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Based on Ph.D. Thesis Submitted to Iowa State University

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ABSTRACT

An iterative method is developed to find an approximation to the least squares solution of minimum norm, f_o, of an operator equation, $Kf = g$, of the first kind. In applications to integral equations, the superfluous oscillations in the final solution of methods in the literature are not an apparent problem. Let $K : H_1 \rightarrow H_2$ be a completely continuous operator from a real separable Hilbert space H_1 to a real Hilbert space H_2. Let $g \in R(K) + N(K^*)$. A sequence $\{f_{\lambda_j}\}_{j=1}^{\infty} \subset N(K)^{\perp}$ converging to f_o is constructed. f_{λ_j} minimizes the functional $Q_{\lambda_j}(f) = ||Kf - g|| + \lambda_j ||f||$, $\lambda_j > 0$. Stability and error bounds are obtained.

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1. INTRODUCTION AND REVIEW OF THE LITERATURE

1.1. Statement of the Problem

Throughout this paper we will assume that H_1 is a real separable (or finite dimensional) Hilbert space, H_2 is a real Hilbert space and K is a compact (or completely continuous) operator from H_1 to H_2. The basic definitions used above can be found in Bachman and Narici [3, Chapter 6 and 17].

We will use the notation:

$R(K)$ = range of K,

$N(K) = \{f \in H_1 : Kf = 0\}$

and

$M^{\perp} = \{f \in H_1 : \langle f,h \rangle = 0 \text{ for all } h \in M \subset H_1\}$

where $\langle , \rangle$ denotes inner product in the Hilbert space. From Strand [20, page 22] we have that for an operator K

$H_2 = N(K^*) \oplus \overline{R(K)}$ where $\overline{R(K)}$ is the closure of $R(K)$,

$H_1 = N(K) \oplus (N(K))^{\perp}$,

and

$\overline{R(K^*)} = [N(K)]^{\perp}$.

We are interested in finding an approximation to the least squares solution of minimum norm, f_o, of the operator equation $Kf = g$ where $g \in H_2$, when such a solution exists. Kammerer and Nashed [11] give the following definitions.

<u>Definition 1.1.1</u> For $f \in H_1$, $g \in H_2$, an element $\bar{f} \in H_1$ is a least squares solution of $Kf = g$ if and only if

(1.1.2) $$\|K\bar{f} - g\| = \inf\{\|Kf - g\| \mid f \in H_1\}.$$

<u>Definition 1.1.2</u> For $f \in H_1$, $g \in H_2$, an element $f_o \in H_1$ is a least squares solution of minimum norm of $Kf = g$ if and only if it is a least squares solution of $Kf = g$ and $\|f_o\| \leq \|\bar{f}\|$ for all least squares solutions $\bar{f}$ of $Kf = g$.

Strand [20, pages 23-24] notes that the least squares solution of minimum norm to $Kf = g$ exists (is unique and in $N(K)^{\perp}$) if and only if $g \in H_2$ can be written in the form

(1.1.3) $$g = g_1 + g_2$$

where $g_1 \in R(K)$ and $g_2 \in N(K^*)$. For this paper we will assume g is always in the form $g = g_1 + g_2$, with $g_1 \in R(K)$. From Strand [20] and Diaz and Metcalf [5], f is a solution of $\|Kf - g\| = \inf\{\|Kh - g\| \mid h \in H_1\}$ if and only if f is a solution to $Kf = g_1$. They also note that the least square solutions to $K^*Kf = K^*g$ and $Kf = g_1$ are the same, this will become important later on in this paper.

The least squares solution of minimum norm of $Kf = g$ can be written as a series in terms of g and the eigenvalues and eigenvectors of KK^* and K^*K, so is known exactly ([20, page 25], [24, page 143]). However, finding each eigenvector of KK^* is as difficult a problem, numerically, as the original problem.

Twomey [25], Phillips [18], Tikhonov [23] and Strand [20] consider various generalizations of the method of regularization for compact operators K. Strand [20] considers the functional

$$(1.1.4) \qquad Q(f) = \|Kf - g\|^2 + \lambda\|f - p\|^2, \ \lambda > 0,$$

where $p \in H_1$ is an estimate for the least squares solution of minimum norm of $Kf = g$. He proves that (1.1.4) is minimized by

$$f_\lambda = (K^*K + \lambda I)^{-1}(K^*g + \lambda p). \tag{1.1.5}$$

Strand [20] gives a proof for the following theorem.

<u>Theorem 1.1.1</u>. $\bar{f} = \lim_{\lambda \to o^+} f_\lambda$ exists and $\bar{f}$ is the unique solution of $K^*Kf = K^*g$ for which $\|f - p\|^2$ is a minimum. We will use this theorem in Chapter 2 for the case where $p \equiv 0$, to obtain an approximation to the least squares solution of minimum norm of $Kf = g$. In general a method for determining a good choice for λ in actual applications and construction of an inverse to represent $(K^*K + \lambda I)^{-1}$ is difficult since $(K^*K + \lambda I)^{-1}$ is generally ill-conditioned for a useful choice for λ. In this dissertation, the problem of a choice for λ will be considered and finding $(K^*K + \lambda I)^{-1}$ will be avoided.

1.2. Iterative Methods

Landweber [13], Diaz and Metcalf [5], Kammerer and Nashed [10,11,12] and Nashed [14,15] study iterative methods for finding the least squares solution of minimum norm of $Kf = g$, by constructing a sequence $\{f_i\}_{i=1}^{\infty}$ which converges to the least squares solution of minimum norm. The sequence $\{f_i\}_{i=1}^{\infty}$ is defined by

$$f_{n+1} = f_n + \alpha_n W_n \tag{1.2.1}$$

where α_n and W_n are chosen to give the method of steepest descent, conjugate gradient or weak steepest descent method. These methods converge to the least squares solution of minimum norm of $Kf = g$ for various choices for a starting vector and restrictions on $g \in H_2$. For perturbations of g by a function ϵ in H_2 we have the operator equation $Kf = g + \epsilon$ which, since $g + \epsilon$ might not be an element of $R(K) + N(K^*)$, may fail to have a least squares solution of minimum norm. Strand [20, page 47] shows that these methods are sensitive to small perturbations of g and may fail to converge to a function near the actual solution. This is a serious problem for a numerical method, since discretization error and roundoff error act like a perturbation of g.

The iterative method considered in this dissertation will depend on operators of the form $(\lambda I + K^*K) : H_1 \to H_1$, $\lambda > 0$. Since $\lambda I + K^*K$ is bounded and positive definite the inverse $(\lambda I + K^*K)^{-1}$ exists and is defined on all of H_1 for each $\lambda > 0$. For λ not too small the solutions of $(K^*K + \lambda I)f = K^*g$ will not be as sensitive to perturbation of g as the equation $K^*Kf = K^*g$ (see section 2.3), but has a solution close to the least squares solution of minimum norm of $K^*Kf = K^*g$ (or $Kf = g$).

Some examples will be given, together with a discussion of some of the problems of discretization, of finding approximations to least square solutions of minimum norm of integral equations of the first kind.

2. DEVELOPMENT OF AN ITERATIVE METHOD FOR CONVERGENCE TO THE LEAST SQUARES SOLUTION OF MINIMUN NORM

2.1. Introduction

First. An iterative method is obtained for an approximate solution to the least squares solution of minimum norm of $Kf = g$, when such a solution exists.

Second. Several theorems will be given which study the properties of changes in the solution to $Kf = g$ when g is replaced with $g + \epsilon$, ϵ an element of H_2, to give an indication of stability.

Third. Theorems relating $\lambda > o$ and $f_\lambda \in H_1$, where f_λ is a solution to $(K^*K + \lambda I)f = K^*g$ will be given, as well as some theorems on error bounds.

2.2. An Iterative Method

The least squares solution of minimum norm, f_o, is an element of $[N(K)]^\perp$. For any element $f \in N(K)$ we have $K(f_o + f) = Kf_o + Kf = Kf_o$, so since $f_o + f$ and f_o are mapped to the same element in the range of K it is important for an iterative procedure to stay on $[N(K)]^\perp$. The goal is to construct a sequence of f_λ's which satisfy the theorem below, converge to f_o and stays in $N(K)^{\perp}$.

Theorem 2.2.1. If $f_\lambda \in H_1$ is the solution to $(K^*K + \lambda I)f = K^*g$, $\lambda > 0$, then $f_\lambda \in R(K^*) \subset N(K)^{\perp}$.

Proof: Since $(K^*K + \lambda I)f_\lambda = K^*g$ we have $\lambda f_\lambda = K^*g - K^*K f_\lambda$ which in turn implies that

$$f_\lambda = \frac{K^*(g - K f_\lambda)}{\lambda}, \quad \text{thus} \quad f_\lambda \in R(K^*) \subset [N(K)]^{\perp}.$$

Now we develop an iterative method related to a steepest descent method for Tikhonov-Twomey regularization, which converges to the least squares solution of minimum norm, when such a solution exists. The method developed allows one to find a solution of $(K^*K + \lambda I)f = K^*g$ for a $\lambda > 0$. When the convergence fails or becomes slow then the advantage of this method is that a new starting vector can be obtained.

We really wish to implement the procedure in Step 1 and 2 following and in doing so the main result we use is given in the corollary on page 20 for Theorem 2.2.2 which follows.

Step 1: Choose a sequence of positive real numbers $\{\lambda_i\}_{i=1}^{\infty}$ such that $\lambda_i \to 0$ as $i \to \infty$, a starting element $f_{1,o} \in H_1$ for the sequence $\{f_{1,n}\}_{n=o}^{\infty}$ converging to the solution, f_1, of $(K^*K + \lambda_1 I)f = K^*g$, as required by Theorem 2.2.2.

Step 2: we now proceed inductively. For $j = 2,3,4,\ldots$ using $f_{j,o} = f_{j-1}$ as a starting vector, obtain the sequence $\{f_{j,n}\}_{n=o}^{\infty}$ converging to f_j, where f_j is the solution to $(K^*K + \lambda_j I)f = K^*g$. Continue until λ_j is less than some pre-assigned value.

Theorem 2.2.2. Let $f_{1,n+1} = f_{1,n} + \alpha_{1,n}W_{1,n}$, $\alpha_{1,n} \in$ reals, for each integer $n \geq 0$ and $f_{1,o} \in H_1$ then the sequence $\{f_{1,n}\}_{n=o}^{\infty}$ converges to the function $f = f_1 \in R(K^*)$ that minimizes $Q_{\lambda_1}(f) = \|Kf - g\|^2 + \lambda_1\|f\|^2$ where

$$W_{1,n} = K^*K\, f_{1,n} + \lambda_1 f_{1,n} - K^*g,$$

$$\alpha_{1,n} = -\frac{\langle W_{1,n}, W_{1,n}\rangle}{\langle KW_{1,n}, KW_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle} \quad \text{for} \quad W_{1,n} \neq 0,$$

and

> if n_1 is the first n such that $W_{1,n} = 0$
> take $\alpha_{1,n} = 0$ and $f_1 = f_{n,1}$ for all $n \geq n_1$.

Proof: From Theorem 2.2.1 we have $f_1 \in R(K^*)$. To choose $\alpha_{1,n}$ consider $f_{1,n+1} = f_{1,n} + \alpha W_{1,n}$ and

$$Q_{\lambda_1}(f_{1,n+1}) = \|K f_{1,n+1} - g\|^2 + \lambda_1 \|f_{1,n+1}\|^2$$

(2.2.1)
$$= \langle K^*K f_{1,n+1}, f_{1,n+1}\rangle - 2\langle K^*g, f_{1,n+1}\rangle + \langle g,g\rangle + \lambda_1 \langle f_{1,n+1}, f_{1,n+1}\rangle$$

$$= \langle K^*K f_{1,n} + \alpha K^*KW_{1,n}, f_{1,n} + \alpha W_{1,n}\rangle - 2\langle K^*g, f_{1,n} + \alpha W_{1,n}\rangle + \langle g,g\rangle + \lambda_1\langle f_{1,n} + \alpha W_{1,n}, f_{1,n} + \alpha W_{1,n}\rangle$$

(2.2.2)
$$= \langle K^*K f_{1,n}, f_{1,n}\rangle + 2\alpha\langle K^*K f_{1,n}, W_{1,n}\rangle + \alpha^2\langle K^*KW_{1,n}, W_{1,n}\rangle - 2\langle K^*g, f_{1,n}\rangle$$

$$- 2\alpha\langle K^*g, W_{1,n}\rangle + \langle g,g\rangle + \lambda_1\langle f_{1,n}, f_{1,n}\rangle$$

$$+ 2\lambda_1\alpha\langle f_{1,n}, W_{1,n}\rangle + \lambda_1\alpha^2\langle W_{1,n}, W_{1,n}\rangle .$$

We now take the derivative of $Q_{\lambda_1}(f_{1,n+1})$ with respect to α and set the derivative equal to zero:

$$\frac{d}{d\alpha} Q_{\lambda_1}(f_{1,n+1}) = 2\langle K^*K f_{1,n}, W_{1,n}\rangle + 2\alpha\langle K^*KW_{1,n}, W_{1,n}\rangle$$

$$- 2\langle K^*g, W_{1,n}\rangle + 2\lambda_1\langle f_{1,n}, W_{1,n}\rangle$$

$$+ 2\lambda_1\alpha\langle W_{1,n}, W_{1,n}\rangle = 0.$$

Note also

$$\frac{d^2}{d\alpha^2} Q_{\lambda_1}(f_{1,n+1}) = 2(\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle) > 0$$

for $W_{1,n} \neq 0$. Solving $\frac{d}{d\alpha} Q_{\lambda_1}(f_{1,n+1}) = 0$ for α gives

$$\alpha(\langle K^*KW_{1,n},W_{1,n}\rangle + \lambda_1\langle W_{1,n},W_{1,n}\rangle$$

$$= \langle K^*g,W_{1,n}\rangle - \langle K^*Kf_{1,n},W_{1,n}\rangle - \lambda_1\langle f_{1,n},W_{1,n}\rangle$$

$$= - \langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle$$

where $W_{1,n} \neq 0$, yielding the result:

$$(2.2.3)\quad \alpha_{1,n} = \alpha = - \frac{\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle}{\langle K^*KW_{1,n},W_{1,n}\rangle + \lambda_1\langle W_{1,n},W_{1,n}\rangle} .$$

Now we write $f_{1,n+1}$ as

$$f_{1,n+1} = f_{1,n} - \frac{\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle}{\langle K^*KW_{1,n},W_{1,n}\rangle + \lambda_1\langle W_{1,n},W_{1,n}\rangle} W_{1,n}$$

for $W_{1,n} \neq 0$.

Now we show that $\{Q_{\lambda_1}(f_{1,n})\}_{n=o}^{\infty}$ is a decreasing sequence of positive real numbers. From the expression (2.2.2), replacing $n+1$ with n in (2.2.1) and α with $\alpha_{1,n}$ in both (2.2.2) and (2.2.1) we get the following expression.

$$Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) = \langle K^*Kf_{1,n}, f_{1,n}\rangle$$

$$+ 2\,\alpha_{1,n}\langle K^*Kf_{1,n}, W_{1,n}\rangle + \alpha_{1,n}^2\langle K^*KW_{1,n}, W_{1,n}\rangle$$

$$- 2\langle K^*g, f_{1,n}\rangle - 2\,\alpha_{1,n}\langle K^*g, W_{1,}n\rangle + \langle g,g\rangle + \lambda_1\langle f_{1,n}, f_{1,n}\rangle$$

$$+ 2\,\lambda_1\alpha_{1,n}\langle f_{1,n}, W_{1,n}\rangle + \lambda_1\alpha_{1,n}^2\langle W_{1,n}, W_{1,n}\rangle$$

$$- \langle K^*Kf_{1,n}, f_{1,n}\rangle + 2\langle K^*g, f_{1,n}\rangle$$

$$- \langle g,g\rangle - \lambda_1\langle f_{1,n}, f_{1,n}\rangle$$

$$= 2\,\alpha_{1,n}\langle K^*Kf_{1,n}, W_{1,n}\rangle + \alpha_{1,n}^2\langle K^*KW_{1,n}, W_{1,n}\rangle$$

$$- 2\,\alpha_{1,n}\langle K^*g, W_{1,n}\rangle + 2\,\lambda_1\alpha_{1,n}\langle f_{1,n}, W_{1,n}\rangle$$

$$+ \lambda_1\alpha_{1,n}^2\langle W_{1,n}, W_{1,n}\rangle$$

$$= 2\,\alpha_{1,n}(\langle K^*Kf_{1,n} - K^*g, W_{1,n}\rangle + \lambda_1\langle f_{1,n}, W_{1,n}\rangle)$$

$$+ \alpha_{1,n}^2(\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle).$$

So we have from the expression for $\alpha_{1,n}$

$$Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) = -\frac{2\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle^2}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle}$$

$$+ \frac{\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle^2}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle}$$

$$= -\frac{\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle^2}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle} .$$

Therefore, we have $Q_{\lambda_1}(f_{1,n+1}) \leq Q_{\lambda_1}(f_{1,n})$ and equality occurs when $\langle K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g, W_{1,n}\rangle = 0$ for some n.

Note that if $K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g = 0$ for some n, say n_1, then $f_{1,n_1} = (K^*K + \lambda_1 I)^{-1}K^*g$ from (1.1.5), where $p \equiv 0$, is the minimizing value for $Q_{\lambda_1}(f)$. Thus if $K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g \neq 0$ for $n > n_1$ and $K^*Kf_{1,n_1} + \lambda_1 f_{1,n_1} - K^*g = 0$ for $n = n_1$ take $\alpha_{1,n} = 0$ for $n \geq n_1$ and $f_{1,n} = f_1$ for $n \geq n_1$. In general take $W_{1,n} = K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g$. We have discussed the case $W_{1,n} \neq 0$ for all n, then (2.2.1) becomes

$$\alpha_{1,n} = -\frac{\langle W_{1,n}, W_{1,n}\rangle}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle}$$

and

(2.2.4)

$$Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) = -\frac{\langle W_{1,n}, W_{1,n}\rangle^2}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle}$$

So $Q_{\lambda_1}(f_{1,n+1}) < Q_{\lambda_1}(f_{1,n})$ for all n. We now have

$$Q_{\lambda_1}(f_{1,n+1}) = Q_{\lambda_1}(f_{1,n}) - \frac{\langle W_{1,n}, W_{1,n}\rangle^2}{\langle K^*KW_{1,n}, W_{1,n}\rangle + \lambda_1\langle W_{1,n}, W_{1,n}\rangle}$$

$$= Q_{\lambda_1}(f_{1,o}) - \sum_{k=o}^{n} \frac{\langle W_{1,k}, W_{1,k}\rangle^2}{\langle K^*KW_{1,k}, W_{1,k}\rangle + \lambda_1\langle W_{1,k}, W_{1,k}\rangle}.$$

So $\left\{\sum_{k=o}^{n} \frac{\langle W_{1,k}, W_{1,k}\rangle^2}{\langle K^*KW_{1,k}, W_{1,k}\rangle + \lambda_1\langle W_{1,k}, W_{1,k}\rangle}\right\}_{n=o}^{\infty}$ is an increasing sequence of positive real numbers bounded above by $Q_{\lambda_1}(f_{1,o})$ and so converges. Now since

$$\langle K^*KW_{1,k}, W_{1,k}\rangle = \langle KW_{1,k}, KW_{1,k}\rangle$$

$$\leq \|K\|^2 \langle W_{1,k}, W_{1,k}\rangle$$

we have

$$\langle K^*KW_{1,k}, W_{1,k}\rangle + \lambda_1\langle W_{1,k}, W_{1,k}\rangle \leq (\|K\|^2 + \lambda_1)\langle W_{1,k}, W_{1,k}\rangle$$

and

(2.2.5)

$$\frac{1}{(\|K\|^2 + \lambda_1)\langle W_{1,k}, W_{1,k}\rangle} \leq \frac{1}{\langle K^*KW_{1,k}, W_{1,k}\rangle + \lambda_1\langle W_{1,k}, W_{1,k}\rangle} .$$

Multiply each side of (2.2.5) by $\langle W_{1,k}, W_{1,k}\rangle^2$ and sum over k, yielding

$$\frac{1}{(\|K\|^2 + \lambda_1)} \sum_{k=o}^{\infty} \langle W_{1,k}, W_{1,k} \rangle$$

$$\leq \sum_{k=o}^{\infty} \frac{\langle W_{1,k}, W_{1,k} \rangle^2}{\langle K^*KW_{1,k}, W_{1,k} \rangle + \lambda_1 \langle W_{1,k}, W_{1,k} \rangle} < \infty .$$

So $\sum_{k=o}^{\infty} \langle W_{1,k}, W_{1,k} \rangle$ converges, thus

(2.2.6) $$\langle W_{1,k}, W_{1,k} \rangle \to 0 \quad \text{as} \quad k \to \infty.$$

We now show $\{f_{1,n}\}_{n=o}^{\infty}$ converges to $f_1 = (K^*K + \lambda I)^{-1}K^*g$. Once this is done the proof will be complete, since this f_1 minimizes $Q_{\lambda_1}(f)$. Since $(K^*K + \lambda_1 I)$ is invertible, there exists a positive real number $M_{\lambda_1} = \|(K^*K + \lambda_1 I)^{-1}\|^{-1}$ such that for all n

$$M_{\lambda_1}\|f_{1,n} - f_1\| \le \|(K^*K + \lambda_1 I)(f_{1,n} - f_1)\|$$

$$= \|K^*Kf_{1,n} + \lambda_1 f_{1,n} - (K^*Kf_1 + \lambda_1 f_1)\|$$

$$= \|K^*Kf_{1,n} + \lambda_1 f_{1,n} - K^*g\|$$

$$= \|W_{1,n}\| .$$

Therefore we have

$$(2.2.7) \qquad \|f_{1,n} - f_1\| \le \frac{\|W_{1,n}\|}{M_{\lambda_1}} \to 0 \quad \text{as} \quad n \to \infty$$

from (2.2.6).

So we have shown that $f_{1,n}$ converges to f_1 in norm as $n \to \infty$. This completes the proof of the theorem.

Now take λ_2 from the sequence $\{\lambda_j\}_{j=1}^{\infty}$ and use $f_1 \in R(K^*)$, from Theorem 2.2.1. as a starting element, i.e. take $f_{2,o} = f_1$, and as in the case for λ_1 construct the sequence $\{f_{2,n}\}_{n=o}^{\infty}$ which converges to $f_2 \in R(K^*)$ such that $f_{2,n+1} = f_{2,n} + \alpha_{1,n}W_{2,n}$ where

$$\alpha_{2,n} = - \frac{\langle W_{2,n}, W_{2,n} \rangle}{\langle K^*KW_{2,n}, W_{2,n} \rangle + \lambda_2 \langle W_{2,n}, W_{2,n} \rangle} ,$$

and

$$W_{2,n} = K^*Kf_{2,n} + \lambda_2 f_{2,n} - K^*g$$

for

$$K^*Kf_{2,n} + \lambda_2 f_{2,n} - K^*g \neq 0$$

and $\alpha_{2,n} = 0$ whenever

$$K^*Kf_{2,n} + \lambda_2 f_{2,n} - K^*g = 0.$$

In general, use f_{j-1} as a starting element in $R(K^*)$, take $f_{j,o} = f_{j-1}$, $j = 2,3,4,\ldots$ and construct the sequence

(2.2.8) $\{f_{j,n}\}_{n=o}^{\infty}$ which converges to f_j

such that $f_{j,n+1} = f_{j,n} + \alpha_{j,n} W_{j,n}$

where $\alpha_{j,n} = -\dfrac{\langle W_{j,n}, W_{j,n}\rangle}{\langle K^*KW_{j,n}, W_{j,n}\rangle + \lambda_j\langle W_{j,n}, W_{j,n}\rangle}$,

$$W_{j,n} = K^*Kf_{j,n} + \lambda_j f_{j,n} - K^*g$$

for

$$K^*Kf_{j,n} + \lambda_j f_{j,n} - K^*g \neq 0$$

and $\alpha_{j,n} = 0$ whenever $K^*Kf_{j,n} + \lambda_j f_{j,n} - K^*g = 0$.

From the remarks at the beginning of section 2.2, we need that the sequence defined by (2.2.8) stay in $[N(K)]^{\perp}$. Since for $j > 1$ and $f_{j,o} = f_{j-1} \in R(K^*)$, we have the following corollary to Theorem 2.2.2..

<u>Corollary 2.2.3</u>. If $j > 1$ then the sequence defined by (2.2.8), $\{f_{j,n}\}_{n=o}^{\infty} \subset R(K^*) \subset [N(K)]^{\perp}$.

<u>Theorem 2.2.4</u>. Let f_o be the least squares solution of minimum norm of $Kf = g$, $f_{1,o} \in H_1$, $\{\{f_{j,n}\}_{n=o}^{\infty}\}_{j=1}^{\infty}$ constructed as in (2.2.8) and let $\epsilon > 0$, then there exists positive integers n and j such that

$\|f_{j,n} - f_o\| < \epsilon$.

Proof: Follows from $\|f_{j,n} - f_o\| \leq \|f_{j,n} - f_j\| + \|f_j - f_o\|$.

2.3. Perturbations of g

We have that $R(K) + N(K^*) \subset H_2$ is dense in H_2. For a perturbation of g by an element ϵ belonging to H_2 it is possible that $g + \epsilon$ will fail to be an element of $R(K) + N(K^*)$, in which case the least squares solution of minimum norm if $Kf = g + \epsilon$ would fail to exist. The goal is to determine relationships of the solutions of

$$(K^*K + \lambda I)f = K^*g,$$

$$(K^*K + \lambda I)f = K^*(g + \epsilon)$$

and the least squares solution of minimum norm of $Kf = g$. Then an approximate solution of minimum norm of $Kf = g$ can be found, provided the perturbation of g is not too large.

Theorem 2.3.1. If $f_\lambda \in H_1$ is the solution to $(K^*K + \lambda I)f = K^*g$ and $f_{\lambda,\epsilon} \in H_1$ is the solution to $(K^*K + \lambda I)f = K^*(g + \epsilon)$ where ϵ belongs to H_2 then

$$\|f_{\lambda,\epsilon} - f_\lambda\| \leq \frac{\|K\|}{\|(K^*K + \lambda I)^{-1}\|^{-1}} \|\epsilon\|$$

Proof: We have that

$$(K^*K + \lambda I) f_\lambda = K^*g,$$

$$(K^*K + \lambda I) f_{\lambda,\epsilon} = K^*(g + \epsilon).$$

Subtracting the above equations, we get

$$(K^*K + \lambda I) f_{\lambda,\epsilon} - (K^*K + \lambda I) f_\lambda = K^*(g + \epsilon) - K^*g,$$

$$(K^*K + \lambda I)(f_{\lambda,\epsilon} - f_\lambda) = K^*\epsilon.$$

Now since $K^*K + \lambda I$ has a bounded inverse, there exists an $M_\lambda = \|(K^*K + \lambda I)^{-1}\|^{-1}$ such that

$$M_\lambda \|f_{\lambda,\epsilon} - f_\lambda\| \leq \|(K^*K + \lambda I)(f_{\lambda,\epsilon} - f_\lambda)\|$$

$$= \|K^*\epsilon\|$$

$$\leq \|K^*\| \|\epsilon\|$$

$$= \|K\| \|\epsilon\|,$$

so $$\|f_{\lambda,\epsilon} - f_\lambda\| \leq \frac{\|K\|}{M_\lambda} \|\epsilon\|.$$

Remark: With reference to Theorem 2.3.1, if we let $\underline{\mu}$ be the minimum (or inf of the) non-negative eigenvalues of K^*K then

$$\|(K^*K + \lambda I)^{-1}\|^{-1} = \underline{\mu} + \lambda.$$

Theorem 2.3.2. If given $\delta > 0$ and

1. f_o is the least squares solution of minimum norm of $Kf = g$,
2. $f_{j,\epsilon}$ is the solution of $(K^*K + \lambda I)f = K^*(g+\epsilon)$ for each j, $j = 1,2,\ldots,$
3. $f_{j,n,\epsilon}$ defined by (2.2.8),
4. ϵ is an element of H_2 then

$$\|f_{\bar{j},\bar{n},\epsilon} - f_o\| \leq \frac{\|K\|\|\epsilon\|}{\|(K^*K + \lambda_{\bar{j}} I)^{-1}\|^{-1}} + \delta \quad \text{for some } \bar{j} \text{ and } \bar{n}.$$

Proof: $\|f_{j,n,\epsilon} - f_o\| = \|f_{j,n,\epsilon} - f_{j,\epsilon} + f_{j,\epsilon} - f_j + f_j - f_o\|$

$$\leq \|f_{j,n,\epsilon} - f_{j,\epsilon}\| + \|f_{j,\epsilon} - f_j\| + \|f_j - f_o\|$$

$$\leq \frac{\|K\|\|\epsilon\|}{\|(K^*K + \lambda_j I)^{-1}\|^{-1}} + \|f_{j,n,\epsilon} - f_{j,\epsilon}\| + \|f_j - f_o\| .$$

Now there exists a j and n such that $\|f_{j,n,\epsilon} - f_{j,\epsilon}\| < \frac{\delta}{2}$ and $\|f_j - f_o\| < \frac{\delta}{2}$, say for $\bar{j} = j$ $\bar{n} = n$ we have

$$\|f_{\bar{j},\bar{n},\epsilon} - f_o\| \leq \frac{\|K\|\|\epsilon\|}{\|(K^*K + \lambda_{\bar{j}} I)^{-1}\|^{-1}} + \delta .$$

2.4. Some Error Bounds and Properties of λ

Now we will prove some theorems which will yield some bounds on $\|f_\lambda - f_o\|$ in terms of $\lambda > 0$ where $f_\lambda \in H_1$ is a solution to $(K^*K + \lambda I)f = K^*g$ and f_o is the least squares solution of minimum norm of $Kf = g$. The bound on $\|f_\lambda - f_o\|$ will depend on knowledge of the singular system

$(U_n, V_n; \mu_n)$ for the compact operator K which is described below. The notation and results follow that of Strand [20, page 19], [21] and Tricomi [24]. Define $\sigma_o(K^*K) = \sigma_o(KK^*) = \{\gamma_n : \gamma_n$ is an eigenvalue of K^*K, $\gamma_n > 0$, $n \in N_k\}$ where $N_k = \{1,2,\ldots,k\}$, (k may be $\mathfrak{N}_o$).

Let

$$U = \{U_n : KK^*U_n = \gamma_n U_n, \ n \in N_k\}$$

and

$$V = \{V_n : K^*KV_n = \gamma_n V_n, \ n \in N_k\}.$$

Since K^*K is compact, symmetric and non-negative definite, assume that the γ_n are ordered such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq \cdots > 0$. Define $\mu_n = \gamma_n^{-1/2}$, $n \in N_k$. Then we have $0 < \mu_1 \leq \mu_2 \leq \cdots$,

$$U_n = \mu_n K V_n$$

and

$$V_n = \mu_n K^* U_n .$$

Now in this notation we can state Picard's Theorem, a proof of which is given in Strand [20, page 25].

<u>Theorem 2.4.1. (Picard)</u>. Let $(U_n, V_n; \mu_n)$ be a singular system for the compact operator $K : H_1 \to H_2$ and $\bar{g} \in H_2$. Then the equation $Kf = \bar{g}$ has a solution $f_o \in H_1$ if and only if $\sum_{n \in N_m} \mu_n^2 |\langle \bar{g}, U_n \rangle|^2 < \infty$ and $\bar{g} \in \overline{R(K)}$. Also for $\bar{g} \in R(K)$ we have $f_o = \sum_{n \in N_m} \langle \bar{g}, U_n \rangle \mu_n V_n$.

Strand [20, page 26] notes that for $g = g_1 + g_2$, $g_1 \in R(K)$ and $g_2 \in N(K^*)$ then $f_o = \sum_{n \in N_m} < g, U_n > \mu_n V_n$ is the least squares solution of minimum norm of the equation $Kf = g$.

<u>Theorem 2.4.2</u>. Let

1. K^*K have finite non-zero spectrum σ_o, $N_k = m$,

2. $f_\lambda \in H_1$ be the least squares solution of minimum norm of $Kf = g$ then

$$\|f_\lambda - f_o\| \leq \frac{\lambda}{\lambda + \gamma_m} \sqrt{\sum_{i=1}^{m} |\langle g, u_i \rangle|^2 \mu_i^2} \ .$$

Proof:

$$\|f_\lambda - f_o\|^2 = \|(K^*K + \lambda I)^{-1} K^*g - f_o\|^2$$

$$= \|(K^*K + \lambda I)^{-1} K^*Kf_o - f_o\|^2$$

$$= \|(K^*K + \lambda I)^{-1} (K^*K - (K^*K + \lambda I)) f_o\|^2$$

$$= \lambda^2 \|(K^*K + \lambda I)^{-1} f_o\|^2$$

$$= \lambda^2 \|(K^*K + \lambda I)^{-1} \sum_{n=1}^{m} \langle g, u_n \rangle \mu_n v_n\|^2$$

$$= \lambda^2 \|\sum_{n=1}^{m} \langle g, u_n \rangle \mu_n (K^*K + \lambda I)^{-1} v_n\|^2 ,$$

and since $(K^*K + \lambda I)^{-1} V_n = \frac{V_n}{\gamma_n + \lambda}$ we have

$$\|f_\lambda - f_o\|^2 = \lambda^2 \| \sum_{n=1}^{m} \langle g, U_n \rangle \mu_n \frac{V_n}{\gamma_n + \lambda} \|^2$$

$$= \lambda^2 \sum_{n=1}^{m} \frac{|\langle g, U_n \rangle|^2 \mu_n^2}{(\gamma_n + \lambda)^2}$$

$$\leq \left(\frac{\lambda}{\gamma_m + \lambda}\right)^2 \sum_{n=1}^{m} |\langle g, U_n \rangle|^2 \mu_n^2 .$$

Now we take the square root of each side of the above inequality, giving

$$\|f_\lambda - f_o\| \leq \frac{\lambda}{\gamma_m + \lambda} \sqrt{\sum_{n=1}^{m} |\langle g, U_n \rangle^2 \mu_n^2} .$$

This completes the proof of the theorem.

Theorem 2.4.2 can be extended to the case where K^*K does not have a finite non-zero spectrum.

Theorem 2.4.3. If under the same hypotheses as Theorem 2.4.2, except that K^*K has non-finite spectrum and an integer $N > 0$, a real number $\delta > 0$ are known such that

$$\sum_{n=N+1}^{\infty} |\langle g, U_n\rangle|^2 \mu_n^2 \leq \delta$$

then

$$\|f_\lambda - f_o\| \leq \sqrt{\left(\frac{\lambda}{\lambda + \gamma_N}\right)^2 \sum_{n=1}^{N} |\langle g, U_n\rangle|^2 \mu_n^2 + \delta}.$$

Proof: Simular to the proof of Theorem 2.4.2 we have

$$\|f_\lambda - f_o\|^2 = \lambda^2 \|\sum_{n=1}^{\infty} \langle g, U_n\rangle \mu_n \frac{V_n}{\gamma_n + \lambda}\|^2$$

$$= \lambda^2 \sum_{n=1}^{\infty} |\langle g, U_n\rangle|^2 \frac{\mu_n^2}{(\gamma_n + \lambda)^2}$$

$$= \lambda^2 \sum_{n=1}^{N} |\langle g, U_n \rangle|^2 \frac{\mu_n^2}{\left(\gamma_n + \lambda\right)^2}$$

$$+ \sum_{n=N+1}^{\infty} |\langle g, U_n \rangle|^2 \mu_n^2 \left(\frac{\lambda}{\gamma_n + \lambda}\right)^2$$

$$\leq \left(\frac{\lambda}{\gamma_N + \lambda}\right)^2 \sum_{n=1}^{N} |\langle g, U_n \rangle|^2 \mu_n^2 + \sum_{n=N+1}^{\infty} |\langle g, U_n \rangle|^2 \mu_n^2$$

$$\leq \left(\frac{\lambda}{\gamma_N + \lambda}\right)^2 \sum_{n=1}^{N} |\langle g, U_n \rangle|^2 \mu_n^2 + \delta .$$

Now we take the square root of each side of the above inequality, giving

$$\|f_\lambda - f_o\| \leq \sqrt{\left(\frac{\lambda}{\gamma_N + \lambda}\right)^2 \sum_{n=1}^{N} |\langle g, U_n \rangle|^2 \mu_n^2 + \delta} .$$

Remark: Let

$$A_\lambda = \begin{cases} \dfrac{\lambda}{\gamma_n + \lambda} \sqrt{\displaystyle\sum_{n=1}^{m} |\langle g, U_n \rangle|^2 \mu_n^2} & \text{if } m < \infty \\ \sqrt{\left(\dfrac{\lambda}{\gamma_n + \lambda}\right)^2 \displaystyle\sum_{n=1}^{N} |\langle g, U_n \rangle|^2 + \delta} & \text{otherwise.} \end{cases}$$

From (2.2.7) and (2.2.8) we have

$$\text{(2.4.1)} \qquad \|f_{j,n} - f_o\| \le \|f_{j,n} - f_j\| + \|f_j - f_o\|$$

$$\le \frac{\|W_{j,n}\|}{\|(K^*K + \lambda_j I)^{-1}\|^{-1}} + A_{\lambda_j} .$$

Remark: Since the eigenvectors of K^*K associated with its non-zero eigenvalues span $N(K)^\perp$ and f_o, f_j, $f_{j,n}$ belong to $N(K)^\perp$ for $j > 1$, we have

$$\gamma\|f_j - f_o\| \le \|K^*K(f_j - f_o)\|$$

and

$$(\gamma+\lambda_j)\|f_{j,n}-f_j\| \leq \|(K^*K+\lambda I)(f_{j,n}-f_j)\|$$

where $\gamma = \inf_{\gamma_n \in \sigma_o(K^*K)} \gamma_n$ (see Helmberg [7, page 225]).

Since $K^*K(f_j - f_o) = \lambda_j f_j$ and $(K^*K+\lambda I)(f_{j,n}-f_j) = w_{j,n}$ we have

$$\|f_j - f_o\| \leq \frac{\lambda_j\|f_j\|}{\gamma}$$

and

$$\|f_{j,n}-f_j\| \leq \frac{\|w_{j,n}\|}{\gamma+\lambda_j}.$$

Thus (2.4.1) can be rewritten as

(2.4.2) $$\|f_{j,n}-f_o\| \leq \frac{\|w_{j,n}\|}{\gamma+\lambda_j} + \frac{\lambda_j\|f_j\|}{\gamma}$$

giving non- a priori error bounds when $\gamma \neq 0$.

Using the same techniques as in the above remark, we can replace $\|(K^*K + I\lambda)^{-1}\|^{-1}$ with $\gamma + \lambda$ in Theorems 2.3.1 and 2.3.2.

<u>Theorem 2.4.4</u>. If f_λ is the solution to $(K^*K + \lambda I)f = K^*g$ then $\|f_\lambda\|$ is strictly decreasing in $\lambda > 0$.

Proof: Since $K^*K : H_1 \to H_1$ is a symmetric, non-negative definite and compact linear operator, there is an orthonormal bases $\{e_i\}$ for H_1 such that $(K^*K + \lambda I)e_i = (\gamma_i + \lambda)e_i$ for all i. Since K^*K is compact and non-negative we may assume $\gamma_i \geq \gamma_{i+1} \geq 0$ for all i. So we have that $K^*K + \lambda I \simeq \text{diag}(\gamma_1 + \lambda, \gamma_2 + \lambda, \ldots)$ relative to $\{e_i\}$. Thus

$$(K^*K + \lambda I)^{-1} \simeq \text{diag}\left(\frac{1}{\gamma_1 + \lambda}, \frac{1}{\gamma_2 + \lambda}, \ldots\right).$$

Since $f_\lambda = (K^*K + \lambda I)^{-1}K^*g$, it follows that

$$f_\lambda = \sum_i \frac{\langle K^*g, e_i \rangle}{\gamma_i + \lambda} e_i$$

and

$$\|f_\lambda\|^2 = \sum_i \frac{|\langle K^*g, e_i\rangle|^2}{(\gamma_i + \lambda)^2} \quad .$$

The theorem follows immediately.

We now prove that $\|Kf_\lambda\|$ is increasing as $\lambda > 0$ is decreasing. The theorem following is of interest, in its own right, when compared to Theorem 2.4.4.

<u>Lemma 2.4.5</u>. If $\alpha > 0$, $\beta > 0$, $f_\alpha \in R(K^*)$ is the solution to $(K^*K + \alpha I)f = K^*g$ and $f_\beta \in R(K^*)$ is the solution to $(K^*K + \beta I)f = K^*g$ then

$$\langle f_\alpha, f_\beta\rangle \geq \frac{\alpha\langle f_\alpha, f_\alpha\rangle + \beta\langle f_\beta, f_\beta\rangle}{\alpha + \beta}$$

and

$$\langle Kf_\alpha, Kf_\beta\rangle \geq \frac{\alpha\langle Kf_\alpha, Kf_\alpha\rangle + \beta\langle Kf_\beta, Kf_\beta\rangle}{\alpha + \beta}$$

Proof: We have the result that

$$K^*Kf_\alpha + \alpha f_\alpha = K^*Kf_\beta + \beta f_\beta$$

which implies

$$K^*K(f_\alpha - f_\beta) = \beta f_\beta - \alpha f_\alpha . \tag{2.4.3}$$

Now taking the inner product of each side of (2.4.3) with $f_\alpha - f_\beta$ we obtain the following result.

$$0 \leq \langle K^*K(f_\alpha - f_\beta), f_\alpha - f_\beta \rangle$$

$$= \langle \beta f_\beta - \alpha f_\alpha, f_\alpha - f_\beta \rangle$$

$$= \beta \langle f_\beta, f_\alpha - f_\beta \rangle - \alpha \langle f_\alpha, f_\alpha - f_\beta \rangle$$

$$= \beta \langle f_\beta, f_\alpha \rangle - \beta \langle f_\beta, f_\beta \rangle - \alpha \langle f_\alpha, f_\alpha \rangle + \alpha \langle f_\alpha, f_\beta \rangle$$

$$= (\alpha + \beta) \langle f_\alpha, f_\beta \rangle - (\beta \langle f_\beta, f_\beta \rangle + \alpha \langle f_\alpha, f_\alpha \rangle).$$

So we have

$$\frac{\beta\langle f_\beta, f_\beta\rangle + \alpha\langle f_\alpha, f_\alpha\rangle}{\alpha + \beta} \le \langle f_\beta, f_\alpha\rangle .$$

The result

$$\frac{\beta\langle Kf_\beta, Kf_\beta\rangle + \alpha\langle Kf_\alpha, Kf_\alpha\rangle}{\alpha + \beta} \le \langle Kf_\alpha, Kf_\beta\rangle$$

has a similar proof, starting with

$$K^*K(K^*K(f_\alpha - f_\beta)) = K^*K(\beta f_\beta - \alpha f_\alpha)$$

and then taking the inner product of each side with $f_\alpha - f_\beta$.

<u>Theorem 2.4.5</u>. If $0 < \beta < \alpha$, $f_\alpha \in R(K^*)$ is the solution to $(K^*K + \alpha I)f = K^*g$ and $f_\beta \in R(K^*)$ is the solution to $(K^*K + \beta I)f = K^*g$ then $\|Kf_\beta\| \ge \|Kf_\alpha\|$.

Proof: Since we have

$$(K^*K + \alpha I)f_\alpha = (K^*K + \beta I)f_\beta ,$$

take the inner product of each side with f_α yielding the following result.

$$\langle K^*Kf_\alpha + \alpha f_\alpha, f_\alpha\rangle = \langle K^*Kf_\beta + \beta f_\beta, f_\alpha\rangle,$$

$$\langle K^*Kf_\alpha, f_\alpha\rangle + \alpha\langle f_\alpha, f_\alpha\rangle = \langle K^*Kf_\beta, f_\alpha\rangle + \beta\langle f_\beta, f_\alpha\rangle,$$

$$\langle Kf_\alpha, Kf_\alpha\rangle + \alpha\langle f_\alpha, f_\alpha\rangle = \langle Kf_\beta, Kf_\alpha\rangle + \beta\langle f_\beta, f_\alpha\rangle.$$

In a similar way, taking the inner product with f_β yields:

$$\langle Kf_\alpha, Kf_\beta\rangle + \alpha\langle f_\alpha, f_\beta\rangle = \langle Kf_\beta, Kf_\beta\rangle + \beta\langle f_\beta, f_\beta\rangle.$$

Now using the previous lemma we have

$$\begin{aligned}\langle Kf_\alpha, Kf_\alpha\rangle + \alpha\langle f_\alpha, f_\alpha\rangle &= \langle Kf_\beta, Kf_\alpha\rangle + \beta\langle f_\beta, f_\alpha\rangle \\ &\geq \frac{\beta\langle Kf_\beta, Kf_\beta\rangle + \alpha\langle Kf_\alpha, Kf_\alpha\rangle}{\alpha + \beta} + \beta\frac{(\beta\langle f_\beta, f_\beta\rangle + \alpha\langle f_\alpha, f_\alpha\rangle)}{\alpha + \beta}\end{aligned}$$

When we multiply each side of the previous result by $\alpha + \beta$ and simplify the algebraic expression, we obtain

(2.4.4)

$$\beta\langle Kf_\alpha, Kf_\alpha\rangle + \alpha^2\langle f_\alpha, f_\alpha\rangle \geq \beta\langle Kf_\beta, Kf_\beta\rangle + \beta^2\langle f_\beta, f_\beta\rangle.$$

In a similar way from

$$\langle Kf_\alpha, Kf_\beta\rangle + \alpha\langle f_\alpha, f_\beta\rangle = \langle Kf_\beta, Kf_\beta\rangle + \beta\langle f_\beta, f_\beta\rangle$$

we obtain:

(2.4.5)

$$\alpha\langle Kf_\beta, Kf_\beta\rangle + \beta^2\langle f_\beta, f_\beta\rangle \geq \alpha\langle Kf_\alpha, Kf_\alpha\rangle + \alpha^2\langle f_\alpha, f_\alpha\rangle.$$

Now add equations (2.4.4) and (2.4.5), obtaining the inequality:

(2.4.6)
$$\beta\langle Kf_\alpha, Kf_\alpha\rangle + \alpha\langle Kf_\beta, Kf_\beta\rangle \geq \beta\langle Kf_\beta, Kf_\beta\rangle + \alpha\langle Kf_\alpha, Kf_\alpha\rangle.$$

So (2.4.6) yields the result:

$$(\alpha - \beta)\langle Kf_\beta, Kf_\beta\rangle \geq (\alpha - \beta)\langle Kf_\alpha, Kf_\alpha\rangle .$$

Since $0 < \beta < \alpha$ we have

$$\langle Kf_\beta, Kf_\beta\rangle \geq \langle Kf_\alpha, Kf_\alpha\rangle .$$

Thus we have $\|Kf_\beta\| \geq \|Kf_\alpha\|$ and the theorem.

3. IMPLEMENTING THE ALGORITHM

3.1. Calculation of a Starting Element

For the basic algorithm discribed in section 3.2 we require an initial choice for λ, λ_1, and any starting element $f_{1,o} \in H_1$.

In practice we will choose any initial starting element $f \in H_1$ such that $f \neq 0$ and $\|K^*Kf\| \neq 0$. Then we determine an actual starting vector $f_{1,o} \in H_1$ and a initial λ, λ_1, such that $f_{1,o}$ is, in some sense, close to a solution of $K^*Kf + \lambda_1 f = K^*g$.

Now to choose the actual $f_{1,o}$, consider the element

$$f_s = \frac{K^*Kf}{\|K^*Kf\|} \in R(K^*) .$$

The element f_s is chosen normalized so as to control the size of the norm of the actual starting element $f_{1,o}$, which will be either f_s or $-f_s$.

Choose $\lambda_1 \geq 0$ so that

$$\hat{Q}_\lambda(f_s) = \|(K^*K + \lambda I)f_s - K^*g\|^2$$

is minimized. Note that if $\hat{Q}_{\lambda_1}(f_s) = 0$ for some choice for $\lambda_1 > 0$ then we would have obtained an equation

$$(K^*K + \lambda_1 I) f = K^*g$$

which has f_s as its solution.

To minimize $\hat{Q}_\lambda(f_s)$ with respect to λ, we write $\hat{Q}_\lambda(f_s)$ as

$$\hat{Q}_\lambda(f_s) = \|K^*Kf_s - K^*g\|2$$

$$+ 2\lambda\langle f_s, K^*Kf_s - K^*g\rangle + \lambda^2\|f_s\|^2.$$

Taking the derivative $\hat{Q}_\lambda(f_s)$ with respect to λ and setting $\frac{d\hat{Q}_\lambda(f_s)}{d\lambda} = 0$ we get the result

$$\frac{d\hat{Q}_\lambda(f_s)}{d\lambda} = 2\langle f_s, K^*Kf_s - K^*g\rangle + 2\lambda\|f_s\|^2$$

$$= 2\langle f_s, K^*Kf_s - K^*g\rangle + 2\lambda = 0.$$

Note also that $\frac{d^2\hat{Q}(f_s)}{d\lambda^2} = 2\langle f_s, f_s\rangle > 0$, so that the solution to $\frac{d\hat{Q}(f_s)}{d\lambda} = 0$ yields a minimum.

By solving $\frac{d\hat{Q}_\lambda(f_s)}{d\lambda} = 0$ for $\lambda = \bar{\lambda}$, the minimum, we have the result:

(3.1.1)
$$\begin{aligned}\bar{\lambda} &= -\langle f_s, K^*Kf_s - K^*g\rangle \\ &= \langle g - Kf_s, Kf_s\rangle \\ &= \langle Kf_s, g\rangle - \langle Kf_s, Kf_s\rangle .\end{aligned}$$

By substituting $\lambda = \bar{\lambda}$ in $\hat{Q}_\lambda(f_s)$ we obtain

$$\hat{Q}_{\bar{\lambda}}(f_s) = \|K^*Kf_s - K^*g\|^2 - \langle Kf_s, g - Kf_s\rangle^2 .$$

For the choice for λ_1 we need $\lambda_1 \geq 0$, so consider the following three cases:

<u>Case 1</u>. If $\bar{\lambda} = 0$, take $\lambda_1 = \bar{\lambda}$ and we have computed an exact solution, $f_s \in R(K^*) \subset N(K)^\perp$, to $K^*Kf = K^*g$.

Since K^* is one-to-one on the range of K we have that

$$K^*(Kf_s - g_1) = 0,$$

so $Kf_s = g_1$. Thus we have computed the least squares solution of minimum norm, $f_s = f_o$, of $Kf = g$.

<u>Case 2</u>. If $\bar{\lambda} > 0$ then take $\lambda_1 = \bar{\lambda}$ and $f_{1,o} = f_s$ for a starting element.

<u>Case 3</u>. If $\bar{\lambda} < 0$ then replace f_s with $-f_s$ in (3.1.1), since then the new value of $\bar{\lambda}$ is

$$(3.1.2) \qquad \bar{\lambda} = \langle K(-f_s), g - K(-f_s)\rangle$$

$$= \langle - Kf_s, g + Kf_s\rangle$$

$$= - \langle Kf_s, g\rangle - \langle Kf_s, Kf_s\rangle,$$

as compared with equation (3.1.1). So if $\langle Kf, g - Kf\rangle$ is negative for f_s and positive for $-f_s$ take $f_{1,o} = -f_s$ and $\lambda_1 = \bar{\lambda} = - \langle Kf_s, g\rangle - \langle Kf_s, Kf_s\rangle$.

If $\langle Kf_s, g\rangle = 0$ then a new starting element f must be chosen.

Now from here on suppose that $\langle Kf_s, g\rangle \neq 0$.

If $\langle Kf, g - Kf\rangle$ is negative for both f_s and $-f_s$ then from (3.1.1) and (3.1.2) there exists a real number $a > 1$ such that for the scaled equation

$$Kf = ag$$

we have $\bar{\lambda} = \langle Kf, ag - Kf\rangle \geq 0$ for f one of the elements f_s or $-f_s$. In the above computation it was critical that $\langle Kf_s, g\rangle \neq 0$. So we take $\lambda_1 = \bar{\lambda}$ and $f_{1,o}$ to be the element f_s or $-f_s$ which makes $\bar{\lambda} \geq 0$. The new equation $Kf = ag$ is then solved for af_o, the least squares solution of minimum norm of $Kf = ag$, and scaled back to the least squares solution of minimum norm, f_o, of the original equation $Kf = g$ at the end of the algorithm.

In summary, for each $f \neq 0$ where $\|K^*Kf\| \neq 0$ and $f \in H_1$ a starting element $f_{1,o} \in R(K^*)$ and a $\lambda_1 \geq 0$ can be chosen so that $\|(K^*K + \lambda I)f_s - aK^*g\|$ is a minimum (here $a \geq 1$). Thus we start as close as possible (in norm) to a solution of $(K^*K + \lambda I)f - aK^*g = 0$.

Once λ_1 is chosen, a sequence $\{\lambda_j\}_{j=1}^{\infty}$ can be constructed such that $\lambda_j > 0$, $\lambda_j \to 0$ as $j \to \infty$, by taking $\lambda_{j+1} = x\lambda_j$ where $0 < x < 1$. In practice the sequence $\{\lambda_j\}_{j=1}^{\infty}$ is terminated when λ_j is less than some pre-assigned value μ. For the first λ_j such that $\lambda_j \leq \mu$ take $\lambda_j = \mu$.

x is determined by numerical experimentation.

3.2. The Sequence $\{f_{j,n}\}_{n=o}^{\infty}$

The sequence $\{f_{j,n}\}_{n=o}^{\infty}$, for j fixed, is constructed by taking

$$f_{j,n+1} = f_{j,n} + \alpha_{j,n} W_{j,n}$$

where $\alpha_{j,n}$ and $W_{j,n}$ are given by (2.2.8). The construction of the sequence $\{f_{j,n}\}_{n=o}^{\infty}$ is terminated when a measure of convergence fails, due to rounding errors in computation, to give an approximate solution f_j to the equation $(K^*K + \lambda_j I)f = K^*g$ (possibly scaled). The measure of convergence used is to note that from (2.2.4),

$$Q_{\lambda_j}(f_{j,n}) - Q_{\lambda_j}(f_{j,n+1}) = \frac{\langle W_{j,n}, W_{j,n}\rangle^2}{\langle KW_{j,n}, KW_{j,n}\rangle + \lambda_j \langle W_{j,n}, W_{j,n}\rangle}$$

for $W_{j,n} \neq 0$, i.e. $Q_{\lambda_1}(f_{j,n+1}) < Q_{\lambda_j}(f_{j,n})$, an approximation to f_j is chosen to be the least value of n for which $Q_{\lambda_j}(f_{j,n+1}) < Q(f_{j,n})$ fails in the computations, or $\|W_{j,n}\|$ is less than some pre-assigned value; which ever occurs first.

We use $\|W_{j,n}\|$ as a measure of $\|f_{j,n} - f_o\|$, since

$$\|f_{j,n} - f_o\| \leq \frac{\|W_{j,n}\|}{\|(K^*K + \lambda I)^{-1}\|^{-1}} + \|f_j - f_o\|$$

(see 2.4.1). Ideally we first choose λ so that $\|f_j - f_o\|$ is small and then make $\|W_{j,n}\|$ small. In practice it is difficult to make $\|f_j - f_o\|$ small with out some knowledge of the solution.

3.3. A Flow Diagram of the Basic Algorithm

The following diagram, given in Figure 3.3.1, is a basic description of the algorithm to find the least squares solution of minimum norm of the operator equation $Kf = g$. The flow diagram describes the program in Appendix A for matrix equations and the subroutine BITER for discretized integral equations in Appendix B.

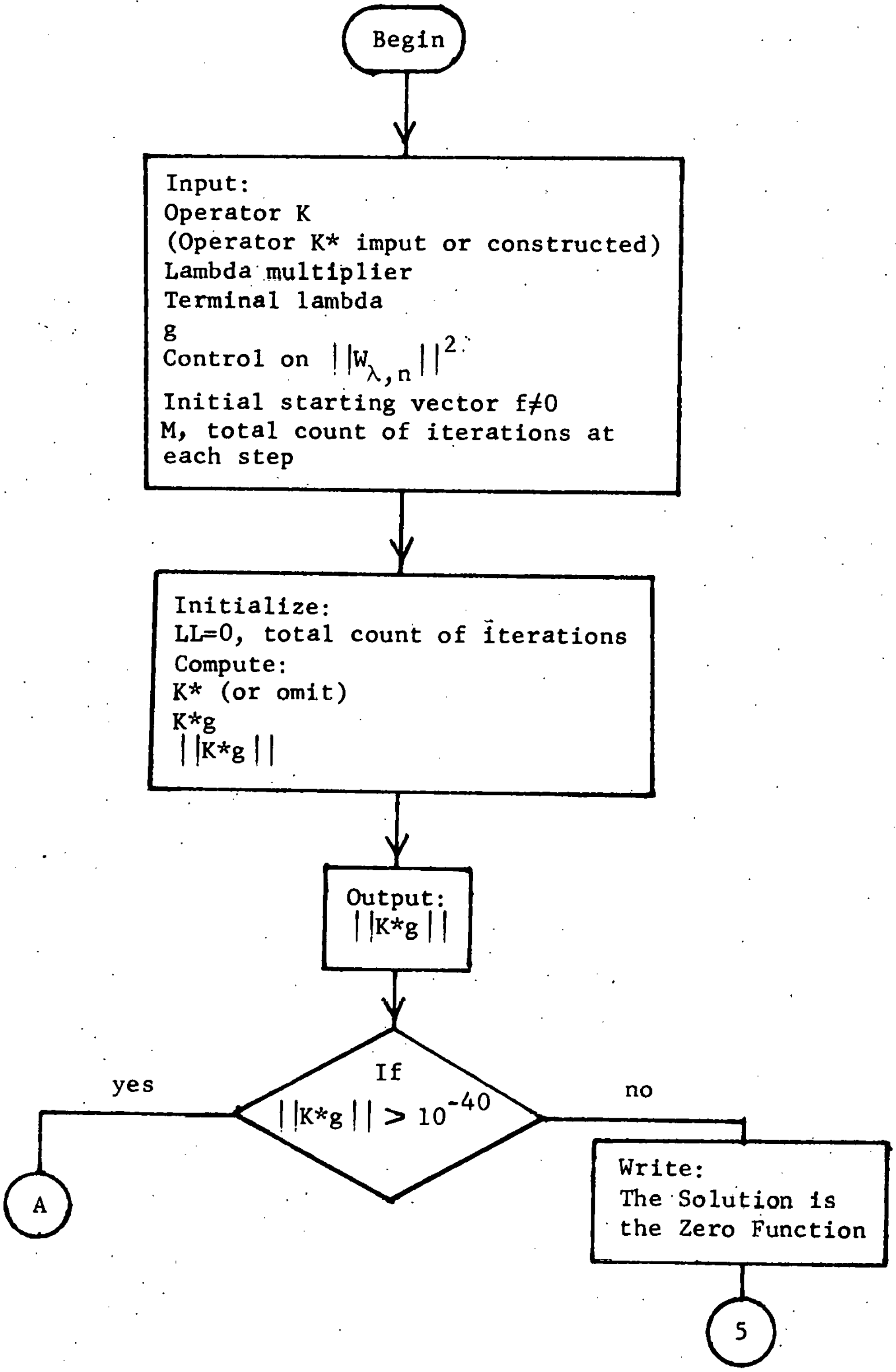

Figure 3.3.1. Flow Chart of the Basic Algorithm

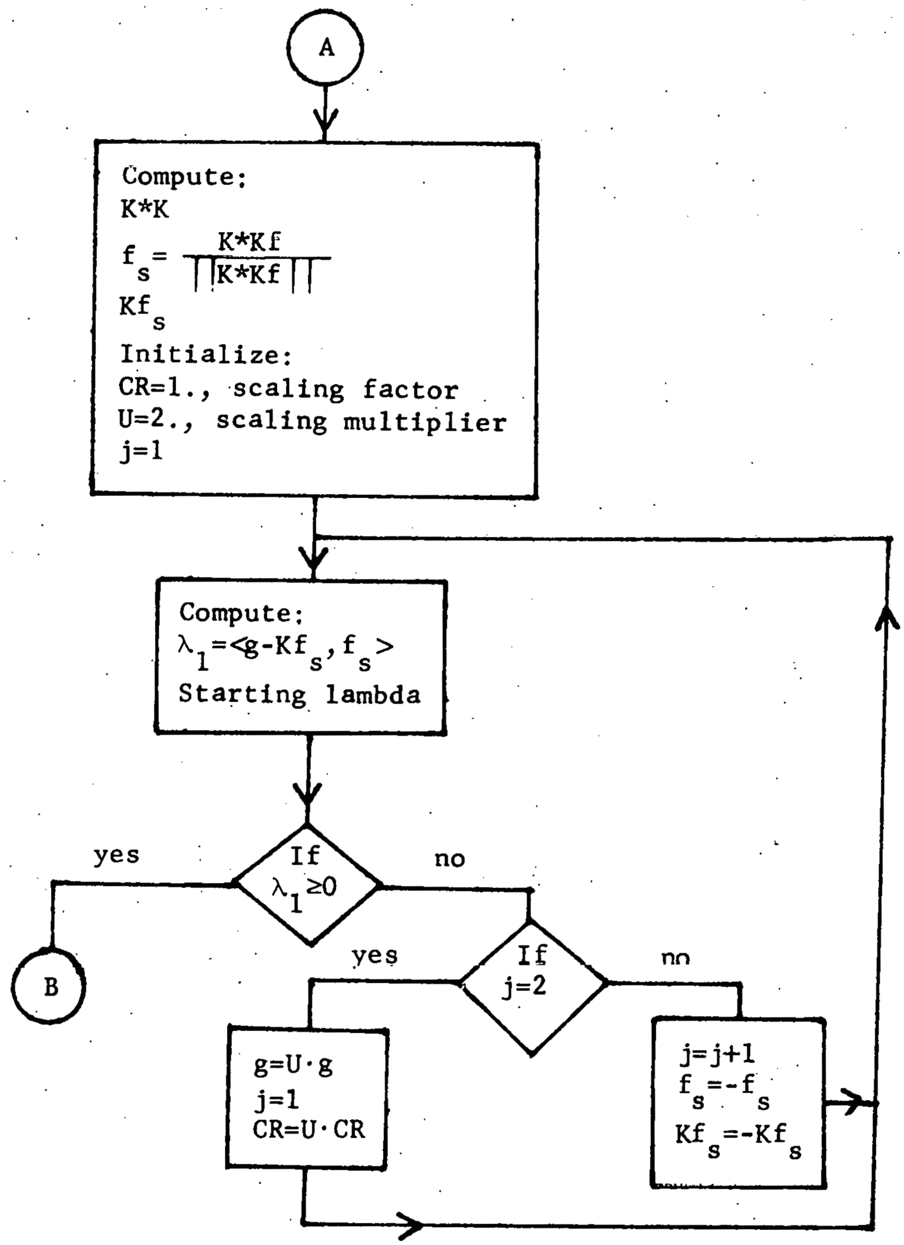

Figure 3.3.1. (Continued)

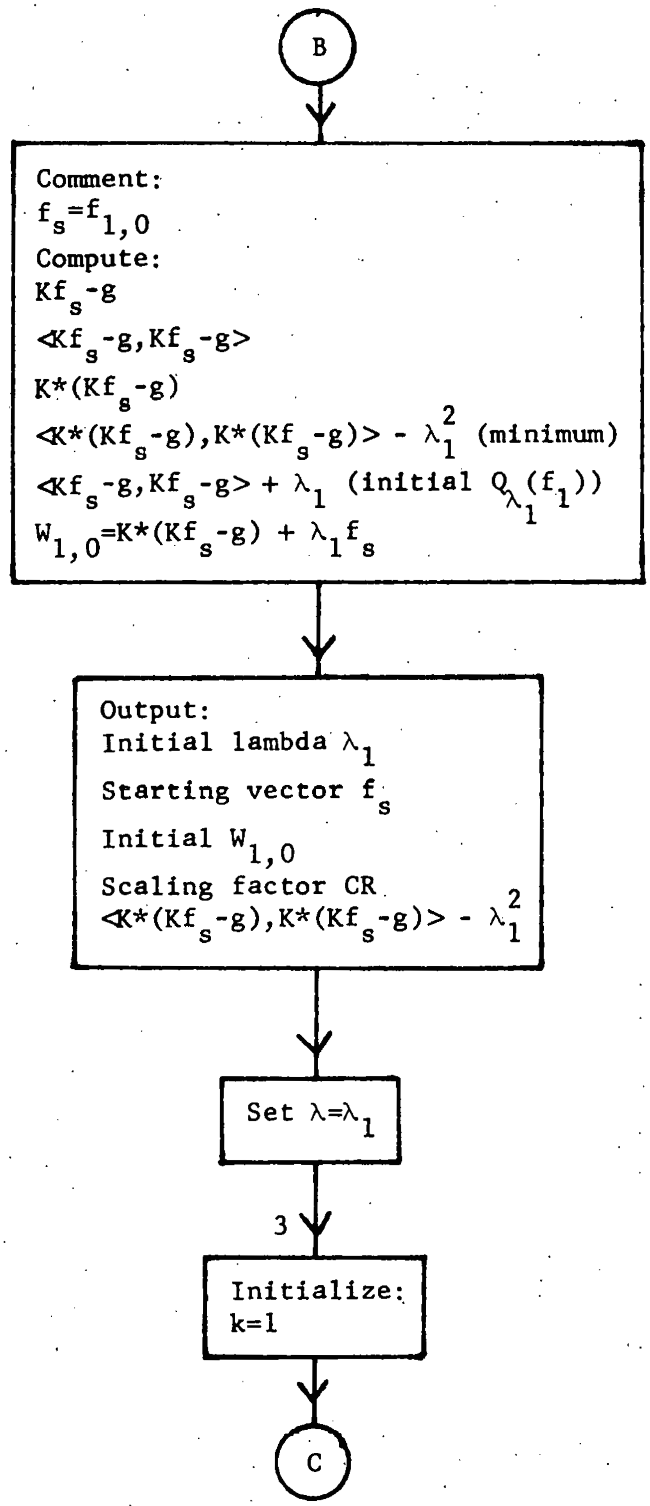

Figure 3.3.1. (Continued)

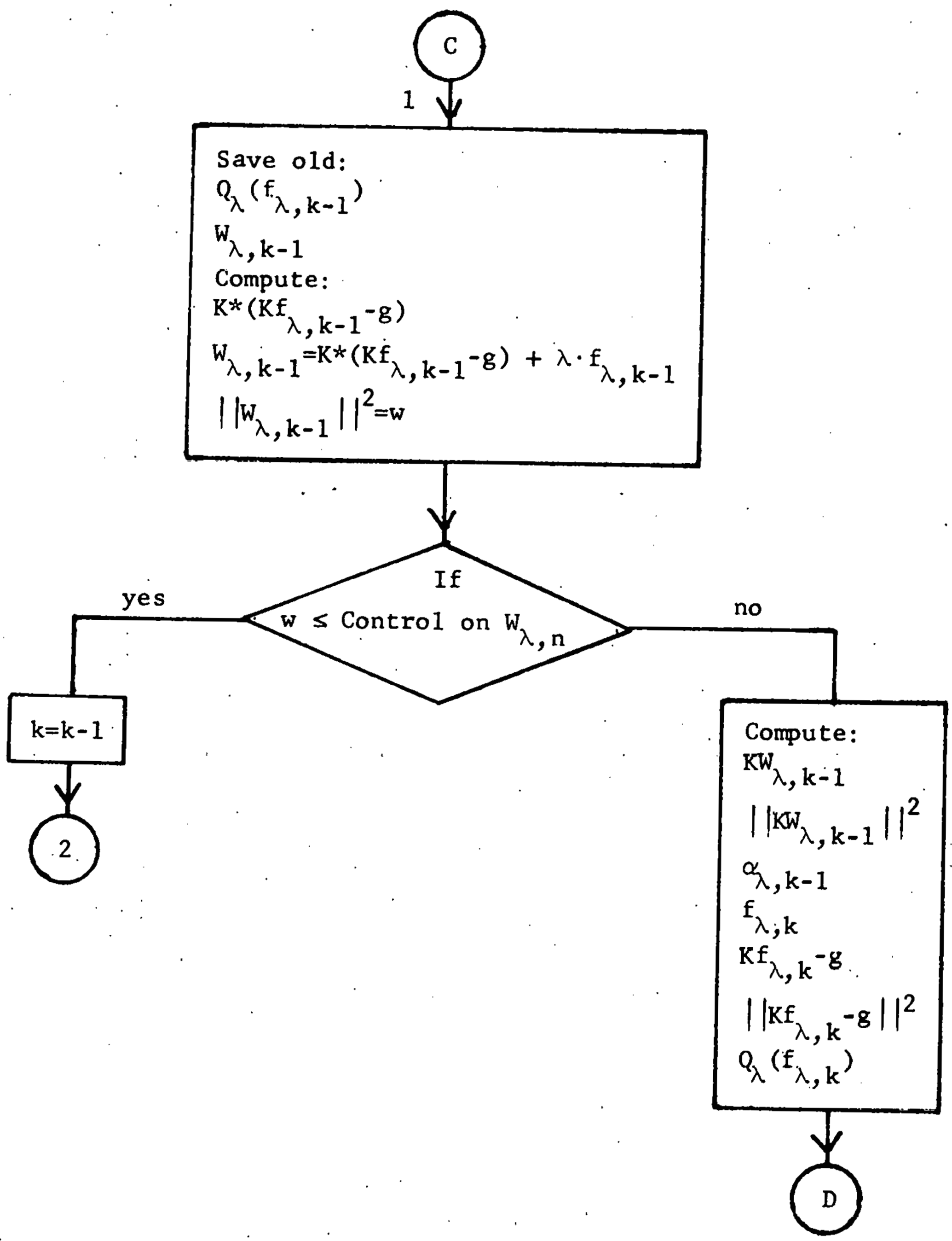

Figure 3.3.1. (Continued)

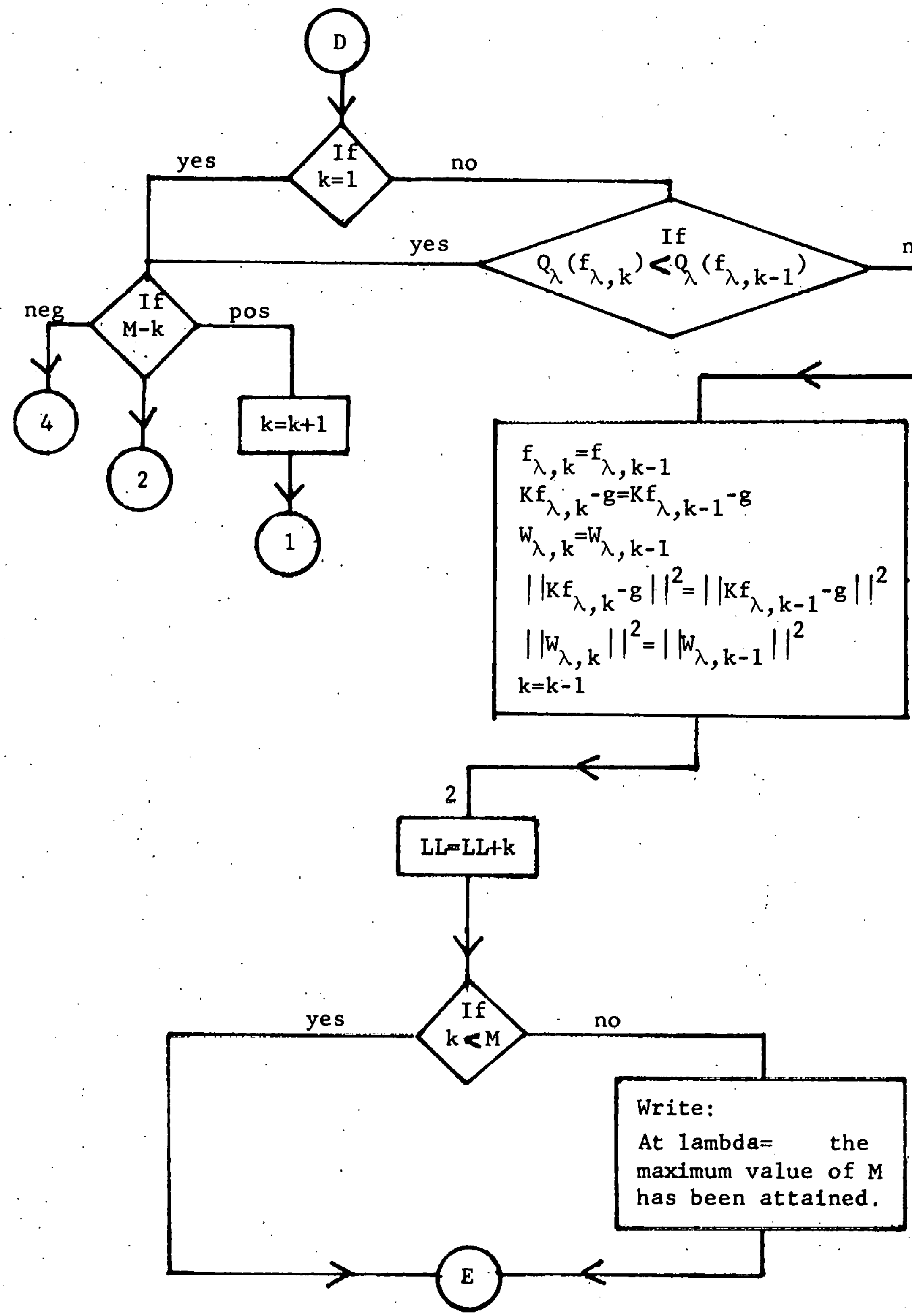

Figure 3.3.1. (Continued)

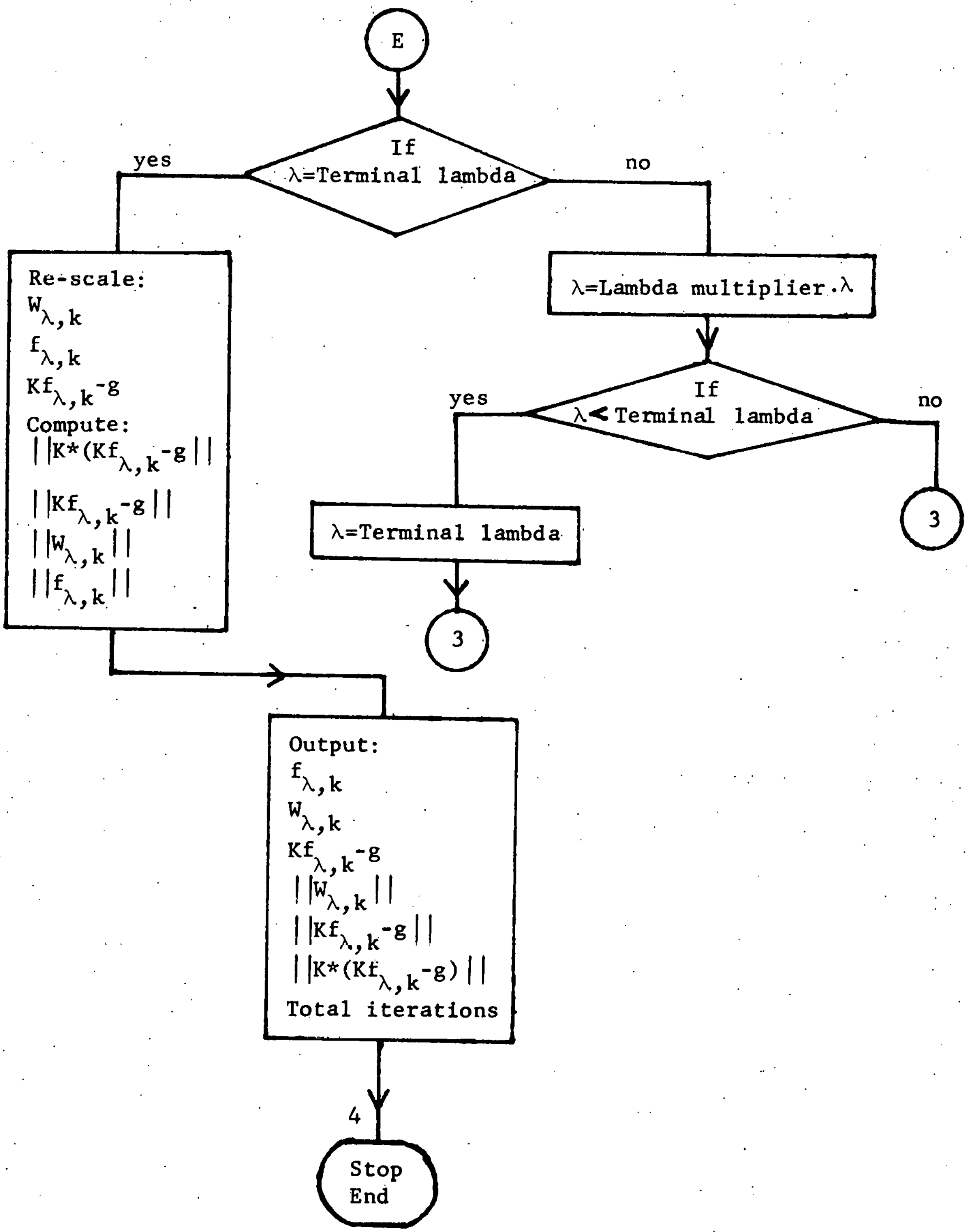

Figure 3.3.1. (Continued)

4. EXAMPLES

4.1. Matrix Examples

Let the real Hilbert spaces $H_1 = E^n$, $H_2 = E^m$ where $E^n = \{(x_1,\ldots,x_n)^T : x_i \text{ real}\}$ with the inner product

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i, \quad x,y \in E^n$$

be given. In this finite dimensional case each linear transformation $A : H_1 \to H_2$ has a unique matrix representation

$$A = (a_{i,j})_{\substack{i=1,m\\ j=1,n}}$$

with respect to each bases $\{v_1,v_2,\ldots,v_n\}$ in H_1 and $\{w_1,w_2,\ldots,w_m\}$ in H_2. We shall always take our bases vectors to be the canonical bases. Since H_1 and H_2 are finite dimensional, all linear transformations are bounded and compact. Since A is of finite rank we have $R(A) = \overline{R(A)}$, so the algorithm (see section 2.2) will converge to the least squares solution of minimum norm, x_o, of $Ax = g$ for all $g \in H_2$.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 3 & 3 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 \end{bmatrix} .$$

Then A has rank three.

Consider $Ax = g$ for the following two cases.

<u>Case 1</u>. $g_1 = (10,12,13,22,43,35)^T \in R(A)$.

<u>Case 2</u>.

$$g_2 = g_1 + (-5,-2,-2,1,1,1)^T$$

$$= (5,10,11,23,24,36)^T$$

where $(-5,-2,-2,1,1,1)^T \in N(A)$.

In both Case 1 and Case 2 the least squares solution of minimum norm, x_o, of $Ax = g_i$, $i = 1,2$ is

$$\vec{x}_o = (17/6, 43/12, 43/12, 29/6, 49/12, 49/12) \in N(A)^{\perp}.$$

$\vec{x}_o$ was constructed by finding a basis

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

for $N(A)$. Then a basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{is found for } N(A)^{\perp}.$$

Since $\vec{x}_o$ is contained in $N(A)^{\perp}$, $\vec{x}_o$ must be a linear combination of the above three vectors. Therefore to solve $Ax = g_i$ we need only solve

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ -x_1 + x_2 + x_3 \\ x_3 \\ x_3 \end{bmatrix} = g_i, \quad i = 1,2 \quad .$$

The algorithm in Appendix A was used to find the approximate least squares solution of minimum norm for Case 1 and Case 2 with the results given in Table 1. Comparisons with the actual solution were computed separately.

For Run 1 Case 1 we will satisfy the error bounds of Statement (2.4.1) so that $\|f_{j,n} - f_o\| < 10^{-6}$. We will assume that we know the eigenvalues of A^TA, which were computed to be 56.96,4.72,2.32,0,0,0. Also we will assume that we have an approximation for $\|f_o\|$, the norm of the least squares solution of minimum norm, say 9.5. Now from (2.4.1) we have

$$\|f_{j,n} - f_o\| \leq \frac{\|w_{\lambda_j,n}\|}{\lambda_j + 2.32} + \frac{9.5\lambda_j}{\lambda_j + 2.32}$$

We will choose terminal lamda, $\mu = \lambda_j$ and $\|W_{j,n}\|$ so that

$$\frac{9.5\mu}{\mu + 2.32} < \frac{10^{-6}}{2}$$

and

$$\frac{\|W_{j,n}\|}{\mu + 2.32} < \frac{10^{-6}}{2} \quad .$$

We actually used $\mu = 1. \times 10^{-7}$ and made $\|W_{j,n}\|^2 \leq 10^{-12}$.

The results of several runs of Case 1 and Case 2 are given in Table 1. For each run we used the initial starting vector $F = (1.,1.,1.,0,0,0)^T$. We computed $\|A^T g_i\| \simeq 470.14$.

Table 1. Matrix Example

	Run 1	Run 2	Run 3	Run 4
g	g_1	g_1	g_1	g_2
M	400	400	400	400
Tλ	1.(-7)	1.(-16)	1.(-16)	1.(-16)
W1	1.(-12)	1.(-10)	1.(-20)	1.(-10)
λm	1.(-5)	1.(-5)	1.(-5)	1.(-5)
Iλ	411.68	411.68	411.68	411.68
EN	4.2(-7)	3.3(-6)	3.3(-11)	3.3(-6)
E1	1.(-6)	4.4(-6)	4.4(-11)	4.4(-6)
E2	1.(-6)	4.4(-6)	4.4(-11)	4.4(-6)
E3	2.3(-7)	1.9(-6)	1.8(-11)	1.9(-6)
W2	8.8(-7)	8.6(-6)	8.7(-11)	8.6(-6)
Re	6.5(-7)	5.(-6)	5.1(-11)	6. + 2.(-12)
TI	132	104	293	104
Ex	3.19	2.60	6.93	1.66

M = maximum number of allowable iterations for a given step.

Tλ = terminal Lambda.

W1 = terminal choice for $\|W_{j,n}\|^2$.

λm = Lambda multiplier.

Iλ = computed initial λ_1, approximately.

EN ≃ computed $\|f_{T\lambda,n} - f_o\|$.

EI = error bound computed from (2.4.1).

E2 = error bound computed from (2.4.2).

E3 = computed $\max|f_{T\lambda,n} - f_o|$.

W2 = computed $\|(A^TA + \lambda I)f_{T\lambda,n} - A^Tg\|$.

Re = computed $\|(Af_{T\lambda,n} - g)\|$.

TI = total iterations.

Ex = execution time, seconds, WATFIVE.

4.2. An Inner Product Space

We will consider the real n-dimensional space R^n with scalar field the real numbers. Define a mapping from $R^n \times R^n$ to R by

$$(u,v) \to \langle u,v \rangle = \sum_{j=1}^{n} T_j u_j v_j$$

where $u = (u_1,u_2,\ldots,u_n)^T$, $v = (v_1,v_2,\ldots,v_n)^T$ and $T_j > 0$, $T_j \in R$ for all $j = 1,\ldots,n$. Thus we have a Hilbert space, which will be denoted by H_T^n. Let $\{e_1^n,e_2^n,\ldots,e_n^n\}$ be the orthogonal (not necessarily normalized) bases for H_T^n where $e_j^n = (0,0,\ldots,1_j,\ldots,0)^T$, $j = 1,n$.

If $A : H_T^m \to H_S^n$ then $A^* : H_S^n \to H_T^m$ will denote the adjoint.

<u>Theorem 4.2.1</u>. If $A : H_T^m \to H_S^n$ is $(T_j a_{i,j})_{\substack{i=1,n \\ j=1,m}}$ then $A^* = (S_i a_{i,j})_{\substack{j=1,m \\ i=1,n}}$.

Proof: We will show that the above A^* is the adjoint of A.

$$\langle Ae_k^m, e_\ell^n \rangle = \langle \sum_{i=1}^{n} T_k a_{i,k} e_i^n, e_\ell^n \rangle$$

$$= T_k a_{\ell,k} \langle e_\ell^n, e_\ell^n \rangle$$

$$= T_k S_\ell a_{\ell,k}$$

and

$$\langle e_k^m, A^* e_\ell^n \rangle = \langle e_k^m, \sum_{j=1}^{m} S_\ell a_{\ell,j} e_j^m \rangle$$

$$= S_\ell a_{\ell,k} \langle e_k^m, e_k^m \rangle$$

$$= S_\ell T_k a_{\ell,k}$$

$$= \langle Ae_k^m, e_\ell^n \rangle$$

for all k and ℓ, therefore A^* is the adjoint of A.

4.3. Discretizing Integral Equations

We will obtain a discretized form of Fredholm integral equations of the first kind,

(4.3.1)
$$Kf(y) = \int_0^1 k(y,x)f(x)dx = g(y), \quad 0 \leq y \leq 1$$

where $g \in R(K) + N(K^*) \subset L_2[0,1]$, $f \in L_2[0,1]$ and $k(y,x)$ is square integrable with respect to the product Lebesque measure on $[0,1] \times [0,1]$.

Akhiezer and Glazman [1] prove that if $\int_0^1\int_0^1 |k(y,x)|^2 dy\,dx < \infty$ then K is compact.

For $Kf(s) = \int_0^1 k(s,x)f(x)dx$ Bachman and Narici [3, page 403] note that the adjoint to K is K^*, given by

$$K^*f(s) = \int_0^1 k(x,s)f(x)dx.$$

Of particular interest is the equation

(4.3.2)
$$\int_0^1 k(y,s)\int_0^1 k(y,x)f(x)\,dx\,dy + \lambda f(s) = \int_0^1 k(y,s)g(y)\,dy$$

for $\lambda > 0$.

For numerical purposes, we will restrict the discussion to the functions $k(y,x)$, g and f_λ where we have Riemann integration.

The methods of discretization of the integrals will initially parallel that found in Anselone [2, page 13] and Isaacson and Keller [9].

We have that

(4.3.3)
$$\int_0^1 k(y,s)g(y)\,dy = \sum_{j=1}^{m} S_j k(y_j,s)g(y_j) + R_m(s)$$

where $R_m(s)$ is the discretization error. Here $S_j > 0$ $j = 1,m$, $\sum_{j=1}^{m} S_j = 1$ are the quadrature weights for $0 = y_1 < y_2 < \ldots < y_m = 1$, a partition of the interval $[0,1]$, chosen equally spaced for the Newton-Cotes closed integration formulae.

For $\int_0^1 k(y,s)\int_0^1 k(y,x)f(x)\,dx\,dy$ we have

$$\text{(4.3.4)}\quad \int_0^1 k(y,s)\int_0^1 k(y,x)f(x)\,dx\,dy = \sum_{j=1}^{m} S_j k(y_j,s)\sum_{i=1}^{n} T_i k(y_j,x_i)f(x_i) + \bar{R}(s)$$

where $\bar{R}(s)$ is the discretization error. Here $T_i > 0$ for $i = 1,n$, $\sum_{i=1}^{n} T_i = 1$ are the quadrature weights for $0 = x_1 < x_2 < \ldots < x_n = 1$, a partition of the interval $[0,1]$ chosen equally spaced for the Newton-Cotes closed integration formulae. The partition and weights for the y_j's are chosen as for (4.3.3).

Thus from (4.3.3) and (4.3.4), (4.3.2) becomes

$$\text{(4.3.5)}\quad \sum_{j=1}^{m} S_j k(y_j,s)\sum_{i=1}^{n} T_i k(y_j,x_i)f(x_i) + \lambda f(s) + \bar{R}(s) = \sum_{j=1}^{m} S_j k(y_j,s)g(y_j) + R_m(s).$$

Now, (4.3.5) can be written in the equivalent form

$$(S_j k(y_j,s))^T_{j=1,m} (T_i k(y_j,x_i))_{\substack{j=1,m \\ i=1,n}} (f(x_i))_{i=1,n} + \lambda f(s) + \bar{R}(s)$$

$$= \sum_{j=1}^{m} S_j k(s,y_j) g(y_j) + R_m(s).$$ Where $(v_i)_{i=1,\ell}$ denotes an ℓ-dimensional column vector.

Now partition the interval $[0,1]$ for s such that $s_i = x_i$ for $i = 1,n$. Thus we obtain the linear system

(4.3.6)

$$(S_j k(y_j,x_i))_{\substack{i=1,n \\ j=1,m}} \cdot (T_i k(y_j,x_i))_{\substack{j=1,m \\ i=1,n}} (f(x_i))_{i=1,n}$$

$$+ \lambda (f(x_i))_{i=1,n} = (S_j k(y_j,x_i))_{\substack{i=1,n \\ j=1,m}} (g(y_j))_{j=1,m}$$

$$+ (R_m(x_i) - \bar{R}(x_i))_{i=1,n}.$$

Let

$$K = (T_i k(y_j, x_i))_{\substack{j=1,m \\ i=1,n}}, \quad K : H_T^n \to H_S^m ;$$

$$K^* = (S_j k(y_j, x_i))_{\substack{i=1,n \\ j=1,m}}, \quad K^* : H_S^m \to H_T^n ;$$

$$\vec{f} = (f(x_i))_{i=1,n}, \quad \vec{f} \in H_T^n ;$$

$$\vec{g} = (g(x_j))_{j=1,m}, \quad \vec{g} \in H_S^m$$

and

$$\vec{R}_1 = (R_n(x_i) - \bar{R}(x_i))_{i = 1,n}, \quad \vec{R} \in H_T^n .$$

Thus (4.3.6) can be written in the form

(4.3.7) $$K^* K \vec{f} + \lambda \vec{f} = K^* \vec{g} + \vec{R}_1 .$$

Similar to (4.3.5) we have the integral equation

(4.3.8)
$$\int_0^1 k(y,s) \int_0^1 k(y,x) f(x)\, dx\, dy = \int_0^1 k(y,s) g(y)\, dy$$

and its discretized form

(4.3.9) $$K^*K\vec{f} = K^*\vec{g} + \vec{R}_1,$$

where $\vec{R}_1$ is the discretization error vector. To solve equation (4.3.8) numerically we consider the system

(4.3.10) $$K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g}$$

for $\lambda > 0$ and small.

4.4. Error Bounds

Throughout this section we will assume that $\gamma = \inf_{\gamma_n \in \sigma_o(K^*K)} \gamma_n$. For $\lambda > 0$, first we will develop relationships between the solutions of the following equations

$$K^*K\vec{f}_\lambda + \lambda\vec{f}_\lambda = K^*\vec{g} + \vec{R}_\lambda,$$

$$K^*K\vec{f}_o = K^*\vec{g} + \vec{R}_o$$

and $$K^*K\vec{\vec{f}}_\lambda + \lambda\vec{\vec{f}}_\lambda = K^*\vec{g}$$

where $\vec{R}_\lambda$, $\vec{R}_o$ are the discretization error vector. $\vec{f}_\lambda$, $\vec{f}_o$ and $\vec{\vec{f}}_\lambda$ are the respective solutions to the above equations. These relationships will depend on an estimate for $\|\vec{R}_\lambda\|$, $\|\vec{R}_o\|$ and γ.

Assuming the above notation we have the following two theorems.

Theorem 4.4.1.

$$\|\vec{f}_\lambda - \vec{\vec{f}}_\lambda\| \leq \frac{\|\vec{R}_\lambda\|}{\underline{\mu} + \lambda} \tag{4.4.1}$$

where $\underline{\mu} = \inf\limits_{\gamma_n \in \sigma(K^*K)} \gamma_n$.

Proof: Since $(K^*K + \lambda I)\vec{f}_\lambda = K^*\vec{g} + \vec{R}_\lambda$ and $(K^*K + \lambda I)\vec{\vec{f}}_\lambda = K^*\vec{g}$ we have $(K^*K + \lambda I)(\vec{f}_\lambda - \vec{\vec{f}}_\lambda) = \vec{R}_\lambda$. Thus

$$\begin{aligned}(\underline{\mu} + \lambda)\|\vec{f}_\lambda - \vec{\vec{f}}_\lambda\| &= \|(K^*K + \lambda I)^{-1}\|^{-1}\|\vec{f}_\lambda - \vec{\vec{f}}_\lambda\| \\ &\leq \|(K^*K + \lambda I)(\vec{f}_\lambda - \vec{\vec{f}}_\lambda)\| \\ &= \|\vec{R}_\lambda\| .\end{aligned}$$

Thus we have the theorem.

Remark: Since $K^*K\vec{f}_o = K^*\vec{g} + \vec{R}_o$ we have $K^*(K\vec{f}_o - \vec{g}) = \vec{R}_o$. Thus $\vec{R}_o \in R(K^*) \subset N(K)^\perp$. Therefore $\vec{f}_o \in N(K^*K)^\perp$.

Theorem 4.4.2.

(4.4.2) $$\|K^*K(\vec{f}_o - \vec{\vec{f}}_\lambda)\| \leq \|\vec{R}_o\| + \lambda\|\vec{\vec{f}}_\lambda\|,$$

(4.4.3) $$\|\vec{f}_o - \vec{\vec{f}}_\lambda\| \leq \frac{\|\vec{R}_o\| + \lambda\|\vec{\vec{f}}_\lambda\|}{\gamma} \quad \text{for} \quad \gamma \neq 0,$$

(4.4.4) $$\|\vec{f}_o - \vec{\vec{f}}_\lambda\| \leq \frac{\|\vec{R}_o\| + \lambda\|\vec{f}_o\|}{\lambda + \gamma}$$

and

(4.4.5) $$\|\vec{f}_o - \vec{\vec{f}}_\lambda\| \leq \frac{(\lambda + \gamma)\|\vec{R}_o\| + \lambda\|K^*\vec{g}\|}{\gamma(\lambda + \gamma)} \quad \text{for} \quad \gamma \neq 0.$$

Proof: For (4.4.2): Since $K^*K\vec{\vec{f}}_\lambda + \lambda\vec{\vec{f}}_\lambda = K^*\vec{g}$ and $K^*Kf_o = K^*g + \vec{R}_o$ we have $K^*K\vec{\vec{f}}_\lambda + \lambda\vec{\vec{f}}_\lambda = -\vec{R}_o + K^*K\vec{f}_o$. Thus $K^*K(\vec{f}_o - \vec{\vec{f}}_\lambda) = \vec{R}_o + \lambda\vec{\vec{f}}_\lambda$. Taking the norm of each side and using the triangular inequality we have

$$\|K^*K(\vec{f}_o - \vec{\bar{f}}_\lambda)\| \leq \|\vec{R}_o\| + \lambda\|\vec{\bar{f}}_\lambda\|.$$

<u>For (4.4.3)</u>: Since $\gamma\|\vec{f}_o - \vec{\bar{f}}_\lambda\| \leq \|K^*K(\vec{f}_o - \vec{\bar{f}}_\lambda)\|$, the result follows from (4.4.2).

<u>For (4.4.4)</u>: Since $K^*K(\vec{f}_o - \vec{\bar{f}}_\lambda) = \vec{R}_o + \lambda\vec{\bar{f}}_\lambda$ we have $(K^*K + \lambda I)(\vec{f}_o - \vec{\bar{f}}_\lambda) = \vec{R}_o + \lambda\vec{f}_o$. Taking norms of each side of the above and using the triangular inequality yields $\|(K^*K + \lambda I)(\vec{f}_o - \vec{\bar{f}}_\lambda)\| \leq \|\vec{R}_o\| + \lambda\|\vec{f}_o\|$. Since $(\lambda + \gamma)\|\vec{f}_o - \vec{\bar{f}}_\lambda\| \leq \|(K^*K + \lambda I)(\vec{f}_o - \vec{\bar{f}}_\lambda)\|$ (4.4.4) follows.

<u>For (4.4.5)</u>: Since $\|(K^*K + \lambda I)\vec{\bar{f}}_\lambda\| = \|K^*g\|$ and $(\lambda + \gamma)\|\vec{\bar{f}}_\lambda\| \leq \|(K^*K + \lambda I)\vec{\bar{f}}_\lambda\|$ we have $\|\vec{\bar{f}}_\lambda\| \leq \dfrac{\|K^*\vec{g}\|}{\lambda + \gamma}$. Thus from (4.4.3) the result follows.

Now take $\vec{\bar{f}}_{j,n}$ as a term in the sequence (2.2.8) converging to the least squares solution of minimum norm, $\vec{f}_o$, of $K^*K\vec{f} = K^*\vec{g} + \vec{R}_o$. The following error bounds on $\|\vec{f}_o - \vec{\bar{f}}_{j,n}\|$ can be obtained.

Theorem 4.4.3.

(4.4.6)
$$\|\vec{f}_o - \vec{\vec{f}}_{j,n}\| \le \frac{\|\vec{R}_o\| + \|K^*K\vec{\vec{f}}_{j,n} - K^*\vec{g}\|}{\gamma} \quad \text{if} \quad \gamma \neq 0,$$

(4.4.7)
$$\|\vec{f}_o - \vec{\vec{f}}_{j,n}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_o\| + \lambda\|\vec{\vec{f}}_j\|}{\gamma} \quad \text{if} \quad \gamma \neq 0,$$

(4.4.8)
$$\|\vec{f}_o - \vec{\vec{f}}_{j,n}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_o\| + \lambda_j\|\vec{f}_o\|}{\lambda_j + \gamma},$$

and

(4.4.9)
$$\|\vec{f}_o - \vec{\vec{f}}_{j,n}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{(\lambda_j + \gamma)\|\vec{R}_o\| + \lambda_j\|K^*\vec{g}\|}{\gamma(\lambda_j + \gamma)} \quad \text{if} \quad \gamma \neq 0.$$

Proof: For (4.4.6): Since $K^*K\vec{f}_o = K^*\vec{g} + \vec{R}_o$, we have $K^*K(\vec{f}_o - \vec{\vec{f}}_{j,n}) = K^*\vec{g} - K^*K\vec{\vec{f}}_{j,n} + \vec{R}_o$. Take the norm of each side of the above result and use the triangle inequality, yielding $\|K^*K(\vec{f}_o - \vec{\vec{f}}_{j,n})\| \le \|\vec{R}_o\| + \|K^*\vec{q} - K^*K\vec{\vec{f}}_{j,n}\|$. The result (4.4.6) follows from $\gamma\|\vec{f}_o - \vec{\vec{f}}_{j,n}\| \le \|K^*K(\vec{f}_o - \vec{\vec{f}}_{j,n})\|$.

<u>For (4.4.7), (4.4.8), (4.4.9)</u>: These bounds on $\|\vec{f}_o - \vec{\bar{f}}_{j,n}\|$ follow from the triangle inequality $\|\vec{f}_o - \vec{\bar{f}}_{j,n}\| \le \|\vec{\bar{f}}_{j,n} - \vec{\bar{f}}_j\| + \|\vec{\bar{f}}_j - \vec{f}_o\|$, Theorem 4.4.2 and $(\lambda_j + \gamma)\|\vec{\bar{f}}_{j,n} - \vec{\bar{f}}_j\| \le \|(K^*K + \lambda_j I)(\vec{\bar{f}}_{j,n} - \vec{\bar{f}}_j)\| = \|W_{j,n}\|$.

4.5. <u>Perturbations of g</u>

In the case where g is not known exactly, we have the integral equation

$$(4.5.1) \qquad \int_0^1 k(y,x)f(x)dx = g(y) + \epsilon(y),$$

where $\epsilon(y) \in L_2[0,1]$. (4.4.10) may or may not have a solution (see 2.3). We follow the procedure and notation of section 4.3 and obtain the matrix representation for (4.5.1). Thus

$$(4.5.2) \qquad K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g} + K^*\vec{\epsilon} + \vec{R}_2$$

where $\vec{R}_2$ is the discretization error vector. The system of equations actually solved is

(4.5.3) $$K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g} + K^*\vec{\epsilon}.$$

Let $\vec{\bar{f}}_{\lambda,\epsilon}$ be the solution to (4.5.3).

$\|\vec{\bar{f}}_\lambda - \vec{\bar{f}}_{\lambda,\epsilon}\| \leq \frac{\|K\|\|\vec{\epsilon}\|}{\gamma + \lambda}$ from Theorem 2.3.1 and statement (4.3.10).

Let $\vec{\bar{f}}_{j,n,\epsilon}$ be a term in the sequence defined by (2.2.8). If a bound for $\|K^*\vec{\epsilon}\|$ or $\|\vec{\epsilon}\|$ can be found, we can obtain bounds on $\|\vec{f}_o - \vec{\bar{f}}_{j,\epsilon}\|$ and $\|\vec{f}_o - \vec{\bar{f}}_{j,n,\epsilon}\|$. Theorems 4.4.2 and 4.4.3 can be extended, with similar proof, to the following theorems.

Theorem 4.5.1.

(4.5.3) $$\|K^*K(\vec{f}_o - \vec{\bar{f}}_{\lambda,\epsilon})\| \leq \|\vec{R}_o\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{\bar{f}}_{\lambda,\epsilon}\|$$

(4.5.4)
$$\|\vec{f}_o - \vec{\bar{f}}_{\lambda,\epsilon}\| \leq \frac{\|\vec{R}_o\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{\bar{f}}_{\lambda,\epsilon}\|}{\gamma} \quad \text{if} \quad \gamma \neq 0$$

(4.5.6) $$\|\vec{f}_o - \vec{\bar{f}}_{\lambda,\epsilon}\| \leq \frac{\|\vec{R}_o\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{f}_o\|}{\lambda + \gamma}$$

(4.5.7)

$$\|\vec{f}_o - \vec{\bar{f}}_{\lambda,\epsilon}\| \le \frac{(\lambda+\gamma)\|\vec{R}_o\| + \lambda\|K^*(\vec{g}+\vec{\epsilon})\|}{\gamma(\lambda+\gamma)} \quad \text{if} \quad \gamma \neq 0.$$

<u>Theorem 4.5.2</u>.

(4.5.8)

$$\|\vec{f}_o - \vec{\bar{f}}_{j,n,\epsilon}\| \le \frac{\|\vec{R}_o\| + \|K^*K\vec{\bar{f}}_{j,n,\epsilon} - K^*(\vec{g}+\vec{\epsilon})\|}{\gamma} \quad \text{if} \quad \gamma \neq 0,$$

(4.5.9)

$$\|\vec{f}_o - \vec{\bar{f}}_{j,n,\epsilon}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_o\| + \|K^*\vec{\epsilon}\| + \lambda_j\|\vec{\bar{f}}_{j,\epsilon}\|}{\gamma} \quad \text{if} \quad \gamma \neq 0,$$

(4.5.10)

$$\|\vec{f}_o - \vec{\bar{f}}_{j,n,\epsilon}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_o\| + \|K^*\vec{\epsilon}\| + \lambda_j\|\vec{\bar{f}}_o\|}{\lambda_j + \gamma},$$

(4.5.11)

$$\|\vec{f}_o - \vec{\bar{f}}_{j,n,\epsilon}\| \le \frac{\|W_{j,n}\|}{\lambda_j + \gamma} + \frac{(\lambda_j+\gamma)\|\vec{R}_o\| + \lambda_j\|K^*(\vec{g}+\vec{\epsilon})\|}{\gamma(\lambda_j+\gamma)} \quad \text{if} \quad \gamma \neq 0.$$

It should be noted that the error bounds given by (4.4.6) and (4.5.8) are numerically not as nice as they look. Even for $\|\vec{R}_o\| = \|K^*\vec{\epsilon}\| = 0$ and γ fairly large, making $\|K^*K\vec{\vec{f}}_{j,n} - K^*\vec{g}\|$ small is not necessarily good measure of convergence for the following reason. The methods of numerical quadrature used to obtain the matrix representation of the integral equation are essentially piecewise polynomial approximations to the integrand. From numerical experimentation, making $\|K^*K\vec{\vec{f}}_{j,n} - K^*\vec{g}\|$ as small as possible, will cause oscillations in the final result. This is due to computer convergence. An apparent measure of this effect can be noted in the output column AF-G of the algorithm in Appendix B.

4.6. Integral Equation Examples

For a first example we will consider the integral equation with symmetric kernel $k(x,y) = x + y$,

(4.6.1)

$$Kf(y) = \int_0^1 (x+y)f(x)\,dx = \frac{1}{3} + \frac{y}{2} \in R(K), \quad y \in [0,1].$$

K^*K has non-zero eigenvalues

$$\frac{7}{12} + \frac{\sqrt{3}}{3} = 1.160683 \quad \text{and} \quad \frac{7}{12} - \frac{\sqrt{3}}{3} \simeq .005983.$$

For (4.6.1) the least squares solution of minimum norm is $f_o(x) = x$.

The algorithm described in Chapter 2 and listed in Appendix B was used to obtain the following approximate solutions. Simpson's rule was used to determine the quadrature weights. In this example $\|\vec{R}_o\| = \|\vec{R}_1\| = 0$, the norm of the discretization error vectors. The interval [0,1] was partitioned:

$$\vec{y} = \vec{x} = \begin{bmatrix} 0.00 \\ 0.25 \\ 0.30 \\ 0.75 \\ 1.00 \end{bmatrix} .$$

A 5 x 5 matrix represention for the integral equation (4.4.1) gives

$$K = \begin{bmatrix} 0.0000 & 0.0833 & 0.0833 & 0.2500 & 0.0833 \\ 0.0208 & 0.1667 & 0.1250 & 0.3333 & 0.1042 \\ 0.0417 & 0.2500 & 0.1667 & 0.4167 & 0.1250 \\ 0.0625 & 0.3333 & 0.2083 & 0.5000 & 0.1458 \\ 0.0833 & 0.4167 & 0.2500 & 0.5833 & 0.1667 \end{bmatrix}$$

and

$$K^*K = \begin{bmatrix} 0.0278 & 0.1528 & 0.0972 & 0.2361 & 0.0694 \\ 0.0382 & 0.2153 & 0.1389 & 0.3403 & 0.1007 \\ 0.0486 & 0.2778 & 0.1806 & 0.4444 & 0.1319 \\ 0.0590 & 0.3403 & 0.2222 & 0.5486 & 0.1632 \\ 0.0694 & 0.4028 & 0.2639 & 0.6528 & 0.1944 \end{bmatrix} .$$

The non-zero eigenvalues of $K^*K : H_T^5 \to H_T^5$ were computed to be 1.160677 and 0.005983. We assumed that an estimate for $\|\vec{f}_o\|$ is known, $\|\vec{f}_o\| \simeq .577$ from (4.4.8)

$$\|\vec{f}_o - \vec{\bar{f}}_{j,n}\| \leq \frac{\|w_{j,n}\|}{\lambda_j + 0.005983} + \frac{.577\lambda_j}{\lambda_j + 0.005983} .$$

λ_j = terminal $\lambda = 1. \times 10^{-9}$ and $\|w_{j,n}\|^2 = 10^{-8}$ are determined so that $\|\vec{f}_o - \vec{\bar{f}}_{j,n}\| < 10^{-6}$. We used $M = 600$, maximum iterations per change in λ, $\lambda_{i+1} = .0001\lambda_i$, and initial starting vector $(f_i)_{i=1,5}$, $f_i = 1$, $i = 1,5$. The algorithm computed an initial starting λ, $\lambda_1 = .133889238$. After 225 iterations, we found the $\vec{\bar{f}}_j$ approximate least squares solution of minimum norm to be

$$\begin{bmatrix} \bar{f}_j(0) \\ \bar{f}_j(.25) \\ \bar{f}_j(.50) \\ \bar{f}_j(.75) \\ \bar{f}_j(1.0) \end{bmatrix} = \begin{bmatrix} .000000176 \\ .250000097 \\ .500000023 \\ .749999948 \\ .999999873 \end{bmatrix} .$$

Also the following information was computed:

$$\|\vec{\bar{f}}_{j,n} - \vec{f}_o\| < 9 \times 10^{-8} < 10^{-6},$$

$$\|\vec{\bar{f}}_{j,n} - \vec{f}_o\| \leq \frac{\|K^*K\vec{\bar{f}}_{j,n} - K^*\vec{g}\|}{\gamma} < 10^{-6},$$

and

$$\max_{i=1,5} |\bar{f}_{j,n}(x_i) - f_o(x_i)| < 1.8 \times 10^{-7}.$$

The same problem was run a second time using the same initial information, except to terminate when $\lambda_j = 10^{-15}$ and $\|w_{j,n}\|^2 < 10^{-28}$ obtaining, after 745 iterations,

$$\|\vec{\bar{f}}_{j,n} - \vec{f}_o\| < 8. \times 10^{-13}$$

and

$$\max_{i=1,5} |\bar{f}_{j,n}(x_i) - f_o(x_i)| < 1.6 \times 10^{-12}.$$

As a second example consider the integral equation

(4.6.2)
$$Kf(y) = \int_0^1 (y-x)^2 f(x)\,dx = \frac{y^2}{2} - \frac{2y}{3} + \frac{1}{4} \in R(K)$$

which Bellman et al. [4, page 159] used for testing purposes of an algorithm he developed for finding the least squares solution of minimum norm. (4.6.2) has least squares solution of minimum norm $f_o(x) = x$. The quadrature method Bellman used was Simpson's rule, with 11 equally spaced points, as we will use.

A bound for the norm of the quadrature error was found to be $\|\vec{R}_o\| = 0$, since Simpson's rule integrates $\int_0^1 (y-x)^2 f(x)\,dx$ for $f(x) = x$ exactly and the errors for

$$\int_0^1 (y-x)^2 \int_0^1 (y-x)^2 f(x)\,dx\,dy = \int_0^1 (y-x)^2 g(y)\,dy$$

subtract identically when $f(x) = x$.

In Bellman's example, also based on a variation of regularization, he used $\lambda = 10^{-7}$ and required a good initial approximation to the solution. To obtain somewhat comparable results, the algorithm in Appendix B with his λ was used.

The matrix operator K^*K; $K^*K : H_T^{11} \to H_T^{11}$, has computed non-zero eigenvalues .038101, .027778 and .000814. To compare the results of the algorithm in Appendix B with his we used λ_j = Terminal Lambda $= 10^{-7}$, control on $\|(K^*K + \lambda_i I)f - K^*g\|^2 = 10^{-16}$, lambda multiplier $= 10^{-3}$, maximum steps for each $\lambda_i = 300$ and initial starting vector $(f_i)_{i=1,11}$ where $f_i = 0$ for $i = 1,10$, $f_{11} = 1$.

The algorithm given in Appendix B choses a new starting vector and gave the results given in Table 2 for Run 1. The bound for $\|\vec{f}_{j,n} - \vec{f}_o\|$ was computed using $\gamma = .000814$ and .577 as an estimate for $\|\vec{f}_o\|$ in (4.4.8). The results were about the same as Bellman obtained.

Now Bellman also looked at the same example (4.6.2), but rounds the data representing $\vec{g}$ correct to three places. We ran the same example with the same starting information as for Run 1, to obtain two place accuracy compared to none for Bellman (see Table 2, Run 2). For Run 2, the bound for $\|\vec{\bar{f}}_{j,n,\epsilon} - \vec{f}_o\|$ was computed using $\gamma = .000814$, $.577$ as an estimate for $\|\vec{f}_o\|$ and

$$\|K^*\vec{\epsilon}\| \leq \|K^*\|\|\vec{\epsilon}\| \simeq \sqrt{.038101} \cdot .0005 \simeq .0001$$

in (4.5.10).

For a third run of a discretized version of (4.6.2) the 11-point trapezoidal rule was used as the quadrature method. In this case, the matrix operator K^*K has computed non-zero eigenvalues .040182, .028900 and .000926. An upper bound on the norm of the discretization error vector was computed to be $\|\vec{R}_o\| \leq .056$. To compare with Run 1, the same starting information was used and the computed error determined by using (4.4.8).

Table 2. Integral Equation Example 2

	Run 1	Run 2	Bellman	Run 3
Total iterations	339	339		341
$\|\vec{\bar{f}}_{j,n} - \vec{f}_o\|$	2.7(-5)	1.9(-3)		1.2(-2)
$\max_i \|\bar{f}_{j,n} - f_o\|$	4.6(-5)	3.(-3)	.9	3.1(-2)
Computed error	1.(-4)	.11		61.

For a last example we will consider the integral equation of the first kind with kernel

$$k(x,y) = \begin{cases} (1-y)x & \text{if } 0 \le x \le y \le 1 \\ (1-x)y & \text{if } 0 \le y \le x \le 1 . \end{cases}$$

Tricomi [24, page 116] notes that the eigenvalues associated with

$$K^*K(f)(x) = \int_0^1 k(x,y)\int_0^1 k(x,y)f(x)\,dx\,dy = \gamma f(x)$$

are $\gamma_1 = \frac{1}{\pi^4}$, $\gamma_2 = \frac{1}{(2\pi)^4}, \ldots, \gamma_n = \frac{1}{n^4\pi^4}, \ldots$.

Strand [20, page 71] shows that for

$g(y) = y(3-5y^2+3y^4-y^5)/30 \in R(K)$,

$Kf(y) = \int_0^1 k(x,y)f(x)\,dx = g(x)$ has least squares solution

of minimum norm $f_o(x) = x - 2x^3 + x^4$.

For the discretization of the integral equation

$$\int_0^1 k(x,y)\int_0^1 k(x,y)f(x)\,dx\,dy = \int_0^1 k(x,y)g(y)\,dy$$

the error for a 51 point Simpson's rule was estimated to be (upper bound) 1.4×10^{-7}.

The matrix representation for K^*K has minimum positive non-zero eigenvalue $\gamma = 7.915 \times 10^{-9}$ (computed). Since γ is small a choice for the terminal λ_j, from the error bounds is not practical (see Theorem 4.4.3). Considering the expression

$$\|\vec{f}_j - \vec{f}_o\| \leq \frac{\lambda_j \|\vec{\bar{f}}_j\| + \|\vec{R}_o\|}{\gamma} ,$$

numerical experimentation indicates that a reasonable choice for terminal λ_j is for the order of magnitude of $\lambda_j \|\vec{\bar{f}}_j\|$ and $\|\vec{R}_o\|$ be about the same. Assuming $\|\vec{\bar{f}}_j\| \simeq .22 \simeq \|\vec{f}_o\|$ we get a choice for terminal λ_j, $\lambda_j = 6.4 \times 10^{-7}$. Using $\|K^*K\vec{\bar{f}}_{j,n} + \lambda\vec{\bar{f}}_{j,n} - K^*\vec{g}\|^2 = 10^{-16}$ we get a computed error bound: $\|\vec{\bar{f}}_{j,n} - \vec{f}_o\| < .46$ from (4.4.8). We allowed a maximum of 500 iterations per step and set the lambda multiplier $= .001$. The algorithm in Appendix B generated its own starting vector from the initial starting vector

$$\vec{f}(x_i) = \begin{cases} 1 & \text{if } 17 \le i \le 35 \\ 0 & \text{otherwise} \end{cases} .$$

and initial $\lambda_1 \simeq .00795$. After 52 iterations we obtained actual results:

$$\|\vec{\vec{f}}_{j,n} - \vec{f}_o\| < 1.332 \times 10^{-4}$$

$$\max_{i=1,52} |f_{j,n}(x_i) - f_o(x_i)| < 3.024 \times 10^{-4},$$

and

$$\|K^*(K\vec{\vec{f}}_{j,n} - \vec{f}_o)\| < 7.121 \times 10^{-6}.$$

5. SUMMARY AND FUTURE RESEARCH

5.1. Summary

The advantage of the iterative method developed in the paper is primarily that it avoids the calculation of $(K^*K + \lambda I)^{-1}$ directly and that the iterative process is stable. The unwanted oscillations in the final solution that often appear in solutions by other methods in the literature do not occur. In some applications the error bounds obtained become large numerically when an upper bound on $\|\vec{R}_o\|$ or $\|\vec{\epsilon}\|$ is large or when γ is near zero. An upper bound for $\|\vec{R}_o\|$ can be difficult to obtain unless some information about the solution is known.

When comparing with other iterative methods in the literature, we generally have at least two additional significate figures. The iterative method here converges for examples where other methods in the literature fail to converge to a solution.

5.2. Future Research

It would be desirable to modify the algorithms in Appendix A and B so that γ and the parameters for convergence are computed within the algorithm.

A method of discretization for integral equations with non-uniform mesh size should be developed that better describes the properties of a particular kernel and g.

It would be nice to find a method of determining the weights for quadrature in representing the kernel, independent of any information about the form of the solution.

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8. APPENDIX

8.1. Appendix A, Linear System

The following program is written to find the least squares solution of minimum norm of an NN by N system of equations, $Ax = g$. The basic flow chart for the program is given in Figure 3.2.1.

The user must input the following information in the order given:

1. NN = the number of rows of the matrix A. The format for reading NN is given in line SYST0280.

2. N = the number of columns of the matrix A. The format for reading N is given in line SYST0280.

3. M = the maximum number of iterations per change in lambda. The format for reading M is given in line SYST0280.

4. BB = terminal lambda. The format for reading M is given in line SYST0320.

5. DD = control on $\|ATAF + LAMBDA*F - ATG\|^2$. The format for reading DD is given in line SYST0320.

2

6. The matrix A is read by rows. The format for reading A is given in line SYST0340.

7. Read the vector g. The format for reading g is given in line SYST0340.

8. Read the initial starting vector F. The format is given in line SYST0340.

9. XLAM = lambda multiplier. The format for reading XLAM is given in line SYST0340.

The output of the starting information begins at line SYST1440. The output of the final solution begins at line SYST2370.

```
C
C    THE FOLLOWING IS A PROGRAM TO SOLVE AN NN BY N SYSTEM OF
C    ECUATIONS, AF=G, FOR THE LEAST SQUARES SOLUTION OF MINIMUM
C    NORM.  IF N OR NN IS GREATER THAN 10, THE DIMENSION
C    STATEMENT MUST BE CHANGED TO AT LEAST A(NN,N),G(NN),F(N),
C    AT(N,NN),ATA(N,N),R(NN),T(N),V(NN),W(N),S(NN), AND WW(N).
C
C    INPUT:
C
C          N= THE NUMBER OF COLUMNS OF THE MATRIX A .
C
C          NN=THE NUMBER OF ROWS OF THE MATRIX A.
C
C          M=THE MAXIMUM NUMBER OF ITERATIONS PER CHANGE IN
C          LAMBDA.
C
C          BB=TERMINAL LAMBDA.
C
C          DD=CONTROL ON ||ATAF+LAMBDA*F-ATG||**2.
C
C          XLAM=LAMBDA MULTIPLIER.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSICN A(10,10),G(10),F(10),AT(10,10),ATA(10,10),R(10)
      DIMENSION T(10),V(10),W(10),S(10),WW(10)
      REAC(5,11) NN
      READ(5,11) N
   11 FORMAT(I5)
      READ(5,11) M
      READ(5,21) BB
      READ(5,21) DD
   21 FORMAT(D10.5)
   31 FORMAT(8F10.5)
      DO 10 I=1,NN
C
C    READ MATRIX A BY ROWS.
C
```

```
   10 READ(5,31) (A(I,J),J=1,N)
C
C   READ VECTOR G OF AF=G.
C
      READ(5,31) (G(I),I=1,NN)
C
C   READ INITIAL STARTING VECTOR F.
C
      READ(5,31) (F(I),I=1,N)
      READ(5,31) XLAM
      WRITE(6,12) NN,N,M
   12 FORMAT('0','NN=',I5,5X,'N=',I5,5X,'M=',I5)
      WRITE(6,26) DD
   26 FORMAT('0','CONTROL ON || ATAF+LAMBDA*F-ATG ||**2=',D24.16)
      WRITE(6,22) BB
   22 FORMAT('0','CONTROL ON LAMBDA=',D24.16)
      WRITE(6,24) XLAM
   24 FORMAT('0','LAMBDA MULTIPLIER=',D24.16)
      WRITE(6,32)
   32 FORMAT('0','MATRIX A=',/)
      DO 20 I=1,NN
   20 WRITE(6,42) (A(I,J),J=1,N)
   42 FORMAT(' ',10(F10.5,1X))
      WRITE(6,52)
   52 FORMAT('0','G=',/)
      WRITE(6,42) (G(I),I=1,NN)
      WRITE(6,62)
   62 FORMAT('0','STARTING VECTOR F=',/)
      WRITE(6,42) (F(I),I=1,N)
      LL=0
      DO 30 I=1,N
      DO 30 J=1,NN
   30 AT(I,J)=A(J,I)
      DO 410 I=1,N
      W(I)=0.0D0
      DO 410 J=1,NN
  410 W(I)=W(I)+AT(I,J)*G(J)
```

```
      ZZ=0.0D0
      DO 420 I=1,N
  420 ZZ=ZZ+W(I)*W(I)
      ZZ=DSQRT(ZZ)
      WRITE(6,29) ZZ
   29 FORMAT('0','|| ATG ||=',D24.16)
      IF(ZZ.GE.1.D-40) GO TO 430
      WRITE(6,421)
  421 FORMAT('0',6X,'SOLUTION IS THE ZERO VECTOR',/)
      GO TO 600
C
C   OBTAIN A STARTING VECTOR AND AN INITIAL LAMBDA.
C
  430 DO 40 I=1,N
      DO 40 J=1,N
      ATA(I,J)=0.0D0
      DO 40 L=1,NN
   40 ATA(I,J)=ATA(I,J)+AT(I,L)*A(L,J)
      DO 50 I=1,N
      T(I)=0.0D0
      DO 50 J=1,N
   50 T(I)=T(I)+ATA(I,J)*F(J)
      ZZ=0.0D0
      DO 60 I=1,N
   60 ZZ=ZZ+T(I)*T(I)
      ZZ=DSQRT(ZZ)
      DO 70 I=1,N
   70 T(I)=T(I)/ZZ
      DO 80 I=1,NN
      R(I)=0.0D0
      DO 80 J=1,N
   80 R(I)=R(I)+A(I,J)*T(J)
      CR=1.0D0
      C=2.0D0
      J=1
   90 X=0.0D0
      DO 100 I=1,NN
```

```
  100 X=X+(G(I)-R(I))*R(I)
      IF(X.GE.0.0D0) GO TO 1000
      IF(J.EQ.2) GO TO 1010
      DO 110 I=1,N
  110 T(I)=-T(I)
      DO 115 I=1,NN
  115 R(I)=-R(I)
      GO TO 90
 1010 DO 120 I=1,NN
  120 G(I)=U*G(I)
      J=1
      CR=CR*U
      GO TO 90
 1000 DO 130 I=1,NN
  130 R(I)=R(I)-G(I)
      RR=0.0D0
      DO 140 I=1,NN
  140 RR=RR+R(I)*R(I)
      DO 150 I=1,N
      W(I)=0.0D0
      DO 150 J=1,NN
  150 W(I)=W(I)+AT(I,J)*R(J)
      ZZ=0.0D0
      DO 160 I=1,N
  160 ZZ=ZZ+W(I)*W(I)
      YY=ZZ-X*X
      AB=RR+X
      DO 170 I=1 ,N
  170 W(I)=W(I)+X*T(I)
C
C   OUTPUT STARTING INFORMATION.
C
      WRITE(6,72) X
   72 FORMAT('0','INITIAL LAMBDA=',D24.16)
      WRITE(6,82)
   82 FORMAT('0',3X,'STARTING VECTOR F',10X,'ATAF+LAMBDA*F-ATG',/)
      DO 180 I=1,N
```

```
  180 WRITE(6,92) T(I),W(I)
   92 FORMAT(' ',3D24.16)
   96 FORMAT(' ',D24.16)
      WRITE(6,84)
   84 FORMAT('0',10X,'AF-G',/)
      DO 182 I=1,NN
  182 WRITE(6,96) R(I)
      WRITE(6,94) CR
   94 FORMAT('0','SCALING FACTOR=',D24.16)
      WRITE(6,102) YY
  102 FORMAT('0','MINIMUM VALUE=',D24.16)
      XX=X
C
C    MAIN ITERATIONS OF THE ALGORITHM BEGIN.
C
 1020 X=XX
      K=1
 1030 AA=AB
      YY=ZZ
      DO 190 I=1,N
      WW(I)=W(I)
      W(I)=0.0D0
      DO 190 J=1,NN
  190 W(I)=W(I)+AT(I,J)*R(J)
      DO 200 I=1,N
  200 W(I)=W(I)+X*T(I)
      ZZ=0.0D0
      DO 210 I=1,N
  210 ZZ=ZZ+W(I)*W(I)
      IF(ZZ.LT.DD) GO TO 1040
      DO 220 I=1,NN
  220 V(I)=R(I)
      DO 225 I=1,N
  225 F(I)=T(I)
      VV=RR
      DO 230 I=1,NN
      S(I)=0.0D0
```

```
      DO 230 J=1,N                                                      SYST1860
  230 S(I)=S(I)+A(I,J)*W(J)                                             SYST1870
      Y=0.0D0                                                           SYST1880
      DO 240 I=1,NN                                                     SYST1890
  240 Y=Y+S(I)*S(I)                                                     SYST1900
      Y=Y+X*ZZ                                                          SYST1910
      B=ZZ/Y                                                            SYST1920
      DO 250 I=1,N                                                      SYST1930
  250 T(I)=T(I)-B*W(I)                                                  SYST1940
      DO 260 I=1,NN                                                     SYST1950
      R(I)=0.0D0                                                        SYST1960
      DO 260 J=1,N                                                      SYST1970
  260 R(I)=R(I)+A(I,J)*T(J)                                             SYST1980
      DO 270 I=1,NN                                                     SYST1990
  270 R(I)=R(I)-G(I)                                                    SYST2000
      RR=0.0D0                                                          SYST2010
      DO 280 I=1,NN                                                     SYST2020
  280 RR=RR+R(I)*R(I)                                                   SYST2030
      AB=0.0D0                                                          SYST2040
      DO 290 I=1,N                                                      SYST2050
  290 AB=AB+T(I)*T(I)                                                   SYST2060
      AB=RR+X*AB                                                        SYST2070
      IF(K.EQ.1) GO TO 300                                              SYST2080
      IF(AB.LT.AA) GO TO 300                                            SYST2090
      DO 310 I=1,N                                                      SYST2100
      T(I)=F(I)                                                         SYST2110
  310 W(I)=WW(I)                                                        SYST2120
      DO 295 I=1,NN                                                     SYST2130
  295 R(I)=V(I)                                                         SYST2140
      RR=VV                                                             SYST2150
      ZZ=YY                                                             SYST2160
      K=K-1                                                             SYST2170
      GO TO 330                                                         SYST2180
  300 IF(M-K) 600,330,340                                               SYST2190
  340 K=K+1                                                             SYST2200
      GO TO 1030                                                        SYST2210
 1040 K=K-1                                                             SYST2220
```

```
  330 LL=LL+K
      IF(K.LT.M) GO TO 440
      WRITE(6,112) X
  112 FORMAT('0',3X,'AT LAMBDA=',D24.16,3X,'THE MAXIMUM VALUE OF M HAS
     CBEEN ATTAINED')
  440 IF(X.EQ.BB) GO TO 360
      XX=XLAM*X
      IF(XX.LT.BB) GO TO 350
      GO TO 1020
  350 XX=BB
      GO TO 1020
C
C   RE-SCALE THE COMPUTATIONS AND OUTPUT.
C
  360 WRITE(6,122)
  122 FORMAT('0',3X,'FINAL SOLUTION F',10X,'ATAF+LAMBDA*F-ATG',/)
      DO 370 I=1,N
      T(I)=T(I)/CR
      W(I)=W(I)/CR
  370 WRITE(6,92) T(I),W(I)
      WRITE(6,84)
      DO 375 I=1,NN
      R(I)=R(I)/CR
  375 WRITE(6,96) R(I)
      ZZ=ZZ/(CR*CR)
      RR=RR/(CR*CR)
      ZZ=DSQRT(ZZ)
      RR=DSQRT(RR)
      TT=0.0D0
      DO 380 I=1,N
  380 TT=TT+T(I)*T(I)
      TT=DSQRT(TT)
      WRITE(6,132)
  132 FORMAT('0',8X,'|| F ||',12X,'|| ATAF+LAMBDA*F-ATG ||',10X,
     C'|| AF-G ||',/)
      WRITE(6,92) TT,ZZ,RR
      DO 390 I=1,N
```

```
      V(I)=0.0D0
      DO 390 J=1,NN
  390 V(I)=V(I)+AT(I,J)*R(J)
      VVV=0.0D0
      DO 400 I=1,N
  400 VVV=VVV+V(I)*V(I)
      VVV=DSQRT(VVV)
      WRITE(6,144) VVV
  144 FORMAT('0',8X,'|| AT(AF-G) ||=',D24.16)
      WRITE(6,142) LL
  142 FORMAT('0','TOTAL ITERATIONS=',I6)
C
C    IF THE ACTUAL SOLUTION IS KNOWN, A PROGRAM TO COMPARE THE
C    ACTUAL SOLUTION WITH THE COMPUTED SOLUTION CAN BE PLACED
C    HERE. THE COMPUTED SOLUTION IS GIVEN BY THE VECTOR T. THE
C    VECTOR F CAN BE ASSIGNED THE ACTUAL SOLUTION.
C
  600 STOP
      END
```

8.2. Appendix B, Subroutine BITER

Subroutine BITER is a program written to find a λ-approximate least squares solution of minimum norm of an integral equation of the first kind. The subroutine BITER assumes that the integral equation is of the form

$$\int_0^1 k(y,x)f(x)dx = g(y), \quad y \in [0,1].$$

A driver program must be written which obtains the following information:

1. Partitions the interval integrated over, [0,1], into N - 1 equal subintervals for an N-point closed Newton-Cotes integration formula. The N-points constructed are stored in the vector XV.
2. Partitions the domain of g, [0,1], into NN - 1 equal subintervals for an NN-point closed Newton-Cotes integration formula. The NN-points constructed are stored in the vector YV.
3. Construct the vector GV(I) = g(YV(I)) I = 1,NN.

4. Construct the matrix representing the kernel $k(x,y)$, $A(k(XV(I),YV(j))_{\substack{j=1,NN \\ I=1,N}}$.

5. Construct the vector WTK of weights for the kernel using a closed N-point Newton-Cotes formula.

6. Construct the vector WTKT of weights for integration of the adjoint of the kernel using a closed NN-point Newton-Cotes formula.

7. The driver program must be dimensioned as the dimension statement for the subroutine BITER.

8. The driver program must supply the controls on convergence given by

   XLMDA = Lambda multiplier,

   BB = Terminal lambda,

   and

   DD = control on $\|ATAF + LAMBDA*F - ATG\|^2$.

The output section of the starting vector and initial lambda begins at line BITR1080. The output section for the final results begins at line BITR1990.

```
      SUBROUTINE BITER(N,NN,A,GV,WTK,WTKT,XV,YV,F,XLMDA,M,BB,
     CAT,ATA,R,T,V,W,WW,S,DD)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NN,N),AT(N,NN),WTK(N),WTKT(NN),XV(N),YV(NN),F(N)
      DIMENSION ATA(N,N),GV(NN),R(NN),T(N),V(NN),W(N),WW(N),S(NN)
      LL=0
C
C   CONSTRUCT THE ADJOINT=AT TO KERNAL=A.
C
C
C   TRANSPOSE THE UNWEIGHTED KERNEL.
C
      DO 510 I=1,NN
      DO 510 J=1,N
  510 AT(J,I)=A(I,J)
C
C   CONSTRUCT THE WEIGHTED KERNEL.
C
      DO 520 I=1,NN
      DO 520 J=1,N
  520 A(I,J)=A(I,J)*WTK(J)
C
C   CONSTRUCT THE WEIGHTED KERNEL TRANSPOSE.
C
      DO 530 I=1,N
      DO 530 J=1,NN
  530 AT(I,J)=AT(I,J)*WTKT(J)
C
C   TEST FOR A ZERO SOLUTION.
C
      DO 810 I=1,N
      W(I)=0.0D0
      DO 810 J=1,NN
  810 W(I)=W(I)+AT(I,J)*GV(J)
      ZZ=0.0D0
      DO 820 I=1,N
  820 ZZ=ZZ+WTK(I)*(W(I)*W(I))
```

```
      ZZ=DSQRT(ZZ)
      WRITE(6,29) ZZ
   29 FORMAT('0','|| ATG ||=',D24.16)
      IF(ZZ.GE.1.D-40) GO TO 830
      WRITE(6,821)
  821 FORMAT('0',6X,'SOLUTION IS THE ZERO FUNCTION',/)
      GO TO 620
C
C     CONSTRUCT THE WEIGHTED KERNEL TRANSPOSE*WEIGHTED KERNEL.
C
  830 DO 540 I=1,N
      DO 540 J=1,N
      ATA(I,J)=0.0D0
      DO 540 L=1,NN
  540 ATA(I,J)=ATA(I,J)+AT(I,L)*A(L,J)
C
C     OBTAIN AN ACTUAL STARTING VECTOR AND INITIAL LAMBDA.
C
      DO 20 I=1,N
      T(I)=0.0D0
      DO 20 J=1,N
   20 T(I)=T(I)+ATA(I,J)*F(J)
      ZZ=0.0D0
      DO 30 I=1,N
   30 ZZ=ZZ+WTK(I)*(T(I)*T(I))
      ZZ=DSQRT(ZZ)
      DO 40 I=1,N
   40 T(I)=T(I)/ZZ
      DO 50 I=1,NN
      R(I)=0.0D0
      DO 50 J=1,N
   50 R(I)=R(I)+A(I,J)*T(J)
      CR=1.0D0
      U=2.0D0
      J=1
   92 X=0.0D0
      DO 60 I=1,NN
```

```
   60 X=X+(GV(I)-R(I))*(R(I)*WTKT(I))                                   BITR0750
      IF(X.GE.0.0D0) GO TO 1000                                         BITR0760
      IF(J.EQ.2) GO TO 1001                                             BITR0770
      J=J+1                                                             BITR0780
      DO 70 I=1,N                                                       BITR0790
   70 T(I)=-T(I)                                                        BITR0800
      DO 80 I=1,NN                                                      BITR0810
   80 R(I)=-R(I)                                                        BITR0820
      GO TO 92                                                          BITR0830
 1001 DO 450 I=1,NN                                                     BITR0840
  450 GV(I)=U*GV(I)                                                     BITR0850
      J=1                                                               BITR0860
      CR=CR*U                                                           BITR0870
      GO TO 92                                                          BITR0880
 1000 DO 90 I=1,NN                                                      BITR0890
   90 R(I)=R(I)-GV(I)                                                   BITR0900
      RR=0.0D0                                                          BITR0910
      DO 100 I=1,NN                                                     BITR0920
  100 RR=RR+WTKT(I)*(R(I)*R(I))                                         BITR0930
      DO 110 I=1,N                                                      BITR0940
      W(I)=0.0D0                                                        BITR0950
      DO 110 J=1,NN                                                     BITR0960
  110 W(I)=W(I)+AT(I,J)*R(J)                                            BITR0970
      ZZ=0.0D0                                                          BITR0980
      DO 120 I=1,N                                                      BITR0990
  120 ZZ=ZZ+WTK(I)*(W(I)*W(I))                                          BITR1000
      YY=ZZ-X*X                                                         BITR1010
      AB=RR+X                                                           BITR1020
      DO 130 I=1,N                                                      BITR1030
  130 W(I)=W(I)+X*T(I)                                                  BITR1040
C                                                                       BITR1050
C   OUTPUT SECTION FOR THE STARTING INFORMATION.                        BITR1060
C                                                                       BITR1070
      WRITE(6,2) X                                                      BITR1080
    2 FORMAT('0','INITIAL LAMBDA=',D24.16)                              BITR1090
      WRITE(6,3)                                                        BITR1100
    3 FORMAT('0',12X,'X=',18X,'STARTING=',12X,'ATAF+LAMBDA*F-ATG=',/)   BITR1110
```

```
      DO 140 I=1,N
  140 WRITE(6,4) XV(I),T(I),W(I)
    4 FORMAT(' ',3D24.16)
      WRITE(6,5)
    5 FORMAT('0',12X,'Y=',18X,'AF-G=',/)
      DO 150 I=1,NN
  150 WRITE(6,6) YV(I),R(I)
    6 FORMAT(' ',2D24.16)
      WRITE(6,7) CR
    7 FORMAT('0','SCALING FACTOR=',D24.16)
      WRITE(6,8) YY
    8 FORMAT('0','MINIMUM VALUE=',D24.16)
      XX=X
C
C    THE MAIN ITERATIONS OF THE ALGORITHM.
C
  171 X=XX
      K=1
   41 AA=AB
      YY=ZZ
      DO 160 I=1,N
      WW(I)=W(I)
      W(I)=0.0D0
      DO 160 J=1,NN
  160 W(I)=W(I)+AT(I,J)*R(J)
      DO 170 I=1,N
  170 W(I)=W(I)+X*T(I)
      ZZ=0.0D0
      DO 180 I=1,N
  180 ZZ=ZZ+WTK(I)*(W(I)*W(I))
      IF(ZZ.LT.DD) GO TO 403
      DO 190 I=1,NN
  190 V(I)=R(I)
      DO 200 I=1,N
  200 F(I)=T(I)
      VV=RR
      DO 210 I=1,NN
```

```
      S(I)=0.0D0                                                        BITR1490
      DO 210 J=1,N                                                      BITR1500
  210 S(I)=S(I)+A(I,J)*W(J)                                             BITR1510
      Y=0.0D0                                                           BITR1520
      DO 220 I=1,NN                                                     BITR1530
  220 Y=Y+WTKT(I)*(S(I)*S(I))                                           BITR1540
      Y=Y+X*ZZ                                                          BITR1550
      B=ZZ/Y                                                            BITR1560
      DO 230 I=1,N                                                      BITR1570
  230 T(I)=T(I)-B*W(I)                                                  BITR1580
      DO 240 I=1,NN                                                     BITR1590
      R(I)=0.0D0                                                        BITR1600
      DO 240 J=1,N                                                      BITR1610
  240 R(I)=R(I)+A(I,J)*T(J)                                             BITR1620
      DO 250 I=1,NN                                                     BITR1630
  250 R(I)=R(I)-GV(I)                                                   BITR1640
      RR=0.0D0                                                          BITR1650
      DO 260 I=1,NN                                                     BITR1660
  260 RR=RR+WTKT(I)*(R(I)*R(I))                                         BITR1670
      AB=0.0D0                                                          BITR1680
      DO 270 I=1,N                                                      BITR1690
  270 AB=AB+WTK(I)*(T(I)*T(I))                                          BITR1700
      AB=RR+X*AB                                                        BITR1710
      IF(K.EQ.1) GO TO 201                                              BITR1720
      IF(AB.LT.AA) GO TO 201                                            BITR1730
      DO 280 I=1,N                                                      BITR1740
      T(I)=F(I)                                                         BITR1750
  280 W(I)=WW(I)                                                        BITR1760
      DO 290 I=1,NN                                                     BITR1770
  290 R(I)=V(I)                                                         BITR1780
      RR=VV                                                             BITR1790
      ZZ=YY                                                             BITR1800
      K=K-1                                                             BITR1810
      GO TO 401                                                         BITR1820
  201 IF(M-K) 620,401,301                                               BITR1830
  301 K=K+1                                                             BITR1840
      GO TO 41                                                          BITR1850
```

```
403 K=K-1
401  LL=LL+K
     IF(K.LT.M) GO TO 440
     WRITE(6,1) X
   1 FORMAT('0',3X,'AT LAMBDA=',D24.16,3X,'THE MAXIMUM VALUE OF M HAS
    CBEEN ATTAINED')
440 IF(X.EQ.BB) GO TO 600
     XX=XLMDA*X
     IF(XX.LT.BB) GO TO 610
     GO TO 171
610 XX=BB
     GO TO 171
C
C    OUTPUT SECTION FOR THE FINAL RESULTS.
C
600 WRITE(6,23)
  23 FORMAT('0',12X,'X=',14X,'FINAL SOLUTION F=',10X,
    C'ATAF+LAMBDA*F-ATG=',/)
     DO 320 I=1,N
     T(I)=T(I)/CR
     W(I)=W(I)/CR
320 WRITE(6,4) XV(I),T(I),W(I)
     WRITE(6,5)
     DO 330 I=1,NN
     R(I)=R(I)/CR
330 WRITE(6,6) YV(I),R(I)
     ZZ=ZZ/(CR*CR)
     RR=RR/(CR*CR)
     ZZ=DSQRT(ZZ)
     RR=DSQRT(RR)
     WRITE(6,26) ZZ,RR
  26 FORMAT('0','|| ATAF+LAMBDA*F-ATG ||=',D24.16,5X,'|| AF-G ||=',
    CD24.16)
     WRITE(6,28) LL
  28 FORMAT('0','TOTAL ITERATIONS=',I6)
     CC=0.0D0
     DO 340 I=1,N
```

```
340 CC=CC+WTK(I)*(T(I)*T(I))                                    BITR2230
    CC=DSQRT(CC)                                                BITR2240
    WRITE(6,9) CC                                               BITR2250
  9 FORMAT('0',' NORM OF THE SOLUTION=',D24.16)                 BITR2260
    DO 390 I=1,N                                                BITR2270
    V(I)=0.0D0                                                  BITR2280
    DO 390 J=1,NN                                               BITR2290
390 V(I)=V(I)+AT(I,J)*R(J)                                      BITR2300
    VVV=0.0D0                                                   BITR2310
    DO 400 I=1,N                                                BITR2320
400 VVV=VVV+WTKT(I)*(V(I)*V(I))                                 BITR2330
    VVV=DSQRT(VVV)                                              BITR2340
    WRITE(6,144) VVV                                            BITR2350
144 FORMAT('0',8X,'|| AT(AF-G) ||=',D24.16)                     BITR2360
620 RETURN                                                      BITR2370
    END                                                         BITR2380
```