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**DYNAMICS OF A FAST REACTOR WITH
IN-CORE THERMIONIC CONVERTERS**

Second Annual Report

**ENGINEERING EXPERIMENT STATION
COLLEGE OF ENGINEERING
THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA**

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DYNAMICS OF A FAST REACTOR WITH IN-CORE
THERMIONIC CONVERTERS

Second Annual Report

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SUMMARY

This report summarizes activities for the period September 1, 1967 to August 31, 1968 under USAEC Contract AT(04-3)-670. The major objective of this contract is the investigation of the dynamic behavior of space reactors utilizing in-core thermionic diodes. This includes inherent stability of the reactor system, start-up behavior from zero to full power operation, and control during operation. The studies are intended to have general applicability in the sense that they are not restricted to any single specific reactor concept.

The efforts of the first report period under this contract (Engineering Experiment Station Report, November 1967) were primarily devoted to preliminary stability investigations utilizing both analytical and computer methods. Related activities included studies of diode electrical behavior and the initiation of an effort to develop a computer code for solving the detailed dynamic equations representing a thermionic reactor system.

The efforts of this report period have been devoted primarily to three areas: (1) the detailed development of the two-region stability model used in the previous report period, together with a study of the model's limitations and the consequences for stability when a spectrum of plausible thermionic reactor concepts is considered, (2) the development of a computer program describing an in-core thermionic reactor system in detail, including a realistic representation of the electrical characteristics of thermionic diodes (this code will be

used to study problems in system start-up and control), and (3) generalized studies in nonlinear reactor dynamics with particular application to the two-region model of the thermionic reactor.

Each of the above three areas is developed fully in this report. Progress is as follows:

1) The two region model used for stability studies provides a reasonably conservative estimate of the restriction on temperature coefficients for inherent reactor stability. Over plausible ranges of operating power density levels and reactor and diode materials and dimensions, the restriction is:

$$\alpha_f + \alpha_e + (0.06 \text{ to } 0.19) \alpha_c > 0$$

where the subscripts f, e, and c refer to the fuel, emitter, and collector respectively, and where a positive α_i corresponds to negative temperature feedback.

2) The system dynamic equations and the theoretical equations describing the electrical characteristics of the diodes have been programmed for the digital computer and checked out, and preliminary start-up computations have been performed.

3) Nonlinear aspects have been attacked by two methods. The direct method of Liapunov has been used to demonstrate nonlinear stability for a restricted domain of the parameters of interest, and a Fourier series method has been utilized in searching for autonomous periodic solutions to the nonlinear dynamic equations. Both of these efforts have met with limited success.

The studies pertaining to item (1) are considered completed. Activities under item (2) are continuing and will form the basis for much of the work during the next report period in start-up and control. A limited amount of further activity on the nonlinear studies, item (3), will also continue through the next report period, with primary emphasis on the Fourier series expansion period.

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I. LINEAR STABILITY STUDIES

A. SUMMARY

The two-region stability model, used in the previously reported stability analysis,⁽¹⁾ is developed in detail so that a thorough understanding of the assumptions and limitations of the model are obtained. The time constants characterizing the two-region model are derived in terms of the material properties, dimensions, and operating conditions of the reactor. The stability requirements for various plausible combinations of diode dimensions, compositions, and power density levels are determined in terms of allowable temperature coefficients for the fuel, emitter, and collector. The linear stability requirements simplify to the single requirement:

$$\alpha_f + \alpha_e + (0.06 \text{ to } 0.19)\alpha_c > 0$$

where the subscripts f, e, and c refer to fuel, emitter, and collector respectively and where a positive α_i corresponds to negative feedback. This linear requirement is compared with a nonlinear result obtained on the analog computer for the multi-region model and is shown to be conservative because the feedback due to coolant and clad heating is neglected.

The requirement above indicates that the stability of the reactor is not strongly affected by the specific design of the diode. The study also indicates that small electrical perturbations will not significantly affect system temperatures and that further refinements to heat transport processes downstream of the collector are unnecessary for stability considerations.

The validity of lumping the fuel and emitter into a single region has been investigated. The assumption that the emitter temperature follows the

average fuel temperature without time delay (and hence that the fuel and emitter temperature coefficients may simply be added) has been found to be correct within 10% for perturbing frequencies up to a_f/R_f^2 , where a_f and R_f are the thermal diffusivity and radius of the fuel cylinder, respectively.

The introduction of a finite thermal resistance between fuel and emitter has been found to increase the effective fuel temperature coefficient by an amount approximately equal to the ratio of the temperature difference across the resistance to the temperature difference across the gas gap between emitter and collector.

The effect on stability of fuel with circular cross-section holes along the centerline is shown to change one of the time constants only slightly and to have no effect on the reactor stability.

B. INTRODUCTION

In the stability studies reported in the previous annual report⁽¹⁾ a two-temperature region model was utilized. Back emission from the collector was neglected so that the dynamic equations for the average temperature in each of the two regions could be assumed to be given by:

$$\frac{dT_1}{dt} = b(n-n_o) - \eta_1 T_1 \quad (1)$$

$$\frac{dT_2}{dt} = \eta_2 T_1 - \eta_3 T_2 \quad (2)$$

where subscripts 1 and 2 denote the fuel-emitter and collector regions respectively, b is an appropriate reciprocal heat capacity for the fuel-emitter region, the η 's are the reciprocal time constants for heat transfer into or out of the regions, T_1 and T_2 are the temperature differences from

the equilibrium or steady state values, n is the fuel power density, and n_0 is the steady state (initial) power density. The inclusion of collector back emission in this linear model would simply add a term proportional to T_2 to Eq. (1).

As of the last reporting period, the stability studies had been general, with no attempt made to determine the relative validity of the above heat transfer model nor to evaluate the range of values of the various time constants for representative thermionic concepts. This section summarizes the result of efforts in the latter two directions.

First, the derivation of Eqs. (1) and (2) is carried out so that the reciprocal time constants $\tau_{1,2,3}$ are defined. In the process, the limitations and assumptions of the model are discussed, as is the general flexibility of the model. Second, the effect on the linear stability of fuel materials, diode materials, dimensions, coolants, and power density operating levels are presented.

C. DERIVATION OF THE EQUATIONS FOR THE TWO-TEMPERATURE REGION MODEL

The steady-state temperature distribution in a cylindrical internally-fueled diode appears as shown schematically in Fig. 1. The temperature profile within the fuel is parabolic under the assumption of a radially uniform heat generation rate. The temperature change across each of the metallic regions--emitter, collector, clad, and coolant--is relatively small, 5°-10°C, because the regions are thin and have high thermal conductivity (heat transfer coefficient, in the case of the coolant). The primary temperature drop occurs across the gap separating emitter and collector and typically

Figure 1. Steady-state temperature distribution in a cylindrical internally-fueled diode.

may be of the order of 1000°C. A secondary drop occurs across the electrical insulation separating collector and clad and typically is of the order of 50°-100°C. End effects and electrical lead losses are neglected. Normally, ideal thermal bonding between materials is also assumed. This latter assumption is investigated subsequently and is found to be relatively unimportant with regard to reactor stability.

The parabolic temperature distribution within the fuel is given by the solution of the steady state heat conduction equation in infinite cylindrical geometry and can be written in terms of the fuel centerline temperature as

$$T_f(r) = T_{\text{c}} - \frac{n_o r^2}{4k_f} \quad (3)$$

where $T_f(r)$ is the radial temperature distribution, °K

T_{c} is the fuel centerline temperature, °K

n_o is the steady state or initial power density, watts/cc

r is the radial dimension, cm

k_f is the average fuel thermal conductivity, assumed constant, watts/cm-°K.

The spatially averaged fuel temperature, $\bar{T}_f$, can be shown to be

$$\bar{T}_f = T_{\text{c}} - \frac{n_o R_f^2}{8k_f} \quad (4)$$

where R_f is the radius of the fuel region.

If the contact resistance between fuel and emitter and the temperature drop across the emitter are both assumed negligible, then the average emitter temperature is approximately equal to the fuel surface temperature

$$\bar{T}_e \cong T_f(R_f) = T_{\text{c}} - \frac{n_o R_f^2}{4k_f} \quad (5)$$

where the subscript e denotes the emitter. The difference in average fuel and emitter temperatures is

$$\bar{T}_f - \bar{T}_e = \frac{n_o R_f^2}{8k_f} \quad (6)$$

The steady state heat transfer rate from fuel to emitter is

$$Q_{f \rightarrow e} = -2\pi R_f L k_f \left. \frac{dT_f}{dr} \right|_{r=R_f}$$

or

$$Q_{f \rightarrow e} = 8\pi L k_f (\bar{T}_f - \bar{T}_e) \quad (7)$$

where use of Eqs. (3) and (6) has been made and where L is the length of the fuel in cm.

In transient analyses, the assumption that the fuel spatial temperature distribution remains unchanged from its initial parabolic distribution implies that the heat transferred from fuel to emitter is always given by Eq. (7) with the average fuel and emitter temperatures being now time-dependent. This is the model used by Gronroos⁽²⁾ in his multi-node analog computer simulation. The adequacy of representing the fuel region by a single node or average temperature has also been given limited investigation by Gronroos. He divided the fuel into six regions (in addition to retaining the emitter, collector, insulator, clad, coolant, and heat transfer loops as separate regions) and calculated the temperature profile within the fuel during the extreme case of an open circuit for which the cooling of the emitter is suddenly reduced by ~50%. Although this study demonstrated the adequacy of the single-region parabolic-average temperature model for the particular case investigated, it did not encompass reactor power transients, as opposed

to the electrical transient, or the use of oxide fuels, both of which can introduce effects of more significance in terms of reactor stability, as will be shown.

The analysis proceeds by writing an energy balance on each region utilizing energy loss rates from that region by approximations similar to Eq. (7). The exception to this occurs for the emitter region which, because of the electron cooling, must be treated somewhat differently.

For the fuel region, the energy balance equation is

$$\pi R_f^2 L \rho_f c_p \frac{d\bar{T}_f(t)}{dt} + 8\pi L k_f [\bar{T}_f(t) - \bar{T}_e(t)] = \pi R_f^2 L n(t) \quad (8)$$

where $\rho_f c_p$ is the volumetric heat capacity of the fuel material.

Similarly for the emitter region

$$2\pi R_f t_e L \rho_e c_p \frac{d\bar{T}_e}{dt} + 2\pi R_f L q_e = 8\pi L k_f (\bar{T}_f - \bar{T}_e) \quad (9)$$

where $\rho_e c_p$ is the volumetric heat capacity of the emitter, t_e is the thickness of the emitter, and q_e is the net energy transfer rate between the emitter and collector due to thermal radiation, electron emission, cesium conduction, etc. With the minor approximations of representing the volume of the emitter region by $2\pi R_f t_e L$ and the outer surface area by $2\pi R_f L$, Eqs. (8) and (9) are the equations used by Gronroos in his simulation.

The simplification to the two-region model is obtained by cancelling the common factors in Eqs. (8) and (9) and adding to give

$$\rho_f c_p \frac{d\bar{T}_f}{dt} + \frac{2t_e}{R_f} \rho_e c_p \frac{d\bar{T}_e}{dt} + \frac{2q_e}{R_f} = n(t) \quad (10)$$

At this point, the assumption is made that the temperature response of the emitter follows that of the average fuel without delay. This is equivalent

to taking the time derivative of Eq. (6) noting that the right hand side is just a constant. This approximation would be expected to be good for the case of a fuel material with relatively high thermal conductivity, such as UC, together with an ideal thermal bond between fuel and emitter. For UO_2 and/or a relatively poor thermal bond between regions, i.e., if the thermal resistance to heat flow is significantly increased, there would be a response delay time between regions and consequently a breakdown of the validity of the model. This aspect will be treated subsequently. Introducing into Eq. (10) the assumption that

$$\frac{d\bar{T}_f}{dt} = \frac{d\bar{T}_e}{dt} \quad (6')$$

we obtain

$$\left(\rho_f c_p + \frac{2t_e}{R_f} \rho_e c_p \right) \frac{d\bar{T}_e}{dt} + \frac{2q_e}{R_f} = n(t) \quad (11)$$

where the equation is written in terms of the average emitter temperature (as opposed to the fuel temperature) since the heat loss term, q_e , is determined by this temperature. Clearly, this equation could have been written down immediately, since the temperature change in each region is the same and its coefficient is just the combined heat capacity.

The linearization of the heat loss, q_e , is assumed to be of the form

$$H(\bar{T}_e - T_b) \quad (12)$$

where T_b is an empirical constant. The justification for this assumption arises from observations noted in two independent studies. Gronroos⁽²⁾ used a Richardson-Dushman type equation to describe electron emission from emitter and collector, introducing empirical "patchiness" factors for both surfaces to account for all the shortcomings of this model. In addition to the electron cooling, his model included thermal radiation, end and lead losses, and

cesium conduction. The result of his study, reproduced here as Fig. 2, indicates that in spite of the highly nonlinear nature of the equations describing the various energy transfer processes, the total energy transfer can very adequately be represented by a linear function of the absolute emitter temperature over a reasonably wide temperature band about the operating temperature.

Figure 2. Emitter heat flow rate vs. emitter temperature.

Wilkins⁽³⁾ has observed a similar behavior both experimentally and calculationaly, the latter from a considerably more detailed theoretical model than that used by Gronroos.

The heat transfer coefficient, H , is assumed to be a constant given by its steady state value of

$$H = \frac{q_{eo}}{\bar{T}_{eo} - T_b} = \frac{R_f}{2} \frac{n_o}{\bar{T}_{eo} - T_b} \quad (13)$$

the latter relation stemming from the steady state energy balance on the fuel. Here $\bar{T}_{eo}$ is the initial steady-state emitter temperature.

Defining T_1 as the emitter temperature deviation from its steady state value

$$T_1(t) = \bar{T}_e(t) - \bar{T}_{eo} \quad (14)$$

Eq. (11) may be written

$$\left(\rho_f c_p f + \frac{2t_e}{R_f} \rho_e c_p e \right) \frac{dT_1}{dt} = (n - n_o) - \frac{n_o}{\bar{T}_{eo} - T_b} T_1 \quad (15)$$

Comparison of Eqs. (1) and (15) reveals that

$$b = \left(\rho_f c_p f + \frac{2t_e}{R_f} \rho_e c_p e \right)^{-1} \quad (16)$$

$$\eta_1 = \frac{n_o b}{\bar{T}_{eo} - T_b} = \frac{2Hb}{R_f} \quad (17)$$

For the collector side of the diode, it is assumed that an energy balance on the collector alone suffices, the thermal energy loss being through the combined thermal resistance of the thin layers of insulator, clad, coolant film, and the electrical loss being a fraction of the net energy input from the emitter.

$$\begin{aligned} t_c \rho_c c_p c \frac{d\bar{T}_c}{dt} + U(\bar{T}_c - T_o) &= q_e (1 - \eta_e) \\ &= H(1 - \eta_e)(\bar{T}_e - T_b) \end{aligned} \quad (18)$$

Here, subscript c refers to the collector material only, U is the overall heat transfer coefficient for heat loss to the coolant, η_e is the electrical efficiency of the diode, and T_o is the bulk coolant temperature (constant temperature reservoir).

The electrical efficiency, η_e , is just the ratio of the net electrical energy from the diode to the total energy transmitted from the emitter and is of order 10-20%.* Since the electrical energy is small by comparison with the thermal heat flux, the neglect of Joule heating and electrical load perturbations should not introduce any significant error with regard to the stability problem.

The overall heat transfer coefficient, U , is given by

$$U = \left[\frac{1}{2} \frac{t_c}{k_c} + \frac{t_i}{k_i} + \frac{t_{cl}}{k_{cl}} + \frac{1}{h} \right]^{-1} \quad (19)$$

where the subscripts i and cl refer to insulator and clad, and h is the film heat transfer coefficient between clad and coolant. The factor of $\frac{1}{2}$ preceding the collector thermal resistance arises from assuming the average collector temperature lies at the half-thickness plane.

Defining T_2 as the deviation in average collector temperature from its steady state value

$$T_2(t) = \bar{T}_c(t) - \bar{T}_{co} \quad (20)$$

and incorporating (20) into (18) noting that at steady state

$$U[\bar{T}_{co} - T_o] = H(1 - \eta_e)(\bar{T}_{co} - T_b) \quad (21)$$

* This is not to be confused with the electron cooling fraction which is more like 50-60%.

we obtain

$$t_c \rho_c c_{p_c} \frac{dT_2}{dt} + UT_2 = H(1-\eta_e)T_1$$

or

$$\frac{dT_2}{dt} = \frac{H(1-\eta_e)}{t_c \rho_c c_{p_c}} T_1 - \frac{U}{t_c \rho_c c_{p_c}} T_2 \quad (22)$$

Comparison of (22) with (2) yields

$$\eta_2 = \frac{H(1-\eta_e)}{t_c \rho_c c_{p_c}} \quad (23)$$

$$\eta_3 = \frac{U}{t_c \rho_c c_{p_c}} \quad (24)$$

Finally, in this model, the reactivity feedback is given by the sum of the fuel and emitter coefficients times T_1 for region 1, and by the collector coefficient times T_2 for region 2:

$$\rho_f = (\alpha_f + \alpha_e)T_1 + \alpha_c T_2 = \alpha_1 T_1 + \alpha_2 T_2 \quad (25)$$

where the α 's are the temperature coefficients of reactivity of the various major regions. The summation of fuel and emitter coefficients simply implies that in this model the fuel and the emitter act as an entity, consistent with Eq. (6').

The deficiencies of this model are not as significant as they might at first appear. More importantly, the deficiencies can be shown to lead to conservative results, but not so much so that the value of the model is destroyed. Finally, we note that inadequacies and inaccuracies of the input parameters manifest themselves in the time constants $\eta_{1,2,3}$. By examining the manner in which these constants enter into the stability criteria, an

appreciation for the influence of these inadequacies, as well as the influence of various materials, diode dimensions, operating conditions, etc., is rapidly and easily obtained.

D. LINEAR STABILITY REQUIREMENTS FOR VARIOUS POSSIBLE COMBINATIONS OF DIODE DIMENSIONS, COMPOSITIONS, AND POWER DENSITY LEVELS.

In this section are presented the stability criteria, derived earlier,⁽¹⁾ in terms of the parameters of the two-temperature region model. The criteria are evaluated for minimum, maximum, and reference values of the diode parameters determined from considerations of plausible ranges of dimensional, compositional, and operational power density level variations. For the reference values, a comparison of the linear stability regimes obtained from the use of the prompt-jump and effective-lifetime neutron dynamics models is given. Stability results are presented in terms of requirements on the temperature coefficients of reactivity of the combined fuel-emitter and the collector. Finally, a comparison and discussion of the temperature coefficient requirements presented here with those obtained by Gronroos⁽²⁾ for the JPL multi-region model are given.

Stability Requirements

The stability requirements in this model arise from applying the Routh-Hurwitz stability criterion to the linearized dynamic equations. This was presented in detail in the previous report⁽¹⁾, and only a brief review will be given here.

The temperature dynamic equations are given by Eqs. (1) and (2). The neutron dynamics are approximated by the point reactor linearized "prompt-jump" (PJ) approximation, an approximation known to be extremely good for

reactors characterized by a small prompt neutron lifetime, i.e., fast reactors. The effect of delayed neutrons is determined by comparing results obtained using the PJ approximation with those obtained from a simpler approximation, the "effective lifetime" (EL) approximation. This latter model attempts to correct for ignoring delayed neutrons through the substitution of a delayed-neutron lifetime for the prompt neutron lifetime. The reactor and feedback transfer functions are determined through Laplace transformations of the dynamic equations, and the Routh-Hurwitz stability criterion is applied to give the following linear stability requirements.

Prompt-Jump Approximation

$$\frac{\alpha_1 b n_0}{\beta} + \eta_1 + \eta_3 > 0 \quad (26)$$

$$\frac{\alpha_1 b n_0}{\beta} \left[\lambda + \eta_3 + \frac{\alpha_2}{\alpha_1} \eta_2 - \frac{\lambda(\eta_3 + \frac{\alpha_2}{\alpha_1} \eta_2)}{\frac{\alpha_1 b n_0}{\beta} + \eta_1 + \eta_3} \right] + \eta_1 \eta_3 > 0 \quad (27)$$

$$\alpha_1 \eta_3 + \alpha_2 \eta_2 > 0 \quad (28)$$

Effective Lifetime Approximation

$$\eta_1 \eta_3 + \frac{\lambda b n_0}{\beta(\eta_1 + \eta_3)} (\alpha_1 \eta_1 - \alpha_2 \eta_2) \quad (29)$$

$$\alpha_1 \eta_3 + \alpha_2 \eta_2 > 0 \quad (28)$$

where β is the effective delayed neutron fraction and λ is an appropriate mean decay constant for delayed neutron precursors. These requirements are based on the sign convention that the reactivity feedback is the negative of the reactivity insertion, and therefore a positive temperature coefficient actually corresponds to negative reactivity feedback, consistent with earlier work.

Eq. (28), which is the static power coefficient requirement, results from either model. Condition (29), linear in α_1 and α_2 , replaces the two conditions, (26) and (27). Condition (26) is a linear requirement on α_1 only, while (27) is hyperbolic in α_1 and α_2 .

Two-region Parameters

The parameters appearing in conditions (26)-(29) are estimated for ranges of configurations, compositions, and operational power densities. A "reference" diode is indicated for comparative purposes. These ranges are given in Table I, and all apply to cylindrical, centrally-fueled diodes.

TABLE I
DIODE PARAMETERS

		Reference Material	Dimensions, Inches	Reference Dimensions Inches
Fuel	UC, UO ₂	UO ₂	0.5-1.1 OD	1.10 OD
Emitter	W	W	0.03-0.05	0.03
Collector	Nb, Mo, Ni	Nb	0.02-0.03	0.02
Insulator	Al ₂ O ₃	Al ₂ O ₃	0.01-0.02	0.01
Clad	Nb	Nb	0.02-0.03	0.02
Coolant	Li, NaK	Li		
		<u>Range</u>		<u>Reference Value</u>
Fuel Power Density, w/cc		100-500		100
Emitter Surface Heat Flux, w/cm ²		50-150		70
Emitter Temperature, °K		2000		2000
Coolant Temperature, °K		1000		1000
Electrical Efficiency, %		5-15		12

The property values used are given in the appendix at the end of this section, together with the diode combinations of materials, dimensions, and power density levels resulting in the extremum values of the parameters appearing in the stability requirements. These values are summarized in Table II.

TABLE II
PARAMETERS FOR STABILITY EQUATIONS

	<u>Min</u>	<u>Max</u>	<u>Ref</u>
η_1, sec^{-1}	0.031	0.12	0.031
η_2, sec^{-1}	0.172	1.35	0.55
η_3, sec^{-1}	1.66	8.65	8.65
η_2/η_3	0.06	0.19	0.06
$bn_o, ^\circ\text{K-sec}^{-1}$	26	100	26
β	0.0027	0.007	0.007
λ, sec^{-1}	0.054	0.077	0.077

The minimum β and λ values arise from the consideration of U-233 as a possible fuel material. The maximum and reference values are for U-235. The mean decay time λ^{-1} derives from

$$\frac{1}{\lambda} = \frac{1}{\beta} \sum_i \frac{\beta_i}{\lambda_i}$$

The ratio η_2/η_3 is determined on a consistent basis of materials, etc., and hence does not necessarily correspond simply to the ratio of the extreme values of η_2 and η_3 separately.

Using conditions (26) - (29), a stability plot is developed in Fig. 3 for the reference value parameters and for the two neutron dynamics approximations. The coordinates are taken as previously defined:

$$x = \alpha_1 \frac{bn_o}{\beta} \quad y = \alpha_2 \frac{\eta_2}{\eta_3} \frac{bn_o}{\beta}$$

From this plot, it is immediately obvious that the EL model is unnecessarily restrictive in indicating regions of linear stability for incore thermionic reactors. This conclusion is also verified by analog computer studies.

Restricting our attention to the prompt-jump approximation, further simplifications can be introduced by incorporating information with regard to the anticipated magnitudes of the temperature coefficients.

Considering first the stability requirement of Eq. (26): if α_1 is positive (corresponding to a negative combined fuel-emitter temperature coefficient), then (26) is always satisfied, since all time constants and $\frac{bn_o}{\beta}$ are all positive. If α_1 is negative, then its magnitude must satisfy

$$|\alpha_1| < \frac{\beta}{bn_o} (\eta_1 + \eta_3) .$$

On using the minimum values for β , η_1 , and η_3 , and the maximum bn_o , we find that

$$|\alpha_1| < 45 \times 10^{-6}$$

This value is at least an order of magnitude and perhaps two orders larger than any estimate made to date for the combined coefficient.^(2,4) Therefore, Eq. (26) may be considered to be satisfied under all probable circumstances.

Considering next Eq. (27), we note from the remarks with regard to (26) and Table II that

$$\eta_1 \ll \eta_3$$

Figure 3. Regions of linear stability for thermionic fast-spectrum reactors.

$$\frac{\alpha_1 b n_o}{\beta} \ll \eta_3$$

so that (27) may be written approximately as

$$\alpha_1 \eta_3 + \alpha_2 \eta_2 \left(1 - \frac{\lambda}{\eta_3}\right) + \frac{\beta}{b n_o} \eta_1 \eta_3 > 0 \quad (30)$$

From Table II, we observe that

$$\frac{\lambda}{\eta_3} \ll 1$$

so that (27) actually reduces to a less stringent stability requirement than (28), since $\frac{\beta}{b n_o} \eta_1 \eta_3$ is always positive.

In other words, for the values of the parameters of interest for thermionic reactor concepts, the stability requirements reduce to the single requirement given by Eq. (28), which translates physically to the requirement that the static power coefficient be negative. This result was postulated earlier for the concept studied by Gronroos⁽²⁾; here it is proven for all cases of interest.

Since the collector materials always introduce negative reactivity feedback, α_2 is always positive. Consequently, linear stability is assured if α_1 is positive (again, this actually means a negative combined fuel emitter temperature coefficient) or if negative, its magnitude is less than

$$|\alpha_1| < \frac{\eta_2}{\eta_3} \alpha_2$$

Inserting the values of the ratio $\frac{\eta_2}{\eta_3}$ from Table II, we see that this corresponds to

$$|\alpha_1| < (0.06 \text{ to } 0.19) \alpha_2 \quad (31)$$

or that the stabilizing effect of the collector coefficient is at most less than a 20% correction to considering just the sum of the fuel plus emitter coefficients, α_1 , alone.

A comparison of this simple result with that obtained by Gronroos⁽²⁾ with his considerably more detailed computer solution is shown in Fig. 4. This is for a UC-fueled diode with a fuel radius of 1.50 cm and emitter thickness of 0.100 cm. The ratio η_2/η_3 is calculated to be ~ 0.11 . The three curves plotted are for different values of the emitter coefficient as indicated.

Note first that the result given by Eq. (31) is conservative as postulated, except at the limit of zero collector feedback coefficient where they are equal. Next note that Gronroos' results pertain to the case of no coolant transport delay. This means that the coolant temperature is changing with time (instantaneously from reactor outlet to reactor inlet) and consequently is introducing reactivity feedback effects from clad and coolant which enhance the collector-side reactivity feedback. This has the overall effect of allowing a more positive (in the real sense) combined fuel-emitter temperature coefficient. This is, of course, a highly unrealistic situation and cannot be used for design considerations. Gronroos has also performed limited calculations⁽⁵⁾ with an infinite coolant transport delay which correspond more directly with the assumptions behind Eq. (31). These results are unreported and unavailable but tend toward the result given by Eq. (31). The other interesting feature of this comparison is that the asymptotic slope at large values of the collector coefficient is essentially identical to that predicted by Eq. (31). In other words, for small collector

Figure 4. Comparison of stability requirements for the JPL UC-fueled concept (no coolant transport delay time).

temperature coefficients, stabilization is achieved principally by the feedback from clad and coolant, this feedback being considerably enhanced by the no-coolant-transport-delay assumption. As the collector feedback coefficient is increased in magnitude, it becomes predominant in determining the allowable fuel-emitter coefficient, and the asymptotic nature predicted by Eq. (31) is realized.

The stability requirement given by Eq. (31) has several important implications:

- 1) The design of the diode has little effect on the stability of the reactor provided that the sum of the fuel and emitter temperature coefficients is positive (negative feedback).
- 2) Refinements to the energy transfer processes on the collector side of the diode are unnecessary for stability considerations. This includes such considerations as Joule heating in the collector, thermal bonding between collector, insulator, and clad, and the magnitude of the coolant film heat transfer coefficient.
- 3) The neglect of the temperature coefficients of clad and coolant is not only conservative but also unimportant for the case of constant coolant bulk temperature. For the case of coolant loop feedback, these coefficients tend to increase the stability, i.e., increase the allowable combined fuel-emitter coefficient.
- 4) Because of the importance of requiring the combined fuel-emitter coefficient to provide essentially zero or a net negative reactivity feedback, a more detailed examination of the assumptions of the lumped-fuel model and the ideal thermal bond between fuel and emitter is warranted.

APPENDIX TO SECTION I D

1) Material properties (evaluated at reference temperatures)

<u>Material</u>	<u>ρ, gm/cc</u>	<u>c_p, $\frac{\text{watt-sec}}{\text{gm-}^\circ\text{C}}$</u>	<u>k, $\frac{\text{watt}}{\text{cm-}^\circ\text{C}}$</u>	<u>μ, $\frac{\text{gm}}{\text{cm-sec}}$</u>
UO ₂	10.9	0.314	0.017	
UC	13.6	0.272	0.340	
W	18.6	0.167		
Nb	8.4	0.31	0.63	
Mo	9.0	0.31	1.17	
Ni	8.6	0.63	0.28	
Al ₂ O ₃			0.069	
Li	0.465	4.15	0.485	0.00268
NaK	0.705	0.88	0.256	0.00145

2) Minimum, maximum, and reference values of the parameters b , U , η_1 , η_2 , and η_3

$$b = [\rho_f c_{p_f} + \frac{2t_e}{R_f} \rho_e c_{p_e}]^{-1}$$

	<u>$b_{\min}$</u>	<u>$b_{\max}$</u>	<u>b_{ref}</u>
Value, $\frac{\text{cc-}^\circ\text{C}}{\text{watt-sec}}$	0.20	0.26	0.26
Fuel	UC	UO ₂	UO ₂
R_f , cm	0.635	1.4	1.4
t_e , cm	0.127	0.076	0.076

$$U = \left[\frac{1}{2} \frac{t_c}{k_c} + \frac{t_i}{k_i} + \frac{t_{cl}}{k_{cl}} + \frac{1}{h} \right]^{-1}$$

	<u>U_{min}</u>	<u>U_{max}</u>	<u>U_{ref}</u>
Value, watts/cm ² -°C	0.68	1.15	1.15
Collector Material	Ni	Nb	Nb
Collector Thickness, cm	0.076	0.051	0.051
Insulator Material	Al ₂ O ₃	Al ₂ O ₃	Al ₂ O ₃
Insulator Thickness, cm	0.051	0.0254	0.0254
Clad Material	Nb	Nb	Nb
Clad Thickness, cm	0.076	0.051	0.051
Coolant Material	Li(laminar)	Li(turbulent)	Li(turbulent)
Coolant Nussult No.	3.0	6.7	6.7

$$\eta_1 = \frac{n_o b}{\bar{T}_{eo} - T_b} \quad \bar{T}_{eo} = 2000^\circ K \quad T_b = 1160^\circ K^*$$

Note: For minimum η_1 , the product of $(n_o b)$ must be minimum. For consistency, this turns out to require $(\eta_o)_{\min} b_{\max}$ for the minimum product due to the limitation of surface heat flux. Similarly, maximum η_1 will be given by $(n_o)_{\max} b_{\min}$.

	<u>η_1 min</u>	<u>η_1 max</u>	<u>η_1 ref</u>
Value, sec ⁻¹	0.031	0.12	0.031
b, $\frac{\text{cc-}^\circ\text{C}}{\text{watt-sec}}$	0.26	0.20	0.26
n_o , $\frac{\text{watts}}{\text{cc-fuel}}$	100	500	100

* From Gronroos⁽²⁾, Fig. 30.

$$\eta_2 = \frac{R_f n_o (1 - \eta_e)}{2 t_c \rho_c c_{p_c} (T_{eo} - T_b)}$$

	$\eta_2 \text{ min}$	$\eta_2 \text{ max}$	$\eta_2 \text{ ref}$
Value, sec^{-1}	0.172	1.35	0.55
Collector Material	Ni	Nb	Nb
Collector Thickness, cm	0.076	0.051	0.051
R_f , cm	1.40	0.635	1.40
n_o , watts/cc	100	500	100
η_e , %	15	5	12

$$\eta_3 = \frac{U}{t_c \rho_c c_{p_c}}$$

	$\eta_3 \text{ min}$	$\eta_3 \text{ max}$	$\eta_3 \text{ ref}$
Value, sec^{-1}	1.66	8.65	8.65
Collector Material	Ni	Nb	Nb
Collector Thickness, cm	0.076	0.051	0.051
U	U_{min}	U_{max}	U_{ref}

$$\eta_2 / \eta_3 = \frac{R_f n_o (1 - \eta_e)}{2 U (T_{eo} - T_b)}$$

	$\left(\frac{\eta_2}{\eta_3}\right) \text{ min}$	$\left(\frac{\eta_2}{\eta_3}\right) \text{ max}$	$\left(\frac{\eta_2}{\eta_3}\right) \text{ ref}$
Value	0.06	0.19	.063
R_f , cm	1.4	0.635	1.4
n_o , watts/cc	100	500	100
η_e , %	15	5	12
U	U_{max}	0.94*	U_{ref}

* Minimum U with turbulent flow to accomodate high power density.

E. FREQUENCY LIMITATION OF THE LUMPED-FUEL MODEL

In Section C the assumption was made that the average fuel temperature was given by the spatially averaged steady-state parabolic distribution. The emitter temperature was assumed equal to the fuel surface temperature and related to the average fuel temperature by the steady-state value according to Eq. (5). By these assumptions the fuel and emitter temperatures rise and fall in complete unison and by equal amounts. For this reason the fuel and emitter can be treated as a single entity, simply adding heat capacities and temperature coefficients.

In rapid transients or power perturbations, the fuel surface temperature and consequently the emitter temperature may actually be quite different from that obtained under the above assumption due to the finite thermal diffusivity of the fuel material or a poor thermal bond between fuel and emitter. If the transient gives rise to a fuel-average temperature change considerably different from the corresponding emitter temperature change, and if the fuel temperature coefficient leads to positive reactivity feedback, then an instability may arise. This effect will now be considered.

The procedure is to transform the exact space-time heat conduction equation to the Laplace or frequency domain, solve the resulting equation for the spatial dependence, compute the frequency-dependent average temperature, and then determine under what conditions this solution compares with the corresponding lumped-fuel solution. As before, infinite cylindrical geometry, constant properties, and a spatially constant heat generation rate are assumed. Initially, a perfect thermal bond between fuel and emitter are also assumed. Under these conditions, the conduction equation may be written

$$\frac{\partial T_f(r,t)}{\partial t} - \frac{a_f}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_f}{\partial r} \right) = b_f n(t) \quad (32)$$

where $a_f = k_f / \rho_f c_{p_f}$ is the thermal diffusivity and b_f is the reciprocal volumetric heat capacity of the fuel, $b_f = (\rho_f c_{p_f})^{-1}$.

Let

$$\begin{aligned} \delta T_f(r,t) &= T_f(r,t) - T_f(r,0) \\ \delta n(t) &= n(t) - n_0 \end{aligned} \quad (33)$$

In these variables Eq. (32) becomes

$$\frac{\partial \delta T_f(r,t)}{\partial t} - \frac{a_f}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta T_f}{\partial r} \right) = b_f \delta n \quad (34)$$

where the steady state relation between $T_f(r,0)$ and n_0 has been employed.

Define the Laplace transforms of δT_f and δn by

$$\begin{aligned} \theta_f(r,s) &= L\{\delta T_f(r,t)\} \\ N(s) &= L\{\delta n(t)\} \end{aligned} \quad (35)$$

Transforming Eq. (34) utilizing the definitions (35), we obtain

$$\frac{d^2 \theta_f(r,s)}{dr^2} + \frac{1}{r} \frac{d\theta_f}{dr} - p(s) \theta_f = - \frac{N(s)}{k_f} \quad (36)$$

where

$$p(s) = s/a_f$$

Eq. (36) is a modified Bessel equation of order zero which can be shown to have the general solution

$$\theta_f(r,s) = C_1(s) I_0(\sqrt{p} r) + C_2(s) K_0(\sqrt{p} r) + \frac{b_f N(s)}{s} \quad (38)$$

where I_0 and K_0 are the modified Bessel function of the first and second kinds respectively.

If the fuel extends to $r = 0$ (i.e. is not annular between some finite minimum radius and R_f), then $C_2(s) = 0$ since $K_o(x) \rightarrow \infty$ as $x \rightarrow 0$. The function $C_1(s)$ is evaluated in terms of the surface temperature. Setting $r = R_f$ in Eq. (38) and solving for $C_1(s)$, we obtain

$$C_1(s) = [\theta_f(R_f, s) - \frac{b_f N(s)}{s}] \frac{1}{I_o(\sqrt{p} R_f)} \quad (39)$$

so that (38) becomes

$$\theta_f(r, s) = \frac{I_o(\sqrt{p} r)}{I_o(\sqrt{p} R_f)} [\theta_f(R_f, s) - \frac{b_f N(s)}{s}] + \frac{b_f N(s)}{s} \quad (40)$$

In turn, $\theta_f(R_f, s)$ is determined from an energy balance on the emitter, still assuming no temperature drop across the emitter and continuity of temperature and heat flux between fuel and emitter.

The energy balance is given by

$$t_e \frac{dT_f(R_f, t)}{dt} = -k_f b_e \left. \frac{dT_f(r, t)}{dr} \right|_{r=R_f} - b_e H [T_f(R_f, t) - T_b] \quad (41)$$

Here b_e is the reciprocal volumetric heat capacity of the emitter and H is as defined in Section C.

For simplicity define

$$\left. \begin{aligned} c_1 &= \frac{k_f b_e}{t_e} \\ c_2 &= \frac{b_e H}{t_e} \end{aligned} \right\} \quad (42)$$

Utilizing the definitions (33) and (42), Eq. (41) becomes

$$\frac{d\delta T_f(R_f, t)}{dt} = -c_1 \left. \frac{\partial \delta T_f(r, t)}{\partial r} \right|_{r=R_f} - c_2 \delta T_f(R_f, t) \quad (43)$$

On taking the Laplace transform of (43)

$$s\theta_f(R_f, s) = -c_1 \left. \frac{\partial \theta_f(r, s)}{\partial r} \right|_{r=R_f} - c_2 \theta_f(R_f, s) \quad (44)$$

Taking the spatial derivative of (40), evaluating at $r = R_f$, inserting this result into (44) and solving for $\theta_f(R_f, s)$ results in

$$\theta_f(R_f, s) = \frac{c_1 b_f N(s)}{s} \sqrt{p} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)} [s + c_2 + c_1 \sqrt{p} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)}]^{-1} = \ell(s) \frac{b_f N(s)}{s} \quad (45)$$

where

$$\ell(s) = c_1 \sqrt{p} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)} [s + c_2 + c_1 \sqrt{p} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)}]^{-1} \quad (46)$$

Finally, inserting (45) into (40) gives for the transformed spatially dependent fuel temperature

$$\theta_f(r, s) = \frac{b_f N(s)}{s} \left\{ 1 + \frac{I_0(\sqrt{p} r)}{I_0(\sqrt{p} R_f)} [\ell(s) - 1] \right\} \quad (47)$$

The spatially averaged temperature is defined by

$$\bar{T}_f(t) = \frac{2}{R_f} \int_0^{R_f} T(r, s) r dr \quad (48)$$

and the corresponding Laplace transform of the deviation from the steady state value by

$$\bar{\theta}_f(s) = \frac{2}{R_f} \int_0^{R_f} \theta_f(r, s) r dr \quad (49)$$

Inserting (47) into (49) and performing the integration results in

$$\bar{\theta}_f(s) = \frac{b_f N(s)}{s} \left\{ 1 + [\ell(s) - 1] \frac{2}{R_f \sqrt{p}} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)} \right\} \quad (50)$$

Since the emitter is assumed to have a spatially constant temperature, Eq. (45) already represents the transformed average emitter temperature. By Eq. (6') these two transformed temperatures are assumed equal in the lumped fuel model. Comparison of (45) and (50) shows that for the exact model to approach the lumped-fuel model,

$$1 + [\ell(s)-1] \frac{2}{R_f \sqrt{p}} \frac{I_1(\sqrt{p} R_f)}{I_0(\sqrt{p} R_f)} = \ell(s)$$

which is realized only for small values of the Bessel function arguments.

For small arguments, the following expansions are valid

$$I_0(x) \approx 1$$

$$I_1(x) \approx x/2.$$

Introduction of these values into (45) and (50) produces the lumped-fuel equations previously used.

A plot of $\frac{I_1(x)}{I_0(x)}$ vs x is shown in Fig. 5 together with the asymptotic value for small x of $x/2$. For $x = 1$, the error between actual and asymptotic values is seen to be approximately 10%. Arbitrarily taking this error value as an upper limit for the validity of the lumped-fuel model results in the requirement that the perturbation frequency satisfy the condition

$$\omega < \frac{a_f}{R_f^2}$$

where ω is the frequency of the perturbation in radians/sec. For UC, a_f is of the order of $0.1 \text{ cm}^2 \text{ sec}^{-1}$ so that for a 1 cm radius fuel element, the lumped-fuel model begins to break down at frequencies of order 0.1 radians/sec. For oxide fuels the thermal diffusivity is an order of magnitude smaller and consequently (for the same size fuel rod) the lumped-fuel model becomes invalid at frequencies of order 0.01 radians/sec. These frequencies are, of course, comparable to the decay constants of the delayed neutron precursors and consequently indicate that the influence of this effect should be examined in more detail. There is, too, the implication that since fuel temperature changes are not accompanied by simultaneous emitter temperature changes of equal magnitude, the stability requirement of the lumped fuel

Figure 5. $I_1(x)/I_0(x)$ and $x/2$ vs. x .

model, Eq. (31), is no longer a conservative requirement. These aspects must be examined utilizing Eq. (50) to describe the average fuel temperature behavior as a function of frequency rather than the lumped-fuel average temperature equation of the previous section. This will be carried out during the next report period.

F. THE EFFECT ON STABILITY OF A FINITE CONTACT THERMAL RESISTANCE BETWEEN FUEL AND EMITTER

The thermal resistance between the fuel and emitter will depend on the method of fabrication or assembly and may well differ significantly from diode to diode. Before attempting to estimate the bounds of this resistance, we shall first investigate how such a resistance enters into the stability equations. Intuitively, the resistance would appear to isolate further the fuel from the rest of the diode, causing the positive reactivity feedback from the fuel to have an increased significance over the delayed negative feedback from emitter and collector.

This effect can be investigated rather easily by means of the model derived in Section C. Instead of the emitter temperature being equated to the fuel surface temperature, however, a temperature "jump" given by the boundary condition

$$H_R [T_f(R_f, t) - \bar{T}_e(t)] = -k_f \frac{dT_f(R_f, t)}{dr} \quad (51)$$

is assumed. Here, H_R is the contact conductance.

We assume that H_R is time-independent and given by the steady state value of heat loss from the diode:

$$H_R [T_f(R_f, 0) - \bar{T}_{e0}] = H(\bar{T}_{e0} - T_b) \quad (52)$$

Thus,

$$\bar{T}_{eo} = \frac{H_R}{H_R + H} T_f(R_f, 0) + \frac{H}{H + H_R} T_b \quad (53)$$

that is, the emitter temperature approaches the limit of the fuel surface temperature as H_R approaches infinity (no contact resistance).

Using (53) and the previously developed equations, the stability requirements can be shown to reduce to

$$\alpha_e + \alpha_f \left(1 + \frac{H}{H_R}\right) > 0 \quad (54)$$

when the collector feedback is ignored. In other words, the fuel coefficient is effectively increased by the factor $1 + \frac{H}{H_R}$ over the case of perfect thermal contact because of the higher average fuel temperature.

From Eq. (52)

$$\frac{H}{H_R} = \frac{T_f(R_f, 0) - \bar{T}_{eo}}{\bar{T}_{eo} - T_b}$$

so that for an emitter temperature of 2000°K , $T_b = 1160^\circ\text{K}$, and a temperature drop across the resistance of 200°K , the apparent effect is to increase the positive feedback by about 25%. This is still a relatively small correction and we conclude that a finite contact resistance between fuel and emitter will have only a minor effect on the stability requirements given previously by Eq. (30).

G. EFFECTS ON STABILITY INTRODUCED BY ANNULAR CYLINDRICAL FUEL SHAPES AS OPPOSED TO SOLID FUEL

For purposes of power shaping and maintaining fuel centerline temperatures below melting, holes may be incorporated in the fuel cylinders along the centerline. The effect on stability of this modification has been investigated by methods similar to the previous sections.

Without detailing the analysis, the previous results all apply provided the reciprocal heat capacity, b , and time constant, η_1 , are modified to reflect the change in volume of the fuel material.

$$b' = [\rho_f c_{pf} + \frac{2R_f t_e}{R_f^2 - R_0^2} \rho_e c_{pe}]^{-1} \quad (16')$$

$$\eta_1' = \frac{2HR_f b}{(R_f^2 - R_0^2)} \quad (17')$$

where R_0 is the hole radius and the primes denote modified values of the corresponding quantities previously defined.

Clearly, for a given R_f , η_1 is increased by the factor $[1 - (\frac{R_0}{R_f})^2]^{-1}$, while b is changed by less than half this amount. For example, for a 1" diameter rod with a 1/4" diameter hole, the effect is to increase η_1 by only 6%. Even for a 1/2" diameter rod and a 1/4" diameter hole, the effect amounts only to a 30% increase in η_1 and a 10% increase in b . These increases do not in any event change the stability requirement given by Eq. (28), i.e., they are not sufficient to introduce the additional requirements of Eqs. (26) and (27). Since Eq. (28) does not involve either b or η_1 , the stability requirement for the fuel with a hole is identical to that for solid fuel.

H. FUTURE WORK

The two-region model appears to retain the important dynamic features necessary to provide an adequate representation for stability studies while remaining sufficiently simple algebraically that physical insight

is not lost. The linear stability studies indicate that a single criterion, condition (31), suffices to dictate temperature coefficient requirements for inherent reactor stability. The effects on stability of design modifications have been shown to be simply evaluated using the two-region model. The only area not completely investigated concerns the effect on stability of relatively high frequency reactivity perturbations for the case of the fuel material having a low thermal diffusivity or being thermally insulated from the emitter material (in the sense of some form of contact resistance). This effect will be considered in subsequent studies, but is considered to be of relative unimportance and hence of low priority.

II. THE SECOND METHOD OF LIAPUNOV

A. SUMMARY

The variable gradient method is used to obtain a Liapunov function for the point reactor with delayed neutrons and one temperature region of reactivity feedback. The result is completely satisfactory and in agreement with the known stability requirement for this model. The method is then used to generate a Liapunov function for the two-temperature region model, also with delayed neutrons. The resulting nonlinear stability requirements are shown to be identical with those obtained from applying the Welton criterion.

For the time constants characteristic of thermionic reactors, the nonlinear criteria are highly restrictive. The Liapunov method does appear to offer the possibility of increasing the domain of stability and hence can conceivably be a less restrictive analysis technique than is Welton's criterion.

B. INTRODUCTION

During the last report period, three methods of generating Liapunov functions in a systematic manner were investigated. The objective was to determine the relative merits with regard to a possible application to the two-region nonlinear problem. The first of these methods, the method of Aizerman, was demonstrated to result in a highly restrictive domain of stability in the case of the simple reactor model whose stability domain was known. The second method, that of Szego, gave the correct exact answer, but it was observed that for systems containing more than two or

possibly three equations, the method becomes extremely complex and tends to lose its systematic nature. This method has been deferred in favor of the third method, the variable gradient method, which also gave the correct exact answer and in addition gave promise of retaining the systematics when applied to higher order systems.

C. SOLUTION FOR POINT REACTOR WITH DELAYED NEUTRONS

The next step, that of adding delayed neutrons, was initiated and the system equations to be studied were given by Eqs. (66), (67), and (68) of Ref. 1*, repeated here for convenience as Eqs. (55), (56), and (57):

$$\dot{x}_1 = -\omega x_1 + \lambda x_2 - \xi(n_0 + x_1)x_3 \quad (55)$$

$$\dot{x}_2 = \omega x_1 - \lambda x_2 \quad (56)$$

$$\dot{x}_3 = Kx_1 - \gamma x_3 \quad (57)$$

Before proceeding with the solution, a statement of the applicable stability theorem for the variable gradient method is in order.

Theorem (Ref. 6)

If for the system of equations $\dot{\underline{x}} = \underline{X}(\underline{x})$, with $\underline{X}(\underline{0}) = \underline{0}$, there exists a region Ω and a real vector function ∇V , with elements ∇V_i , such that

$$1) \quad (\partial \nabla V_i / \partial x_j) = (\partial \nabla V_j / \partial x_i),$$

$$2) \quad \nabla V \text{ is not zero at any point in } \Omega, \text{ except the origin,}$$

*Eq. (66) of Ref. 1 should read: $-\xi(n_0 + x_1)x_3 \dots$

$$3) \quad dV/dt = \nabla V' \dot{\underline{x}} \leq 0 \text{ for } \underline{x} \neq \underline{0}$$

$$= 0 \quad \text{for } \underline{x} = \underline{0}$$

$$4) \quad dV/dt \text{ is not identically zero in } \Omega \text{ on a solution of the system, other than at } \underline{x} = \underline{0},$$

and if the scalar function V , formed by a line integral of ∇V , as

$$V = \int_0^{\underline{x}} \nabla V' d\underline{x}$$

has the properties

$$5) \quad V(\underline{x}) > 0 \quad \underline{x} \neq 0, \text{ and } V(\underline{0}) = 0 \text{ (positive definite),}$$

$$6) \quad \text{one of the surfaces, } V = K_1, \text{ bounds } \Omega,$$

then the given system $\dot{\underline{x}} = X(\underline{x})$ is asymptotically stable in Ω .

We now proceed to apply this method to the system represented by Eqs. (55) - (57). The gradient ∇V is taken to be the vector

$$\nabla V = \begin{vmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \\ \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 \end{vmatrix}$$

We note from the previous example without delayed neutrons that the off-diagonal elements α_{ij} , $i \neq j$, were zero ($\alpha_{12} = \alpha_{21} = 0$). Since this introduces considerable simplification to the expression for $\dot{V}$, it is reasonable to attempt the same simplification here. Setting $\alpha_{ij} = 0$, $i \neq j$, the expression for $\dot{V} = \nabla V' \dot{\underline{x}}$ reduces to:

$$\dot{V} = \alpha_{11}x_1[-\xi(n_0 + x_1)x_3 - \omega x_1 + \lambda x_2] + \alpha_{22}x_2(\omega x_1 - \lambda x_2) + \alpha_{33}x_3(Kx_1 - \gamma x_3)$$

Dealing first with the coefficient of the x_1x_3 term and requiring it to be identically zero (as in the previous example with the x_1x_2 term) results again in

$$\alpha_{11} = \frac{K}{\xi} \frac{\alpha_{33}}{n_0 + x_1}$$

Again, as previously, let $\alpha_{33} = \frac{\xi}{K}$. The expression for $\dot{V}$ now becomes

$$\dot{V} = -\left[\frac{x_1}{n_0 + x_1} - \alpha_{22}x_2\right](\omega x_1 - \lambda x_2) - \frac{\xi\gamma}{K}x_3^2$$

One way in which the first term can be made positive definite is to determine α_{22} (which can be a function of $\underline{x}$) such that the numerator of the first bracketed term equals the second bracketed term while the denominator remains positive definite. This is accomplished by letting

$$\alpha_{22}(x_2) = \frac{f(x_2)}{g(x_2)}$$

The term in the first brackets becomes

$$\frac{x_1}{n_0 + x_1} - \frac{f(x_2)}{g(x_2)} x_2 = \frac{x_1 g(x_2) - n_0 f(x_2) x_2 - x_1 x_2 f(x_2)}{(n_0 + x_1) g(x_2)}$$

Requiring the numerator to equal the second bracketed term:

$$x_1 [g(x_2) - x_2 f(x_2)] - n_0 f(x_2) x_2 = \omega x_1 - \lambda x_2$$

Clearly,

$$n_0 f(x_2) = \lambda$$

$$f(x_2) = \lambda / n_0$$

$$g(x_2) - \frac{x_2 \lambda}{n_0} = \omega$$

$$g(x_2) = \omega + \frac{\lambda}{n_0} x_2$$

$$\alpha_{22} = \frac{\lambda / n_0}{\omega + \frac{\lambda}{n_0} x_2} = \frac{1}{\frac{\omega n_0}{\lambda} + x_2}$$

Note that $\frac{\omega n_0}{\lambda} + x_2 > 0$ in exactly the same way $n_0 + x_1 > 0$ (from consideration of the values of x_1 and x_2 at the equilibrium point). Hence, the denominator of the first term is always positive definite and $\dot{V}$ is given by

$$\dot{V} = - \frac{n_0 (\omega x_1 - \lambda x_2)^2}{(n_0 + x_1)(\omega n_0 + \lambda x_2)} - \frac{\xi \gamma}{K} x_3^2 \quad (58)$$

The desired V-function is

$$V = \int_0^{x_1} \frac{x_1'}{n_0 + x_1'} dx_1' + \int_0^{x_2} \frac{x_2'}{\frac{\omega n_0}{\lambda} + x_2'} dx_2' + \frac{\xi}{K} \int_0^{x_3} x_3' dx_3'$$

or

$$V = x_1 - n_0 \log(1 + \frac{x_1}{n_0}) + x_2 - \frac{\omega}{\lambda} n_0 \log(1 + \frac{\lambda x_2}{\omega n_0}) + \frac{\xi}{2K} x_3^2 \quad (59)$$

By checking Eqs. (58) and (59) against the conditions 1-6 given in the theorem, the stability requirements are ascertained. Condition 1 is satisfied since all off-diagonal matrix elements of ∇V were set equal to zero. Condition 2 is satisfied since none of the diagonal elements of ∇V are zero. Condition 3 is satisfied for positive physical parameters n_0 , ω , λ , γ , and K and the requirement that $\xi \geq 0$. Skipping condition 4 for the moment, condition 5 requires that V be positive definite. If, in Eq. (59), $\xi = 0$, then V is only positive semi-definite, and condition 5 is not fulfilled. Hence, the stability requirement on ξ reduces to

$$\xi > 0$$

that is, the temperature coefficient of reactivity feedback must always be positive (resulting in negative feedback in the convention used here), a result which is, of course, well known for this system. From Eqs. (58) and (59), we observe that the $V = K_1$ bound on Ω occurs at infinity, that is, V approaches infinity either as $\underline{x} \rightarrow \infty$ or as $x_1 \rightarrow -n_0$ or $x_2 \rightarrow -\frac{\omega n_0}{\lambda}$.

Thus, prior to consideration of condition 4, the derived V -function indicates system stability in the half-space of interest, namely, for all positive values of the reactor power, provided only that the temperature coefficient is positive. The shutdown point, $(x_1 = -n_0, x_2 = -\frac{\omega n_0}{\lambda}, x_3 = -Kn_0/\gamma)$ is excluded from the stability domain.

To prove that condition 4 is met requires that $\dot{V}$ be set equal to zero and the resulting solution inserted into Eqs. (55) - (57). For the stability theorem to apply, $\dot{V}$ must equal zero in Ω only at the origin, $\underline{x} = 0$ (corresponding to the initial steady state power and temperature).

With $\dot{V}$ in Eq. (58) set equal to zero, then, since each term is positive definite, we require

$$\begin{aligned}\omega x_1 &= \lambda x_2 \\ x_3 &= 0\end{aligned}\quad (\gamma \neq 0) \tag{60}$$

Note that if γ is zero, the requirement that $x_3 = 0$ does not arise from $\dot{V} = 0$. (The possibility of $\xi = 0$ has already been eliminated.) This case will be treated subsequently. Considering the case of $\gamma \neq 0$, and substituting (60) into (55) - (57),

$$\dot{x}_1 = 0 \tag{61}$$

$$\dot{x}_2 = 0 \tag{62}$$

$$\dot{x}_3 = Kx_1 \tag{63}$$

But (61) implies $x_1 = \text{constant}$, and hence (63) can be integrated to give

$$x_3 = Kx_1 t = 0 \tag{64}$$

since $x_3 = 0$ from (60). Therefore

$$x_1 = x_2 = x_3 = 0$$

is the only point at which $\dot{V} = 0$ and condition 4 is satisfied.

If $\gamma = 0$, the results are altered just slightly, but are still consistent. Now, only the first of Eqs. (60) can be used and we have

$$\dot{x}_1 = -\xi(n_0 + x_1)x_3 \quad (\xi \neq 0) \tag{61'}$$

$$\dot{x}_2 = 0 \tag{62'}$$

$$\dot{x}_3 = Kx_1 \tag{63'}$$

But (60) implies $\dot{x}_1 = \frac{\lambda}{\omega} \dot{x}_2 = 0$, so that (61') becomes

$$(n_0 + x_1)x_3 = 0$$

That is, either $(n_0 + x_1) = 0$ or $x_3 = 0$.

If $x_1 = -n_0$, the reactor is shut down, and this point has already been excluded from the stability region Ω by condition 6. On the other hand, if $x_3 = 0$, then the result above for $\gamma \neq 0$ applies. Thus, condition 4 is satisfied for all $\gamma \geq 0$ and the conclusions stated above are unmodified.

The derived V-function and the stability requirements are not new. The requirement on the temperature coefficient has been demonstrated by other investigators and by other methods. (7,8) The V-function derived here is similar to that given by Ref. 7, where the V-function was simply presented without derivation and was presumably obtained by the investigator's intuition and experience. His result corresponds to the variable gradient approach with the a priori assumption that all off-diagonal coefficients are zero. While this happens to work for the single region problem given here, it does not, in general, work for higher order systems such as the two-temperature region model, to be considered next. The intuitional approach has purposely been avoided here, since the desire is to develop a systematic approach to generating V-functions applicable to reactor stability problems in general.

D. THE TWO-TEMPERATURE REGION PROBLEM

We now apply the variable gradient method to the reactor model of Sec. B, that of two spatial regions each characterized by its own temperature

coefficient of reactivity feedback, heat transfer from region 1 to region 2 (neglecting back emission) and from region 2 to a constant temperature reservoir, with the neutron dynamics represented by the point reactor model and one group of delayed neutrons. In terms of the real state variables:

$$\dot{n} = - \frac{(\beta - \rho)}{\ell} n + \lambda c \quad (65)$$

$$\dot{c} = \frac{\beta}{\ell} n - \lambda c \quad (66)$$

$$\dot{T}_1 = b(n - n_0) - \eta_1 T_1 \quad (67)$$

$$\dot{T}_2 = \eta_2 T_1 - \eta_3 T_2 \quad (68)$$

with the feedback given by

$$\rho = - \alpha_1 T_1 - \alpha_2 T_2 \quad (69)$$

Defining

$$x_1 = n - n_0 \quad (70)$$

$$x_2 = c - c_0 \quad (71)$$

$$x_3 = T_1 \quad (72)$$

$$x_4 = T_2 \quad (73)$$

$$\xi_1 = \alpha_1 / \ell \quad (74)$$

$$\xi_2 = \alpha_2 / \ell \quad (75)$$

$$\omega = \beta / \ell \quad (76)$$

Eqs. (65) - (68) become

$$\dot{x}_1 = - \omega x_1 + \lambda x_2 - (\xi_1 x_3 + \xi_2 x_4) (n_0 + x_1) \quad (77)$$

$$\dot{x}_2 = \omega x_1 - \lambda x_2 \quad (78)$$

$$\dot{x}_3 = bx_1 - \eta_1 x_3 \quad (79)$$

$$\dot{x}_4 = \eta_2 x_3 - \eta_3 x_4 \quad (80)$$

where the equilibrium value of $c_0 = \frac{\omega n_0}{\lambda}$ has been incorporated.

The gradient ∇V is now the vector

$$\nabla V = \begin{vmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{41}x_1 + \dots + \alpha_{44}x_4 \end{vmatrix}$$

and

$$\begin{aligned} \dot{V} = & (\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4) [-\omega x_1 + \lambda x_2 - (\xi_1 x_3 + \xi_2 x_4)(n_0 + x_1)] \\ & + (\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4) (\omega x_1 - \lambda x_2) \\ & + (\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \alpha_{34}x_4) (bx_1 - \eta_1 x_3) \\ & + (\alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4) (\eta_2 x_3 - \eta_3 x_4) \end{aligned}$$

where the α_{ij} may, in general, be functions of any or all of the variables in $\underline{x}$.

Although the form of Eq. (77) differs from Eq. (55) by an additional term, it nonetheless is reasonable to assume that α_{11} , α_{22} , and α_{33} will be the same as previously since the x_1 , x_2 , x_3 dependent portions of the first three equations are unchanged.

$$\alpha_{11} = \frac{1}{n_0 + x_1} \quad \alpha_{22} = \frac{1}{\frac{\omega n_0}{\lambda} + x_2} \quad \alpha_{33} = \frac{\xi_1}{b}$$

Assuming further that the coefficients of all triple cross product terms

$(x_i \ x_j \ x_k)$ are zero requires $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0$. In practice one assumes the α_{ij} are constants unless or until it proves necessary to do otherwise. Making this assumption, the curl requirement results in $\alpha_{21} = \alpha_{31} = \alpha_{41} = 0$. At this point $\dot{V}$ has been simplified to:

$$\begin{aligned} \dot{V} = & - \frac{n_0(\omega x_1 - \lambda x_2)^2}{(n_0 + x_1)(\omega n_0 + \lambda x_2)} - \frac{\xi_1 \eta_1}{b} x_3^2 - \xi_2 x_1 x_4 \\ & + (\alpha_{23} x_3 + \alpha_{24} x_4)(\omega x_1 - \lambda x_2) \\ & + (\alpha_{32} x_2 + \alpha_{34} x_4)(b x_1 - \eta_1 x_3) \\ & + (\alpha_{42} x_2 + \alpha_{43} x_3 + \alpha_{44} x_4)(\eta_2 x_3 - \eta_3 x_4) \end{aligned} \quad (81)$$

There is now only a single term involving the cross product $x_1 x_2$. Setting the coefficient $\alpha_{32} = 0$ eliminates this term, and again assuming the α_{ij} are constant, set $\alpha_{23} = 0$. This in turn leaves only a single term in $x_2 x_3$; hence set $\alpha_{42} = \alpha_{24} = 0$. Requiring the coefficient of $x_1 x_4$ to be zero results in

$$\alpha_{34} = \alpha_{43} = \frac{\xi_2}{b}$$

so that Eq. (81) becomes

$$\begin{aligned} \dot{V} = & - \frac{\eta_0 (\omega x_1 - \lambda x_2)^2}{(n_0 + x_1)(\omega n_0 + \lambda x_2)} \\ & - \left\{ \frac{(\xi_1 \eta_1 - \xi_2 \eta_2)}{b} x_3^2 + [(\eta_1 + \eta_3) \frac{\xi_2}{b} - \eta_2 \alpha_{44}] x_3 x_4 + \eta_3 \alpha_{44} x_4^2 \right\} \end{aligned} \quad (82)$$

Note that, as opposed to all the previous examples, this system requires a non-zero off-diagonal term, α_{34} . Further, note that it is not essential for the coefficient of the $x_3 x_4$ term to be zero since both x_3^2 and x_4^2 terms are present. Rather, Eq. (82) requires α_{44} to be determined such

that Sylvester's theorem, for the quadratic expression in x_3 and x_4 to be positive definite, is satisfied.

If the coefficient of x_3x_4 is simply set to zero, one can easily show that for stability the temperature coefficients α_1 and α_2 must both be positive, a highly restrictive requirement. By retaining the x_3x_4 term, the stability domain can be increased significantly.

The line integration of $\nabla V'$ yields:

$$V = x_1 - n_0 \log \left(1 + \frac{\xi_1}{n_0}\right) + x_2 - \frac{\omega n_0}{\lambda} \log \left(1 + \frac{\lambda x_2}{\omega n_0}\right) + \frac{\xi_1}{2b} x_3^2 + \frac{\xi_2}{b} x_3 x_4 + \frac{1}{2} \alpha_{44} x_4^2 \quad (83)$$

presuming α_{44} does not depend on $\underline{x}$. From the requirement that $V > 0$:

$$a) \quad \alpha_{44} > 0$$

$$b) \quad \xi_1 > 0$$

$$c) \quad \xi_1 \alpha_{44} > \xi_2^2 / b \quad (\text{from Sylvester's theorem})$$

while from $\dot{V} \leq 0$:

$$d) \quad \xi_1 \eta_1 - \xi_2 \eta_2 \geq 0$$

$$e) \quad \frac{1}{b} (\xi_1 \eta_1 - \xi_2 \eta_2) \eta_3 \alpha_{44} - \frac{1}{4} \left[\frac{\xi_2}{b} (\eta_1 + \eta_3) - \alpha_{44} \eta_2 \right]^2 \geq 0$$

We seek a value of α_{44} which, in addition to satisfying conditions

(a), (c), and (e), results in the largest region of stability.

Let

$$\gamma^2 = \frac{4}{b} (\xi_1 \eta_1 - \xi_2 \eta_2) \eta_3$$

$$\delta = \frac{\xi_2}{b} (\eta_1 + \eta_3)$$

then condition (e) becomes

$$e') \quad \gamma^2 \alpha_{44} - (\delta^2 - 2\delta\eta_2 \alpha_{44} + \eta_2^2 \alpha_{44}^2) \geq 0$$

Consider the locus of values α_{44} such that the equality of (e') is satisfied. Then

$$\alpha_{44}^{\pm} = \frac{1}{2\eta_2} [\gamma^2 + 2\delta\eta_2 \pm \sqrt{(\gamma^2 + 2\delta\eta_2)^2 - 4\delta^2\eta_2^2}]$$

where the $\alpha_{44}^{\pm}$ indicates the two possible roots. The argument of the square root simplifies to:

$$\gamma^2(\gamma^2 + 4\delta\eta_2)$$

For this argument to be non-negative (so that the square root will be real) requires

$$\gamma^2 + 4\delta\eta_2 \geq 0$$

or

$$f') \quad \xi_1\eta_3 + \xi_2\eta_2 \geq 0$$

which, we recall, is equivalent to the static power coefficient requirement obtained from the linear system study.

Next, we observe that α_{44}^+ must result in a larger range of allowable negative values for δ , while still fulfilling condition (a), than does α_{44}^- . Hence, let

$$\alpha_{44} = \alpha_{44}^+ + \varepsilon > 0$$

and substitute into (e'). After cancelling those terms which result in

the equality in (e'), there arises the following requirement for ε :

$$\varepsilon < \frac{1}{2} \frac{\sqrt{\gamma^2 (\gamma^2 + 4\delta\eta_2)}}{\eta_2}$$

Since ε can be any negative value, select

$$\varepsilon = -\frac{1}{4} \frac{\gamma^2}{\eta_2}$$

The expression for α_{44} becomes

$$\alpha_{44} = \alpha_{44}^+ - \frac{1}{4} \frac{\gamma^2}{\eta_2} = \frac{1}{2\eta_2} \left[\frac{1}{2} (\gamma^2 + 4\delta\eta_2) + \sqrt{\gamma^2 (\gamma^2 + 4\delta\eta_2)} \right] \quad (84)$$

which is always positive or zero because of condition (f'). By removing the equality in condition (f') so that

$$f) \quad \xi_1 \eta_3 + \xi_2 \eta_2 > 0$$

then $\alpha_{44} > 0$ and hence both conditions (a) and (e) are satisfied. It remains to be demonstrated that (c) is also satisfied by this choice of α_{44} .

Since both terms in α_{44} are positive definite, we can write

$$\alpha_{44} \geq \frac{1}{4\eta_2} (\gamma^2 + 4\delta\eta_2)$$

so that if

$$\frac{\xi_1}{4\eta_2} (\gamma^2 + 4\delta\eta_2) > \frac{\xi_2^2}{b}$$

than condition (c) will be satisfied. By substituting the expressions for γ^2 and δ into this inequality, it can be shown that the above requirement reduces to condition (d) and hence (c) is indeed satisfied.

In terms of the temperature coefficients, the stability requirements obtained from this analysis are

$$(1) \quad \alpha_1 \eta_1 - \alpha_2 \eta_2 \geq 0$$

$$(2) \quad \alpha_1 \eta_3 + \alpha_2 \eta_2 > 0$$

The addition of these two requirements gives condition (b), $\alpha_1 > 0$.

Interestingly, the Welton criterion⁽⁹⁾ which requires the real part of the feedback transfer function, $H(j\omega)$, to be always positive:

$$R\{H(j\omega)\} = R\{\alpha_1 b \frac{(j\omega) + \eta_3 + \frac{\alpha_2}{\alpha_1} \eta_2}{(j\omega + \eta_1)(j\omega + \eta_3)}\} > 0$$

gives the same two requirements. Popov's criterion for the reactor with no delayed neutrons,⁽⁹⁾ which requires

$$R\{H(j\omega)\} + \frac{q}{\omega} \text{Im}\{H(j\omega)\} > 0$$

for arbitrary positive q , turns out to be least restrictive for $q = 0$, i.e., when it coincides with Welton's criterion.

A general plot of the stability region is indicated in Fig. 6. For the thermionic reactor for which $\alpha_2 > 0$ and for the time constants of the previous section, condition (1) requires

$$\alpha_1 > (3 \text{ to } 20) \alpha_2$$

a condition unlikely to be met in any of the proposed concepts. This result points up the shortcoming of the Liapunov method and Welton's criterion: they both provide sufficient but not necessary criteria for asymptotic nonlinear stability. Clearly, as the perturbations of the reactor state variables decrease in magnitude, the linear model criteria must be approached. This is equivalent to stating that the stability boundary

Figure 6. Region of nonlinear stability for the two-temperature model according to sufficient but not necessary criteria.

$\alpha_1 = \frac{\eta_2}{\eta_1} \alpha_2$ must rotate counterclockwise about the origin to the $\alpha_1 = -\frac{\eta_2}{\eta_3} \alpha_2$ boundary in the second quadrant as the perturbation magnitude decreases.

This transitional stability region might possibly be accommodated by the Liapunov method (by including cross-product terms such as $x_1 x_3$), and this aspect is continuing to be investigated.

III. START-UP ANALYSIS

A. SUMMARY

The equations describing the dynamic behavior of a representative in-core thermionic reactor system have been programmed for the digital computer, and preliminary comparisons with previous analog results are described. The current-voltage relationship for the diode has been successfully simulated digitally and is in the process of being incorporated directly into the system description. Several trial cases of open circuit start-up have been studied. In these cases, the reactor system is assumed to be at prescribed initial conditions with the electrical circuit open. At closure, reactivity is simultaneously inserted to bring the system to rated electrical output, and the open-loop response is obtained.

B. INTRODUCTION

Preliminary to investigating different possible start-up modes, a relatively detailed system simulation was required, including reasonable relationships for the diode electrical characteristics. Because of the size and complexity of the overall system description and, to some extent, because of previous familiarity with some of the problems encountered in a comparable analog simulation at JPL, it was decided to perform the simulation on the digital computer.

An integration scheme capable of handling sets of simultaneous first-order differential equations was obtained and adapted to the system equations of interest. The current-voltage and heat-transfer characteristics of a thermionic diode were simulated in three different ways: first, with simple

Richardson-Dushman type theoretical expressions; second, with curve fits to some experimental data; and finally (and presently), with the more detailed theoretical expressions used by Wilkins in SIMCON.⁽¹⁰⁾ This latter simulation was by far the most complex and difficult to perform, but was deemed necessary to provide a realistic representation of the variables over the wide operating range anticipated during the start-up transients.

C. SYSTEM DESCRIPTION AND ANALYSIS

For the initial studies, the system is described in terms of a single diode, primary and secondary coolant loops with an intermediate heat exchanger, and a waste-heat radiator in a manner very similar to that of Gronroos.⁽²⁾ The diode is assumed to consist of a cylinder of fuel with the emitter, collector, electrical insulator, and outer clad sheath forming successive cylindrical annuli concentric with the fuel as indicated schematically in Fig. 1. Each of these regions is characterized by its spatially-averaged temperature. The neutron dynamics are described by the prompt-jump approximation.

In trial problems, it was found that one of the limiting factors in the speed of the integration routine was due to the large time constant associated with the electrical insulator region. Computer running time was reduced considerably when the insulator and cladding were collapsed into a single region with a composite describing equation. The collector could likewise have been collapsed into this region, but it was felt that a more accurate description would be obtained with the collector separate. The resulting nine-region model was tested and compared with the previous ten-region case. The loss of information and generality were found to be

negligible when compared to the considerable improvement in computational time.

The equations used to describe the system are:

Reactor Power, (n)

$$\frac{dn}{dt} = \frac{(d\rho/dt + \lambda\rho)}{\beta - \rho} n \quad (85)$$

$$\rho = \sum_{i=1}^7 \alpha_i \Delta T_i + \rho_0(t) \quad (86)$$

Average Fuel Temperature, ($\bar{T}_f$)

$$\rho_f V_f c_{p_f} \frac{d\bar{T}_f}{dt} = V_f n - \frac{4\pi (\bar{T}_f - \bar{T}_e)}{\left[\frac{1}{K_f L_f} + \frac{\log(R_e/R_f)}{K_e L_e} \right]} \quad (87)$$

Average Emitter Temperature, ($\bar{T}_e$)

$$\rho_e V_e c_{p_e} \frac{d\bar{T}_e}{dt} = \frac{4\pi (\bar{T}_f - \bar{T}_e)}{\left[\frac{1}{K_f L_f} + \frac{\log(R_e/R_f)}{K_e L_e} \right]} - 2\pi R_e L_e q_e \quad (88)$$

Average Collector Temperature, ($\bar{T}_c$)

$$\rho_c V_c c_{p_c} \frac{d\bar{T}_c}{dt} = 2\pi R_e L_e (q_e - I_v) - \frac{4\pi (\bar{T}_c - \bar{T}_I)}{\left[\frac{\log(R_c/R_{gap})}{K_c L_c} + \frac{\log(R_I/R_c)}{K_I L_I} + \frac{\log(R_{cl}/R_I)}{K_{cl} L_{cl}} \right]} \quad (89)$$

Average Insulator-Cladding Temperature, ($\bar{T}_I$)

$$(\rho_I V_I c_{p_I} + \rho_{cl} V_{cl} c_{p_{cl}}) \frac{d\bar{T}_I}{dt} = \frac{4\pi (\bar{T}_c - \bar{T}_I)}{\left[\frac{\log(R_c/R_{gap})}{K_c L_c} + \frac{\log(R_I/R_c)}{K_I L_I} + \frac{\log(R_{cl}/R_I)}{K_{cl} L_{cl}} \right]} \quad (\text{Cont.})$$

$$- \frac{4\pi (\bar{T}_I - \bar{T}_{co})}{\left[\frac{\log(R_I/R_c)}{K_I L_I} + \frac{\log(R_{cl}/R_I)}{K_{cl} L_{cl}} + \frac{2}{h R_{cl} L_{cl}} \right]} \quad (90)$$

Average Reactor Coolant Temperature, ($\bar{T}_{co}$)

$$\begin{aligned} N_{ds}^0 c_{co} V_{co} c_{p_{co}} \frac{d\bar{T}_{co}}{dt} &= \frac{N_{ds} 4\pi (\bar{T}_I - \bar{T}_{co})}{\left[\frac{\log(R_I/R_c)}{K_I L_I} + \frac{\log(R_{cl}/R_I)}{K_{cl} L_{cl}} + \frac{2}{h R_{cl} L_{cl}} \right]} \\ &\quad - 2w_{co} c_{p_{co}} (\bar{T}_{co} - \bar{T}_{co_i}) \end{aligned} \quad (91)$$

$$\bar{T}_{co} = \frac{1}{2} (\bar{T}_{co_i} + \bar{T}_{co_e}) \quad (92)$$

$$\tau_{co} \frac{d\bar{T}_{co_i}}{dt} = \bar{T}_{co_i} - \bar{T}_{p_e} \quad (93)$$

Primary Loop Temperature, ($\bar{T}_p$)

$$M_p C_{p_p} \frac{d\bar{T}_p}{dt} = N_{dp} 2w_{co} c_{p_{co}} (\bar{T}_{p_i} - \bar{T}_p) - U_h A_h (\bar{T}_p - \bar{T}_s) \quad (94)$$

$$\bar{T}_p = \frac{1}{2} (\bar{T}_{p_i} + \bar{T}_{p_e}) \quad (95)$$

$$\tau_p \frac{d\bar{T}_{p_i}}{dt} = \bar{T}_{p_i} - \bar{T}_{co_e} \quad (96)$$

Secondary Loop Temperature, ($\bar{T}_s$)

$$M_s c_{p_s} \frac{d\bar{T}_s}{dt} = U_h A_h (\bar{T}_p - \bar{T}_s) - N_{rp} 2w_{rc} c_{p_{rc}} (\bar{T}_s - \bar{T}_{s_i}) \quad (97)$$

$$\bar{T}_s = \frac{1}{2} (\bar{T}_{s_i} - \bar{T}_{s_e}) \quad (98)$$

$$\tau_s \frac{d\bar{T}_{s_i}}{dt} = \bar{T}_{s_i} - \bar{T}_{r_e} \quad (99)$$

Average Radiator Temperature, ($\bar{T}_r$)

$$\rho_r L_r A_{rc} c_{p_r} \frac{d\bar{T}_r}{dt} = 2w_{rc} c_{p_{rc}} (\bar{T}_{ri} - \bar{T}_r) - \epsilon_r \sigma A_{rs} \bar{T}_r^4 \quad (100)$$

$$\bar{T}_r = \frac{1}{2} (\bar{T}_{ri} + \bar{T}_{re}) \quad (101)$$

$$\tau_r \frac{d\bar{T}_{ri}}{dt} = \bar{T}_{ri} - \bar{T}_{se} \quad (102)$$

The symbols in Eqs. (85) - (102) are defined as follows:

- ρ = reactivity ($\delta k/k$)
- β = delayed neutron fraction
- α_i = feedback coefficient of i^{th} region ($\delta k/k/^\circ K$)
- V_i = volume of i^{th} region (cm^3)
- ρ_i = density of i^{th} region (gm/cm^3)
- c_{p_i} = heat capacity of i^{th} region ($\text{watt-sec}/\text{gm}^\circ K$)
- L_i = length of i^{th} region (cm)
- k_i = thermal conductivity of i^{th} region ($\text{watt}/\text{cm}^\circ K$)
- R_i = radius of i^{th} region (cm)
- I = net converter current (amp/cm^2)
- v = converter voltage (volts)
- h = heat transfer coefficient from collector-insulator-cladding surface to reactor coolant ($\text{watt}/\text{cm}^2^\circ K$)
- N_{ds} = number of diodes in series
- w_{co} = reactor coolant flow rate past one series of diodes (gm/sec)
- M_i = mass of i^{th} side of the heat exchanger (gm)
- N_{dp} = number of diode stacks in parallel
- U_h = overall heat transfer coefficient of the heat exchanger ($\text{watt}/\text{cm}^2^\circ K$)

A_h = area of the heat exchanger (cm^2)

N_{rp} = number of radiator pipes

w_{rc} = radiator coolant flow rate through one radiator pipe (gm/sec)

A_{rc} = cross sectional area of a radiator pipe (cm^2)

ϵ_r = radiator emissivity

σ = Stefan-Boltzmann constant ($\text{w/cm}^2\text{°K}^4$)

A_{rs} = surface area of one radiator pipe (cm^2)

τ_i = coolant transport delay time for i^{th} region (sec).

$\bar{T}_{ji}$ = average coolant inlet temperature in j^{th} region ($^{\circ}\text{K}$)

$\bar{T}_{je}$ = average coolant exit temperature in j^{th} region ($^{\circ}\text{K}$)

The q_e term in the emitter and collector equations represents the heat flux removed from the emitting surface due to electron and ion cooling, thermal radiation, and cesium gas conduction, and has dimensions of watt/cm^2 . The electron and ion cooling rates are determined by one of three methods, namely: by the Richardson-Dushman approach; by use of empirical current-voltage data; or by a complex iterative scheme which is subsequently discussed. The radiation term is a straightforward fourth power of temperature dependence, and the cesium gas conduction contribution is an empirical expression following Wilkins. (10)

It should be noted that the present system model is lacking some of the finer complexities of a realistic description. For instance, the introduction of liquid metal pumps and pumping losses, and the effects of coolant transport delay have not yet been included. At present, the system assumes that temperature changes in the primary coolant are immediately experienced by the radiator and vice versa. Also, the studies to date have been done

with an uncontrolled system. This means that only the inherent feedback characteristics of the system are utilized to control it.

For the digital solution of this set of equations, a flexible and highly accurate digital program for the numerical integration of a system of differential equations was obtained.⁽¹¹⁾ This program is designed to solve a system of first-order differential equations with given initial conditions by finding the fifth-degree polynomial approximation to the solution of the equations. A practical feature of the program is the automatic increase or decrease of the internal step size during the integration. This is possible because the routine continually tests the solution to guarantee that the integration scheme is numerically stable throughout the integration.

The digital program was extensively tested to insure its usefulness for the problem of start-up dynamics. Two representative cases are included here which show comparisons of digital output with the corresponding results obtained from analog computer simulations. The first case depicts a third-order model with prompt-jump neutronics and two temperature feedback regions. The graph (see Fig. 7) shows reactor power vs temperature in Region 1. The comparison is seen to be excellent. Although the actual situation depicted is a very unrealistic case, it demonstrates the accuracy of the computer program over an extremely wide range of variables.

The second representative comparison is with a study made by Gronroos⁽²⁾ at JPL. His analog model included prompt-jump neutronics, with six delayed groups, as well as fuel, emitter, collector, insulator, cladding, coolant, heat exchanger, and radiator regions. The heat transport across the gap was by Richardson-Dushman type electron cooling, cesium conduction, and thermal radiation. A digital program was constructed which exactly duplicated

Figure 7. Comparison of solution to third order nonlinear problem from digital and analog computer.

this model and resulted in a system of fifteen first-order nonlinear differential equations. The time constants and reactivity feedback coefficients used by Gronroos were used in the digital simulation. The results are shown in Fig. 8 for a positive reactivity insertion of 2 cents with the system previously operating at equilibrium.

As can be seen, the comparison is not as close as might be hoped. Although some end values are dissimilar, the reactivity insertion is exactly compensated in both systems (identical feedback coefficients were used). The digital simulation was closely checked, and it is believed that the numerical integration results are accurate.

The model was also tested against other analog simulations with similar success. After the multi-region comparison cases were completed, the insulation-clad region collapsing was incorporated and tested, and several trials of simulated start-ups were run. A sample of the system output for one of these start-up trials is shown in Fig. 9. In this case, the reactor and coolant system are brought up to an assumed equilibrium state for the conditions given in the figure and with the electrical circuit open. The circuit is then closed and a simultaneous reactivity insertion rate of 1 cent/sec is initiated for ten seconds. For this simulation the materials used were: uranium-carbide fuel; tungsten emitter; molybdenum collector; aluminum dioxide insulator; niobium cladding; lithium coolant in both primary and secondary loops; and a stainless steel radiator. Reactivity feedbacks were included for the fuel, emitter, collector, reactor coolant, and radiator regions. The system was uncontrolled with no coolant transport delay, the output current was taken from empirical data, and the output voltage was assumed constant.

Figure 8. Comparison study of large thermionic reactor system dynamics from digital and analog computer.

Figure 9. Example of simulated start-up of a thermionic reactor system.

As can be seen in the figure, the system exhibits good uncontrolled behavior for the given conditions. The variables rise smoothly to their respective operating points and level off rapidly. However, the heat transfer mechanism within the diode is highly simplified, and in a realistic start-up the output voltage would not be constant.

D. DESCRIPTION OF THE CONVERTER CURRENT-VOLTAGE ANALYSIS

As previously mentioned, before a realistic study of the simulated start-up of a thermionic converter could be made, a more accurate representation of the actual plasma processes (than could be obtained from the simple Richardson-Dushman equation approach) had to be incorporated. Thus, after a working model of the whole system had been devised and successfully tested, considerable effort was spent on improving the current-voltage and transport phenomena descriptions for simulating an operating thermionic diode.

The basis of this work is a digital program by Wilkins⁽¹⁰⁾ for computing thermionic-converter performance characteristics. Although it could not be used directly in the dynamics integration model, it did provide an excellent starting point from which to devise a current-voltage scheme.

Wilkins' program requires that the net diode current as well as thermal conditions at the emitter and collector surfaces and certain properties of these surfaces and the interelectrode gas be known. Since for the start-up analysis, the converter current is one of the quantities to be calculated for a given set of thermal conditions, Wilkins' procedure, as such, is not acceptable for direct incorporation into the digital simulation. However, a known parameter for the start-up problem is the heat input at the emitting surface. Thus, this parameter could be initially used in an iterative routine to converge upon the current and voltage profile.

To use the present method, the emitter, collector, and cesium reservoir temperatures, diode gap, and certain known properties of the emitter, collector, and interelectrode gas are specified. Knowing these conditions, the cesium pressure, surface coverage, and fractional charge transfer can be calculated at both electrodes. This in turn enables the cesiated work functions to be determined and, therefore, the electron and ion emission currents from each surface may be calculated.

Next, an initial guess is made for the electron temperature in the plasma. Using the electron and ion current balances at the emitter and collector sheaths, an iterative scheme calculates the sheath voltage drops and random currents. Once these are known to a specified tolerance, the net electron and ion currents may be determined, and the net output current may be calculated. Also, the plasma voltage drop follows directly from this analysis and determines the output voltage. This sheath analysis includes both Schottky effects and surface phenomena.

Finally, an energy balance is performed, and if this does not fall within the specified tolerances, the initial assumption for the electron temperature is at fault. The electron temperature is then adjusted and the whole procedure repeated until the desired tolerances are met.

This routine has been run for various test cases to compare results with those of Wilkins. Several trial comparisons are shown in Fig. 10. As can be seen, the results are in excellent agreement with those of Wilkins, and in good agreement with the corresponding experimental measurements for the entire range of converter operating conditions.

Figure 10. Comparison of present computed current-voltage data with Wilkins' data and with experiments.

Work is presently being continued in this area to incorporate the current-voltage procedure into the system simulation. There are currently some minor difficulties at low heat input values where the iterative method will occasionally fail to converge to a solution. This situation is being investigated, and present indications are that the problem can be circumvented without undue difficulty.

E. FUTURE WORK

Work in the immediate future will be aimed at completing the incorporation of the current-voltage program into the main system simulation. Then studies with the one-diode model under simulated start-up can be made using this new routine. Specific cases will be chosen and run with the one-diode model. Since lengthy computer runs can be anticipated with this large program, these cases will be carefully chosen.

A start-up procedure like that previously discussed will be tested; i.e., assume that the system is initially at some steady state operating condition with the electrical components off, then close the electrical circuit and simultaneously initiate various reactivity insertions to bring the system to a desired final operating condition. It is anticipated that simulated diode instabilities during start-up transients can be inserted during this type of procedure and the consequences to the system computed. Upon development of a suitable control scheme, the present system will be modified to incorporate a controller (or controllers, for multiple input-output control).

Looking beyond the one-diode model, it appears possible to use the current program to form a basis for multi-diode studies. By working with several diodes in a stack and many stacks in a reactor, spatial considerations may be investigated.

IV. NONLINEAR OSCILLATIONS AND STABILITY

A. SUMMARY

Autonomous periodic solutions to the nonlinear equations of reactor dynamics are sought by means of Fourier series expansions. Such solutions correspond to stable limit cycles if the system is linearly unstable, and to unstable limit cycles if the system is linearly asymptotically stable. In particular, a linearly stable system may become unstable for large enough disturbances.

The Fourier series method is applied to different formulations of reactor dynamics, and the results are compared. Equations for higher-order approximations are derived. The method is illustrated in a special case of centers in a one-region nonlinear reactor model.

The application to the two-temperature reactor illustrates a fundamental difference between the effective-lifetime model and reactor systems with delayed neutrons. For the effective-lifetime model, limit cycles are found only when the equilibrium power exceeds the critical power for linear stability (stable limit cycles). With delayed neutrons, limit cycles occur when the equilibrium power is less than the critical power (unstable limit cycles). Linearly stable systems, with or without delayed neutrons, can become unstable for sufficiently large perturbations in the presence of prompt positive and delayed negative feedback. Near the threshold of linear instability, with delayed neutrons included, this takes the form of an unstable limit cycle, as demonstrated by the Fourier series method and by digital computations. Realistic operating conditions for the thermionic reactor are not close to the threshold of linear instability, and the perturbation required to induce instability appears to be extremely large.

B. INTRODUCTION

In the previous annual report,⁽¹⁾ the analytical method was referred to as the method of Sandmeier and Torlin. In this report it is designated as the Fourier series method.

The first application of the method in reactor dynamics appears to be due to Rumsey,⁽¹⁾ who obtained a first-order correction to a reactor-system transfer function by solving for Fourier coefficients of the neutron density in terms of the coefficients of a periodic reactivity input. Nelkin⁽¹³⁾ used essentially the same approach. Both authors recognized that a negative reactivity bias is necessary to maintain a steady nonlinear oscillation, but neither recognized the need to distinguish between the equilibrium power level and the time-average power in a nonlinear oscillation. Akcasu⁽¹⁴⁾ also investigated the negative reactivity bias in a nonlinear oscillation, but without including reactivity feedback.

Sandmeier⁽¹⁵⁾ obtained a consistent set of equations for forced oscillations by including the power-level shift. Wasserman⁽¹⁶⁾ reviewed the existing work on the Fourier series method and discussed its relation to the usual describing-function technique. Smets^(17,18) distinguished between "direct" and "indirect" methods of deriving the describing function, and was apparently the first author to suggest the use of Fourier series for studying autonomous nonlinear oscillations in reactor systems. However, Smets discussed only the possibility of stable limit cycles.

Torlin⁽¹⁹⁾ obtained a formula for the magnitude of the first harmonic of an autonomous reactivity oscillation, assuming one harmonic in reactivity and two harmonics in power, and using an iterative scheme. He recognized the possibility of either stable or unstable limit cycles. Unfortunately, the particular reactor model he selected for illustration was not analyzed

correctly in his paper. Torlin's contribution was the procedure for application of Fourier series to autonomous oscillations: Derive a formula for the magnitude of a Fourier component in terms of the frequency, ω , and then find the regions in parameter space that simultaneously correspond to real positive numbers for ω^2 and for the magnitude of the Fourier component.

Akcasu and Shotkin^(20,21) have made recent contributions to the theory of describing functions. However, their application to autonomous oscillations did not include delayed neutrons. They succeeded in proving the nonexistence of unstable limit cycles for the case of linear feedback without delayed neutrons (i.e., the effective-lifetime model).

In the remainder of this report, we outline the formulation of Torlin's approximation and apply it to the two-temperature reactor model, using both the effective-lifetime model and the prompt-jump approximation. Preliminary results were recently published;^(22,23) except for Ref. 1, these appear to be the first published predictions of unstable limit cycles in a two-temperature reactor system. Next, we derive equations for some higher-order approximations that are being investigated. Following this, an alternative procedure that is free from iterations is proposed. Finally, we illustrate the application of various approximations to the classic one-region problem of Ergen and Weinberg⁽²⁴⁾ which exhibits a center (sustained nonlinear oscillations following any size perturbation).

C. PERIODIC SOLUTIONS USING TORLIN'S FORMULATION

Following the notation of B. Z. Torlin⁽¹⁹⁾ except that ℓ is used instead of T for the neutron generation time, the equations representing

the point-reactor model may be written:

$$\left. \begin{aligned} \frac{dN}{dt} &= \beta \frac{\rho - 1}{\ell} N + \sum_j \lambda_j c_j \\ \frac{dc_j}{dt} &= \frac{\beta_j}{\ell} N - \lambda_j c_j \end{aligned} \right\} \quad (103)$$

where ρ is reactivity in dollars. As in Torlin's paper, we write

$$N = \bar{N}(1 + n) \quad (104)$$

$$W_1(p)n = \rho(1 + n) \quad (105)$$

$$W_1(p) = p \left(\frac{\ell}{\beta} + \sum_j \frac{\alpha_j}{p + \lambda_j} \right) \quad (106)$$

$$W_2(p)n \frac{\bar{N}}{N_0} = \rho \quad (107)$$

where $\bar{N}$ is the time-average neutron density, N_0 is the equilibrium neutron density, and $\alpha_j = \beta_j/\beta$. Eq. (104) defines n as relative fluctuation about $\bar{N}$. Eq. (105) is the nonlinear point-reactor model written in differential-operator form where

$$W_1(p) = W_1(d/dt)$$

and where $W_1^{-1}(s)$ is the source-free zero-power transfer function for a critical reactor. In the applications we will specialize to the prompt-jump approximation

$$W_1(p) = p \sum_j \frac{\alpha_j}{p + \lambda_j} \quad \text{or} \quad \frac{p}{p + \lambda} \quad (108)$$

where

$$\frac{1}{\lambda} = \sum_j \frac{\alpha_j}{\lambda_j}$$

and also to the case of no delayed neutrons

$$W_1(p) = p \frac{\ell}{\beta} \quad \text{or} \quad \frac{p}{\lambda} \quad (109)$$

The latter option is the effective-lifetime model, in which ℓ is interpreted as an effective lifetime β/λ in order to match the prompt-jump approximation at low frequencies. This matching permits a realistic appraisal of the effect of delayed neutrons in slow transients.^(1,22)

The feedback as expressed by Eq. (107) need not be linear, but we consider $W_2(p)$ to be a linear operator in what follows. The factor $\bar{N}/N_0$ is introduced because $W_2(p)$ is related to the equilibrium power N_0 while n is the relative fluctuation about $\bar{N}$. Obviously $\bar{N} = N_0$ for the linearized system.

Regard Eqs. (105) and (107) as defining an autonomous system and postulate

$$\rho(\omega t) = a_0 + \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t} \quad (k \neq 0) \quad (110)$$

$$n(\omega t) = \sum_{\ell=-\infty}^{\infty} A_\ell e^{i\ell\omega t} \quad (\ell \neq 0) \quad (111)$$

Note the absence of a constant term A_0 in Eq. (111). The constant a_0 in the reactivity represents the negative bias needed to keep $\bar{N}$ from drifting.

Substitute (110) and (111) into (105):

$$\sum_{-\infty}^{\infty} A_{\ell} W_1(i\ell\omega) e^{i\ell\omega t} = a_0 + \sum_{-\infty}^{\infty} a_k e^{ik\omega t} + a_0 \sum_{-\infty}^{\infty} A_{\ell} e^{i\ell\omega t} \\ + \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_m A_n e^{i(m+n)\omega t}$$

Here $m \neq 0$ and $n \neq 0$, and the nonlinearity resides in the last two terms. Matching coefficients of like frequencies and noting that the double summation contributes when $m+n = \ell$,

$$A_{\ell} W_1(i\ell\omega) = a_{\ell} + \sum_{-\infty}^{\infty} a_k A_{\ell-k}$$

where now $k=0$ is included in the summation. Solving for A_{ℓ} yields

$$A_{\ell} = \frac{a_{\ell} + \sum_{-\infty}^{\infty} a_k A_{\ell-k}}{W_1(i\ell\omega)} \quad (112)$$

An iterative scheme is suggested by writing

$$A_{\ell}^{(\nu+1)} = \frac{a_{\ell} + \sum_{-\infty}^{\infty} a_k A_{\ell-k}^{(\nu)}}{W(i\ell\omega)} \quad (113)$$

This is Torlin's Eq. (8) with the translator's misprint corrected. The index $\nu = 0$ is to represent the linear approximation.

For $\ell = 0$, Eq. (112) yields

$$a_0 = - \sum_{-\infty}^{\infty} a_k A_{-k}$$

Here the $k=0$ term is absent because $A_0 = 0$. Since $A_{-k} = A_k^*$

$$a_0 = -2\text{Re} \sum_{k=1}^{\infty} a_k A_k^*$$

Torlin's Eq. (9) is

$$a_0 = -2\text{Re} \sum_{\ell=1}^{\infty} a_{\ell} A_{-\ell} \quad (114)$$

His Eq. (10) should be written

$$a_0^{(\nu+1)} = -2\text{Re} \sum_{\ell=1}^{\infty} a_{\ell} A_{-\ell}^{(\nu)} \quad (115)$$

and its purpose is to indicate consistent values for a_0 when iterating with Eq. (113).

The steady-state power coefficient is $W_2(0)$, and it must be negative in this formulation. By considering the reactivity needed to shift between two steady states N_0 and $\bar{N}$, where $\bar{N} > N_0$, we find

$$\bar{N} = N_0 [1 + a_0 W_2^{-1}(0)] \quad (116)$$

which Torlin writes as

$$\bar{N} = N_0 (1 - a_0 \kappa_0) \quad (117)$$

Using the linear approximation in Eq. (114) we find

$$a_0 = -2|a_1|^2 \text{Re} W_1^{-1}(i\omega) \quad (118)$$

correct to order a_1^2 . This is essentially Torlin's Eq. (15), and it turns out to be correct to order a_1^2 in all iterations whether one retains two harmonics in reactivity or truncates at one harmonic as does Torlin.

Eq. (118) is consistent with the observation that $\bar{N} = N_0$ and $a_0 = 0$ if

delayed neutrons are neglected, because in that case $W_1(i\omega)$ has no real part.

Combining Eqs. (107), (117), and (118),

$$\frac{a_1}{A_1} = W_2(i\omega) [1 + 2\kappa_0 |a_1|^2 \operatorname{Re} W_1^{-1}(i\omega)] \quad (119)$$

Torlin retains one harmonic in ρ and two in n . Using the linear approximation for $v=0$, he generates expressions for A_1 in terms of a_0 and a_1 from Eq. (113). These are combined with Eq. (119) to eliminate A_1 . Terms of order a_1^4 are neglected in the process. The first iteration yields

$$|a_1|^2 = \frac{1 - \frac{W_2(i\omega)}{W_1(i\omega)}}{\frac{W_2(i\omega)}{W_1(i\omega)} [2\kappa_0 \operatorname{Re} W_1^{-1}(i\omega) - \frac{2\operatorname{Re} W_1^{-1}(i\omega)}{W_1(i\omega)}]} \quad (120)$$

The second iteration yields Eq. (17) of Torlin's paper:

$$|a_1|^2 = \frac{1 - \frac{W_2(i\omega)}{W_1(i\omega)}}{\frac{W_2(i\omega)}{W_1(i\omega)} [2\kappa_0 \operatorname{Re} W_1^{-1}(i\omega) - \frac{2\operatorname{Re} W_1^{-1}(i\omega)}{W_1(i\omega)} + \frac{W_1^{-1}(2i\omega)}{W_1(i\omega)}]} \quad (121)$$

Note that the critical condition (threshold of linear instability, corresponding to a linear center) is

$$1 - \frac{W_2(i\omega)}{W_1(i\omega)} = 0 \quad (122)$$

so that the numerator of Eq. (120) or (121) must vanish on any critical boundary in parameter space (if such a boundary exists).

Periodic solutions are sought by noting that Eq. (121) must yield a real positive number. By demanding that the right-hand side of (121) be

the ratio of two complex numbers of the same phase, one generates a polynomial whose roots are ω^2 . All complex and negative real roots ω^2 are discarded, and the positive real roots are tested in Eq. (121) to determine if they yield $|a_1|^2 > 0$. The latter presumably represent limit cycles, stable if the linear system is unstable, and vice versa.

The truncation of the Fourier series and the neglect of terms of order $|a_1|^4$ both serve to restrict the range of validity of the method to small $|a_1|^2$. Since $|a_1|^2 \rightarrow 0$ at a critical boundary, this suggests that the method should predict limit cycles accurately in a small neighborhood near critical (equilibrium power near the critical power for linear instability).

D. APPLICATION TO THE TWO-TEMPERATURE REACTOR

The two-temperature point-reactor model for a thermionic fast reactor is discussed in detail in Ref. 1, and a brief review appears in Section I of this report. The incremental temperatures T_1 and T_2 , corresponding to prompt and delayed regions respectively, are given by Eqs. (1) and (2) of Section I.B. A brief summary of the linear stability criteria appears in Section I.D. The reactivity is

$$\rho = -\alpha_1 T_1 - \alpha_2 T_2 \quad (123)$$

where ρ is in dollars if each α is in dollars per $^{\circ}\text{C}$.

The relation between Torlin's notation and the notation of Ref. 1 is as follows:

$$W_1^{-1}(p) = \frac{\beta}{n_0} G(p) \quad (124)$$

$$W_2(p) = - \frac{n_0}{\beta} H(p) \quad (125)$$

$$1 - \frac{W_2(p)}{W_1(p)} = 1 + G(p)H(p) \quad (126)$$

For exhibiting the linear stability parameter space as in Fig. 11, we retain the parameters x and y of Ref. 1; see, for example, Fig. 3 of this report. The co-ordinates x and y are essentially the negative of prompt and delayed reactivity coefficients respectively; each is also proportional to the equilibrium power n_0 (N_0 in Torlin's notation).

Fig. 11, which shows regions of linear stability for the first example we used to illustrate the Fourier series method, is taken from Fig. 2 of Ref. 1, and represents Case I of that report. The reciprocal time constants are $\eta_1 = 0.5 \text{ sec}^{-1}$, $\eta_2 = 0.25 \text{ sec}^{-1}$, $\eta_3 = 0.2 \text{ sec}^{-1}$, and $\lambda = 0.1 \text{ sec}^{-1}$. The more realistic thermionic reactor model of Fig. 3 will be analyzed in a later report.

In the present notation, the feedback transfer function of Ref. 1 becomes

$$W_2(i\omega) = - K \frac{i\omega + \xi}{(i\omega + \eta_1)(i\omega + \eta_3)} \quad (127)$$

where

$$K = \frac{\alpha_1 b n_0}{\beta} \quad (128)$$

and

$$\xi = \eta_3 + \eta_2 \frac{\alpha_2}{\alpha_1} \quad (129)$$

as in Ref. 1. We use the second approximation, Eq. (121), in the form

Figure 11. Reactivity-coefficient parameter space for thermionic fast reactor; x and y are proportional to negative of prompt and delayed coefficients respectively. Shaded area destabilized by delayed neutrons.

$$|a_1|^2 = \frac{W_1(i\omega)W_2^{-1}(i\omega) - 1}{2[\kappa_0 - W_1^{-1}(i\omega)] \operatorname{Re} W_1^{-1}(i\omega) + W_1^{-1}(i\omega)W_1^{-1}(2i\omega)} \quad (130)$$

Effective-Lifetime Approximation

From Eq. (109), using the effective lifetime β/λ ,

$$W_1(i\omega) = i\omega/\lambda \quad (131)$$

Inspection of Eq. (130) shows that the denominator is real, so that the condition for real $|a_1|^2$ is identical to the condition that $W_1(i\omega)W_2^{-1}(i\omega)$ be real. From (127) and (131),

$$\frac{W_1(i\omega)}{W_2(i\omega)} = - \frac{\omega}{K\lambda} \frac{\eta_1\eta_3 - \omega^2 + i(\eta_1 + \eta_3)\omega}{\omega - i\xi} \quad (132)$$

This expression is real if the numerator and denominator have the same phase:

$$\frac{(\eta_1 + \eta_3)\omega}{\eta_1\eta_3 - \omega^2} = - \frac{\xi}{\omega}$$

Solving for ω^2 yields

$$\omega^2 = \frac{\eta_1\eta_3\xi}{\xi - \eta_1 - \eta_3} \quad (133)$$

Eq. (133) must yield a real number for ω . This requires

$$\frac{\eta_1\eta_3\xi}{\xi - \eta_1 - \eta_3} > 0$$

We conclude (1) if $\xi > 0$ we require $\xi > \eta_1 + \eta_3$ and (2) if $\xi < 0$ the condition is automatically satisfied. From Ref. 1,

$$\xi = \eta_3 \left(1 + \frac{y}{x}\right) \quad (134)$$

The requirements for positive ω^2 are therefore

$$\frac{y}{x} > \frac{\eta_1}{\eta_3} \quad \text{and} \quad 1 + \frac{y}{x} < 0 \quad (135)$$

Part of this region in the x, y plane may be ruled out by the requirement that the steady-state power coefficient must have the proper sign for stability. From Ref. 1, this condition is $K\xi > 0$ where $K = x$. In terms of x and y , this is

$$x + y > 0 \quad (136)$$

The combined conditions (135) and (136) determine a triangular region in the x, y plane lying above the two lines

$$x + y = 0 \quad \text{and} \quad y = \frac{\eta_1}{\eta_3} x \quad (137)$$

Next we test the ω^2 in this region and reject any subregion for which $|a_1|^2$ is not positive. By substituting into Eq. (130) we derive a formula for $|a_1|^2$ that is valid when $|a_1|^2$ is real.

$$|a_1|^2 = \frac{2\omega^2}{K\lambda^3} (\eta_1\eta_3 + K\lambda - \omega^2) \quad (138)$$

Using Eq. (133), this becomes

$$|a_1|^2 = \frac{2\eta_1\eta_3\xi[K\lambda\xi - (\eta_1 + \eta_3)(K\lambda + \eta_1\eta_3)]}{K\lambda^3(\xi - \eta_1 - \eta_3)^2} \quad (139)$$

It is easily verified that K and ξ have the same sign throughout the region above the lines given by (137). Therefore, the requirement for positive $|a_1|^2$ is

$$K\lambda\xi > (\eta_1 + \eta_3)(K\lambda + \eta_1\eta_3) \quad (140)$$

With an equality sign, (140) is the equation of the critical line (resonance line) in the x, y plane, so that positive $|a_1|^2$ occurs only above the critical line. Since x and y are both proportional to the equilibrium power, the limit cycles are predicted only when the equilibrium power is greater than the critical power for linear instability (stable limit cycles). The region for $|a_1|^2 > 0$ in this approximation is shown in Fig. 12.

A traverse across the shaded region of Fig. 12 at $y = 1.5$ is shown in Fig. 13, where ω^2 from Eq. (133) and $|a_1|^2$ from Eq. (139) are plotted as functions of x .

Partial verification was obtained using the analog computer. Stable limit cycles were found in the shaded region of Fig. 12, but only for $x > -1$. This is consistent with Shotkin's observation⁽²⁵⁾ which, in terms of the present notation, states that the only limit cycles are stable limit cycles between $x = -1$ and the critical line. We observed that, as x was increased from -1 toward the critical line, the limit cycles decreased in size and vanished at the critical line. This is consistent with Fig. 13. Further, no limit cycles were found outside the shaded region, in agreement with Akcasu and Shotkin.⁽²⁰⁾

Figure 12. Region of stable limit cycles in the second approximation (effective-lifetime model).

Figure 13. Frequency and amplitude of stable limit cycles in the second approximation for $y = 1.5$ (effective-lifetime model).

Complete correspondence between the exact solutions and the Fourier series method in this approximation is certainly not to be expected. It is not surprising that the correspondence is poor far away from the critical line. In particular, the behavior near $x = -1.5$ in Fig. 13 where $|a_1|^2 \rightarrow 0$ has no significance; the linear system is not oscillatory in that neighborhood, and it is difficult to see how nonlinearities could produce a stable limit cycle there.

Prompt-Jump Approximation

We retain Eq. (127) for the feedback, but with the neutron dynamics obtained from Eq. (108):

$$W_1(i\omega) = \frac{i\omega}{i\omega + \lambda} \quad (141)$$

Use Eq. (121) for $|a_1|^2$, with

$$\kappa_0 = -W_2^{-1}(0) = \frac{\eta_1 \eta_3}{K\xi} \quad (142)$$

as obtained from Eqs. (116), (117), and (127). We find

$$|a_1|^2 = - \frac{\eta_1 \eta_3 - \omega^2 + K(\lambda + \xi) + i\omega(\eta_1 + \eta_3 + K - \frac{K\lambda\xi}{\omega^2})}{[K(\lambda + \xi) + i\omega K(1 - \frac{\lambda\xi}{\omega^2})] [\frac{2\eta_1 \eta_3}{K\xi} - 1 - \frac{\lambda^2}{2\omega^2} + i \frac{\lambda}{2\omega}]} \quad (143)$$

The denominator of (143) may be written as

$$\begin{aligned} & \frac{2\lambda\eta_1\eta_3}{\xi} + 2\eta_1\eta_3 - K\xi - \frac{3}{2}K\lambda - \frac{K\lambda^3}{2\omega^2} \\ & + i \left[\frac{2\eta_1\eta_3\omega}{\xi} - K\omega - \frac{2\lambda\eta_1\eta_3}{\omega} + \frac{3K\lambda\xi}{2\omega} + \frac{K\lambda^3\xi}{2\omega^3} \right] \end{aligned}$$

Real $|a_1|^2$ requires that the numerator and denominator of (143) have the same phase. This yields a cubic equation in ω^2 :

$$\begin{aligned}
 & \left(\frac{2\eta_1\eta_3}{\xi} - K \right) \omega^6 + \left\{ \frac{2\eta_1\eta_3}{\xi} [(\eta_1 + \eta_3)(\lambda + \xi) - \eta_1\eta_3 - \lambda\xi] \right. \\
 & \quad \left. + [\eta_1\eta_3 + \frac{3}{2}\lambda\xi - (\eta_1 + \eta_3)(\xi + \frac{3}{2}\lambda)]K - \frac{1}{2}\lambda K^2 \right\} \omega^4 \\
 & \quad + \left\{ 2\lambda\eta_1^2\eta_3^2 + \frac{1}{2}\lambda [\lambda^2(\xi - \eta_1 - \eta_3) - 3\eta_1\eta_3\xi]K \right. \\
 & \quad \left. - \frac{1}{2}\lambda(\xi^2 + \lambda^2)K^2 \right\} \omega^2 - \frac{1}{2}K\lambda^3\xi(\eta_1\eta_3 + K\xi) = 0
 \end{aligned} \tag{144}$$

Each real ω^2 from (144) corresponds to a real number $|a_1|^2$ given by

$$|a_1|^2 = \frac{\omega^2 - \eta_1\eta_3 - K(\lambda + \xi)}{\frac{2\lambda\eta_1\eta_3}{\xi} + 2\eta_1\eta_3 - K\xi - \frac{3}{2}K\lambda - \frac{K\lambda^3}{2\omega^2}} \tag{145}$$

We test all real roots of (144) for which $\omega^2 > 0$, and retain those for which $|a_1|^2 > 0$ in (145).

The product of roots of Eq. (144) is

$$\omega_1^2 \omega_2^2 \omega_3^2 = \frac{K\lambda^3\xi^2(\eta_1\eta_3 + K\xi)}{2(2\eta_1\eta_3 - K\xi)} \tag{146}$$

We can look for sign changes of a root ω^2 by seeking cases where (146) is 0 or ∞ . Four cases are distinguished:

1)	$K\xi = 2\eta_1\eta_3,$	$x+y = 1,$	$\omega^2 \rightarrow \infty,$	$ a_1 ^2 \rightarrow \infty$
2)	$K = 0 \ (\xi \rightarrow \infty),$	$x = 0,$	$\omega^2 \rightarrow \infty,$	$ a_1 ^2 \rightarrow \infty$
3)	$K\xi = 0,$	$x+y = 0,$	$\omega^2 \rightarrow 0,$	$ a_1 ^2 \rightarrow 0$
4)	$K\xi = -\eta_1\eta_3,$	$x+y = -\frac{1}{2},$	$\omega^2 \rightarrow 0$	$ a_1 ^2 \rightarrow 0$

Case 4 is irrelevant because it is in the range of positive steady-state power coefficient.

In addition, we have the critical line where $\omega \rightarrow \omega_c$ and $|a_1|^2 \rightarrow 0$. It is verified by numerical computation that one root ω^2 yields $|a_1|^2 > 0$ in the region bounded by Cases 1, 2, and 3 and the critical line. This region, the shaded area in Fig. 14, is linearly stable, so this approximation predicts unstable limit cycles.*

Numerical calculations for this $|a_1|^2$ and the corresponding ω^2 are shown in Fig. 15 for $y = 1$. The critical line passes through the point $x = -0.633$, $y = 1$ where $\omega^2 \rightarrow \omega_c^2 = 0.110$ and $|a_1|^2 \rightarrow 0$. Between $x = -0.633$ and $x = -1$, this $|a_1|^2$ is negative, returning to zero at $x = -1$ as predicted by Case 3 above. Of course, the large values of $|a_1|^2$ for x not near the critical line must be discarded as outside the scope of this approximation.

With the analog computer it was found that unstable limit cycles exist only in part of the shaded region of Fig. 14, near the critical line. As one moves away from the critical line, the limit cycle grows, as suggested by Fig. 15, but not far from the critical line the separatrix between stable and unstable regions in the state space becomes open to ∞ . This is illustrated by the analog traces in Fig. 16, obtained at $x = -0.5$, $y = 1$ (Point A of Ref. 1).

Analog traces demonstrating an unstable limit cycle very close to the critical line (near Point B of Ref. 1) are shown in Fig. 17. The presence of the limit cycle is deduced from the fact that unstable trajectories are seen to wind completely around a closed region before going off to ∞ .

*Eq. (145) also predicts $|a_1|^2 > 0$ in the unstable region above the line $x+y = 1$, but that region was later ruled out by investigating the sign of $|A_1|^2$. This will be discussed in a later report.

Figure 14. Region of unstable limit cycles in the second approximation (prompt-jump model).

Figure 15. Frequency and amplitude of unstable limit cycles in the second approximation for $y = 1$ (prompt jump model).

Figure 16. Trajectories illustrating one type of nonlinear instability in the linearly stable region (analog computations).

Figure 17. Unstable limit cycle slightly below critical power (analog computations).

Subsequent studies have indicated that the limit cycles disappear smoothly as the critical line is approached. It was erroneously inferred in Ref. 1 that the disappearance was abrupt. It is very difficult to simulate precisely the resonance condition with the analog computer, and the curves in Ref. 1 labeled Point B are now known to be slightly below critical.

In addition to predicting some limit cycles where none exist (e.g., Fig. 16), this order of analytic approximation fails to predict some that do exist. Unstable limit cycles were also found near the critical line but above the line $x + y = 1$ in Fig. 14. The next section discusses some higher-order analytic approximations that are being investigated.

E. HIGHER APPROXIMATIONS

We have extended Torlin's formulation⁽¹⁹⁾ by retaining two harmonics in the reactivity oscillation. The first iteration yields Eq. (120). The second iteration is similar to Eq. (121) or (130) but with an extra term in the denominator:

$$|a_1|^2 = \frac{W_1(i\omega)W_2^{-1}(i\omega) - 1}{2[\kappa_0 - W_1^{-1}(i\omega)]\text{Re } W_1^{-1}(i\omega) + W_1^{-1}(i\omega)W_1^{-1}(2i\omega)Q} \quad (147)$$

where

$$Q = 1 + W_1^{-1}(-i\omega)W_2(2i\omega)$$

In the third iteration, Q changes to

$$Q = [1 + W_1^{-1}(-i\omega)W_2(2i\omega)][1 + W_1^{-1}(2i\omega)W_2(2i\omega)]$$

For the fourth iteration,

$$Q = [1 + W_1^{-1}(-i\omega)W_2(2i\omega)][1 + W_1^{-1}(2i\omega)W_2(2i\omega) + W_1^{-2}(2i\omega)W_2^2(2i\omega)]$$

The iterative scheme generates a geometric series. If the series converges, the limiting case would be Eq. (147) with Q given by

$$Q = \frac{1 + W_1^{-1}(-i\omega)W_2(2i\omega)}{1 - W_1^{-1}(2i\omega)W_2(2i\omega)} \quad (148)$$

Unfortunately, there are cases in the range of interest for which the convergence is not guaranteed. This question is being studied. Also, direct methods are being considered which do not involve iteration.

One interesting property of Eqs. (147) and (148) follows from Eq. (122) for the critical or resonance frequency:

$$1 - W_1^{-1}(i\omega_c)W_2(i\omega_c) = 0 \quad (149)$$

This guarantees that $|a_1|^2 \rightarrow 0$ at critical. However, when $\omega \rightarrow \frac{1}{2}\omega_c$, Eqs. (147) and (148) yield $Q \rightarrow \infty$ and $|a_1|^2 \rightarrow 0$. Since Eq. (122) determines the critical boundary independently of ω , it follows that a subharmonic frequency is predicted by Eqs. (147) and (148). This presumably appears as one of the new roots introduced by the higher approximation, which increases the degree of the polynomial in ω^2 . It is unlikely, however, that subharmonic limit cycles would be meaningful.

F. AN ALTERNATIVE FORMULATION

To circumvent the iterative process, an alternative approach may be considered. We eliminate the reactivity from Eqs. (105) and (107), recalling that $W_1(p)$ and $W_2(p)$ are differential operators, and obtain

$$W_1(p)n = \frac{\bar{N}}{N_0} [W_2(p)n + nW_2(p)n] \quad (150)$$

Postulate the periodic solution Eq. (111), substitute into Eq. (150), and match coefficients. We find

$$A_\ell W_1(i\ell\omega) = \frac{\bar{N}}{N_0} [A_\ell W_2(i\ell\omega) + \sum_{m=-\infty}^{\infty} A_m A_{\ell-m} W_2(im\omega)] \quad (151)$$

Retaining two harmonics in each series, we find for $\ell = 1$

$$A_1 W_1(i\omega) = \frac{\bar{N}}{N_0} \{A_1 W_2(i\omega) + A_{-1} A_2 [W_2(-i\omega) + W_2(2i\omega)]\} \quad (152)$$

and for $\ell = 2$

$$A_2 W_1(2i\omega) = \frac{\bar{N}}{N_0} [A_2 W_2(2i\omega) + A_1^2 W_2(i\omega)] \quad (153)$$

Solve (153) for A_2 , substitute into (152), and solve the resulting equation for $|A_1|^2$ to find

$$|A_1|^2 = \frac{\left[\frac{N_0 W_1(i\omega)}{\bar{N} W_2(i\omega)} - 1 \right] \left[\frac{N_0 W_1(2i\omega)}{\bar{N} W_2(2i\omega)} - 1 \right]}{1 + \frac{W_2(-i\omega)}{W_2(2i\omega)}} \quad (154)$$

where we have used $A_{-1} = A_1^*$.

The ratio $N_0/\bar{N}$ may be derived from Eqs. (117) and (118). Its contribution to the numerator of Eq. (154) is important in the first factor, which is small, but it may be set equal to 1 in the second factor.

Reconciliation of Eqs. (147) and (154) is not yet complete,^{*} though they should be consistent when (154) is regarded as the limit of an iterative process by using (148) for Q.

Another direct approach is under investigation using an integral-equation formulation of reactor dynamics.

G. APPLICATION TO A ONE-REGION EXAMPLE

The Fourier series method should work for centers as well as for limit cycles. Following a suggestion by Shotkin,⁽²⁶⁾ the various expressions for $|a_1|^2$ were tested with the classic problem of Ergen and Weinberg⁽²⁴⁾

$$\frac{dN}{dt} = \frac{\rho}{\ell} N \quad (155)$$

$$\rho = -\alpha T \quad (156)$$

$$\frac{dT}{dt} = K(N - N_0) \quad (157)$$

$$\frac{dN}{dt} = -\frac{\alpha}{\ell} NT \quad (158)$$

The linearized case may be written

$$\frac{d}{dt} \begin{vmatrix} N - N_0 \\ T \end{vmatrix} = \begin{vmatrix} 0 & -\alpha N_0/\ell \\ K & 0 \end{vmatrix} \begin{vmatrix} N - N_0 \\ T \end{vmatrix} \quad (159)$$

^{*} Reconciliation has recently been accomplished by adding the bias reactivity a_0 to the right-hand-side of Eq. (107) before combining with Eq. (105). In Eq. (154), $W_1(i\omega)$ should be replaced by $W_1(i\omega) - a_0$.

The characteristic equation is

$$s^2 + \frac{\alpha K n_0}{\ell} = 0$$

and one identifies the "reactor natural frequency"

$$\Omega^2 = \frac{\alpha K N_0}{\ell}$$

Using an effective lifetime β/λ

$$\Omega^2 = \frac{\lambda \alpha K N_0}{\beta} \quad (160)$$

Using Torlin's notation, one has

$$W_1(p) = \frac{p}{\lambda} \quad (161)$$

$$W_2(p) = -\frac{\Omega^2}{\lambda p} \quad (162)$$

It is well known that the system (157) and (158) admits a first integral, representing closed orbits in the N, T plane provided $\alpha > 0$:

$$\log \frac{N}{N_{\min}} - \left(\frac{N}{N_0} - \frac{N_{\min}}{N_0} \right) = \frac{\alpha T^2}{2K\ell N_0} \quad (163)$$

illustrating a center for which $\bar{N} = N_0$ and for which the frequency approaches Ω for $|\delta N| \ll N_0$.

We first consider Torlin's second iteration with $a_2 = 0$. Eq. (121) yields

$$|a_1|^2 = \frac{2\omega^2}{\lambda^2} \left(1 - \frac{\omega^2}{\Omega^2} \right) \quad (164)$$

Treat (164) as a quadratic in ω^2 and solve with small a_1 :

$$\omega^2 = \begin{cases} \frac{1}{2} \lambda^2 |a_1|^2 \\ \Omega^2 - \frac{1}{2} \lambda^2 |a_1|^2 \end{cases} \quad (165)$$

The second root yields a frequency less than Ω such that $\omega \rightarrow \Omega$ as

$a_1 \rightarrow 0$. It may be expressed in terms of $|A_1|^2$ by means of Eq. (119), whence

$$|A_1|^2 = |a_1|^2 |W_2^{-1}(i\omega)|^2 \quad (166)$$

$$|A_1|^2 = \frac{\lambda^2 \omega^2}{\Omega^4} |a_1|^2 \quad (167)$$

and, correct to first order for ω near Ω ,

$$|A_1|^2 = \frac{\lambda^2}{\Omega^2} |a_1|^2 \quad (168)$$

The frequency is now given by

$$\omega^2 = \Omega^2 (1 - \frac{1}{2} |A_1|^2) \quad (169)$$

Next, we try the second iteration with a_2 included. Using Eq. (147) with

$$Q = 1 + W_1^{-1}(-i\omega)W_2(2i\omega)$$

We find

$$|a_1|^2 = \frac{2\omega^2}{\lambda^2} (1 - \frac{\omega^2}{\Omega^2}) (1 - \frac{\Omega^2}{2\omega^2})^{-1} \quad (170)$$

This yields a cubic in ω^2 which for small a_1 has roots

$$\omega^2 = \left\{ \begin{array}{l} \pm \frac{1}{2} \lambda \Omega |a_1| \\ \Omega^2 - \frac{1}{4} \lambda^2 |a_1|^2 \end{array} \right. \quad (171)$$

Using (168) we find

$$\omega^2 = \Omega^2 (1 - \frac{1}{4} |A_1|^2) \quad (172)$$

For the third example, consider the limit of the iterative process, Eq. (147) with Q given by Eq. (148). One finds

$$|a_1|^2 = \frac{2\omega^2}{\lambda^2} \left(1 - \frac{\omega^2}{\Omega^2}\right) \left(1 - \frac{\Omega^2}{2\omega^2}\right)^{-1} \left(1 - \frac{\Omega^2}{4\omega^2}\right) \quad (173)$$

This yields another cubic in ω^2 which for small a_1 has roots

$$\begin{aligned} \lambda^2 |a_1|^2 \\ \omega^2 = \left\{ \frac{1}{4} \Omega^2 - \frac{2}{3} \lambda^2 |a_1|^2 \right. \\ \left. \Omega^2 - \frac{1}{3} \lambda^2 |a_1|^2 \right\} \end{aligned} \quad (174)$$

Note the appearance of the extraneous subharmonic, for which $\omega \rightarrow \frac{1}{2} \Omega$ as $a_1 \rightarrow 0$.

Again using (168), the largest root may be written as

$$\omega^2 = \Omega^2 \left(1 - \frac{1}{3} |A_1|^2\right) \quad (175)$$

Note that (169), (172) and (175) are similar, the coefficients of the perturbation being respectively $1/2$, $1/4$, and $1/3$.

Finally, if there is to be consistency, we hope that (175), the limit of the iterative process, should agree with the direct approach. Using Eq. (154) with $\bar{N} = N_0$, we find

$$\omega^2 = \Omega^2 \left[1 - \frac{\Omega^2 |A_1|^2}{4\omega^2 - \Omega^2}\right] \quad (176)$$

from which

$$4\omega^4 - 5\Omega^2\omega^2 + \Omega^4(1 + |A_1|^2) = 0 \quad (177)$$

The approximate roots of (177) are

$$\omega^2 = \left\{ \begin{array}{l} \frac{1}{4} \Omega^2 (1 + \frac{4}{3} |A_1|^2) \\ \Omega^2 (1 - \frac{1}{3} |A_1|^2) \end{array} \right. \quad (178)$$

It is interesting that the subharmonic $\frac{1}{2} \Omega$ appears once again. It is highly encouraging that the physically significant root in Eq. (178) is identical to (175). We conclude that Eq. (154) for $|A_1|^2$ in the direct method is consistent with Eqs. (147) and (148), the limit of the iterative process, at least for a simple example without delayed neutrons where $\bar{N}/N_0 = 1$ and $a_0 = 0$.

Subsequently we plan to compare the various formulations, including the integral-equation formulation, and carry forward the comparison of higher-order Fourier series approximations with analog and digital computations.

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