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Some Singular Integral Equations in Linear Transport Theory

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SOME SINGULAR INTEGRAL EQUATIONS IN LINEAR TRANSPORT THEORY

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ABSTRACT

Solutions to the steady-state transport equation for one speed, isotropic scattering in a half-space can be found from singular integral equations with Cauchy principal-value integrals. Galerkin methods for these equations are studied with special emphasis on sensitivity of approximation errors to choice of basis.

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SOME SINGULAR INTEGRAL EQUATIONS IN NEUTRON TRANSPORT THEORY

I. Introduction

The theory of singular integral equations with Cauchy principal-value integrals⁽¹¹⁾ has been a useful tool in solving one-dimensional problems of radiative transfer,^(1,16) and is fundamental in the Case method of singular eigenfunction expansions in neutron transport theory.⁽²⁾

For homogeneous media, the singular equations can be reduced by methods of analytic function theory to quadratures (for half-space problems) and to Fredholm integral equations (for slab problems).⁽¹⁾ For media with the phase function depending on optical depth, singular equations have been derived,^(9,10,7) but classical methods do not seem to regularize the equations.

Even for homogeneous media, the analytical manipulations are rather complex, often involving a Wiener-Hopf factorization.⁽⁵⁾ It is suggested then that numerical methods be applied directly to the singular equations.^(10,13,14,5a) This approach should be contrasted both with analytical methods and with numerical methods for treating directly the transport equation.⁽¹²⁾

In this paper we continue investigation of the albedo problem for a half-space.^(6,7,10) Galerkin methods are studied analytically and numerically. We clarify difficulties of some previous numerical

calculations with singular equations, (6,7,15) and present numerical results.

II. The Albedo Problem

For a given function w , with $0 \leq w(x) < 1$, we consider the one-dimensional steady-state transport equation

$$u \frac{\partial \psi}{\partial x} + \psi = \phi ,$$

$$\phi(x) = \frac{w(x)}{2} \int_{-1}^1 \psi(x, v) dv, \quad 0 \leq x < \infty, \quad -1 \leq u \leq 1, \quad (1a)$$

with boundary conditions

$$\psi(0, u) = \psi_0(u), \quad 0 \leq u \leq 1 ,$$

$$\psi(x, u) \rightarrow 0 \quad \text{as } x \rightarrow \infty . \quad (1b)$$

The half-space will be called homogeneous if w in Eq. 1a is a constant function.

We concentrate on computing the albedo of the half-space defined by the input data ψ_0 as

$$\text{Alb} = \frac{\int_0^1 u \psi(0, -u) du}{\int_0^1 u \psi_0(u) du} . \quad (2)$$

This is determined if we compute the outgoing flux

$$S(u) = u \psi(0, -u), \quad 0 \leq u \leq 1 . \quad (3)$$

With results of a previous paper,⁽¹⁰⁾ it is possible to extend the numerical methods below to compute $\psi(x,u)$ for any fixed x , $0 \leq x < \infty$. We indicate this extension in the last section, but for simplicity and comparison with other numerical results, we restrict attention to computing the albedo.

We use the fact that Eqs. 1 are equivalent to the familiar equations

$$\psi(x,u) = \psi_0(u)e^{-x/u} + \frac{1}{u} \int_0^x e^{-(x-y)/u} \phi(y) dy, \quad (4a)$$

$$\psi(x,-u) = \frac{1}{u} \int_x^\infty e^{-(y-x)/u} \phi(y) dy, \quad 0 \leq u \leq 1, \quad (4b)$$

and

$$\phi(x) = \frac{w(x)}{2} \int_0^1 \psi_0(v) e^{-x/v} dv + \frac{w(x)}{2} \int_0^\infty E_1(x-y) \phi(y) dy. \quad (5)$$

In terms of ϕ , we have S expressed by

$$S(u) = \int_0^\infty e^{-y/u} \phi(y) dy. \quad (6)$$

Singular integral equations for various functions and special choices of w can be derived from Eq. 5 by transform methods, and also from Eqs. 1 by an extension of the Case method of singular eigenfunctions.^(6,7,10) With the choice of

$$w(x) = \int_0^{\infty} e^{-x/\eta} c(\eta) d\eta, \quad 0 \leq x < \infty. \quad (7)$$

satisfying $0 \leq w(x) \leq c < 1$, the Laplace transform of ϕ in Eq. 5 gives, with Eq. 6, the result^(10, Eq. 24)

$$\begin{aligned} S(u) = & \frac{1}{2} \int_0^1 v \psi(v) \int_0^{\infty} \frac{c(\eta)p}{p+v} d\eta dv \\ & + \frac{1}{2} \int_0^{\infty} c(\eta)p \int_0^1 \left[\frac{S(p) - S(v)}{p-v} + \frac{S(p)}{p+v} \right] dv d\eta, \end{aligned} \quad (8)$$

with

$$p = \frac{u\eta}{u+\eta}. \quad (9)$$

If $c(\eta)$ in Eq. 8 is a finite sum of δ -functions, the equation perhaps offers a simpler computation for the outgoing flux $S(u)$ than direct methods for the transport equation.⁽¹²⁾ When $c(\eta) = \delta(\eta-\infty)$, Wiener-Hopf methods⁽⁵⁾ or methods of invariance⁽³⁾ are most often used to express S in terms of Chandrasekhar's H-function.

III. Existence and Uniqueness

A well-known result⁽⁵⁾ for Eqs. 1 is unique solvability in L^1 if $\|w\|_{\infty} < 1$. Eq. 8 is a necessary condition on this unique solution, so existence is trivial; however, uniqueness is not.

For computation of the outgoing flux $S(u)$, the variable u is restricted to $0 \leq u \leq 1$. Eq. 8 has been studied in the Hilbert spaces $L^2[0,1]$ and $L^2[0,1; u^2 du]$ by using classical properties of the Hilbert transform.^(6,9) Unique solvability is guaranteed^(9,10) if

$$\int_0^\infty \frac{v}{1+v} \left[\pi + \sqrt{\frac{1+v}{v}} + \ln(1+2v) \right] c(v) dv < 1. \quad (10)$$

Nonuniqueness of solutions to Eq. 8 in $L^2[0,1]$ is well known for $c(\eta) = c\delta(\eta-\infty)$, that is $w(x) = c$ in Eq. 7. This result follows by classical methods for singular integral equations.^(1,11) If w is not constant and Eq. 10 is violated, a spectral analysis in $L^2[0,1]$ is needed for the operator on the right-hand side of Eq. 8 to determine when 1 is in the spectrum. For the simpler Eq. 12 below, it is not difficult to find conditions on c and s sufficient for 1 to be in the spectrum; however, further classification of this spectral point is obvious only when $s = \infty$.

Numerical computations have been done with Eq. 8 for the special choice

$$c(\eta) = c\delta(\eta - s), \quad 0 \leq c < 1, \quad 0 < s, \quad (11)$$

that is,

$$S(u) = \frac{cp}{2} \int_0^1 \frac{\psi_0(v) dv}{p+v} + \frac{cp}{2} \int_0^1 \left[\frac{S(p) - S(v)}{p-v} + \frac{S(v)}{p+v} \right] dv, \quad (12)$$

$$p = \frac{us}{u+s}.$$

Both Galerkin and collocation methods have been used to determine the A_n if the approximation

$$S_N(u) = \sum_{n=1}^N A_n P_n(u) \quad (13)$$

is used in Eq. 12 for some choice of basis functions $\{P_n\}$. Numerical results have been compared with those from the ANISN code.⁽⁴⁾ For polynomial bases, good results are obtained until the condition in Eq. 10 is sufficiently violated,^(6,7,15) e.g., $c = .99$ and $s = 50$.

These numerical difficulties are not surprising when Eq. 12 is considered in $L^2[0,1]$ for c and s in the gap between uniqueness, Eq. 10, and nonuniqueness, $s = \infty$. The main observation of this paper is the following.

With S in Eq. 6 the Laplace transform of the solution ϕ to Eq. 5, we should consider Eq. 8 not in $L^2[0,1]$, but in a space of functions analytic in a half-plane. There is a classical setting for this which is more natural if we make the change of variables in Eq. 8

$$z = \frac{1}{u}, \quad R(z) = S\left(\frac{1}{z}\right) \quad (14)$$

The Hardy space H^2 is the space of Laplace transforms of functions in $L^2[0, \infty]$. The inner product is given by

$$\langle s_1, s_2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_1(it) \bar{s}_2(it) dt \quad (15)$$

The norm defined by this inner product makes H^2 and $L^2[0, \infty]$ isometrically isomorphic. (11a)

It is well known⁽⁵⁾ that Eq. 5 has a unique solution in $L^2[0, 1]$ if $\|w\|_{\infty} < 1$. Eq. 8 therefore has a unique solution in H^2 if $\|w\|_{\infty} < 1$.

We use this observation in the next section to study numerical methods for Eq. 12 with basis functions in H^2 .

IV. Numerical Results

We denote the Laplace transform by

$$\tilde{\phi}(z) = \int_0^{\infty} e^{-zx} \phi(x) dx, \quad (16)$$

so that R in Eq. 14 is

$$R(z) = \tilde{\phi}(z) \quad (17)$$

For the special choices

$$\psi_0(v) \equiv 1, \quad w(x) = ce^{-x/s}, \quad (18)$$

we write Eq. 12 in terms of R and $a = 1/s$ as

$$R(z) = \frac{c}{2} \int_0^1 \frac{v dv}{1 + v(z+a)} + \frac{c}{2} \int_1^\infty \left[\frac{R(z+a) - R(t)}{t - z - a} + \frac{R(z+a)}{t + z + a} \right] \frac{dt}{t} . \quad (19)$$

This will be written symbolically as

$$R = F + L(R) , \quad (20)$$

and identified as the Laplace transform of Eq. 5. For $0 \leq c < 1$, this has a unique solution in H^2 . For $a > 0$, it is also true that the operator L is compact in H^2 , because the integral operator in Eq. 5 is Hilbert-Schmidt in $L^2[0, \infty]$. For $a = 0$, L is self-adjoint and positive definite, again by Eq. 5.

For $\{r_n\}$, an orthonormal basis in $L^2[0, \infty]$, we choose the approximation

$$R_N = \sum_{n=1}^N A_n \tilde{r}_n \quad (21)$$

in Eq. 20 to get the equations

$$\sum_{n=1}^N A_n \left[\delta_{n,m} - \langle L(\tilde{r}_n), \tilde{r}_m \rangle \right] = \langle F, \tilde{r}_m \rangle . \quad (22)$$

These have a unique solution for N sufficiently large, and $\{S_N\}$ converges in H^2 to the unique solution for Eq. 20. This is a familiar result for Galerkin methods ($a > 0$) and Ritz methods ($a = 0$).⁽⁸⁾

We choose the special functions

$$r_n(x) = \sqrt{2\gamma} L_n(2\gamma x) \quad (23)$$

for the Laguerre functions

$$L_n(x) = \frac{e^{x/2}}{(n-1)!} \frac{d^{(n-1)}}{dx^{(n-1)}} \left[x^{(n-1)} e^{-x} \right], \quad (24)$$

and γ a positive parameter. The approximation in Eq. 21 is in terms of the rational functions

$$\tilde{r}_n(z) = \sqrt{2\gamma} \frac{(z-\gamma)^{n-1}}{(z+\gamma)^n}, \quad (25)$$

so that the outgoing flux is approximated by

$$S_N(u) = \sum_{n=1}^N A_n \sqrt{2\gamma} \frac{u(1-u\gamma)^{(n-1)}}{(1+u\gamma)^n}, \quad (26)$$

and the albedo by

$$\text{Alb}_N = \frac{2\sqrt{2}}{\gamma^{3/2}} \sum_{n=1}^N A_n \int_b^1 v^{(n-1)} \frac{1-v}{(1+v)^2} dv, \quad (27)$$

where $b = (1-\gamma)/(1+\gamma)$ if $0 < \gamma \leq 1$.

By Eq. 15, the matrix entries in Eq. 22 are

$$\langle L(\tilde{r}_n), \tilde{r}_m \rangle = \frac{-\sqrt{2\gamma}}{2\pi i} \int_{-i\infty}^{i\infty} L(\tilde{r}_n)(z) \frac{(z+\gamma)^{(m-1)}}{(z-\gamma)^m} dz. \quad (28)$$

The Cauchy theorem and the fact that $L(\tilde{r}_n)$ is in H^2 , give

$$\langle L(\tilde{r}_n), \tilde{r}_m \rangle = \frac{\sqrt{2\gamma}}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[(z+\gamma)^{(m-1)} L(\tilde{r}_n)(z) \right]_{z=\gamma} \quad (29)$$

For $a = 0$ in Eq. 19, the matrix in Eq. 29 is symmetric. An elementary but tedious calculation gives for $0 < \gamma \leq 1$

$$\langle L(\tilde{r}_n), \tilde{r}_m \rangle = \begin{cases} \frac{c}{\gamma} \ln(1+\gamma), & n = m \\ -\frac{c}{2\gamma} \int_b^1 v^{|n-m|-1} \frac{1-v}{1+v} dv, & |n-m| \geq 1 \end{cases} \quad (30)$$

and from Eqs. 19 and 20,

$$\langle F, \tilde{r}_m \rangle = \frac{c}{2\gamma^{3/2}} \int_b^1 v^{(m-1)} \frac{1-v}{(1+v)^2} dv \quad (31)$$

The parameter γ is still free to be chosen subject to $0 < \gamma \leq 1$. Numerical results based on Eqs. 22 and 30 in Table 1 show sensitivity to the parameter γ .

Table 1

The Albedo for Isotropic Incidence with $w(x) = c$

c	.9	.95	.97	.99	.99	.999	.999	.9999	.9999
γ	.5	.5	.5	.5	.1	.5	.1	.5	.1
N									
2	.4776	.5962	.6691	.7773	.7927	.8486	.9287	.8570	.9630
4	.4779	.5966	.6711	.7936	.7944	.9043	.9297	.9214	.9743
6	.4780	.5966	.6711	.7945	.7945	.9202	.9297	.9445	.9765
8	.4780	.5967	.6711	.7946	.7945	.9259	.9297	.9560	.9770
10	.4780	.5967	.6711	.7946	.7945	.9282	.9297	.9628	.9771

.7949 (ANISN)*

For $a > 0$, the matrix in Eq. 29 is not symmetric. Entries contain terms such as

$$\frac{1}{(m-1)!} \int_1^\infty \frac{d^{(m-1)}}{dz^{(m-1)}} \left[\frac{(z+\gamma)^{(m-1)} (z+a-\gamma)^{(n-1)}}{(z+a+\gamma)^n (t+(z+a))} \right]_{z=\gamma} \frac{dt}{t},$$

which can be reduced to an integral on $[0,1]$ once the derivative is evaluated. Since these become lengthy even for moderate n and m , we have done numerical computations for $a \geq 0$ by a choice of weighted residuals different from Eq. 22.

We have only numerical evidence that a good approximation to the albedo results for

$$\sum_{n=1}^N A_n \left[\langle \tilde{r}_n, \psi_m \rangle - \langle L(\tilde{r}_n), \psi_m \rangle \right] = \langle F, \psi_m \rangle \quad (32)$$

with $\text{Re}(z_m) > 0$ and

$$\psi_m(z) = \frac{1}{z + z_m} \quad (33)$$

This choice, by the inner product in Eq. 15, is equivalent to collocation, so that Eq. 32 becomes

$$\sum_{n=1}^N A_n \left[\tilde{r}_n(z_m) - L(\tilde{r}_n)(z_m) \right] = \frac{c}{2} \int_0^1 \frac{v dv}{1 + v(z_m + a)} \quad (34)$$

The component $(I - P_N) \psi_m$ of ψ_m orthogonal to $\tilde{r}_1, \dots, \tilde{r}_N$ has norm, for real z_m ,

$$\|(I - P_N) \psi_m\| = \|\psi_m\| \left(\frac{z_m - \gamma}{z_m + \gamma} \right)^N.$$

We make the ad hoc decision to keep $|z_m - \gamma|/|z_m + \gamma| < \frac{1}{2}$ so that this projection for large N is small, uniformly in m. Numerical results based on Eq. 34 are given in Table 2 for $N = 2, 4, 6, 8, 10$, $\gamma = .5 + 1/s$, and

$$z_m = \gamma \frac{19 + m}{21 - m}$$

Table 2

The Albedo for Isotropic Incidence with $w(x) = .99 e^{-x/s}$

N \ s									
	0.5	1.0	10	50	100	500	10^3	10^4	$\frac{10^6}{10^8}$
2	.1887	.2614	.5145	.6518	.6959	.7545	.7647	.7748	.7760
4	.1894	.2615	.5146	.6527	.6960	.7599	.7742	.7909	.7931
6	.1894	.2615	.5146	.6528	.6961	.7599	.7743	.7919	.7944
8	.1895	.2615	.5146	.6528	.6961	.7599	.7743	.7920	.7945
10	.1895	.2615	.5146	.6528	.6961	.7599	.7743	.7920	.7945
ANISN*	.1899	.2620	.5149	.6530	.6963	.7602	.7746	.7923	

*Numerical simulation: finite slab of 50 mean free paths with 500 intervals; S-32 Gauss quadrature.

Error estimates and a study of convergence as N goes to infinity would require an analysis of the change of basis from $\{\tilde{r}_1, \dots, \tilde{r}_N\}$ to $\{P_N \psi_1, \dots, P_N \psi_N\}$. We have not attempted this analysis.

V. Conclusion

Eq. 12 for $0 \leq u \leq 1$ can be written in terms of a Cauchy principal-value integral as

$$s(u) = \frac{cp}{2} \int_0^1 \frac{\psi_0(v) dv}{p+v} + s(p) \frac{cp}{2} \ln \frac{1+p}{1-p} + \frac{cp}{2} \text{p.v.} \int_0^1 \frac{S(v) dv}{v-p}, \quad (35)$$

with $p = us/(u+s)$, and s a positive parameter.

For $s = \infty$, this equation is of classical type,⁽¹¹⁾ and numerical methods are discussed in Reference 5a. For $s < \infty$, the equation is not classical, and this theory does not seem to apply. By interpreting Eq. 12 in the Hardy space H^2 , these difficulties are avoided.

We have restricted attention to computing the outgoing flux $S(u) = u\psi(0, -u)$. We conclude by indicating applicability of these computational methods to interior fluxes.

For $\psi_0(v) = \delta(v-u)$, $w(x) = ce^{-x/s}$, and ϕ the solution to Eq. 5, we denote dependence on u by setting $J(x, u) = \phi(x)$. Then for x a parameter, we have from Eq. 17 of Reference 10, with appropriate changes in Eqs. 2a and 4 of the same reference,

$$J(x, u) = \frac{ce^{-x/p}}{2} + \frac{cp}{2} \int_0^1 \left[\frac{J(x, p) - J(x, v)}{p-v} + \frac{J(x, p)}{p+v} \right] dv, \quad (36)$$

with $p = \frac{us}{u+s}$.

With fixed x and u_0 , $0 \leq u_0 \leq 1$, and $F(u)$ defined by Eq. 4b as

$$F(u) = \int_x^\infty e^{-(y-x)/u_0} J(y,u) dy ,$$

we obtain from Eq. 36

$$F(u) = \frac{cpu_0}{2(p+u_0)} e^{-x/p} + \frac{cp}{2} \int_0^1 \left[\frac{F(p) - F(v)}{p - v} + \frac{F(p)}{p + v} \right] dv, (37)$$

which is an equation in H^2 . This determines the flux $u_0 \psi(x, -u_0)$ in its dependence on the input angle, $u = \cos \theta$. A similar equation can be derived for the flux in Eq. 4a. The methods of the previous section can be used to compute interior fluxes.

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