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**A Numerical Study of the System
of Differential Equations for the
Drift Wave in Tokamaks**

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A NUMERICAL STUDY OF THE SYSTEM OF DIFFERENTIAL
EQUATIONS FOR THE DRIFT WAVE IN TOKAMAYS

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A NUMERICAL STUDY OF THE SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE DRIFT WAVE IN TOKAMAKS

Julius Smith
J. C. Whitson

ABSTRACT

The following report discusses the authors' numerical study of a form of the drift wave equations which makes use of the usual quadratic approximation for the ion Z-function term and the usual exponential approximation for the electron Z-function term. Our recent work has shown that the electron Z-function term is not well represented by the exponential. Nevertheless, the numerical methods discussed in this report are similar to those which are applicable in the general case using the Z-function for both electrons and ions, although the use of the ion Z-function term changes the asymptotic behavior of the solutions.

I. INTRODUCTION

Our starting point is the system of ordinary differential equations for the drift wave in a finite- β inhomogeneous plasma in a sheared magnetic field

$$\begin{aligned} \frac{d^2 E}{dx^2} - (y + \kappa)E &= \frac{yq}{x} B \\ \frac{d^2 B}{dx^2} - \kappa B &= -\frac{1}{x} \left(\frac{d^2 E}{dx^2} - \kappa E \right) \end{aligned} \quad (1)$$

The plasma density is assumed to vary in the x direction, E is the component of the perturbed electric field in a direction of shear perpendicular to x , and B is the component of the perturbed magnetic field in a direction parallel to x . x is given in dimensionless units while the other quantities (also dimensionless) are given by the equations

$$g = \lambda - \mu^2 x^2 + i\sigma \quad (2)$$

$$\lambda = \frac{\omega - \omega_\star}{(\tau+1)\omega_\star} \quad \mu = \frac{\mu_\star \omega_\star}{\omega}$$

$$\sigma = \lambda \sqrt{\pi} \frac{x_e}{x} e^{-\frac{x_e^2}{x^2}},$$

where the real constants κ , ν , $\omega_\star$, τ , $\mu_\star$, and x_e are given by

$$\begin{aligned} \mu_\star &= \frac{L_n}{\tau L_s} & \omega_\star &= \frac{L_s \mu_\star}{k v_i a_i} \\ \kappa &= k^2 a_i^2 & \nu &= \beta_i(\tau)(\tau+1) \frac{L_s^2}{L_n^2} \\ \tau &= \frac{T_e}{T_i} & x_e &= \sqrt{\frac{\tau m}{2M}} \frac{L_s}{L_n} \end{aligned} \quad (3)$$

Eliminating ω from equations (2) yields

$$\lambda = \frac{\mu_\star}{(\tau+1)\mu} - \frac{1}{\tau+1} \quad (4)$$

In Equation (1) we now take κ , ν , and the other real constants of (3) to be given. Thus the coefficients of (1) are all determined as functions of x and the complex parameter μ . The basic eigenvalue problem is then to find a value of the complex parameter μ and complex solutions of (1), E , B , on the interval $0 \leq x < \infty$ which satisfy

$$\text{Im} \mu < 0, \quad (5)$$

$$E'(0) = 0, \quad (E \text{ even}) \quad (6)$$

$$B(0) = 0, \quad (B \text{ odd}) \quad (7)$$

$$E \text{ decays to } 0 \text{ as } x \rightarrow \infty, \quad (8)$$

$$B \text{ decays to } 0 \text{ as } x \rightarrow \infty. \quad (9)$$

It is clear that the coefficients of (1) depend on μ in a nonlinear

manner, and hence many of the standard numerical techniques for finding eigenvalues are not applicable.

For purposes of analysis, it is useful to allow the first equation (1) to be substituted in the right-hand side of the second to obtain

$$\begin{bmatrix} E \\ B \end{bmatrix}'' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E \\ B \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} a_{11} &= g + \kappa & a_{12} &= \frac{vq}{x} \\ a_{21} &= -\frac{q}{x} & a_{22} &= \kappa - \frac{vq}{x^2} \end{aligned} \quad (11)$$

II. THE ELECTROSTATIC CASE

We first examine briefly the case $v = 0$, (electrostatic case) which has been treated by Gladd and Horton [2]. In this case the equations uncouple, and we are left to solve a single second-order equation

$$E'' - (g + \kappa)E = 0, \quad (12)$$

subject to

$$E'(0) = 0, \quad E'(\infty) = 0. \quad (13)$$

One of the known techniques for such a problem which is not affected by the nonlinearity in μ is the shooting method. A straightforward application of this method would be to choose a value of E at $x = 0$, say $E = 1$. (E is determined by (12) and (13) only up to an arbitrary constant) If a value of μ is selected, then a solution with initial conditions $E(0) = 1$, $E'(0) = 0$ may be determined by using an initial

value code. The value, $E(L, \mu)$, for some "large" finite length L may then be determined. We wish to force $f(\mu) = E(L, \mu) = 0$. Thus $f(\mu) = E(L, \mu)$ may be viewed as an analytic function of the complex parameter μ and our initial value code merely as a means for evaluating this complex function. The problem is then reduced to finding the zeroes of a complex valued function and may be accomplished with the aid of a complex root-finding routine. Unfortunately, shooting in the forward direction on this equation leads to severe numerical problems. It is known that for $x \rightarrow \infty$ the solution E of (12) is given by

$$E \sim c_1 x^p e^{-\frac{i\mu x^2}{2}} + c_2 x^{-(1+p)} e^{\frac{i\mu x^2}{2}}. \quad (14)$$

(See Appendix.)

It follows that, in the case $\text{Im} \mu < 0$, $e^{-\frac{i\mu x^2}{2}}$ decays while $e^{\frac{i\mu x^2}{2}}$ grows as $x \rightarrow \infty$. Since we are trying to calculate the decaying solution, we would have to arrange for c_2 to be precisely equal to zero in order to obtain the decaying solution. It is clear that this degree of precision cannot be accomplished on a computer. Thus, in practice, the solutions to (12) always have some part of the growing solution mixed in with the decaying solution. Hence, after a certain interval length, solutions are grossly in error, and we cannot isolate the decaying mode with this technique. The problem is referred to in the numerical analysis literature as inherent instability (see Fox [1]).

Fortunately, the difficulties in this case may be alleviated by shooting backwards from some distance L . Intuitively this may be explained by observing that in shooting backwards the mode we desire is now increasing while the nondesired mode is decaying. Hence, the

term $c_2 x^p e^{\frac{i\mu x^2}{2}}$ which is small to begin with is not amplified by the shooting process. A more rigorous mathematical error analysis may be given for this process. Suffice it to say that the method works in practice as may be verified by using the test problem, $y'' + (1 - x^2)y = 0$, to compute the solution $y = e^{-\frac{x^2}{2}}$.

Shooting backwards to obtain the eigenvalues may be carried out either with or without the aid of the asymptotic solution. In the first case, we set $E = x^p e^{-\frac{i\mu x^2}{2}}$ and $E' = (\frac{p}{x} - i\mu x)E$ at $x = L$, whereas in the second case we set $E = 0$, $E' = \epsilon$ at $x = L$. Here ϵ is selected in such a manner that the solution does not grow too large. (Most initial-value problem solvers have difficulties if the numbers grow large.)

In either case, we use our differential equation solver to calculate $f(\mu) = E'(\mu, 0)$, the slope of E at zero. Again, we desire to have $f(\mu) = 0$. This may be accomplished by using a complex root finder to call the function $f(\mu)$ generated by the initial value problem.

We find the complex secant method adequate for our purposes, i.e.,

$$\mu_n = \mu_{n-1} - \frac{f(\mu_{n-1})}{f(\mu_{n-1}) - f(\mu_{n-2})} (\mu_{n-1} - \mu_{n-2}). \quad (15)$$

Here, we select an initial guess μ_0 and a second guess $\mu_1 = (1 + s)\mu_0$ where s is a small number. The initial value solver is used to calculate $f(\mu_0)$ and $f(\mu_1)$. A corrected value of μ is then obtained from (15), and the process repeated. In practice, rapid convergence (from 5-20 iterations) is obtained for six-place accuracy in μ if one starts with a good guess. In particular, modes near marginal stability were obtained by taking the initial guess $\text{Re} \mu = \mu_*$ and $\text{Im} \mu = \text{small negative number}$, for κ small.

III. THE FULL SYSTEM OF EQUATIONS

The full system of equations may be treated in a similar manner shooting backwards as follows: we start at some large finite length L with two linearly independent solutions of the system (1), $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$, which satisfy $E_i(L) = 0$, $B_i(L) = 0$, $i = 1, 2$. This may be accomplished, for example, by setting

$$E_1 = 0, B_1 = 0, E'_1 = \epsilon_1, B'_1 = 0,$$

and

(16)

$$E_2 = 0, B_2 = 0, E'_2 = 0, B'_2 = \epsilon_2.$$

A similar procedure using asymptotic values of $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$ may be attempted; however, we coded the simpler procedure listed above. We then observe that any solution of (1) which satisfies $E(L) = 0$, $B(L) = 0$ is a linear combination of $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$. Thus, in order to satisfy conditions (6) and (7), we require (Observe that we cannot shoot all the way to 0 since the equations (1) are singular at $x = 0$. Therefore, we satisfy the boundary conditions (6) and (7) for x small rather than zero. In practice we used $x = 10^{-4}$ as our "zero" point.)

$$\begin{aligned} E'(0) &= c_1 E'_1(0) + c_2 E'_2(0) = 0 \\ B(0) &= c_1 B_1(0) + c_2 B_2(0) = 0 \end{aligned} \quad (17)$$

In order for (17) to have a non-trivial solution, it is necessary that

$$\det \begin{bmatrix} E'_1(0) & E'_2(0) \\ B_1(0) & B_2(0) \end{bmatrix} = 0 \quad (18)$$

A reasonable procedure for finding the eigenvalues is to use the initial value solver to calculate E_1 , B_1 , E_2 , and B_2 satisfying (16), for a given value of μ .

This yields a value for

$$f(\mu) = \det \begin{bmatrix} E_1'(\mu, 0) & E_2'(\mu, 0) \\ B_1(\mu, 0) & B_2(\mu, 0) \end{bmatrix} . \quad (19)$$

A root finder may then be used to locate a value of μ for which $f(\mu) = 0$. Once a value of μ for which (18) is satisfied has been obtained, we may solve (17) to obtain

$$c_1 = -cB_2(0) \quad c_2 = cB_1(0) . \quad (20)$$

The desired eigenfunction is then

$$\begin{aligned} E &= c_1 E_1 + c_2 E_2 \\ &= -cB_2(0)E_1 + cB_1(0)E_2 . \end{aligned}$$

We normalize $E(0) = 1$ so $-cB_2(0)E_1(0) + cB_1(0)E_2(0) = 1$ thus

$$c = \frac{1}{B_1(0)E_2(0) - B_2(0)E_1(0)} , \quad (21)$$

and

$$c_1 = \frac{-B_2(0)}{B_1(0)E_2(0) - B_2(0)E_1(0)} \quad c_2 = \frac{B_1(0)}{B_1(0)E_2(0) - B_2(0)E_1(0)} . \quad (22)$$

Thus E may be recovered by taking as initial conditions (at $x=L$)

$$E = 0, B = 0, E' = c_1 E_1', B' = c_2 E_2' . \quad (23)$$

Although we obtained some results with the above procedure, we encountered some numerical difficulties as follows. With v turned on we are able to find a solution to our problem for a length L which was not too large. As we attempted to increase L , however, the accuracy with which the initial value solver had to be tuned increased an order of magnitude. If this was not done, the determinant (19) was not computed with sufficient precision to allow the root finder to calculate a μ for which $f(\mu) = 0$. In fact we ultimately reached lengths for which the accuracy of the differential equation solver on the machine could not be increased sufficiently to compute a value for which $f(\mu) = 0$. This left us in doubt as to whether we were satisfying conditions (8) and (9) at a sufficient distance from the origin. This was reinforced by the fact that in some cases μ was still changing as we increased the length of the interval.

IV. INVARIANT IMBEDDING

At this point, we turned to the method of invariant imbedding as described in Scott [3]. In the electrostatic case this amounts to defining $s = \frac{E'}{E}$ and $r = \frac{E}{E'} = \frac{1}{s}$.

With this change of variables we have

$$s' = \frac{E E'' - E'^2}{E^2} = \frac{E''}{E} - \left(\frac{E'}{E}\right)^2 = (g + \kappa) - s^2,$$

thus

$$s' + s^2 = g + \kappa \tag{24}$$

$$-r^{-2}r' + r^{-2} = g + \kappa$$

$$r' + (g + \kappa)r^2 = 1. \tag{25}$$

Both (24) and (25) are examples of Riccati equations. Our condition $E'(0) = 0$ translates into

$$s(0) = 0 \quad . \quad (26)$$

In our first implementation of the method of invariant imbedding, we attempted to shoot with (24), (25) in the forward direction. This was accomplished by starting with the s equation together with the initial condition $s = 0$ to generate a solution via an initial value solver.

The values of s are monitored in such a way that, if $|s| > 1$, we switch to the r -equation (25). We observe that (see appendix) $s \sim -i\mu x$ corresponds to the decaying mode. Correspondingly we have $r \sim \frac{1}{i\mu x}$ (grow) and $r \sim \frac{1}{-i\mu x}$ (decay).

Thus it is clear that, for sufficiently large x , the equation will switch to r . We observe that the forward shooting method remains unstable even for the invariant imbedding equations. It was noticed that, when a known eigenvalue was inserted for μ , the equations would stay on the $r \sim \frac{1}{-i\mu x}$ mode for a while but would always switch to the $r \sim \frac{1}{i\mu x}$ mode at just about the same point the forward shooting method (without imbedding) began to be swamped by the growing mode. Again, however, the problem may be cured by shooting backwards, starting with $r = \frac{1}{-i\mu x}$ at some finite length $x = L$ and switching to s when $|r| > 1$.

This leaves us with a value $s = s(\mu, 0)$ at the origin. If we take $f(\mu) = s(\mu, 0)$, we may use a root finder to locate where $f(\mu) = 0$. The method also works if we take $r = 0$ at $x = L$ (corresponds to $E(L) = 0$). The asymptotic values $r = \frac{1}{-i\mu x}$ lead to somewhat smoother solutions and

more rapid computation times for small values of $\text{Im}\mu$. Starting with $r = 0$ in these instances leads to oscillations in r for large x , before the differential equation solver latches onto the right solution. In the electrostatic case, the gain from using imbedding is modest. One may use very large shooting lengths L since the r and s equations do not involve particularly rapid decay or oscillations while the original equations may contain both.

To recover our eigenfunctions in the invariant imbedding procedure, we observe that in an interval where r is calculated we have

$$E = rE' , \quad (27)$$

$$E'' = (g + \kappa)E = (g + \kappa)rE' .$$

Thus, letting $m = E'$ we have

$$m' = (g + \kappa)rm , \quad (28)$$

a first-order equation which may be initialized by

$$m(L) = \epsilon . \quad (29)$$

m grows as we shoot inward, hence it is advantageous to calculate $q = \frac{1}{m}$ which decays. We find

$$m' = \frac{q'}{q^2} ,$$

$$\frac{-q'}{q^2} = (g + \kappa)r \frac{1}{q} .$$

Thus we add to our system of equations

$$q' = -(g + \kappa)r q , \quad (30)$$

$$q(L) = \frac{1}{\epsilon} . \quad (31)$$

From q we have

$$E' = \frac{1}{q} \quad \text{and} \quad E = rE' = \frac{r}{q} . \quad (32)$$

In the s -interval we have

$$E' = sE . \quad (33)$$

Again we compute with

$$k = \frac{1}{E} ,$$

so that

$$\frac{-k'}{k^2} = s \frac{1}{k} ,$$

and we add to our system of equations

$$k' = -sk . \quad (34)$$

From k we have

$$E = \frac{1}{k} , \quad E' = \frac{s}{k} , \quad (35)$$

the initial value for k being obtained from the formula $k = \frac{q}{r}$.

In switching from s to r , we have

$$q = \frac{k}{s} . \quad (36)$$

An extra shot may be used to normalize taking

$$q(L) = \frac{1}{\varepsilon k(0)} . \quad (37)$$

With regard to the full electromagnetic equations, however, the imbedding removes the precision problems encountered in computing the determinant (19) and gives satisfactory results for both eigenvalues and eigenfunctions. The amount of computing time increases only modestly with increasing length L and the error criterion for the initial value

solver need not be extremely small and does not have to be lowered for increased shooting length.

The basic program of imbedding for systems is given in Scott [3], Chapter VIII.

Just as in the shooting code, we need two linearly independent solutions,

$$\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2 \\ B_2 \end{bmatrix} ,$$

to (10).

The solutions may be combined to give the matrix equation

$$\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}'' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} . \quad (38)$$

In analogy to the quantity r introduced for the single equation (12), we introduce the matrix quantity R defined by

$$\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \begin{bmatrix} E_2 \\ B_2 \end{bmatrix} = [R] \begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \begin{bmatrix} E_2' \\ B_2' \end{bmatrix} . \quad (39)$$

This yields

$$\begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \begin{bmatrix} E_2' \\ B_2' \end{bmatrix} = [R]' \begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \begin{bmatrix} E_2' \\ B_2' \end{bmatrix} + R \begin{bmatrix} E_1'' \\ B_1'' \end{bmatrix} \begin{bmatrix} E_2'' \\ B_2'' \end{bmatrix} ,$$

or from (38)

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \begin{bmatrix} E_2' \\ B_2' \end{bmatrix} = [R]' \begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \begin{bmatrix} E_2' \\ B_2' \end{bmatrix} + [R] [A] \begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$$

$$= R' \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} + RAR \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} .$$

Thus, if we assume $\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}$ to be nonsingular, we obtain

$$R' = I - RAR, \quad (40)$$

which is a first-order system of equations in four complex unknowns r_{ij} , the components of the matrix R . To fit the boundary conditions (6) and (7), we must use a matrix S which is not the inverse of R .

In fact, if we define $E' = u$ and $B' = v$, we may rewrite the system (10) in the form

$$E' = u$$

$$u' = a_{11}E + a_{12}B$$

$$B' = v$$

$$v' = a_{21}E + a_{22}B ,$$

or

$$\begin{bmatrix} u \\ B \end{bmatrix}' = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ B \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ v \end{bmatrix}$$

$$\begin{bmatrix} E \\ v \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} u \\ B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E \\ v \end{bmatrix} .$$

It follows that

$$\begin{bmatrix} E' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E' \\ B' \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ B' \end{bmatrix} \quad (41)$$

$$\begin{bmatrix} E \\ B' \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} E' \\ B' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E \\ B' \end{bmatrix} .$$

We may then introduce the matrix S by the formula

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = [S] \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}, \quad (42)$$

where $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$ are the same independent solutions given before.
Thus

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix}' = [S]' \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} + [S] \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}'.$$

Using (41) we find that

$$\begin{aligned} & \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} \\ &= S' \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} + S \left\{ \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} \right\}. \end{aligned}$$

Employing (42) we obtain

$$\begin{aligned} & \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} \\ &= S' \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} + S \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} \\ &+ S \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}. \end{aligned}$$

Assuming that $\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}$ is nonsingular, we find

$$S' = -S \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S - S \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} S + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix}, \quad (43)$$

which is again a first-order system in the four unknowns s_{ij} of the matrix S . Relations may be obtained which relate the elements of R and S as follows:

We have from (39) and (42)

$$\begin{bmatrix} E_i' \\ B_i' \end{bmatrix} = S \begin{bmatrix} E_i \\ B_i \end{bmatrix}, \quad i = 1, 2$$

$$\begin{bmatrix} E_i \\ B_i \end{bmatrix} = R \begin{bmatrix} E_i' \\ B_i' \end{bmatrix}.$$

Thus

$$E_i' = s_{11} E_i + s_{12} B_i'$$

$$B_i' = s_{21} E_i + s_{22} B_i'$$

$$E_i = r_{11} E_i' + r_{12} B_i'$$

$$B_i = r_{21} E_i' + r_{22} B_i'.$$

Solving the last two equations for E_i' and B_i' , we have

$$E_i' = \frac{1}{r_{11}} E_i - \frac{r_{12}}{r_{11}} B_i'$$

$$B_i = r_{21} \left(\frac{1}{r_{11}} E_i - \frac{r_{12}}{r_{11}} B_i' \right) + r_{22} B_i'$$

$$B_i = \frac{r_{21}}{r_{11}} E_i + \left(r_{22} - \frac{r_{12}r_{21}}{r_{11}} \right) B_i'.$$

Hence, the connecting equations are

$$\begin{aligned} s_{11} &= \frac{1}{r_{11}} & s_{12} &= \frac{-r_{12}}{r_{11}} \\ s_{21} &= \frac{r_{21}}{r_{11}} & s_{22} &= r_{22} - \frac{r_{12}r_{21}}{r_{11}}, \end{aligned} \quad (44)$$

which may be solved backwards to yield

$$\begin{aligned} r_{11} &= \frac{1}{s_{11}} & r_{12} &= \frac{-s_{12}}{s_{11}} \\ r_{21} &= \frac{s_{21}}{s_{11}} & r_{22} &= s_{22} - \frac{s_{12}s_{21}}{s_{11}} \end{aligned} \quad (45)$$

The procedure we use is similar to that employed for the single equation. The decaying solution (see appendix) has asymptotic values:

$$\begin{aligned} r_{11} &\sim -\frac{1}{i\mu x} \\ r_{12} &\sim \frac{v}{\sqrt{k} x} \\ r_{21} &\sim \frac{-1}{\sqrt{k} x} \\ r_{22} &\sim \frac{-1}{\sqrt{k}} \end{aligned} \quad (46)$$

We use these to start the backwards shot on R at some large length L. (Again the values $R=0$ may be used but the results are not as smooth for small $\text{Im}\mu$.)

The code is set to switch to S when $|r_{11}| > 1$. We observe from (42) that if

$$\det \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = 0 ,$$

which implies that

$$\begin{aligned} C_1 E_1' + C_2 E_2' &= 0 , \\ C_1 B_1 + C_2 B_2 &= 0 , \end{aligned}$$

then $\det S = 0$ if

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix}$$

is nonsingular. Thus, we set $f(\mu) = \det S$ at $x = 0^+$, and arrange for a root finder to locate a μ for which $f(\mu) = 0$.

The solutions E and B may be recovered from the imbedding equations once we have determined an eigenvalue μ . This may be done as follows. First we find $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$, linearly independent solutions of (10) then

$$\begin{bmatrix} E \\ B \end{bmatrix} = c_1 \begin{bmatrix} E_1 \\ B_1 \end{bmatrix} + c_2 \begin{bmatrix} E_2 \\ B_2 \end{bmatrix} .$$

From (39) we have

$$\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} = R \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} , \quad (47)$$

and from (38) we have

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}' = A \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} . \quad (48)$$

Therefore

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}' = AR \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} .$$

Thus, the matrix

$$M = \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} \quad (49)$$

satisfies the first-order system

$$M' = A R M . \quad (50)$$

* Actually we choose x at some small non-zero value say $x = 10^{-4}$ since the system (10) is singular at $x = 0$ (see (11)).

We may choose initial conditions so that

$$\begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2' \\ B_2' \end{bmatrix}$$

are independent, e.g.

$$\begin{aligned} E_1'(L) &= \epsilon_1 & B_1'(L) &= 0 & E_2'(L) &= 0 \\ B_2'(L) &= \epsilon_2 & & & \epsilon_1, \epsilon_2 &\neq 0 \end{aligned}$$

Hence, M satisfies the initial conditions

$$M(L) = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \quad (51)$$

Numerically, the solution M grows as we shoot from L inward so that we may with advantage solve instead for Q , the inverse of M .

We have

$$\begin{aligned} QM &= I \\ Q'M + QM' &= 0 \\ Q'M + QARM &= 0 \end{aligned}$$

and since M is nonsingular,

$$Q' = -QAR \quad (52)$$

The sign in the equation for Q has been reversed, and we are thus solving for decaying modes. From (51) we have the initial condition

$$Q(L) = \begin{bmatrix} \epsilon_1^{-1} & 0 \\ 0 & \epsilon_2^{-1} \end{bmatrix} \quad (53)$$

Thus, we solve equation (52) for Q subject to (53). We then make use of

$$M = Q^{-1}, \quad (54)$$

and finally defining

$$N = \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} , \quad (55)$$

we may calculate from (47)

$$N = RM . \quad (56)$$

Therefore, in the regime where R is calculated, we know the values of $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$.

In the regime where S is calculated, we have

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = S \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} , \quad (57)$$

as well as

$$\begin{bmatrix} E_1' & E_2' \\ S_1 & B_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} .$$

It follows that

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} .$$

Hence, if we define

$$L = \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} , \quad (58)$$

we have

$$L' = \left(\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \right) L . \quad (59)$$

Again, defining $PL = I$

$$P'L + PL' = 0 ,$$

$$P'L + P \left\{ \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \right\} L = 0 ,$$

$$P' = -P \left\{ \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \right\} . \quad (60)$$

We thus calculate P and then

$$L = P^{-1} . \quad (61)$$

Finally, if we define

$$K = \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} , \quad (62)$$

we have

$$K = SL . \quad (63)$$

Thus, in K and L we have all the quantities E_1 , E_2 , B_1 , B_2 , E_1' , E_2' , B_1' , and B_2' . The initial conditions for P may be determined from the values of these same quantities in the R regime as follows:

$$L = \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (64)$$

$$P = L^{-1} . \quad (65)$$

A switch back to R may be accomplished by

$$M = \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ L_{21} & L_{22} \end{bmatrix} , \quad (66)$$

which yields initial conditions for $Q = M^{-1}$.

One shot inward is then used to evaluate^{*} $E_1(0)$, $E_2(0)$, $B_1(0)$, $B_2(0)$, $E_1'(0)$, and $E_2'(0)$. Now if $\begin{bmatrix} E \\ B \end{bmatrix} = c_1 \begin{bmatrix} E_1 \\ B_1 \end{bmatrix} + c_2 \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$,

^{*} See the remarks on the bottom of page 17 regarding the "zero" point.

$$\begin{aligned}
E &= c_1 E_1 + c_2 E_2 , \\
B &= c_1 B_1 + c_2 B_2 , \\
E' &= c_1 E_1' + c_2 E_2' , \text{ and} \\
B' &= c_1 B_1' + c_2 B_2' .
\end{aligned}
\tag{67}$$

Hence we may set $E'(0)$ and $B(0) = 0$, which yields

$$\begin{aligned}
c_1 E_1'(0) + c_2 E_2'(0) &= 0 \\
c_1 B_1(0) + c_2 B_2(0) &= 0 .
\end{aligned}
\tag{68}$$

We have already arranged for a μ which makes $\det S=0$ so that $\det \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = 0$. Thus, we need only select

$$\begin{aligned}
c_1 &= -cB_2(0) \\
c_2 &= cB_1(0)
\end{aligned}
\tag{69}$$

and if we normalize to $E(0) = 1$, we have

$$\begin{aligned}
1 &= E(0) = c_1 E_1(0) + c_2 E_2(0) \\
1 &= -cB_2(0)E_1(0) + cB_1(0) E_2(0) ,
\end{aligned}$$

so that

$$\begin{aligned}
c &= \frac{1}{B_1(0) E_2(0) - B_2(0) E_1(0)} \\
c_1 &= \frac{-B_2(0)}{B_1(0) E_2(0) - B_2(0) E_1(0)} \\
c_2 &= \frac{B_1(0)}{B_1(0) E_2(0) - B_2(0) E_1(0)}
\end{aligned}$$

or written in terms of K and L

$$\begin{aligned}
c_1 &= \frac{-K_{22}(0)}{K_{21}(0) L_{12}(0) - K_{22}(0) L_{11}(0)} \\
c_2 &= \frac{K_{21}(0)}{K_{21}(0) L_{12}(0) - K_{22}(0) L_{11}(0)} .
\end{aligned}
\tag{70}$$

If we write (67) in matrix form we have

$$\begin{bmatrix} E \\ B \end{bmatrix} = N \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (71)$$

$$\begin{bmatrix} E' \\ B' \end{bmatrix} = M \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} ,$$

or alternatively

$$\begin{bmatrix} E \\ B' \end{bmatrix} = L \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (72)$$

$$\begin{bmatrix} E' \\ B \end{bmatrix} = K \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, after a first shot which determines c_1 and c_2 , we may use (71) and (72) on a second shot to determine the fields E, B, E', B' in the appropriate regime.

V. STABLE MODES

As the shear rate increases, the drift wave passes from an unstable ($\text{Im } \mu < 0$) to a stable ($\text{Im } \mu > 0$) region. The calculation of the eigenvalues in this region presents further computational problems. We address ourselves to the electrostatic case. In the course of changing from an unstable to a stable mode, the solutions change from decay as $x \rightarrow \infty$ to exponential growth as $x \rightarrow \infty$. Specifically, we would like to calculate the solution

$$E \sim c x^p e^{-\frac{i\mu x^2}{2}}$$

as $x \rightarrow \infty$ for which $E'(0) = 0$ ($\text{Im } \mu > 0$). If we shoot forward either with the ordinary shooting procedure or imbedding, the solution is always mixed

with some combination of the decaying mode. Shooting backwards from large L , with the conditions

$$E \sim x^p e^{-\frac{i\mu x^2}{2}}$$

or

$$r \sim \frac{1}{-i\mu x}$$

results in numerical instabilities similar to those encountered in Section II for shooting forward for the decaying mode. (The growing solution actually decays as we shoot backwards). One approach is to shoot backwards from a moderate size length L using the appropriate WKB solution as an initial condition as suggested in Gladd and Horton [2]. This process yields a fairly accurate (within a few percent) value for the eigenvalue μ . The problem in using this method for the stable modes is that it is not obvious when an optimal approximation is achieved. One cannot use arbitrarily large lengths to test the process since the previously mentioned instabilities will arise if the shooting length L is chosen to be too large. The approximation will also fail if L is too small. An alternative procedure is the following.

We view equation (12) as a differential equation in the complex plane

$$\frac{d^2 E}{dz^2} - (g + \kappa)E = 0 \quad (73)$$

where we now take

$$g = \lambda - \mu^2 z^2 + i\sigma, \quad (74)$$

and

$$\sigma(z) = \lambda\sqrt{\pi} \frac{x_e}{z} e^{-\frac{x_e^2}{z^2}}. \quad (75)$$

The asymptotic form for E is

$$E \sim c_1 z^p e^{-\frac{i\mu z^2}{2}} + c_2 z^{-(p+1)} e^{\frac{i\mu z^2}{2}} .$$

Now for $\text{Im } \mu > 0$

$$c_1 z^p e^{-\frac{i\mu z^2}{2}}$$

is a growing mode for $z \rightarrow \infty$ along the x-axis. It decays, however, for $z \rightarrow \infty$ along an appropriately chosen contour. We have had success with the contour shown below.

It is clear that the solution decays to zero for $z \rightarrow \infty$ along the negative imaginary axis. We assume that our solution is continued into the complex plane and we integrate the imbedding equations along the given contour. The last contour is on the x-axis since we would like to impose the condition $E' = 0$ as $z \rightarrow 0$ along the x-axis. The imbedding equations may be obtained by defining

$$s = \frac{dE}{dz} / E . \quad (76)$$

We have

$$\frac{ds}{dz} = -s^2 + (g + \kappa) \quad (77)$$

and for $r = \frac{1}{s}$

$$\frac{dr}{dz} = 1 - (g + \kappa)r^2 \quad (78)$$

If we introduce a parameterization of a given contour by the complex function $z = h(t)$ of the real parameter, the imbedding equations (77) and (78) become along the contour

$$\frac{ds}{dt} = \frac{ds}{dz} \cdot \frac{dz}{dt} = h'(t)(-s^2 + g + \kappa) \quad (79)$$

$$\frac{dr}{dt} = \frac{dr}{dz} \cdot \frac{dz}{dt} = h'(t)(1 - (g + \kappa)r^2) \quad (80)$$

Since we desire to calculate that solution E which is of the form

$$E \sim c_1 e^{\frac{-i\mu z^2}{2}} z^p$$

for $z \rightarrow \infty$, we have $r \sim \frac{1}{-i\mu z}$ along the contour which tends to ∞ . In particular, if we parameterize the part of the contour in the figure which lies along the imaginary axis by $z = -it$ from $t = L$ to $t = y_0$, we have $r \sim \frac{1}{-\mu t}$ to impose for large t .

For an appropriate choice of α , the contour $z = \alpha t$ may be used to start the solution at ∞ . This yields $E \sim c_1 e^{-i\alpha^2 \mu t^2 / 2}$. Thus the factor α^2 may be chosen to make the solution decay at ∞ . This can be done even in the case where the solution does not grow or decay along the x -axis (i.e., $\text{Im } \mu = 0$). We point out only that starting on the wrong contour may yield the wrong branch of s , i.e., one which does not continue the values obtained for $\text{Im } \mu < 0$ with contour being the real axis. This may be checked by calculating the value $s(\mu, 0)$ and seeing if it remains continuous as we vary μ from $\text{Im } \mu < 0$ to $\text{Im } \mu > 0$.

VI. NUMERICAL PROCEDURES

To obtain a guess for μ , we use a model problem for the electrostatic case which may be solved exactly. We take $\sigma = \lambda\sigma_0$ for σ_0 a constant which may be chosen to simulate the function $\sqrt{\pi} \frac{x}{x} e^{-\frac{x^2}{x^2}}$. For example σ_0 may be chosen as the maximum value of this function. We then solve the electrostatic problem numerically using $\sigma = \sigma_\epsilon$ where

$$\sigma_\epsilon = \lambda\sigma_0 + \epsilon(\sigma - \lambda\sigma_0) \quad . \quad (81)$$

For $\epsilon = 0$ we have the model problem, for $\epsilon = 1$ the problem we desire to solve. The values of ϵ are then increased from zero to one using the values of μ for the previous ϵ as a guess for the next μ .

The model problem equation is

$$\frac{d^2 E}{dx^2} - (\lambda + \kappa + i\lambda\sigma_0 - \mu^2 x^2)E = 0 \quad . \quad (82)$$

Making the substitution $E = e^{\frac{-i\mu x^2}{2}} \bar{E}$, we have

$$\frac{d^2 \bar{E}}{dx^2} - 2i\mu x \frac{d\bar{E}}{dx} - (\lambda + \kappa + i\lambda\sigma_0 + i\mu)\bar{E} = 0 \quad (83)$$

or defining $\alpha = -2i\mu$, $\beta = -(\lambda + \kappa + i\lambda\sigma_0 + i\mu)$ and n by $\beta = -n\alpha$, there results

$$\frac{d^2 \bar{E}}{dx^2} + \alpha x \frac{d\bar{E}}{dx} - n\alpha \bar{E} = 0 \quad . \quad (84)$$

Making the substitution $\bar{x} = \sqrt{-\frac{\alpha}{2}} x$ we have

$$\frac{d^2 \bar{E}}{d\bar{x}^2} - 2\bar{x} \frac{d\bar{E}}{d\bar{x}} + 2n \bar{E} = 0 \quad , \quad (85)$$

the Hermite equation.

In order to make $e^{\frac{-i\mu x^2}{2}}$ dominate for $\text{Im } \mu < 0$, we must choose $n = 0, 1, \dots$, thus making $\bar{E}$ a polynomial of the form $\bar{E} = H_n(\sqrt{i\mu}x)$ and

$$E = H_n(\sqrt{i\mu}x) e^{\frac{-i\mu x^2}{2}} \quad n = 0, 1, \dots \quad (86)$$

where $\beta = -n\alpha$, $n = 0, 1, \dots$ so that

$$-[\lambda(1+i\sigma_0) + \kappa + i\mu] = 2i\mu n$$

and

$$\lambda(1+i\sigma_0) + \kappa + (2n+1)i\mu = 0, \quad n = 0, 1, 2, \dots \quad (87)$$

This equation is quadratic in μ , and for even modes we take $n = 0, 2, 4, \dots$

Employing (4), we have

$$\left(\frac{0}{(i+1)\mu} - \frac{1}{i+1} \right) (1 + i\sigma_0) + \kappa + (2n+1)i\mu = 0,$$

and hence

$$(2n+1)\mu^2 + \left(\frac{-i\sigma_0}{i+1} + \left(\frac{1}{i+1} - \kappa \right) i \right) \mu + \frac{0}{i+1} - \frac{1}{i+1} = 0 \quad (88)$$

together with

$$E_n = H_n(\sqrt{i\mu}x) e^{\frac{-i\mu x^2}{2}} \quad (89)$$

for $n = 0, 1, 2, \dots$

We used as our primary initial value solvers the routines KUTTA which is a variable step integrator based on the Runge-Kutta-Merson technique and DE, described in the book of Shampine and Gordon [4]. The more sophisticated DE was especially useful in integrating near the singularities at the origin.

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APPENDIX

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APPENDIX

We consider the equation

$$s' + s^2 = g + \kappa \quad (90)$$

$$g = \lambda - \mu^2 x^2 + i\sigma$$

$$\sigma = \sqrt{\pi} \frac{x e^{-x^2}}{x} ,$$

$$\sim \frac{\sqrt{\pi} x e^{-x^2}}{x} - \sqrt{\pi} \frac{x^3 e^{-x^2}}{x^3} .$$

Thus

$$g \sim -\mu^2 x^2 + \lambda + \kappa + \frac{\sigma_0}{x} \text{ as } x \rightarrow \infty , \quad (91)$$

where

$$\sigma_0 = i\sqrt{\pi} x e^{-x^2} .$$

If we insert the series

$$s \sim a_1 x + a_0 + \frac{a_{-1}}{x} + \dots$$

in (90), we obtain

$$a_1 - \frac{a_{-1}}{x^2} - \dots + (a_1 x + a_0 + \frac{a_{-1}}{x} + \dots)^2 = -\mu^2 x^2 + \lambda + \kappa + \frac{\sigma_0}{x} .$$

It follows that

$$a_1^2 = -\mu^2 \quad (92)$$

$$a_1 = \pm i\mu ,$$

$$2a_0 a_1 = 0 \quad (93)$$

$$a_0 = 0 ,$$

$$a_1 + a_0^2 + 2a_1 a_{-1} = \lambda + \kappa, \quad (94)$$

$$a_{-1} = \frac{\lambda + \kappa}{2a_1} - \frac{1}{2}.$$

Thus we have two distinct solutions

$$s \sim i\mu x + \frac{p_1}{x} \quad (95)$$

$$p_1 = \frac{\lambda + \kappa}{2i\mu} - \frac{1}{2}$$

and

$$s \sim -i\mu x + \frac{p}{x} \quad (96)$$

$$p = \frac{-\lambda - \kappa}{2i\mu} - \frac{1}{2} = -(1 + p_1).$$

Thus to the first order we have

$$r \sim \pm \frac{1}{i\mu x} \quad (97)$$

Since $s = \frac{E'}{E}$ we have the solutions

$$E \sim x^p e^{\frac{-i\mu x^2}{2}}, \quad (98)$$

$$E \sim x^{-(1+p)} e^{\frac{i\mu x^2}{2}},$$

where

$$p = \frac{-(\lambda + \kappa)}{2i\mu} - \frac{1}{2},$$

so that

$$E \sim c_1 x^p e^{-\frac{i\mu x^2}{2}} + c_2 x^{-(1+p)} e^{\frac{i\mu x^2}{2}}.$$

For the full system we have

$$R' = I - RAR \quad (99)$$

where I is the identity matrix, and

$$A = A_0 x^2 + A_1 x + A_2 + \dots \quad (100)$$

where

$$A_0 = \begin{bmatrix} -\mu^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (101)$$

$$A_1 = \begin{bmatrix} 0 & -v\mu^2 \\ \mu^2 & 0 \end{bmatrix} \quad (102)$$

$$A_2 = \begin{bmatrix} \lambda + \kappa & 0 \\ 0 & \kappa + v\mu^2 \end{bmatrix} . \quad (103)$$

We insert the series

$$R = R_0 + \frac{1}{x} R_1 + \frac{1}{x^2} R_2 + \dots \quad (104)$$

into (99) and equate coefficients to obtain

$$R_0 A_0 R_0 = 0 . \quad (105)$$

If we write

$$R_0 = \begin{bmatrix} r_{11}^0 & r_{12}^0 \\ r_{21}^0 & r_{22}^0 \end{bmatrix} , \quad (106)$$

equation (105) reduces to

$$-\mu^2 (r_{11}^0)^2 = 0 \quad (107)$$

$$-\mu^2 r_{12}^0 r_{21}^0 = 0 .$$

Hence, if $\mu \neq 0$, the matrix R_0 is of the form

$$R_0 = \begin{bmatrix} 0 & 0 \\ r_{21}^0 & r_{22}^0 \end{bmatrix} \quad (108)$$

or

$$R_0 = \begin{bmatrix} 0 & r_{12}^0 \\ 0 & r_{22}^0 \end{bmatrix} .$$

If we insert the cases $r_{21}^0 \neq 0$ or $r_{12}^0 \neq 0$ into the later equations, we find eventually that both these cases lead to nonsolvable equations.

Thus, it may be shown that

$$R_0 = \begin{bmatrix} 0 & 0 \\ 0 & r_{22}^0 \end{bmatrix} . \quad (109)$$

The next equation is

$$R_0 A_0 R_1 + R_0 A_1 R_0 + R_1 A_0 R_0 = 0 . \quad (110)$$

We take

$$R_1 = \begin{bmatrix} r_{11}^1 & r_{12}^1 \\ r_{21}^1 & r_{22}^1 \end{bmatrix} .$$

It is easy to see that (110) is already satisfied for R_0 of the form (109).

The next equation is

$$\begin{aligned} & R_0 A_0 R_2 + R_0 A_1 R_1 + R_0 A_2 R_0 \\ & + R_1 A_0 R_1 + R_1 A_1 R_0 + R_2 A_0 R_0 = I \end{aligned} \quad (111)$$

where I is the identity matrix. If we take

$$R_2 = \begin{bmatrix} r_{11}^2 & r_{12}^2 \\ r_{21}^2 & r_{22}^2 \end{bmatrix} ,$$

we obtain the equations

$$\begin{aligned}
 -\mu^2(r_{11}^1)^2 &= 1 \\
 \mu^2 r_{22}^0 r_{11}^1 - \mu^2 r_{21}^1 r_{11}^1 &= 0 \\
 -\mu^2 r_{11}^1 r_{12}^1 - r_{11}^1 r_{22}^0 v \mu^2 &= 0 \\
 \mu^2 r_{22}^0 r_{12}^1 + (r_{22}^0)^2 (\kappa + v \mu^2) - \mu^2 r_{12}^1 r_{21}^1 - v \mu^2 r_{21}^1 r_{22}^0 &= 1 .
 \end{aligned} \tag{112}$$

This yields (for decaying solutions and $\text{Im } \mu < 0$)

$$\begin{aligned}
 r_{11}^1 &= \frac{-1}{i\mu} , \\
 r_{21}^1 &= r_{22}^0 , \\
 r_{12}^1 &= -v r_{22}^0 , \\
 \kappa (r_{22}^0)^2 &= 1 ,
 \end{aligned} \tag{113}$$

so that (again for decaying solutions, $\text{Im } \mu < 0$)

$$\begin{aligned}
 r_{11}^1 &= \frac{-1}{i\mu} \\
 r_{12}^1 &= \frac{v}{\sqrt{\kappa}} \\
 r_{21}^1 &= -\frac{1}{\sqrt{\kappa}} \\
 r_{22}^0 &= -\frac{1}{\sqrt{\kappa}} .
 \end{aligned} \tag{114}$$

This gives us all the highest order terms in the equation for R

so that

$$r_{11} \sim -\frac{1}{i\mu x} \quad (115)$$

$$r_{12} \sim \frac{\nu}{\sqrt{\kappa x}}$$

$$r_{21} \sim -\frac{1}{\sqrt{\kappa x}}$$

$$r_{22} \sim -\frac{1}{\sqrt{\kappa}} \quad .$$