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OPTIMUM FILTERS WITH TIME WIDTH CONSTRAINTS
FOR LIQUID ARGON TOTAL-ABSORPTION DETECTORS*

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Abstract

Optimum filter responses are found for triangular current input pulses occurring in liquid argon ionization chambers used as total absorption detectors. The filters considered are subject to the following constraints: finite width of the output pulse having a prescribed ratio to the width of the triangular input current pulse and zero area of a bipolar antisymmetrical pulse or of a three lobe pulse, as required for high event rates. The feasibility of pulse shaping giving an output equal to, or shorter than, the input one is demonstrated. It is shown that the signal-to-noise ratio remains constant for the chamber interelectrode gap which gives an input pulse width (i.e., electron drift time) greater than one third of the required output pulse width (i.e., resolving time).

Introduction

Electronic noise and electrode configuration in liquid argon total absorption detectors have been studied in Ref. 1. It has been shown that the lower limit for electronic noise is determined only by the chamber volume, electron drift velocity, event resolving time and the unity-gain frequency of the field effect transistor. This limit is approximately the same in the case of equal output waveforms for multiple-plate ionization chambers and large gap ionization chambers where the electron drift is much longer than the event resolving time. In all cases it has been assumed that the ionization chamber capacitance is matched to the amplifier capacitance regardless of the method used to achieve this condition. However, the behavior of multiple chambers in which the drift time is approximately equal to the resolving time (i.e., output pulse width) showed a large increase of noise, due to the assumption in the analysis of a weighting function of a fixed shape and the width equal to the difference between the resolving time λ_m and the electron drift time t_d .

This filter is satisfactory for nearly ballistic excitation (electron drift time less than one half of the resolving time). For the region of significant ballistic deficit and especially when the electron drift time is comparable with resolving time (output response) one needs a general treatment based on the search of an optimum filter.

In this paper we investigate the behavior of the optimum ENC_s as a function of the ratio α of the drift time t_d and of the finite output pulse width λ_m for the cases of unipolar output response $w_A(t)$, for zero

area three-lobe response $w_C(t)$ and for bipolar antisymmetrical response $w_B(t)$.

First, we shall consider (Fig. 3) the case of unipolar output $w_A(t)$ because it leads to a simpler treatment: we shall give thereafter the results for three lobe pulse $w_C(t)$ and bipolar output pulse $w_B(t)$. In all cases, we shall see that ENC_s has at least one minimum for t_d/λ_m between 0 and 1; however, ENC_s increases slightly with respect to the minimum, moving to the right and, for $t_d/\lambda_m = 1$, ENC_s is finite and practically equal to the ENC_s obtained with large gaps: $t_d/\lambda_m \gg 1$ (Figs. 4, 9, and 13).

Representation of Signals and Noise Power

Let us consider (Fig. 1) the triangular waveform of the input current:

$$\frac{I_0}{p} \left[1 - (1 - e^{-pt_d}) \frac{1}{pt_d} \right] = Q_s \frac{2}{pt_d} \left[1 - \frac{1 - e^{-pt_d}}{pt_d} \right] \quad (1)$$

where Q_s is the pulse charge $I_0 t_d / 2 = Q_s$.

We shall synthesize the responses of the optimum filters $h_A(t)$, $h_B(t)$, $h_C(t)$ for the input current (1) by means of several steps: first let us shape the input pulse (1) by means of a filter D , into a δ -pulse $Q_s \delta(t)$. From Eq. (1) we obtain the transfer function of filter D ,

$$D(p) = \frac{1}{2} \frac{pt_d}{1 - \frac{1 - e^{-pt_d}}{pt_d}} \quad (2)$$

This transfer function can be implemented by the feedback arrangement shown in Fig. 1.

The series noise of the preamplifier can be thought of as an input current noise of bilateral spectral density,

$$4C_d C_a e_n^2 w^2 / 2 \quad (3)$$

where e_n is the rms equivalent series noise voltage (i.e., unilateral density) of the field effect transistor ($V/\text{Hz}^{1/2}$). The input noise current is a current

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formed by the superposition of doublets in the Campbell's theorem representation,² and it produces at the output of filter D a noise having the bilateral spectral density given by,

$$\frac{dN^2(\omega)}{df} = C_d C_a \frac{e_n^2}{2} \frac{\omega^6 t_d^4}{2(1 - \cos \omega t_d) + \omega^2 t_d^2 - 2\omega t_d \sin \omega t_d} \quad (4)$$

As a second step, let us now look for a filter W to be cascaded to D giving a δ response of finite width λ_m , expressed within the interval $0 \rightarrow \lambda_m$, by

$$w(t) = \sum_{n=-\infty}^{\infty} F_n e^{i2\pi n t \lambda_m} \quad (n \text{ integer}) \quad (5)$$

Its Laplace transform is given by,

$$W(p) = (1 - e^{-p\lambda_m}) \sum_{n=-\infty}^{\infty} F_n \frac{1}{p - i2\pi n/\lambda_m} \quad (6)$$

The noise at the filter output is, then,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |w(i\omega)|^2 \frac{dN^2(\omega)}{df} d\omega \quad (7)$$

We can express $|w(i\omega)|^2$ as a double sum,

$$|W(i\omega)|^2 = 2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F_m F_n (1 - \cos \omega \lambda_m) \frac{1}{\omega - \frac{2\pi m}{\lambda_m}} \frac{1}{\omega - \frac{2\pi n}{\lambda_m}} \quad (8)$$

By using this expansion in Eq. (7) we note, upon inspection of Eq. (4), that the individual terms are divergent. This fact suggests to introduce an additional low-pass filtering term in order to proceed with filter synthesis.

We can design the output waveforms $w_A(t)$, $w_B(t)$, $w_C(t)$ indirectly, by means of their second derivatives, by defining first a filter L, as shown in Fig. 2. This is obtained simply by following the filter D by a double integrator. In the same figure the δ -response of the filter L is plotted:

$$L(t) = \frac{t_d}{2} e^{-\frac{t}{t_d}} \sum_{k=0}^{\infty} \frac{e^{-k}}{k!} \left(k - \frac{t}{t_d}\right)^k 1\left(\frac{t}{t_d} - k\right) \quad (9)$$

Filter L produces, as output waveform, the ramp function $Q_s t \mathbf{1}(t)$ when fed by the input triangular pulse, Eq. (1). At its output the bilateral noise density

$$\frac{e_n^2 C_d C_a}{2} \frac{\omega^2 t_d^4}{2(1 - \cos \omega t_d) + \omega^2 t_d^2 - 2\omega t_d \sin \omega t_d} \quad (10)$$

is present, due to the input noise current, Eq. (3).

The next step will be that of finding suitable

transfer functions $G(p)$, cascaded to L, as shown in Fig. 3(b), in order to produce the optimum output pulses $w_A(t)$, or $w_B(t)$, or $w_C(t)$.

The most general filter, having a δ -response of total width λ_m , can be defined by a transfer function,

$$G(p) = (1 - e^{-p\lambda_m}) \sum_{n=-\infty}^{\infty} G_n \frac{1}{p - i \frac{2\pi n}{\lambda_m}} \quad (11)$$

But the transfer function $G(p)$ excited by a ramp function does not give a response confined to the width λ_m without additional conditions. In fact, the output pulse is given, in the Laplace transform domain by,

$$W(p) = \frac{Q_s}{p^2} (1 - e^{-p\lambda_m}) \sum_{n=-\infty}^{\infty} G_n \frac{1}{p - i \frac{2\pi n}{\lambda_m}} = Q_s (1 - e^{-p\lambda_m}) \left[\sum_{\substack{n \neq 0 \\ n \neq 0}} \frac{G_n}{\frac{4\pi^2 n^2}{\lambda_m^2}} \left(\frac{-1}{p - i \frac{2\pi n}{\lambda_m}} + \frac{1}{p} \right) + \sum_{n \neq 0} \frac{i G_n}{\frac{2\pi n}{\lambda_m}} \frac{1}{p^2} + \frac{G_0}{p^3} \right] \quad (12)$$

It is apparent from this expression that, if $w(t)$ has to be zero outside of the interval $0 \rightarrow \lambda_m$, we must add the conditions,

$$G_0 = 0 \quad \text{and} \quad \sum_{\substack{n \neq 0 \\ n \neq 0}} \frac{G_n}{n} = 0 \quad (13)$$

The expression for $W(p)$ becomes,

$$W(p) = Q_s (1 - e^{-p\lambda_m}) \sum_{\substack{n \neq 0 \\ n \neq 0}} \frac{G_n}{\frac{4\pi^2 n^2}{\lambda_m^2}} \left(\frac{1}{p} - \frac{1}{p - i \frac{2\pi n}{\lambda_m}} \right) \quad (14)$$

Comparing Eq. (14) with Eq. (5), we can evaluate the terms of the Fourier representation of $w(t)$ as functions of all G_n ,

$$F_0 = \frac{\lambda_m^2}{4\pi^2} \sum_{\substack{n \neq 0 \\ n \neq 0}} \frac{G_n}{n^2}, \quad F_n = \frac{-\lambda_m^2}{4\pi^2 n^2} G_n \quad (15)$$

In time domain, the output pulse can be written as follows:

$$w(t) = Q_s \left[\sum_{\substack{n \neq 0 \\ n \neq 0}} \frac{G_n}{\frac{4\pi^2 n^2}{\lambda_m^2}} \left(1 - e^{i \frac{2\pi n t}{\lambda_m}} \right) \right] [1(t) - 1(t - \lambda_m)] \quad (16)$$

We note that $\{w(t) = 0\}$ for $t \rightarrow 0_+$ and $t \rightarrow \lambda_m^-$.

These properties of $w(t)$ do not represent a loss of generality because a discontinuity in $w(t)$ or $w'(t)$ at $t = 0$ or $t = \lambda_m$ would imply (with the noise spectrum given by Eq. (4)) at the input of W an infinite noise amplitude at the filter output. These continuity requirements are, therefore, compulsory for $w(t)$.

The output noise will be given, recalling Eq. (10) and (11), by

$$N^2 = \frac{e_n^2 C_d C_a}{2} \int_{-\infty}^{\infty} \left\{ \frac{\omega^2 t_d^4 2(1-\cos\omega\lambda_m)}{2(1-\cos\omega t_d) + \omega^2 t_d^2 - 2\omega t_d \sin\omega t_d} \right. \\ \left. \cdot \left| \sum_{n \neq 0} G_n \frac{1}{i(\omega - \frac{2\pi n}{\lambda_m})} \right|^2 \frac{d\omega}{2\pi} \right\} \quad (17)$$

By inverting the order of sums and integral operations, we obtain,

$$N^2 = \frac{e_n^2 C_d C_a}{2\pi} \sum_{n \neq 0} \sum_{m \neq 0} G_n \bar{G}_m \int_{-\infty}^{\infty} \left\{ \frac{1-\cos\omega\lambda_m}{(\omega - \frac{2\pi n}{\lambda_m})(\omega - \frac{2\pi m}{\lambda_m})} \right. \\ \left. \cdot \frac{\omega^2 t_d^4}{2(1-\cos\omega t_d) + \omega^2 t_d^2 - 2\omega t_d \sin\omega t_d} \right\} d\omega \quad (18)$$

Introducing the following notation,

$$\alpha = \frac{t_d}{\lambda_m} \cdot \kappa = \omega \lambda_m \quad (19)$$

we can write,

$$N^2 = \frac{e_n^2 C_d C_a \lambda_m^3 \alpha^2}{2\pi} \sum_{n \neq 0} \sum_{m \neq 0} G_n \bar{G}_m \int_{-\infty}^{\infty} \left\{ \frac{(1-\cos\kappa)}{(\kappa - 2\pi n)(\kappa - 2\pi m)} \right. \\ \left. \cdot \frac{\alpha^2 \kappa^2}{2(1-\cos\alpha\kappa) + \alpha^2 \kappa^2 - 2\alpha\kappa \sin\alpha\kappa} d\kappa \right\} \quad (20)$$

and finally,

$$N^2 = \frac{e_n^2 C_d C_a \lambda_m^3 \alpha^2}{2\pi} \sum_{n \neq 0} \sum_{m \neq 0} G_n \bar{G}_m W_{nm}(\alpha) \quad (21)$$

where

$$W_{nm}(\alpha) = \int_{-\infty}^{\infty} \frac{\alpha^2 \kappa^2}{2(1-\cos\alpha\kappa) + \alpha^2 \kappa^2 - 2\alpha\kappa \sin\alpha\kappa} \frac{(1-\cos\kappa)}{(\kappa - 2\pi n)(\kappa - 2\pi m)} d\kappa \quad (22)$$

We note that $W_{nm}(\alpha)$ is now finite for every integer value of m and n . Let us now determine $G(p)$ for special uses of interest here which lead, respectively, to the optimum output waveforms $w_A(t)$, $w_C(t)$, $w_B(t)$.

Optimum Unipolar Output Waveform $w_A(t)$

Imposing symmetry about $\lambda_m/2$ we have,

$$G_n = G_{-n} \quad (23)$$

Further, we note that the second condition from Eq. (13) is included in Eq. (23).

From Eq. (16), the maximum output, which occurs for $\frac{t}{\lambda_m} = 0.5$ is,

$$w_A\left(\frac{\lambda_m}{2}\right) = \frac{Q_s \lambda_m^2}{4\pi^2} \sum_{n \neq 0} \frac{G_n}{n^2} (1 - (-1)^n) \quad (24)$$

It is convenient to put the square of this expression in the form,

$$\left(w_A\left(\frac{\lambda_m}{2}\right)\right)^2 = \frac{Q_s^2 \lambda_m^4}{16\pi^4} \sum_{n \neq 0} \sum_{m \neq 0} \frac{G_n G_m}{n^2 m^2} (1 - (-1)^n)(1 - (-1)^m) \quad (25)$$

From Eqs. (21) and (25), the signal-to-noise ratio squared is,

$$\frac{\left(w_A\left(\frac{\lambda_m}{2}\right)\right)^2}{N^2} = \frac{Q_s^2 \lambda_m^2}{8\pi^3 \alpha^2} \frac{1}{e_n^2 C_d C_a}$$

$$\frac{\sum_{n \neq 0} \sum_{m \neq 0} \frac{G_n G_m}{n^2 m^2} (1 - (-1)^n)(1 - (-1)^m)}{\sum_{n \neq 0} \sum_{m \neq 0} G_n G_m W_{nm}(\alpha)} \quad (26)$$

and, calling μ the ratio

$$\mu = \frac{\sum_{n \neq 0} \sum_{m \neq 0} \frac{G_n G_m}{n^2 m^2} (1 - (-1)^n)(1 - (-1)^m)}{\sum_{n \neq 0} \sum_{m \neq 0} G_n G_m W_{nm}(\alpha)} \quad (27)$$

the equivalent noise charge becomes,

$$ENC_s = \frac{2.2^{1/2} \pi^{3/2} \alpha e_n C_d^{1/2} C_a^{1/2}}{\lambda_m^{1/2} \mu^{1/2}} \quad (28)$$

Introducing, with notations of Ref. 1 the volume V of the ionization chamber, the drift velocity v_e of the electrons, and the dielectric constant ϵ , we get,

$$C_d = \frac{\epsilon V}{C_d^2 v_e^2} \quad (29)$$

and the equivalent noise charge can be written:

$$ENC_s = \frac{2.2^{1/2} \pi^{3/2}}{\mu^{1/2}} \frac{e_n C_a^{1/2} \epsilon^{1/2} v_e^{1/2}}{\lambda_m^{3/2} v_e} = \zeta \frac{e_n C_a^{1/2} \epsilon^{1/2} v_e^{1/2}}{\lambda_m^{3/2} v_e} \quad (30)$$

We maximize μ as given by Eq. (27) by an optimum choice of the G_m . It is immediately seen, by standard variational methods that, if

$$\sum_{m \neq 0} G_m \frac{(1-(-1)^n)(1-(-1)^m)}{n^2 m^2} = \mu \sum_{m \neq 0} G_m W_{nm}(\alpha) \quad (31)$$

for every integer n , different from 0, the ratio (27) is stationary for variations of the G_m .

Recalling that $G_m = G_{-m}$, according to Eq. (23), the set of equations (31) can be written,

$$\sum_{m=1}^{\infty} \left[\frac{2(1-(-1)^n)(1-(-1)^m)}{m^2 n^2} - \mu(W_{nm} + W_{n,-m}) \right] G_m = 0 \quad (32)$$

Noting that the coefficient in brackets is invariant for a change of n into $-n$, we can symmetrize Eq. (32) with respect to m and n and write,

$$\sum_{m=1}^{\infty} \left[\frac{4(1-(-1)^m)(1-(-1)^n)}{m^2 n^2} - \mu(W_{mn} + W_{-mn} + W_{m,-n} + W_{-m,-n}) \right] G_m = 0 \quad (33)$$

We define Q_{mn} as,

$$Q_{mn} = W_{mn} + W_{-mn} + W_{m,-n} + W_{-m,-n} \quad (34)$$

Then the set of equations (31) becomes,

$$\sum_{m=1}^{\infty} \left[\frac{4(1-(-1)^m)(1-(-1)^n)}{m^2 n^2} - \mu Q_{mn} \right] G_m = 0 \quad (35)$$

$n = 1, 2, \dots$

where $Q_{mn}(\alpha)$ is given, according to Eqs. (22) and (34), by

$$Q_{mn}(\alpha) = \frac{4}{\alpha^2} \int_{-\infty}^{\infty} \left\{ \frac{1 - \cos \kappa}{(\kappa^2 - 4\pi^2 m^2)(\kappa^2 - 4\pi^2 n^2)} \cdot \frac{\alpha^4 \kappa^4}{2(1 - \cos \alpha \kappa) + \alpha^2 \kappa^2 - 2\alpha \kappa \sin \alpha \kappa} d\kappa \right\} \quad (36)$$

$Q_{mn}(\alpha)$ is tabulated for $\alpha = 0.6$ and $\alpha = 1$ in Table I. In the same table the eigenvalues $\mu(\alpha)$ for which the system of Eq. (35) has a nontrivial solution and the corresponding eigenvectors $G_m(\alpha)$ are given. The $F_m(\alpha)$, according to Eq. (15), are also tabulated.

In Fig. 4, the coefficient $\zeta = 15.75/\mu^{1/2}$, to be used in Eq. (30) in order to calculate the ENC_s , is plotted as a function of α . The asymptotic value 9.79 for $\alpha \gg 1$ can be compared with $8\sqrt{2} = 11.31$ as given by Eq. (17a) of Ref. 1. In Table I and Fig. 4, the number of harmonics considered allows to estimate $\mu(\alpha)$ with an accuracy better of one per cent. $\zeta(\alpha)$ shows a minimum for $\alpha = 0.6$ and it is independent of α for $\alpha > 1$. For $\alpha \rightarrow \infty$, Eq. (22) gives: $Q_{mn} = \delta_{mn} \cdot 2\pi$, where δ_{mn} is the Kronecker symbol; the eigenvalue μ is $8/\pi(1+1/81+1/625+\dots) = 2.584$, and the corresponding eigenvector is $G_1=1, G_3=1/9, G_5=1/25, G_7=1/49, \dots$

$G_2=G_4=G_6=G_8 \dots = 0$. This asymptotic filter (for $\alpha \gg 1$) corresponds, as expected, to a $g(t)$ of triangular shape as shown in Fig. 5. The output pulse $w_A(t)$ is given by,

$$w_A(t) = 12 \left(\frac{t}{\lambda_m} \right)^2 - 16 \left(\frac{t}{\lambda_m} \right)^3 \quad \text{for } 0 \leq t \leq 1/2 \lambda_m \quad (37)$$

and is symmetrical about $\frac{\lambda_m}{2}$ as plotted in Fig. 5 together with the overall weighting function $h_A(t) = \mathcal{L}(t) * g_A(t)$ of the whole filter, H_A . In Figs. 6 and 7, $g_A(t)$, $w_A(t)$ and weighting function of the whole filter H_A are plotted for $\alpha = 1$ and for $\alpha = 0.60$.

In Fig. 8, the outputs of the whole filter (designed for $\alpha = 1$) are plotted for triangular input current pulses of equal charge but shorter or longer by a factor 0.9 or 1.1 from the expected $t_d = \lambda_m$. One may note the sensitivity of the peak amplitude to the drift time t_d , while the amplitude and width of the tails give an estimate of the sensitivity, as far as the tails are concerned, to an input shape different from the expected one. These data give, indirectly, an estimate of the sensitivity (for an ideal input pulse) to inaccuracies in the response of the designed filter.

As expected, the peak amplitude sensitivity is quite high: 0.70 (sensitivity 1 is peculiar of $\alpha \rightarrow \infty$, while the sensitivity 0, for zero ballistic deficit, is obtained for $\alpha \rightarrow 0$). For $\alpha = 0.6$, sensitivity is 0.46.

Optimum Three Lobe Output Waveform $w_C(t)$

We have to add the condition of zero area for $w_C(t)$ to the requirements of Eq. (13) and the symmetry condition of Eq. (23). According to Eq. (15), this condition is,

$$F_0 = \sum_{m=1}^{\infty} \frac{G_m}{m^2} = \sum_{m=1}^{\infty} F_m = 0 \quad (38)$$

The maximum of $(w_C(t))^2$, occurring at $\frac{t}{\lambda_m} = 0.5$, is according to Eq. (16),

$$\left[w_C\left(\frac{\lambda_m}{2}\right) \right]^2 = \frac{\lambda_m^4}{16\pi^4} Q_s^2 \sum_{m \neq 0} \sum_{n \neq 0} \frac{G_n G_m}{m^2 n^2} (-1)^{m+n} \quad (39)$$

From Eqs. (21) and (39), the signal-to-noise ratio squared is,

$$\frac{\left(w_C\left(\frac{\lambda_m}{2}\right) \right)^2}{N^2} = \frac{Q_s^2 \lambda_m^4}{8\pi^3 \alpha^2} \frac{1}{e_n^2 C_d C_a} \frac{\sum_{n \neq 0} \sum_{m \neq 0} \frac{G_n G_m}{m^2 n^2} (-1)^{m+n}}{\sum_{n \neq 0} \sum_{m \neq 0} G_n G_m W_{nm}} \quad (40)$$

The ENC_s is given by Eq. (30), having defined as μ the ratio,

$$\mu = 4 \frac{\sum_{n \neq 0} \sum_{m \neq 0} \frac{G_n G_m}{m^2 n^2} (-1)^{m+n}}{\sum_{n \neq 0} \sum_{m \neq 0} G_n G_m Q_{nm}} \quad (41)$$

The optimum waveform $w_C(t)$ is found again by maximizing Eq. (41) by variations of the G_m , taking into account the constraint, Eq. (38).

In this equation let us substitute G_n as a function of all harmonics, G_r of lower order exploiting Eq. (38). We get,

$$G_n = -n^2 \sum_{r=1}^{n-1} \frac{G_r}{r^2} \quad (42)$$

and Eq. (41) becomes,

$$\mu = \frac{4 \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \frac{G_i G_k}{i^2 k^2} ((-1)^n - (-1)^i) ((-1)^n - (-1)^k)}{\sum_{i=1}^{n-1} \sum_{k=1}^{n-1} G_i G_k \left(Q_{ik} + \frac{n^4}{i^2 k^2} Q_{nn} - \frac{n^2}{k^2} Q_{in} - \frac{n^2}{i^2} Q_{kn} \right)} \quad (43)$$

μ is maximum for the set of $G_1, G_2, \dots, G_{n-1}$, which is a nontrivial solution of the following homogeneous system of $n-1$ equations,

$$\sum_{k=1}^{n-1} \left\{ \frac{((-1)^n - (-1)^k) ((-1)^n - (-1)^i}{i^2 k^2} - \frac{n^4}{i^2 k^2} Q_{nn} - \frac{n^2}{k^2} Q_{in} - \frac{n^2}{i^2} Q_{kn} \right\} G_k = 0 \quad (44)$$

$$i = 1, 2, \dots, n-1$$

G_n is given by Eq. (42), once the system of Eq. (44) has been solved by finding the eigenvalue μ and the corresponding eigenvector $G_1, G_2, \dots, G_{n-1}$.

In Table II, the eigenvalues and the eigenvectors are given for $\alpha = 0.36$ and $\alpha = 1$, and in Fig. 9 the ζ value, used in Eq. (30) in order to calculate the ENC, is plotted as a function of α . ζ shows a minimum for $\alpha = 0.36$ and it is independent of α for $\alpha > 1$. The number of harmonics considered allows to estimate $\zeta(\alpha)$ with an accuracy of better than one per cent. In Figs. 10 and 11, $w_C(t)$ and $h_C(t) = \mathcal{L}(t) * g_C(t)$ are plotted for $\alpha = 1$ and $\alpha = 0.36$. Figure 12 gives the output responses of the filter H_C , designed for $\alpha = 1$, for input pulse width $\alpha = 0.9$ and $\alpha = 1.1$. Sensitivities are 0.93 and 0.68, respectively, for filter designed for $\alpha = 1$ and $\alpha = 0.36$.

Optimum Bipolar Antisymmetrical Output Waveform $w_B(t)$

We have to add the requirement of antisymmetry* with respect to $\lambda_m/2$, to the requirements of Eq. (13), which is expressed by

$$G_n = -G_{-n} \quad (45)$$

Equation (16) becomes, then,

$$w_B(t) = Q_s \sum_{n=1}^{\infty} \frac{G_n}{\frac{4\pi^2 n^2}{\lambda_m^2}} \left(e^{i \frac{2\pi n}{\lambda_m} t} - e^{-i \frac{2\pi n}{\lambda_m} t} \right) \quad (46)$$

and by introducing

$$S_n = -iG_n \quad (47)$$

the waveform $B(t)$ can be written,

$$w_B(t) = \frac{2Q_s}{4\pi^2} \sum_{n=1}^{\infty} \frac{S_n}{n^2} \sin \frac{2\pi n}{\lambda_m} t \quad (48)$$

where all S_n are real. By using Eq. (48), the maximum of the output pulse squared can be written as:

* One could synthesize optimum zero-area bipolar filters without the requirement of antisymmetry and this would lead to an output waveform similar to Deighton's filters.³ This development has not been dealt with in this paper.

$$[w_B(t)]^2 = \frac{4Q_s^2 \lambda_m^4}{16\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_m S_n}{m^2 n^2} \sin\left(\frac{2\pi}{\lambda_m} m t_{\max}\right) \sin\left(\frac{2\pi}{\lambda_m} n t_{\max}\right), \quad (49)$$

where $t_{\max}$ has to be found.

The noise squared, recalling Eqs. (10), (11), and the definition of the S_n , can be written,

$$N^2 = \frac{e^2 C_d C_a \lambda_m^3 \alpha^2}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S_m S_n W_{mn}, \quad (50)$$

where W_{mn} is given by Eq. (22). By Eq. (45), the sums can be extended only to positive integers, and consequently N^2 becomes,

$$N^2 = \frac{e^2 C_d C_a \lambda_m^3 \alpha^2}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} R_{mn} S_m S_n, \quad (51)$$

where R_{mn} is,

$$R_{mn} = W_{mn} + W_{-m, -n} - W_{-m, n} - W_{m, -n} \quad (52)$$

By means of Eq. (22), R_{mn} can be written

$$R_{mn}(\alpha) = 4 \int_{-\infty}^{\infty} \left\{ \frac{(1 - \cos \kappa) mn}{(\kappa^2 - 4\pi^2 n^2)(\kappa^2 - 4\pi^2 m^2)} \cdot \frac{4\pi^2 \alpha^2 \kappa^2 d\kappa}{2(1 - \cos \kappa) + \alpha^2 \kappa^2 - 2\alpha \kappa \sin \kappa} \right\} \quad (53)$$

The signal-to-noise ratio can be written,

$$\frac{[w_B(t_{\max})]^2}{N^2} = \frac{Q_s^2 \lambda_m}{8\pi^3 e^2 C_d C_a \alpha^2} \mu \quad (54)$$

where μ is given by,

$$\mu = \frac{4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{S_n S_m}{n^2 m^2} \sin\left(\frac{2\pi}{\lambda_m} n t_{\max}\right) \sin\left(\frac{2\pi}{\lambda_m} m t_{\max}\right)}{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} S_n S_m R_{mn}} \quad (55)$$

We introduce for brevity the notation,

$$\sin\left(\frac{2\pi}{\lambda_m} n t_{\max}\right) = \beta_n \quad (56)$$

If only n harmonics are considered, Eq. (55) can be written,

$$\mu = \frac{4 \sum_{i=1}^n \sum_{k=1}^n \frac{S_i S_k}{i^2 k^2} \beta_i \beta_k}{\sum_{i=1}^n \sum_{k=1}^n S_i S_k R_{ik}} \quad (57)$$

Expressing the n th harmonic as a function of all the lower order ones by means of Eq. (13) (recalling Eq. (47)), we have,

$$S_n = -n \sum_{k=1}^{n-1} \frac{S_k}{k}, \quad (58)$$

and Eq. (57) becomes,

$$\mu = \frac{4 \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} S_i S_k \left(\frac{\beta_i \beta_k}{i^2 k^2} - \frac{\beta_i \beta_n}{i^2 k n} - \frac{\beta_k \beta_n}{k^2 i n} + \frac{\beta_n^2}{i k n^2} \right)}{\sum_{i=1}^{n-1} \sum_{k=1}^{n-1} S_i S_k \left(R_{ik} - \frac{n}{k} R_{in} - \frac{n}{i} R_{kn} + \frac{n^2}{i k} R_{nn} \right)} \quad (59)$$

In order to maximize μ , we introduce a tentative value for $t_{\max}/\lambda_m$ and calculate the set of β_k according to Eq. (56). μ is maximum when Eq. (59) is stationary for a variation of S_k . This happens when the S_k are a nontrivial solution of the following system of $n-1$ equations in $S_1, S_2, \dots, S_{n-1}$,

$$\sum_{k=1}^{n-1} \left\{ \left(\frac{\beta_i \beta_k}{i^2 k^2} - \frac{\beta_i \beta_n}{i^2 k n} - \frac{\beta_k \beta_n}{k^2 i n} + \frac{\beta_n^2}{i k n^2} \right) - \frac{\mu}{4} \left(R_{ik} - \frac{n}{k} R_{in} - \frac{n}{i} R_{kn} + \frac{n^2}{i k} R_{nn} \right) \right\} S_k = 0 \quad (60)$$

$i = 1, 2, \dots, n-1$

To solve this system of equations, the determinant is equated to zero and the eigenvalue μ is found, as well as the corresponding eigenvector $S_1, S_2, \dots, S_{n-1}$. (S_n is calculated by means of Eq. (58).) Using Eq. (48), the waveform is calculated, in particular the location $t_{\max}/\lambda_m$ of the maximum, and so a new set of β_k is calculated and the whole process is iterated. The process is rapidly convergent and few iterations are generally sufficient.

In Table III the coefficients R_{mn} , the eigenvalues and the eigenvectors are given for several values of α , and in Fig. 13 the ζ value is plotted. The asymptotic value 30, for $\alpha > 1$, can be compared with 36 as given by Eq. (17b) of Ref. 1 and the value 26.3, for $\alpha_{\text{opt}} = 0.42$, with the value 33.84 as given by Eq. (5) of Ref. 1 for $\alpha_{\text{opt}} = 0.33$. For Table III and Fig. 14, the number of

harmonics considered allows to estimate $\zeta(\alpha)$ with an accuracy better than one per cent. In Figs. 14 and 15, $w_B(t)$ and $h_B(t) = g_B(t) * i(t)$ are plotted respectively for $\alpha = 1$, $\alpha = 0.42$.

Figure 16 gives the output responses of the filter H_B designed for $\alpha = 1$, for input pulse width $\alpha = 0.9$ and $\alpha = 1.1$. Sensitivities are 0.85 and 0.57, respectively, for $\alpha = 1$ and $\alpha = 0.42$.

Conclusions

It has been shown that the signal-to-noise ratio remains constant for the chamber interelectrode gap which gives an input pulse width (i.e., electron drift time) greater than about one third of the required output pulse width (i.e., resolving time). Sensitivity to pulse shape, which could arise from changes in drift times, increases from 0 to 1, passing from $\alpha = 0$ to $\alpha = \infty$. Sensitivity is given for α_{opt} and $\alpha = 1$ for filters H_A , H_B , H_C . The ratio of the tail amplitude to the peak amplitude is roughly equal, for $\alpha = 1$, to the relative deviation of actual drift time from the design's nominal one. The triangular input current assumed is well representative of the current pulse delivered by all types of liquid argon total absorption detectors (either sampling or nonsampling) and, therefore, the results are useful for filter design in all these cases.

Filter D can be easily implemented according to Fig. 1. In order to avoid instabilities and overload due to pileup (before the final differentiator), an exponentially decaying pulse is produced instead of a current step by suitably lowering the feedback loop gain. The final differentiator is substituted by a zero pole cancellation differentiator. Thus an exponentially decaying waveform with a time constant much shorter than the width of the input triangular pulse is obtained at the output.

This unit inserted between a conventional charge preamplifier and a conventional gaussian shaping amplifier allows us to approximate easily the calculated filters.

References

1. V. Radeka; High Resolution Liquid Argon Total Absorption Detectors: Electronic Noise and Electrode Configuration. IEEE Trans. Nucl. Sci., NS-24, no. 1, 293-298 (1977).
2. V. Radeka and N. Karlovac; Time Variant Filters for High Rate Pulse Amplitude Spectrometry Semiconductor Nuclear-Particle Detectors and Circuits. Proceedings of the Conference of Gatlinburg. Publication 1593. National Academy of Sciences, 1969.
3. M. O. Deighton; Minimum-Noise Filter with Good Low-Frequency Rejection. Proceedings Ispra Nuclear Electronics Symposium, May 1969, Euratom.

TABLE I

 $Q_{nm}(0.6)$

5.02413,+00	2.33254,-01	1.04945,-01	-3.77540,-02	5.28985,-02	-2.17091,-02
2.33254,-01	6.92882,+00	2.79590,-02	-1.05507,-01	2.71334,-02	-4.56121,-02
1.04945,-01	2.79590,-02	5.85671,+00	-2.00839,-01	2.68753,-02	-5.92365,-02
-3.77540,-02	-1.05507,-01	-2.00839,-01	6.44814,+00	2.03984,-01	-9.67553,-03
5.28985,-02	2.71334,-02	2.68753,-02	2.03984,-01	6.28443,+00	-1.84490,-01
-2.17091,-02	-4.56121,-02	-5.92365,-02	-9.67553,-03	-1.84490,-01	6.13154,+00

 $\mu(0.6)=3.21440$ $\zeta(0.6)=8.78$

$G_1 = 1.00000,+00$
 $G_2 = -3.39315,-02$
 $G_3 = 7.77611,-02$
 $G_4 = 6.99204,-03$
 $G_5 = 2.32945,-02$
 $G_6 = 4.75132,-03$

$F_0 = -1.0016579$
 $F_1 = 1.00000,+00$
 $F_2 = -8.48289,-03$
 $F_3 = 8.64012,-03$
 $F_4 = 4.37003,-04$
 $F_5 = 9.31779,-04$
 $F_6 = 1.31981,-04$

 $Q_{nm}(1)$

6.27041,+00	-1.27346,-02	-1.27368,-02	-1.27397,-02	-1.27435,-02	-1.27484,-02
-1.27346,-02	6.27040,+00	-1.27382,-02	-1.27411,-02	-1.27448,-02	-1.27497,-02
-1.27368,-02	-1.27382,-02	6.27040,+00	-1.27432,-02	-1.27469,-02	-1.27519,-02
-1.27397,-02	-1.27411,-02	-1.27432,-02	6.27040,+00	-1.27499,-02	-1.27549,-02
-1.27435,-02	-1.27448,-02	-1.27469,-02	-1.27499,-02	6.27043,+00	-1.27591,-02
-1.27484,-02	-1.27497,-02	-1.27519,-02	-1.27549,-02	-1.27591,-02	6.27038,+00

 $\mu(1)=2.58893$ $\zeta(1)=9.79$

$G_1 = 1.00000,+00$
 $G_2 = 2.35638,-03$
 $G_3 = 1.13206,-01$
 $G_4 = 2.35732,-03$
 $G_5 = 4.22635,-02$
 $G_6 = 2.35892,-03$

$F_0 = -1.015070$
 $F_1 = 1.00000,+00$
 $F_2 = 5.89095,-04$
 $F_3 = 1.25785,-02$
 $F_4 = 1.47332,-04$
 $F_5 = 1.69054,-03$
 $F_6 = 6.55255,-05$

TABLE II

 $Q_{mn}(0.36)$

7.11176,+00	5.99263,-01	4.17201,-01	-1.23327,-02	8.63438,-02	6.55755,-02
5.99263,-01	4.67036,+00	3.07962,-01	-1.65234,-01	1.30693,-02	1.55421,-02
4.17201,-01	3.07962,-01	6.95460,+00	-5.03236,-01	-7.90861,-02	-3.86105,-02
-1.23327,-02	-1.65234,-01	-5.03236,-01	6.46971,+00	2.50804,-01	1.24007,-01
8.63438,-02	1.30693,-02	-7.90861,-02	2.50804,-01	5.63745,+00	2.02628,-02
6.55755,-02	1.55421,-02	-3.86105,-02	1.24007,-01	2.02628,-02	6.80214,+00

 $\mu(0.36)=0.216338$ $\zeta(0.36)=33.86$

$G_1 = 1.00000,+00$
 $G_2 = -3.86311,+00$
 $G_3 = 1.54022,-01$
 $G_4 = -7.50064,-01$
 $G_5 = 7.51727,-02$
 $G_6 = -2.68650,-01$

$F_1 = 1.00000,+00$
 $F_2 = -9.65779,-01$
 $F_3 = 1.71136,-02$
 $F_4 = -4.68790,-02$
 $F_5 = 3.00691,-03$
 $F_6 = 7.46251,-03$

 $Q_{mn}(1)$ (See Table I)

$G_1 = 1.00000,+00$
 $G_2 = -3.77856,+00$
 $G_3 = 1.28780,-01$
 $G_4 = -9.51306,-01$
 $G_5 = 4.17458,-02$
 $G_6 = -4.27753,-01$

 $\mu(1)=0.158$ $\zeta(1)=39.62$

$F_1 = 1.00000,+00$
 $F_2 = -9.44640,-01$
 $F_3 = 1.43089,-02$
 $F_4 = -5.94567,-02$
 $F_5 = 1.66983,-03$
 $F_6 = -1.18820,-02$

TABLE III

 $R_{mn}(0.42)$

1.24924,+01	3.58717,+00	2.19536,+00	1.62968,+00	1.31248,+00	1.07449,+00
3.58717,+00	6.37697,+00	5.68257,-01	4.87026,-01	4.37666,-01	3.32657,-01
2.19536,+00	5.68257,-01	7.82415,+00	4.40555,-01	4.17703,-01	2.75691,-01
1.62968,+00	4.87026,-01	4.40555,-01	5.98256,+00	4.19020,-01	1.87853,-01
1.31248,+00	4.37666,-01	4.17703,-01	4.19020,-01	6.78266,+00	-8.73150,-02
1.07449,+00	3.32657,-01	2.75691,-01	1.87853,-01	-8.73150,-02	6.35910,+00

$$\begin{aligned}
S_1 &= 1.00000,+00 \\
S_2 &= -1.32828,+00 \\
S_3 &= -5.11778,-01 \\
S_4 &= -2.39709,-01 \\
S_5 &= -2.81635,-01 \\
S_6 &= -2.94086,-01
\end{aligned}$$

$$\mu(0.42) = 0.360$$

$$\frac{t_{\max}}{\lambda_m} = 0.308 \quad \zeta(0.42) = 26.25$$

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= -3.3207,-01 \\
F_3 &= -5.6864,-02 \\
F_4 &= -1.4981,-02 \\
F_5 &= -1.1265,-02 \\
F_6 &= -8.1690,-03
\end{aligned}$$

 $R_{mn}(1)$

6.27844,+00	-2.68998,-03	-2.17829,-03	-2.05836,-03	-2.11441,-03	-2.28483,-03
-2.68998,-03	6.28113,+00	-2.12921,-03	-2.45632,-03	-2.91194,-03	-3.48519,-03
-2.17829,-03	-2.12921,-03	6.28053,+00	-3.27514,-03	-4.06991,-03	-5.01348,-03
-2.05836,-03	-2.45632,-03	-3.27514,-03	6.27890,+00	-5.40231,-03	-6.73195,-03
-2.11441,-03	-2.91194,-03	-4.06991,-03	-5.40231,-03	6.27627,+00	-8.67043,-03
-2.28483,-03	-3.48519,-03	-5.01348,-03	-6.73195,-03	-8.67043,-03	6.27230,+00

$$\begin{aligned}
S_1 &= 1.00000,+00 \\
S_2 &= -1.18406,+00 \\
S_3 &= -6.94363,-01 \\
S_4 &= -2.30453,-01 \\
S_5 &= -3.16921,-01 \\
S_6 &= -3.33105,-01
\end{aligned}$$

$$\mu(1) = 0.275$$

$$\zeta(1) = 30.03$$

$$\frac{t_{\max}}{\lambda_m} = 0.30$$

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= -2.9601,-01 \\
F_3 &= -7.7151,-02 \\
F_4 &= -1.4403,-02 \\
F_5 &= -1.2676,-02 \\
F_6 &= -9.2529,-03
\end{aligned}$$

Figure Captions

- Fig. 1. Filter D for transformation of the input current triangular pulse into a $\delta(t)$ pulse.
- Fig. 2. Filter L obtained by cascading a double integrator to filter D. δ -response of filter L.
- Fig. 3. Filter H obtained by cascading to D or L a suitable "optimum" filter W or G in order to obtain waveforms of the type $w_A(t)$, $w_B(t)$, $w_C(t)$ with the lowest equivalent noise charge ENC_s .
- Fig. 4. Parameter ζ proportional to ENC_s (to be used in Eq. (30)) versus $\alpha = \frac{t_d}{\lambda_m}$ (ratio drift time t_d and output pulse width λ_m) for the unipolar output pulses $w_A(t)$.
- Fig. 5. Input pulse $i(t)$, output pulse $w_A(t)$, response $g_A(t)$, and overall δ -response $h_A(t)$ (weighting function) for $\alpha \rightarrow \infty$.
- Fig. 6. Input pulse $i(t)$, output pulse $w_A(t)$, response $h_A(t)$ (weighting function) for $\alpha = 1$.
- Fig. 7. Input pulse $i(t)$, output pulse $w_A(t)$, response $h_A(t)$ (weighting function) for $\alpha = 0.6$ (α value for which minimum ENC_s is obtained).
- Fig. 8. Output pulses $w_A(t)$ for input pulses having width t_d , shorter ($0.9 \lambda_m$), equal, and longer ($1.1 \lambda_m$), than the nominal value for the filter H_A designed for $\alpha = 1$, that is $t_d = \lambda_m$. Sensitivity of mismatch referred to peak amplitude and tail amplitude can be observed.
- Fig. 9. Parameter ζ proportional to ENC_s (to be used in Eq. (30)) versus $\alpha = \frac{t_d}{\lambda_m}$ (ratio of drift time t_d and output pulse width λ_m) for three-lobe-zero-area output pulses $w_C(t)$.
- Fig. 10. Input pulse $i(t)$, output pulse $w_C(t)$, response $h_C(t)$ (weighting function) for $\alpha = 1$.
- Fig. 11. Input pulse $i(t)$, output pulse $w_C(t)$, response $h_C(t)$ (weighting function) for $\alpha = 0.36$ (α -value for which, at constant λ_m , minimum ENC_s is obtained).

- Fig. 12. Output pulses $w_C(t)$ for input pulses having width t_d , shorter ($0.9 \lambda_m$), equal, and larger ($1.1 \lambda_m$), than the nominal value for the filter H_C designed for $\alpha = 1$, that is $t_d = \lambda_m$. Sensitivity of mismatch referred to peak amplitude and tail amplitude can be observed.
- Fig. 13. Parameter ζ proportional to ENC_s (to be used in Eq. (30)) versus $\alpha = \frac{t_d}{\lambda_m}$ (ratio of drift time t_d and output pulse width λ_m) for the two lobes antisymmetrical pulses $w_B(t)$.
- Fig. 14. Input pulse $i(t)$, output pulse $w_B(t)$, δ -response $h_B(t)$ (weighting function) for $\alpha = 1$.
- Fig. 15. Input pulse $i(t)$, output pulse $w_B(t)$, δ -response $h_B(t)$ (weighting function) for $\alpha = 0.42$ (α value for which, at constant λ_m , minimum ENC_s is obtained).
- Fig. 16. Output pulses $w_B(t)$ for input pulses having width t_d , shorter ($0.9 \lambda_m$), equal, and larger ($1.1 \lambda_m$), than the nominal value for the filter H_B designed for $\alpha = 1$, that is $t_d = \lambda_m$. Sensitivity of mismatch referred to peak amplitude and tail amplitude can be observed.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6

FIGURE 7

FIGURE 8

FIGURE 9

FIGURE 10

FIGURE 11

FIGURE 12

FIGURE 13

FIGURE 14

FIGURE 15

FIGURE 16