This perturbation neutronic transfer function is

$$G_1(s) = \frac{\Delta N}{\Delta K}(s) = \frac{n_0(s + \lambda)}{ls(s + \frac{\beta}{l} + \lambda)},$$

where n_0 is the equilibrium neutron level of interest and the other terms are defined in the previous section.

In practice, a simple neutronic control loop may have the form shown in Figure 4, with ϕ representing a non-linear characteristic of a control actuator. k_1 may be a controller gain and $G_2(s)$ a controller or actuator transfer function. Lumping the transfer functions, $G(s) = G_1(s)G_2(s)$, and gains together as shown in Figure 5, Popov's stability technique may be utilized.

Roughly, the Popov theorem states that for this system to be absolutely stable, it is sufficient that a straight line (Popov line) exists which passes through the $-1/k$ point in the $G^*(\omega)$ plane so that a plot of $G^*(\omega)$ lies entirely to the right of the line (Reference 10).

Here

$$0 < \frac{\phi(\sigma)}{\sigma} \leq k, \quad \sigma \neq 0,$$

and $G^*(\omega)$ is related to the frequency response $G(i\omega)$ by

$$x = R_e G^*(\omega) = R_e G(i\omega)$$

$$y = I_m G^*(\omega) = \omega G(i\omega)$$

Figure 4. Simple Neutronic Control Loop

Figure 5. Control Loop for Popov Analysis

Figure 6. Popov Diagram for Stable Neutronic Loop

Figure 7. Popov Diagram for Marginal Neutronic Loop

For the above neutronic control loop with $G_2(s) = \frac{1}{\tau s + 1}$,

$$x = R_e G^*(\omega) = k_3 \frac{[p_1 p_2 - p_3 (p_1 + p_2) - \omega^2]}{(\omega^2 + p_1^2)(\omega^2 + p_2^2)},$$

and

$$y = I_m G^*(\omega) = \frac{-k_3 p_1 p_2 p_3 + \omega^2 (p_1 + p_2 - p_3)}{(\omega^2 + p_1^2)(\omega^2 + p_2^2)},$$

where

$$k_3 = k_1 n / l \tau, \quad p_1 = 1/\tau, \quad p_2 = \beta/l \quad \text{and} \quad p_3 = \lambda.$$

Then it is readily noted that the locus $G^*(\omega)$ cannot cross the negative x axis, since, with $v = \omega^2$,

$$\frac{y(v)}{x(v)} \rightarrow p_1 + p_2 - p_3 > 0, \\ v \rightarrow \infty.$$

and $x(v)$ and $y(v)$ are negative for v very large but finite. Thus, the locus converges to the origin in the third quadrant as shown in Figure 6.

Of course, added heat exchange and actuator dynamics may make the system unstable at high gain or large neutron level. And, for the locus shown by Figure 7, the Popov technique guarantees system stability for $k \leq k_c$ only.

To remove the dependence of stability on neutron level n_0 , the loop is sometimes closed on $\log n$ or, perturbation-wise, on $\Delta n/n_0$.

Though only a simple neutronic loop is analyzed here, it is apparent that computer stability analysis by means of complicated reactor transfer functions could well include certain system nonlinearities by use of a simple Popov analysis.

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COMPUTATION OF OPTIMAL CONTROLS
FOR A NUCLEAR ROCKET REACTOR

by

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ABSTRACT

Various optimal reactor control problems with complicated constraints are solved for a nuclear rocket engine. The technique utilizes a digital computer with the numerical algorithm formed by a succession of linear programming problems. Each programming problem has a solution which is used to relinearize the system's dynamical equations.

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COMPUTATION OF OPTIMAL CONTROLS FOR A NUCLEAR ROCKET REACTOR

Introduction

This paper considers the computation of the optimal controls for a nuclear rocket reactor. Optimal controls have been calculated for a nuclear rocket reactor in Reference 1 by means of the maximum principle and its extensions for state constraints. In that study, it was necessary to keep the constraints fairly simple. Utilizing a programming technique the present study considers more complicated constraints.

Most numerical procedures for the calculation of optimal control fall into the general categories of either gradient methods, generalized Newton methods or methods based on the maximum principle. In almost all cases, Lagrange multiplier functions are used to addend the constraint differential equations to the optimality criterion. In many cases, as for the neutron kinetics, the natural instability of the system makes a practical application of the preceding techniques very difficult.

The numerical procedure used here, however, is based on a iterative sequence of linear programming problems (Reference 2). In this context the distinction between state and control loses much of its significance, and inequality constraints, which form an integral part of linear programming, are as easily applied to the state as

to the control. Additionally, no Lagrange multipliers or penalty functions are usually needed, since each iterate can easily be forced to satisfy all inequality constraints and boundary conditions. The fact that each iterate satisfies all inequality constraints eliminates the instability problems associated with the nuclear reactor.

Nuclear Rocket Reactor

The nuclear rocket reactor to be considered is one which makes use of hydrogen as the propelling medium. Figure 1 is a schematic illustration of a nuclear rocket propulsion system. Here, hydrogen is pumped over the reactor core, where heat is transferred to the propellant. The hot hydrogen gas then exits through a nozzle to furnish the thrust. It is assumed that the only variables available to control the nuclear rocket reactor are the hydrogen flow-rate, u_2 , and the reactivity u_1 .

The differential equations which approximate the nuclear reactor behavior are

$$\dot{Q} = \left(\frac{u_1 - \beta}{l} \right) Q + \lambda C, \quad (1)$$

$$\dot{C} = \frac{\beta}{l} Q - \lambda C; \quad (2)$$

where: Q is the reactor power level,
 C is the neutron precursor level,
 u_1 is the reactivity,
 u_2 is the hydrogen flow rate

β is the proportion of neutrons which are delayed
 due to the precursors,
 l is the mean generation time of neutrons, and
 λ is the average decay constant of the neutron
 precursors.

Equations 1 and 2 are the classical one-lump, spatially-independent neutron kinetics, which are discussed in Reference 3.

The heat transfer between the reactor core and the hydrogen is approximated by

$$\dot{T} = \frac{Q}{M_C} - a u_2 T; \quad (3)$$

where: T is the average core temperature,
 M_C is the effective core mass heat capacity,
 a is a heat-transfer coefficient.

Equation 3 is a one-lump heat transfer process, which is discussed in Reference 1.

Optimal Control Problem

The specific problem considered is that of reactor startup. It is assumed that the reactor is operating at a relatively low power level -- sufficient to supply power to various auxiliary systems. It is desired to transfer the reactor from this low power level to a power level and temperature which corresponds to maximum thrust. During this transfer, the state and control variables are subject to inequality magnitude and rate constraints for mechanical

and safety reasons. The totality of constraints which are considered are

$$\begin{aligned}
 0 &\leq Q \leq Q_{\max}, \\
 0 &\leq C \\
 0 &\leq T \leq T_{\max}, \\
 \dot{T} &\leq \dot{T}_{\max}, \\
 -0.9\beta &\leq u_1 \leq 0.9\beta, \\
 \dot{u}_1 &\leq \dot{u}_{1\max}, \\
 u_{2\min} &\leq u_2 \leq u_{2\max}.
 \end{aligned} \tag{4}$$

The initial boundary conditions $(Q(0), C(0), T(0))$ are specified by the steady-state low power level. The terminal values of power and temperature $(Q(t_1), T(t_1))$ are fixed by the maximum thrust criteria. The terminal value of precursor level $(C(t_1))$, as well as the terminal time (t_1) itself, are allowed to be free, subject of course to the restrictions

$$\begin{aligned}
 0 &\leq t_1, \\
 0 &\leq C(t_1).
 \end{aligned}$$

The criterion of most interest in the operation of the nuclear rocket reactor is the quantity of hydrogen consumed in the process of transferring from one operating condition to another. A small percentage saving in hydrogen consumed can result in a large percentage increase in payload. This is particularly true since not only is a saving affected in the hydrogen proper, but also a saving

is affected in the size of the on-board storage tanks. For this reason, the optimality criteria is that the quantity J , defined by

$$J = \int_{t_0}^{t_1} u_2(t) dt, \quad (5)$$

have minimum value.

Computational Method

The numerical procedure used to obtain solutions to the optimal nuclear rocket reactor problem makes use of an iterative sequence of linear programming problems. Figure 2 is a schematic illustration of the algorithm. A convergence proof for a very restricted class of problems is presented in Reference 2. The first step is to approximate the continuous differential equations and the integral performance criterion by a set of discrete algebraic equations. There are of course many methods to obtain these approximations, most of which are patterned after numerical integration procedures. The technique used here for the process differential equations is the second-order approximation

$$Q_{k+1} - Q_k = \frac{\Delta t}{2} \left\{ \left(\frac{u_{1k} - \beta}{\ell} \right) (Q_{k+1} + Q_k) + \lambda (C_{k+1} + C_k) \right\} \quad (6)$$

$$C_{k+1} - C_k = \frac{\Delta t}{2} \left\{ \frac{\beta}{\ell} (Q_{k+1} + Q_k) - \lambda (C_{k+1} + C_k) \right\} \quad (7)$$

$$T_{k+1} - T_k = \frac{\Delta t}{2} \left\{ \frac{1}{M_c} (Q_{k+1} + Q_k) - a u_{2k} (T_{k+1} + T_k) \right\} \quad (8)$$

In obtaining Equations 6, 7 and 8, it is assumed that the controls are stepwise approximations; that is, the controls u_1 and u_2 are constant over each time interval Δt . Since the approximations are second-order, the truncation error of the Taylor series, over any interval t to $t+\Delta t$, (or k to $k+1$) is proportional to Δt^3 . The interval Δt , while variable, is determined by

$$\Delta t = \frac{(t_1 - t_0)}{N} ; \quad (9)$$

where: N is the number of time increments in the interval $(t_1 - t_0)$.

Due to the fact that the controls are piecewise constant over each Δt , the integral performance criterion can be represented algebraically by

$$J = \Delta t \sum_{k=1}^N u_{2k} \quad (10)$$

Equations 6, 7, 8, and 10 are nonlinear, and thus are not directly suitable for a linear programming problem. The general linear programming problem is to determine a vector $\underline{x}$ such that the linear objective function

$$\phi(\underline{x}) = \underline{c}^T \underline{x}; \quad (11)$$

where: $\underline{c}^T$ is a constant row vector,

has minimum (maximum) value. The vector $\underline{x}$ must also satisfy the constraints

$$A \underline{x} (\leq) \text{ or } (=) \text{ or } (\geq) \underline{b} ; \quad (12)$$

where: A is a constant matrix,

$\underline{b}$ is a constant vector,

and

$$a_i \leq x_i \leq \beta_i ; \quad (13)$$

where: a_i and β_i are constants, and

i ranges over all components of $\underline{x}$.

A complete discussion of linear programming can be found in many sources; see for example Reference 4.

In order to execute linear programming procedures, the nonlinear Equations 6, 7, 8, and 10 must be linearized to conform to Equations 11 and 12. The linearization is accomplished by retaining the first two terms (the constant and linear terms) of the Taylor series expansion for the nonlinear equations. Since the terminal time is free, Δt is treated as a variable in the Taylor series and in the ensuing linear programming problems.

Except for the rate constraints, linearization completes the formulation of the linear programming problem. From a practical standpoint, the rate constraints are easily formulated as linear inequality constraints of the form shown in Equation 12. That is, the inequality constraints

$$u_{1,k+1} - u_{1,k} \leq \Delta t \dot{u}_1 \max \quad (14)$$

and

$$T_{k+1} - T_k \leq \Delta t \dot{T}_{\max} \quad (15)$$

will prevent the reactivity and temperature from increasing too rapidly.

As can be seen from Figure 2, the solution to the non-linear problem is obtained as the limit of a sequence of linear programming problems with each new problem in the sequence resulting from a linearization about the previous solution. Because of the generalized Newtonian character of this process, convergence, if it occurs, will be quadratic.

Most large scale digital computers have general linear programming systems available which considerably simplify the programming effort required for the solution of problems in this class. The calculations in this paper were performed on an IBM-360/40 digital computer located at the University of New Mexico Computing Center. The Mathematical Programming System/360 (MPS/360) is a general system available for solution of linear programming problems on the IBM-360 series of computers. Using the source language of MPS/360, only sixteen statements were required for the solution of the linear programming problems associated with the nuclear rocket reactor. This program includes the reinsertion of the previous optimal basis as the starting basis for each new iteration.

The matrix of coefficients and other data associated with the linearized algebraic equations are illustrated in

Figure 3. The recursive nature of the algebraic equations and the inequality rate constraints generates a blocked matrix. The non-zero coefficients of this matrix, together with the coefficients in the right-hand-side column and the data on bounds for each variable, form the input data for the MPS/360 program. On each iteration these coefficients must be generated in the proper format for MPS/360. Since the matrix is blocked, the data may be generated by a series of DO-loops in a Fortran program. The complete algorithm shown in Figure 2 was formed by alternately running the Fortran and MPS/360 programs. Convergence was indicated when the input data to the MPS/360 program was optimal in accordance with the standard tolerances set in the MPS/360 program.

Computational Results for the Nuclear Rocket Reactor

Computational results for the nuclear rocket reactor have been obtained using the linear programming algorithm previously described. For numerical calculations, the parameter values, constraints and initial conditions were assumed to be those shown in Table I.

Table I
Numerical Values

<u>Parameter</u>	<u>Value</u>
β	0.64×10^{-2}
ℓ	0.3×10^{-4}
λ	0.4 (sec)^{-1}
M_c	$0.555 \times 10^{-2} \text{ MJ/k}^\circ$
a	$0.18 \times 10^{-1} \text{ }^\circ\text{K}^{-1}$
$u_{1\min}$	-0.9β
$u_{1\max}$	0.9β
$u_{2\min}$	$2.0 \text{ }^\circ\text{K/sec}$
$u_{2\max}$	$400.0 \text{ }^\circ\text{K/sec}$
$\dot{u}_{1\max}$	$0.15 \times 10^{-1} \text{ }^\circ\text{K/sec}$
$T_{\max}$	$1600.0 \text{ }^\circ\text{K/sec}$
$Q_{\max}$	20.0 MW
$T_{\max}$	$2500.0 \text{ }^\circ\text{K}$
$Q(0)$	0.01 MW
$c(0)$	5.332
$T(0)$	$277.8 \text{ }^\circ\text{K}$
$Q(t_1)$	20.0 MW
$c(t_1)$	free
$T(t_1)$	$2500.0 \text{ }^\circ\text{K}$

Figure 4 shows the trajectories obtained in the absence of the reactivity and temperature rate constraints. The data in this figure confirm the results of Mohler (Reference 1) that the minimum propellant process is also one of minimum time. That is, the optimal policy is to insert maximal reactivity until the power level constraint is reached. The reactivity must then decrease in order to hold the power level on its constraint boundary. The constant power level condition continues until the terminal temperature boundary condition is satisfied. During the entire startup process, the propellant flow rate remains on its lower boundary. The process is certainly an absolute minimum, since the integral performance criteria assumes its absolute minimum value under these conditions. The computation time taken to obtain the data is shown in Table II.

The data illustrated in Figure 5 is similar to that of Figure 4; however, the effective, core, mass heat capacity was artificially lowered by a factor of ten so that the temperature constraint was reached prior to the power constraint. In this case, the optimal process is also one of minimal time; but, the propellant consumption is much higher than that of Figure 4. The higher propellant consumption results from the fact that the hydrogen flow rate must increase in order to keep the temperature on its boundary, during the latter portion of the trajectory. The optimality of the trajectory shown

in Figure 5 might be questioned, since there is an alternative policy which can keep the temperature on its boundary.

That is, one could allow the propellant flow rate to remain on its lower bound and use reactivity variations to vary the power level, Q , so that the temperature differential equation is satisfied. The temperature rate on the temperature boundary is zero, however, so that the power level would have to be constant, i.e.,

$$\dot{T} = 0 = \frac{Q}{M_c} - au_{2\min} T_{\max}.$$

If the power level remains constant, the terminal power level boundary condition cannot be satisfied.

One could postulate an optimal policy which allowed the temperature to remain interior to its boundary. It is shown in Appendix A that interior to state constraints the optimal control policy must be bang bang; that is, a singular condition is not feasible for a startup trajectory. Using this fact, combined with the fact that temperature increases faster than power, (for this non-typical case) one can conclude that the trajectory shown is optimal. The computation time required to obtain the data of Figure 5 is similar to that shown for Figure 4 in Table II.

Figure 6 illustrates the results of calculations with all constraints imposed. It is seen that the optimal policy is to insert reactivity at maximum rate until the

reactivity upper bound is attained. At this time the reactivity rate drops to zero so that the reactivity remains on its upper bound until the temperature rate intersects its upper bound. Control reactivity is then decreased so that temperature increases at the maximum allowable rate. The constant rate of temperature increase requires an almost constant (slightly increasing ramp) power level. Sometime prior to the end of the trajectory, reactivity is again inserted at maximum rate to bring the power level to its terminal boundary value. The propellant flow rate remains on its lower bound until the final portion of the trajectory, where the power level is rapidly increasing. The flow rate must increase at this time so that the temperature rate constraint is not exceeded. It is obvious that this process is also one of minimum time.

An interesting feature of the trajectory is the apparent oscillatory behavior of power level during the latter portion. In an effort to more closely evaluate this characteristic, the trajectory of Figure 7 was calculated with approximately twice the definition of Figure 6; that is, the value of Δt is approximately halved. It can be seen that the frequency of this oscillatory behavior approximately doubled and its amplitude decreased slightly. It is concluded that the oscillatory behavior is due to the fact that the algebraic equations used in the calculations are only approximations to the continuous differ-

ential equations. In fact, in the limit as Δt approaches zero, the oscillations must disappear, since the constraint on reactivity rate will eventually prohibit fast variations in power level. The oscillatory behavior does illustrate, however, that discrete approximations to differential equations must be used with some caution.

The propellant consumption for the smoothed versions of the trajectories shown in Figures 6 and 7 is very close to the absolute minimum consumption possible. For this reason, it is concluded that the control policy shown is indeed optimal. The computation times required to obtain the data of Figure 6 is shown in Table II.

Conclusions

The optimal control and state trajectories, in the presence of control, control-rate, state and state-rate constraints, have been calculated for a nuclear rocket reactor. Although the optimality criteria was assumed to be minimum propellant consumption, the process is also one of minimal time.

The linear programming algorithm is very powerful for problems of this nature, although it has received little attention in the optimal control literature. In addition, the computation time for problems of this size is not excessive, and the programming effort is minimal.

Table II

Nuclear Rocket Engine MPS/360 Computation Time (1)

FIGURE	NO. OF ITER. TO FEASIBLE	NO. OF ITER TO OPTIMAL (3)	MAJOR ITERATION	COMPUTING TIME (MIN) (2)
4	13	---	1	3.12
	---	11	1	2.69
	1	---	2	2.81
	---	2	2	2.80
	Feasible at start	---	3	2.65
	---	1	3	3.68
	Feasible and optimal at start			2.60
			TOTAL	20.35
6	21	---	1	5.63
	---	40	1	3.03
	34	---	2	10.42
	---	29	2	7.25
	16	---	3	4.25
	---	14	3	2.92
	19	---	4	5.05
	---	17	4	4.38
	Feasible and optimal to start			2.53
			TOTAL	45.46

- (1) The starting trajectories were obtained by integrating the differential equations with maximum reactivity and minimum propellant flow rate for 1.5 seconds.
- (2) These computation times do not include running of the Fortran program between each major iteration. One run of this program requires approximately 1.50 minutes.
- (3) Optimal here refers to optimal for the linear program.

Figure 1. Schematic Illustration of Nuclear Rocket

Figure 2. Schematic Illustration of the Linear-Programming Algorithm for Optimal Control Problems

NOTE: SUPERSCRIPT REFERS TO COMPONENT NUMBER

NOTE: SUPERSCRIPT REFERS TO COMPONENT NUMBER

(2)

(1)

x_1^1	x_1^2	x_1^3	u_1^1	u_1^2	x_2^1	x_2^2	x_2^3	u_2^1	u_2^2	x_3^1	x_3^2	x_3^3	u_3^1	u_3^2	x_4^1	x_4^2	x_4^3	u_4^1	u_4^2	x_5^1	x_5^2	x_5^3	u_5^1	u_5^2	Δt
X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	
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(1) OBJECTIVE FUNCTION ROW

(2) RIGHT HAND SIDE

Figure 3. Illustration of Blocked Character of Work Matrix for Maximal Entries with Second-Order Approximation

Figure 4. Nuclear Rocket Reactor Trajectories with 20-Megawatt Power-Level Constraint

Figure 5. Nuclear Rocket Reactor Trajectories with Fast Temperature Response

Figure 6. Nuclear Rocket Reactor Trajectories with Temperature-Rate Constraint

Figure 7. Nuclear Rocket Reactor Trajectories with Increased Definition

APPENDIX A

POTENTIALLY SINGULAR SUBARCS FOR THE

NUCLEAR ROCKET REACTOR

A.1 Introduction

Potentially singular subarcs, interior to the state constraint boundaries, are investigated for the nuclear rocket engine. It is shown that a zero-power-level subarc along which the average core temperature is decreasing is the only singular condition permissible, if the problem is normal. From physical considerations it does not seem plausible that a zero-power-level subarc forms part of the optimal trajectory for the nuclear rocket reactor. However, if one investigates the problem of reactor shutdown, then this singular subarc might form a portion of the optimal trajectory.

A.2 Nuclear Rocket Reactor Singularity

Theorem 25 of Reference 5 states that interior to the state constraint boundaries, the solution to an optimization problem must satisfy the maximum principle.

The differential equations which approximate the behavior of the nuclear rocket engine are repeated here for convenience

$$\dot{Q} = \left(\frac{u_1 - \beta}{l} \right) Q + \lambda C, \quad (A.1)$$

$$\dot{C} = \frac{\beta}{l} Q - \lambda C, \quad (A.2)$$

$$\dot{T} = \frac{Q}{M_C} - a u_2 T. \quad (A.3)$$

The criteria for optimization is minimal propellant consumption; thus, trajectories along which

$$J = \int_{t_0}^{t_1} u_2(t) dt$$

has minimum value are required.

The Hamiltonian for this system is

$$\begin{aligned} H(\underline{p}, \underline{x}, \underline{u},) &= p_0 u_2 + p_1 \left(\left(\frac{u_1^{-\beta}}{\ell} \right) Q + \lambda C \right) + p_2 \left(\frac{\beta}{\ell} Q - \lambda C \right) + p_3 \left(\frac{Q}{M_c} - a u_2 T \right) \\ &= (p_0 - p_3 a T) u_2 + p_1 \frac{Q}{\ell} u_1 + p_1 \left(- \frac{\beta}{\ell} Q + \lambda C \right) + p_2 \left(\frac{\beta}{\ell} Q - \lambda C \right) + p_3 \frac{Q}{M_c} ; \end{aligned} \quad (A.4)$$

where: $p_0, p_1, p_2, p_3, Q, C, T, u_1$ and u_2 represent values along the optimal trajectory.

The corresponding costate equations are

$$\dot{p}_0 = 0 \quad (A.5)$$

$$\dot{p}_1 = - \left(\frac{u_1^{-\beta}}{\ell} \right) p_1 - \frac{\beta}{\ell} p_2 - \frac{1}{M_c} p_3, \quad (A.6)$$

$$\dot{p}_2 = -\lambda p_1 + \lambda p_2, \quad (A.7)$$

$$\dot{p}_3 = a u_2 p_3. \quad (A.8)$$

The Hamiltonian, Equation A.4, indicates that this problem may be u_2 singular (i.e., maximum H is independent of u_2) if

$$p_0 - p_3 aT \equiv 0;$$

u_1 singular, if

$$p_1 \frac{Q}{\ell} \equiv 0;$$

or both. The possibility that these conditions may exist is investigated in the following material.

Case I; Assume that the problem is u_2 singular; that is,

$$p_0 - p_3 aT \equiv 0. \quad (A.9)$$

Equation A.9 implies that

$$\frac{d}{dt} (p_0 - p_3 aT) = \dot{p}_0 - p_3 \dot{aT} - \dot{p}_3 aT \equiv 0 \quad (A.10)$$

Substituting the values for $\dot{p}_0$ and $\dot{p}_3$ from Equations A.5 and A.8 respectively, it follows that

$$-p_3 \dot{aT} - p_3 a^2 u_2 T \equiv 0. \quad (A.11)$$

It follows from equation A.11 that either

$$\dot{T} \equiv a u_2 T, \quad (A.12)$$

or

$$p_3 \equiv 0.$$

Equation A.12 together with Equation A.3 imply that

$$Q \equiv 0, \quad (A.13)$$

which in turn implies that

$$\dot{Q} \equiv 0.$$

It follows from Equation A.1 that

$$C \equiv 0.$$

Thus, it seems that a zero-power-level singular u_2 solution may exist. Equation A.12 requires that the temperature decrease along this trajectory, since both a and u_2 are greater than zero. The condition

$$T \equiv 0$$

is trivial, since it results in a time-stationary point.

Now consider

$$P_3 \equiv 0. \quad (A.14)$$

Equation A.14 and A.9 imply that

$$P_0 \equiv 0.$$

Therefore, the problem is non-normal (as defined in the calculus of variations).

Case II: Assume that the problem is u_1 singular; that is,

$$p_1 \equiv 0 \quad (A.15)$$

Equation A.7 indicates that if equation A.15 is to be true, then

$$\dot{p}_2 \equiv \lambda p_2,$$

which in turn implies that

$$\dot{p}_1 \equiv 0,$$

so that Equation A.6 requires that

$$p_3 \equiv -\left(\frac{M_c \beta}{\ell}\right) p_2 = -\left(\frac{M_c \beta}{\ell}\right) A \exp(\lambda t). \quad (A.16)$$

the problem has free terminal time; therefore, the Hamiltonian must be zero. It follows from this that the constant A in Equation A.16 cannot be zero, since if A is zero Equation A.16 requires that

$$p_2 \equiv p_3 \equiv 0,$$

and Equation A.4 requires that

$$p_0 \equiv 0.$$

All of the components of the costate are therefore identically zero which violates the maximum principle.

The preceding discussion indicates that

$$p_3 \equiv -\left(\frac{M_c \beta}{\ell}\right) A \exp(\lambda t), \quad (\text{A.17})$$

so that

$$\dot{p}_3 \equiv -\left(\frac{M_c \beta}{\ell}\right) A \lambda \exp(\lambda t). \quad (\text{A.18})$$

If the identities of Equations A.17 and A.18 are substituted into Equation A.8, the result implies that

$$u_2 = \frac{\lambda}{a}. \quad (\text{A.19})$$

The problem must therefore also be u_2 singular, since in general Equation A.19 is not a u_2 bound. Again, the Hamiltonian, Equation A.4, requires

$$p_0 - p_3 a T \equiv 0 \quad (\text{A.20})$$

for u_2 singularity. But

$$p_3 \equiv -\left(\frac{M_c \beta}{\ell}\right) A \exp(\lambda t),$$

so that

$$p_0 - \left(\frac{M_c \beta}{\ell}\right) A \exp(\lambda t) a T \equiv 0. \quad (\text{A.21})$$

Now if $T \neq 0$, Equation A.21 implies that

$$T \equiv -\left(\frac{\ell p_0}{M_c \beta A a}\right) \exp(-\lambda t),$$

so that

$$\dot{T} \equiv -\left(\frac{\ell p_0 \lambda}{M_c \beta A a}\right) \exp(-\lambda t) \quad (\text{A.22})$$

If Equation A.22 is inserted into Equation A.3, it is easily seen that

$$Q \equiv 0,$$

which in turn implies that

$$\dot{Q} \equiv 0.$$

Equation A.1 thus requires

$$C \equiv 0.$$

Again a zero-power-level solution with decreasing temperature is obtained. In addition, if $T \neq 0$, Equation A.20 requires

$$p_0 \equiv 0,$$

so that the problem must be non-normal in this case, too.

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