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DESCRIPTION OF A THERMONUCLEAR REACTOR BASED
ON THE USE OF A LAYER OF RELATIVISTIC ELECTRONS
TO CONFINE AND HEAT THE PLASMA

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DESCRIPTION OF A THERMONUCLEAR REACTOR BASED ON
THE USE OF A LAYER OF RELATIVISTIC ELECTRONS TO
CONFINE AND HEAT THE PLASMA

by

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A long layer of rotating relativistic electrons is employed in the proposed scheme of a thermonuclear reactor to perform the following functions:

(a) The magnetic field of the rotating relativistic electrons, in combination with an external magnetic field, provides a closed pattern of magnetic field lines.

(b) The rotating electrons ionize neutral gas, initiating the plasma, and subsequently this plasma, which is trapped within the pattern of closed magnetic lines, is heated up to fusion temperature by scattering with the relativistic electrons. Hence the electrons continuously lose energy and as they have finite lifetimes they are replaced continuously; thus the energy of the layer (hereafter called E-layer) is maintained at a constant level.

The equilibrium of the plasma under these conditions has been investigated and solutions have been found satisfying Maxwell's, hydrodynamic, and diffusion equations in steady state.

The stability of this equilibrium distribution has been investigated, the results being that only stable solutions are admitted in the case of the so-called "flute instability" and "exchange of magnetic lines."

Parameters of a production machine are presented as well as parameters of an experimental model which will be built under an experimental program just initiated in Livermore.

The general requirements for an ideal scheme intended to confine a plasma in adequate density and temperature in order to produce thermonu-

clear reactions in large scale, can be summarized as follows:

(1) Establishing a magnetic field pattern of completely closed magnetic lines before injecting any plasma into this pattern.

(2) Since, as is well known, it is not possible to initiate a plasma within a uniform magnetic field, due to random motion, or Fermi drift, this pattern of closed magnetic lines should be established in such a way that (a) in a certain region the field has a minimum value, and (b) as one moves from this region outward one encounters an increasing magnetic field. In this fashion the plasma tends to drift inward; only after a density gradient is established will the plasma tend to move slowly outward at a rate as allowed by diffusion.

(3) The third necessary requirement is that, after establishing this magnetic field pattern, means are required to ionize neutral particles within this pattern, thus initiating the plasma.

(4) After initiating the plasma, means are required within this magnetic field pattern to heat the plasma up to a temperature where fusion reactions can take place at an adequate rate, so that the thus-released thermonuclear energy will be higher than the energy loss by diffusion and radiation.

The above requirements can be accomplished in the proposed scheme only by one "means" which creates the field pattern and ionizes and heats the plasma up to fusion temperature. This "means" is a "layer of rotating relativistic electrons." The required energy of these electrons is about 3 Mev for a model reactor and 20 to 30 Mev for a production machine. This electron layer, or E-layer, is established as follows:

Within an evacuated cylindrical vessel a magnetic field is established by means of the coils (Fig. 1, Nos. 1, 2, and 3). The direction of this field is substantially parallel to the axis of the cylinder, converging at both ends in order to reflect the electrons. The electrons are injected almost tangentially between two coaxial cylinders (Fig. 1, No. 4). In the gap between the two cylinders, a traveling magnetic field wave is established by means of two helices (Fig. 1 No. 5). The velocity of this wave is about one thousandth of the light velocity. The electrons are injected periodically in short pulses, then they are trapped in the trough of the wave and are transferred to the region where the layer is to be established. The electron injection starts when the field B_0 , far from the ends, is equal to mc^2/eR . The vector potential of the external field is

$$A = B_0 \left[\frac{r}{2} + \frac{a}{k} J_1(kr) \cdot \frac{\cosh(kz)}{\cosh(kL)} \right] \quad (1)$$

where $kr_1 = 1.84$, a is a constant, and r_1 and $2L$ are the desired radius and length of the layer, respectively. By such a field shape it can be achieved that the gyration radius of the electrons remains constant along

the layer, although the azimuthal momentum of the electrons varies along said layer. The electrons encounter, at the point of their ejection from the traveling wave, a radial field which accelerates them in the axial direction. The ratio of the total momentum (p) to the azimuthal momentum (p_θ), far from the ends, depends on the value of the constant " α " in the expression in the vector potential, namely,

$$\frac{p}{p_\theta} = 1 + \alpha J_0(kr_i) . \quad (2)$$

The value of " α " will be chosen such that this ratio will be almost 1.4. Thus the electrons spiral back and forth at a frequency which is not much lower than their gyrofrequency.

As soon as the electrons are concentrated in an appreciable number within the layer, their own magnetic field starts to depress the magnetic field within the volume enclosed by the layer. When the charge per unit length q becomes equal to their energy V (both expressed in esu), or, expressed differently, when the number N of the electrons per unit length becomes equal to

$$N = \frac{mc^2}{e} = \frac{m}{m_0} \cdot \frac{1}{r_e} \quad (3)$$

where m is the electron relativistic mass, m_0 the rest mass and r_e the classical electron radius, then the field inside the enclosed volume is reduced to zero. By increasing somewhat more the charge per unit length of the layer, we reverse the field within the enclosed volume. At that moment the combination of the layer field and the external field provide the pattern of closed magnetic lines.

Now if we have a neutral gas within the vessel, this gas is ionized by the electrons, and thus the plasma is initiated. The electrons continuously lose energy by scattering and consequently they have a finite lifetime. Therefore, their injection continues in order to replace the electrons lost during this process. However, the energy loss of the electrons is an energy gain for the plasma. Consequently the plasma starts to be heated. This energy gain is proportional to the plasma density and the energy of the electrons. The diffusion and radiation losses are proportional to the density and to the square of the density, respectively, and also are functions of the temperature. Consequently, for a given plasma density and electron energy there corresponds a limiting temperature which can thus be realized.

For example, with 3-Mev electrons a plasma density of 10^{12} and a temperature of 50 kv are possible if the rate of diffusion obeys, at that temperature, the classical equation. With electron energy of 20 Mev and plasma density 5×10^{14} , a temperature of 20 kv can be obtained, and so on.

As soon as the plasma temperature starts to rise, the plasma starts to diffuse outward. The diffusion current in turn creates a pressure

gradient and a Hall current. The Hall current in turn modifies this magnetic field pattern of closed lines. This modification, however, tends to increase the density of the magnetic lines of the pattern and not to destroy them. Of course during the build-up of the plasma the intensity of the external magnetic field should be increased so that at any moment $B_{ex}^2 = 8\pi p + B_e^2$; where p is the plasma pressure, B_e the field at the surface of the layer, far from the ends, and B_{ex} the external magnetic field, also far from the ends.

An immediate question that now arises is whether or not a self-consistent equilibrium of the plasma exists under these conditions. In order to solve this problem, I started with the assumption that in a cross section of the plasma, on a plane normal to the axis of symmetry, far from the ends, the magnetic field goes through zero within the plasma at an unknown radius $r_i < r_o$, where r_o is the radius of the boundary of the plasma, and further that the flux through this cross section between r_i and r_o is equal and opposite to the flux between 0 and r_i ; the Larmor radius has been assumed negligible relative to the physical dimensions of the system. Then with the aid of Maxwell equations, hydrodynamic equations, and the diffusion equation, a class of solutions has been obtained for the cylindrical part, far from the ends, and one solution has been worked out for the whole volume including the ends (see Appendix I). The solution for the vector potential in that case is

$$A_\theta = \left[c_1 J_1(k_1 r) + c_2 J_1(k_2 r) \right] \frac{\cosh(kz)}{\cosh(kL)} + A(r) \quad (4)$$

and

$$A(r) = \frac{c_3}{r} e^{-\lambda r^2} \quad \text{for } r < r_i$$

$$A(r) = \frac{c_3}{r} e^{\lambda(r^2 - r_o^2)} \quad \text{for } r_i < r < r_o$$

This solution requires at $r = r_i$ a field discontinuity where the field jumps from $-B_o e^{-\lambda r_i^2}$ to $+B_o e^{-\lambda r_i^2}$. This discontinuity can be realized only by the presence of a sheath of rotating charged particles and where within the thickness of this sheath the field goes through zero. The existence, for other reasons, of the E-layer makes possible the existence of this solution, otherwise from the investigation of such an equilibrium of the plasma it would have been possible to discover the necessity of the E-layer, had not this layer been postulated, long before the above mathematical solution had been obtained. As a matter of fact, a crude description of the proposed E-layer reactor was delivered by the author at the Sherwood Meeting on April 7, 1953, in Berkeley.

In a numerical example, with the aid of the above-given solution, the shape of the plasma, the distribution of the plasma pressure and the

magnetic field have been computed. The result is presented in the graphs (Figs. 2 and 3). In Fig. 2 a cross section on a plane through the axis of symmetry is shown. The magnetic lines (which also are equipressure lines) are shown as well as the shape of the plasma at the ends. The line where $p = 0$ is the last closed magnetic line and consequently the boundary of the plasma. A line just outside this boundary line is open and it goes out very close to the axis. Thus the diffused plasma is guided out through lines very close to the axis of symmetry forming two collimated beams. In that way the diverters, which have been proved necessary in the Stellerator experiments, are built-in inherently.

The above given solution is continuous outside of the plasma and it has only two singular points at the intersection of the boundary line ($p = 0$) with the axis of symmetry. Since there is no plasma out of the boundary line and hence no Hall current, the necessary current as required by the vector potential will be provided by material coils.

In Fig. 3 a cross section of the plasma on a plane normal to the axis of symmetry, far from the ends, is shown. In the upper graph (Fig. 3(a)) the variation of the plasma density is plotted against the radius. We see that the density is zero at the axis of symmetry and the boundary, and it is maximum at the region of the layer. In the lower graph (Fig. 3(b)) the magnetic field distribution is shown. Since the magnetic field is reversed within the plasma volume, the quantity $\beta = 8\pi p/B^2 = 1$ by definition. Further due to the fact that the maximum pressure is not at the axis, the quantity η , where

$$\eta = 2\pi \int_0^{r_0} n^2 r dr / \pi n_0^2 r_0^2, \quad (5)$$

which is proportional to the yield of the thermonuclear reactions, is about 3 times higher than in a parabolic distribution.

In the above solution $\nabla \cdot (\rho V)$ is not zero. The $\nabla \cdot (\rho V) = -S$ in the region $0 < r < r_1$ and $+S$ in the region $r_1 < r < r_0$. If we now assume that all the diffused particles are replaced by neutrals in the region of the maximum pressure, one can say how the sources and sinks required by the solution can be realized. However, on closer examination, one can see that they cancel one another automatically. Since no sources or sinks are present, what actually will happen is this: inside, due to the lack of sinks, the density will tend to increase at a rate S . Along the same magnetic line outside the layer, the density will tend to decrease at the same rate. This rate of increase or decrease of the density is very slow and the change of the pressure is negligible during the time required for the particles to travel along a line. As a result of this density change, a small pressure gradient is created along the lines and the particles which are superfluous inside appear as a source in the same line outside, thus automatically the inside sinks appear outside as sources.

The first problem that arises as soon as the equilibrium distribution is known is the investigation of the stability of this equilibrium.

We can distinguish the possible perturbations in two classes:

(1) Perturbation where the magnetic lines are tilted or where Alfvén waves are propagated along the magnetic lines. In this case the Alfvén waves are always characterized by a Poynting vector traveling along the lines. It is easy to prove that along the cylindrical part (where the lines are parallel to the axis of symmetry) any such motion is stable. Unfortunately, I have not proved this for the whole volume including the ends. However, I can say this: the value of ω^2 resulting from a virtual displacement ξ depends on the integral of the energy change over the whole volume. If we divide this integral into two parts, one restricted in the region where the lines are parallel to the axis and the other over the rest of the volume including the ends, then the first one contributes to positive stability and the lowest root of $|\omega| > S/r$, where S is the sound velocity and r the radius of the plasma volume. If we assume now that the second integral results in a positive value of ω^2 and, hence, instability, we know from other cases that the instability grows the most as S/r . Since the cylindrical volume is more than 10 times the volume of the ends, one can by inference conclude that the over-all result will be stability, except if the perturbation is localized. In this latter case then, a Poynting vector propagating along the lines does not exist and then this perturbation belongs to the second class which is defined by $P \cdot B = 0$, which I shall examine hereafter (where P is the Poynting vector and B the equilibrium magnetic field). Although the above reasoning on the first class of perturbations appears sound, it is not completely satisfactory and consequently an effort will be made to find a more rigorous proof in this case.

(2) The second class is characterized by the absence of a Poynting vector along the lines; or $P \cdot B = 0$.

If E, h are the perturbed electric and magnetic fields, respectively, then the expression: $P \cdot B = E \times h \cdot B = E \cdot h \times B = 0$. In the case where $E = 0$, one finds only trivial motions along the lines or trivial rotation of the plasma. Hence the case which makes sense is

$$B \times h = 0.$$

Now if we assume a perturbation of the type:

$$q = q(r, z) \sin(m\theta) e^{\omega t},$$

then with the aid of the equations:

$$E + V \times B = 0, \tag{6}$$

$$\nabla \times E = -\dot{h}, \tag{7}$$

the above condition ($B \times h = 0$) leads to the general solution

$$\text{for } v_{\theta} = r \cdot f(rA_{\theta}) \sin(m\theta) e^{\omega t} \quad (8)$$

$$E_{\theta} = \frac{1}{r} \phi(rA_{\theta}) \cos(m\theta) e^{\omega t} \quad (9)$$

where f and ϕ are arbitrary functions of rA_{θ} . On the other hand rA_{θ} is constant along a magnetic line. It is now obvious that at the axis of symmetry, $E_{\theta} = 0$. Then it follows that $\phi(rA_{\theta}) = 0$. However, the line that coincides with the axis runs along the boundary surface. Consequently $\phi(rA_{\theta}) = 0$ and hence $E_{\theta} = 0$ also at $r = r_0$, and in general on the boundary surface. Thus, one can see that this kind of perturbation is a completely internal motion, whereas the surface remains at rest. This in turn is due to the fact that all the boundary lines return through the axis of symmetry.

As we move, far from the ends, rA_{θ} degenerates to a function of r only. Then the same holds for f and ϕ and the perturbation degenerates to the type

$$q = q(r) \sin(m\theta) e^{\omega t}.$$

This perturbation has been investigated with the normal mode method. If we define as $\varpi = B \cdot h + \bar{p}$ the perturbation of the energy density (where $\bar{p}$ = the perturbed pressure), then from the linearized equations of motion we derive the equation (see Appendix II):

$$\frac{\partial^2 \varpi}{\partial r^2} + \frac{\partial \varpi}{r \partial r} - \frac{\partial \varpi}{\partial r} \cdot \frac{\partial \rho_0}{\rho_0 \partial r} - \left(\frac{\omega^2}{S^2} + \frac{m^2}{r^2} \right) \varpi = 0 \quad (10)$$

This equation is valid for any disturbance where the perturbed velocity is not constant in time. In the present case where all the plasma particles are subject to rotation due to the Hall current, any radial displacement ξ_r will change V since all the particles are continuously subject to a centrifugal force field. Hence $\xi_r = \dot{v}_r \neq 0$. A thorough investigation of the above equation in the case where v_r (the radial component of the perturbed velocity) and hence E_{θ} becomes zero for two different values of the radius, gave the result that the eigenvalue of ω^2 is always negative, which means stability.

The above investigation of the general solution for $B \times h = 0$ shows that E_{θ} becomes zero for $r = 0$ and $r = r_0$. Consequently all the perturbations of the type $B \times h = 0$ are stable in the present scheme. This result is due to the fact that (1) the plasma is always subject to a centrifugal force field and a case where $v = 0$ is not possible, and (2) the flux and hence (rA_{θ}) is symmetric about the E-layer, resulting in symmetric values of the perturbed quantities about said layer. It should be noted at this point that this class of perturbation " $B \times h = 0$ " includes the

so-called "flute instability" and the rotation of flux tubes. If the latter were neutral or unstable this could result in the so-called line interchange with rapid motion of plasma outward. Since in our case these motions are stable and since these two perturbations are considered as the most dangerous, one can start being somewhat confident as to the stability of the proposed scheme.

Now, I would like to give an example of the dimensions and parameters of a production machine. As mentioned above, due to the fact that the maximum density is occurring far from the axis of symmetry and that $\beta = 1$, the yield in the present case is the highest possible for a given field. Consequently a magnetic field of only 30,000 gauss is adequate to operate a production machine with T-D reactions. The dimensions of the plasma volume in this machine will be 4 ft diameter and 60 ft length; electron energy 20 Mev, average electron current 0.5 amp; plasma temperature 20 kv, plasma density $5 \times 10^{14}/\text{cm}^3$; total heat energy release 1500 megawatts.

The total net electric power, after subtracting the power necessary for the electron accelerator and excitation of the field, is approximately 200 megawatts.

The tritium consumed is completely recovered by $\text{Li}^6 + n = \text{T} + \text{He}^4$ reactions. The Li^6 is placed in the form of Li^6NO_3 in an aqueous solution circulating in cupro-nickel tubes placed about the fusion volume. About 95% of the 14-Mev neutrons released from the T-P reactions are captured by the Li^6 yielding tritium, the other 5% comes from the $(n, 2n)$ reactions with the Cu^{63} . The Cu^{63} $(n, 2n)$ reaction for 13-14 Mev neutrons is of the order of 300 mb. Detail calculations on the subject show that $(n, 2n)$ reactions overcompensate these 5% losses, and it is then possible to recover completely the tritium consumed in the reactor.^a

The Li^6NO_3 solution is pressurized to about 500 psi and its temperature is about 230°C. Circulating through a heat-exchanger produces saturated steam of 120 psi pressure. The steam is superheated by two superheaters placed at each end of the reactor. These superheaters are in the form of hollow cylinders, the walls of which are the superheater elements. The two collimated beams that leave the reactor through both ends are diverted and directed against the superheater tubes. Although the steam is superheated, the over-all conversion efficiency, because of the low steam pressure, is only 20%.

^aSee NCC-5, pp 83-110, by the present author.

A development up to a full-scale reactor is considered possible in ten years if the present experimental program proves successful. This program, which has just been initiated at this Laboratory, aims at the construction of an accelerator and model reactor in about two years. At present about 15 people participate in this program.

The data of the accelerator and reactor are:

- (a) Accelerator: 3 Mev, 100 amp pulsed
 Pulse duration: 0.5 μ sec
 Repetition rate: 60 p.p.s.
 Beam diameter: 1/2"
 Random angular distribution: $\pm 2 \times 10^{-3}$ radians
- (b) Model reactor:
 Tank diameter: 3 ft
 Tank length: 30 ft
 E-layer diameter: 2 ft
 E-layer length: 20 ft
 Maximum plasma density: $10^{12}/\text{cm}^3$ approximately
 Plasma temperature: 10-100 kv
 Maximum external magnetic field: 3000 gauss

If the effort of this experimental program will continue at the present rate, we hope that in about two years we will be ready to start injecting electrons in the tank and the first attempt will be made to establish the E-layer.

The first object of this experiment is to demonstrate the possibility of establishing the E-layer, depressing the field and reversing the field in order to obtain the pattern of closed magnetic lines. If this first step is successful, then the initiation and heating of the plasma will be examined, the rate of fusion and diffusion will be measured and in general an extensive study of hot plasma will be possible under steady-state conditions. Of course, one can expect that many new effects will show up in such an experiment, some useful, others harmful. The aim then is to utilize the useful effects and minimize the harmful ones. Although one deals with a completely unknown field, making difficult any positive predictions, I would like to say that it is my belief that it is within the realm of possibility to develop within five years a model reactor capable of producing more power than it consumes.

APPENDIX I

As mentioned above, the vector potential governing the equilibrium of the plasma, in the proposed reactor, should satisfy the requirement within the plasma that

$$\int_F B_z dF = 0$$

on any plane $z = \text{constant}$, where z is the axis of symmetry and by F is understood the area which on any such plane is occupied by the plasma.

The equations that should be satisfied by a steady-state equilibrium solution are:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{I-1})$$

$$\frac{4\pi}{c} \mathbf{j} = \nabla \times \nabla \times \mathbf{A} \quad (\text{I-2})$$

$$\nabla p = \frac{1}{c} \cdot \mathbf{j} \times \mathbf{B} \quad (\text{I-3})$$

$$\mathbf{v} = -\nabla p \cdot \frac{M}{m\omega^2 \tau_p} \quad (\text{I-4})$$

$$S = \frac{\delta p}{\delta t} + \nabla \cdot (\rho \mathbf{v}) \quad (\text{I-5})$$

and for steady state

$$S = \nabla \cdot (\rho \mathbf{v}) \quad (\text{I-5a})$$

The equilibrium solution is assumed symmetric about the z axis, the co-ordinates are cylindrical: r, θ, z ; the vector potential $\mathbf{A}$ and the current $\mathbf{j}$ have components only in the azimuthal direction.

Further definitions are:

- $\mathbf{B}$ = the magnetic field having components in the radial and z directions
- p = the scalar pressure of the plasma
- ρ = the mass density of the plasma
- $\mathbf{v}$ = the diffusion velocity
- M, m = the ion and electron mass respectively
- τ = the mean time between Coulomb collisions
- $\omega = \frac{e}{mc} \sqrt{B_r^2 + B_z^2}$, the electron gyro-frequency, and
- S = the strength of sources or sinks within the plasma volume.

It is assumed that the diffused particles are replaced by neutrals at the region of highest pressure. Consequently within the plasma volume the condition

$$\int_V S dV = 0$$

should be satisfied by the solution. The solution for the vector potential is assumed to be in the form

$$rA = \phi(r) + f(r) \cdot \frac{\cosh(kz)}{\cosh(kL)} \quad (I-6)$$

and k to be of the order of r_0^{-1} (r_0 is the boundary radius of the plasma), whereas $2L$ (the length of the plasma) is assumed at least $20 r_0$. Thus the second function vanishes very fast as one moves from the end inward, parallel to the z axis. Consequently in this internal region the magnetic lines are parallel to the axis of symmetry. It is then possible to derive first the solution for the infinite cylinder, namely the function $\phi(r)$ only, and thereafter to modify the solution by adding the z -dependent function. In the cylindrical part (far from the ends) we observe that the pressure is

$$p = \frac{B_o^2 - B_z^2}{8\pi} \quad (I-7)$$

The pressure is constant along a magnetic line as well as the function $\psi = rA$. In order to satisfy the basic requirement that

$$\int_0^{r_0} B_z r dr = 0$$

it is obvious that in a region where the radius is less than a value r_i , the value of B is positive and in the remaining region (where $r_i < r < r_0$) the value of B is negative.

From Eq. (I-7), solving B , we obtain

$$B_z = \pm \sqrt{B_o^2 - 8\pi p} \quad (I-7a)$$

which indicates that the same line has a value $+\sqrt{B_o^2 - 8\pi p}$ in the region where $r < r_i$ and the value $-\sqrt{B_o^2 - 8\pi p}$ in the regions $r_i < r < r_0$.

From the equation $B = \nabla \times A$ we have

$$B_z = \frac{\partial(rA)}{r\partial r} \quad (I-1a)$$

Let $\psi_i = rA_i$ in the region $r < r_i$
 $\psi_o = rA_o$ in the region $r_i < r < r_0$.

For every value of $r = r_m$ where $\psi = \psi_i(r_m)$ in the internal region, there corresponds a value r_n in the external region where r_n is determined from the equation

$$\psi_o(r_n) = \psi_i(r_m) \quad (I-8)$$

where $r_m < r_i$ and $r_i < r_n < r_o$

From Eqs. (I-7) and (I-1a) it follows that

$$\frac{\partial \psi(r_n)}{r_n \partial r} = - \frac{\partial \psi(r_m)}{r_m \partial r} \quad (I-8a)$$

The above conditions, (I-8) and (I-8a), are satisfied for any value of r if ψ_o and ψ_i are solutions respectively of the differential equations

$$\frac{\partial \psi_o}{\partial r} - r \psi_o = 0 \quad (I-9)$$

$$\frac{\partial \psi_i}{\partial r} + r \psi_i = 0 \quad (I-9a)$$

Solutions satisfying these differential equations are

$$\psi_o = c_o e^{\lambda r^2} \quad (I-10)$$

$$\psi_i = c_i e^{-\lambda r^2} \quad (I-10a)$$

Taking in consideration that the field should decrease as one moves away from the axis of symmetry, we observe that the second solution corresponds to the inside region. At $r = r_o$, $p = 0$ and $B = -B_o$ then

$$\psi_o = -\frac{B_o}{2\lambda} e^{\lambda(r^2 - r_o^2)} \quad (I-11)$$

and the second equation becomes

$$\psi_i = -\frac{B_o}{2\lambda} e^{-\lambda r^2} \quad (I-11a)$$

At $r = r_i$ it is obvious that

$$\psi_o = \psi_i$$

Then

$$r_i^2 - r_o^2 = -r_i^2 \quad \text{or} \quad r_i = r_o/\sqrt{2}.$$

The magnetic field is

$$\text{for } r < r_i \quad B_z = B_0 e^{-\lambda r^2} \quad (\text{I-12})$$

$$\text{for } r_i < r < r_0 \quad B_z = -B_0 e^{+\lambda(r^2 - r_0^2)} \quad (\text{I-12a})$$

As we approach r_i from the inside we find that

$$\lim_{r \rightarrow r_i} B = B_0 e^{-\lambda r_i^2}$$

as we approach r_i from outside

$$\lim_{r \rightarrow r_i} B = -B_0 e^{-\lambda r_i^2}$$

Consequently the existence of a field jump or field discontinuity is required at $r = r_i$. The value of the intensity of the field jump is

$$B_e = 2B_0 e^{-\lambda r_i^2} \quad (\text{I-12b})$$

The only way to create such a field jump at $r = r_i$ is to provide a current sheath at that radius. Since this current sheath is inside a high temperature plasma, such a current can only be created by organized motion of charged particles. Hence the relativistic electron layer will fulfill this requirement and create the field jump.

$$\text{Let} \quad \lambda = k_e / 2r_i \quad (\text{I-13})$$

$$\text{Then} \quad k_e r_0 = 2 \ln \left(\frac{2B_0}{B_e} \right) \quad (\text{I-14})$$

$$\text{and} \quad B_e = \frac{4\pi}{c} j_e \quad (\text{I-15})$$

where j_e is the current per cm of the electron layer.

The rate of diffusions is

$$\text{for } r < r_i \quad \rho v_r = -\frac{Mmc^2}{e^2} \cdot \frac{r}{r_i^2} 2 \ln \left(\frac{2B_0}{B_e} \right) \quad (\text{I-16})$$

$$\text{for } r_i < r < r_0 \quad \rho v_r = +\frac{Mmc^2}{e^2} \cdot \frac{r}{r_i^2} 2 \ln \left(\frac{2B_0}{B_e} \right) \quad (\text{I-16a})$$

The strength of the required distributed sources inside the plasma are (assuming τ constant)

$$\text{for } r < r_i \quad S = \nabla \cdot (\rho v) = -\frac{Mp_o}{m\omega_o} \cdot \frac{k_e}{r_i} = -S_o \quad (\text{I-17})$$

$$\text{for } r_i < r < r_o \quad S = +\frac{Mp_o}{m\omega_o} \cdot \frac{k_e}{r_i} = +S_o \quad (\text{I-17a})$$

$$\text{and} \quad \int_0^{r_o} S r dv = -S_o \frac{r_i^2}{2} + S_o \frac{r_o^2 - r_i^2}{2} = 0$$

The complete solution with the z -dependent function is assumed in the form for $r > r_i$

$$A_o = -\frac{B_o}{k} \left[c_1 J_1(k_1 r) + c_2 J_1(k_2 r) \right] \frac{\cosh(kz)}{\cosh(kL)} - \frac{B_o}{2\lambda r} e^{-\lambda(r^2 - r_o^2)} \quad (\text{I-18})$$

$$\text{for } r < r_i \quad A_i = -\frac{B_o}{k} \left[c_1 J_1(k_1 r) + c_2 J_1(k_2 r) \right] \frac{\cosh(kz)}{\cosh(kL)} - \frac{B_o}{2\lambda r} e^{-\lambda r^2} \quad (\text{I-18a})$$

The boundary condition which determine the constants are

$$(1) \text{ at } r = r_i \quad r_i A_o = r_i A_i = -B_o/2\lambda$$

$$z = L$$

$$(2) \text{ at } r = r_i \quad B_z \text{ goes through zero, or}$$

$$c_1 k_1 J_0(k_1 r_i) + c_2 k_2 J_0(k_2 r_i) = 0$$

$$(3) \text{ at } r = r_i \quad \partial p / \partial r = 0 \quad (\text{since } B_z = 0)$$

Then

$$\int_L^0 \left(\frac{\partial p}{\partial z} \right) dz = - \int_0^\infty j B r dz = \frac{B_o^2}{8\pi}$$

$$(4) \text{ at } r = r_i \quad B_r = \epsilon B_o$$

$$z = L \text{ where } \epsilon \text{ is given}$$

Furthermore, the arguments $k_1 r_i$ and $k_2 r_i$ should be selected thus so B_r is always finite for every value of r between zero and r_0 except $r = 0$ where it becomes zero. The complete solution for $k_e r_i = 10$, and $\epsilon = 0.184$ is:

For $r > r_i$

$$-\frac{k_e}{B_0 r_i} (r A_0) = 5.435 \times \left[1.666 J_1(2.12x) + 2.136 J_1(5.385x) \right] \frac{\cosh(1.84 \frac{z}{r_i})}{\cosh(1.84 \frac{L}{r_i})} + e^{5(x^2 - 2)} = a$$

For $r < r_i$

$$-\frac{k_e}{B_0 r_i} (r A_i) = 5.435 \times \left[1.665 J_1(2.12x) + 2.136 J_1(5.385x) \right] \frac{\cosh(1.84 \frac{z}{r_i})}{\cosh(1.84 \frac{L}{r_i})} + e^{-5x^2} = a$$

where $x = r/r_i$, $a = 1$ at the boundary line where $p = 0$ and $a = 0$ for $p = p_0 = B_0^2/8\pi$.

The field configuration and the plasma shape have been computed from the above numerical example and plotted in Fig. 2.

The solution is continuous for $a > 1$. This implies that the current distribution as given by said vector potential is finite beyond the plasma boundary where no plasma exists to create such current as required by the vector potential.

Consequently, in order that the above solution can be physically realized, material coils should be placed beyond the plasma boundary. Such coils should be energized in such a way that the current distribution inside those coils is as given by the vector potential. At a certain distance (towards the negative z) from the plasma boundary, the vector potential function should be substituted by another function of the form

$$A = g(r)e^{+kz} + j(r)$$

so that the vector potential vanishes at infinity.

As mentioned above, the solution given here has been obtained in the frame of certain approximations. This approximation breaks near the boundary within the last few Larmor radii.

In a further approximation there should be taken into consideration that:

- (1) the mean time between collisions, τ , is proportional to $1/\rho$,
- (2) that the temperature decreases as the electrons approaching the boundary and encountering higher field lose energy by radiation.,
- (3) the Hall current density at the boundary line goes to zero, since there are no particles there to create such current.

A further approximation in the cylindrical part can be expressed in the form:

$$\text{for } 0 < r < r_i \quad \psi_i = rA_i = \sum_{n=i} c_n e^{-(\lambda_n r^2)^n}$$

$$\text{for } r_i < r < r_o \quad \psi_o = rA_o = \sum_{n=i} c_n e^{+[\lambda_n (r^2 - r_o^2)]^n}$$

and accordingly, thereafter, to determine the z -dependent functions.

APPENDIX II

In the investigation of the stability of the equilibrium state of the plasma, I classify possible perturbations into two categories, the criterion being whether or not a Poynting vector (P) travels along the magnetic lines. This can be expressed as

$$(1) \quad P \cdot B \neq 0 \quad (\text{Alfvén waves}) \quad (\text{II-1})$$

$$(2) \quad P \cdot B = 0 \quad (\text{II-1a})$$

In the present appendix our attention will be restricted to the second category.

Equations, assumptions, and approximations:

The equations of the problem are:

$$\rho \frac{dv}{dt} = -\nabla \hat{p} + \hat{j} \times \frac{B}{c} + j_o \times \frac{h}{c} \quad (\text{II-2})$$

$$E + v \times B = 0 \quad (\text{II-3})$$

$$\nabla \times E = -\frac{1}{c} \frac{h}{t} \quad (\text{II-4})$$

$$\nabla \times h = \frac{4\pi j}{c} \quad (\text{II-5})$$

$$\frac{dp}{p dt} = \gamma \cdot \frac{d\rho}{\rho dt} \quad (\text{II-6})$$

$$\frac{\partial \hat{p}}{\partial t} + \nabla \cdot (\rho v) = 0 \quad (\text{II-7})$$

where B , p , ρ are the equilibrium (undisturbed) values of the magnetic field, plasma pressure, and plasma density, respectively.

E , h the perturbed values of the electric and magnetic field, respectively

j the Hall current

$\hat{j}$ the perturbed current

$\hat{p}$, $\hat{\rho}$ the perturbed values of the plasma pressure and density, respectively, and v the perturbed mass velocity of the plasma.

From Eqs. (II-6) and (II-7) we eliminate $\partial \rho / \partial t$ and we obtain

$$\frac{\partial \hat{p}}{\partial t} = -v \cdot \nabla p - \gamma p \nabla \cdot v \quad (\text{II-8})$$

and we introduce a new variable

$$\hat{\mathcal{O}} = \hat{p} + (B \cdot h / 4\pi) \quad (\text{II-9})$$

From the system of 6 equations (II-2, -3, -4, -5, -8, and -9) we can determine the six unknown quantities

$$v, \hat{p}, \hat{\mathcal{O}}, h, E, \hat{j}.$$

The plasma has been assumed of infinite conductivity insofar as the ohmic term (not shown here) in Eq. (II-2) is concerned, so that this term can be considered negligible. However Coulomb collisions are assumed to Maxwellianize the plasma and their frequency is assumed to be such as to allow the representation of the stress-tensor by the gradient of a scalar pressure.

The solutions are assumed in the form

$$\hat{q} = q(r, z) e^{im\theta + \omega t} \quad (\text{II-10})$$

where $\hat{q}$ is any of the perturbed quantities. Further it is assumed that $|\omega| < eB/Mc$, where M is the ion mass.

The expression (II-1a) can be written

$$E \times h \cdot B = E \cdot h \times B = 0 \quad (\text{II-11})$$

which implies

$$E = 0 \quad (\text{II-11a})$$

or

$$h \times B = 0 \quad (\text{II-11b})$$

The first case yields only trivial motion along the magnetic lines or trivial rotation of the plasma and is not of interest. Consequently the case which makes sense is the second, namely

$$\mathbf{h} \times \mathbf{B} = 0.$$

Expanding this equation we obtain

$$\frac{h_r}{h_z} = \frac{B_r}{B_z} \quad (\text{II-12})$$

and

$$h_\theta = 0. \quad (\text{II-12a})$$

Combining Eqs. (II-12) and (II-12a) with Eqs. (II-3) and (II-4) we obtain two sets of equations:

$$(a) \quad \frac{\partial E_r}{\partial z} = \frac{\partial E_z}{\partial r} \quad (\text{II-13})$$

$$E_r = -V_\theta B_z \quad (\text{II-13a})$$

$$E_z = V_\theta B_r \quad (\text{II-13b})$$

which after elimination of E_r and E_z yields the partial differential equation

$$B_r \frac{\partial V_\theta}{\partial r} + B_z \frac{\partial V_\theta}{\partial z} = V_\theta \frac{B_r}{r} \quad (\text{II-14})$$

(b)

$$\frac{h_r}{h_z} = \frac{B_r}{B_z} \quad (\text{II-15})$$

$$-\frac{\omega}{c} h_r = \frac{\partial E_z}{r \partial \theta} - \frac{\partial E_\theta}{\partial z} = i \frac{m}{r} E_z - \frac{\partial E_\theta}{\partial z} \quad (\text{II-15a})$$

$$-\frac{\omega}{c} h_z = \frac{\partial(r E_\theta)}{r \partial r} - \frac{\partial E_r}{r \partial \theta} = \frac{\partial(r E_\theta)}{r \partial r} - i \frac{m}{r} E_r \quad (\text{II-15b})$$

which in turn yields the partial differential equation:

$$B_r \frac{\partial E_\theta}{\partial r} + B_z \frac{\partial E_\theta}{\partial z} = -B_r \frac{E_\theta}{r}. \quad (\text{II-16})$$

The general solutions of the partial differential Eqs. (II-14) and (II-16) are, respectively,

$$V_\theta = r \cdot f(r A_\theta) e^{im\theta + \omega t} \quad (\text{II-14a})$$

$$E_\theta = \frac{1}{r} \cdot \phi(r A_\theta) e^{im\theta + \omega t} \quad (\text{II-16a})$$

where f and ϕ are arbitrary functions of (rA_θ) ; the value of rA_θ within the plasma is given by the equilibrium solution, Equation (4), and according to this solution $rA_\theta = f_0(r, z)$. However as soon as we move from the ends inward along the z -direction, the z -dependent function vanishes exponentially; thus resulting in that (rA_θ) is practically a function of r only in most of the plasma volume, far from the ends.

Consequently the perturbed quantities v_θ and E_θ degenerate in this region to a function of r only, and in general the perturbation is degenerated to the form

$$\hat{q} = q(r) e^{im\theta + \omega t}$$

Since rA_θ is constant along a magnetic line, it follows that $f(rA_\theta)$ and $\phi(rA_\theta)$ are constants along a magnetic line. Then if we find the solutions of Eqs. (II-2, -3, -4, -5, -8, and -9) in the region where rA_θ is practically a function of r only, then the values of E_θ , v_θ and the other perturbed quantities can be easily obtained for the whole volume. Consequently it is enough to solve the system of 6 equations (II-2, -3, -4, -5, -8, and -9) for a perturbation of the type

$$\hat{q} = q(r) e^{im\theta + \omega t}$$

This form of perturbation, after eliminating from the system of 6 equations the quantities p , j , h , and E , yields the equations

$$\rho \omega v_r = -\frac{\partial \varpi}{\partial r} \quad (\text{II-17})$$

$$\rho \omega r_\theta = -i \frac{m}{r} \varpi \quad (\text{II-18})$$

$$\left(\frac{\gamma p_0}{\rho} + \frac{B_z^2}{4\pi\rho} \right) \rho \nabla \cdot \mathbf{v} = -\omega \varpi \quad (\text{II-19})$$

where $\left(\frac{\gamma p_0}{\rho} + \frac{B_z^2}{4\pi\rho} \right) = S^2$, in which S is the sound velocity in the medium.

By eliminating v_θ from Eqs. (II-18) and (II-19), we obtain

$$\omega \rho \frac{\partial(r v_r)}{r \partial r} = -\left(\frac{\omega^2}{S^2} + \frac{m^2}{r^2} \right) \varpi \quad (\text{II-20})$$

Finally from Eqs. (II-17) and (II-20) by eliminating v_r we obtain

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial \varphi}{r \partial r} - \left(\frac{\partial \varphi}{\partial r} \cdot \frac{\partial \rho}{\rho \partial r} \right) - \left(\frac{\omega^2}{S^2} + \frac{m^2}{r^2} \right) \varphi = 0 \quad (\text{II-21})$$

From Eqs. (II-14a, -16a, and II-17, -18) we have

$$+ \frac{\partial \varphi}{\partial r} = \frac{\rho \omega}{r B_z} \phi(r A_\theta) e^{im\theta + \omega t} \quad (\text{II-21a})$$

$$i \frac{m}{r} \varphi = \rho \omega r f(r A_\theta) e^{im\theta + \omega t} \quad (\text{II-21b})$$

The last two equations impose restrictions in the behavior of φ along the radius. Our task now is to determine boundary conditions satisfying those restrictions, as well as the condition that the value of the velocity should be finite at the axis, and thereafter determine the sign of the eigenvalues of ω^2 that can satisfy these conditions.

At first we examine the behavior of φ near the axis of symmetry ($r = 0$). In this region the density of the plasma

$$\rho = \rho_0 (1 - e^{-2\lambda r^2}) \rightarrow 2\lambda r^2 \cdot \rho_0$$

as $r \rightarrow 0$

Hence $\frac{\partial \rho}{\rho \partial r} \rightarrow \frac{2}{r}$ and $\frac{\omega^2}{S^2}$ becomes proportional to r^2 and thus it can be neglected; then as r tends to zero, Eq. (II-21) degenerates to

$$\frac{\partial^2 \varphi}{\partial r^2} - \frac{\partial \varphi}{r \partial r} - \frac{m^2}{r^2} \varphi = 0 \quad (\text{II-22})$$

yielding solutions

$$\varphi \propto r^n \quad (\text{II-22a})$$

where

$$n = \pm \sqrt{m^2 + 1} + 1 \quad (\text{II-23})$$

From Eq. (II-17) we have

$$v_r = -\frac{1}{\rho \omega} \frac{\partial \varphi}{\partial r} \propto r^{n-3} \quad (\text{II-22b})$$

Since v_r should be finite at the origin it follows that

$$n-3 = \sqrt{m^2 + 1} - 2 > 0 \quad (\text{II-24})$$

or

$$m^2 > 3$$

Thus the modes $m = 0$ and $m = 1$ are excluded. The first admissible mode

is $m = 2$, resulting in that

$$n \geq 3.26 \quad (\text{II-24a})$$

$$V_r = 0, \quad \frac{\partial \mathcal{W}}{\partial r} = 0 \quad \text{at } r = 0$$

and by Eq. (II-21a)

$$V_r = 0, \quad \frac{\partial \mathcal{W}}{\partial r} = 0 \quad \text{at } r = r_0.$$

Now we have established the boundary conditions and we can proceed to determine the admissible eigenvalues of ω^2 in Eq. (II-21).

By changing the variable $\mathcal{W}$ by u where

$$u = \left(\frac{r}{\rho}\right)^{1/2} \cdot \mathcal{W}, \quad (\text{II-25})$$

Eq. (II-21) becomes

$$\frac{\partial^2 u}{\partial r^2} - \left\{ \frac{\omega^2}{S^2} + \frac{m^2 - .25}{r^2} - \left[\frac{\partial^2 \rho}{2\rho \partial r^2} - \frac{3}{4\rho} \left(\frac{\partial \rho}{\partial r}\right)^2 + \frac{1}{2r} \cdot \frac{\partial \rho}{\partial r} \right] \right\} u = 0 \quad (\text{II-26})$$

which can be written as

$$\frac{\partial^2 u}{\partial r^2} - \left[\frac{\omega^2}{S^2} + k^2 + \frac{m^2 + 2 - .25}{r^2} - \psi \right] u = 0 \quad (\text{II-26a})$$

where

$$\psi = \frac{\partial^2 \rho}{2\rho \partial r^2} - \frac{3}{4\rho} \left(\frac{\partial \rho}{\partial r}\right)^2 + \frac{1}{2r} \left(\frac{\partial \rho}{\partial r}\right) + k^2 - \frac{\mu^2}{r^2}. \quad (\text{II-27})$$

For

$$\rho = \frac{1}{C_\mu^2(kr)}, \quad \psi = 0 \quad (\text{II-28})$$

where

$$C_\mu(kr) = c_1 I_\mu(kr) + c_2 K_\mu(kr) \quad (\text{II-29})$$

and

$$I_\mu(kr) = (-i)^\mu \cdot J_\mu(ikr)$$

$$K_\mu(kr) = (-i)^{\mu+1} H_\mu^{(1)}(ikr)$$

By this substitution we not only simplify Eq. (II-26a) but also we can represent piecewise, by appropriate selection of the order and argument

of the modified Bessel's functions, any density distribution where the ~~density distribution where~~ the density is zero at the axis of symmetry, and at a boundary radius $r = r_0$.

Then Eq. (II-26) becomes

$$\frac{\partial^2 u}{\partial r^2} - \left(\frac{\omega^2}{S^2} + k^2 + \frac{m^2 + \mu^2 - 0.25}{r^2} \right) u = 0. \quad (\text{II-27})$$

At $r = 0$, $u = \frac{\partial u}{\partial r} = 0$ (from Eqs. II-22a, II-24a, II-25).

From Eqs. (II-20) and (II-25) we have

$$\omega \frac{\partial(rv_r)}{\partial r} = - \left[\frac{\omega^2}{S^2} + \frac{m^2}{r^2} \right] \left(\frac{r}{\rho} \right)^{1/2} u. \quad (\text{II-28})$$

The function $\partial(rv_r)/\partial r$ is alternating in the region $0 < r < r_0$, as

$$\int_0^{r_0} \frac{\partial(rv_r)}{\partial r} dr = r \cdot v_r(r_0) = 0.$$

If we now assume that ω^2 is positive then by Eq. (II-28) "u" should be an alternating function. However, for positive ω^2 , it results from Eq. (II-27) that u is a function starting from zero at $r = 0$ increasing monotonely with increasing radius, as $\partial^2 u / \partial r^2$ remains always positive. This contradicts the requirement imposed by Eq. (II-28).

Consequently, solutions are possible only for negative eigenvalues of ω^2 , indicating that all the perturbations of the type

$$h \times B = 0$$

are positively stable.

An investigation of the lowest possible value of $|\omega|$ resulted in that

$$|\omega| > S_0 / r_0$$

where $S_0 = \gamma p / \rho$, the sound velocity in the plasma in the absence of magnetic field.

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Figure Captions for "Description of a Thermonuclear Reactor Based on the Use of a Layer of Relativistic Electrons to Confine and Heat the Plasma," by Nicholas Christofilos.

Fig. 1. General outline of the proposed reactor.

Fig. 2. Steady-state equilibrium of the plasma in the E-layer reactor.

Fig. 3. Steady-state equilibrium of the plasma in the E-layer reactor. (a) Plasma density distribution in a cross section normal to the axis of symmetry. (b) Magnetic field distribution in a cross section normal to the axis of symmetry.