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TRANSIENT FREE-CONVECTION HEAT TRANSFER IN LIQUID METALS IN VERTICAL RECTANGULAR CHANNELS

by

Roger P. Heinisch

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TRANSIENT FREE-CONVECTION HEAT TRANSFER
IN LIQUID METALS IN
VERTICAL RECTANGULAR CHANNELS

by

Roger P. Heinisch

Reactor Engineering Division

Based on a thesis submitted to
the Faculty of Purdue University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

May 1968

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NOMENCLATURE

English

a	Constant
a_i	Approximating-function coefficients (see Eq. 4.25)
a_a	Limit on x in Eq. B.7
a^*	Defined by Eq. C.20
B	Body force
b	Constant
b^*	Parameter appearing in Eq. C.21
b_i	Approximating-function coefficients (see Eq. 4.26)
b_a	Limit on x in Eq. B.7
C_i	Coefficients in collocation scheme (see Eq. 4.31)
c	Specific heat
c_a	Limit on y in Eq. B.7
c_k	Approximating-function coefficients (see Eq. B.4)
c_p	Specific heat at constant pressure
D(u)	Differential or integral operator on u
d_n	Approximating-function coefficients (see Eq. 4.28)
d_a	Limit on y in Eq. B.7
e_n	Approximating-function coefficients (see Eq. 4.29)
erfc	Complimentary error function
F_i	Inhomogeneous portion of Eq. B.7
f_i	Weighting functions (see Eq. 4.17)
f'_i	Derivative of f_i
f_u	Local radiative flux
$f_{u,0}$	Normal radiative flux
Gr	Grashof number, $g\beta\Delta T L^3/\nu^2$
Gr_L^*	Grashof number (modified), $g\beta q_w L^4/\nu^2$
Gr_x	Grashof number, $g\beta\Delta T x^3/\nu^2$
Gr_x^*	Grashof number (modified), $g\beta q_w x^4/\nu^2$
g	Acceleration due to gravity
g_i	Weighting function (see Eqs. 4.18)
g'_i	Derivative of g_i
H(y)	Weighting function (see Eq. B.8)
h	Heat-transfer coefficient
h^*	Finite-difference increment
K	Thermal conductivity of the foil
k	Thermal conductivity of the fluid
L	Height of the plate
L(u)	Differential operator on u
$\mathcal{L}$	Foil thickness
M	Order of approximation
M_0	Parameter appearing in Eq. C.22
N	Order of approximation
Nu_L	Nusselt number, hL/k
Nu_x	Nusselt number, hx/k
P_i	Function of dependent variables in Eq. B.7
Pr	Prandtl number, $\mu c_p/k$
p	Pressure
Q	Defined by Eq. C.16
$\bar{Q}$	Defined by Eq. C.22
Q_i	Function of dependent variables in Eq. B.7
q	Dimensional heat flux
q_R	Dimensional radiant heat flux
Ra_L^*	Rayleigh number, $g\beta q_w L^4/k\alpha\nu$
S	Width of foil
s	Surface
T	Dimensional temperature
$\bar{T}$	Defined by Eq. C.21
T^*	Approximating function for temperature
T_L	Limiting temperature
t	Dimensional time

English

U	Dimensional velocity in the X direction
U_1	Dimensional characteristic velocity, $U_1 = U/(RY/\Delta)$
U_1^*	Characteristic velocity, $U_1 Gr^{-1/2}$
U_0	Characteristic velocity (see Eq. 3.30)
u	Dimensionless velocity
u_1	Dimensionless characteristic velocity, $Gr^{-1/2}(U_1 L/\alpha)$
u_2	Defined by Eq. A.5
u_j	Value of u at $x = j\Delta x$
V	Dimensional velocity on Y direction
v	Dimensionless velocity, VL/α
v^*	Arbitrary function
W	Test-section width
w	Weighting function
X	Dimensionless distance from leading edge of channel
x	Dimensionless distance from leading edge of channel, X/L
Y	Dimensional distance from foil
y	Dimensionless distance from foil, Y/W
Z	Dimensional distance horizontally along foil

Greek

α	Thermal diffusivity
α_i	Approximating-function coefficients (see Eq. 4.41)
β	Coefficient of thermal expansion
β_i	Approximating-function coefficients (see Eq. 4.42)
Γ_i	Interpolation functions for modified method of lines
Δ	Dimensional boundary-layer thickness
Δ^*	Boundary-layer thickness, $[Gr^{1/4} \Delta(x, t)]$
Δx	Finite-difference increment on x
δ	Dimensionless boundary-layer thickness, $Gr^{1/4}(\Delta/L)$
ϵ_n	Total normal emissivity
η	Dimensionless length, Y/Δ
θ	Dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$
Λ	Approximating functions (see Eqs. B.6)
λ	Wavelength
μ	Dynamic viscosity
ν	Kinematic viscosity
ξ	Transformation variable, $x/(1+x)$
ρ	Density
τ	Dimensionless time, $Gr^{1/2} \alpha t/L^2$
τ^*	Modeling time (see Eq. 5.1)
ϕ_i	Approximating functions (see Eq. B.1)
ψ_i, ψ_ℓ	Approximating functions (see Eq. B.14)

Subscripts

A	Refers to $y = 0$
B	Refers to $y = 1$
C	Cold wall
c	Centerline
conv.	Convection
f	Properties evaluated at average temperature, i.e., $(T + T_w)/2$
H	Hot wall
inst.	Instantaneous
k	Number of dependent variables
m	Number of dependent variables
n	Order of approximation
q.s.	Quasi-steady
T	Thermal
v	Viscous
w	Wall
x	Refers to x location
∞	Ambient conditions

TRANSIENT FREE-CONVECTION HEAT TRANSFER IN LIQUID METALS IN VERTICAL RECTANGULAR CHANNELS

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SUMMARY

This report presents an analytical and experimental investigation of transient free convection in a narrow, vertical, parallel-plane channel following a step change in uniform wall heat flux.

In the analysis, the time-dependent boundary-layer equations are formulated for two-dimensional flow and heat transfer with no internal heat generation and constant physical properties, except in the body-force term. These equations are solved by a technique generally referred to as the method of integral relations. The analysis is checked by using the method of lines to ascertain the validity of the results. The results obtained by the two methods are generally in good agreement. Solutions are obtained for various Prandtl numbers. Two boundary conditions are considered in the analysis: constant wall temperature and constant wall heat flux.

The construction of an experimental apparatus and instrumentation for various test sections is described. Three test sections were used to form a heated rectangular channel 6 x 1 x 10 in. Test Section I consisted of both 6 x 10-in. faces heated by a uniform heat flux with mercury as the fluid; Test Section II, the same boundary conditions as Test Section I, with lead as a heat-transfer medium to thermally model the mercury; and Test Section III, one 6 x 10-in. surface heated by a uniform heat flux, with the second surface maintained at constant ambient temperature using mercury as the fluid.

The experiments were performed with mercury since its small Prandtl number is typical of other liquid metals presently under consideration for use as fast-reactor coolants. Transient and steady-state heat-transfer results are reported. Initially the fluid was stagnant and at a uniform temperature. The transient was initiated by using radiant heating to suddenly increase the wall heat flux from zero to some constant pre-selected value. Temperature-time histories of the wall and fluid were recorded during the transient and steady-state periods at several locations on the wall and in the flow field. Developing local and average Nusselt numbers were determined for various operating conditions. The theoretical and experimental steady-state limiting cases were in good agreement.

The results show that when the wall heat flux is sufficiently large on both faces, the mechanism for energy transport in the fluid is predominantly by conduction. For lower values of imposed wall heat flux or when one wall is maintained at a constant temperature, the steady-state condition was reached with convection as the predominant mechanism of heat transfer. Turbulence levels inferred from temperature-time traces in the fluid indicated that the fluid experienced a greater degree of local turbulence for Test Section III than for Test Section I. Approximate values of Rayleigh number are obtained for the transition for laminar to turbulent flow.

Measurements in the inlet section indicate that axial conduction is quite small for all test sections used. Dimensionless temperature profiles are obtained for various axial locations in the duct for all test sections used.

I. INTRODUCTION

Transient fluid-flow problems are among the more difficult ones to analyze in engineering sciences and yet are often encountered in modern-day technology. Until quite recently, unsteady flows have received relatively little attention. However, transient convection heat transfer is presently becoming increasingly important. Hence there is a great need for a more complete understanding of the phenomenon. The literature contains little experimental data on the steady-state and even less on transient free convection in liquid metals.

This report is concerned with an experiment on and an analysis of two-dimensional transient and steady-state free-convection heat transfer. Detailed understanding of transient free convection resulting from a step change in surface temperature or wall heat flux has become increasingly important because of the many applications in the design of advanced engineering systems where the above conditions may be approximated. Thus, in the experiment, the constant* uniform-heat-flux boundary condition is approximated, while in the analysis, both boundary conditions are considered.

In the analytical part of the work, the transient free-convection heat transfer from a vertical, semi-infinite flat plate is investigated. The problem is formulated using Prandtl's boundary-layer theory. In essence, the theory provides approximate solutions to the Navier-Stokes equations for the viscous external or internal flow, where the flow is induced by a pressure force and/or a body force such as gravity.

Because of the mathematical complexity involved in the solutions of the boundary-layer equations, only a few exact numerical solutions

* In this report, "constant" refers to temporal invariance, and "uniform" implies spacial homogeneity.

have been reported in the literature. Under special instances (such as uniform and nonuniform wall temperature or wall-heat-flux boundary conditions), the steady-state problem can be reduced to a system of ordinary nonlinear differential equations by means of a transformation of dependent and independent variables. These solutions are called similar solutions, but often they do not describe a physically realistic problem. They are useful for obtaining initial conditions for nonsimilar flows and for gaining insight into the phenomenon. Although formal similarity variables have been obtained for transient free convection from a semi-infinite vertical flat plate,¹⁻³ no numerical results for that problem have been presented in the literature. Thus, solutions to the system of nonlinear, coupled, partial differential equations which describe transient free-convection heat transfer over a vertical flat plate are attempted using approximate techniques. These solutions are obtained by two different methods, and comparisons are made to provide confidence in the results.

The system of partial differential equations in three independent variables is reduced by an integral technique to a system in two independent variables. The resultant coupled, nonlinear system of partial differential equations is again reduced by two separate methods. The first of these is referred to as the method of integral relations. The other method is known as the method of lines, an explicit finite-difference scheme. Approximate temperature and velocity distributions are obtained by both methods.

The experiment described here was concerned with the measurement of transient laminar and turbulent free-convection heat transfer of mercury in rectangular vertical channels with two different types of boundary conditions: (a) uniform heat flux imposed on each face; and (b) uniform heat flux imposed on one surface, while the other surface was maintained at constant temperature. With a thin foil for the heated surface, it was possible to obtain practically a step change in uniform heat flux in a somewhat unique manner by using thermal-radiation heating.

To obtain the greatest amount of information from the experiment, local fluid and wall temperature had to be measured. From the knowledge of the heat flux, modified Grashof and Rayleigh numbers were computed; and from the measurement of temperatures, local and average Nusselt numbers were calculated.

To isolate the different mechanisms of energy transport, the mercury in the test section was replaced by a block of lead instrumented with thermocouples. To thermally model the mercury in the test section, the time scale in the lead system was corrected to account for property variations. In this manner, the deviation from conduction was measured.

An attempt was made to establish an empirical correlation for the heat transfer and a criterion for the stability of these flows. In the present context, the stability refers to the transition from laminar to turbulent flow. Since heat-transfer coefficients differ considerably in the laminar- and the turbulent-flow regimes, the ability to predict the transition from laminar to turbulent flow is of practical importance in applications where heat transfer is due primarily to free convection.

Chapter II presents a literature survey to establish a relation of the present investigation to previous studies reported in the literature. Chapter III formulates the problem and simplifies the governing equations. Chapter IV develops the approximate methods used in solving the conservation equations. The problem is reduced for solution on the digital computer, and limiting analytical solutions are presented. Chapter V describes the experimental apparatus and the design criteria used. Chapter VI presents and compares results of the analysis and the experiment. Chapter VII presents the conclusions based on the study and makes recommendations for further investigations.

II. REVIEW OF THE LITERATURE

A. Steady-state Studies

During the past century, many studies have been devoted to free-convection heat transfer. Most of these investigations have been concerned with steady-state heat transfer. Only recently has work been reported on transient free convection. This is primarily due to the previous apparent lack of interest, the mathematical complexity of the problem, and the experimental difficulties involved in studies of this kind. Because of the many studies available, a comprehensive literature review will not be attempted. Instead, reference will be made to existing surveys.

Oberbeck⁵ in 1879 was the first to obtain the general governing equations for the general free-convection problem. In 1881, Lorenz⁶ simplified the equations derived by Oberbeck by introducing additional assumptions, which included steady state. His results for the average heat-transfer coefficient are given by

$$\text{Nu} = 0.548(\text{GrPr})^{1/4}. \quad (2.1)$$

This simple analysis achieved an excellent degree of success. It showed the manner in which the variables affect the heat-transfer coefficient and predicted the proper functional relation. The numerical constant and exponent are essentially correct, even when compared with the exact results. These were just the first attempts in a long list of studies dealing with steady-state free-convection heat transfer. Two of the more comprehensive surveys of the important steady-state contributions in the English literature are those by Ede.^{7,8} Ostrach⁹ also gave an extensive review. Ostroumov^{10,11} surveyed the Russian literature on transient and steady-state free-convection heat transfer.

Of particular note is a survey article of the Russian experimental work by Fedynskiy¹² discussing a number of steady-state correlations for external natural convection of liquid metals over tubes, plates, and spheres. The metals used were mercury, tin, bismuth-lead alloy, and sodium. Finally, the early work of Saunders¹³ on the steady-state free-convection of mercury should be noted in which a vertical flat plate was submerged in a large mass of fluid. The comparison between experiment and existing theory was quite good.

B. Transient Analytical Studies

One of the first analytical studies of transient free convection appears to be that by Illingsworth¹⁴ in 1950. This analysis used the boundary-layer approach for the flow of a perfect gas over an infinite vertical flat plate, which is suddenly subjected to a step change in wall temperature. The leading-edge effects were neglected. With this assumption,

the equations reduced to the form of the transient heat-conduction equation and were then solved by standard procedures.¹⁵

In 1951, Sugawara and Michiyoshi¹⁶ neglected the convective terms in the energy equation and solved the partial differential equation

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (2.2)$$

by the method of successive approximations. Using the results of the first approximation, they solved the energy and momentum equations with the convective terms included. This calculation is concerned primarily with the early phases of boundary layer growth. One controversial point was discussed by Siegel¹⁷ in a comparison of his results with those predicted by Sugawara and Michiyoshi. Siegel noted that the temperature gradients calculated near the wall at large distances from the leading edge are greater than those given by the solution of Eq. 2.2. Thus in Ref. 16, either the method is not carried out sufficiently far to satisfy convergence criteria or numerical errors exist in the analysis.

Using the Kármán-Pohlhausen integral method, Siegel¹⁷ reduced the governing equations for the transient free-convection problem from a vertical infinite flat plate from three to two independent variables for both a step change in wall temperature and wall heat flux. He used several assumed temperature and velocity profiles to reduce the original equations. The limiting solutions were compared with the exact solution to study the effect of the various profiles assumed. The resultant nonlinear hyperbolic partial-differential equations were amenable to closed-form solution using the method of characteristics. The characteristic equations describing the limits of the initial transient, pure transient, and steady-state regimes were solved, but the original equation for intermediate values of time were not solved. The simple analytical expressions obtained for the defining relations may be considered a classical solution of an engineering problem.

At approximately the same time, there appeared a paper by Sevruck¹⁸ on transient convection in a fluid between two concentric spheres of only slightly different diameters. The two spheres were suddenly brought to, and maintained at, temperatures different from the initial fluid temperature. Sevruck assumed a solution in a form of a power series in the Rayleigh number and, considering only the zeroth and first perturbations, arrived at a solution for the temperature and velocity distributions in terms of the stream function. The stream function, in turn, was expressible as a series of integrals of Bessel functions.

Sparrow and Gregg¹⁹ presented a solution to a nearly quasi-steady free-convection problem. This terminology refers to the temperature and

velocity distributions in a boundary layer as being identical to those of completely steady state for the given instantaneous boundary conditions, even though the actual boundary conditions change with time. This mathematical situation would be best approximated physically when the time variation in the boundary conditions is relatively small. Sparrow and Gregg expanded the dimensionless temperature and stream function in the form of a series about the quasi-steady-state solution. When the first derivative of the wall temperature with respect to time is much greater than high-order derivatives, the ratio of local instantaneous heat-transfer coefficient to local quasi-steady heat-transfer coefficient for $Pr = 0.72$ is

$$\frac{h_{inst.}}{h_{q.s.}} = 1 + 1.53 \frac{d\theta}{dt} \left(\frac{x}{g\beta\theta} \right)^{1/2} + \dots \quad (2.3)$$

For quasi-steady relationships to be valid, $(d\theta/dt)(x/g\beta\theta)^{1/2} \leq 0.033$. This constant was found to depend on the Prandtl number.

Yang² obtained similarity solutions for unsteady free convection for the following cases: (a) the surface temperature varying inversely as a linear combination of the distance from the leading edge and time, (b) the surface temperature varying directly with a linear function of the above space coordinate and inversely with the square of a linear function of time, and (c) the surface temperature being kept uniform but varying with time at large distances from the leading edge.

Gebhart²⁰ analyzed the problem of transient free convection from flat vertical plates using his own dimensionless variables together with an integral method of analysis. The assumption of similarity between the temperature and velocity profiles led to two simultaneous differential equations, which were solved numerically. Interpretation of his results is somewhat difficult.

Bodoia and Osterle²¹ investigated the transient development of free convection between heated vertical plates. The governing equations were expressed in a finite-difference form and then solved numerically. Chung and Anderson²² used a perturbation technique for quasi-steady boundary-layer flows and reported solutions for the transient free-convection flow over a vertical plate and around a horizontal cylinder. Their work is primarily an extension of Ref. 19.

Hellums²³ described the first numerical solution of the transient free-convection problem from a vertical semi-infinite plate. The original purpose of his study was to obtain only the steady-state solutions, but he also obtained the transient solution to the free-convection problem for $Pr = 0.72$. There is some doubt as to how he obtained the initial conditions for starting the numerical integration scheme. Neither his thesis nor the paper²⁴ based on it discussed how to start the computations. Yang²⁵ was

also unable to determine the starting procedure. Thus, although the steady-state values agree with published results, there is doubt concerning the transient results presented. Churchill²⁶ gave an excellent summary of the finite-difference solutions obtained for various free-convection problems.

Simultaneously, Menold and Yang²⁷ and Schetz and Eichhorn²⁸ obtained the exact transient free-convection solutions that are recognized as being in the general class of physical problems known as "parallel" flows.²⁹ These solutions can be used as standards of comparison for approximate analytical methods.

Gebhart^{30,31} extended his original analysis to include the effects of the heat capacity of the wall on the transient and to predict the boundary between the regimes of true convection and quasi-steady processes. Gebhart *et al.*³²⁻³⁴ experimentally verified his previously discussed analysis. In general, there was good agreement with the analysis for a wide range of thermal capacities.

The above brief survey is not complete and does not include all the analytical solutions known to have been published on free-convection heat transfer. Only the studies on the transient free-convection flow from a vertical flat plate have been referred to. Hopefully, the preceding serves as a guide to the techniques used for the solution of problems of this type. An obvious conclusion is that simple closed-form solutions exist only in the most trivial case. To obtain solutions to practical problems, rigor and accuracy must be sacrificed for ease of mathematical computation. Thus, one is led to approximate methods for solving this type of problem.

C. Transient Experimental Studies

The experimental investigations of transient free convection should also be mentioned. The two earliest studies of transient free convection were those by Ostroumov³⁵ and Klei.³⁶ In the former work, the geometry used was that of a horizontal cylinder formed by a 0.1-mm-dia platinum wire immersed in ethyl alcohol, water, or air. The wire was heated by a suddenly applied direct current. A plot of the transient wire temperature showed that when the fluid used was alcohol, the maximum was greater than the steady-state value. In water, a similar, but less pronounced, overshoot was observed; in air, no such overshoot could be detected. This was the first observation of the "temperature overshoot" effect that has received so much discussion since 1958. An obvious fault overlooked by Ostroumov was that in air the heat capacity of the wire is large in comparison with that of the fluid in the boundary layer. Thus, the assumed boundary condition of a step input in the heat flux is only poorly approximated. This failure to meet the desired boundary condition could account for the lack of an observed overshoot in the wire temperature.

Klei's work was conducted on a vertical 0.0005-in.-thick platinum foil, 1 in. high and 18 in. long, immersed in air. Although the experimental technique was relatively crude, the trends of the data also indicate a temperature overshoot. The results obtained in this work are briefly discussed by Siegel.¹⁷

Goldstein and Eckert³⁷ used the Mach-Zender interferometer in a study of the transient and steady-state free-convection boundary layers in air and water about a vertical plate. The 0.001-in.-thick stainless steel foil was heated electrically. No apparent overshoot as reported by previous investigators was observed, but near the end of the conduction phase the determination of $d\theta/dy|_{y=0}$ by extrapolation was difficult. This apparently introduced inaccuracies. These are evidenced by scatter of the data in Figs. 10 and 11 of Ref. 37. The scatter of data is confined only to the regime of the temperature profile where overshoot would occur.

Lillie³⁸ experimentally investigated the steady-state free-convection flow of a sodium-potassium alloy (NaK). Two parallel vertical plates formed the test section, which was heated by imposing a uniform heat flux electrically on one face while maintaining the second face at constant temperature. Because of a mechanical failure, no local temperature measurements in the fluid were obtained. The wall temperatures were measured, and a bulk temperature of the fluid was inferred from them.

Novotny and Eckert³⁹ obtained experimental data on laminar free convection in a parallel-plate channel using a Mach-Zender interferometer. The fluid was initially stagnant and at a uniform temperature. Heat was suddenly generated in the fluid. Good agreement was obtained between existing theory and experimental data for the range of parameters studied.

Maahs⁴⁰ analytically and experimentally investigated the transient natural convection in a horizontal cylinder. The analytical portion of his work consisted of an application of the Kármán-Pohlhausen integral method. The two resulting nonlinear partial differential equations were solved by successive approximations. This approach was checked by a finite-difference scheme. Experimentally, a wide range of fluids, Prandtl numbers from 300 to 0.7, were used to fill a long horizontal copper cylinder, which was heated by applying a constant circumferentially uniform heat flux to the outside wall.

The literature survey indicates that there is a lack of information available on the steady-state and transient free-convection heat transfer for fluids characterized by a low Prandtl number. Experimental data on the steady-state flow of low-Prandtl-number fluids have been presented for external flows only. No temperature profiles within channels have yet been reported for liquid metals. A determination of the temperature overshoot and its magnitude is desirable for a narrow channel containing

a liquid metal. In particular, a determination of the heat-transfer mechanisms involved in the free convection of low-Prandtl-number fluids is of primary importance.

The transient free-convection problem of a low-Prandtl-number fluid in a vertical channel has not, to my knowledge, been investigated either experimentally or analytically.

III. MATHEMATICAL FORMULATION OF PROBLEM

A. Introduction

The state and motion of a fluid are specified by its velocity vector and two independent thermodynamic properties as functions of position and time. These functions are, in turn, based on the differential equations deduced from the principles of the conservation of mass, momentum, and energy. These equations are given below without a derivation of the mathematical description; instead reference is made to the published literature.^{9,29,42} For completeness, the more important assumptions upon which the derivations depend are discussed.

B. Basic Equations

To attain appreciable compression of liquids (and solids), extremely high pressures of the order of thousands of atmospheres are required. For this reason, liquids can generally be treated as incompressible fluids. The density changes being small, the velocity of a liquid in free-convection flow is much smaller than the speed of sound. Thus, the speed of sound can serve as the characteristic velocity scale in the description of continuous media.

In the present problem, the flow of a nuclear-reactor coolant is of particular interest. The geometry chosen is one of a vertical parallel-plate channel. A rectangular Cartesian-coordinate system chosen is fixed with respect to the fluid as shown in Fig. 1, such that the Y axis is perpendicular to the plate and the X axis is in the plane of, and in a direction parallel to, the plate. The fluid motion is in the X-Y plane, and the dependent variables are assumed to be independent of the Z direction.

Fig. 1
Physical Model and
Coordinate System

The following assumptions are made in the present analysis:

1. The fluid is continuous, isotropic, Newtonian, and incompressible.
2. The flow is two-dimensional.
3. There is no internal heat generation.
4. Thermal radiation is negligible compared to conduction or convection.
5. Viscous dissipation is negligible compared to conduction or convection.
6. The physical properties are independent of temperature, except in the formulation of the buoyancy force.
7. Molecular-heat conduction is given by the Fourier-Biot law.

The results obtained experimentally and analytically for steady-state free convection indicate that Assumption 2 is valid. Assumption 1 is discussed above, and Assumption 3 is an imposed condition that is experimentally controllable. Since the fluids of interest in this study (liquid metals) are opaque to thermal radiation, idealization 4 is a justifiable one for the temperature ranges under consideration. Ostrach⁹ by the application of dimensional analysis showed that viscous dissipation under normal free-convection flow conditions can be neglected; thus Assumption 5 is also justifiable under present conditions. Assumption 7 is actually the phenomenological basis of derivation for the energy equation and has been shown to give an accurate description of molecular-heat conduction under a wide range of conditions.

For an incompressible fluid, the density is constant with respect to space and time. Thus, the conservation of mass or the continuity equation can be written in vectorial form as⁴²

$$\nabla \cdot \bar{u} = 0. \quad (3.1)$$

The conservation-of-momentum equations for an incompressible fluid is written as⁴²

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}(\nabla \cdot \bar{u}) = \bar{B} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \bar{u}. \quad (3.2)$$

The left-hand side represents inertial forces; the terms on the right-hand side represent the body, pressure, and viscous forces, respectively.

The conservation-of-energy equation is written as⁴²

$$\rho c \left(\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T \right) = \nabla \cdot (k \nabla T) - T \left(\frac{\partial p}{\partial T} \right)_v (\nabla \cdot \bar{u}). \quad (3.3)$$

The left-hand side of this equation represents the total rate of change of all sensible forms of energy (internal energy) at a given point in space. The first term on the right-hand side represents the rate of energy change due to conduction; the second term accounts for the rate of work done by the pressure forces.

Equations 3.1-3.3, together with the equation of state, are sufficient to determine the velocity field and the temperature distribution once initial and boundary conditions are specified. Because of the nonlinear character and a very complex nature of the system of equations, additional assumptions are required before further progress can be made.

For a vertical semi-infinite flat plate, which is being used as a model in this investigation, the boundary-layer theory yields reasonably accurate results.⁹ This method considers the flow field about a body as

being divided into two regimes. Near the body, the viscous effects are assumed to predominate; in the outer flow, the fluid is assumed inviscid. An analogy with the forced-convection flow situation was made by Schmidt and Beckman,⁴³ who derived the boundary-layer equations for steady-state free convection. They assumed that the motion is confined within the boundary layer only. Far from the plate, the medium is assumed to be stagnant. Under these conditions, the equation for conservation of momentum in the X direction reduces to

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = B_x - \frac{1}{\rho} \frac{\partial p}{\partial X} + \nu \frac{\partial^2 U}{\partial Y^2}, \quad (3.4)$$

and the equation for the conservative of momentum in the Y direction becomes

$$\frac{\partial p}{\partial Y} = 0. \quad (3.5)$$

The energy equation is reduced to

$$\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \alpha \frac{\partial^2 T}{\partial Y^2} \quad (3.6)$$

when axial heat conduction is neglected in comparison to transverse heat conduction. For liquid metals, this restriction may be invalid but is made to render the equation more tractable mathematically. In the experimental portion of this study, data will be presented to substantiate this assumption for the flow conditions arising in the actual systems. The body force B_x for the system under consideration is

$$-B_x = g. \quad (3.7)$$

Finally, evaluating the axial pressure gradient at the free-stream (stagnant) conditions, we obtain

$$\frac{\partial p}{\partial X} = -\rho_\infty g. \quad (3.8)$$

Expanding the density in a Taylor's series as a function of temperature about the free-stream density results in

$$\rho = \rho_\infty + \left. \frac{\partial \rho}{\partial T} \right|_{T_\infty} (T - T_\infty) + \left. \frac{\partial^2 \rho}{\partial T^2} \right|_{T_\infty} \frac{(T - T_\infty)^2}{2} + \dots \quad (3.9)$$

For small variations in the density characteristic of free convection, one obtains, after retaining only the first two terms of Eq. 3.9 the approximation

$$\frac{\rho - \rho_\infty}{\rho} = \frac{1}{\rho} \frac{\partial \rho}{\partial T} (T - T_\infty) = -\beta(T - T_\infty), \quad (3.10)$$

where the coefficient of thermal expansion, β , is defined by

$$\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \Big|_p.$$

The resultant equations to be solved under the assumptions set forth above are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (3.11)$$

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = g\beta(T - T_\infty) + \nu \frac{\partial^2 U}{\partial Y^2}, \quad (3.12)$$

and

$$\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \alpha \frac{\partial^2 T}{\partial Y^2}. \quad (3.13)$$

In the above system, there are three equations and three unknowns in three independent variables. Thus, the problem is mathematically well defined and could be solved once the appropriate boundary and initial conditions are imposed. Two separate cases are of particular physical interest: (a) the plate is suddenly raised to a uniform temperature; and (b) the plate is suddenly heated by a uniform heat flux.

The boundary conditions for Case a are expressed as

$$U(X, 0, t) = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.14)$$

$$U(X, \infty, t) = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.15)$$

$$\frac{\partial U}{\partial Y} \Big|_{Y \rightarrow \infty} = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.16)$$

$$U(0, Y, t) = 0, \quad 0 < Y < \infty, \quad t > 0; \quad (3.17)$$

$$T(X, 0, t) = T_w, \quad 0 < X < \infty, \quad t > 0; \quad (3.18)$$

$$T(X, \infty, t) = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.19)$$

and

$$\frac{\partial T}{\partial Y}(X, \infty, t) = 0, \quad 0 < X < \infty, \quad t > 0. \quad (3.20)$$

For Case b the boundary conditions are expressed as

$$U(X, 0, t) = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.21)$$

$$U(X, \infty, t) = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.22)$$

$$\left. \frac{\partial U}{\partial Y} \right|_{Y \rightarrow \infty} = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.23)$$

$$U(0, Y, t) = 0, \quad 0 < Y < \infty, \quad t > 0; \quad (3.24)$$

$$-k \left. \frac{\partial T}{\partial Y} \right|_{Y \rightarrow 0} = q_w, \quad 0 < X < \infty, \quad t > 0; \quad (3.25)$$

$$\left. \frac{\partial T}{\partial Y} \right|_{Y \rightarrow \infty} = 0, \quad 0 < X < \infty, \quad t > 0; \quad (3.26)$$

and

$$T(X, \infty, t) = 0, \quad 0 < X < \infty, \quad t > 0. \quad (3.27)$$

The initial conditions are the same for both cases and may be written as

$$U(X, Y, 0) = 0, \quad 0 < X < \infty, \quad 0 < Y < \infty; \quad (3.28)$$

and

$$T(X, Y, 0) = 0, \quad 0 < X < \infty, \quad 0 < Y < \infty. \quad (3.29)$$

Equations 3.15, 3.16, 3.22, and 3.23 represent the mathematical conditions imposed by the physical assumption of the existence of a boundary layer. These conditions are probably not satisfied exactly in the experiment, except in the entrance portion of the test section. Thus, this solution is presumed to be valid only in the "starting length" and is at best an approximation elsewhere. This assumption probably is a severe restriction of the analysis.

C. Dimensionless Equations

The nondimensionalization of the governing equations is accomplished by letting

$$L = \left[\frac{\alpha^2}{g\beta(T_w - T_\infty)} \right]^{1/3}; \quad U_0 = [\alpha g(T_w - T_\infty) \beta]^{1/3};$$

and

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}; \quad (3.30)$$

(Contd.)

$$u = \frac{U}{U_0}; v = \frac{V}{U_0}; \tau = \frac{\alpha t}{L^2};$$

$$\text{Pr} = \frac{\nu}{\alpha}. \quad \text{(Contd.)} \quad (3.30)$$

The justification for using the above dimensional characteristic length and velocity is discussed in Appendix A. In general, a reduction to the least possible number of parameters is desirable. Other non-dimensionalization schemes have been used for transient free-convection problems.^{13,17,20} Hellums and Churchill,⁴⁴ however, indicated that this reduction is of primary importance in the analysis of any physical-mathematical problem and presented a detailed systematic method for obtaining dimensionless variables. The governing equations in terms of the above dimensionless variables are written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3.31)$$

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \theta + \text{Pr} \frac{\partial^2 u}{\partial y^2}, \quad (3.32)$$

and

$$\frac{\partial \theta}{\partial \tau} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial y^2}. \quad (3.33)$$

The coupling of the velocity and temperature fields in free-convection flow is evidenced by Eqs. 3.32 and 3.33. Before proceeding further, we should discuss the mathematical character of the governing equations. The system of equations above is nonlinear and parabolic. A singularity exists at the leading edge of the plate, which causes the solutions of the boundary-layer equations to be invalid very close to $x = y = 0$.

IV. ANALYTICAL METHODS

A. Introduction

No general methods of solution are presently available for the solution of nonlinear partial differential equations. In fact, the numerical treatment of these problems has not been presented in even a remotely exhaustive manner. To date the literature has presented specific problems in certain applied fields, their solutions, and the specific techniques developed for their solution. The only book even approaching a comprehensive compendium of these available methods is that of Collatz.⁶⁸

To quote Collatz is perhaps the best way to demonstrate the rationale behind the analytical portion of this work:

"I have always observed that graduate mathematicians and physicists are very well acquainted with theoretical results, but have no knowledge of the simplest approximate methods. If approximate methods were more well known, perhaps many problems would be solved with their aid which hitherto have simply not been tackled, despite the fact that interest in their solution has existed throughout."

The problem defined by Eqs. 3.31-3.33, together with appropriate boundary and initial conditions (Eqs. 3.14-3.29), is complete, but the solution of this second-order, coupled, nonlinear system of parabolic partial-differential equations is a formidable task. Rigor and some accuracy can be sacrificed for the sake of ease of manipulation and simplicity in performing the calculations. An account is presented here of two heuristic methods, which have been most useful in the foregoing context. Neither method is supported by any body of rigorous mathematical analysis, nor is either of such a nature that they can be suggested a priori. Both take a certain art in their application. Each appears to be useful over a broad range of conditions, and each is successful in solving the present problem.

The first of these methods is the method of integral relations, often called Dorodnitsyn's method.⁵⁴ The Kármán-Pohlhausen method has been shown to be⁴ a subset of the method of integral relations. This technique has been used to solve many⁵³⁻⁵⁶ complex problems of fluid mechanics and heat transfer. The desire for an engineering-type solution, together with the past successes of the method of integral relations, has led to its use in this work.

In the method of integral relations, one seeks to obtain an approximate solution of a partial differential equation or a system of partial differential equations by assuming the solution in functional form such that impose boundary conditions on the approximated dependent variable are satisfied. Approximate satisfaction of the given equation is obtained by attempting to minimize the error between the assumed functional dependence and the exact dependence.

The second method used to check the approximate solution of the once-reduced problem is the method of lines⁵⁸ (often referred to in the English literature as the differential-difference method). This numerical method is explicit in nature and thus tends to be unstable. However, the relative ease of application of the method permits its use with minimum effort. Thus it can be extremely useful when approximate computations require cursory examination to ascertain their validity. An advanced formulation of the method of lines is also obtained and used in Appendix B to demonstrate improved convergence.

B. Approximate Solution by the Method of Integral Relations

The system of Eqs. 3.31-3.33 is solved approximately in this section. The solutions proposed are obtained from the integral forms of the above system by the method of integral relations. Appendix B contains a detailed discussion of the method of integral relations and a brief demonstration of other related approximate techniques. Essentially, the initial technique used to reduce the problem from the x, y, τ plane to the x, τ plane is recognized as a one-parameter method of the general class of integral relations that follow closely the work of Siegel.¹⁷ Approximate velocity and temperature profiles are assumed so that solutions for short and long time approach the exact limiting solutions most accurately. In this section, the problem in the x, τ plane is also reduced to the τ plane by an additional application of the method of integral relations.

A comment is in order regarding the selection of the functional forms for the approximating functions. In the literature, few attempts have been made to establish general criteria by which to choose the "best" functional for use in the method of integral relations. The best functional, in the language of the calculus of variations, requires that the chosen functions should be admissible for the particular problem. This is a restriction only in the sense that the function must satisfy the boundary conditions of the problem, in addition to other mathematical condition like continuity and smoothness. The initial choice of the functions has a great influence on the degree of complexity of the subsequent computation. Therefore any successful application of the method depends to a great extent upon these functions.

Perhaps the only requirement placed on the trial functions in the method of integral relations is that they be a mathematically complete set for the convergence of the approximate solution to the exact solution. That a mathematically complete set is a necessary condition for the convergence of the approximate solution was demonstrated for specific problems and then generalized by Bethel and Abbott.⁴ In fact, they obtain solutions to several incompressible laminar boundary-layer flow problems that had been previously solved by similarity or other techniques such that the results were known to be either exact or numerically accurate. They were essentially

able to obtain as accurate a solution as required by taking higher- and higher-order approximations. However, the two most pertinent conclusions they reached were: (a) "the accuracy of the GKD[†] method in the third and fourth order of approximation is comparable with that obtained at relatively high expenditures with the implicit finite difference method..." and, (b) "...the second order solution with the GKD method is more accurate than some of the solutions obtained with the finite difference method."

If the work of Dorodnitsyn⁵⁴ and subsequent authors⁴ were followed more closely, a change of variables would transform the independent variables from (x, y) to (x, u) . The purpose of this transformation is to define the location of "infinity" more accurately. For free convection, however, the velocity vanishes at the wall as well as at the edge of the boundary layer. Changing the independent variables in an analogous manner to forced flow leads only to a trivial range of an independent variable; that is, zero. Thus no change of variable is effected; instead, the boundary-layer thickness concept is used.²⁹

After integrating the momentum and energy equations with respect to y across the boundary layer, one must assume a form for the functions approximating the velocity and temperature profiles. Because of the asymptotic nature of the velocity and temperature as $y \rightarrow \infty$, exponential and/or error functions may be a better approximation than polynomials. In fact, some attempts were made in this respect. Polynomials yielded poorer results compared to the profiles used in this work. When polynomials are used, the number of the resultant systems of partial-differential equations also render further computations unwieldy. The forms of the approximate profiles adopted result in only a pair of coupled, nonlinear partial differential equations. They are an exact solution of the pure conduction and a good representation of the steady-state limiting solutions.

1. Initial Reduction from x, y, τ to x, τ Plane

a. Constant-wall-temperature Boundary Condition. For the boundary condition given by Eq. 3.18 (that is, the plate suddenly subjected to a sudden uniform change in temperature), the profiles assumed are similar to those originally used by Rutkowski⁶⁴ and extended later by Goldstein in a discussion of Ref. 17:

$$\frac{U}{U_1} = \eta \left[\frac{e^{-\eta^2}}{\sqrt{\pi}} - \operatorname{erfc}(\eta) \right] \quad (4.1)$$

[†]Galerkin, Kantorovich, and Dorodnitsyn.

and

$$\frac{\theta}{\theta_w} = \operatorname{erfc}(\eta), \quad (4.2)$$

where

$$\eta = \frac{Y}{\Delta(x,t)} = \frac{y}{\delta}.$$

The so-called weighting functions are unity, and $\Delta(x, t)$ is the boundary-layer thickness. Bonilla⁶⁵ justified the assumption that the thermal and velocity boundary-layer thicknesses are equal. He showed that this assumption leads to very small errors for the free-convection problem. Later in this report (Figs. 4 and 5), the approximate steady-state velocity and temperature profiles are compared with the exact solutions for low-Prandtl-number fluids. The conclusions of this comparison in general coincide with those of Bonilla; that is,

$$\Delta \cong \Delta_T \cong \Delta_v.$$

Substituting the profiles given by Eqs. 4.1 and 4.2 into Eqs. 3.11-3.13 and integrating with respect to Y from $Y = 0$ to $Y = \infty$, one obtains

$$\frac{1}{6} \frac{\partial u_1 \delta}{\partial \tau} + a \frac{\partial u_1^2 \delta}{\partial x} = \operatorname{Pr}^2 \delta - \operatorname{Pr} \frac{u_1}{\delta} \quad (4.3)$$

and

$$\frac{\partial \delta}{\partial \tau} + b \frac{\partial u_1 \delta}{\partial x} = \frac{2}{\delta}, \quad (4.4)$$

where

$$a = \frac{3\sqrt{2}}{20} - \frac{1}{5}$$

and

$$b = \frac{\sqrt{2} - 1}{6}.$$

(1) Exact Solution for Small Times (Pure-conduction Regime). For the one-dimensional time-dependent problem, δ and u are not functions of the axial space coordinate. The small-time solutions of Eqs. 4.3 and 4.4 are

$$\delta = 2\tau^{1/2} \quad (4.5)$$

and

$$u_1 = \frac{4Pr^2\tau}{1+Pr} \quad (4.6)$$

(2) Exact Solution for Large Times (Steady-state Regime).

Under steady-state conditions, u and δ are no longer functions of time. The solutions of Eqs. 4.3 and 4.4 for the steady-state limiting conditions are

$$\delta = \frac{\Delta^*}{L} = Pr^{-1/2} \left(\frac{8}{3b}\right)^{1/4} \left(\frac{5}{9} + Pr\right)^{1/4} x^{1/4}, \quad (4.7)$$

and

$$u_1 = \frac{U_1^* L}{\alpha} = Pr \left(\frac{8}{3b}\right)^{1/2} \left(\frac{5}{9} + Pr\right)^{-1/2} x^{1/2}. \quad (4.8)$$

Using Eq. 4.7 for $\delta(x)$, one can then compare the assumed temperature and velocity profiles with the exact similar solutions.⁸⁷

For $Pr = 1$, these comparisons are shown in Figs. 2 and 3. The temperature profiles are in excellent agreement.

Fig. 2. Comparison of Approximate with Exact Steady-state Temperature Profiles for $Pr = 1.0$ for the Constant-wall-temperature Boundary Condition

Fig. 3. Comparison of Approximate with Exact Steady-state Velocity Profiles for $Pr = 1.0$ for the Constant-wall-temperature Boundary Condition

The slope at the wall appears to approximate the exact solution very well. The approximate velocity profile differs considerably from the exact solution, but is still seen to be an adequate representation over the entire boundary-layer thickness, and particularly near the wall, where $\eta = 0$. This region is considered important. The wall shear stress predicted from the approximate velocity profile is nearly equal to the exact result. For low-Prandtl-number fluids (i.e., in the liquid-metal range), Figs. 4 and 5 compare the exact⁹⁰ and approximate profiles. Similar to $Pr = 1$, Fig. 4 shows that the temperature profile again agrees exceptionally well with the exact steady-state solution. The velocity profile is a poorer approximation, but is still adequate. Figure 5 compares the temperature distributions for $Pr \pm 0.003$. The predicted profile is a poorer approximation of the exact solution than, say, for $Pr = 1$, but is still a reasonably good representation of the steady-state profile.

Fig. 4. Comparison of Approximate with Exact Steady-state Temperature and Velocity Profiles for $Pr = 0.03$ for the Constant-wall-temperature Boundary Condition

Fig. 5. Comparison of Approximate with Exact Steady-state Temperature Profiles for $Pr = 0.003$ for the Constant-wall-temperature Boundary Condition

The initial conduction-controlled transient and the steady-state limiting solutions give a good approximation of both regimes. It is therefore felt that the one-parameter method for the reduction from three independent variables to two independent variables is a physically reasonable compromise between the desired accuracy and the effort required to obtain solutions.

b. Constant-heat-flux Boundary Condition. For the boundary condition of a suddenly imposed constant wall heat flux, the following profiles were assumed:

$$\frac{U}{U_1} = \eta \left[\frac{e^{-\eta^2}}{\sqrt{\pi}} - \eta \operatorname{erfc}(\eta) \right], \quad (4.9)$$

and

$$\theta = \frac{q\Delta\sqrt{\pi}}{2k} \operatorname{erfc}(\eta). \quad (4.10)$$

Again, substituting Eqs. 4.9 and 4.10 into Eqs. 3.11-3.13 and integrating with respect to Y from $Y = 0$ to $Y = \infty$ yields

$$\frac{1}{6} \frac{\partial u_1 \delta}{\partial \tau} + a \frac{\partial u_1^2 \delta}{\partial x} = \frac{\operatorname{Pr}^2}{\sqrt{2\pi}} \delta^2 - \operatorname{Pr} \frac{u_1}{\delta} \quad (4.11)$$

and

$$\frac{\partial \delta^2}{\partial \tau} + b \frac{\partial u_1 \delta^2}{\partial x} = 2, \quad (4.12)$$

where a and b are the same numerical constants as in Eqs. 4.3 and 4.4.

(1) Exact Solution for Small Times (Pure-conduction Regime). In a similar manner to that of the previous subsection, the small-time solutions of Eqs. 4.11 and 4.12 are

$$\delta = \sqrt{2} \tau^{1/2} \quad (4.13)$$

and

$$u_1 = \frac{3\operatorname{Pr}^2 \tau^{3/2}}{\sqrt{\pi}(1 + 3\operatorname{Pr}/2)}. \quad (4.14)$$

(2) Exact Solution for Large Times (Steady-state Regime). The steady-state solutions of Eqs. 4.11 and 4.12 are

$$\delta = \left(\frac{28a}{b^2} + \frac{4\operatorname{Pr}}{b} \right)^{1/5} (\sqrt{\pi}\operatorname{Pr})^{-1/5} x^{1/5} \quad (4.15)$$

and

$$u_1 = \frac{2}{b} (\sqrt{\pi}\operatorname{Pr}^2)^{2/5} \left(\frac{28a}{b^2} + \frac{4\operatorname{Pr}}{b} \right)^{-2/5} x^{3/5}. \quad (4.16)$$

Using the above expression for $\delta(x)$, one can again compare the approximate and exact steady-state velocity and temperature profiles. The exact⁷⁸ and approximate temperature profiles are compared here only for Prandtl numbers of 1, 0.1, and 0.03, in Figs. 6, 7, and 8, respectively. Results and conclusions are similar to those for the constant-wall-temperature boundary condition. The exact solution for small time does not coincide with the approximate small-time solution for this problem.

Fig. 6. Comparison of Approximate with Exact Steady-state Temperature Profiles for $Pr = 1.0$ for the Uniform-heat-flux Boundary Condition

Fig. 8. Comparison of Approximate with Exact Steady-state Temperature Profiles for $Pr = 0.03$ for the Uniform-heat-flux Boundary Condition

Fig. 7. Comparison of Approximate with Exact Steady-state Temperature Profiles for $Pr = 0.1$ for the Uniform-heat-flux Boundary Condition

as it did for the constant-wall-temperature boundary condition. Figure 9 compares the exact solution for the temperature distribution in a semi-infinite solid subjected to a constant-wall-heat flux given by Jakob⁶⁷ and the approximate profile. There is a 9% difference in the heat-transfer coefficient between the two results.

Fig. 9

Comparison of Approximate Small Time with Exact Solutions for the Uniform-heat-flux Boundary Condition

2. Second Reduction from x, τ to τ Plane

a. Constant-wall-temperature Boundary Condition. To further reduce Eqs. 4.3 and 4.4 to ordinary differential equations, we again use the method of integral relations. Integrating Eqs. 4.3 and 4.4 with respect to x from $x = 0$ to $x = 1$, we obtain

$$\begin{aligned} \frac{1}{6} \frac{d}{d\tau} \int_0^1 f_i u_1 \delta \, dx + a f_i u_1^2 \delta \Big|_{x=1} - a \int_0^1 f_i u_1^2 \delta \, dx = \\ \Pr^2 \int_0^1 f_i \delta \, dx - \Pr \int_0^1 f_i \frac{u_1}{\delta} \, dx \end{aligned} \quad (4.17)$$

and

$$\frac{d}{d\tau} \int_0^1 g_i \delta \, dx + b g_i u_1 \delta \Big|_{x=1} - b \int_0^1 g_i u_1 \delta \, dx = 2 \int_0^1 \frac{g_i}{\delta} \, dx, \quad (4.18)$$

where f_i and g_i are the weighting functions. We must still judiciously approximate the dependence of velocity and boundary-layer thickness on x . We must also assume a form for the weighting function, substitute these into Eqs. 4.17 and 4.18, and carry out the indicated integrations. With a trial-and-error procedure, the "best" approximations from the point of view of stability and convergence of the resultant ordinary non-linear differential equations, for a single-parameter profile, are of the form

$$u_{1,1} = a_1 x^{1/2}, \quad \delta_1 = b_1 x^{1/4}; \quad (4.19)$$

$$f_1 = x, \quad g_1 = x^{1/2}; \quad (4.20)$$

Upon substituting Eqs. 4.19 and 4.20 into Eqs. 4.17 and 4.18 and performing the integrations, we obtain the following ordinary nonlinear differential equations:

$$\frac{da_1 b_1}{d\tau} = \frac{22}{3} \text{Pr} \left(b_1 \text{Pr} - \frac{a_1}{b_1} \right) - \frac{55}{6} a a_1^2 b_1 \quad (4.21)$$

and

$$\frac{db_1}{d\tau} = \frac{14}{5b_1} - \frac{21}{20} b a_1 b_1 \quad (4.22)$$

with the initial conditions $a_1 = b_1 = 0$. The numerical solution of the system of ordinary nonlinear differential equations given by Eqs. 4.21 and 4.22 then results in a complete one-parameter approximate solution of Eqs. 3.11-3.13.

b. Constant-heat-flux Boundary Condition. Similarly, the profiles used for the constant-heat-flux boundary condition in a one-parameter approximation are

$$u_{1,1} = d_1 x^{3/5},$$

$$\delta_1 = e_1 x^{1/5},$$

and

$$f_1 = g_1 = x.$$

Substituting the above approximations in Eqs. 4.11 and 4.12 results in the following nonlinear ordinary differential equations for a one-parameter representation:

$$\frac{de_1 d_1}{d\tau} = 7\text{Pr} \left(\frac{\text{Pr} e_1^2}{\sqrt{2\pi}} - \frac{d_1}{e_1} \right) - \frac{49}{5} a d_1^2 e_1 \quad (4.23)$$

and

$$\frac{de_1}{d\tau} = \frac{6}{5e_1} - \frac{3b}{5} d_1 e_1, \quad (4.24)$$

with the initial conditions $d_1 = e_1 = 0$.

c. Summary. In review, for example, the following sequence of operations was used to reduce the original two-dimensional, transient boundary-layer equations to ordinary differential equations for the constant-wall-temperature boundary condition:

	<u>Equation Numbers</u>
1. Governing equations (x, y, τ)	3.11, 3.12, 3.13
↓	
2. Reduced by integral method to (x, τ) using profiles	4.1, 4.2
↓	
3. Reduced by integral method to (τ) using profiles	4.19, 4.20
↓	
4. Result in ordinary nonlinear differential equations	4.21, 4.22

The reduction from Step 1 to Step 2 in the above chart was sufficiently accurate so that higher-order approximations were considered unnecessary. However, it was felt necessary to check the reduction from Step 2 to Step 3. Initially, higher-order approximations in the method of integral relations were used. Finally, the method of lines was used to go from Step 2 to Step 3. These approaches are described below.

3. Higher-order Approximations by the Method of Integral Relations

To obtain higher-order approximate solutions of Eqs. 4.3 and 4.4, one can use the general concept of the method of integral relations as presented in Appendix B. For an n th-order approximation in the constant-wall-temperature boundary condition,

$$u_{1,n} = a_1 x^{1/2} + a_2 x + \dots + a_n x^{n/2}, \quad (4.25)$$

$$\delta_n = b_1 x^{1/4} + b_2 x^{1/2} + \dots + b_n x^{n/4} \quad (4.26)$$

and

$$f_n = x, x^2, \dots, x^n, \quad g_n = x^{1/2}, x, \dots, x^{n/2} \quad (4.27)$$

are substituted into Eqs. 4.17 and 4.18.

Similarly, for an n th-order approximation for the constant-wall-heat-flux boundary condition, the following expressions are used in the integrated forms of Eqs. 4.11 and 4.12:

$$u_{1,n} = d_1 x^{3/5} + d_2 x^{6/5} + \dots + d_n x^{3n/5}, \quad (4.28)$$

$$\delta_n = e_1 x^{1/5} + e_2 x^{2/5} + \dots + e_n x^{n/5}, \quad (4.29)$$

and

$$f_i = g_i = x + x^2 + \dots + x^i. \quad (4.30)$$

To effectively carry out the numerical computations for higher-order approximations, one must use a collocation scheme to convert the $1/\delta$ term in Eqs. 4.17 and 4.18 to a more useful form. To determine the validity of a collocation scheme to accomplish this conversion, a study similar to the work of Pavlovski,⁶⁶ Dorodnitsyn,⁵⁵ and Bethel and Abbott⁴ was made. This scheme generates a simple analytical expression for inverse polynomials. An example of this is the two-parameter approximation,

$$\frac{1}{b_1(\tau)x^{1/4} + b_2(\tau)x^{1/2}} = C_1 x^{1/4} + C_2 x^{1/2}. \quad (4.31)$$

The new constants, C_1 and C_2 , are then evaluated at two points in the interval $0 \leq x \leq 1$, say $x = 1$ and $x = 1/2$. The solution was obtained using this approximation and compared with the exact expression. For several cases considered, the differences for the function $b_n(\tau)$ obtained by the two methods did not exceed 6% and in general was much smaller than this maximum. Thus, because of the computational ease afforded by the application of this method and the small amount of error introduced by it, the collocation method was used wherever the method of integral relations was employed.

C. Approximate Solution by the Method of Lines

The application of the method of integral relations to reduce the independent variables (x, y, τ) to (x, τ) is considered accurate enough from an engineering point of view so that higher-order approximations need not be considered. However, the reduction of independent variables (x, τ) to (τ) alone requires a separate check.

To obtain an independent solution, Eqs. 4.3 and 4.4 were solved by two different methods: (a) by use of the method of lines, and (b) by the method of lines using the Kantorovich interpolation scheme to reduce the number of strips required. The basis for these two methods is discussed briefly in Appendix B.

1. Standard Method-of-lines Solution

Initially, for Method a the differentials were replaced by central differences; that is,

$$u \frac{\partial u}{\partial x} = u_j \left(\frac{u_{j+1} - u_{j-1}}{2\Delta x} \right). \quad (4.32)$$

This scheme was found to be unconditionally unstable. Some work on the stability of simple nonlinear problems has been reported by Hellums,²³ where, for $u \geq 0$, the expression

$$u \frac{\partial u}{\partial x} = u_j \left(\frac{u_j - u_{j-1}}{\Delta x} \right) \quad (4.33)$$

was found to be stable. Equation 4.32, which intuitively would be expected to be preferred, was also found by Hellums to be unstable and completely useless.

Using Eq. 4.33 to approximate the spacial derivatives in Eqs. 4.3 and 4.4, one obtains the system of ordinary nonlinear differential equations

$$\begin{aligned} \frac{du_{1,i}\delta_i}{d\tau} = & 6Pr^2\delta_i - 6Pr \frac{u_{1,i}}{\delta_i} \\ & - 6a \left[2u_{1,i}\delta_i \left(\frac{u_{1,i} - u_{1,i-1}}{\Delta x} \right) + u_{1,i}^2 \left(\frac{\delta_i - \delta_{i-1}}{x} \right) \right]; \end{aligned} \quad (4.34)$$

$$\frac{d\delta_i}{d\tau} = \frac{2}{\delta_i} - b \left[u_{1,i} \left(\frac{\delta_i - \delta_{i-1}}{\Delta x} \right) + \delta_i \left(\frac{u_{1,i} - u_{1,i-1}}{\Delta x} \right) \right]. \quad (4.35)$$

Equations 4.34 and 4.35 are the mathematical statements of the problem using the ordinary method of lines. The above system of equations was solved numerically using a fifth-order Runge-Kutta forward-integration scheme.

2. Method-of-lines Solution Using Kantorovich Interpolation Scheme

As an example of the Kantorovich interpolation for three lines, the same problem using the improved method of lines is formulated in the following manner. We assume that

$$u^*(x, \tau) = \psi_0 + \psi_1 x + \psi_2 x^2 + \psi_3 x^3$$

and

$$\delta^*(x, \tau) = \Gamma_0 + \Gamma_1 x + \Gamma_2 x^2 + \Gamma_3 x^3,$$

where the ψ_i 's and Γ_i 's are determined by

$$u^*(0, \tau) = 0 = \psi_0,$$

$$\delta^*(0, \tau) = 0 = \Gamma_0,$$

$$u^*(0.25, \tau) = u_1(\tau),$$

$$\delta^*(0.25, \tau) = \delta_1(\tau),$$

$$u^*(0.5, \tau) = u_2(\tau),$$

$$\delta^*(0.5, \tau) = \delta_2(\tau),$$

$$u^*(0.75, \tau) = u_3(\tau),$$

and

$$\delta^*(0.75, \tau) = \delta_3(\tau).$$

From the above expressions, differential equations for the unknown functions ψ_i and Γ_i were obtained and solved numerically. The advantage of the polynomial interpolation across the strips is discussed in Appendix B.

D. Technique for Generating Initial Conditions

Using the method of integral relations, one must solve the system of ordinary, nonlinear, first-order differential equations such as Eqs. 4.3 and 4.4 for a one-parameter approximation. The solution of this system would appear to be straightforward if, say, a typical forward-integration scheme were used to solve the system numerically. The equations are singular at $\tau = 0$, so that starting any numerical integration scheme would appear impossible. The fifth-order, double-precision, Runge-Kutta forward-integration scheme was used. Initial conditions used to initiate the integration were determined in the following manner. An exact analytical short-time solution for the constant-wall-temperature boundary condition was given by Eqs. 4.5 and 4.6. Numerical values for the dependent variables were computed from the short-time solution for a very small time (say 10^{-6}). The numerical integration was then completed and compared with the solution obtained in a like manner for a second value of time (say 10^{-8}). If the two solutions obtained for the same time increment agreed to the fifth significant figure, the initial conditions were assumed to be correct. At this point, the

step size in the forward-integration scheme was reduced by a factor of ten and the solutions were again obtained and compared with the previous values. Using time increments of the order of 10^{-4} ensured an insignificant difference. The results were assumed to be the exact numerical solutions of the reduced differential equations.

Previous investigators^{23, 24} have not discussed a technique used to initiate the integration of the boundary-layer equations. This point was considered essential to an accurate representation of the transient regime. Further verification of the importance of the initial conditions was demonstrated by the numerical calculations. No matter what initial conditions were chosen (as long as the numerical value was a reasonable one), the forward integration resulted in the same steady-state value. Thus, it was seen that steady-state results are independent of initial conditions. For an accurate solution of the transient regime, a logical approach must be used to generate initial conditions.

E. Transformation of Variable on Axial Coordinate

The limits imposed on the independent variable x should be mentioned. The limits of integration with respect to x are between 0 and 1; this implies that the length used to nondimensionalize the axial coordinate is the height of the plate. Rigorously, the formulation of the boundary-layer equations that express the conservation of mass, momentum, and energy stipulate that the vertical flat plate is of infinite width and semi-infinite length. Hence,

the limits of integration should extend from 0 to ∞ in the x direction. To circumvent this problem, the following transformation of independent variables was used:

$$x, \tau \rightarrow \xi, \tau,$$

where

$$\xi = \frac{x}{1+x}, \quad 0 \leq x \leq \infty, \quad 0 \leq \xi \leq 1.0. \quad (4.36)$$

One point should be mentioned regarding the transformation used. (Figure 10 is a plot of ξ versus x .) Although the region of interest is $0 \leq x \leq \infty$, the independent variable ξ is double-valued for negative values of x between zero and minus one. From the above transformation, Eqs. 4.3 and 4.4 become

Fig. 10

Graphical Representation of
Coordinate Transformation

$$\frac{1}{6} \frac{\partial u_1 \delta}{\partial \tau} + a(1 - \xi^2) \frac{\partial u_1^2 \delta}{\partial \xi} = \text{Pr}^2 \delta - \text{Pr} \frac{u_1}{\delta} \quad (4.37)$$

and

$$\frac{\partial \delta}{\partial \tau} + b(1 - \xi^2) \frac{\partial u_1 \delta}{\partial \xi} = \frac{2}{\delta}, \quad (4.38)$$

where the range of the independent variables is $0 \leq \xi \leq 1.0$ and $0 \leq \tau < \infty$. Integrating Eqs. 4.37 and 4.38 after multiplying them by the weighting functions yields

$$\begin{aligned} \frac{d}{d\tau} \int_0^1 \frac{f_i u_1 \delta}{6} d\xi + 2a \int_0^1 \xi f_i u_1^2 \delta d\xi - a \int_0^1 (1 - \xi^2) f_i u_1^2 \delta d\xi = \\ \int_0^1 f_i \text{Pr}^2 \delta d\xi - \int_0^1 f_i \text{Pr} \frac{u_1}{\delta} d\xi \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} \frac{d}{d\tau} \int_0^1 g_i u_1 \delta d\xi + 2b \int_0^1 \xi g_i u_1 \delta d\xi - b \int_0^1 (1 - \xi^2) g_i u_1 \delta d\xi = \\ 2 \int_0^1 \frac{g_i}{\delta} d\xi. \end{aligned} \quad (4.40)$$

By a trial-and-error procedure, the following approximating and weighting functions were chosen for Eqs. 4.39 and 4.40 in the (τ, ξ) problem by the method of integral relations:

$$u_{1,N} = \sum_{i=1}^N \alpha_i(\tau) \xi^i; \quad (4.41)$$

$$\delta_N = \sum_{i=1}^N \beta_i(\tau) \xi^i; \quad (4.42)$$

$$f_i = g_i = \xi + \xi^2 + \dots + \xi^i. \quad (4.43)$$

Substitution of Eqs. 4.41-4.43 into Eqs. 4.39 and 4.40 results in N simultaneous, coupled, nonlinear ordinary differential equations for the unknown functions $\alpha_i(\tau)$ and $\beta_i(\tau)$.

V. EXPERIMENTAL APPARATUS

A. General Considerations

To conduct an experimental investigation of transient and steady-state free-convection heat transfer, a simple liquid-metal system was designed and built. The primary aim of the investigation was to obtain data to check the analytical predictions and to achieve a better understanding of the physical phenomenon. A secondary objective was to quantitatively describe the transition from laminar to turbulent flow. The basic philosophy guiding the design of the experimental apparatus was to obtain data that could contribute to an isolation of, and thus an understanding of, the mechanisms governing the transient free-convection heat transfer in liquid metals. For this, a careful choice of techniques and materials was necessary.

The first choice to be made was that of the heat-transfer fluid. Of the readily available liquid metals, mercury appeared to be the most attractive because of its relative ease of handling. Many studies dealing with heat transfer in mercury have been reported in the literature, and thus a technology has been developed. Equally important, the physical properties of mercury are fairly well known. The use of mercury as a medium for heat transfer has certain disadvantages. Among these are its high toxicity, especially at elevated temperature, its difficulty to contain, and its nonwetting of most surfaces. Mercury is not under consideration as a coolant for fast reactors, but has a low Prandtl number characteristic of the liquid metals.

Detailed knowledge of the transient free-convection heat-transfer characteristics resulting from a step change in the surface temperature or heat flux has become increasingly important because the above boundary conditions may be approached in the many circumstances in modern engineering technology. Most applications can best be practically approximated by a constant wall heat flux rather than a temperature boundary condition. Thus, this type of boundary condition is examined in this report. Uniform, constant heat flux was applied by means of thermal-radiation heating.

The primary objectives were in the determination of transient local temperatures, steady-state heat-transfer coefficients, and the transition from laminar to turbulent flow. A determination of these quantities requires knowledge of the values of local wall temperatures and the input heat flux. The local transient fluid and wall temperatures are usually difficult to measure with a desired accuracy, mainly because of heat-capacity effects. To my best knowledge, no experiments dealing with transient free-convection heat transfer in liquid metals have been reported in the literature.

B. Overall System Design

The overall view of the experimental system is sketched in Fig. 11. The large storage tank contains the mercury. A 50-ft-long, 1/2-in.-OD coiled nickel tube is placed inside the tank to serve as a heat exchanger. Water was circulated through the tube so that the mercury could be maintained at ambient room temperature. Before being filled with mercury, the tank was helium leak-tested to ensure that the tank was leakproof. This practice is recommended because mercury penetrates the voids in a weld joint, and if further welding is required, the heat causes mercury vapor to contaminate the surroundings. Mercury was free to flow through the rectangular channel forming the test section and to return by means of a 1-in.-OD line to the bottom of the storage tank. A typical movable thermocouple is shown positioned for the measurement of local fluid temperature

Fig. 11. Schematic Diagram of Experimental Apparatus

in the test section. Also shown is the positioning device designed to locate the movable thermocouples in the test section. Heaters are shown mounted on rollers so that they can be readily positioned, either for heating the test section or for calibration by the heat flux transducers.

Figure 12 is a photograph of the experimental apparatus. The black rubber hose to the storage tank is used for cooling the mercury; those to the lamps are for cooling the units by water and air. The bellows tubing connecting the riser to the bottom of the tank is readily seen. The lamp units are shown in position for calibration. The black paint on the foil is seen through the Vycor plate. Stainless steel clamps around the test section are also readily visible. The stainless steel frame and the Lucite paneling that form the hood surround the apparatus.

Figure 13 is a detailed photograph of the lamp units in operation and the heat-flux transducers. The rails and nylon wheels upon which the lamps are mounted are also visible. Lamp units 3 and 4 are shown in front of the test section; the other lamp units are in a position for calibration by the heat-flux transducer.

C. Assembly and Design of Test Section

The design of the test section determines the values of the length-to-width ratio (L/W) and the characteristic dimension in obtaining the overall Grashof or Nusselt numbers. Once the fluid has been chosen, the other pertinent parameters, such as the Prandtl number, are fixed. Thus, the overall dimensions of the test section were determined by the limiting values of the physical parameters governing the problem. However, for the transient experiments, the walls of the channel must have negligible thermal-heat capacity to isolate the time dependence involved in heating the fluid from that required to heat the test-section wall. The design considerations governing the choice of wall material and thickness are given in Appendix C. As a result of these calculations and the desire for the fluid to wet the surface, the test-section wall (henceforth called the foil) was chosen to be pure nickel foil either 0.001 or 0.0005 in. thick for various tests.

The uniformity of each foil was checked before use with a P&W supermicrometer by determining the thickness at different locations. With care, reproducible foils with maximum variations of 0.00003 in. in thickness were obtained. Sheets with variations greater than ± 0.00003 in. were discarded. Care was taken to match two foils for a particular assembly into the test section so that they were as similar as possible using the available equipment.

Structural integrity of the foil is nonexistent when mounted in a vertical position. Thus, the foil must be supported by a rigid material that will transmit most of the lamp-emitted thermal energy used to heat the

Fig. 12

Overall View of
Test Apparatus

1. Back face of lamp unit
2. Front face of lamp unit
3. Test section
4. Thermocouple bracket
5. Cross-feed table used to position thermocouples
6. Bellows tube connecting riser and storage tank
7. Electrical-power leads
8. Cooling-water and air hoses
9. Storage tank
10. Clamps
11. Hood-enveloping apparatus
12. Rod for translating lamps

112-9144

112-9142

Fig. 13. View of Lamps in Operation

foil. A glass, quartz, or other synthetic transmitting material would be the obvious choice. The thermal shock resistance and transmission characteristics of the individual materials govern the selection. Of the available materials, Corning Vycor, fused quartz, and calcium fluoride are obvious choices. Calcium fluoride will not withstand thermal shock, and fused quartz is expensive and not as available as Corning Vycor. Thus, Vycor ($11\frac{7}{8} \times 8 \times 1/4$ in.) was chosen as the window material that supported the foil. Figure 14 gives the transmission characteristics of 7905 Vycor glass

Fig. 14. Transmission Characteristics of 7905 Vycor Used in Test Section

used. With this window, approximately 97.5% of the radiant energy is transmitted from a source at 4500°F. The linear coefficient of thermal expansion is 12.6×10^{-7} in./in.°F, which is relatively small and enables the Vycor to withstand thermal shocks.

The Vycor was held in place with three bolts on each 8-in. face. One-sixteenth-inch silicon rubber gaskets were used between the Vycor and a 1 x 6-in. stainless steel flange, which formed the entrance to the duct. Similar gaskets were used between the Vycor and the 1-in. Vycor end pieces, which formed the 1-in. face of the channel. Stainless steel stock, flat to 0.001 in. on the Vycor face, was used on top and bottom to hold the Vycor uniformly on the flange. This flat stock was polished to a 1-micron surface finish to make it highly reflective to radiative energy. Side clamps were used to hold the $11\frac{7}{8}$ -in. face of the Vycor plate rigidly against the Vycor end pieces. The end pieces used to form the $1 \times 1 \times 11\frac{7}{8}$ -in. face of the duct were Vycor, primarily so that the linear coefficients of expansion of all the faces would be equal. However, there was an advantage in this design of the test section since the Vycor surrounding the foil provided an excellent means of insulation so that heat losses were minimized. Details on installation of the Vycor test section are given in Appendix D.

A 3-in.-high 1 x 6-in. stainless steel riser was used to extend the test section. A $3/4$ -in. pipe was welded in the center of one 1 x 3-in. face and connected by a bellows tubing to the bottom of the storage vessel. On the opposite 1 x 3-in. face of the riser, a Swagelok fitting was welded $1/2$ in. from the top so that a $1/4$ -in. stainless steel overflow tube could be attached to collect all mercury overflow due to thermal expansion.

D. Heating of the Test Assembly

A uniform-heat-flux boundary condition is usually imposed by resistance heating. When the liquids studied are metals, however, this technique is impractical because of the high electrical conductivity of the fluid. Thus, thermal-radiation heating of the test section was chosen. The foil was painted with a black paint that has a high emissivity and is independent of temperature and wavelength. The paint used was Tempil Pyromark high-temperature paint, the emissivity of which is independent of wavelength and temperature as shown in Figs. 15 and 16. To determine if the nickel foil painted with Pyromark emits diffusely in accordance with Lambert's cosine law, total radiant flux was measured from the specimens for angles ranging from 0 to 62° and for 300-850°F. The data are shown in Fig. 17 and are practically identical with the results of Wade⁶⁸ for Pyromark painted on stainless steels and Inconel substrates. The close agreement obtained indicates that the diffuse assumption was valid. It was also determined experimentally that the radiant heat flux was uniform across the plate to within 2.25%. The results of this investigation are presented in Appendix E.

Fig. 15
Variation of Total Normal Emissivity
of Pyromark Paint on a Nickel Sub-
strate at 600°F with Wavelength

Fig. 16
Variation of Total Normal Emissivity of
Pyromark Paint on a Nickel Substrate
with Temperature

Fig. 17
Comparison of Experimental Measurements
with Lambert's Cosine Law for Pyromark
Paint on Nickel

The radiant-energy sources were Model 5208 high-density radiant heaters manufactured by Research Incorporated. Their heated surface area was 3 x 10 in. Each unit used six tungsten-filament tubular-quartz lamps in compact array to produce up to 5×10^5 Btu/hr-ft². Both faces of the test section were capable of being heated by two lamp units operating side by side. This resulted in a 6 x 10-in. heated surface area on each side of the test section. To maintain a specular reflecting surface on the lamp reflectors and to cool the lamp endseals as well as the lamp housing, air and water cooling were provided for each unit. These attachments are seen in Fig. 18 as the black hoses with quick-disconnect couplers attached. The lamp units were fastened to an aluminum plate and hung on 1 x 1 x 1/8-in. stainless steel angle by nylon wheels, which enabled each unit to be translated 8 in. in a direction parallel to the heated face of the test section. The purpose of this motion was twofold, first to provide a step input of heat flux with time as a boundary condition, and second to provide a means of calibration of the radiant-heat-flux output of both lamp units while completely removed from the test section.

Fig. 18

View of Lamps and Transducer

1. Bracket connecting lamp units
2. Lamps
3. Heat-flux transducer
4. Bracket to hold Vycor and transducer
5. Riser
6. Channel for lamps to ride on
7. Overflow line
8. Circulation line

112-9145

Two electrical systems were used as power sources for the lamps. A 220-V, 60 Amp supply was used in conjunction with two Variacs for the control of low-heat-flux runs. A system of three saturable reactors was used to supply and control 300 Amps at 440 V to the lamps. An interlock

system was designed so that the saturable reactors could not be turned on unless the cooling water to the lamps was on and other safety considerations were met.

E. Instrumentation

Seven thermocouples were used to measure temperatures in the test section. They were all copper-constantan, stainless-steel-sheathed, grounded-junction, 10-mil-OD thermocouples. All thermocouples were calibrated in a constant-temperature silicon oil bath using a National Bureau of Standards platinum resistance thermometer as a standard. Calibration between 70 and 600°F was carried out in 50° increments. While the bath was being heated between calibration points, the thermocouples were taken from the silicon oil bath and placed into an ice bath to observe the transient response and check the output at 32°F. The time response was excellent, and the output at 32°F was as expected. Five of the thermocouples had 90° bends 3/16 in. from the measuring junction to reduce lead conduction errors. The thermocouples were silver-soldered to the foil at 1, 3, 5, 8, and 10 in. from the leading edge. After the thermocouples were soldered, the junction was coated with thin (~0.0001-in.) coat of DuPont Pyre-M.L. high-temperature varnish to prevent the mercury from reacting with the silver solder. The other two thermocouples could be automatically positioned in two dimensions (x and y) within an accuracy of 1 mil. One of these thermocouples was reinforced by placing a 0.032-in.-OD quartz tube over it about 1 in. from the measuring junction, and then was bent 90° about 3/16 in. from the end to reduce conduction errors. The step between the thermocouple and the quartz tube was filled with Corning Pyrocera cement and smoothed to present an aerodynamic profile for the fluid flowing past it. This couple was used to measure local fluid temperatures. The second thermocouple had a 90° bend 3/4 in. from the measuring junction and was butted against the foil by a uniform torque exerted by using a small spring attached to a 1 3/4-in. pulley. This thermocouple was likewise reinforced with a 0.032-in.-OD quartz tube over it within 1 in. of the 90° bend. With this thermocouple and the ones welded to the foil, data were obtained on the temperature differences between the attached and butted thermocouples caused by thermal contact resistance and other effects.

The thermocouples were positioned by a 2-ft-long aluminum arm bolted directly to a universal compound vise used as a standard drill-press attachment. The table of the compound vise could be positioned within 0.001 in. in two dimensions. The crank handles were removed, and precision gears were attached to the shaft so that the shaft could be rotated automatically by means of a high-torque synchronous motor. Figure 19 is a schematic diagram of the drive train. Figure 20 shows the entire positioning device. Direct readout was possible by means of flexible shafts connected to direct-drive counters on the control console. In this manner, the position of the thermocouples could be changed without any hazard of

Fig. 19

Schematic Diagram of Drive Train
for Traversing Mechanism

NOTE:

MAXIMUM TRAVEL ON RAPID ADVANCE = 3 in./min
TOTAL MAXIMUM TRAVEL = 6 in./min

Fig. 20

View of Traversing Mechanism

1. Cross-feed table
2. Drive motor
3. Gear drive
4. Counter takeoff
5. Mounting block
6. Cantilever arm

mercury vapors. This operation was considered best because of the number of times the thermocouples were repositioned and the possibility of high mercury-vapor concentrations under the hood.

An additional thermocouple was used to measure the temperature of the mercury in the storage tank. To measure axial conduction effects, a stainless-steel-sheathed copper constantan thermocouple was placed 1 in. below the heated portion of the test section, at the centerline of the duct.

Transducers used for calibration of the lamp units were manufactured by Hy-Cal Engineering and were of the Gardon Gage⁷⁶ or foil-radiometer design. The transducers were calibrated against a secondary reference standard calibrated by the National Bureau of Standards. For all heat-flux ranges, full-scale output was 10 mV. Transducers used had heat-flux ranges of 12, 50, 120, and 300 Btu/ft²-sec, which overlapped for a double check on accuracy. Each transducer had $\pm 1\%$ full-scale accuracy with $\pm 1/2\%$ reproducibility and was water-cooled for continuous operation. The transducers were mounted in a 1/8-in. stainless steel holder, which simultaneously held a piece of 7905 Vycor glass in front of the transducer to duplicate the transmission characteristics at the test section. In this manner, the incident heat flux on the foil was determined before the lamp unit was slid in front of the test section.

The outputs from those thermocouples inserted directly into the test section were recorded on Honeywell Model 15 two-pen strip-chart recorders. Full-scale response of each channel was 0.5 sec; accuracy was ± 0.03 mV. All other thermocouple output and the heat-flux transducer output were recorded on a Honeywell Model 15 six-point strip-chart recorder. The accuracy for this instrument was also ± 0.03 mV.

A hot-wire anemometer system was used to measure local fluid velocities. The hot wire was coated with a high-temperature varnish to electrically insulate it from the mercury. Because of the low velocities, local velocity measurements could not be obtained with this system.

Attempts were also made to measure bulk-fluid velocities. A 7000-Gauss electromagnetic flowmeter manufactured by the Arnold Engineering Company was placed on the pipe connecting the riser to the storage tank through which the bulk of the mercury circulated. However, the output signal was undetectable. This was due to the extremely low flow rates.

A second attempt to measure the bulk flow was made by calorimetric means. A water-cooled jacket was placed around the bellows tubing connecting the riser and pipe from the storage tank. Inlet and outlet temperatures of both mercury and water were measured so that the flow rate could be estimated from the energy balance. This approach was also unsuccessful because of the extremely low flow rates. Because of the above failures, no fluid-velocity measurements are presented in this report.

F. Safety and Miscellaneous Considerations for Working with Mercury Vapors

The top of the riser was open to the atmosphere so that thermocouples could be inserted vertically into the test section. This permitted the mercury to be in direct contact with the ambient room air. Because of the resultant toxicity and obvious health hazard, a Lucite hood was designed to enclose the complete apparatus. The Lucite was mounted on a 4-ft x 4-ft 10-in. x 1-ft stainless steel pan, which served as a catch basin in case of a spill. An electric motor was used to constantly drive a 150-cfm blower, which circulated the ambient air from around the hood, through the equipment area, and out through a 6-in. duct on top of the hood. This air then was vented through a series of baffles, which condensed some of the mercury vapor when the test section was at high temperature levels and vented the residual air to the outside of the laboratory. A narrow-band photoelectric cell and microammeter measured the absorption of light in the ultraviolet mercury resonance line at 2537 Å to continuously evaluate the quantity of mercury vapor present in the atmosphere of the enclosure.

Care was taken to maintain a system free from grease or oil film and absent of extraneous foreign matter both before and during the tests. The tank was cleaned by dipping in an acid bath after fabrication. All other parts were thoroughly cleaned with acetone before installation. Nickel surfaces were washed with a dilute acid solution to ensure the wetting of surface by mercury.

G. Boundary Conditions for Test-section Designs

Three types of test sections were designed. All the geometries were identical, the inner dimensions being 6 x 1 x 10 in. A second test section was designed from a solid material to thermally model the first test section used for determining the dominant heat-transfer mechanisms. The third test section was designed late in the program to more closely approximate the analytical boundary conditions used in the model.

All test-section materials were chosen to be compatible with mercury. Pure nickel was selected for the heated walls, primarily because of its ability to be wetted by mercury at the temperature encountered during operation. Other materials such as copper are also able to be wetted by mercury, but they amalgamate and react chemically with it.

The following procedure was used to obtain the data for Test Section I. Lamps were moved to the calibration position, and cooling water and air were turned on. The thermocouples were positioned to the desired locations. The power was applied to the lamps and adjusted according to transducer output, and the recorders were energized. The lamps were

rapidly translated to simultaneously heat the test section when their output was constant in time and equal to a prescribed heat flux. To prevent boiling of mercury for all experiments, the wall temperature was limited to $T_L = 550^\circ\text{F}$, 10 in. from the leading edge. At this temperature, any run was terminated if steady-state conditions had not been met.

Mechanically, the operation of Test Section II was identical with that of Test Section I. However, for the lead test section, a limiting wall temperature of 525°F was prescribed to prevent melting. Test Section III was operated in the same manner as Test Sections I and II, except that only one lamp unit was in operation.

All data were reduced using a FORTRAN program on the IBM 360-50/75 system. Property values were evaluated at an average temperature and then computed using equations presented in Appendix F for the reduction of raw data to reduced form.

1. Test Section I; Both Walls Heated

The first rectangular test section was symmetrical about the centerline. Each heated face consisted of a Vycor plate and a painted black foil. Only one foil was instrumented with thermocouples. A uniform heat flux that was independently controlled could be imposed on both faces of the test channel. Most runs were taken for uniform, constant, and equal heat flux on each side. However, with the flexibility designed in the system, several runs were also made with unsymmetrical heating to observe the effects on temperature distribution and heat-transfer characteristics due to the nonsymmetry.

2. Test Section II; Thermal Modeling of Test Section I

To isolate the different mechanisms of heat transfer and thus gain more detailed understanding of the results obtained with Test Section I, a second test section was designed and installed. The mercury was drained from the first test section, and the painted foils were removed. Then a lead block was cast to the exact inner dimensions of the rectangular duct so that the mercury in the duct was completely replaced by lead. During the casting process, thermocouples were positioned in the lead 1 and 5 in. from the "leading edge"; i.e., $x = 0$. At each of these locations, two thermocouples were placed $1/64$ and $1/2$ in. within the lead from the heated surface. These thermocouple positions corresponded to y coordinate values of essentially 0 and $1/2$. After the lead block was machined to size, the two 6×10 -in. faces were painted with black Pyromark paint in a manner analogous to the mercury tests. The mercury experiments were then physically modeled as close as practical, so that identical experimental conditions existed for both tests.

Thermally the lead was chosen for the following reasons. In nondimensionalizing the independent time variable, one uses the thermal diffusivity and a characteristic length in the following manner:

$$\tau^* = \frac{\alpha t}{L^2}. \quad (5.1)$$

Since both systems have the same physical dimensions, the length for the mercury was equal to that for the lead. Thus, the only remaining parameter was the thermal diffusivity. For all practical purposes, the specific heats of mercury and lead are equal. Lead and mercury are both dense; in fact, mercury is about 1.2 times as dense as lead. The thermal conductivity of lead is about four times that of mercury. Thus, the differences in density and thermal conductivity partially cancel. This results in a smaller difference between the thermal diffusivities of lead and mercury.

Using the above relationship for nondimensional time, one can correlate the model time in the lead system with that of the mercury. In this manner, the effects of convection and conduction as mechanisms affecting the heat transfer in the rectangular duct can be isolated.

3. Test Section III; One Wall Heated, Second Wall Maintained at Constant Temperature

A third test section was constructed for obtaining closer physical correspondence with the boundary conditions used with the theoretical model. One wall was identical to that used in Test Section I. It was heated by the lamps, resulting in a constant heat flux being applied at the painted foil. The second wall in this test section consisted of a 1/16-in.-thick nickel plate $11\frac{7}{8}$ in. high by 8 in. wide. This plate was flat to 0.005 in. on the $11\frac{7}{8} \times 8$ -in. face. Butted directly to the nickel plate was a hollow copper block with outer dimensions of $11\frac{7}{8} \times 8 \times 1\frac{1}{4}$ in. The copper was 1/8 in. thick. Silver-brazing three 1-in.-OD copper tubes to the back face allowed water to be circulated through the block. This arrangement enabled the wall to be maintained essentially at the constant uniform temperature of the mercury in the storage tank. To monitor the cool-wall temperature, two copper-constantan thermocouples were tack-welded to the mercury side of the nickel plate.

VI. DISCUSSION OF RESULTS

A. Introduction

This section presents the results of the approximate analytical solutions described in Section IV for the model formulated in Section III. The results of the two approximate methods of solution are compared, and good agreement is noted. The experimental data are also presented in reduced form for the steady-state situations of interest. Where possible, these data are compared with existing empirical correlations and the approximate solutions presented in this report. Transient experimental data are presented both graphically and in tabular form for ease of interpretation and comparison with the analysis. Appendix G presents an error analysis on the experimental measurements.

B. Analytical Results

This section presents the results obtained by the approximate methods discussed in Section IV. Only typical results of many calculations that demonstrate salient features of the phenomenon are included.

Figures 21-24 depict the approximate solutions to the nonlinear first-order ordinary differential equations obtained numerically on an IBM 360-50/75 digital computer resulting from the one-parameter solution of the method of integral relations for the constant-wall-temperature

Fig. 21. One-parameter Solution by the Method of Integral Relations for $Pr = 1.0$ for Constant-wall-temperature Boundary Condition

Fig. 22. One-parameter Solution by the Method of Integral Relations for $Pr = 0.1$ for Constant-wall-temperature Boundary Condition

Fig. 23. One-parameter Solution by the Method of Integral Relations for $Pr = 0.01$ for Constant-wall-temperature Boundary Condition

Fig. 24. One-parameter Solution by the Method of Integral Relations for $Pr = 0.003$ for Constant-wall-temperature Boundary Condition

boundary condition. Figures 25-28 present the numerical results for the constant-heat-flux boundary condition for the one-parameter approximation. Decreasing the Prandtl number increases the time required to reach the steady state. This agrees with the predictions of Siegel.¹⁷ The solutions for all Prandtl numbers exhibit an overshoot. Table I gives the values of overshoot for the solutions presented in the figures. Whether this overshoot is a direct result of the modeling of the physical phenomenon or is due to some mathematical anomaly remains to be determined. Of particular interest is the fact that the overshoot occurs mathematically, even in approximate solutions where the variables have been averaged by integration.

Fig. 25. One-parameter Solutions by the Method of Integral Relations for $Pr = 1.0$ for Uniform-heat-flux Boundary Condition

Fig. 26. One-parameter Solution by the Method of Integral Relations for $Pr = 0.1$ for Uniform-heat-flux Boundary Condition

Fig. 27. One-parameter Solution by the Method of Integral Relations for $Pr = 0.01$ for Uniform-heat-flux Boundary Condition

Fig. 28. One-parameter Solution by the Method of Integral Relations for $Pr = 0.003$ for Uniform-heat-flux Boundary Condition

TABLE I. Percent Overshoot Obtained by the Method of Integral Relations for the One-parameter Approximation

Pr	Type of Boundary Condition			
	Constant Wall Temperature		Constant Heat Flux	
	u	δ	u	δ
1.0	0.57	0.89	0.02	0.10
0.1	1.10	6.10	0.35	0.50
0.01	1.16	7.50	0.45	1.01
0.003	1.17	7.78	0.53	1.50

1. Discussion of Overshoot

In the selection of a technique by which the mathematical problem could be solved, the following information was obtained that may contribute to the understanding of the overshoot question.

An exact analytical solution of the boundary-layer equations for a special case of transient free convection has been given in Refs. 27 and 28 and belongs to the general class of physical problems widely referred to as parallel flows. The method of solution of the problems is based upon the following physical argument recognized as being valid for forced as well as free convection.²⁹ Initially, the boundary layer has zero thickness; at the outset of flow, the effect of viscous and body forces predominate over the inertia forces. Thus, for the initial stages of motion,

the temperature field is essentially determined by the diffusion of heat normal to the plate; i.e., the temperature field is governed by

$$\frac{\partial T^0}{\partial t} = \alpha \frac{\partial^2 T^0}{\partial Y^2}, \quad (6.1)$$

while the flow field is governed by the diffusion of velocity in the direction perpendicular to the plate

$$\frac{\partial U^0}{\partial t} = \nu \frac{\partial^2 U^0}{\partial Y^2} + g\beta(T^0 - T_\infty^0) \quad (6.2)$$

where the superscript 0 indicates the zeroth approximation. The two-dimensional effects cause the boundary-layer growth to vary in the direction of the flow. This growth gradually propagates from the leading edge and alters the one-dimensional flow and temperature fields at a different time for each position along the plate.

To account for the fact that the body is of finite length or to determine the flow and temperature fields at later instants of time based upon the boundary-layer equations, the solution of Eqs. 6.1 and 6.2 for T^0 and U^0 may be perturbed and the boundary-layer equations linearized. One can then solve these equations and extend the perturbation to any desired order. The PLK*⁶⁹ method can also be used to perturb the independent variable to obtain solutions approximately valid throughout the entire region.

It would appear that this technique could be used for the complete and general solution of the problem (if the perturbation scheme is convergent). In an application of the above method, however, one finds that this approach fails. Here, the perturbations of the above analytical solution vanish identically. This indicates that there may be a region downstream along the plate in which the fluid does not experience the effect of the leading edge, and in this region the flow is described by Eqs. 6.1 and 6.2. However, the physical phenomenon occurring makes it apparent that in the upstream portion of the flow near the leading edge, the flow must vary with the distance from the leading edge. Thus, the fact that the regular perturbation fails to introduce the x dependence into the solution may be interpreted as an indication that the analysis fails to account for the leading-edge effect and indicates the possible existence of an essential singularity in the solution.

This was first discussed by Stewartson⁷⁰ in an analysis of the Rayleigh problem. He determined an essential singularity (which moves downstream and away from the plate) to be at the outer edge of the boundary layer from the initial position of the leading edge. This essential singularity

*Poincaré-Lighthill-Kuo.

connects the Rayleigh solution with the steady-state (Blasius) solution. Up to the present, the occurrence of the type of essential singularity mentioned was rare in the analysis of physical phenomena, and no physical understanding has been developed.

Cheng⁷¹ attempted to analyze the nature of the flow field of the Rayleigh problem in the vicinity of the essential singularity, but failed to obtain information on how to "join" the solutions for the two regions. He independently determined that the essential singularity occurs at the same location as that determined by Stewartson, even when the plate starts initially from rest and then moves in a continuous manner. This ensured that the development of the essential singularity had little to do with the initial discontinuity of velocity under the impulsive start.

Using the analogy with forced convection as described above, one can conclude that apparently the same type phenomenon occurs in transient free convection as in forced convection. This was further established by Yang⁷² who indicated the possibility of an essential singularity and attempted mathematically to establish its location. As a result, Yang confirmed the criterion obtained by Goldstein and Briggs⁷³ by assuming that for small times there exists a penetration distance in the fluid. The significant point of these analyses is that they indicate the occurrence of the singularity close to the end, if not at the end, of the conduction regime.

Because of the lack of theory on essential singularities, a rigorous mathematical treatment of the problem in the vicinity of such a singular point seems to be rather hopeless. Thus, one is initially led to a gross analysis of the physical problem.

Initially, the solution to the problem is as described in Refs. 27 and 28. In the early stage of flow development, the fluid behaves as if it were a semi-infinite solid. The next stage is the initiation of a velocity in the fluid that propagates upward from the leading edge. After enough time has elapsed, the fluid reaches some steady-state condition. This leaves only the region between the conduction and steady-state solutions open to further examination. If there is indeed an "overshoot" in the heat-transfer coefficient as has been predicted in the literature,^{17,23,36} then perhaps there is also a definite relation between the unusual mathematical condition due to the essential singularity and the physical phenomenon referred to as an overshoot. If such is the case, then the conclusions of Yang⁷² substantiate this argument, since the singularity deduced by him lies in the middle of the overshoot, if the overshoot occurs. A final argument follows from the pure-conduction and steady-state solutions obtained by Siegel¹⁷ and those obtained in this work. The pure-conduction solution is a function only of y and t ; the steady-state solution is a function only of y and x . Thus, in the interim period, the mathematical model must transpose from the y - t plane to the y - x plane. This appears to be a mathematical rather than a physical singularity.

It follows from the above discussion that the overshoot is postulated to occur during the "pure transient" between the pure-conduction and steady-state regimes. Other evidence of this phenomenon is given by Siegel,¹⁷ Hellums,²³ and Goldstein and Briggs.⁷³ Siegel and Goldstein and Briggs only indicated the possibility by extrapolation of the limiting cases; Hellums, by a finite-difference calculation scheme for $Pr = 0.733$, has observed an overshoot.

2. Comparison of Analytical Solutions

To estimate the accuracy and rate of convergence of one of the approximate methods of solution used, limiting conditions were discussed in Section IV.D. To make available a further measure of convergence, the methods will be compared in the next few paragraphs. Results have been obtained by the two- and three-parameter approximations to the problem. A solution has also been obtained by the method of lines.

For $Pr = 1.0$, Fig. 29 compares solutions obtained by the method of lines and by the one-parameter approximation. The two methods of solution are in good agreement except in the region just after the end of the pure conduction regime (i.e., the "overshoot" regime) and for small values of the axial coordinate where the dependent variables are not functions of that coordinate. The superposition of all solutions for small values of time for the method of lines indicates the independence of the axial co-

Fig. 29. Comparison of Solutions Obtained by the Method of Integral Relations and the Method of Lines for $Pr = 1.0$

ordinate during the conduction regime, while the steady-state conditions are also clearly demonstrated. This solution illustrates vividly the transfer from the y, τ coordinate system to the x, τ coordinate system, together with the attendant numerical difficulties. The numerical instabilities that occur for large x in the method of lines should be regarded as stability errors due to either the essential singularity in the differential equations or some other instability in the numerical method of finite-differencing the axial space coordinate.

In the solutions obtained by the method of lines, ten lines were chosen for each equation. Use of 40 lines per equation yielded identical numerical results up to the fifth significant figure. The improved formulation of the method of lines was used for the solution of one case. A three-line approximation using the improved

method of lines yielded results within 2.7% of the ten-line method, except for the numerically unstable region where instabilities also occurred. Thus, the improved method speeded up the convergence, at least for the present problem.

The method of lines was used to investigate the error in the one-parameter approximation of the method of integral relations and as a guide in choosing weighting and approximating functions in the following manner. A solution was obtained by the method of lines. After the Prandtl number was specified, the approximating functions in the integral method were then arbitrarily chosen to be various polynomials and transcendental functions. The approximating functions obtained from the steady-state solution gave the best agreement for a two-parameter approximation with exact steady-state results and yielded the closest agreement with the solution based on the method of lines. If the approximating functions were chosen randomly, errors as large as 42% could be encountered for a comparison between the two-parameter approximation and the method of lines.

The preceding discussion assumes that the method of lines yields the "exact" numerical solution of the partial-differential equations. When a three-parameter solution was obtained and compared with solutions based on the method of lines, the results agreed more closely, with the exception of the overshoot regime. Where the overshoot starts, the integration scheme diverged as if a local singularity were encountered. Using extrapolation for the numerical values, the integration was started at some later time and solutions were obtained up to and including steady state.

This numerical difficulty occurred no matter what initial conditions were calculated, or no matter how small or large the time increment was. This numerical difficulty would apparently testify to the questions raised regarding the existence of an essential singularity discussed earlier in Section VI.B.1. However, some questions still remain unanswered such as: Why does this singularity not occur for lower-order approximations? Should not the averaging introduced by the integral approach smooth out any such numerical disturbance?

Fig. 30. Effect of Higher-order Approximations on the Convergence of the Method of Integral Relations to the Method-of-lines Solution

Figure 30 compares the solutions obtained by higher-order approximations using the method of integral relations and those obtained by the method of lines for $Pr = 1.0$ at $0.9 \tau_{\infty}$.

(where τ_{∞} is the time required to reach steady state). The approximate solutions tend uniformly toward the method-of-lines solution with increasing order of approximation. The convergence of the solutions for other approximating functions such as higher- or lower-order polynomials was not uniform; thus the form given by Eqs. 4.41, 4.42, 4.53, and 4.54 was used.

An important aspect of any approximate method is the determination of an order of approximation that is a good compromise between the requirements for accuracy and computational simplicity. The first-order approximation yields results that, from an engineering point of view, are satisfactory. Application of the results or any mathematical manipulation that may be required is straightforward. Interpretation is also easy, and this may be the most important feature for understanding the physical problem. Finally, the numerical problems encountered in the third- and higher-order approximate solutions render a portion of the results invalid. Consequently, only results from the first approximation are presented.

C. Experimental Results

This section presents the results of the experimental investigation and includes raw and reduced data for the three test sections. Detailed data are tabulated in Appendix H. The experimental and analytical results are compared wherever possible.

Fig. 31. Transient Temperature Histories Obtained in the Mercury and Lead Systems Compared with Theoretical Prediction for $q_w = 3600 \text{ Btu/hr-ft}^2$

1. Test Section II

Figure 31 compares Test Sections I and II and the analytical predictions for the temperature-time history at a specific location in the duct for a low heating rate under which steady-state conditions were attained. During the initial stages of the transient, heat transfer by conduction predominates over convection, and the data for Test Sections I and II nearly coincide. Later, heat is convected away from the surfaces by the mercury, while the lead temperature in Test Section I increases to a much greater value.

For a moderately large heat flux, Fig. 32 compares the temperature-time history of Test Sections I and II. There is perhaps a 10-15% difference possibly caused

Fig. 32. Transient Temperature Histories Obtained in the Mercury and Lead Systems for $q_w = 43,200 \text{ Btu/hr-ft}^2$

could be applied on each face, with free convection, conduction from the duct walls, and possible axial conduction in the fluid the only possible mechanisms for energy transport from the test-section walls. This resulted in steady-state conditions for only relatively low values of heating rates. For symmetrical heating above a limiting heat flux of $q \approx 4000 \text{ Btu/hr-ft}^2$, the mercury in the test section attained the limiting temperature, T_L , which was the highest permissible. Above this limiting value of heat flux, there existed another range where both conduction and free convection occurred up to a heat flux of $q \approx 9200 \text{ Btu/hr-ft}^2$. Above this value of heat flux, the temperature-time traces were essentially linear from time $\tau = 0$ to the time when the heating was terminated because the limiting temperature, T_L , was reached.

Figure 33 illustrates the steady-state local heat-transfer characteristics based on the physical properties evaluated at the mean fluid temperature as a function of distance from the leading edge. The experimental values are too high as the leading edge is approached. Two combined effects can possibly cause this. Because of edge effects, the heat flux is not constant near the entrance to the channel, and there are heat losses through the bolts holding the test section together.

by the uncertainty of the physical properties of both mercury and lead. For reduction of time in the lead system to that in the mercury system, the physical properties of mercury were taken from Appendix F, and the best average lead properties from Ref. 79.

2. Test Section I

Before the results are compiled, the trends determined will be discussed. Test Section I had a constant, uniform heat flux applied on the two boundaries. For these boundary conditions, a heat flux between 0 and $5 \times 10^5 \text{ Btu/hr-ft}^2$

Fig. 33. Steady-state Local Experimental and Theoretical Heat-transfer Characteristics as a Function of Distance from the Leading Edge

Also, the value of the local Grashof number, Gr_x , becomes quite small so that the theoretical values calculated using boundary-layer assumptions are probably invalid. Although there is some scatter of experimental data, theory and experiment demonstrate good agreement at a reasonable distance from the leading edge. The steady-state analytical result has been obtained using the model of a vertical plate submerged in a semi-infinite volume of fluid, and it is not for a vertical rectangular channel. The good agreement gives confidence in the use of the physical approximation made in the theoretical model.

Figure 34 compares the theoretical predictions and typical experimental data obtained at steady-state conditions. Although considerable scatter is evident, there is a general agreement between theory and experiment.

Fig. 34. Comparison of Theoretical Predictions and Experimental Data for Steady-state Heat Transfer

Fig. 35. Comparison of Theoretical and Experimental Wall-temperature Distributions

Figure 35 compares steady-state wall temperatures with the theoretical predictions of Sparrow and Gregg⁷⁸ based on boundary-layer theory using similarity methods. In general, there is qualitative agreement with the theory. For Test Section I, the influence of the second heated surface causes the wall temperature of the first surface to increase by a maximum of 10.5%. In Test Section III, the cold wall decreases the overall effect and brings the experimental data in closer agreement with the theoretical prediction.

At steady-state conditions, the temperature profiles were measured for three axial positions in the duct. Figure 36 presents the results of these

measurements in dimensionless form. The high values of thermal conductivity flatten the temperature profiles in the duct. The change in the profiles as one moves up the duct shows that the flow is only developing thermally in the test section. At $x = 1.0$, the temperature profile appears to be changing very little with the axial position, so that for this experimental condition a value of $L/W = 10$ practically results in a fully developed temperature profile.

Fig. 36
Steady-state Temperature Profiles
at Three Axial Locations for Both
Surfaces Heated Uniformly

Figure 37 shows the wall-temperature variation as a function of the axial space coordinate with t as a parameter from the start of the transient to steady state. The development of the profile with time demonstrates the effect of free convection on the wall temperature. For small time, the axial temperature profiles are fairly flat; this indicates that there is little free convection in the channel. However, at later times, free-convection effects cause the warm fluid to be carried up the duct, with a net result of increasing the wall temperature at positions farther up from the inlet of the duct. A rough extrapolation of the curves presented yields relatively low values of temperature at the leading edge, $x = 0$; this indicates the absence of appreciable axial heat conduction.

Figure 38 presents transient wall temperatures as a function of the axial space coordinate for a moderately high heat flux. In general, convection has a small effect on the wall temperature in comparison with results given for the lower heating rate of Fig. 37.

Figure 39 indicates the gross effect of axial heat conduction. The data obtained by a thermocouple placed at the centerline of the duct and one inch below the heated section show that the relative temperature

Fig. 37. Development of Transient Wall-temperature Profile for $q_w = 3500$ Btu/hr-ft²

Fig. 38. Development of Transient Wall-temperature Profile for $q_w = 42,200$ Btu/hr-ft²

Fig. 39
Centerline Temperature-Time History
One inch below Heated Section for
 $q_w = 3500$ Btu/hr-ft²

increase is much smaller than the fluid temperature in contact with the wall (see Fig. 38). The experimental data give information on the assumption that the axial conduction is negligible compared to the transverse heat conduction in the development of the analytical model.

Fig. 40. Effects of Unsymmetrical Heating on Centerline Temperature

Effects of unsymmetrical heating are illustrated in Fig. 40. The temperatures measured are at the channel centerline and are assumed to be the most meaningful for comparison. When the same amount of total energy is applied to the system, the fluid centerline-temperature histories are amazingly similar, but when the second surface has half the original energy applied the centerline temperature is affected

drastically as is demonstrated in the figure. Application of other fractional amounts of the original energy input to the second face resulted in a decrease of the centerline temperature in proportion to the reduction.

Figure 41 gives the differences between the temperatures measured by the attached thermocouples on the one mil foil and thermocouples butted against the foil. The solid points refer to data obtained during the transient portion of a run; the open points indicate steady-state temperatures. The welded thermocouples 10 in. from the leading edge experienced some malfunction, which could have been caused by (1) the lacquer covering the soldered thermocouple, (2) included flux in the junction, (3) an extra contact resistance due to foreign matter, or (4) just a nonwetting of the butted thermocouple by the mercury. Whatever the reason, that set of readings appears to be high in comparison with other values. The highest value at each location shown is the greatest value obtained in all the available data. The maximum difference obtained was 4.6%. Absolute measurements of temperature can be made with good accuracy, particularly in

Fig. 41. Effects of Attached Thermocouple and Thermocouple Butted against Foil on Temperature Measurements

higher-temperature ranges. However, when attempts are made to measure temperature differences of say 5-10°F, such as are needed in the determination of local heat-transfer coefficients, the errors introduced by this method are of the order of the quantity measured, and the technique appears to be invalid.

Bayley⁸⁰ found that the steady-state heat-transfer rates predicted by the theory presented in his paper for mercury during turbulent free convection from a vertical flat plate represent a direct extrapolation of those calculated by Saunders¹³ under laminar conditions. This is because the molecular conduction is the principal mode of heat transfer in liquid metals even though there may be significant macroscopic fluid motion. This effect is further substantiated by comparing data obtained by modeling mercury with lead.

The actual temperature-time plots yielded about 5° variations about a mean temperature, which is plotted in Fig. 32. These variations occurred when a modified Rayleigh number based on x in a range between 5×10^8 and 5×10^9 was exceeded. Although the effect of the flow is not discernible, these oscillations are assumed to be local eddies caused by turbulent fluctuations. The transition to turbulence in a moving vertical fluid layer is considered to result from hydrodynamic instability, in distinction to the thermal instability of the Benard problem.⁹

The two factors that tend to promote turbulence are: (a) Somewhere in the channel the shear has a maximum, and (b) for liquid metals such as mercury, the relatively low viscosity minimizes the damping and opposes the transition from laminar to turbulent flow. On the other hand, considering a liquid metal as a fluid with a high thermal conductivity would in general decrease the temperature differences in the fluid and thus would retard the transition to turbulent flow.

3. Test Section III

Test Section III differed from Test Section I in that one surface of Test Section III was maintained at the constant uniform temperature of the mercury in the storage tank. With this boundary condition, energy is transferred not only to the fluid but also to the cool wall.

The test section could be operated with and without circulation. The noncirculating (or closed-system) situation has been studied by several investigators⁸⁰⁻⁸⁶ under steady-state conditions. In one study,⁷⁷ transient free-convection data have been obtained using air, water, and silicone oils as test fluids. Distinction is made between two situations: (a) when there is a net circulation of the fluid between the top of the test section and the bottom of the storage tank, and (b) when there is no such circulation; the test sections are designated IIIa and IIIb, respectively.

Figure 42 shows the dimensionless steady-state temperature profiles at various axial locations in the fluid for Test Section IIIa. Under the noncirculating conditions, the temperature has a local maximum near the cold wall and a local minimum near the hot wall. This effect was also observed by Eckert and Carlson,⁸² Elder,⁸⁴ and Landis and Yanowitz.⁷⁷

Fig. 42. Steady-state Temperature Profiles at Three Axial Locations for Test Section IIIa

This type of effect was predicted numerically by de Vahl Davis and Kettleborough.⁸⁵ The secondary flows reported by Elder probably superimpose on a moving central core of fluid of nearly constant vorticity to promote this temperature inversion. Batchelor⁸¹ was the first to suggest this region of constant vorticity in an approximate analysis of this problem. He also postulated an isothermal core, but Eckert and Carlson's experimental investigations showed this assumption to be erroneous.

Figure 43 presents developing temperature profiles for Test Section IIIb. Initially, the temperature distribution appears to exhibit a transient behavior similar to the boundary-layer growth from a heated vertical plate. After sufficient time has elapsed, fully developed temperature profiles are established and the local maximum and minimum previously discussed are evident. Figure 44 compares the steady-state heat-transfer characteristics obtained for Test Section IIIb with empirical correlations obtained by other investigators for various fluids and test sections having different L/W values. The agreement with other studies is fairly good. The effect of decreasing the Prandtl number by at least two, and in some cases by about four, orders of magnitude may account for the difference between the results. Hence there appears to be a slight Prandtl-number effect on the Nusselt number. The data can be correlated by the equation $Nu_{w,f} = 0.173(Ra_{w,f})^{0.28}$ over the range of $Ra_{w,f} = 3 \times 10^3$ to 10^8 . A comparison of this equation with the

Figure 43 presents developing temperature profiles for Test Section IIIb. Initially, the temperature distribution appears to exhibit a transient behavior similar to the boundary-layer growth

Fig. 43. Development of Transient-temperature Profile for Test Section IIIb

Fig. 44. Comparison of Present Correlation with Previous Correlations for Other Fluids

proposed correlation by Landis and Yanowitz⁷⁷ shows that the exponents on $Ro_{w,f}$ are essentially identical and the constants are nearly the same.

To show the dependence of the Prandtl number on the heat transfer, the steady-state experimental results of different investigators have been plotted in Fig. 45 together with the steady-state results of Test Section IIIa in the functional form

$$\frac{Nu_{L,f}}{Gr_f^{1/4}} = f(Pr).$$

Figure 45 illustrates the effect of the heat-transfer characteristics on the Prandtl number and the thermo-physical properties of the fluid. This function $f(Pr)$ is not linear, however. Thus, the exact functional relation must be calculated graphically. This results in the form of an equation originally presented by Fedynskiy,¹²

$$\frac{Nu_{L,f}}{Gr_f^{1/4}} = Pr^{(0.3+0.02/Pr^{1/3})}. \quad (6.4)$$

Figure 46 shows steady-state dimensionless temperature profiles for Test Section IIIb. The increase in the temperature due to net flow is monotonic at any given transverse position, y . Steep gradients are in evidence near the cold wall. This is attributed to the large rate of heat transfer between the relatively warm flowing mercury and the cool wall.

The nearly linear portions of the temperature profile indicate that conduction heat transfer predominates in certain regions of the channel. A comparison of Figs. 46 and 42 reveals significant differences, which indicate the general effect of circulation on the temperature distribution, particularly in the region near the cold wall. Test Section IIIa provides a convective sink for the hot mercury; in Test Section IIIb, the hot mercury must flow past the cool wall resulting in an increase in the local fluid temperature.

Fig. 45. Demonstration of Dependence of Heat-transfer Characteristics on Prandtl Number

Fig. 46
Steady-state Temperature Profiles
at Three Axial Locations for
Test Section IIIb

Figure 47 presents a typical centerline temperature-time history in the channel. The effect of the cold wall is evident when temperature-time histories for Test Section III are compared with those for Test Section I. The constant-temperature wall serves as a heat sink, which in turn reduces the local duct temperatures.

Fig. 47
Transient-temperature Profile at the
Channel Centerline for Test Section IIIa,
 $q_w = 31,200 \text{ Btu/hr-ft}^2$

VII. SUMMARY AND CONCLUSIONS

The problem of transient free-convection heat transfer in liquid metals has been studied both analytically and experimentally. The general system of conservation equations was reduced by an integral method from a three- to a two-independent-variable system of coupled, nonlinear partial differential equations. The solutions compared favorably with exact results when the equations in two independent variables were solved for limiting conditions; i.e., $t \rightarrow 0$ and $t \rightarrow \infty$. On this basis, higher-order approximations were unnecessary, and the system of equations in two independent variables was again reduced to a system of ordinary differential equations by an integral method. To ascertain the validity of this method, the former system of equations was also solved by the method of lines, an explicit numerical scheme. A comparison of the solutions obtained by the two methods showed that the results based on the integral method were converging uniformly to the solution obtained by the method of lines. Numerical instabilities were experienced in one region of time for both methods. These instabilities were isolated to the pure transient regime where the equations depend on all three independent variables. The possible existence of an essential singularity has been indicated, and the instabilities are attributed to the mathematical and not the physical singularity. The so-called temperature overshoot, sometimes observed experimentally, also exists in that region. Whether the overshoot does or does not exist physically is a question yet to be answered.

Only the solutions for a one-parameter approximation using the method of integral relations were presented. Because of the ease of manipulation and the relative straightforward manner in which the results can be interpreted, the one-parameter solution is recommended for applications where high accuracy is not needed. In the computations, the so-called temperature overshoot is present, but it is felt that some basic questions regarding it remain unanswered. The method-of-lines solution is also obtained using an interpolation scheme where it is found that time to achieve convergence is speeded by a factor of more than three.

An experimental investigation was performed with three different test sections to obtain various boundary conditions and to thermally model the heat transfer. The results were compared with those predicted by the analysis. Developing local and overall steady-state Nusselt numbers were obtained and compared with existing correlations for other fluids and geometrically similar channels. Steady-state temperature profiles were obtained and presented in dimensionless form. Wall-temperature profiles were compared with those predicted by similarity methods. The experimental data and the analytical prediction agreed qualitatively. Temperature profiles measured for one wall maintained at a constant temperature agreed with the analytical predictions better than those for the wall with a constant heat flux applied. This improved agreement is attributed to more accurately modeling the analytical boundary conditions experimentally.

Wall temperatures measured by attached thermocouples and by thermocouples butted against 1-mil nickel foil were compared. Generally, the two measurements agreed below a maximum of 2.1%. However, using the butted thermocouple in the determination of temperature differences was considered unsatisfactory.

Transient local-temperature histories are presented in dimensional form to demonstrate the heat-transfer situations for different test sections. Only general agreement between the analysis and experimental results was obtained for the pure transient regime. The differences could at least partially be accounted for by the failure to achieve the idealizations desired in the experimental design.

When both walls were heated by a constant, uniform heat flux, three separate heat-transfer regimes existed. The first regime, which consisted of the transient run complete to steady state, was attained when the applied heat flux was small. At higher heat-transfer rates, the second regime appeared where both conduction and convection were the mechanisms of heat transfer, but steady state was not reached because of possible boiling of the mercury. For relatively high heat fluxes, the primary mechanism for heat transfer was molecular conduction. This conclusion was the result of the thermal modeling of the mercury within the test section by lead. Temperature-time histories demonstrated the occurrence of local turbulent eddies. These eddies were inferred from temperature fluctuations as large as 11°F on temperature-time traces. However, the net effect of turbulence on the overall heat transfer was practically negligible.

Changing one boundary condition for the same geometry to a constant, uniform wall temperature resulted in a system in which the boundary conditions formulated in the analytical model were more closely approximated experimentally. This flow situation was much more unstable, at least if turbulence levels were inferred from the temperature-time histories. Fluctuations of 10-12°F were more frequent than for the other boundary conditions. Some fluctuations of 18-19°F were noted. Apparently, the formation of a warm boundary layer from the lower edge of the heated wall interacts violently with the boundary layer that begins growing at the top of the cold wall. This interaction results in the onset of turbulence much more readily, and when the turbulence occurs, the result is a level of turbulence of greater magnitude.

There was a significant change in steady-state temperature profiles between the situations when the fluid was allowed to recirculate back to the storage tank and when the top of the fluid was a free surface. However, the change in boundary conditions from two isothermal walls as used by previous investigators did not significantly affect the steady-state results. The length-to-width ratio of the channel was not specifically studied in this investigation, but it may present a small effect on both the transient and steady-state results.

Based on the investigation presented in this study, the following recommendations regarding further work on this problem are made:

(1) Using a duct four to five times as wide as the present value would probably improve the test section design considerably so that it would better represent the mathematical model. Not only would the analytical model be approximated more closely, but also, if the turbulence is partially caused by the interaction of the two boundary layers, this effect could be reduced.

(2) The experience gained in the design and operation of the present test sections shows that a test section with adjustable width could not be designed and operated in practice.

(3) The effect of the length-to-width ratio of the channel on the heat-transfer characteristics should be specifically investigated.

(4) Further attempts should be made to obtain either local or average velocity measurements.

(5) If possible, the system should be redesigned using sodium as a test fluid so that free convection, superheat, and boiling could be measured during the same test run.

(6) The effect of relaxing the assumption that the thermal and velocity boundary-layer thicknesses are equal should be investigated further.

(7) The two-dimensional momentum equations and the energy equation with the axial heat-conduction term retained should be solved to obtain a more realistic solution of transient free convection in a vertical channel. Appropriate boundary conditions for the channel under consideration should be used.

APPENDIX A

Justification of the Choice of Dimensionless Variables

Upon reduction of the partial differential equations that describe the conservation of mass, momentum, and energy for a constant-wall-temperature boundary condition, one obtains

$$\frac{1}{6} \frac{\partial(u_2 \delta_1)}{\partial \tau_1} + a \frac{\partial(u_2^2 \delta_1)}{\partial x} = Gr_L Pr^2 \delta_1 - Pr \frac{u_2}{\delta_1} \quad (A.1)$$

and

$$\frac{\partial \delta_1}{\partial \tau_1} + b \frac{\partial(u_2 \delta_1)}{\partial x} = \frac{2}{\delta_1}. \quad (A.2)$$

In the above, the following dimensionless variables have been used:

$$u_2 = \frac{U_1 L}{\alpha}, \tau_1 = \frac{t \alpha}{L^2}, \text{ and } \delta_1 = \frac{\Delta}{L}. \quad (A.3)$$

Using a one-parameter integral approach and assuming that

$$u_2 = a_1(\tau_1) x, \text{ and } \delta_1 = b_1(\tau_1) x, \quad (A.4)$$

Eqs. A.1 and A.2 were solved for various values of the Grashof number for fixed values of Prandtl number. Figures A.1-A.4 illustrate these solutions. These graphs show that $a_1(\tau_1)$, $b_1(\tau_1)$, and τ_1 are multiples of each other for the various Grashof numbers used as the parameter in the problem.

Fig. A.1. Solution of Eqs. A.1 and A.2 for
Pr = 0.01, Gr = 10^7

Fig. A.2. Solution of Eqs. A.1 and A.2 for
Pr = 0.01, Gr = 10^9

Fig. A.3. Solution of Eqs. A.1 and A.2 for
Pr = 0.01, Gr = 10^{11}

Fig. A.4. Solution of Eqs. A.1 and A.2 for
Pr = 0.01, Gr = 10^{13}

Thus, the above equations were nondimensionlized so that the Grashof number was absorbed in the dimensionless variables defined by Eq. A.3. This is the most appropriate method of nondimensionlization, but because of the precedent set in the literature, it was not used initially.

Introducing the dimensionless variables

$$\left. \begin{aligned} u_2 &= Gr^{1/2} u_1, \\ \delta_1 &= Gr^{-1/4} \delta, \\ \text{and} \\ \tau_1 &= Gr^{-1/2} \tau, \end{aligned} \right\} \quad (A.5)$$

in Eqs. A.1 and A.2, the system of equations presented in Section IV.B.1.a is obtained

$$\frac{1}{6} \frac{\partial(u_1 \delta)}{\partial \tau} + a \frac{\partial(u_1^2 \delta)}{\partial x} = Pr^2 \delta - Pr \frac{u_1}{\delta}; \quad (A.6)$$

$$\frac{\partial \delta}{\partial \tau} + b \frac{\partial(u_1 \delta)}{\partial x} = \frac{2}{\delta}. \quad (A.7)$$

APPENDIX B

Analytical Techniques

1. Integral Methods

Before the method of integral relations is described in detail, related methods belonging to the same general class should be mentioned. The methods of Ritz and Galerkin seek to approximate solutions of ordinary differential and integral equations with given boundary conditions by assuming an approximate solution in terms of simple functions that satisfy the desired boundary conditions. Galerkin's method was extended to partial differential equations by Kantorovich.⁴⁵ When applied to partial differential equations, the method has been designated in some of the literature as Kantorovich's method. To make the present discussion complete and to compare the method of integral relations with the Ritz, Galerkin and Kantorovich methods, they are briefly outlined below.

It is desirable to mention also the classical Karman-Pohlhausen method.²⁹ Strictly speaking, these methods should not be called approximate because approximate methods refer to methods containing an algorithm, allowing in principle to obtain a solution with any degree of accuracy. The present technique primarily differs from the conventional Karman-Pohlhausen method in that the approximating functions are required to be the first N members of a complete set of functions, where N is the Nth approximation required. The weighting functions are any set of linearly independent functions judiciously chosen to permit one to speed convergence and promote ease of computation.

a. Ritz Method

Ritz⁴⁶ suggested a direct approximate solution of ordinary differential or integral equations denoted by the operator $D(u)$ by reducing them to systems containing a finite number of equations and the same number of unknowns. Application of the technique suggested by Ritz assumes the form of the unknown function $u(x)$ to be a certain class of functions represented by the finite series

$$u(x) = \sum_{i=1}^n a_i \phi_i(x), \quad (B.1)$$

where a_i are real coefficients and ϕ_i is an assumed orthogonal set of functions. No rational technique is available for choosing these functions a priori, but Miklin⁷⁴ has contributed toward the solution of this problem for linear systems. The equation degenerates into a function in a finite number of variables; that is, the coefficients $a_1, a_2, \dots, a_n$, must satisfy the relation

$$D_n(u) = D_n(a_i). \quad (B.2)$$

One seeks a sequence of the a_n coefficients that result in the minimum of the function $D_n(a_i)$. The coefficients a_i are found from the solutions to the system of n equations in n unknowns

$$\frac{\partial D_n}{\partial a_i} = 0, \quad i = 1, 2, 3, \dots, n. \quad (B.3)$$

From a variational standpoint, the one advantage of the Ritz method, in comparison with other approximate methods, is that if a true minimum does exist, then the approximate value of D_n is an upper bound on the true minimum of $D(u)$. For if D_n is lower than the true minimum, then $D(u)$ is not its lowest value. This fact is noted by Finlayson and Scriven.⁴⁷ On the other hand, if the variational scheme is not a true minimum, so that the sign of the second variation is either positive or negative, then this method has no advantage.

b. Galerkin Method

Another technique that can be used to solve various differential equations is that of Galerkin. If the solution of the ordinary differential equation denoted by the general operator $L(u) = 0$ is desired, then an approximation to the solution can be obtained by using the functions $w_1, w_2, \dots, w_k$ and assuming an approximate solution of the form

$$u = \sum_{k=1}^n c_k w_k. \quad (B.4)$$

This satisfies the prescribed boundary conditions where the coefficients c_k are determined from the n equations

$$\int_s L(u) w_k ds = 0. \quad (B.5)$$

In Eq. B.5, w_k are the so-called weighting functions.

If u is the exact solution, $L(u)$ vanishes identically. If u is not the exact solution, then $L(u)$ denotes the "error in the differential equation," often referred to as the residual, and the integral in Eq. B.5 is a weighted average integral of the error.

The Galerkin method is said to be of great generality because it does not depend on the existence of a variational or minimum principle. The solution obtained is in the form of functions, and this is distinctly

advantageous in comparison with finite-difference methods. Galerkin's method allows closer contact with the physical problem; thus its use may provide insight into the effects of the governing parameters.

Singer⁴⁸ showed that the Galerkin and Ritz schemes are identical for some cases. Finlayson and Scriven⁴⁹ used the Galerkin approach to construct approximate solutions of differential and integral equations. By defining the weighting functions in five different ways, they showed that one can obtain the methods of collocation, least squares, subdomain, and moments and the pure Galerkin method. Following the work of Crandall,⁵⁰ they define this general approach as the method of weighted residuals.

c. Kantorovich Method

The Ritz and Galerkin methods have been extended with varied degrees of success to problems of more than one independent variable.⁵¹ These extensions have been accomplished primarily because of the efforts of Kantorovich.⁵²

Kantorovich^{45, 53} assumed approximate solutions of the form

$$u(x, y) = \sum_{i=1}^n \Lambda_i(x, y) \psi_i(x), \quad (\text{B.6})$$

where the set of functions $\Lambda(x, y)$ is chosen a priori to satisfy the prescribed boundary conditions. For partial differential equations in two independent variables, this method reduces the problem to a system of n ordinary differential equations.

d. Method of Integral Relations

Various other methods have been suggested.^{47, 49, 54, 68} The practical advantage of these methods is that they approximate the system of partial differential equations by a system of ordinary differential, algebraic, or transcendental equations. However, they are offshoots of the three basic techniques discussed above. One such method is known as the method of integral relations originally proposed by Dorodnitsyn.⁵⁴ This method is one of the more general methods of solution of partial differential equations and is applicable to equations of various types. To increase its effectiveness, a generalized method of integral relations was proposed.⁵⁵ By the introduction of a weighting function, chosen in accordance with the expected behavior of the unknown function, one obtains a system of ordinary differential equations that are relatively smoother than those in the original method.⁵⁴ By this technique, greater accuracy can be obtained with a smaller number of interpolation functions.

The idea of the generalized method of integral relations, hereafter called the method of integral relations, is illustrated in the following manner. Suppose that in some two-dimensional rectangular domain, $a_a \leq x \leq b_a$, $c_a \leq y \leq d_a$, there is given a system of equations of the "divergent" type

$$\frac{\partial}{\partial x} P_i(x, y; u_1, \dots, u_m) + \frac{\partial}{\partial y} Q_i(x, y; u_1, \dots, u_m) = F_i(x, y; u_1, \dots, u_m),$$

$$i = 1, 2, 3, \dots, m. \quad (B.7)$$

Multiplying Eq. B.7 by a weighting function $H(y)$ and carrying out the integration with respect to y over the interval $y = c_a$ to $y = d_a$, we obtain the following integral relation that is the starting point for the method:

$$\frac{d}{dx} \int_{c_a}^{d_a} P_i H dy + Q_{i,d_a} H(d_a) - Q_{i,c_a} H(c_a) - \int_{c_a}^{d_a} Q_i H' dy = \int_{c_a}^{d_a} F_i H dy. \quad (B.8)$$

The system of weighting functions $H(y)$ is chosen from any set of linearly independent functions. Approximate expressions must also be chosen for P_i , Q_i , and F_i .

$$P_i \approx \sum_{\ell=1}^n P_{i,\ell} \psi_{\ell}(y), \quad (B.9)$$

and similarly for Q_i and F_i . Here $\psi_{\ell}(y)$ is an interpolation polynomial, whose actual form depends on the choice of interpolation formula. (It may contain not only powers of y , but also transcendental functions.)

Replacing P_i , Q_i , and F_i by their approximate expressions in Eq. B.8, we obtain a system of $n \times m$ ordinary differential equations for $m(n+1)$ unknown functions $u_{k,\ell}$, where

$$u_{k,\ell} = u_k(x, y_{\ell}). \quad (B.10)$$

The m boundary conditions provide the remaining equations to complete the system.

The basic equations do not have to be in "divergence" form as in the formulation of Dorodnitsyn. This form is useful and convenient in that the integration with respect to one of the variables can be carried out exactly. Bethel and Abbott⁴ presented a parallel formulation in such a way that it more closely follows the Galerkin and Kantorovich method.

There is one more possible application⁵⁶ of the method of integral relations. In two-dimensional problems, the integration can be carried out in two directions simultaneously, approximating the function with respect to both variables. When this is done, the original problem is approximated by a system of algebraic or transcendental equations.

As can be seen from the above discussion, together with recent survey articles concerning applications of the method of integral relations, little effort has been expended on the solution of multidimensional problems. To use the method in the solution of transient free convection from a vertical plate, it must be generalized to partial differential equations in three independent variables. Two approaches are possible. In the first approach, the original system of equations is multiplied by a weighting function of two of the variables under consideration and is simultaneously integrated with respect to these variables. A quadrature formula is applied to the two-dimensional integral thus obtained, and we are left with a system of ordinary differential equations in the third variable. This approach is analogous to the reduction of a two-dimensional problem to algebraic or transcendental equations. In the second approach, the original system of equations is first integrated with respect to one independent variable, and the dependent variables are approximated with respect to one independent variable by some expression that depends on the remaining two variables. The approximate equations thus obtained are a system of partial differential equations with respect to the remaining two independent variables, which can then be solved by the method of integral relations developed for two-dimensional problems or by some other method.

The latter extension of the method will be used in the present work because limiting solutions, which can be used to check the accuracy of the approximation, are possible. Before proceeding further in the application of this method or in the development of other methods, we should discuss the advantages and disadvantages involved in the use of the method of integral relations.

A principal advantage of the method of integral relations is that an integrand is being approximated; the loss in accuracy from the approximation is thereby reduced. The integral is a smoother function than the integrand and as a result, for a low-order approximation, a reasonable representation for the integral is obtained more accurately than could be for the integrand function. In addition, the integral admits a continuous representation, even when the integrand function has a singularity of the first order. Furthermore, the form of the method is often convenient, at least in fluid mechanics and related problems, since the conservation equations are automatically satisfied, and calculations may therefore be completed in the presence of discontinuities.

A distinct disadvantage of the scheme is that the results of the method depend upon the choice of the coordinate system. In addition, there is some arbitrariness in the interpolation functions chosen and, in some instances, in the quantities being approximated by them. It remains more an art than a science to correctly choose coordinates, independent variables, weighting functions, and dependent variables. The convergence question will arise when the method is applied to a specific problem, but considerable effort has been expended, with excellent results, on the convergence of the method when applied to problems of fluid mechanics that have known exact solutions.⁴ On this basis, the method should yield a reasonable approximation. Serious difficulties in the application of the method of integral relations (in particular, for nonlinear equations) can arise when one needs to solve a boundary-value problem for an approximation of high order.

2. Method of Lines

The second method used to verify some of the approximate calculations is the method of lines. This is a numerical method that has been somewhat neglected by engineers except for the solution of two-dimensional partial differential equations on analog computers.⁵⁷ In the solution of partial differential equations with two independent variables, the method of lines is used to finite-difference one independent variable. This results in a system of ordinary differential equations. The nature of the system of ordinary differential equations depends both on the form of the original partial differential equation and on the type of boundary conditions imposed on the problem. A fairly complete bibliography and derivation of the basic difference equations is given in Ref. 58.

Since Ref. 58 was written, several other articles on the method of lines have been published. Hicks and Wei⁵⁹ modified the method for the one-dimensional diffusion equation using noncentral differences and found the scheme to be much faster in terms of computer time than conventional finite-difference methods. Fisher⁶⁰ demonstrated the use of higher-order differences as a method for reducing the finite-difference errors. Howe and Haneman⁶¹ solved the wave equation and the two-dimensional diffusion equation with excellent results. Krylov and Liskovets⁶² estimated the error involved when the method is applied to one-dimensional, linear hyperbolic, and parabolic partial differential equations.

Slobodyansky⁶³ considered a technique that increases the accuracy of the solution without increasing the number of equations to be solved. The expression obtained by Slobodyansky can be derived in the following manner. Expand the function $v^*(y)$ in a Taylor's series so that the difference equation can be expressed in terms of the second derivatives of $v^*(y)$ with respect to y at the points $y + h^*$, y , and $y - h^*$ (where h^* is the increment used), thus obtaining

$$v^*(y+h^*) = v^*(y) + h^* \frac{dv^*(y)}{dy} + \frac{h^{*2}}{2} \frac{d^2v^*(y)}{dy^2} + \frac{h^{*3}}{6} \frac{d^3v^*(y)}{dy^3} + \frac{h^{*4}}{24} \frac{d^4v^*(y)}{dy^4} + \dots$$

and

$$v^*(y-h^*) = v^*(y) - h^* \frac{dv^*(y)}{dy} + \frac{h^{*2}}{2} \frac{d^2v^*(y)}{dy^2} - \frac{h^{*3}}{6} \frac{d^3v^*(y)}{dy^3} + \frac{h^{*4}}{24} \frac{d^4v^*(y)}{dy^4} - \dots$$

By adding the above expansions, one obtains

$$v^*(y+h^*) - 2v^*(y) + v^*(y-h^*) = h^{*2} \frac{d^2v^*(y)}{dy^2} + \frac{h^{*4}}{12} \frac{d^4v^*(y)}{dy^4} + O(h^{*6}). \quad (B.11)$$

The accuracy of this expansion is to the order of h^{*6} . From Ref. 58, we have

$$\frac{d^2v^*(y)}{dy^2} = \frac{v^*(y+h^*) - 2v^*(y) + v^*(y-h^*)}{h^{*2}}.$$

Now

$$\frac{d^2v^*(y+h^*)}{dy^2} = \frac{d^2v^*(y)}{dy^2} + h^* \frac{d^3v^*(y)}{dy^3} + \frac{h^{*2}}{2} \frac{d^4v^*(y)}{dy^4} + \dots$$

and

$$\frac{d^2v^*(y-h^*)}{dy^2} = \frac{d^2v^*(y)}{dy^2} - h^* \frac{d^3v^*(y)}{dy^3} + \frac{h^{*2}}{2} \frac{d^4v^*(y)}{dy^4} + \dots$$

so that

$$h^{*2} \frac{d^4v^*(y)}{dy^4} = \frac{d^2v^*(y+h^*)}{dy^2} - 2 \frac{d^2v^*(y)}{dy^2} + \frac{d^2v^*(y-h^*)}{dy^2} + O(h^{*4}). \quad (B.12)$$

Substitution of Eq. B.12 into Eq. B.11 yields

$$v^*(y+h^*) - 2v^*(y) + v^*(y-h^*) = \frac{5}{6} h^{*2} \frac{d^2v^*(y)}{dy^2} + \frac{h^{*2}}{12} \frac{d^2v^*(y+h^*)}{dy^2} + O(h^{*6}). \quad (B.13)$$

This expression is fundamental to the present work.

An interpolation scheme for the method of lines using the basic ideas of the Kantorovich's method⁵³ is illustrated below. The solution of a general two-dimensional partial differential equation is assumed according to the form

$$T(x, y) = \sum_{i=1}^n \Lambda_i(x, y) \psi_i(x), \quad (B.14)$$

where $\Lambda(x, y)$ are functions chosen in advance according to the equation and the type of prescribed boundary conditions. The function $\psi_i(x)$ is determined from a solution of the resultant system of differential equations. The conditions on $\psi_i(x)$ are that $T(x, y)$ satisfies the partial differential equation on certain specified lines throughout the region.

This approach may be constructed by approximate interpolation techniques in the following manner. Introduce the approximate values of $T_k(x)$ on certain lines as unknowns. Using an interpolation formula, one may then calculate approximate values of $T^*(x, y)$. Assuming that $T^*(x, y)$ satisfies the partial differential equation on the lines chosen and the system boundaries, one can determine the unknowns, $T_k(x)$.

With an interpolation scheme, this method is written in the approximate form as

$$T^*(x, y) = \sum_{i=1}^M \Lambda_i(y) \psi_i(x) \quad (B.15)$$

for a given $\Lambda_i(y)$. When there are N intermediate lines, there exist, for a second-order differential equation, $N + 4$ conditions for the functions $\psi_i(x)$. These are one for each intermediate line and two for each boundary line, one satisfying the governing equation, and one satisfying the boundary condition. In this manner, we obtain $M = N + 4$ equations. Thus, using a simple polynomial interpolation and four intermediate lines, one finds that $M = 8$, so that a seventh-degree polynomial can be used. For two intermediate lines ($M = 6$), a fifth-degree polynomial can be used.

The above scheme may best be illustrated by means of a simple example. Consider the Laplace equation

$$\nabla^2 T(x, y) = 0 \quad (B.16)$$

with the boundary conditions

$$T(x, 0) = T_A = \text{constant};$$

$$T(x, 1) = T_B = \text{constant}. \quad (B.17)$$

Appropriate boundary conditions for x can be prescribed. Using a single intermediate line, one obtains $M = 5$, so that in polynomial form there results

$$T^*(x, y) = \psi_0(x) + \psi_1(x) y + \psi_2(x) y^2 + \psi_3(x) y^3 + \psi_4(x) y^4. \quad (B.18)$$

The ψ_i 's are determined from

$$T(x, 0) = T_A, \quad (B.19)$$

$$T(x, 1) = T_B, \quad (B.20)$$

$$\left. \frac{\partial^2 T}{\partial y^2} \right|_{y=0} = - \left. \frac{\partial^2 T}{\partial x^2} \right|_{y=0} = - \frac{\partial^2 T_A}{\partial x^2} = 0, \quad (B.21)$$

$$\left. \frac{\partial^2 T}{\partial y^2} \right|_{y=1} = - \left. \frac{\partial^2 T}{\partial x^2} \right|_{y=1} = 0, \quad (B.22)$$

and

$$T^*(x, 1/2) = T_0(x), \quad (B.23)$$

where the unknown function $T_0(x)$ is determined from the condition that the polynomial satisfies Eq. B.15 on the line $y = 1/2$. Evaluating the ψ_i 's in Eq. B.17 by use of Eqs. B.18-B.22, we obtain

$$T^*(x, y) = (1 - y) T_A + \frac{T_B}{9} (32y^3 - 7y - 16y^4) + \frac{16}{9} T_0(y^4 - 2y^3 + y). \quad (B.24)$$

Evaluating Eq. B.16 on $y = 1/2$ yields

$$\left. \frac{\partial^2 T^*}{\partial y^2} \right|_{y=1/2} = - \frac{d^2 T_0(x)}{dx^2}. \quad (B.25)$$

Using Eqs. B.25 and B.13, the ordinary differential equation for $T_0(x)$ obtained is

$$\frac{5}{6} \frac{d^2 T_0}{dx^2} + 4(T_A - 2T_0 + T_B) = 0. \quad (B.26)$$

The solution of this equation determines the function $T_0(x)$, and its substitution into Eq. B.24 completes the approximate solution. Of particular note is the fact that Eq. B.26 is identical in form to that obtained by Slobodyansky. The obvious advantage is the polynomial interpolation across the strips afforded by the application of Kantorovich's method combined with the sixth-order accuracy in h^* obtained by Slobodyansky.

The preceding is an improved formulation of the method of lines using Kantorovich's method to interpolate between the lines to accelerate convergence and thus permit the use of a minimum number of lines for any problem. In this report, use is also made of the ordinary method of lines, a fairly complete derivation and discussion of which are contained in Ref. 58 and needs no further elaboration here.

APPENDIX C

Design Considerations

For the geometry chosen in this report, the effect of foil material and its thickness must be taken into account in selection of a material. Initially the concept of forming an absorbing layer was as follows. The Vycor glass plate was to be painted to a uniform thickness of about 0.0001-0.0002 in. on one face with Pyromark high-temperature paint. Then by a deposition or evaporation method, 0.0003 in. of nickel was to be plated on the paint. This procedure would have resulted in a 0.0005- to 0.0006 in. layer, which would absorb the incident thermal radiation and adhere to the Vycor very well. Initial attempts to produce this type of bond on a test piece were not successful. To make a sufficiently strong bond, the cost would have been prohibitive. For this reason, a very thin foil, painted on one side, was chosen for the test-section wall.

Addition of heat to the foil by exposing it to a uniform radiative flux on one face results in several competing mechanisms of energy transfer. Thus, one must consider conduction through the foil, conduction and steady or transient convection in the boundary layer, thermal radiation incident upon the foil, and finally absorption of energy by the foil due to finite heat capacity if the temperature of the foil changes with time.

When a heat balance is made on the foil, the following assumptions apply:

1. Conduction through the foil is negligible; i.e., this requires a small Biot number $= h\ell/2K$.
2. Incident radiation is homogeneous and uniform (obtained data experimentally).
3. The painted foil is a "gray" body.
4. Radiation emitted into the mercury can be neglected because of the opaqueness of the mercury.
5. Radiation leaving the foil in the direction of the heat source (lamps) can be neglected.
6. Axial conduction in the fluid may be neglected.

An energy balance on a foil of width S , height dX , and thickness $\ell/2$ yields

$$q_R S dX = \rho c_p \frac{\ell}{2} \frac{\partial T}{\partial t} S dX - k \left. \frac{\partial T}{\partial Y} \right|_{Y=0} S dX - K \frac{\ell}{2} \frac{\partial^2 T}{\partial X^2} S dX$$

or

$$q_R = \rho c_p \frac{\ell}{2} \frac{\partial T}{\partial t} - k \left. \frac{\partial T}{\partial Y} \right|_{Y=0} - K \frac{\ell}{2} \frac{\partial^2 T}{\partial X^2}. \quad (C.1)$$

From this equation, information can be obtained regarding the design parameters of interest if it is assumed that the imposed heat flux on the face of the foil is uniform and the temperature is independent of time. For steady-state conditions,

$$q_R = -k \left. \frac{\partial T}{\partial Y} \right|_{Y=0} - \frac{K \ell}{2} \frac{\partial^2 T}{\partial X^2}. \quad (C.2)$$

As the distance from the leading edge increases, $\partial T / \partial X$ and $\partial^2 T / \partial X^2$ decrease with X ; this decreases the importance of the last term in Eq. C.2. Goldstein³⁷ was the first to analyze this problem in the following manner. He assumed a uniform heat flux out of the plate and used theoretical predictions⁴¹ to obtain an approximate value for the heat-transfer coefficient. Next he compared the relative magnitude of the terms in Eq. C.2. He estimated the plate-conduction term from the theoretical wall-temperature distribution for constant wall heat flux. For a fluid of given Pr , this yields that the ratio of heat conducted down the plate to the heat input is proportional to

$$\frac{K}{k} \frac{\ell}{2X} \left(\frac{1}{Gr_x^*} \right)^{1/5} \quad (C.3)$$

The above expression shows that if the thermal conductivity of the fluid is small, the conduction of heat along the length of the foil will be negligible for distances far away from the leading edge. In fact, for a nickel foil 1 in. from the leading edge in the range of heat fluxes used, in the experiments the conduction is about 1% of the total applied heat flux. As a result of the above calculations, the heat-conduction term in Eq. C.1 can be neglected. The resultant equation after the above assumptions reduces to

$$q_R = \rho \frac{c_p \ell}{2} \frac{\partial T}{\partial t} - k \left. \frac{\partial T}{\partial X} \right|_{Y=0} \quad (C.4)$$

Equation C.4 can be simplified by using the relation

$$-k \left. \frac{\partial T}{\partial X} \right|_{Y=0} = h \Delta T, \quad (C.5)$$

where the heat-transfer coefficient, h , is assumed to be constant in space and time. Since this is unreal physically, some approximation is required. Finally, assuming a step change in q_R from zero to some finite but constant value equal to

$$\rho \frac{c_p \ell}{2} \frac{\partial T}{\partial t} + h \Delta T, \quad (C.6)$$

i.e.,

$$h \Delta T = q_R = 0, \quad t < 0, \quad q_R = \text{constant}, \quad t > 0, \quad (C.7)$$

the differential equation reduces to

$$\rho \frac{c_p \ell}{2} \frac{\partial T}{\partial t} = -h \Delta T. \quad (C.8)$$

The solution of Eq. C.8 together with Eq. C.7 is

$$\Delta T = q_R \frac{1}{h} \left[1 - \exp\left(-\frac{2h}{\rho c_p \ell} t\right) \right]. \quad (C.9)$$

Now,

$$q_{\text{conv.}} = h \Delta T = q_R \left[1 - \exp\left(-\frac{2h}{\rho c_p \ell} t\right) \right]. \quad (C.10)$$

The quantity indicating how closely the heat flux leaving the foil approaches q_R is

$$\frac{q_{\text{conv.}}}{q_R} = 1 - \exp\left(-\frac{2h}{\rho c_p \ell} t\right) \quad (C.11)$$

The above equation yields a system time constant, $\tau_c = \rho c_p \ell / 2h$, which is a measure of how closely the system approaches a step change in the boundary condition. Using actual property data and a very conservative value for the heat-transfer coefficient, h , the time constant for 0.0005-in.-thick foil is 0.2 sec, which is very low. In comparison with other similar studies^{35,37,77} for air, water, and oils, this value is very good.

A better approximation for a step function in radiation heat-flux input could be made by considering one-dimensional conduction in the fluid. During the initial portion of the transient, this assumption is exact. This is the period during which the heat flux from the foil should be stabilizing at the constant input value.

Following the analysis of Goldstein³⁷ for a semi-infinite medium (the fluid) and an infinite plane (the sheet), one has

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial Y^2}, \quad (C.12)$$

with

$$\lim_{Y \rightarrow \infty} T = 0, \quad (C.13)$$

$$q_R = \frac{\rho c_p \ell}{2} \frac{\partial T}{\partial t} - k \frac{\partial T}{\partial Y} \Big|_{Y=0} = \text{constant}, \quad t > 0, \quad (C.14)$$

and

$$T = 0, \quad t = 0. \quad (C.15)$$

Using the Laplace Transform, Goldstein solved the above problem for the temperature distribution in the fluid from which the wall-temperature variation, heat conducted from the wall, and the heat-transfer coefficient can be obtained. Thus,

$$\begin{aligned} Q = q_R \frac{\alpha^{1/2}}{k} \left\{ 2 \left(\frac{t}{\pi} \right)^{1/2} \left\{ \exp \left(-\frac{Y^2}{4\alpha t} \right) \right\} - \frac{Y^2}{\alpha^{1/2}} \operatorname{erfc} \left[\frac{Y}{2(\alpha t)^{1/2}} \right] \right. \\ \left. + \frac{1}{a^*} \exp \left(a^* \frac{Y}{\alpha^{1/2}} \right) \exp(a^* t) \operatorname{erfc} \left[a^* t^{1/2} + \frac{Y}{2(\alpha t)^{1/2}} \right] \right. \\ \left. - \frac{1}{a^*} \operatorname{erfc} \left[\frac{Y}{2(\alpha t)^{1/2}} \right] \right\}, \quad (C.16) \end{aligned}$$

$$\frac{\Delta T k a^*}{q_R \alpha^{1/2}} = 2 a^* \left(\frac{t}{\pi} \right)^{1/2} + \exp(a^* t) \operatorname{erfc}(a^* t^{1/2}) - 1, \quad (C.17)$$

$$\frac{q_{\text{conv.}}}{q_R} = \left[1 - \exp(a^* t) \operatorname{erfc}(a^* t^{1/2}) \right], \quad (C.18)$$

and

$$\frac{h \alpha^{1/2}}{k a^*} = \frac{1 - \exp(a^* t) \operatorname{erfc}(a^* t^{1/2})}{\frac{2}{\pi^{1/2}} a^* t^{1/2} + \exp(a^* t) \operatorname{erfc}(a^* t^{1/2}) - 1}, \quad (C.19)$$

where

$$a^* = \frac{2k}{\ell \rho c_p \sqrt{\alpha}} \quad (C.20)$$

From Eq. C.18, the design data can be obtained in concise closed form; that is, the larger the value of the parameter a^* , the closer a step change in heat flux is approximated. For the nickel-mercury system used, a^{*2} was about 1000 sec^{-1} . To obtain a more rigorous comparison with previous experimental systems, Goldstein³⁷ has an a^{*2} of 0.05 sec^{-1} for a brass-air system and about 773 sec^{-1} for his stainless steel-water system. The results obtained by interferometric measurements further substantiated the closer approximation of a step function for system with high a^{*2} values. For the preceding reasons, the nickel-mercury system used should be satisfactory in approximating a step input in wall heat flux.

Finally, as a secondary check, it was feasible to use the results of an analysis by Gebhart²⁰ to confirm the above conclusions. In terms of Gebhart's dimensionless variables,

$$\bar{T} = \frac{\alpha t}{L^2} [b^* Ra_L^*]^{2/5} \quad (C.21)$$

measures the effect of the wall and fluid on the experimentally determined step function in wall heat flux, and

$$\bar{Q} = \frac{c''}{\rho c_p L M_0} [b^* Ra_L^*]^{1/5} \quad (C.22)$$

determines the relation of a finite heat capacity of the wall to that of the heat capacity of the boundary layer. In the above notation, the parameters b^* and M_0 are functions of the Prandtl number and the temperature and velocity distributions. They have been evaluated numerically by Gebhart.

For this experiment, one obtains

$$\bar{Q} \cong 0.0000338$$

for the highest heat flux obtained, which would indicate that the thermal capacity of the wall may be neglected. Likewise, one obtains

$$\bar{T} \cong 6.1 \times 10^{-8} t/\text{sec}, \quad (C.23)$$

which when combined in ratio form, $\bar{T}/\bar{Q}$, gives an indication of the system time constant,

$$\bar{T}/\bar{Q} \cong 1.8 \times 10^{-3} t/\text{sec}, \quad (C.24)$$

which again appears satisfactory.

APPENDIX D

Installation of Vycor

Because of the extreme difficulties experienced in installation of the Vycor glass used to construct the test-section walls, the installation procedures will be discussed here.

Initially, any contact of metal to Vycor resulted in an immediate failure. Thus, gasketing was used on all interfaces. Durabla 1/64 in. thick was used first, but silicon rubber 1/16 in. thick finally was adapted to effectively cushion the interface and was satisfactory.

To ensure a uniform pressure at the top and bottom of the Vycor, stainless steel stock 1/2 in. thick was machined flat to 0.001 in. The clamps on the 10-in. edge of the test section were also machined flat to within 0.001 in. At the onset of installation, it was believed that 30 in.-lb of torque would be required to contain mercury within the test section. This torque was obtained by starting at 5 in.-lb with the center bolt on each face and working outward, and after the sequence was complete, increasing the torque by 5-in.-lb increments until a 30-in.-lb torque was uniformly applied all around. After many failures, a trial-and-error approach was used; that is, hand-tighten the bolts, fill with mercury, and tighten slowly until no leaks occurred. In this manner 8 in.-lb of uniform torque at room temperature was successful in containing the mercury. After the first 11 times, the test section was brought up to 500°F and the thermal contraction upon cooling caused the torque to reduce to 1/2 in.-lb, which resulted in a leak. Immediate retorquing to 8 in.-lb closed all leaks.

A visual inspection of the Vycor plates with a polariscope (i.e., polarized light) showed that residual stresses existed in each plate. Every plate was initially found to be in a relatively high state of stress. The largest stress concentrations occurred around the holes due to drilling, but the edges also had appreciable stresses due to cutting. The centers had minor stresses apparently due to polishing. To remove these residual stresses, the personnel in the glass shop at Argonne recommended that the Vycor be annealed. The strain point is approximately 710°C. Thus, the glass shop recommended heating the plate to 910°C and holding it at that temperature for 1½ hr. The plates were then cooled slowly to 500°C and removed. After the annealing, a check with a polariscope revealed that all residual stresses were removed. The holes drilled through each plate were also fire-polished to close up drilling checks, which could be potential points of stress concentration.

Finally, if the stainless steel tank was struck with a wrench, the resulting vibrations would cause the Vycor glass to shatter. Thus, the

bolts were insulated with nylon sleeves and the entire apparatus covered with tissue paper to prevent any such vibrations. Upon installation of one of the final sections, a sonic boom caused by an aircraft flying at high speed shattered the Vycor. After the U.S. Air Force offered their cooperation to prevent any overflights, the test section remained intact.

APPENDIX E

Measurements of Radiative Flux versus Position

The homogeneity and uniformity of the radiative flux emitted from the lamp units were determined in the following manner. The heat-flux sensors were moved vertically (in the x direction) while the distance from the vertical edge of the lamps (z direction) was held constant. The heat flux was held constant during this mapping of the transducer output. Finally, the entire procedure was repeated for several separate values of heat flux.

A typical set of measurements from a single constant heat flux is plotted in Fig. E.1. The curve for $z = 1/2$ is higher than the other two curves because the lamp units join at that point and a thin solid bar prevents the installation of a single lamp there. However, uniform radiation from the units on each side combine to present a relatively reasonable incident flux.

Fig. E.1

Heat Flux as a Function of Position for
Lamp Unit at $q_w = 4200 \text{ Btu/hr-ft}^2$

The conclusion from the measurements is that the radiative flux emitted from the lamps is nearly uniform and may be considered as uniform in the calculations.

APPENDIX F

Physical Property Data of Mercury

To make approximate analytical expressions available for use on the digital computer for reduction of raw data, least-squares polynomial-curve fits were made of the available property data. An iterative procedure was carried out until the curve fits corresponded with the input data points at a preset number of temperature levels usually specified to 1° increments. Standard and mean deviations were computed along with the mean percent deviation so that the overall accuracy was described quantitatively. The final curve fits were within 0.08% of the input data in the range of 70-800°F.

Expressions were obtained in the above manner using the data tabulated by Weatherford, Tyler, and Ku.⁸⁸ The following expressions were used:

$$\begin{aligned} \text{DENSITY} = & -0.95269274 \times 10^{-14}T^6 + 0.26846987 \times 10^{-10}T^5 \\ & -0.29559722 \times 10^{-7}T^4 + 0.15943073 \times 10^{-4}T^3 \\ & -0.42944549 \times 10^{-2}T^2 + 0.43478051T + 0.82974720 \times 10^3; \end{aligned}$$

$$\begin{aligned} \text{THER COND} = & -0.18682275 \times 10^{-18}T^7 + 0.56415186 \times 10^{-15}T^6 \\ & -0.6766859 \times 10^{-12}T^5 + 0.40985698 \times 10^{-9}T^4 \\ & -0.13218156 \times 10^{-6}T^3 + 0.2070148 \times 10^{-4}T^2 \\ & +0.30108259 \times 10^{-2}T + 4.5928302; \end{aligned}$$

$$\begin{aligned} \text{SPEC HT CONST PRESS} = & -0.7529142 \times 10^{-16}T^5 + 0.17372781 \times 10^{-12}T^4 \\ & -0.15511758 \times 10^{-9}T^3 + 0.71788743 \times 10^{-7}T^2 \\ & -0.1930361 \times 10^{-4}T + 0.034842871; \end{aligned}$$

$$\begin{aligned} \text{ABS VISC} = & -0.4787065 \times 10^{-14}T^4 - 0.26304791 \times 10^{-8}T^3 \\ & +0.64872204 \times 10^{-5}T^2 - 0.622212 \times 10^{-2}T + 4.1266451; \end{aligned}$$

where T is in °F.

APPENDIX G

Error Analysis

The following analysis of the probable errors in calculated quantities will use the statistical approach that describes the probable error of a function in terms of the probable errors of the quantities on which the function depends. Briefly, this result is expressible as follows: The probable error $E(F)$ of the function

$$F = F(x_1, x_2, \dots, x_n)$$

is related to the probable errors $E(x_i)$ of the mean values of the measured quantities, x_i , using the standard root-mean-square concept

$$E(F) = \left[\sum_{i=1}^n \left(\frac{\partial F}{\partial x_i} \right)^2 E^2(x_i) \right]^{1/2} \quad (G.1)$$

The estimated errors in the experimentally measured quantities are:

1. Temperature

Calibration error	$\pm 0.10^\circ\text{F}$
Reading error	$\pm 0.25^\circ\text{F}$
Total	$\pm 0.35^\circ\text{F}$

2. Heat flux

Transducer calibration error	$\pm 432 \text{ Btu/hr-ft}^2$
Reading error	$\pm 40 \text{ Btu/hr-ft}^2$
Total	$\pm 472 \text{ Btu/hr-ft}^2$

3. Axial location

Calibration error	$\pm 0.0156 \text{ in.}$
Reading error	$\pm 0.005 \text{ in.}$
Total	$\pm 0.0206 \text{ in.}$

4. The Grashof number based on L is defined as

$$\text{Gr}_{L,f} = \frac{g\beta\Delta TL^3}{\nu^2}$$

Thus, assuming physical property values are exact,

$$\frac{E(\text{Gr}_{L,f})}{\text{Gr}_{L,f}} = \left[\frac{E^2(\Delta T)}{\Delta T^2} \right]^{1/2} = \frac{0.35}{10} = \pm 0.035.$$

5. The Grashof number based on x and $\text{Gr}_{x,f}$ is

$$\begin{aligned} \frac{E(\text{Gr}_{x,f})}{\text{Gr}_{x,f}} &= \left[\frac{E^2(\Delta T)}{\Delta T^2} + \frac{E^2(x)}{x^6} \right]^{1/2} \\ &= \left(0.001225 + 2 \times 10^{-12} \right)^{1/2} \\ &= \pm 0.035. \end{aligned}$$

6. The modified Grashof number based on L and $\text{Gr}_{L,f}^*$ is

$$\frac{E(\text{Gr}_{L,f}^*)}{\text{Gr}_{L,f}^*} = \left[\frac{E^2(q)}{q^2} \right]^{1/2} = \pm 0.118.$$

7. The modified Grashof number based on x and $\text{Gr}_{x,f}^*$ is

$$\frac{E(\text{Gr}_{x,f}^*)}{\text{Gr}_{x,f}^*} = \left[\frac{E^2(q)}{q^2} + \frac{E^2(x)}{x^8} \right]^{1/2} = \pm 0.035.$$

8. The Nusselt number based on L and $\text{Nu}_{L,f}$ is

$$\frac{E(\text{Nu}_{L,f})}{\text{Nu}_{L,f}} = \left[\frac{E^2(q)}{q^2} + \frac{E^2(\Delta T)}{\Delta T^2} \right]^{1/2} = \pm 0.133.$$

9. The Nusselt number based on x and $\text{Nu}_{x,f}$ is

$$\frac{E(\text{Nu}_{x,f})}{\text{Nu}_{x,f}} = \left[\frac{E^2(q)}{q^2} + \frac{E^2(\Delta T)}{\Delta T^2} + \frac{E^2(x)}{x^2} \right]^{1/2} = \pm 0.133.$$

APPENDIX H

Tabulation of Data

Selected basic experimental data and analytical results are tabulated in this appendix. The nomenclature and procedures for data reduction have been presented elsewhere in this report and will not be repeated here.

TABLE II. Experimental Data for Test Section I at Moderate Heat Flux

$$q_w = 41,760 \text{ Btu/hr-ft}^2$$

t, sec	T(X = 1 in., Y = 0), °F	T(X = 1 in., Y = 1/2 in.), °F	T(X = 1 in., Y = 1/2 in.), °F	T(X = 5 in., Y = 0), °F	T(X = 8 in., Y = 0), °F
0	81	81	81	81	81
5	120	92	81	129	146.5
10	128	104	81	159	166
15	140.5	120	81.5	174.5	188
20	155	138.5	81.5	200	212
25	171.5	149.5	82	223	232
30	191	167	82	246.5	255.5
35	210	197	83	267.5	275.5
40	228	215.5	83	288	296
45	246	232	84	316.5	326
50	253	251	85	330.5	341
55	281	273	86	346	349
60	300	289	87	371	383
65	318.5	309	89	392	401
70	332	324	91	410	418
75	350.5	343	93	432	432.5
80	369	366.5	96	446	456.5
85	386.5	379	99.5	463	481.5
90	405	395.5	103.5	489	489
95	422.5	414	108	503	516
100	435	426.5	112	520	537

TABLE III. Experimental Data for Test Section I at Low Heat Flux

$$q_w = 3600 \text{ Btu/hr-ft}^2$$

t, min	T(X = 1 in., Y = 0), °F	T(X = 1 in., Y = 1/2 in.), °F	t, min	T(X = 1 in., Y = 0), °F	T(X = 1 in., Y = 1/2 in.), °F
0	65	65	39	321	313
1	90	78	40	323.5	315
2	97	90	41	325.5	317
3	106.5	101	42	328	320
4	116	113	43	330	321
5	126.5	124.5	44	332.5	324
6	137	135	45	334.5	326
7	147	144.5	46	336.5	327
8	157	154	47	338	328
9	166	163.5	48	340	330
10	176	172	49	341.5	331
11	185.5	181	50	344	334
12	194	189	51	345.5	335
13	202	198	52	346	336
14	210	205	53	348	337.5
15	217.5	214	54	350	339
16	224.5	220	55	351.5	340.5
17	231	226.5	56	353	342
18	238	233	57	354	342.5
19	243.5	237	58	355	345
20	249.5	243	59	356	346
21	254.5	250	60	357	347
22	260.5	254.5	61	358	347.5
23	265.5	259.5	62	359	348
24	270	264	63	359.5	348.5
25	274.5	268	64	360.5	349
26	280	274	65	361	350
27	281	275	66	362	351
28	287	282	67	362.5	352
29	291	287	68	364	353
30	294	290	69	366	353.5
31	298.5	293	70	366	354
32	302	295	71	367	354.5
33	304.5	297.5	72	367.5	355
34	308	300	73	368	354.5
35	310.5	303	74	368	355.5
36	313.5	305.5	75	368.2	355
37	316	308	76	368	355
38	319.5	311			

TABLE IV. Experimental Data for Test Section IIIa

$$q_w = 28,800 \text{ Btu/hr-ft}^2$$

t, sec	T(X = 1.0 in., Y = 0), °F	T(X = 1.0 in., Y = 0.5 in.), °F	t, sec	T(X = 1.0 in., Y = 0), °F	T(X = 1.0 in., Y = 0.5 in.), °F
0	69	69	510	162	177
30	78	83	540	161	177.5
60	86.5	91.5	570	167	178
90	97.5	104	600	162	176
120	109	115	630	162.5	175
150	118	120	660	166	175
180	126	132	690	168	174
210	132.5	138	720	168	178
240	138.5	145	750	178	179.5
270	141	148	780	167.5	180
300	149	156.5	810	173	180
330	149	158	840	177.5	182
360	150	159	870	170.5	183
390	156	160	900	175.5	180
420	155	170	930	168.5	176.5
450	160	174	960	167	177.5
480	161	175.5	990	171	180

TABLE V. Analytical Results for One-parameter Approximation,
Pr = 1.0, Constant-wall-temperature Boundary Condition,
Runge-Kutta Step Size 0.5×10^{-3}

τ	$a_1(\tau)$	$b_1(\tau)$	τ	$a_1(\tau)$	$b_1(\tau)$
0	10^{-4}	10^{-3}	5.0	4.95619	2.80209
0.5	1.30404	1.66065	5.5	4.94869	2.79930
1.0	2.45954	2.23636	6.0	4.94249	2.79795
1.5	3.39797	2.55839	6.5	4.93851	2.79744
2.0	4.08209	2.72471	7.0	4.93640	2.79733
2.5	4.52636	2.79722	7.5	4.93547	2.79737
3.0	4.77983	2.81973	8.0	4.93518	2.79745
3.5	4.90322	2.820145	8.5	4.93517	2.79752
4.0	4.95061	2.81361	9.0	4.93524	2.79756
4.5	4.96061	2.8068			

TABLE VI. Analytical Results for One-parameter Approximation,
Pr = 0.003, Constant-wall-temperature Boundary Condition,
Runge-Kutta Step Size 0.1×10^{-1}

τ	$a_1(\tau)$	$b_1(\tau)$	τ	$a_1(\tau)$	$b_1(\tau)$
0	$\sim 10^{-4}$	$\sim 10^{-1}$	600	0.0204241	42.83418
50	0.0022635	16.80496	700	0.0220049	42.76886
100	0.0044140	23.49313	800	0.0231116	42.36913
150	0.0065202	28.36606	900	0.0238328	41.84491
200	0.0085546	32.15522	1000	0.0242913	41.27601
250	0.0104954	35.14462	1100	0.0244965	41.83511
300	0.0123248	37.48923	1200	0.0245697	40.51999
350	0.0140284	39.29468	1300	0.0244921	40.30215
400	0.0155960	40.64465	1400	0.0243406	40.28740
450	0.0170207	41.61147	1500	0.024297	40.26319
500	0.0182997	42.26023			

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