UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

NUMERICAL SOLUTION OF THE ONE-DIMENSIONAL
LAGRANGIAN HYDRODYNAMIC EQUATIONS

John G. Trulio and Kenneth R. Trigger

January 1961

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:
A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.
As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

BLANK PAGE

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

CONTENTS

	<u>Page No.</u>
Abstract	7
I. Introduction	9
II. Transformation Properties of the von Neumann-Richtmyer Equations	10
Definitions of Variables	10
Transformation of the von Neumann-Richtmyer Equations Among Themselves to Produce Difference Analogs of the Equations for Conservation of Energy and Mass	11
III. Form Requirements and Accuracy of the Difference Equations	17
Criteria for Differencing, Which, When Satisfied for Conservation Equations, Yield Reasonably Accurate Difference Equations Which are Ex- actly Conservative; Rough Definition of a Good Difference Analog	17
IV. Necessary Properties of the Lagrangian Difference Equations	20
The Basic Conservation Principles in Integral and Differential Form	20
Derivation of Equivalent Differential Equations for Which Good Difference Analogs Should be Obtained by Transformations Among the Basic Difference Equations	21

CONTENTS (Continued)

	<u>Page No.</u>
V. Modified von Neumann-Richtmyer Equations	24
Energy Conservation on Shock Regions Obtained by Treating $(P + Q)$ as a Single Variable	24
Introduction of a New Velocity Variable and Change in Time Centering of Momentum Equation (Explicit and Implicit Forms)	25
Transformations Among the Implicit Equations Leading to Good Difference Analogs of the Conservation Laws	28
VI. Variable Δt , Δa , ρ_0	34
Failure of the von Neumann-Richtmyer Equations to Conserve Energy, with or without the $(P + Q)$ Modification	34
Transformations Among the Implicit Equations Leading to Good Difference Analogs of the Conservation Laws	37
VII. Interfaces and Free Surfaces; Boundary Conditions	39
Special Features of the Interface Problem	39
Equations for the Motion of Interfaces and Free Surfaces Which Preserve the Conservation Properties of the Interior Equations Even in Spall and Join Processes	42
Reduction of the Interface Equations to the Interior Equations at a Fictitious Interface	44

CONTENTS (Continued)

	<u>Page No.</u>
VIII. Cylindrical and Spherical Flow	46
Integral and Differential Forms of the Conser-	
vation Principles for Spherical Systems	46
Derivation of a Kinetic Energy Analog from the	
Difference Equations for Mass and Mo-	
mentum Conservation	50
Derivation of a Force Analog from the Require-	
ment that the Difference Equations for	
Kinetic and Internal Energy Imply Exact	
Conservation of a Total Energy Analog	54
Summary of Equations	62
(i) Interior of Single Fluid	62
(a) Implicit Equations	62
(b) Explicit Equations	64
(ii) Interface Equations	65
Appendix A. Definitions of Form Preservation and <u>M</u> th-Order	
Analog; Application of Definitions to Conserva-	
tion of Volume	66
Appendix B. Deduction of Good Difference Analogs of Eqs. (9a)-	
(20a) from Eqs. (1e), (2e), (21e), (13e), with	
(Δt , Δa , ρ_0) Constant	72
Appendix C. Deduction of Analogs of Eqs. (9a)-(20a) from the	
(g)-Equations for Arbitrary $\left(\Delta^{n+\frac{1}{2}} t, \Delta^{j+\frac{1}{2}} a, \rho^0_{j+\frac{1}{2}} \right)$	78

CONTENTS (Continued)

	<u>Page No.</u>
Appendix D. Reduction of the Interface Equations to the Normal Interior Equations at a Fictitious Interface . . .	82
Appendix E. The Conservation Equations for Spherical Systems	85
Appendix F. Deduction of Analogs of Eqs. (9a)-(15a) in the Case of Spherical Symmetry	91

Numerical Solution of the One-Dimensional
Lagrangian Hydrodynamic Equations*

John G. Trulio and Kenneth R. Trigger

Lawrence Radiation Laboratory, University of California
Livermore, California

January 1961

ABSTRACT

It is shown that on non-shock regions the difference equations of von Neumann and Richtmyer for one-dimensional fluid flow provide a good analog of the total energy density which is exactly conserved, if a uniform space-time mesh is used. Implicit and explicit modifications of these equations are developed which exactly conserve good analogs of energy, mass, momentum, and volume on both shock and non-shock regions, and for an arbitrary space-time mesh. The equations are generalized to describe one-dimensional cylindrical and spherical flow, again satisfying exact conservation theorems. Difference equations are also presented for calculating the motion of interfaces and free surfaces. These equations maintain the conservation properties of the interior difference equations, even in collision and cavitation processes. Lagrangian coordinates are used throughout.

BLANK PAGE

Numerical Solution of the One-Dimensional
Lagrangian Hydrodynamic Equations

John G. Trulio and Kenneth R. Trigger

Lawrence Radiation Laboratory, University of California
Livermore, California

I. INTRODUCTION

In all but a few simple cases, numerical methods must be used to solve the equations describing the one-dimensional motion of a fluid.¹ A large number of numerical schemes for this purpose have been tested with regard to accuracy, generality, and convenience in application. By these standards, perhaps the best procedure available is that of von Neumann and Richtmyer.^{2,3} With their work as a point of departure, the objectives of this paper are

- i) to show that the Lagrangian difference equations of von Neumann and Richtmyer possess certain transformation properties that are fundamental to the description of fluid motion.⁴ For example, although none of the von Neumann-Richtmyer equations explicitly expresses conservation of total energy, these equations can be combined to give a good difference analog of the differential equation for total energy conservation in non-shock regions.
- ii) to list the transformation properties which should be incorporated as completely as possible into any finite-difference representation of the conservation principles of fluid mechanics.
- iii) to modify the von Neumann-Richtmyer equations so that they acquire more fully the desired transformation properties, especially in

connection with the use of a variable time-step and the treatment of material boundaries.

- iv) to present difference equations for one-dimensional spherical and cylindrical motion, which have the desired transformation properties.

One-dimensional motion is discussed explicitly in this paper in Lagrangian coordinates only. However, the criteria given here for choosing Lagrangian difference equations are also valid for Eulerian coordinates and for arbitrary time-dependent coordinates.

II. TRANSFORMATION PROPERTIES OF THE VON NEUMANN-RICHTMYER EQUATIONS

Definitions of Variables

The definitions of von Neumann and Richtmyer are followed closely.

U = material velocity,

V = specific volume,

$\rho = 1/V$ = density,

P = pressure,

Q = artificial viscosity,²

E = specific internal energy,

X = Cartesian position coordinate,

a = Lagrangian coordinate defined by

$$a = [X(a, t)]_{t=0}.$$

All variables are considered to be functions of (a, t) . To integrate the equations of motion numerically, these continuous independent variables are replaced by the discrete variables (a_j, t^n) , where

$$j = 0, 1, \dots, J; \quad n = 0, 1, 2, \dots$$

Then, by definition, we have

$$V_{j+\frac{1}{2}}^n = V\left(a_{j+\frac{1}{2}}, t^n\right), \text{ etc.}$$

Also:

$$P_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left(P_{j+\frac{1}{2}}^{n+1} + P_{j+\frac{1}{2}}^n \right),$$

$$PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} = P_{j+\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

$$PQ_{j+\frac{1}{2}}^n = P_{j+\frac{1}{2}}^n + Q_{j+\frac{1}{2}}^{n-\frac{1}{2}},$$

$$E_j^n = \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right).$$

For all $z(a, t)$:

$$(\partial z / \partial t)_a \equiv z_t; \quad (\partial z / \partial a)_t \equiv z_a.$$

Transformation of the von Neumann-Richtmyer Equations Among Themselves to Produce Difference Analogs of the Equations for Conservation of Energy and Mass

For the one-dimensional linear motion of a fluid, the basic differential equations and their difference analogs are given by von Neumann and Richtmyer as follows:

$$\rho_0 U_t = - (P + Q)_a, \quad (1a)$$

$$\rho_0 \left(U_j^{n+\frac{1}{2}} - U_j^{n-\frac{1}{2}} \right) / \Delta t = - \left(PQ_{j+\frac{1}{2}}^n - PQ_{j-\frac{1}{2}}^n \right) / \Delta a, \quad (1b)$$

} Conservation of momentum

$$\rho_0 V_t = U_a, \quad (2a)$$

$$\rho_0 \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta t = \left(u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}} \right) / \Delta a, \quad (2b)$$

$$E_t + (P + Q)V_t = 0, \quad (3a)$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta t + \left(P_{j+\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta t = 0. \quad (3b)$$

* Internal energy equation.

We will now form differential equations for the rate of change of kinetic energy and total energy, using Eqs. (1a), (2a), (3a). It will then be shown that when Q is equal to 0, the corresponding difference equations (Eqs. (1b), (2b), (3b)) can be combined in a closely analogous way to give good analogs of the differential equations for kinetic and total energy.

Eliminating V_t between Eqs. (2a) and (3a), we get

$$\rho_0 E_t + (P + Q)U_a = 0. \quad (4a)$$

Also, multiplying Eq. (1a) by U gives

$$\rho_0 \left(\frac{1}{2} U^2 \right)_t + U(P + Q)_a = 0; \text{ Kinetic energy equation.} \quad (5a)$$

Adding Eqs. (4a) and (5a), we get

$$\rho_0 \left(\frac{1}{2} U^2 + E \right)_t + [U(P + Q)]_a = 0,$$

or:

$$H_t + W_a = 0; \text{ Conservation of energy,} \quad (6a)$$

where

$$H = \rho_0 \left(\frac{1}{2} U^2 + E \right) = \text{total energy per unit volume,}$$

$W = U(P + Q)$ = rate of work on a plane surface of
unit area normal to direction of motion.

If $\left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n\right)/\Delta t$ is eliminated between Eqs. (2b) and (3b), we get

$$\rho_0 \left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n\right)/\Delta t + \left(P_{j+\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}}\right)/\Delta a = 0. \quad (4b)$$

From Eq. (1b):

$$\begin{aligned} \frac{1}{2} U_j^{n+\frac{1}{2}} \left\{ \left[\rho_0 \left(U_j^{n+\frac{1}{2}} - U_j^{n-\frac{1}{2}}\right)/\Delta t + \left(PQ_{j+\frac{1}{2}}^n - PQ_{j-\frac{1}{2}}^n\right)/\Delta a \right] \right. \\ \left. + \left[\rho_0 \left(U_j^{n+3/2} - U_j^{n+\frac{1}{2}}\right)/\Delta t + \left(PQ_{j+\frac{1}{2}}^{n+1} - PQ_{j-\frac{1}{2}}^{n+1}\right)/\Delta a \right] \right\} = 0, \end{aligned}$$

or:

$$\begin{aligned} \rho_0 \left(\frac{1}{2} U_j^{n+3/2} U_j^{n+\frac{1}{2}} - \frac{1}{2} U_j^{n+\frac{1}{2}} U_j^{n-\frac{1}{2}} \right)/\Delta t \\ + U_j^{n+\frac{1}{2}} \left[\frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n \right) - \frac{1}{2} \left(PQ_{j-\frac{1}{2}}^{n+1} + PQ_{j-\frac{1}{2}}^n \right) \right] \\ / \Delta a = 0. \quad (5b) \end{aligned}$$

Equation (4b) implies:

$$\begin{aligned} \rho_0 \left(E_j^{n+1} - E_j^n\right)/\Delta t + \left[\left(P_{j+\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}}\right) \right. \\ \left. + \left(P_{j-\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \left(U_j^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}}\right) \right] / 2\Delta a = 0. \end{aligned}$$

Adding this last equation to Eq. (5b), and applying the definition of $\left(PQ_{j+\frac{1}{2}}^n\right)$, gives

$$\begin{aligned} \rho_0 \left\{ \left[\frac{1}{2} U_j^{n+3/2} U_j^{n+\frac{1}{2}} + \frac{1}{2} \left(E_{j+\frac{1}{2}}^{n+1} + E_{j-\frac{1}{2}}^{n+1} \right) \right] - \left[\frac{1}{2} U_j^{n+\frac{1}{2}} U_j^{n-\frac{1}{2}} \right. \right. \\ \left. \left. + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right] \right\} / \Delta t + \left\{ \left[\frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right] \right. \\ \left. - \left[\frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j-\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}} \right) \right] \right\} / \Delta a \\ + \left[\frac{1}{2} \left(Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - Q_{j+\frac{1}{2}}^{n-\frac{1}{2}} \right) \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) + \frac{1}{2} \left(Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} - Q_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right) \right. \\ \left. \times \left(U_j^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}} \right) \right] / \Delta a = 0. \quad (A) \end{aligned}$$

If the Q 's are zero in Eq. (A), then:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta t + \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta a = 0, \quad (6b)$$

where

$$\begin{aligned} H_j^n &= \rho_0 \left[\frac{1}{2} U_j^{n+\frac{1}{2}} U_j^{n-\frac{1}{2}} + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right], \\ W_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \left[\frac{1}{2} \left(P_{j+\frac{1}{2}}^{n+1} + P_{j+\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right]. \end{aligned}$$

The quantities $\left(H_j^n\right)$ and $\left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right)$, and Eq. (6b), are good difference analogs

of the total energy per unit volume H , the rate of work W , and the

differential Eq. (6a) for energy conservation. However, in shock regions ($Q \neq 0$), Eq. (A) cannot be put in the "conservation form" exhibited by Eq. (6b). From our point of view, the von Neumann-Richtmyer equations are deficient in this respect. However, the defect is easily remedied, as we will show later (Sec. V).

In connection with mass conservation, some important comparisons remain to be made between the von Neumann-Richtmyer difference equations and the differential equations. By definition:

$$U = X_t. \quad (7a)$$

By following the general scheme of differencing of the other differential equations, Eq. (7a) becomes:

$$U_j^{n+\frac{1}{2}} = \left(X_j^{n+1} - X_j^n \right) / \Delta t. \quad (7b)$$

Combining Eqs. (7a) and (2a) we get:

$$\rho_0 V_t = X_{ta} = X_{at},$$

or:

$$(\rho_0 V - X_a) = (\rho_0 V - X_a)_{t=0} \equiv \rho(a, 0)V(a, 0) - 1.$$

Since $\rho = 1/V$:

$$\rho_0 V = X_a,$$

or:

$$\rho X_a = (\rho X_a)_{t=0}; \text{ Mass conservation.} \quad (8a)$$

Combining Eqs. (7b) and (2b) gives us:

$$\rho_0 V_j^{n+\frac{1}{2}} - \left(X_{j+1}^{n+1} - X_j^{n+1} \right) / \Delta a = \rho_0 V_{j+\frac{1}{2}}^n - \left(X_{j+1}^n - X_j^n \right) / \Delta a,$$

or, since this equation holds for all (n):

$$\rho_0 V_{j+\frac{1}{2}}^n - \left(X_{j+1}^n - X_j^n \right) / \Delta a = \rho_0 V_{j+\frac{1}{2}}^0 - \left(X_{j+1}^0 - X_j^0 \right) / \Delta a. \quad (B)$$

If the definition of the Lagrangian coordinate $a \equiv X(a, 0)$ is used for the difference equations, then $a_j = X_j^0$, and

$$\left(X_{j+1}^0 - X_j^0 \right) / \Delta a = 1.$$

To continue developing the difference equations in parallel with the differential equations, we must define $\rho_0 = 1/V_{j+\frac{1}{2}}^0$. Then Eq. (B) becomes:

$$\rho_0 V_{j+\frac{1}{2}}^n = \left(X_{j+1}^n - X_j^n \right) / \Delta a.$$

For this equation to be a good difference analog of Eq. (8a), we are forced to define $\rho_{j+\frac{1}{2}}^n = 1/V_{j+\frac{1}{2}}^n$, the definition already adopted for $n = 0$. Then

we have:

$$\rho_{j+\frac{1}{2}}^n \left(X_{j+1}^n - X_j^n \right) / \Delta a = \rho_{j+\frac{1}{2}}^0 \left(X_{j+1}^0 - X_j^0 \right) / \Delta a = \rho_{j+\frac{1}{2}}^0. \quad (8b)$$

It is quite significant that we were compelled to define $\rho_{j+\frac{1}{2}}^n = 1/V_{j+\frac{1}{2}}^n$ in

order to arrive at a strict difference analog of Eq. (8a).⁵ This is an example of the way in which transformation requirements restrict the possible difference representations of the equations of motion. Much of the ambiguity that usually arises in writing difference equations for differential equations is thereby eliminated. That the transformation requirements have been met so far by obvious difference equations is due to the special simplicity of the equations for one-dimensional linear motion in Lagrangian coordinates. For

the one-dimensional nonlinear geometries to be discussed later, the transformation properties are not so trivially obtained.

III. FORM REQUIREMENTS AND ACCURACY OF THE DIFFERENCE EQUATIONS

The object of this section is to present criteria for selecting difference equations to represent given differential equations. Of course, in fluid flow problems our main concern is the finite difference representation of conservation principles, and the statements of conservation are basically relations among integrals. However, the problem of choosing difference equations to represent the differential laws of conservation is much the same as that for the integral conservation laws. Also, the differential forms of the hydrodynamic equations are important in their own right.

Ordinarily, many difference equations can be found to represent a given differential equation. In some cases it is possible to decide that a particular difference analog of a given differential equation is "good," by giving a quantitative estimate of the rate of convergence of the solutions of this difference equation toward the solutions of the differential equation. Unfortunately, difference analogs of the coupled nonlinear differential equations of fluid mechanics are at present too complex to be compared in this way. A criterion for evaluating one set of difference equations relative to other possible sets is therefore bound to be somewhat arbitrary. Until a major advance is made in the field of nonlinear difference equations, our definition of a "good difference analog" must be judged by the numerical results which follow from it.

The notion that a difference equation must "preserve the form" of the differential equation it represents, is important in our method of differencing.

The discussion of this idea will be limited to quasi-linear first-order differential equations, since the equations of fluid mechanics are of this type. That is, we consider difference analogs of differential equations having the form:

$$\sum_{j=1}^J \sum_{k=1}^K A_{jk}(\underline{Y}, \underline{y}) B'_{jk}(\underline{Y}, \underline{y}) + C(\underline{Y}, \underline{y}) = 0, \quad (C)$$

where

$$\underline{y} = (y_1, y_2, \dots, y_K) = \text{a set of independent variables,}$$

$$\underline{Y} = (Y_1, Y_2, \dots, Y_N) = (Y_1(\underline{y}), Y_2(\underline{y}), \dots, Y_N(\underline{y}))$$

= a set of functions of the variables $(\underline{y})$,

$$B'_{jk}(\underline{Y}, \underline{y}) = \partial B_{jk}(\underline{Y}, \underline{y}) / \partial y_k,$$

$$C(\underline{Y}, \underline{y}), \left\{ A_{jk}(\underline{Y}, \underline{y}), B_{jk}(\underline{Y}, \underline{y}) \right\}_{\substack{k=1 \rightarrow K \\ j=1 \rightarrow J}} \text{ are functions of the variables}$$

$(\underline{Y}, \underline{y})$. To define "form preservation" roughly, let the functions

$$\left(C, \left\{ A_{jk}, B'_{jk} \right\}_{\substack{k=1 \rightarrow K \\ j=1 \rightarrow J}} \right) \text{ be called the "elements" of the differential equation (C).}$$

Then a difference equation preserves the form of Eq. (C) if, for each element E of Eq. (C), the difference equation contains one and only one function ϕ_E such that

- i) the difference equation is produced if each element E of Eq. (C) is replaced by the corresponding function ϕ_E ,
- ii) ϕ_E is an explicit function of just those variables which appear explicitly in E ,
- iii) ϕ_E approaches E in the limit of an infinitely fine mesh,
- iv) if E is a first derivative, then ϕ_E is a first difference quotient of

some function ψ_E , with respect to the appropriate independent variable.

A more precise definition of form preservation is given in Appendix A.

As a measure of accuracy, we speak of an "Mth-order analog" of a given differential equation. An "Mth-order analog" is defined as follows:

Let $\underline{y}(0)$ be a point in the space of the independent variables.

Expand the difference analog of a given differential expression in a Taylor's Series about $\underline{y}(0)$. Subtract the differential expression, evaluated at $\underline{y}(0)$, from the Taylor's Series. If the point $\underline{y}(0)$ can be chosen so that the terms of the remainder are of degree M or greater, then the difference expression is said to be an Mth-order analog of the given differential expression.

This definition is stated more precisely in Appendix A.

By definition, a good difference analog of a given differential expression is a form-preserving, second-order analog of the differential expression. Also, an analog of a given differential expression is a "fair" analog if it is form-preserving and first-order.

The significance of form-preservation becomes clear, for example, in the choice of a difference analog of Eq. (6a) for energy conservation. A difference equation which preserves the form of Eq. (6a) necessarily conserves a difference analog of the total energy density, with a source term given by a difference analog of the rate of work. In Appendix A, we apply the definitions of this section to Eq. (2a) and its difference analog, Eq. (2b); Eq. (2a) expresses conservation of volume (see Sec. IV).

IV. NECESSARY PROPERTIES OF THE LAGRANGIAN DIFFERENCE EQUATIONS

The basic laws of fluid mechanics are the principles of mass, momentum, and energy conservation. Therefore, to be acceptable, a numerical procedure for solving hydrodynamic problems must provide a good difference representation of the three conservation principles. For an element $\mathcal{E}$ of mass m , occupying a region of space $\mathcal{V}(t)$ bounded by a closed Lagrangian surface $\mathcal{L}(t)$, the primary equations of motion are:

$$\dot{m}_t = 0; \quad \text{Conservation of mass,} \quad (\text{I})$$

where

$$m = \int_{\mathcal{V}(t)} \rho \, d\tau;$$

$$\dot{\underline{M}}_t = \underline{F}; \quad \text{Conservation of momentum,} \quad (\text{II})$$

where

$$\underline{M} = \int_{\mathcal{V}(t)} \rho \, \underline{U} \, d\tau = \text{momentum of element } \mathcal{E},$$

$$\underline{F} = - \int_{\mathcal{L}(t)} (P + Q) \underline{n} \, d\sigma = \text{force acting on element } \mathcal{E},$$

$$\underline{n} = \text{unit outward normal at a point of } \mathcal{L}(t),$$

$$d\sigma = \text{area of an infinitesimal element of } \mathcal{L}(t);$$

$$H_t + W = 0; \quad \text{Conservation of energy,} \quad (\text{III})$$

where

$$H = \int_{\mathcal{V}(t)} \rho \left(\frac{1}{2} \underline{U}^2 + E \right) d\tau = \text{total energy of element } \mathcal{E},$$

$$W = \int_{\mathcal{L}(t)} (P + Q) \underline{U} \cdot \underline{n} \, d\sigma = \text{rate of work done by element } \mathcal{E}.$$

There is one more conservation equation, whose origin is mainly mathematical. For the volume v of a region $R(t)$, we have:

$$v = v(t) = \int_{R(t)} d\tau.$$

If $\underline{S}$ is the velocity of a point on the closed surface Σ assumed to bound $R(t)$, then:

$$v_t = \int_{\Sigma} \underline{S} \cdot \underline{n} d\tau.$$

This equation takes on physical significance when v is made the physical volume of the element $\mathcal{E}$, in which case $\underline{S}$ can be identified with the material velocity $\underline{U}$ of a point on the surface $\mathcal{L}(t)$. Then:

$$v_t = \int_{\mathcal{L}(t)} \underline{U} \cdot \underline{n} d\sigma; \text{ Conservation of volume. (IV)}$$

Equations (I) - (IV) are of the general type

$$\begin{aligned} & \text{time rate of change of an integral over the region } R(t) \\ & = \text{an integral over the surface } \Sigma(t) \text{ bounding } R(t). \end{aligned}$$

An equation of this kind is said to be in "conservation form."

The differential forms of Eqs. (I) - (IV) are perhaps most conveniently obtained by means of Gauss' Theorem. Let $\underline{X} = \underline{f}(\underline{a}, t)$ define the transformation from Cartesian to Lagrangian coordinates so that, by definition, $\underline{U} = \underline{X}_t$. Under this transformation the Cartesian region $\mathcal{V}(t)$ is mapped into the Lagrangian region $\mathcal{V}_L$, which does not change with time. Then, for any function $z(\underline{X}, t)$:

$$\int_{\mathcal{V}(t)} z(\underline{X}, t) d\tau = \int_{\mathcal{V}_L} z(\underline{f}, t) J d\tau_L,$$

and

$$\frac{d}{dt} \int_{V(t)} z(\underline{X}, t) d\tau = \int_{V_L} \left[z(\underline{f}, t) J \right]_t d\tau_L = \int_{V(t)} \frac{(zJ)_t}{J} d\tau, \quad (D)$$

where the partial derivatives are taken with $\underline{a}$ constant, and:

$$\begin{aligned} J &= \text{Jacobian for the transformation from Cartesian to} \\ \text{Lagrangian coordinates} &= \partial(X, Y, Z)/\partial(a, b, c), \\ d\tau_L &= da db dc. \end{aligned}$$

Applying Eq. (D) to the left-hand side of Eq. (IV), and Gauss' Theorem on the right, we obtain:

$$\int_{V(t)} (J_t/J) d\tau = \int_{V(t)} \nabla \cdot \underline{U} d\tau.$$

Since this holds for an arbitrary Lagrangian region:

$$\nabla \cdot \underline{U} = J_t/J = \lim_{v \rightarrow 0} (v_t/v) = \lim_{v \rightarrow 0} \left(\frac{1}{v} \int_{\mathcal{L}(t)} \underline{U} \cdot \underline{n} d\sigma \right). \quad (12a)$$

Similarly, Eqs. (I), (II), (III) give:

$$(\rho J)_t = 0, \quad \text{Conservation of mass;} \quad (9a)$$

$$(\rho J \underline{U})_t + J \nabla \cdot (\underline{P} + \underline{Q}) = 0, \quad \text{Conservation of momentum;} \quad (10a)$$

$$\left[\rho J \left(\frac{1}{2} \underline{U}^2 + E \right) \right]_t + J \nabla \cdot [(\underline{P} + \underline{Q}) \underline{U}] = 0, \quad \text{Conservation of energy.} \quad (11a)$$

Several assumptions concerning continuity are implicit in this derivation of Eqs. (9a) - (12a). These assumptions are violated in a few physically important cases. Therefore Eqs. (9a) - (12a) are not as general as Eqs. (I) - (IV), which are regarded as the fundamental equations of motion.

As noted above, in Lagrangian coordinates (by definition):

$$\underline{U} = \underline{X}_t. \quad (13a)$$

Also, Eqs. (9a), (10a) imply:

$$\left(\frac{1}{2} \rho J \underline{U}^2\right)_t + J \underline{U} \cdot \underline{\nabla} (P + Q) = 0; \quad \text{Kinetic energy equation.} \quad (14a)$$

Equations (11a), (14a) imply:

$$(\rho J E)_t + J(P + Q) \underline{\nabla} \cdot \underline{U} = 0; \quad \text{Internal energy equation.} \quad (15a)$$

Equations (9a), (12a) imply:

$$\rho_t + \rho \underline{\nabla} \cdot \underline{U} = 0. \quad (16a)$$

Equations (9a), (10a) imply:

$$\rho \underline{U}_t + \underline{\nabla} (P + Q) = 0. \quad (17a)$$

Equations (9a), (11a) imply:

$$\rho \left(\frac{1}{2} \underline{U}^2 + E\right)_t + \underline{\nabla} \cdot \left[(P + Q) \underline{U}\right] = 0. \quad (18a)$$

Equations (9a), (14a) imply:

$$\rho \left(\frac{1}{2} \underline{U}^2\right)_t + \underline{U} \cdot \underline{\nabla} (P + Q) = 0. \quad (19a)$$

Equations (9a), (15a) imply:

$$\rho E_t + (P + Q) \underline{\nabla} \cdot \underline{U} = 0. \quad (20a)$$

Equations (16a), (20a) imply:

$$E_t + (P + Q) \underline{\nabla} \cdot \underline{U} = 0. \quad (21a)$$

The indispensable condition imposed on our difference equations is that they provide form-preserving analogs of Eqs. (I) - (IV). Next in importance, manipulation of these difference equations should yield form-preserving analogs of Eqs. (9a) - (21a). It is also desirable that the form-preserving analogs of Eqs. (I) - (IV) be good analogs of these equations. Finally, we prefer that the difference equations representing Eqs. (9a) - (21a) be good analogs of Eqs. (9a) - (21a).

V. MODIFIED VON NEUMANN-RICHTMYER EQUATIONS

As shown in Sec. II, the von Neumann-Richtmyer equations do not lead to a form-preserving difference analog of Eq. (6a) for energy conservation when Q is not equal to 0. This difficulty can be overcome if Q and P are made to appear together in the internal energy equation (3b) just as they do in the momentum equation (1b). Thus, if Eq. (3b) is changed to

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n\right)/\Delta t + \frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n\right) \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n\right)/\Delta t = 0, \quad (3c)$$

then Eq. (A) becomes:

$$\begin{aligned} \rho_0 \left\{ \left[\frac{1}{2} U_j^{n+3/2} U_j^{n+1/2} + \frac{1}{2} \left(E_{j+\frac{1}{2}}^{n+1} + E_{j-\frac{1}{2}}^{n+1} \right) \right] - \left[\frac{1}{2} U_j^{n+1/2} U_j^{n-1/2} \right. \right. \\ \left. \left. + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right] \right\} / \Delta t + \left\{ \left[\frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_{j+1}^{n+1/2} + U_j^{n+1/2} \right) \right] \right. \\ \left. - \left[\frac{1}{2} \left(PQ_{j-\frac{1}{2}}^{n+1} + PQ_{j-\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_j^{n+1/2} + U_{j-1}^{n+1/2} \right) \right] \right\} / \Delta a = 0, \end{aligned}$$

or

$$\left(H_j^{n+1} - H_j^n\right)/\Delta t + \left(W_{j+\frac{1}{2}}^{n+1/2} - W_{j-\frac{1}{2}}^{n+1/2}\right)/\Delta a = 0, \quad (6c)$$

where

$$\begin{aligned} H_j^n &= \rho_0 \left[\frac{1}{2} U_j^{n+1/2} U_j^{n-1/2} + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right], \\ W_{j+\frac{1}{2}}^{n+1/2} &= \left[\frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n \right) \right] \left[\frac{1}{2} \left(U_{j+1}^{n+1/2} + U_j^{n+1/2} \right) \right]. \end{aligned}$$

Equation (6c) is a good difference analog of Eq. (6a) when the Q 's are zero, and a fair difference analog otherwise, because the Q 's are centered at half-integer times. Thus, if the von Neumann-Richtmyer equations were modified in this way, then a good difference analog of Eq. (3a) would be replaced by a fair difference analog of Eq. (3a). However, we would gain a form-preserving difference analog of Eq. (6a) for total energy conservation, which is much more important from our point of view. Nevertheless, Eqs. (1b), (2b), (3c) are unsatisfactory for two reasons:

- i) When Δt varies with n , Eqs. (1b), (2b), (3c) do not lead to form-preserving analogs of Eq. (6a) or Eq. (III) for total energy conservation (see Sec. VI).
- ii) In asking for form-preserving analogs of Eqs. (I), (II), ..., for cylindrical and spherical flow, we do not obtain equations which reduce to Eqs. (1b), (2b), (3c) in the limit of large radius (see Sec. VIII).

To overcome these objections, and at the same time retain the property of total energy conservation on shock regions possessed by Eqs. (1b), (2b), (3c), we are led to the following equations:

$$\rho_0 \left(U_j^n - U_j^{n-1} \right) / \Delta t + \left[\left(P_{j+\frac{1}{2}}^{n-\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n-\frac{1}{2}} \right) - \left(P_{j-\frac{1}{2}}^{n-\frac{1}{2}} + Q_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right) \right] / \Delta a = 0, \quad (1d)$$

$$U_j^{n+\frac{1}{2}} = 2 U_j^n - U_j^{n-\frac{1}{2}}, \quad (1'd)$$

$$\rho_0 \left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - V_{j+\frac{1}{2}}^n \right) / \Delta t = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta a, \quad (2d)$$

$$\left(E_{j+\frac{1}{2}}^{n+\frac{1}{2}} - E_{j+\frac{1}{2}}^n \right) / \Delta t + \left(P_{j+\frac{1}{2}}^{n+\frac{1}{2}} + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) \left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - V_{j+\frac{1}{2}}^n \right) / \Delta t = 0. \quad (3d)$$

Equation (3d) is a good difference analog of Eq. (3a) (or Eq. (21a)), while Eq. (3c) is only a fair difference analog of Eq. (3a) (or Eq. (21a)). However, experience suggests that this formal advantage of the (d) -equations is not significant in practice.

It will now be shown that the (d) -equations yield a good difference analog of the energy conservation equations (6a), (III), on both shock and non-shock regions, when Δt is constant. Equations (1d), (1'd) imply

$$\frac{\rho_0}{2\Delta t} \left(U_j^{n+3/2} U_j^{n+1/2} - U_j^{n+1/2} U_j^{n-1/2} \right) + U_j^{n+1/2} \left(PQ_{j+1/2}^{n+1/2} - PQ_{j-1/2}^{n+1/2} \right) / \Delta a = 0,$$

where

$$PQ_{j+1/2}^{n+1/2} = P_{j+1/2}^{n+1/2} + Q_{j+1/2}^{n+1/2}.$$

By following the steps which led to Eq. (A), we now find:

$$\begin{aligned} & \rho_0 \left\{ \left[\frac{1}{2} U_j^{n+3/2} U_j^{n+1/2} + \frac{1}{2} \left(E_{j+1/2}^{n+1} + E_{j-1/2}^{n+1} \right) \right] \right. \\ & \left. - \left[\frac{1}{2} U_j^{n+1/2} U_j^{n-1/2} + \frac{1}{2} \left(E_{j+1/2}^n + E_{j-1/2}^n \right) \right] \right\} + \left\{ PQ_{j+1/2}^{n+1/2} \left[\frac{1}{2} \left(U_{j+1}^{n+1/2} + U_j^{n+1/2} \right) \right] \right. \\ & \left. - PQ_{j-1/2}^{n+1/2} \left[\frac{1}{2} \left(U_j^{n+1/2} + U_{j-1}^{n+1/2} \right) \right] \right\} / \Delta a = 0, \end{aligned}$$

or:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta t + \left(W_{j+1/2}^{n+1/2} - W_{j-1/2}^{n+1/2} \right) / \Delta a = 0, \quad (6d)$$

where

$$H_j^n = \rho_0 \left[\frac{1}{2} U_j^{n+1/2} U_j^{n-1/2} + \frac{1}{2} \left(E_{j+1/2}^n + E_{j-1/2}^n \right) \right],$$

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right].$$

Rather than develop further analogs of Eqs. (I) - (IV), (9a) - (21a), we note at this point that the (d)-equations have a serious defect in principle, which shows itself clearly when initial conditions are specified. To start the numerical solution of any flow problem using Eqs. (1d), (1'd), (2d), (3d), velocities must be given for all j at both $t = t^0$ and $t = t^{\frac{1}{2}}$. These initial-value requirements are at odds with both physical intuition and exact mathematical conditions for a properly posed hydrodynamic problem. Special first-cycle procedures are needed to meet this initial-value difficulty. While several schemes might be used, the most satisfactory solution is to get from $t = t^0$ to $t = t^1$ via difference equations that require only one set of velocities among their initial conditions. However, if we can go from t^0 to t^1 with a given set of difference equations, then these equations can be used for all time-steps. Thus, we are led to propose still a third set of difference analogs of Eqs. (1a), (2a), (3a). Although in practice the difference between these equations and the (d)-equations is negligible, from a theoretical standpoint this third set of equations is the most satisfactory one we have devised for the numerical solution of the one-dimensional linear hydrodynamic equations in Lagrangian form. Their transformation properties will therefore be discussed in considerable detail. The equations are:

$$\rho_0 \left(U_j^{n+1} - U_j^n \right) / \Delta t + \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta a = 0. \quad (1e)$$

$$\rho_0 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta t = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta a, \quad (2e)$$

where

$$U_j^{n+\frac{1}{2}} = \frac{1}{2} \left(U_j^{n+1} + U_j^n \right).$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta t = 0. \quad (21e)$$

In addition:

$$U_j^{n+\frac{1}{2}} = \left(X_j^{n+1} - X_j^n \right) / \Delta t. \quad (13e)$$

Equations (21e), (13e) are good difference analogs of Eqs. (21a), (13a). It will now be shown that Eqs. (1e), (2e), (21e), (13e) can be manipulated to produce an entirely equivalent set of difference equations which are good analogs of the basic conservation Eqs. (I) - (IV). It will also be shown that, just as the differential Eqs. (1a), (2a), (21a), (13a) transform among themselves to give the differential Eqs. (9a) - (21a), the difference Eqs. (1e), (2e), (21e), (13e) can be combined algebraically to give good analogs of Eqs. (9a) - (21a).

Following the steps which led to Eq. (8b), it can be seen that Eqs. (2e), (13e) imply:

$$\rho_{j+\frac{1}{2}}^n \left(X_{j+1}^n - X_j^n \right) = \rho_{j+\frac{1}{2}}^0 \left(X_{j+1}^0 - X_j^0 \right) = \rho_0 \Delta a = \text{constant}. \quad (F)$$

The quantity $\left[\rho_{j+\frac{1}{2}}^n \left(X_{j+1}^n - X_j^n \right) \right]$ is a good difference analog of

$$\left(\int_{X_j^n}^{X_{j+1}^n} \rho \, dx \right), \text{ the total mass between } \left(X_j^n \right) \text{ and } \left(X_{j+1}^n \right).^6 \text{ Therefore,}$$

if we define

$$m_{j+\frac{1}{2}}^n = \rho_{j+\frac{1}{2}}^n \left(X_{j+1}^n - X_j^n \right),$$

then by Eq. (F):

$$\left(m_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n \right) / \Delta t = 0. \quad (\text{Ie})$$

Next, combining Eqs. (Ie), (F), we get:

$$\left(M_j^{n+1} - M_j^n \right) / \Delta t = - \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right), \quad (\text{IIe})$$

where

$$M_j^n = m_j^n U_j^n,$$

$$m_j^n = \frac{1}{2} \left(m_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n \right).$$

The quantity M_j^n is a good difference analog of $\left(\int_{X_{j-\frac{1}{2}}^n}^{X_{j+\frac{1}{2}}^n} \rho U dx \right)$, the total

x component of momentum between $\left(X_{j-\frac{1}{2}}^n \right)$ and $\left(X_{j+\frac{1}{2}}^n \right)$. Also,

$\left[- \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \right]$ is a good difference analog of $\left(- \int_{\mathcal{L}(t)} (P+Q)n_x d\sigma \right)$,

the x component of the force acting on this region.

If Eq. (IIe) is multiplied by $\left(U_j^{n+\frac{1}{2}} \right)$, we get

$$\frac{1}{2} \left(U_j^{n+1} + U_j^n \right) \left(m_j^{n+1} U_j^{n+1} - m_j^n U_j^n \right) / \Delta t + U_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0.$$

Since $m_j^{n+1} = m_j^n$, this becomes:

$$\left[\frac{1}{2} m_j^{n+1} (U_j^{n+1})^2 - \frac{1}{2} m_j^n (U_j^n)^2 \right] / \Delta t + U_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (G)$$

Now, combining Eqs. (2e), (2le) gives us:

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \rho_0 \Delta a = 0,$$

or

$$\left(m_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) = 0. \quad (H)$$

Equations (G), (H) imply:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta t + \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (IIIe)$$

where

$$H_j^n = m_j^n \left[\frac{1}{2} (U_j^n)^2 + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right],$$

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right].$$

Here H_j^n is a good difference analog of $\left[\int_{X_{j-\frac{1}{2}}^n}^{X_{j+\frac{1}{2}}^n} \rho \left(\frac{1}{2} U^2 + E \right) dx \right]$, the

total energy on the region between $\left(X_{j-\frac{1}{2}}^n \right)$ and $\left(X_{j+\frac{1}{2}}^n \right)$. The rate at

which the material in this region does work on its surroundings at time

$\left(\begin{smallmatrix} n+1 \\ t \end{smallmatrix} \right)$ would be given exactly by $\left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right)$, were it not for the

PQ and U averages appearing in the W's. The quantity $\left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right)$

is a good difference analog of $\left[\int_{\mathcal{L}(t)} (P+Q) \underline{U} \cdot \underline{n} d\sigma \right]$, the rate of work.

Equations (2e), (9e), (F) imply:

$$\left(\begin{smallmatrix} m^{n+1} \\ j+\frac{1}{2} \end{smallmatrix} \begin{smallmatrix} v^{n+1} \\ j+\frac{1}{2} \end{smallmatrix} - \begin{smallmatrix} m^n \\ j+\frac{1}{2} \end{smallmatrix} \begin{smallmatrix} v^n \\ j+\frac{1}{2} \end{smallmatrix} \right) / \Delta t = U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}},$$

or, since $\rho_{j+\frac{1}{2}}^n v_{j+\frac{1}{2}}^n = 1$:

$$\left[\left(X_{j+1}^{n+1} - X_j^{n+1} \right) - \left(X_{j+1}^n - X_j^n \right) \right] / \Delta t = U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}}. \quad (J)$$

Equation (J) also follows directly from Eq. (13e). Thus:

$$\left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta t = U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}}, \quad (IVe)$$

where

$$v_{j+\frac{1}{2}}^n = X_{j+1}^n - X_j^n = \int_{X_j^n}^{X_{j+1}^n} dx.$$

Thus Eqs. (1e), (2e), (21e), (13e) lead to the conservation of good difference analogs of mass, momentum, energy, and volume, through good difference analogs of Eqs. (I) - (IV). In Appendix B, it is shown how the (e)-equations (i.e., Eqs. (1e), (2e), (21e), (13e)) can be further manipulated to produce good difference analogs of Eqs. (9a) - (20a). Of course, the

transformation properties of Eqs. (1d), (2d), (3d) are exactly the same as those of Eqs. (1e), (2e), (2le) since these two sets of equations are identical in form. However, the complete set of (d) -equations includes Eq. (1'd), which leads to the initial-value difficulty mentioned earlier. Nevertheless, the introduction of Eq. (1'd) greatly simplifies the task of obtaining the solutions of hydrodynamic problems by stepwise advances in time.

In particular, suppose that $\left(U_j^{n-1}, U_j^{n-\frac{1}{2}}, P_{j+\frac{1}{2}}^n, P_{j+\frac{1}{2}}^{n-1}, Q_{j+\frac{1}{2}}^{n-\frac{1}{2}}, V_{j+\frac{1}{2}}^n, E_{j+\frac{1}{2}}^n \right)$ are known for all j . Then for any j , Eq. (1d) is a linear algebraic equation to be solved for the unknown U_j^n , and U_j^n can easily be equated to a known function of known variables. An equation in which an unknown quantity is equated to a known function of known quantities is termed "explicit." An equation which is not explicit is termed "implicit." For any j , it is clearly a trivial matter to put Eq. (1d) into explicit form for the variable U_j^n . Having evaluated U_j^n for all j , Eq. (1'd) is explicit for the unknown $\left(U_j^{n+\frac{1}{2}} \right)$, and $\left(U_j^{n+\frac{1}{2}} \right)$ can be evaluated for all j . It is then easy to write Eq. (2d) in explicit form for $\left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right)$. Hence, for every j , the value of $\left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right)$ can readily be calculated. At this point, following the scheme of von Neumann and Richtmyer,⁷ $\left(Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right)$ can be found for any j through an explicit equation. However, we come next to Eq. (6d), which contains the two unknowns $\left(E_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right)$ and $\left(P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right)$. Equation (6d) is to be

solved simultaneously with an equation of state:

$$P_{j+\frac{1}{2}}^{n+1} = F\left(V_{j+\frac{1}{2}}^{n+1}, E_{j+\frac{1}{2}}^{n+1}\right).$$

Since the function F may be very complicated, and may even be given as a table, it is not generally possible to write explicit equations for $\left(P_{j+\frac{1}{2}}^{n+1}\right)$ and $\left(E_{j+\frac{1}{2}}^{n+1}\right)$. In general then, for any j , Eq. (6d) and the equation of state are a pair of implicit equations for $\left(P_{j+\frac{1}{2}}^{n+1}\right)$ and $\left(E_{j+\frac{1}{2}}^{n+1}\right)$. These two equations can be solved rather simply by various iterative procedures.

If Eq. (1'd) is deleted from the (d) -equations, then, although U_j^n can still be gotten explicitly for all j , there is no way to evaluate $\left(U_j^{n+\frac{1}{2}}\right)$, and we cannot proceed to find $\left(V_{j+\frac{1}{2}}^{n+1}\right)$, etc. This difficulty can be met by using the definition of $\left(U_j^{n+\frac{1}{2}}\right)$ given in the (e) -equations. However, Eq. (1e)

contains $\left(U_j^{n+1}, P_{j+\frac{1}{2}}^{n+1}, P_{j-\frac{1}{2}}^{n+1}, Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}, Q_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)$ as unknowns, and there-

fore cannot be written in explicit form. Of course, there are other equations in the (e) -set relating these variables, but there appears to be no way to combine them to produce explicit equations for any of the unknowns. Therefore the (e) -equations, the Q -equation, and the equation of state constitute a set of implicit equations which must be solved simultaneously for all j

for the unknowns $\left(U_j^{n+1}, V_{j+\frac{1}{2}}^{n+1}, Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}, P_{j+\frac{1}{2}}^{n+1}, E_{j+\frac{1}{2}}^{n+1}\right)$. The job of

solving these equations is not trivial, although it can be done by iterative methods, notably Newtonian iteration.

Thus, the (d) -equations have a considerable advantage of convenience over the (e) -equations, although the gain is achieved by compromising somewhat the physical sense of the equations. It is worth noting that this computational advantage is partly offset by the fact that numerical stability requirements limit Δt for the (d) -equations, while the (e) -equations are unconditionally stable; we will not discuss this point further.

VI. VARIABLE Δt , Δa , ρ_0

Up to now we have assumed, along with von Neumann and Richtmyer, that $\rho_{j+\frac{1}{2}}^0 = \rho_0$, $a_{j+1} - a_j = \Delta a$, and $t^{n+1} - t^n = \Delta t$, for all n, j . The properties of the various difference equations will now be discussed without the restriction of a fixed Δt , and also with variable Δt , Δa , ρ_0 . By definition:

$$\begin{aligned} \Delta_{j+\frac{1}{2}} a &= a_{j+1} - a_j; & \Delta_j a &= a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \\ \Delta_{t^{n+\frac{1}{2}}} &= t^{n+1} - t^n; & \Delta_t^n &= t^{n+\frac{1}{2}} - t^{n-\frac{1}{2}} \end{aligned}$$

If a variable Δt is used, then Eqs. (1b), (2b), (3c) become:

$$\rho_0 \left(U_j^{n+\frac{1}{2}} - U_j^{n-\frac{1}{2}} \right) / \Delta_t^n = - \left(PQ_{j+\frac{1}{2}}^n - PQ_{j-\frac{1}{2}}^n \right) / \Delta a, \quad (1f)$$

$$\rho_0 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta_{t^{n+\frac{1}{2}}} = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta a, \quad (2f)$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta_{t^{n+\frac{1}{2}}} + \frac{1}{2} \left(PQ_{j+\frac{1}{2}}^{n+1} + PQ_{j+\frac{1}{2}}^n \right) \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta_{t^{n+\frac{1}{2}}} = 0.$$

(3f)

From Eq. (1f) we get:

$$\begin{aligned} \frac{1}{2} \rho_0 \Delta a \left(U_j^{n+3/2} U_j^{n+1/2} - U_j^{n+1/2} U_j^{n-1/2} \right) + U_j^{n+1/2} \left(PQ_{j+1/2}^{n+1} - PQ_{j-1/2}^{n+1} \right) \Delta^{n+1}_t \\ + U_j^{n+1/2} \left(PQ_{j+1/2}^n - PQ_{j-1/2}^n \right) \Delta^n_t = 0. \end{aligned}$$

Also, Eqs. (2f), (3f) imply:

$$\begin{aligned} \frac{1}{2} \rho_0 \Delta a \left(E_{j+1/2}^{n+1} + E_{j-1/2}^{n+1} - E_{j+1/2}^n - E_{j-1/2}^n \right) + \frac{1}{2} \left(PQ_{j+1/2}^{n+1} + PQ_{j+1/2}^n \right) \\ \times \left(U_{j+1}^{n+1/2} - U_j^{n+1/2} \right) \Delta^{n+1}_t + \frac{1}{2} \left(PQ_{j-1/2}^{n+1} + PQ_{j-1/2}^n \right) \\ \times \left(U_j^{n+1/2} - U_{j-1}^{n+1/2} \right) \Delta^{n+1}_t = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(H_j^{n+1} - H_j^n \right) / \Delta^{n+1}_t + \frac{U_j^{n+1/2}}{\Delta a} \left[\left(PQ_{j+1/2}^{n+1} - PQ_{j-1/2}^{n+1} \right) \frac{\Delta^{n+1}_t}{\Delta^{n+1}_t} \right. \\ \left. + \left(PQ_{j+1/2}^n - PQ_{j-1/2}^n \right) \frac{\Delta^n_t}{\Delta^{n+1}_t} \right] + \frac{1}{2\Delta a} \left[\left(PQ_{j+1/2}^{n+1} + PQ_{j+1/2}^n \right) \right. \\ \left. \times \left(U_{j+1}^{n+1/2} - U_j^{n+1/2} \right) + \left(PQ_{j-1/2}^{n+1} + PQ_{j-1/2}^n \right) \left(U_j^{n+1/2} - U_{j-1}^{n+1/2} \right) \right] = 0, \end{aligned} \quad (6f)$$

where

$$H_j^n = \rho_0 \left[\frac{1}{2} U_j^{n+1/2} U_j^{n-1/2} + \frac{1}{2} \left(E_{j+1/2}^n + E_{j+1/2}^n \right) \right].$$

Unless $\Delta^{n+1}_t = \Delta^{n+1/2}_t = \Delta^n_t$, Eq. (6f) cannot be put in the form

$$\left(H_j^{n+1} - H_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta a = 0.$$

Hence, Eqs. (1f), (2f), (3f) do not lead to a form-preserving analog of Eq. (6a) for total energy conservation. It can also be seen that these equations provide no form-preserving analog of Eq. (III). The von Neumann-Richtmyer equations are identical with Eqs. (1b), (2b), (3c) on non-shock regions, while on shock regions they do not conserve an analog of the total energy density even when Δt is constant (see Sec. II). Therefore, with a variable Δt the equations of von Neumann and Richtmyer do not lead to a form-preserving analog of Eq. (III) for total energy conservation. Thus, we will not try to extend either the von Neumann-Richtmyer equations or Eqs. (1b), (2b), (3c) to the case of variable Δt , Δa , and ρ_0 .

Equations (1e), (2e), (21e), (13e) contain just the single time difference $t^{n+1} - t^n = \Delta^{n+\frac{1}{2}}_t$. It can be directly verified that no other time difference is introduced in the derivation of Eqs. (1e), (11e), ..., (20e). Therefore, the transformation properties of the (e)-equations are the same for variable Δt as for fixed Δt . This is also true of the (d)-equations, although we should note that the equation

$$U_j^{n+\frac{1}{2}} = \left(X_j^{n+1} - X_j^n \right) / \Delta^{n+\frac{1}{2}}_t$$

must be added to Eqs. (1d), (1'd), (2d), (3d) if the transformation properties of the (d)-equations are to be described as fully as those of the (e)-equations.

As in the case of constant Δt , $\left(U_j^{n+\frac{1}{2}} \right)$ is found by extrapolation according to Eq. (1'd). However, Eq. (1'd) gives only a first-order analog of

$U\left(a_j, t^{n+\frac{1}{2}}\right)$ in the case of variable Δt . In this respect the explicit

(d) -equations are not as accurate as the implicit (e) -equations, although in our experience the difference is negligible.

With variable ρ_0 , Δa , and Δt , the (e) -equations become:

$$\frac{1}{2} \left(\rho_{j+\frac{1}{2}}^0 \Delta_{j+\frac{1}{2}} a + \rho_{j-\frac{1}{2}}^0 \Delta_{j-\frac{1}{2}} a \right) \left(U_j^{n+1} - U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0; \quad (1g)$$

$$\rho_{j+\frac{1}{2}}^0 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta_{j+\frac{1}{2}} a, \quad (2g)$$

where

$$U_j^{n+\frac{1}{2}} = \frac{1}{2} \left(U_j^{n+1} + U_j^n \right);$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = 0; \quad (21g)$$

$$U_j^{n+\frac{1}{2}} = \left(X_j^{n+1} - X_j^n \right) / \Delta^{n+\frac{1}{2}}_t. \quad (13g)$$

As in the derivation of Eq. (Ie), Eqs. (2g) and (13g) give:

$$\left(m_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = 0, \quad (1g)$$

where

$$m_{j+\frac{1}{2}}^n = \rho_{j+\frac{1}{2}}^n \left(X_{j+1}^n - X_j^n \right) = \rho_{j+\frac{1}{2}}^0 \Delta_{j+\frac{1}{2}} a.$$

The steps which led to Eq. (IIe) can be repeated to show that

$$\left(M_j^{n+1} - M_j^n \right) / \Delta^{n+\frac{1}{2}}_t = - \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right), \quad (\text{IIg})$$

where

$$M_j^n = m_j^n U_j^n,$$

$$m_j^n = \frac{1}{2} \left(m_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n \right).$$

By following the procedure of Sec. V, the following equations can be established:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (\text{IIIg})$$

where

$$H_j^n = \left[\frac{1}{2} m_j^n \left(U_j^n \right)^2 + \frac{1}{2} \left(m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n E_{j-\frac{1}{2}}^n \right) \right],$$

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right];$$

$$\left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}}, \quad (\text{IVg})$$

where

$$v_{j+\frac{1}{2}}^n = X_{j+1}^n - X_j^n.$$

The analogs of Eqs. (9a) - (20a) are obtained in Appendix C by further manipulation of the (g)-equations. It is shown there that the (g)-equations do not provide form-preserving analogs of Eqs. (11a), (18a). In some other

cases we get fair difference analogs of the (a) -equations, rather than good difference analogs.

Just as the implicit (e) -equations were generalized to the (g) -equations, we can also modify the explicit (d) -equations to allow for variable Δt , Δa , ρ_0 . The modified (d) -equations have the same transformation properties as

the (g) -equations, except that $\left(U_j^{n+\frac{1}{2}} U_j^{n-\frac{1}{2}} \right)$ replaces $\left(U_j^n \right)^2$.

VII. INTERFACES AND FREE SURFACES; BOUNDARY CONDITIONS

There are some practical problems in fluid mechanics in which there is one equation of state and the density is continuous, over the entire system. However, we are more generally concerned with systems containing density discontinuities that move with fluid velocity. An interface, i. e., a surface of contact between two fluids with different equations of state, is the most common example of this sort of discontinuity. Another important example is a "free surface," which is usually defined as a fluid surface not in contact with any other fluid. It will be seen presently that a free surface can be regarded as a kind of interface. It is convenient to enlarge the definition of a free surface to include any surface that moves with the fluid, on which either the pressure or the particle velocity is known as a function of time at all points. Thus, in presenting equations for the motion of a free surface, we also show how pressure and velocity boundary conditions are included in the solution of fluid flow problems, if these boundary conditions are given on surfaces which move with the fluid. Boundary conditions of this kind are the ones most often met in problems suited to a Lagrangian coordinate description.

In addition to the common Lagrangian boundary conditions, our interface equations handle two special cases of great practical interest:

- i) "spall," i. e., cavitation in a single fluid, in which two free surfaces suddenly appear in the interior of the fluid;
- ii) "join," i. e., the collision of two fluids, in which two free surfaces suddenly become an interface.

When used with interior equations such as the (g) -equations, which provide form-preserving analogs of the conservation laws (I) - (IV), the interface equations presented here describe both spall and join in such a way as to conserve exactly good difference analogs of mass, momentum, energy, and volume. So far as we know, our interface scheme is unique in this respect. The other transformation properties of the (g) -equations, discussed in Sec. VI, are also maintained intact.

Special problems arise in solving the equations of motion in the neighborhood of a density discontinuity. Strictly speaking, the differential equations do not apply at points of discontinuity in density, since most of the derivatives appearing in the equations do not exist at such points. However, these derivatives do exist in the limit as the discontinuity is approached from one side or the other. For example, U_a has finite limiting values as a density discontinuity is approached from one side or from the other, but the two limits are generally different. Most of the derivatives in the equations of motion behave in a similar way.⁸ These discontinuities have the effect of reducing the accuracy of difference analogs of the equations of motion when a finite difference scheme is applied at interfaces and free surfaces. Difference equations which are second-order analogs of the differential equations of motion in the interior of a fluid acquire first-order errors at density discontinuities. The effect is particularly significant for cylindrical and spherical flow, and for big jumps in density.

In shock processes, density discontinuities are associated with pressure discontinuities. In fact, ideally, shocks are pressure discontinuities. However, a shock moves with finite velocity relative to nearby fluid particles, while interfaces and free surfaces are stationary with respect to neighboring particles of fluid. Therefore, a pressure discontinuity cannot coincide with an interface or free surface point for any finite time. This is true in general of the force per unit area, even when it includes contributions from sources other than the pressure, such as the artificial viscosity Q . Thus, a discontinuity in the effective pressure $(P + Q)$ at an interface or free surface would either originate in, or propagate to, the interior of a single fluid. However, $(P + Q)$ must be continuous in the interior of a single fluid, since here the artificial viscosity Q eliminates any pressure discontinuities while being itself continuous. Hence $(P + Q)$ must be continuous at interfaces and free surfaces. Our interface and free-surface (or boundary) calculations are based primarily on the continuity of the effective pressure $(P + Q)$.

While it is a crude approximation to treat an interface point as if it were an ordinary interior fluid point, this procedure obviously has the desirable property that it correctly moves a "fictitious interface," i.e., an interface which is really a surface in the interior of a single fluid. That is, the motion of an interior point is exactly the same whether this motion is calculated from the interface equations or the interior equations. This is consistent with the fact that the interface equations are the more general equations, since they must be valid under fewer continuity restrictions. For this very reason, while interior equations should be special cases of interface equations, it should not be expected that good interior equations will be good interface equations. Our interface and free-surface (or boundary) calculations are based, then, on two principles:

i) Continuity of $(P + Q)$.

ii) Reduction of the interface equations to the interior equations for a fictitious interface.

Using the (g) -equations to calculate normal interior motion, we now give equations for the one-dimensional linear motion of an interface. To meet the requirement that $(P + Q)$ be continuous, the interface is thought of as located at $\left(j + \frac{1}{2}\right)$, which would be the "midpoint" of a normal interior

zone of fluid. The position X_*^n and velocity $U_*^{n - \frac{1}{2}}$ of the interface are kept track of, which is not the case at the midpoint of an ordinary interior

zone. In place of the usual $\left(\rho_{j + \frac{1}{2}}^n, e_{j + \frac{1}{2}}^n, P_{j + \frac{1}{2}}^n, Q_{j + \frac{1}{2}}^{n - \frac{1}{2}}\right)$, we now have

$\left(\rho_+^n, e_+^n, P_+^n, Q_+^{n - \frac{1}{2}}\right)$ and $\left(\rho_-^n, e_-^n, P_-^n, Q_-^{n - \frac{1}{2}}\right)$, which represent the

limiting values of density, etc., as the interface is approached from one side or the other. Also, m_+ and m_- are the masses of the two half-zones of material created by the presence of an interface at a normal zone center.

Then the equations of motion for the interface (i.e., the equations determining X_*^{n+1} and $U_*^{n + \frac{1}{2}}$) are:

$$U_*^{n + \frac{1}{2}} = \left(X_*^{n+1} - X_*^n\right) / \Delta^{n + \frac{1}{2}}_t; \quad (13*)$$

$$\rho_+^{n+1} = m_+ / \left(X_{j+1}^{n+1} - X_*^{n+1}\right), \quad (22P)$$

$$m_+ = \rho_+^n \left(X_{j+1}^n - X_*^n\right), \text{ by definition;}$$

$$\rho_-^{n+1} = m_- / \left(X_*^{n+1} - X_j^{n+1}\right), \quad (22M)$$

$$m_- = \rho_-^n \left(X_*^n - X_j^n \right), \text{ by definition;}$$

$$Q_+^{n+\frac{1}{2}} = 2 C_{0+}^2 \left(U_{j+1}^{n+\frac{1}{2}} - U_*^{n+\frac{1}{2}} \right)^2 / \left(V_+^{n+1} + V_+^n \right) \text{ (see Footnote 2);} \quad (23P)$$

$$Q_-^{n+\frac{1}{2}} = 2 C_{0-}^2 \left(U_*^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right)^2 / \left(V_-^{n+1} + V_-^n \right); \quad (23M)$$

$$m_+ \left(E_+^{n+1} - E_+^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_+^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_*^{n+\frac{1}{2}} \right) = 0; \quad (24P)$$

$$P_+^{n+1} = f_+ \left(\rho_+^{n+1}, E_+^{n+1} \right); \quad (25P)$$

$$m_- \left(E_-^{n+1} - E_-^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_-^{n+\frac{1}{2}} \left(U_*^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) = 0; \quad (24M)$$

$$P_-^{n+1} = f_- \left(\rho_-^{n+1}, E_-^{n+1} \right); \quad (25M)$$

$$PQ_+^{n+\frac{1}{2}} - PQ_-^{n+\frac{1}{2}} = 0. \quad (26)$$

The unknowns in these equations are assumed to be $\left(X_*^{n+1}, U_*^{n+\frac{1}{2}}, \rho_+^{n+1}, \rho_-^{n+1}, Q_+^{n+\frac{1}{2}}, Q_-^{n+\frac{1}{2}}, P_+^{n+1}, P_-^{n+1}, E_+^{n+1}, E_-^{n+1} \right)$. It can be seen that with Eq. (26), which insures the continuity of $(P+Q)$, the number of equations is equal to the number of unknowns. To solve these equations, we guess a value of X_*^{n+1} . This permits the explicit calculation of $\left(U_*^{n+\frac{1}{2}}, \rho_+^{n+1}, \rho_-^{n+1}, Q_+^{n+\frac{1}{2}}, Q_-^{n+\frac{1}{2}} \right)$. Then $\left(E_+^{n+1}, P_+^{n+1} \right)$ are found from Eqs. (24P), (25P) exactly as at a normal interior point. Similarly, Eqs.

(24M), (25M) are solved for $\left(E_-^{n+1}, P_-^{n+1}\right)$. We then form $\left(PQ_+^{n+\frac{1}{2}}\right)$ and $\left(PQ_-^{n+\frac{1}{2}}\right)$, and see whether or not they are equal to each other within a specified accuracy, as required by Eq. (26). Thus, Eq. (26) can be looked at as a single complicated implicit equation in the one unknown X_*^{n+1} , with Eqs. (13*) - (25M) acting as a set of auxiliary equations which give the precise form of the dependence of $\left(PQ_+^{n+\frac{1}{2}} - PQ_-^{n+\frac{1}{2}}\right)$ on X_*^{n+1} . For reasonable equations of state, the function $\left(PQ_+^{n+\frac{1}{2}} - PQ_-^{n+\frac{1}{2}}\right)$ is monotonic in the variable X_*^{n+1} . This insures the existence of just one root X_*^{n+1} of Eq. (26), which can be found by a variety of iterative techniques.

By definition, at a free surface point either $\left(PQ_+^{n+\frac{1}{2}}\right)$ or $\left(PQ_-^{n+\frac{1}{2}}\right)$ is known for all n . For concreteness, assume that $\left(PQ_-^{n+\frac{1}{2}}\right)$ is known. Equation (26) is again considered to be a single equation satisfied by the unknown X_*^{n+1} , which now gives the position of a free surface point. However, since $\left(PQ_-^{n+\frac{1}{2}}\right)$ is known, it is only necessary to specify the dependence of $\left(PQ_+^{n+\frac{1}{2}}\right)$ on X_*^{n+1} . As a result, the set of auxiliary equations to be used with Eq. (26) is reduced to Eqs. (13*), (22P), (23P), (24P), (25P). A free surface is therefore an interface of a particularly simple kind.

At a fictitious interface we have, by definition:

$$\rho_+^n = \rho_-^n = \rho_{j+\frac{1}{2}}^n, \quad Q_+^{n-\frac{1}{2}} = Q_-^{n-\frac{1}{2}} = Q_{j+\frac{1}{2}}^{n-\frac{1}{2}},$$

$$E_+^n = E_-^n = E_{j+\frac{1}{2}}^n, \quad P_+^n = P_-^n = P_{j+\frac{1}{2}}^n,$$

and for all r, s :

$$f_+(r, s) = f_-(r, s).$$

It is shown in Appendix D that if the Q -constants $(C_0 +, C_0 -)$ are so chosen that

$$C_0 + m_+ = C_0 - m_- = C_0 m_{j+\frac{1}{2}}, \quad (27)$$

then Eqs. (13*) - (26) are satisfied by

$$\begin{aligned} X_*^{n+1} &= \left(m_- X_{j+\frac{1}{2}}^{n+1} + m_+ X_j^{n+1} \right) / (m_+ + m_-), \\ \rho_+^{n+1} &= \rho_-^{n+1} = \rho_{j+\frac{1}{2}}^{n+1}, \quad Q_+^{n+\frac{1}{2}} = Q_-^{n+\frac{1}{2}} = Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}, \\ E_+^{n+1} &= E_-^{n+1} = E_{j+\frac{1}{2}}^{n+1}, \quad P_+^{n+1} = P_-^{n+1} = P_{j+\frac{1}{2}}^{n+1}, \end{aligned} \quad (28)$$

where $\rho_{j+\frac{1}{2}}^n$, etc., are normal interior quantities. Thus, in advancing from t^n to t^{n+1} , the normal interior equations and Eqs. (13*) - (26) give identical results. Since n is arbitrary, this is true by induction for all n .

Our calculation of the motion of an interface is based on conservation of mass, and not conservation of momentum. This has been done to avoid the inaccuracy involved in applying Newton's Second Law across a density discontinuity. It is satisfying that this scheme is entirely consistent with momentum conservation in the interior of a fluid. In achieving this compatibility, we were forced to require that $C_0 + m_+ = C_0 - m_- = C_0 m_{j+\frac{1}{2}}$, a fact of some interest. This suggests that in the interior of a single fluid, the Q -constant for any zone should be inversely proportional to the mass of the zone. Then the Q -region will extend over a fixed mass of material

rather than a fixed number of Lagrangian zones. In this way the value of Q is made independent of zoning, as indicated by the fact that $Q_+^{n+\frac{1}{2}}$ is equal to $Q_-^{n+\frac{1}{2}}$ at a fictitious interface, regardless of the values of m_+ , m_- . In general, a difference analog of the force per unit area should depend only on the position and time variables, and not on the way in which space and time are subdivided for numerical purposes.

It is worth noting that Eqs. (22P) - (24P) and Eq. (24M), are only first-order analogs of the corresponding continuum equations. Thus, these equations determine density, etc., most accurately not at the interface itself, but at a point between X_*^{n+1} and X_{j+1}^{n+1} , or between X_*^{n+1} and X_j^{n+1} . We have not found a way to obtain second-order accuracy and, at the same time, to recover the normal interior equations at a fictitious interface.

VIII. CYLINDRICAL AND SPHERICAL FLOW

The deduction of satisfactory difference equations for systems with spherical and cylindrical symmetry provides a very good illustration of the use of transformation properties as guides to differencing. For this reason the equations for one-dimensional spherical flow will be developed here, and not merely stated,

By definition, a spherically symmetric system is one in which all the flow variables are functions only of the distance to a fixed point and of the time.

In the Lagrangian description, a spherically symmetric system is thought of as a series of concentric spherical fluid shells. The motion of each fluid shell is to be determined, subject to proper initial and boundary

conditions. For this purpose, the definitions of Sec. II are retained, except that the Cartesian coordinate X is replaced by the spherical coordinate R , and X_j^n is therefore replaced by R_j^n . If the momentum conservation principle is applied to an entire fluid shell, we reach the correct but uninteresting conclusion that the center of spherical symmetry, which coincides with the center of mass of each and every shell, does not accelerate. To yield useful results, momentum conservation must be brought to bear on just a portion of a spherical shell. Thus it is convenient to consider only that part of the spherical system which lies within the cone $\theta' = \text{constant}$, as indicated in Fig. 1. By definition, the axis $\underline{R}$ of this cone points in the "radial direction." The component of the momentum of a fluid element in the radial direction will be referred to as the "radial momentum of the element," and the component of fluid velocity in the radial direction will be called the "radial velocity U ." Before deriving difference equations which satisfy the conditions discussed in Sec. IV, we recall first the specific forms of several of the equations of motion, in spherical coordinates.

We consider only that portion of the spherical shell lying within the cone $\theta' = \text{constant}$. Let θ and ϕ be the usual spherical coordinates, referred to the axis of the cone; the axis of the cone plays the role of the conventional Cartesian z axis, with the vertex of the cone at the Cartesian origin of coordinates. Also, let

$$R_1 = R(a_1, t) = \text{inner radius of spherical shell,}$$

$$R_2 = R(a_2, t) = \text{outer radius of spherical shell,}$$

$$J_{\text{RAD}} = \frac{1}{3} (R^3)_a,$$

$$m_{\text{RAD}} = \int_{a_1}^{a_2} \rho J_{\text{RAD}} da = \text{mass of spherical shell per unit solid angle,}$$

Fig. 1. Conical section of spherical shell.

$$M_{\text{RAD}} = \int_{a_1}^{a_2} \rho U J_{\text{RAD}} da,$$

$$H_{\text{RAD}} = \int_{a_1}^{a_2} \rho \left(\frac{1}{2} U^2 + E \right) J_{\text{RAD}} da = \text{total energy of shell per unit solid angle,}$$

$$v_{\text{RAD}} = \int_{a_1}^{a_2} J_{\text{RAD}} da = \frac{1}{3} (R_2^3 - R_1^3) = \text{volume of shell per unit solid angle.}$$

Then, as shown in Appendix E, Eqs. (I) - (IV) imply:

$$dm_{\text{RAD}}/dt = 0, \quad (\text{L})$$

$$\begin{aligned} \frac{dM_{\text{RAD}}}{dt} + \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] \right. \\ \left. - \int_{R_1}^{R_2} (P + Q) dR^2 \right\} = 0, \end{aligned} \quad (\text{M})$$

$$\frac{dH_{\text{RAD}}}{dt} + R_2^2 U(R_2) \left[P(R_2) + Q(R_2) \right] - R_1^2 U(R_1) \left[P(R_1) + Q(R_1) \right] = 0, \quad (\text{N})$$

$$\frac{dv_{\text{RAD}}}{dt} = R_2^2 U(R_2) - R_1^2 U(R_1). \quad (\text{P})$$

(In Appendix E, these are Eqs. (EE), (EC), (EP), (EQ), respectively.)

Also:

$$\left(\rho J_{\text{RAD}} U \right)_t + R^2 P_a = 0, \quad (\text{Q})$$

$$\left(\frac{1}{2} \rho J_{\text{RAD}} U^2 \right)_t + J_{\text{RAD}} U(P + Q)_a = 0. \quad (\text{R})$$

(These are Eqs. (EL), (EN) of Appendix E.)

We now turn to the problem of finding difference analogs of Eqs. (I) - (IV), etc., which transform properly among themselves. The derivation of such equations rests mainly on two points:

- i) a form-preserving analog (FPA) of the kinetic energy equation (14a) should be implied by FPA's of the mass and momentum conservation equations (I), (II);
- ii) the FPA of Eq. (14a) should combine with an FPA of the internal energy equation (15a) to give an FPA of Eq. (III) for total energy conservation.

To apply these conditions, let $\mathcal{E}_j$ be the element of fluid lying within the cone $\theta' = \text{constant}$, and between the two spherical surfaces

$R = R\left(a_{j+\frac{1}{2}}, t^n\right) = R_{j+\frac{1}{2}}^n$ and $R = R\left(a_{j-\frac{1}{2}}, t^n\right) = R_{j-\frac{1}{2}}^n$. Also, let $\mu_{j+\frac{1}{2}}^n$ be the mass of the element $\mathcal{E}_{j+\frac{1}{2}}$. Then our difference analog of Eq. (I) is:

$$\left(\mu_{j+\frac{1}{2}}^{n+1} - \mu_{j+\frac{1}{2}}^n\right) / \Delta^{n+\frac{1}{2}}_t = 0. \quad (1h)$$

Hence

$$\left(\mu_j^{n+1} - \mu_j^n\right) / \Delta^{n+\frac{1}{2}}_t = 0,$$

where

$$\mu_j^n = \frac{1}{2} \left(\mu_{j+\frac{1}{2}}^n + \mu_{j-\frac{1}{2}}^n \right).$$

Next, as a difference analog of the radial momentum of the element $\mathcal{E}_j$, we write:

$$M_j^n = \mu_j^n U_j^n,$$

where U_j^n is the radial velocity at $R(a_j, t^n)$. The exact radial momentum of this fluid element is

$$M_R(\mathcal{E}_j, t^n) = \pi \sin^2 \theta' \int_{a_j - \frac{1}{2}}^{a_j + \frac{1}{2}} \rho J_{RAD} U da \quad \begin{array}{l} \text{(see Eqs. (EJ), (EA)} \\ \text{in Appendix E).} \end{array}$$

The mass of the element $\mathcal{E}_j$ is given exactly by

$$\mu_j^n = 4\pi \sin^2 \theta' / 2 \int_{a_j - \frac{1}{2}}^{a_j + \frac{1}{2}} \rho J_{RAD} da \quad \begin{array}{l} \text{(see Eqs. (EJ), (ED)} \\ \text{in Appendix E).} \end{array}$$

Therefore

$$M_R(\mathcal{E}_j, t^n) = \mu_j^n U(\bar{a}_j, t^n) \left[\sin^2 \theta' / \left(4 \sin^2 \frac{1}{2} \theta' \right) \right]; \quad a_j - \frac{1}{2} < \bar{a}_j < a_j + \frac{1}{2}.$$

Thus $M_R(\mathcal{E}_j, t^n) \rightarrow \mu_j^n U(\bar{a}_j, t^n)$ as $\theta' \rightarrow 0$. Only in this limit is M_j^n an

FPA of the exact radial momentum $M_R(\mathcal{E}_j, t^n)$, and we therefore consider

θ' to be a very small angle. Owing to the spherical symmetry of the system, there is no loss of generality involved in this assumption. The simplest FPA of the radial component of Eq. (II) is then:

$$\left(M_j^{n+1} - M_j^n \right) / \Delta^{n+\frac{1}{2}}_t = \mathcal{F}_{R_j}^{n+\frac{1}{2}} \quad (S)$$

where $\left(\mathcal{F}_{R_j}^{n+\frac{1}{2}} \Delta^{n+\frac{1}{2}}_t \right)$ is any FPA of the radial impulse delivered to the

element $\mathcal{E}_j$ between the times t^n and t^{n+1} . Now, since $\mu_j^{n+1} = \mu_j^n$:

$$M_j^{n+1} - M_j^n = \mu_j^{n+1} U_j^{n+1} - \mu_j^n U_j^n = \frac{1}{2} \left(\mu_j^{n+1} + \mu_j^n \right) \left(U_j^{n+1} - U_j^n \right).$$

Therefore, if we define $U_j^{n+\frac{1}{2}} = \frac{1}{2} \left(U_j^{n+1} + U_j^n \right)$, then:

$$U_j^{n+\frac{1}{2}} \left(M_j^{n+1} - M_j^n \right) = \frac{1}{2} \mu_j^{n+1} \left(U_j^{n+1} \right)^2 - \frac{1}{2} \mu_j^n \left(U_j^n \right)^2.$$

Thus we obtain

$$\left[\frac{1}{2} \mu_j^{n+1} \left(U_j^{n+1} \right)^2 - \frac{1}{2} \mu_j^n \left(U_j^n \right)^2 \right] / \Delta^{n+\frac{1}{2}}_t = U_j^{n+\frac{1}{2}} \mathcal{F}_{R_j}^{n+\frac{1}{2}}. \quad (T)$$

The kinetic energy of the element $\mathcal{E}_j$ at time t^n is given exactly by:

$$T(\mathcal{E}_j, t^n) = 4\pi \sin^2 \theta' / 2 \int_{a_{j-\frac{1}{2}}}^{a_{j+\frac{1}{2}}} \frac{1}{2} \rho J_{RAD} U^2 da$$

$$\begin{aligned} &= \frac{1}{2} \left[U(\tilde{a}_j, t^n) \right]^2 4\pi \sin^2 \theta' / 2 \int_{a_{j-\frac{1}{2}}}^{a_{j+\frac{1}{2}}} \rho J_{RAD} da \\ &= \frac{1}{2} \mu_j^n \left[U(\tilde{a}_j, t^n) \right]^2; \quad a_{j-\frac{1}{2}} < \tilde{a}_j < a_{j+\frac{1}{2}}. \end{aligned}$$

Hence, $\frac{1}{2} \mu_j^n (U_j^n)^2$ is an FPA of the exact kinetic energy $T(\mathcal{E}_j, t^n)$. As shown in Appendix E, $T(\mathcal{E}_j, t^n) / T_R(\mathcal{E}_j, t^n) \rightarrow 1$ as $\theta' \rightarrow 0$, where $T_R(\mathcal{E}_j, t^n)$ is the contribution to the kinetic energy of element $\mathcal{E}_j$ from the radial component of velocity. This is consistent with the fact that in Eq. (T), the rate of change of the total kinetic energy depends on only the radial component of the force acting on element $\mathcal{E}_j$.

Equation (T) is the fundamental difference equation for the rate of change of kinetic energy. To obtain Eq. (T) it is sufficient that the momentum of a material element be written explicitly as the product of its mass and

an average velocity, and that the mass be conserved. It is not necessary to specify the way in which the mass and the radial component of force are related to other variables, such as radial coordinates.

To make further progress toward FPA's of Eqs. (14a), etc., conservation of mass and volume must be looked at in some detail. By definition:

$$m_{j+\frac{1}{2}}^n = \mu_{j+\frac{1}{2}}^n / 2\pi(1 - \cos \theta'); \quad m_j^n = \frac{1}{2} \left(m_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n \right),$$

$$J_{j+\frac{1}{2}}^n = \frac{1}{3} \left[\left(R_{j+1}^n \right)^3 - \left(R_j^n \right)^3 \right] / \left(a_{j+1}^n - a_j^n \right),$$

$$v_{j+\frac{1}{2}}^n = \frac{1}{3} \left[\left(R_{j+1}^n \right)^3 - \left(R_j^n \right)^3 \right],$$

$$\rho_{j+\frac{1}{2}}^n = m_{j+\frac{1}{2}}^n / v_{j+\frac{1}{2}}^n,$$

$$v_j^n = \frac{1}{2} \left(v_{j+\frac{1}{2}}^n + v_{j-\frac{1}{2}}^n \right),$$

$$\rho_j^n = m_j^n / v_j^n,$$

$$F_{R_j}^{n+\frac{1}{2}} = \mathcal{F}_{R_j}^{n+\frac{1}{2}} / 2\pi(1 - \cos \theta').$$

Clearly, $J_{j+\frac{1}{2}}^n$ is a good analog of the radial part of the Jacobian

$\left(J_{\text{RAD}} = \frac{1}{3} (R^3)_a \right)$. Also, $\left(m_{j+\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n \right)$ are good analogs of the mass

and volume per unit solid angle of a spherical shell. With these definitions,

Eq. (1h) can be written:

$$\left(m_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n \right) / \Delta t^{n+\frac{1}{2}} = 0. \quad (U)$$

Also, Eq. (T) becomes:

$$\left[\frac{1}{2} m_j^{n+1} (U_j^{n+1})^2 - \frac{1}{2} m_j^n (U_j^n)^2 \right] / \Delta^{n+\frac{1}{2}}_t = U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}}, \quad (V)$$

or

$$\left[\frac{1}{2} \rho_j^{n+1} v_j^{n+1} (U_j^{n+1})^2 - \frac{1}{2} \rho_j^n v_j^n (U_j^n)^2 \right] / \Delta^{n+\frac{1}{2}}_t = U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}},$$

or

$$\left[\frac{1}{2} \rho_j^{n+1} J_j^{n+1} (U_j^{n+1})^2 - \frac{1}{2} \rho_j^n J_j^n (U_j^n)^2 \right] / \Delta^{n+\frac{1}{2}}_t = U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}} / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \right), \quad (W)$$

where

$$J_j^n = v_j^n / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \right) = \frac{1}{6} \left[\left(R_{j+1}^n \right)^3 - \left(R_{j-1}^n \right)^3 \right] / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \right).$$

The left-hand side of Eq. (W) is a good difference analog of $\left(\frac{1}{2} \rho J_{\text{RAD}} U^2 \right)_t$, the first term of the kinetic energy equation, Eq. (R). To complete the construction of a difference analog of Eq. (R), a specific form must now be given

to the radial force $\mathcal{F}_{R_j}^{n+\frac{1}{2}}$, or to $F_{R_j}^{n+\frac{1}{2}}$. An unambiguous definition of

$F_{R_j}^{n+\frac{1}{2}}$ is obtained by requiring that the kinetic energy equation (V) combine

with an internal energy equation to give an FPA of Eq. (III) for total energy conservation. (Equation (III) is equivalent to Eq. (N) in spherical systems.)

Our fundamental difference equation for the rate of change of internal energy is:

$$\left(m_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = 0. \quad (X)$$

Following the line of development of the linear one-dimensional equations, we combine Eqs. (V), (X) to produce the total energy equation:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \mathcal{H}_j^{n+\frac{1}{2}} = 0, \quad (Y)$$

where

$$H_j^n = \frac{1}{2} m_j^n (U_j^n)^2 + \frac{1}{2} \left(m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n E_{j-\frac{1}{2}}^n \right),$$

$$\mathcal{H}_j^{n+\frac{1}{2}} = \frac{1}{2} \left[PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left(v_{j-\frac{1}{2}}^{n+1} - v_{j-\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t \right] - U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}}.$$

The problem now is to define $F_{R_j}^{n+\frac{1}{2}}$ in such a way that $\mathcal{H}_j^{n+\frac{1}{2}}$ becomes an FPA of

$$(R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (R^2)_{j-\frac{1}{2}}^{n+\frac{1}{2}} U_{j-\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}},$$

which is the rate at which work is done by the element $\mathcal{E}_j$ on its surroundings, per unit solid angle (see Eq. (N)). At the same time, it can be seen by comparing Eqs. (S), (M) that in order to have momentum conservation it

is essential that $F_{R_j}^{n+\frac{1}{2}}$ be an FPA of $\phi_j^{n+\frac{1}{2}}$, where

$$\phi_j^{n+\frac{1}{2}} = - (R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} + (R^2)_{j-\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \int_{(R^2)_{j-\frac{1}{2}}^{n+\frac{1}{2}}}^{(R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}}} (P+Q) dR^2. \quad (Z)$$

Because of the definitions of $v_{j+\frac{1}{2}}^n$ and the relation

$$U_j^{n+\frac{1}{2}} = \left(R_j^{n+1} - R_j^n \right) / \Delta^{n+\frac{1}{2}}_t, \quad (\text{AA})$$

the requirement that $\mathcal{W}_j^{n+\frac{1}{2}}$ be an FPA of

$$(R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (R^2)_{j-\frac{1}{2}}^{n+\frac{1}{2}} U_{j-\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}}$$

leads us to a unique choice of $F_{R_j}^{n+\frac{1}{2}}$ and $(R^2)_j^{n+\frac{1}{2}}$, as follows.

First:

$$\begin{aligned} v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n &= \frac{1}{3} \left[\left(R_{j+1}^{n+1} \right)^3 - \left(R_j^{n+1} \right)^3 - \left(R_{j+1}^n \right)^3 + \left(R_j^n \right)^3 \right] \\ &= (R^2)_{j+1}^{n+\frac{1}{2}} \left(R_{j+1}^{n+1} - R_{j+1}^n \right) - (R^2)_j^{n+\frac{1}{2}} \left(R_j^{n+1} - R_j^n \right), \end{aligned}$$

where

$$(R^2)_j^{n+\frac{1}{2}} = \frac{1}{3} \left[\left(R_j^{n+1} \right)^2 + R_j^{n+1} R_j^n + \left(R_j^n \right)^2 \right]. \quad (\text{AB})$$

Hence

$$\left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = (R^2)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}}. \quad (\text{AC})$$

Equation (AC) is a good difference analog of the volume conservation equation (Eq. (IV) or its spherical equivalent, Eq. (P)). To get it, we have been

forced to define $(R^2)_j^{n+\frac{1}{2}}$ through Eq. (AB). Next, in view of Eq. (AC),

the work expression $\mathcal{W}_j^{n+\frac{1}{2}}$ becomes (see Eq. (Y)):

$$\mathcal{W}_j^{n+\frac{1}{2}} = \frac{1}{2} \left\{ \left[(R^2)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] PQ_j^{n+\frac{1}{2}} + \left[(R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} - (R^2)_{j-1}^{n+\frac{1}{2}} U_{j-1}^{n+\frac{1}{2}} \right] PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right\} - U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}}.$$

This equation is very similar to the linear one-dimensional expression for the rate of work (see Eq. (IIIe)). In the linear case:

$$\begin{aligned} \mathcal{W}_j^{n+\frac{1}{2}} &= \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \rho_0 \Delta a \\ &= \frac{1}{2} \left[\left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \left(U_j^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}} \right) PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right] - U_j^{n+\frac{1}{2}} F_{R_j}^{n+\frac{1}{2}}, \end{aligned}$$

where

$$F_{R_j}^{n+\frac{1}{2}} = - \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right).$$

The spherical case is different in that $R^2 U$ appears where U appeared before. Therefore, with this one difference, $\mathcal{W}_j^{n+\frac{1}{2}}$ will take on precisely the form of the linear work expression if we define

$$F_{R_j}^{n+\frac{1}{2}} = - (R^2)_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right). \quad (AD)$$

From this definition, it follows that

$$\begin{aligned} \mathcal{W}_j^{n+\frac{1}{2}} &= \frac{1}{2} \left[(R^2)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} + (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \\ &\quad - \frac{1}{2} \left[(R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} + (R^2)_{j-1}^{n+\frac{1}{2}} U_{j-1}^{n+\frac{1}{2}} \right] PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}}. \end{aligned} \quad (AE)$$

$\mathcal{H}_j^{n+\frac{1}{2}}$ is a good difference analog of

$$(R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (R^2)_{j-\frac{1}{2}}^{n+\frac{1}{2}} U_{j-\frac{1}{2}}^{n+\frac{1}{2}} PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}}.$$

Equation (Y) becomes:

$$\left(H_j^{n+1} - H_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (\text{AF})$$

where

$$H_j^n = \frac{1}{2} m_j^n (U_j^n)^2 + \frac{1}{2} \left(m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n E_{j-\frac{1}{2}}^n \right),$$

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left[(R^2)_{j+\frac{1}{2}}^{n+\frac{1}{2}} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} + (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}}.$$

Equation (AF) is a good difference analog of Eq. (N), the spherical equivalent of Eq. (III) for energy conservation. Also, with $F_{R_j}^{n+\frac{1}{2}}$ defined by Eq. (AD), the momentum conservation equation (S) becomes:

$$\left(m_j^{n+1} U_j^{n+1} - m_j^n U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + (R^2)_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (\text{AG})$$

Equation (AG) is a good difference analog of Eq. (M), the spherical equivalent of Eq. (II) for momentum conservation.

The definition of $F_{R_j}^{n+\frac{1}{2}}$ adopted here is, as far as we know, the only one which yields an FPA of the rate of work while at the same time providing an FPA of the momentum conservation equation (M). We have shown that by using this definition, Eqs. (U), (AG), (AF), (AC) satisfy our essential

transformation requirements in giving FPA's of the conservation equations (I) - (IV). In fact, we obtain good analogs of these equations. Nevertheless, Eqs. (U), (AG), (AF), (AC) lead to FPA's of only a few of the equations (9a) - (21a). Therefore Eqs. (U), (AG), (AF), (AC) are not as satisfactory as Eqs. (1g), (2g), (21g), (13g) for linear one-dimensional motion. To show that this is so, the analogs of Eqs. (9a) - (15a) are examined in Appendix F. It is interesting that the cylindrical equations are better in this respect than the spherical equations.

Before leaving the interior spherical equations, we rearrange them to a form more suitable for numerical calculations, and more like Eqs. (1g), (2g), (21g), (13g) for linear motion. It is easy to show that Eqs. (U), (AG), (AF), (AC) are equivalent to:

$$\frac{1}{2} \left[\rho_{j+\frac{1}{2}}^0 \frac{1}{3} (a_{j+1}^3 - a_j^3) + \rho_{j-\frac{1}{2}}^0 \frac{1}{3} (a_j^3 - a_{j-1}^3) \right] (U_j^{n+1} - U_j^n) / \Delta^{n+\frac{1}{2}}_t + (R^2)_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0; \quad (1h)$$

$$\rho_{j+\frac{1}{2}}^0 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = \left[(R^2)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] / \frac{1}{3} (a_{j+1}^3 - a_j^3), \quad (2h)$$

where

$$U_j^{n+\frac{1}{2}} = \frac{1}{2} (U_j^{n+1} + U_j^n),$$

$$(R^2)_j^{n+\frac{1}{2}} = \frac{1}{3} \left[(R_j^{n+1})^2 + R_j^{n+1} R_j^n + (R_j^n)^2 \right];$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n\right) \Delta^{n+\frac{1}{2}}_t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n\right) \Delta^{n+\frac{1}{2}}_t = 0; \quad (21h)$$

$$U_j^{n+\frac{1}{2}} = \left(R_j^{n+1} - R_j^n\right) \Delta^{n+\frac{1}{2}}_t. \quad (13h)$$

As with linear systems, it is again desirable that interfaces be moved by means of equations which enforce continuity of $(P + Q)$ and reduce to the interior equations at a fictitious interface. Following the argument of Sec. VII, it can be shown that these conditions are met by Eqs. (13*) - (26) for linear, cylindrical, and spherical systems alike, if Eqs. (22P), (23P), (22M), (23M), (24P), (24M) are rewritten as

$$\rho_+^{n+1} = m_+ / v_+^{n+1}, \quad (22P')$$

$$\rho_-^{n+1} = m_- / v_-^{n+1}, \quad (22M')$$

$$Q_+^{n+\frac{1}{2}} = 2 C_0^2 m_+^2 \left(v_+^{n+1} - v_+^n\right)^2 / \left(v_+^{n+1} + v_+^n\right) \left(\Delta^{n+\frac{1}{2}}_t\right)^2, \quad (23P')$$

$$Q_-^{n+\frac{1}{2}} = 2 C_0^2 m_-^2 \left(v_-^{n+1} - v_-^n\right)^2 / \left(v_-^{n+1} + v_-^n\right) \left(\Delta^{n+\frac{1}{2}}_t\right)^2, \quad (23M')$$

$$m_+ \left(E_+^{n+1} - E_+^n\right) \Delta^{n+\frac{1}{2}}_t + PQ_+^{n+\frac{1}{2}} \left(v_+^{n+1} - v_+^n\right) \Delta^{n+\frac{1}{2}}_t = 0, \quad (24P')$$

$$m_- \left(E_-^{n+1} - E_-^n\right) \Delta^{n+\frac{1}{2}}_t + PQ_-^{n+\frac{1}{2}} \left(v_-^{n+1} - v_-^n\right) \Delta^{n+\frac{1}{2}}_t = 0. \quad (24M')$$

This completes our discussion of the spherical equations.

For cylindrical systems, our difference equations are:

$$\left(m_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n\right) \Delta^{n+\frac{1}{2}}_t = 0, \quad (1k)$$

$$\left(M_j^{n+1} - M_j^n\right) / \Delta^{n+\frac{1}{2}}_t + R_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) = 0, \quad (\text{IIk})$$

$$\left(m_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - m_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n\right) / \Delta^{n+\frac{1}{2}}_t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n\right) / \Delta^{n+\frac{1}{2}}_t = 0, \quad (\text{AS})$$

$$U_j^{n+\frac{1}{2}} = \left(R_j^{n+1} - R_j^n\right) / \Delta^{n+\frac{1}{2}}_t, \quad (\text{AT})$$

where, by definition:

$R(a, t)$ = cylindrical radial position coordinate,

$a = R(a, 0)$,

$2\pi m_{j+\frac{1}{2}}^n$ = mass of cylindrical zone,

$$m_j^n = \frac{1}{2} \left(m_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}}^n\right),$$

$$M_j^n = m_j^n U_j^n,$$

$$v_{j+\frac{1}{2}}^n = \frac{1}{2} \left[\left(R_{j+1}^n\right)^2 - \left(R_j^n\right)^2 \right].$$

These equations are more conveniently written:

$$\left[\rho_{j+\frac{1}{2}}^0 \frac{1}{2} (a_{j+1}^2 - a_j^2) + \rho_{j-\frac{1}{2}}^0 \frac{1}{2} (a_j^2 - a_{j-1}^2) \right] \left(U_j^{n+1} - U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + R_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0; \quad (\text{Ik})$$

$$\rho_{j+\frac{1}{2}}^0 \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = \left(R_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - R_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right) / \frac{1}{2} (a_{j+1}^2 - a_j^2), \quad (\text{2k})$$

where

$$U_j^{n+\frac{1}{2}} = \frac{1}{2} \left(U_j^{n+1} + U_j^n \right),$$

$$R_j^{n+\frac{1}{2}} = \frac{1}{2} \left(R_j^{n+1} + R_j^n \right);$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = 0; \quad (21k)$$

$$U_j^{n+\frac{1}{2}} = \left(R_j^{n+1} - R_j^n \right) / \Delta^{n+\frac{1}{2}}_t. \quad (13k)$$

Here

$$\rho_{j+\frac{1}{2}}^n = m_{j+\frac{1}{2}}^n / v_{j+\frac{1}{2}}^n = 1 / v_{j+\frac{1}{2}}^n.$$

SUMMARY OF EQUATIONS

Based on the criteria set forth in this paper, we have arrived at the following difference equations for the Lagrangian description of one-dimensional linear, cylindrical, and spherical systems:

(i) Interior of Single Fluid

(a) Implicit Equations

$$m_j \left(U_j^{n+1} - U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(R^{d-1} \right)_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (R1)$$

$$U_j^{n+\frac{1}{2}} = \left(R_j^{n+1} - R_j^n \right) / \Delta^{n+\frac{1}{2}}_t, \quad (R2)$$

$$m_{j+\frac{1}{2}} \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t = \left(R^{d-1} \right)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - \left(R^{d-1} \right)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}}, \quad (R3)$$

$$Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = 2C_{j+\frac{1}{2}}^2 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right)^2 / \left(V_{j+\frac{1}{2}}^{n+1} + V_{j+\frac{1}{2}}^n \right) \left(\Delta_{t}^{n+\frac{1}{2}} \right)^2, \quad (R4)$$

$$\left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) = 0, \quad (R5)$$

$$P_{j+\frac{1}{2}}^{n+1} = f \left(V_{j+\frac{1}{2}}^{n+1}, E_{j+\frac{1}{2}}^{n+1} \right), \quad (R6)$$

where, by definition:

$R(a, t)$ = Eulerian position coordinate,

$a = R(a, 0)$ = Lagrangian coordinate,

$m_{j+\frac{1}{2}} = \rho^0_{j+\frac{1}{2}} \left(a_{j+1}^d - a_j^d \right) / d$ = mass between (a_j) and (a_{j+1}) ,

$$m_j = \frac{1}{2} \left(m_{j+\frac{1}{2}} + m_{j-\frac{1}{2}} \right),$$

$$U_j^{n+\frac{1}{2}} = \frac{1}{2} \left(U_j^{n+1} + U_j^n \right),$$

$d = 1$ for linear flow, 2 for cylindrical flow, 3 for spherical flow,

$$\left(R^{d-1} \right)_j^{n+\frac{1}{2}} = \begin{cases} 1, & \text{if } d = 1 \\ \frac{1}{2} \left(R_j^{n+1} + R_j^n \right), & \text{if } d = 2 \\ \frac{1}{3} \left[\left(R_j^{n+1} \right)^2 + R_j^{n+1} R_j^n + \left(R_j^n \right)^2 \right], & \text{if } d = 3, \end{cases}$$

$C_{j+\frac{1}{2}}$ = Q-constant,

$$PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left(P_{j+\frac{1}{2}}^{n+1} + P_{j+\frac{1}{2}}^n \right) + Q_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

U_j^n = particle velocity at (a_j, t^n) ,

$$V_{j+\frac{1}{2}}^n = 1/\rho_{j+\frac{1}{2}}^n = \text{specific volume at } \left(a_{j+\frac{1}{2}}, t^n\right),$$

$$E_{j+\frac{1}{2}}^n = \text{specific internal energy at } \left(a_{j+\frac{1}{2}}, t^n\right).$$

Equations (R1) - (R6) conserve the total energy analog

$$H_j^n = \frac{1}{2} m_j \left(U_j^n\right)^2 + \frac{1}{2} \left(m_{j+\frac{1}{2}} E_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}} E_{j-\frac{1}{2}}^n\right),$$

with the rate of work at $R\left(a_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}\right)$ given by:

$$W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\left(R^d - 1\right)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} + \left(R^d - 1\right)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right].$$

(b) Explicit Equations

Same as Eqs. (R1) - (R6) except that Eq. (R1) is replaced by

$$m_j \left(U_j^n - U_j^{n-1} \right) / \Delta^{n-\frac{1}{2}}_t + \left(R^d - 1 \right)_j^{n-\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n-\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right) = 0, \quad (E1)$$

and

$$U_j^{n+\frac{1}{2}} = 2 U_j^n - U_j^{n-\frac{1}{2}}$$

replaces the previous definition of $U_j^{n+\frac{1}{2}}$. The total energy analog which is conserved is

$$H_j^n = \frac{1}{2} m_j U_j^{n+\frac{1}{2}} U_j^{n-\frac{1}{2}} + \frac{1}{2} \left(m_{j+\frac{1}{2}} E_{j+\frac{1}{2}}^n + m_{j-\frac{1}{2}} E_{j-\frac{1}{2}}^n \right).$$

The rate of work $W_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ is the same as in the case of the implicit equations.

(ii) Interface Equations

These equations are used with both the implicit and explicit interior calculations. The interface (*) is considered to be at the normal midpoint of a zone.

$$U_*^{n+\frac{1}{2}} = \left(R_*^{n+1} - R_*^n \right) / \Delta^{n+\frac{1}{2}}_t, \quad (N1)$$

$$\rho_+^{n+1} = m_+ / v_+^{n+1}, \quad (N2)$$

$$\rho_-^{n+1} = m_- / v_-^{n+1}, \quad (N3)$$

$$Q_+^{n+\frac{1}{2}} = 2 C_+^2 \left(v_+^{n+1} - v_+^n \right) / \left(v_+^{n+1} + v_+^n \right) \left(\Delta^{n+\frac{1}{2}}_t \right)^2, \quad (N4)$$

$$Q_-^{n+\frac{1}{2}} = 2 C_-^2 \left(v_-^{n+1} - v_-^n \right) / \left(v_-^{n+1} + v_-^n \right) \left(\Delta^{n+\frac{1}{2}}_t \right)^2, \quad (N5)$$

$$m_+ \left(E_+^{n+1} - E_+^n \right) + PQ_+^{n+\frac{1}{2}} \left(v_+^{n+1} - v_+^n \right) = 0, \quad (N6)$$

$$P_+^{n+1} = f_+ \left(v_+^{n+1}, E_+^{n+1} \right), \quad (N7)$$

$$m_- \left(E_-^{n+1} - E_-^n \right) + PQ_-^{n+\frac{1}{2}} \left(v_-^{n+1} - v_-^n \right) = 0, \quad (N8)$$

$$P_-^{n+1} = f_- \left(v_-^{n+1}, E_-^{n+1} \right), \quad (N9)$$

$$PQ_+^{n+\frac{1}{2}} - PQ_-^{n+\frac{1}{2}} = 0, \quad (N10)$$

where, by definition,

$$m_+ = \rho_+^0 v_+^0,$$

$$m_- = \rho_-^0 v_-^0,$$

$$v_+^n = \frac{1}{d} \left[\left(R_{j+1}^n \right)^d - \left(R_*^n \right)^d \right],$$

$$v_-^n = \frac{1}{d} \left[\left(R_*^n \right)^d - \left(R_j^n \right)^d \right].$$

At a fictitious interface these equations reduce to the interior equations if

$C_+ = C_-$. In this case R_*^{n+1} satisfies the equation

$$m_+ v_-^{n+1} = m_- v_+^{n+1}.$$

APPENDIX A

Definitions of Form Preservation and M th-Order Analogs; Application of Definitions to Conservation of Volume

Let

$\underline{y}(i) = (y_1(i), y_2(i), \dots, y_K(i))$ = a point in the difference mesh,

$\underline{Y}(i) = (Y_1(y(i)), Y_2(y(i)), \dots, Y_N(y(i)))$,

$\{Y\} = Y_1(1), \dots, Y_1(M); Y_2(1), \dots, Y_2(M); \dots; Y_N(1), \dots, Y_N(M)$,

$\{y\} = y_1(1), \dots, y_1(m); y_2(1), \dots, y_2(m); \dots; y_K(1), \dots, y_K(m)$,

where $m \geq M$.⁹

Also, let $\{P_k y\}$ be identical with $\{y\}$ except for a permutation of the quantities $y_k(1), \dots, y_k(m)$ which gives the sequence $y_k(1^*), \dots, y_k(m^*)$. Further, let $\{P_k Y\}$ be identical with $\{Y\}$ except that wherever $y_k(i)$ appears as an argument of a Y -function it is replaced by $y_k(i^*)$. Then:

$$F\left(\{Y\}, \{y\}\right) = 0$$

is a form-preserving difference analog of Eq. (C) if

$$F\left(\{Y\}, \{y\}\right) = \sum_{j=1}^J \sum_{k=1}^K a_{jk} \left(\{Y\}, \{y\} \right) \beta'_{jk} \left(\{Y\}, \{y\} \right) + \gamma \left(\{Y\}, \{y\} \right),$$

where

$$\beta'_{jk} \left(\{Y\}, \{y\} \right) = \left[\beta \left(\{P_k Y\}, \{P_k y\} \right) - \beta \left(\{Y\}, \{y\} \right) \right] / \left[y_k(i_1) - y_k(i_2) \right],$$

and, for $j=1 \rightarrow J$, $k=1 \rightarrow K$, $\alpha_{jk}(\{Y\}, \{y\})$ is a function only of the variables which appear explicitly in $A_{jk}(Y, y)$; likewise $\beta_{jk}(\{Y\}, \{y\})$ is a function only of the variables which appear explicitly in $B_{jk}(Y, y)$, and $\gamma(\{Y\}, \{y\})$ is a function only of the variables which appear explicitly in $C(\{Y\}, \{y\})$.

Also, for any point $y(0)$:

$$\lim \beta'_{jk}(\{Y\}, \{y\}) = \left[\partial B_{jk}(Y, y) / \partial y_k \right]_{y=y(0)},$$

$$\lim \alpha_{jk}(\{Y\}, \{y\}) = \left[A_{jk}(Y, y) \right]_{y=y(0)},$$

$$\lim \gamma(\{Y\}, \{y\}) = \left[C(Y, y) \right]_{y=y(0)}.$$

Where "lim" is defined as the limit in which the difference increments $\left\{ y(i) - y(0) \right\}_{i=1}^m$ approach zero.

Next, let $Y_1(y), \dots, Y_N(y)$ be arbitrary analytic functions, and let

$$D(\{Y\}, \{y\}) = F(\{Y\}, \{y\}) - \left[\sum_{j=1}^J \sum_{k=1}^K A_{jk} B'_{jk} + C \right]_{y=y(0)}.$$

Then $F = 0$ is an M th-order analog of Eq. (C) if there exists a point $y(0)$ such that the first nonzero terms in the Taylor expansion of (D) about $y(0)$ are of a degree $\geq M$ in the difference increments $\left\{ y(i) - y(0) \right\}_{i=1}^m$. The product of any (L) increments of the set $\left\{ y(i) - y(0) \right\}_{i=1}^m$ is considered an L th-degree product.

By definition, the equation $(F = 0)$ is a good difference analog of Eq. (C) if $(F = 0)$ is a second-order, form-preserving analog of Eq. (C). Also, $(F = 0)$ is a fair difference analog of Eq. (C) if $(F = 0)$ is a first-order,

form-preserving analog of Eq. (C). Otherwise $(F = 0)$ is a poor difference analog of Eq. (C).

The above definitions will now be applied to Eq. (2a) and its difference analog, Eq. (2b). Here

$$\underline{y} = (y_1, y_2) = (a, t),$$

$$\underline{Y} = (Y_1, Y_2) = (Y_1(\underline{y}), Y_2(\underline{y})) = (V(a, t), U(a, t)),$$

$$\sum_{j=1}^J \sum_{k=1}^K A_{jk} B'_{jk} + C = \rho_0 V_t - U_a,$$

whence

$$A_{12}(\underline{Y}, \underline{y}) = \rho_0, \quad B_{12}(\underline{Y}, \underline{y}) = V, \quad B'_{12}(\underline{Y}, \underline{y}) = V_t;$$

$$A_{21}(\underline{Y}, \underline{y}) = 1, \quad B_{21}(\underline{Y}, \underline{y}) = U, \quad B'_{21}(\underline{Y}, \underline{y}) = U_a;$$

$$C = 0, \quad A_{11}(\underline{Y}, \underline{y}) = 0, \quad A_{22}(\underline{Y}, \underline{y}) = 0;$$

$$J = 2, \quad K = 2.$$

Also, from Eq. (2b):

$$\begin{aligned} F(\{\underline{Y}\}, \{\underline{y}\}) &= \rho_0 \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right) / \Delta t - \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta a \\ &= \rho_0 \left[V\left(a_{j+\frac{1}{2}}, t^{n+1}\right) - V\left(a_{j+\frac{1}{2}}, t^n\right) \right] / (t^{n+1} - t^n) \\ &\quad - \left[U\left(a_{j+1}, t^{n+\frac{1}{2}}\right) - U\left(a_j, t^{n+\frac{1}{2}}\right) \right] / (a_{j+1} - a_j). \end{aligned}$$

Thus

$$\underline{y}(1) = (y_1(1), y_2(1)) = \left(a_{j+\frac{1}{2}}, t^{n+1}\right);$$

$$\underline{y}(2) = (y_1(2), y_2(2)) = \left(a_{j+\frac{1}{2}}, t^n\right);$$

$$\underline{y}(3) = (y_1(3), y_2(3)) = \left(a_{j+1}, t^{n+\frac{1}{2}} \right),$$

$$\underline{y}(4) = (y_1(4), y_2(4)) = \left(a_j, t^{n+\frac{1}{2}} \right),$$

$$Y_1(1) = V_{j+\frac{1}{2}}^{n+\frac{1}{2}}; Y_2(1) = U_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

$$Y_1(2) = V_{j+\frac{1}{2}}^n; Y_2(2) = U_{j+\frac{1}{2}}^n,$$

$$Y_1(3) = V_{j+1}^{n+\frac{1}{2}}; Y_2(3) = U_{j+1}^{n+\frac{1}{2}},$$

$$Y_1(4) = V_j^{n+\frac{1}{2}}; Y_2(4) = U_j^{n+\frac{1}{2}},$$

$$\left\{ y \right\} = a_{j+\frac{1}{2}}, a_{j+\frac{1}{2}}, a_{j+1}, a_j; t^{n+1}, t^n, t^{n+\frac{1}{2}}, t^{n+\frac{1}{2}},$$

$$\left\{ Y \right\} = V_{j+\frac{1}{2}}^{n+\frac{1}{2}}, V_{j+\frac{1}{2}}^n, V_{j+1}^{n+\frac{1}{2}}, V_j^{n+\frac{1}{2}}; U_{j+\frac{1}{2}}^{n+\frac{1}{2}}, U_{j+\frac{1}{2}}^n, U_{j+1}^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}.$$

If we now choose

$$\alpha_{12}(\{Y\}, \{y\}) = \rho_0; \beta'_{12}(\{Y\}, \{y\}) = \left[V\left(a_{j+\frac{1}{2}}, t^{n+1}\right) - V\left(a_{j+\frac{1}{2}}, t^n\right) \right] / (t^{n+1} - t^n),$$

$$\alpha_{21}(\{Y\}, \{y\}) = -1; \beta'_{21}(\{Y\}, \{y\}) = \left[U\left(a_{j+1}, t^{n+\frac{1}{2}}\right) - U\left(a_j, t^{n+\frac{1}{2}}\right) \right] / (a_{j+1} - a_j),$$

$$\gamma(\{Y\}, \{y\}) = 0; \alpha_{11}(\{Y\}, \{y\}) = 0; \alpha_{22}(\{Y\}, \{y\}) = 0,$$

then

$$F(\{Y\}, \{y\}) = \sum_{j=1}^J \sum_{k=1}^K \alpha_{jk} \beta'_{jk} + \gamma.$$

Hence, Eq. (2b) is a form-preserving difference analog of Eq. (2a), provided that the $(\gamma, \alpha_{jk}, \beta'_{jk})$ as defined above are consistent with the definition of form preservation. Now:

$$\beta'_{12}(\{Y\}, \{y\}) = \left[\beta_{12}(\{P_k Y\}, \{P_k y\}) - \beta_{12}(\{Y\}, \{y\}) \right] / \left[y_2(i_1) - y_2(i_2) \right],$$

where

$$\beta_{12}(\{Y\}, \{y\}) = Y_1(2) = V\left(a_{j+\frac{1}{2}}, t^n\right),$$

$$\{P_k y\} = a_{j+\frac{1}{2}}, a_{j+\frac{1}{2}}, a_{j+1}, a_j; t^n, t^{n+1}, t^{n+\frac{1}{2}}, t^{n+\frac{1}{2}},$$

$$y_2(i_1) = t^{n+1}, y_2(i_2) = t^n.$$

Clearly, β_{12} is a function only of those variables (namely, V) which appear explicitly in B_{12} . Also, for any point $\underline{y}(0)$:

$$\lim_{\{\underline{y}(i) - \underline{y}(0)\}_{i=1}^2 \rightarrow 0} \beta'_{12}(\{Y\}, \{y\}) = \left[\partial V / \partial t \right]_{\underline{y}=\underline{y}(0)} = \left[B'_{12}(\underline{Y}, \underline{y}) \right]_{\underline{y}=\underline{y}(0)}.$$

Thus, $\beta'_{12}(\{Y\}, \{y\})$ satisfies the requirements of the definition of form preservation. Similarly, $\beta'_{21}(\{Y\}, \{y\})$ can be shown to satisfy these requirements. Finally:

$$\lim_{\{\underline{y}(i) - \underline{y}(0)\}_{i=1}^2 \rightarrow 0} \alpha_{12}(\{Y\}, \{y\}) = \rho_0 = \left[A_{12}(\{Y\}, \{y\}) \right]_{\underline{y}=\underline{y}(0)},$$

and (α_{12}) is a function only of those variables appearing explicitly in (A_{12}) . Similar statements hold for $(\alpha_{11}, \alpha_{21}, \alpha_{22}, \gamma)$. Therefore Eq. (2b) is a

form-preserving difference analog of Eq. (2a).

If we let $\underline{y}(0) = \left(a_{j+\frac{1}{2}}, t^{n+\frac{1}{2}} \right)$, then Eqs. (2a) and (2b) give

$$\begin{aligned}
 D(\{Y\}, \{y\}) &= \rho_0 \left[V\left(a_{j+\frac{1}{2}}, t^{n+1}\right) - V\left(a_{j+\frac{1}{2}}, t^n\right) \right] / \Delta t \\
 &\quad - \left[U\left(a_{j+1}, t^{n+\frac{1}{2}}\right) - U\left(a_j, t^{n+\frac{1}{2}}\right) \right] / \Delta a - \left(\rho_0 V_t - U_a \right)_{\substack{t=t \\ a=a_{j+\frac{1}{2}}}}^{n+\frac{1}{2}} \\
 &= \rho_0 \left\{ \frac{1}{2} \left[\left(t^{n+1} - t^{n+\frac{1}{2}} \right) + \left(t^n - t^{n-\frac{1}{2}} \right) \right] (\partial^2 V / \partial t^2)_{y=y_0} \right. \\
 &\quad + \frac{1}{6} \left[\left(t^{n+1} - t^{n+\frac{1}{2}} \right)^2 + \left(t^{n+1} - t^{n+\frac{1}{2}} \right) \left(t^n - t^{n+\frac{1}{2}} \right) \right. \\
 &\quad \left. \left. + \left(t^n - t^{n+\frac{1}{2}} \right)^2 \right] (\partial^3 V / \partial t^3)_{\underline{y}=\underline{y}(0)} \right. \\
 &\quad \left. + \text{higher-order terms} \right\} \\
 &\quad - \left\{ \frac{1}{2} \left[\left(a_{j+1} - a_{j+\frac{1}{2}} \right) + \left(a_j - a_{j-\frac{1}{2}} \right) \right] (\partial^2 U / \partial a^2)_{\underline{y}=\underline{y}(0)} \right. \\
 &\quad + \frac{1}{6} \left[\left(a_{j+1} - a_{j+\frac{1}{2}} \right)^2 + \left(a_{j+1} - a_{j+\frac{1}{2}} \right) \left(a_j - a_{j+\frac{1}{2}} \right) \right. \\
 &\quad \left. \left. + \left(a_j - a_{j+\frac{1}{2}} \right)^2 \right] (\partial^3 U / \partial a^3)_{\underline{y}=\underline{y}_0} \right. \\
 &\quad \left. + \text{higher-order terms} \right\}.
 \end{aligned}$$

Hence Eq. (2b) is a second-order analog of Eq. (2a) if $t^{n+\frac{1}{2}} = \frac{1}{2}(t^{n+1} + t^n)$

and $a_{j+\frac{1}{2}} = \frac{1}{2}(a_{j+1} + a_j)$. Otherwise Eq. (2b) is a first-order analog of

Eq. (2a). If the points $\{a_j, t^n\}$ define a rectangular mesh with spacing

$(\Delta a, \Delta t)$, then $t^{n+\frac{1}{2}} = \frac{1}{2}(t^{n+1} + t^n)$ and $a_{j+\frac{1}{2}} = \frac{1}{2}(a_{j+1} + a_j)$. Von Neumann

and Richtmyer place this restriction on their difference equations. Hence Eq. (2b) is a second-order, form-preserving difference analog of Eq. (2a). As such, Eq. (2b) is a good difference analog of Eq. (2a).

The definitions given here appear complicated, owing mainly to problems of notation. However, the ideas behind them are simple. As a result, little practice is required to identify a difference equation at sight as a poor, fair, or good analog of a given differential equation. Therefore, throughout this paper difference equations are classified without explicit formal justification.

APPENDIX B

Deduction of Good Difference Analogs of Eqs. (9a) - (20a) From

Eqs. (1e), (2e), (21e), (13e), With $(\Delta t, \Delta a, \rho_0)$ Constant

For the region of interest, the surface integral $\left(\int_{\mathcal{Q}(t^{n+\frac{1}{2}})} \underline{U} \cdot \underline{n} \, d\sigma \right)$

is exactly $\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right)$.

From Eq. (1e):

$$\left[\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} (X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_j^n) / \Delta a - \rho_{j+\frac{1}{2}}^n (X_{j+\frac{1}{2}}^n - X_j^n) / \Delta a \right] / \Delta t = 0,$$

or

$$\left(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} J_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n \right) / \Delta t = 0, \quad (9e)$$

where

$$J_{j+\frac{1}{2}}^n = (X_{j+1}^n - X_j^n) / \Delta a.$$

For one-dimensional linear motion, $J_{j+\frac{1}{2}}^n$ is a good difference analog of

the Jacobian $(J = X_a)$.

By Eq. (F):

$$\rho_0 = \rho_{j+\frac{1}{2}}^n (X_{j+1}^n - X_j^n) / \Delta a = \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n. \quad (K)$$

Therefore, Eq. (1e) can be written:

$$\left[(\rho J)_j^{n+1} U_j^{n+1} - (\rho J)_j^n U_j^n \right] / \Delta t + J_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (10e)$$

where

$$X_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{4} (X_{j+1}^{n+1} + X_{j+1}^n + X_j^{n+1} + X_j^n),$$

$$J_j^{n+\frac{1}{2}} = \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta a = \frac{1}{4} \left(J_{j+\frac{1}{2}}^{n+1} + J_{j+\frac{1}{2}}^n + J_{j-\frac{1}{2}}^{n+1} + J_{j-\frac{1}{2}}^n \right),$$

$$(\rho J)_j^n = \frac{1}{2} \left(\rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n + \rho_{j-\frac{1}{2}}^n J_{j-\frac{1}{2}}^n \right).$$

In Eq. (10e), $\left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right)$ is a good difference

analog of $\frac{\partial}{\partial X} (P+Q)$, the x component of the gradient of $(P+Q)$.

Using Eqs. (F), (1e), we can write Eq. (IIIe) in the form:

$$\rho_0 \left\{ \left[\frac{1}{2} (U_j^{n+1})^2 + \frac{1}{2} (E_{j+\frac{1}{2}}^{n+1} + E_{j-\frac{1}{2}}^{n+1}) \right] - \left[\frac{1}{2} (U_j^n)^2 + \frac{1}{2} (E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n) \right] \right\} / \Delta t$$

$$+ \left\{ P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}}) \right] - P_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}) \right] \right\} / \Delta a = 0.$$

Then, following the procedure used to obtain Eq. (10e), we obtain:

$$\left\{ (\rho J)_j^{n+1} \left[\frac{1}{2} (U_j^{n+1})^2 + \frac{1}{2} (E_{j+\frac{1}{2}}^{n+1} + E_{j-\frac{1}{2}}^{n+1}) \right] - (\rho J)_j^n \left[\frac{1}{2} (U_j^n)^2 + \frac{1}{2} (E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n) \right] \right\} / \Delta t$$

$$+ J_j^{n+\frac{1}{2}} \left\{ P_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}}) \right] - P_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}) \right] \right\}$$

$$/ \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (11e)$$

Dividing Eq. (J) by (Δa) gives us:

$$\left(J_{j+\frac{1}{2}}^{n+1} - J_{j+\frac{1}{2}}^n \right) / \Delta t = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \Delta a.$$

Hence:

$$\left(J_{j+\frac{1}{2}}^{n+1} - J_{j+\frac{1}{2}}^n \right) / \Delta t = J_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right),$$

or,

$$\left(J_{j+\frac{1}{2}}^{n+1} - J_{j+\frac{1}{2}}^n \right) / J_{j+\frac{1}{2}}^{n+\frac{1}{2}} \Delta t = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right), \quad (12e)$$

where

$$X_j^{n+\frac{1}{2}} = \frac{1}{2} (X_j^{n+1} + X_j^n),$$

$$J_j^{n+\frac{1}{2}} = \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) / \Delta a = \frac{1}{2} \left(J_{j+\frac{1}{2}}^{n+1} + J_{j+\frac{1}{2}}^n \right).$$

Since $v_{j+\frac{1}{2}}^n = X_{j+1}^n - X_j^n = J_{j+\frac{1}{2}}^n \Delta a$, Eq. (12ae) can also be written:

$$\begin{aligned} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / \left[\frac{1}{2} \left(v_{j+\frac{1}{2}}^{n+1} + v_{j+\frac{1}{2}}^n \right) \Delta t \right] &= \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) \\ &= \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \frac{1}{2} \left(v_{j+\frac{1}{2}}^{n+1} + v_{j+\frac{1}{2}}^n \right). \end{aligned}$$

From the definitions of $\left(m_j^n \right)$, $\left(m_{j+\frac{1}{2}}^n \right)$, and $\left(J_{j+\frac{1}{2}}^n \right)$, we find:

$$m_j^n = \frac{1}{2} \left(\rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n + \rho_{j-\frac{1}{2}}^n J_{j-\frac{1}{2}}^n \right) / \Delta a = (\rho J)_j^n / \Delta a.$$

Combining this equation with Eq. (G) gives us:

$$\left[\frac{1}{2} (\rho J)_j^{n+1} \left(U_j^{n+1} \right)^2 - \frac{1}{2} (\rho J)_j^n \left(U_j^n \right)^2 \right] / \Delta t + U_j^{n+\frac{1}{2}} \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta a = 0,$$

or

$$\begin{aligned} &\left[\frac{1}{2} (\rho J)_j^{n+1} \left(U_j^{n+1} \right)^2 - \frac{1}{2} (\rho J)_j^n \left(U_j^n \right)^2 \right] / \Delta t \\ &+ J_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \end{aligned} \quad (14e)$$

Similarly, since $m_{j+\frac{1}{2}}^n = \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n \Delta a$, Eq. (H) implies

$$\begin{aligned} &\left(\rho_{j+\frac{1}{2}}^{n+1} J_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta t \\ &+ \frac{1}{2} \left(J_{j+\frac{1}{2}}^{n+1} + J_{j+\frac{1}{2}}^n \right) P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) = 0. \end{aligned} \quad (15e)$$

We have $J_{j+\frac{1}{2}}^n = \rho_0 / \rho_{j+\frac{1}{2}}^n$. As a result,

$$J_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \left(\rho_0 / 2 \right) \left(1 / \rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} + 1 / \rho_{j+\frac{1}{2}}^n \right),$$

and Eq. (12e) becomes:

$$\left(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \rho_{j+\frac{1}{2}}^n \right) / \Delta t + \frac{1}{2} \left(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \rho_{j+\frac{1}{2}}^n \right) \left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) = 0. \quad (16e)$$

Combining Eqs. (K), (10e), we obtain:

$$\left(U_j^{n+\frac{1}{2}} - U_j^n \right) / \Delta t + \left(J_j^{n+\frac{1}{2}} / \rho_0 \right) \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0.$$

Now:

$$\begin{aligned} J_j^{n+\frac{1}{2}} / \rho_0 &= \left(J_{j+\frac{1}{2}}^{n+\frac{1}{2}} / \rho_0 + J_{j+\frac{1}{2}}^n / \rho_0 + J_{j-\frac{1}{2}}^{n+\frac{1}{2}} / \rho_0 + J_{j-\frac{1}{2}}^n / \rho_0 \right) / 4 \\ &= \frac{1}{4} \left(1 / \rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} + 1 / \rho_{j+\frac{1}{2}}^n + 1 / \rho_{j-\frac{1}{2}}^{n+\frac{1}{2}} + 1 / \rho_{j-\frac{1}{2}}^n \right) \\ &= \left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} + V_{j+\frac{1}{2}}^n + V_{j-\frac{1}{2}}^{n+\frac{1}{2}} + V_{j-\frac{1}{2}}^n \right) / 4. \end{aligned}$$

We therefore define:

$$V_j^{n+\frac{1}{2}} = J_j^{n+\frac{1}{2}} / \rho_0 = \left(V_{j+\frac{1}{2}}^{n+\frac{1}{2}} + V_{j+\frac{1}{2}}^n + V_{j-\frac{1}{2}}^{n+\frac{1}{2}} + V_{j-\frac{1}{2}}^n \right) / 4$$

and

$$\rho_j^{n+\frac{1}{2}} = 1 / V_j^{n+\frac{1}{2}}.$$

Then:

$$\rho_j^{n+\frac{1}{2}} \left(U_j^{n+1} - U_j^n \right) / \Delta t + \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (17e)$$

By precisely the same argument, Eqs. (11e), (14e) give:

$$\begin{aligned} \rho_j^{n+\frac{1}{2}} \left\{ \left[\frac{1}{2} \left(U_j^{n+1} \right)^2 + \frac{1}{2} \left(E_{j+\frac{1}{2}}^{n+1} + E_{j-\frac{1}{2}}^{n+1} \right) \right] - \left[\frac{1}{2} \left(U_j^n \right)^2 + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right) \right] \right\} / \Delta t \\ + \left\{ PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right] - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}} \right) \right] \right\} / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0 \end{aligned} \quad (18e)$$

and

$$\rho_j^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_j^{n+1} \right)^2 - \frac{1}{2} \left(U_j^n \right)^2 \right] / \Delta t + U_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (19e)$$

Similarly, if we define

$$V_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left(V_{j+\frac{1}{2}}^{n+1} + V_{j+\frac{1}{2}}^n \right),$$

$$\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} = 1/V_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

then Eq. (15e) becomes:

$$\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta t + PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) = 0. \quad (20e)$$

APPENDIX C

Deduction of Analogs of Eqs. (9a) - (20a) From the

(g) -Equations for Arbitrary $\left(\Delta^{n+\frac{1}{2}}_t, \Delta_{j+\frac{1}{2}}^a, \rho_{j+\frac{1}{2}}^0\right)$

Retracing the steps taken in Appendix B and Sec. (V), we obtain:

$$\left(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} J_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n\right) / \Delta^{n+\frac{1}{2}}_t = 0, \quad (9g)$$

where

$$J_{j+\frac{1}{2}}^n = \left(X_{j+1}^n - X_j^n\right) / \Delta_{j+\frac{1}{2}}^a;$$

$$\begin{aligned} & \left[(\rho J)_j^{n+1} U_j^{n+1} - (\rho J)_j^n U_j^n \right] / \Delta^{n+\frac{1}{2}}_t \\ & + J_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \end{aligned} \quad (10g)$$

where

$$X_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{4} \left(X_{j+1}^{n+1} + X_{j+1}^n + X_j^{n+1} + X_j^n \right),$$

$$J_j^{n+\frac{1}{2}} = \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta_j^a,$$

$$(\rho J)_j^n = m_j^n / \Delta_j^a,$$

$$\Delta_j^a = \frac{1}{2} \left(\Delta_{j+\frac{1}{2}}^a + \Delta_{j-\frac{1}{2}}^a \right);$$

$$\begin{aligned}
 & \left\{ \frac{1}{2} \left\{ \rho_{j+\frac{1}{2}}^{n+1} J_{j+\frac{1}{2}}^{n+1} \frac{\Delta_{j+\frac{1}{2}}^a}{\Delta_j^a} \left[\frac{1}{2} (U_j^{n+1})^2 + E_{j+\frac{1}{2}}^{n+1} \right] + \rho_{j-\frac{1}{2}}^{n+1} J_{j-\frac{1}{2}}^{n+1} \frac{\Delta_{j-\frac{1}{2}}^a}{\Delta_j^a} \right. \right. \\
 & \times \left. \left[\frac{1}{2} (U_j^{n+1})^2 + E_{j-\frac{1}{2}}^{n+1} \right] \right\} - \frac{1}{2} \left\{ \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n \frac{\Delta_{j+\frac{1}{2}}^a}{\Delta_j^a} \left[\frac{1}{2} (U_j^n)^2 + E_{j+\frac{1}{2}}^n \right] \right. \\
 & \left. \left. + \rho_{j-\frac{1}{2}}^n J_{j-\frac{1}{2}}^n \frac{\Delta_{j-\frac{1}{2}}^a}{\Delta_j^a} \left[\frac{1}{2} (U_j^n)^2 + E_{j-\frac{1}{2}}^n \right] \right\} \right\} \div \Delta^{n+\frac{1}{2}}_t \\
 & + J_j^{n+\frac{1}{2}} \left\{ PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}}) \right] - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} (U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}) \right] \right\}
 \end{aligned}$$

$$\left/ \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \right. = 0; \quad (11g)$$

$$\left(J_{j+\frac{1}{2}}^{n+1} - J_{j+\frac{1}{2}}^n \right) \left/ J_{j+\frac{1}{2}}^{n+\frac{1}{2}} \Delta^{n+\frac{1}{2}}_t \right. = \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) \left/ \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) \right., \quad (12g)$$

where

$$J_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left(J_{j+\frac{1}{2}}^{n+1} + J_{j+\frac{1}{2}}^n \right),$$

$$X_j^{n+\frac{1}{2}} = \frac{1}{2} \left(X_j^{n+1} + X_j^n \right),$$

$$\begin{aligned}
 & \left[\frac{1}{2} (\rho J)_j^{n+1} (U_j^{n+1})^2 - \frac{1}{2} (\rho J)_j^n (U_j^n)^2 \right] \left/ \Delta^{n+\frac{1}{2}}_t \right. \\
 & + J_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \left/ \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \right. = 0, \quad (14g)
 \end{aligned}$$

$$\begin{aligned} & \left(\rho_{j+\frac{1}{2}}^{n+1} J_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t \\ & + J_{j+\frac{1}{2}}^{n+\frac{1}{2}} P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) = 0; \end{aligned} \quad (15g)$$

$$\left(\rho_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + \frac{1}{2} \left(\rho_{j+\frac{1}{2}}^{n+1} + \rho_{j+\frac{1}{2}}^n \right) \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+1} \right) / \Delta_{j+\frac{1}{2}}^a = 0. \quad (16g)$$

Next, from Eqs. (Ig), (IIg):

$$\left(U_j^{n+1} - U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / m_j^{n+\frac{1}{2}} = 0,$$

where

$$m_j^{n+\frac{1}{2}} = \frac{1}{2} \left(m_j^{n+1} + m_j^n \right).$$

Hence

$$\rho_j^{n+\frac{1}{2}} \left(U_j^{n+1} - U_j^n \right) / \Delta^{n+\frac{1}{2}}_t + \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (17g)$$

where

$$\rho_j^{n+\frac{1}{2}} = m_j^{n+\frac{1}{2}} / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right).$$

Also, Eqs. (Ig), (IIIg) yield:

$$\frac{1}{2} \rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(\frac{X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}}}{X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}}} \right) \left\{ \left[\frac{1}{2} (U_j^n)^2 + E_{j+\frac{1}{2}}^{n+1} \right] - \left[\frac{1}{2} (U_j^n)^2 + E_{j+\frac{1}{2}}^n \right] \right\} / \Delta^{n+\frac{1}{2}}_t$$

$$\begin{aligned}
 & + \frac{1}{2} \rho_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left(\frac{X_j^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}}} \right) \left\{ \left[\frac{1}{2} (U_j^n)^2 + E_{j-\frac{1}{2}}^n \right] - \left[\frac{1}{2} (U_j^n)^2 + E_{j-\frac{1}{2}}^n \right] \right\} / \Delta^{n+\frac{1}{2}}_t \\
 & + \left\{ P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_{j+1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}} \right) \right] - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{2} \left(U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}} \right) \right] \right\} \\
 & / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0, \quad (18g)
 \end{aligned}$$

where

$$\rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left(m_{j+\frac{1}{2}}^{n+1} + m_{j+\frac{1}{2}}^n \right) / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right).$$

Multiplying Eq. (17g) by $\left(U_j^{n+\frac{1}{2}} \right)$, we get:

$$\begin{aligned}
 & \rho_j^{n+\frac{1}{2}} \left[\frac{1}{2} (U_j^{n+1})^2 - \frac{1}{2} (U_j^n)^2 \right] / \Delta^{n+\frac{1}{2}}_t + U_j^{n+\frac{1}{2}} \left(P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} - P Q_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \\
 & / \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0. \quad (19g)
 \end{aligned}$$

Finally, from Eqs. (2g), (21g), (1g):

$$\begin{aligned}
 & \rho_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t + P Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) \\
 & / \left(X_{j+1}^{n+\frac{1}{2}} - X_j^{n+\frac{1}{2}} \right) = 0. \quad (20g)
 \end{aligned}$$

Equations (11g), (18g) are not form-preserving analogs of Eqs. (11a), (18a). Furthermore, there appears to be no way to obtain such analogs from

Eqs. (1g), (2g), (21g), and (13g). Also, many of the difference equations (I_g), (II_g), ..., (20g) are only first-order analogs of the corresponding differential equations because of the variable Δa . Thus, for example:

$$\begin{aligned}
 J_j^{n+\frac{1}{2}} &= \left(X_{j+\frac{1}{2}}^{n+\frac{1}{2}} - X_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \Delta_j a \\
 &= \left[X(a_{j+1}, t^{n+1}) + X(a_{j+1}, t^n) - X(a_{j-1}, t^{n+1}) - X(a_{j-1}, t^n) \right] \\
 &\quad / 2(a_{j+1} - a_{j-1}) \\
 &= X_a(a_j, t^{n+\frac{1}{2}}) + \frac{1}{2}(a_{j+1} - 2a_j + a_{j-1}) X_a(a_j, t^{n+\frac{1}{2}}) + \left\{ \begin{array}{l} \text{higher-} \\ \text{order} \\ \text{terms} \end{array} \right. \\
 &= J(a_j, t^{n+\frac{1}{2}}) + \frac{1}{2}(a_{j+\frac{1}{2}} - 2a_j + a_{j-\frac{1}{2}}) J_a(a_j, t^{n+\frac{1}{2}}) + \left\{ \begin{array}{l} \text{higher-} \\ \text{order} \\ \text{terms} \end{array} \right.
 \end{aligned}$$

Therefore, the (g)-equations are not entirely satisfactory. However, these equations meet our primary differencing requirements, particularly with regard to Eqs. (I) - (IV).

APPENDIX D

Reduction of the Interface Equations to the Normal Interior

Equations at a Fictitious Interface

It is to be proved that if (C_{0+}, C_{0-}) are so chosen that $C_{0+} m_+ = C_{0-} m_-$, then Eqs. (13*) - (26) are satisfied by

$$X_*^{n+1} = \left(m_- X_{j+1}^{n+1} + m_+ X_j^{n+1} \right) / (m_+ + m_-),$$

$$\rho_+^{n+1} = \rho_-^{n+1} = \rho_{j+\frac{1}{2}}^{n+1}; \quad Q_+^{n+\frac{1}{2}} = Q_-^{n+\frac{1}{2}} = Q_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

$$E_+^{n+1} = E_-^{n+1} = E_{j+\frac{1}{2}}^{n+1}; P_+^{n+\frac{1}{2}} = P_-^{n+\frac{1}{2}} = P_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$

where $\rho_{j+\frac{1}{2}}^n$, etc., are normal interior quantities.

First, on eliminating (X_*^{n+1}) from Eqs. (22P), (22M) by means of Eq. (28), we get:

$$\rho_+^{n+1} = \rho_-^{n+1} = (m_+ + m_-) / (X_{j+1}^{n+1} - X_j^{n+1}).$$

From the definitions of (m_+, m_-) :

$$\begin{aligned} m_+ + m_- &= \rho_+^n (X_{j+1}^n - X_*^n) + \rho_-^n (X_*^n - X_j^n) \\ &= \rho_{j+\frac{1}{2}}^n (X_{j+1}^n - X_j^n). \end{aligned}$$

Hence, by comparison with the normal interior equation (Ig), we find:

$$m_+ + m_- = m_{j+\frac{1}{2}},$$

and

$$(m_+ + m_-) / (X_{j+1}^{n+1} - X_j^{n+1}) = \rho_{j+\frac{1}{2}}^{n+1}.$$

Thus:

$$\rho_+^{n+1} = \rho_-^{n+1} = \rho_{j+\frac{1}{2}}^{n+1} \triangleq 1/V_{j+\frac{1}{2}}^{n+1}.$$

Next:

$$\begin{aligned} \left(U_{j+1}^{n+\frac{1}{2}} - U_*^{n+\frac{1}{2}} \right) &= \left(X_{j+1}^{n+1} - X_{j+1}^n - X_*^{n+1} - X_*^n \right) / \Delta t = m_+ \left(\frac{1}{\rho_+^{n+1}} - \frac{1}{\rho_+^n} \right) \\ &= m_+ \left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n \right). \end{aligned}$$

Similarly,

$$U_{*}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} = m_{-} \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right).$$

Therefore,

$$Q_{+}^{n+\frac{1}{2}} = 2 C_{0+}^2 m_{+}^2 \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right)^2 / \left(v_{j+\frac{1}{2}}^{n+1} + v_{j+\frac{1}{2}}^n \right) \left(\Delta^{n+\frac{1}{2}}_t \right)^2,$$

$$Q_{-}^{n+\frac{1}{2}} = 2 C_{0-}^2 m_{-}^2 \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right)^2 / \left(v_{j+\frac{1}{2}}^{n+1} + v_{j+\frac{1}{2}}^n \right) (\Delta t)^2.$$

According to the normal interior equations:

$$Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = 2 C_0^2 m_{j+\frac{1}{2}}^2 \left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right)^2 / \left(v_{j+\frac{1}{2}}^{n+1} + v_{j+\frac{1}{2}}^n \right) (\Delta t)^2.$$

Hence, in view of Eq. (27):

$$Q_{+}^{n+\frac{1}{2}} = Q_{-}^{n+\frac{1}{2}} = Q_{j+\frac{1}{2}}^{n+\frac{1}{2}}.$$

Then Eq. (26), plus the given condition that $P_{+}^n = P_{-}^n$, implies:

$$P_{+}^{n+1} = P_{-}^{n+1}.$$

We must now show that $P_{+}^{n+1} = P_{-}^{n+1} = P_{j+\frac{1}{2}}^{n+1}$ and $E_{+}^{n+1} = E_{-}^{n+1}$

$= E_{j+\frac{1}{2}}^{n+1}$ are consistent with Eqs. (24P) - (25M). Equations (25P) and (25M)

are compatible if and only if $E_{+}^{n+1} = E_{-}^{n+1}$, since $P_{+}^{n+1} = P_{-}^{n+1}$, ρ_{+}^{n+1}

$= \rho_{-}^{n+1}$, and $f_{+}(r, s) = f_{-}(r, s)$. Finally, adding Eqs. (24P) and (24M),

we obtain:

$$m_{j+\frac{1}{2}} \left(E_+^{n+1} - E_+^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_+^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) \\ = m_{j+\frac{1}{2}} \left(E_-^{n+1} - E_-^n \right) / \Delta^{n+\frac{1}{2}}_t + PQ_-^{n+\frac{1}{2}} \left(U_{j+1}^{n+\frac{1}{2}} - U_j^{n+\frac{1}{2}} \right) = 0.$$

By comparing these two equations with the normal interior Eq. (20g) it can be seen that Eqs. (24P) - (25M) are indeed satisfied if $P_+^{n+1} = P_-^{n+1}$

$$= P_{j+\frac{1}{2}}^{n+1} \text{ and } E_+^{n+1} = E_-^{n+1} = E_{j+\frac{1}{2}}^{n+1}.$$

APPENDIX E

The Conservation Equations for Spherical Systems

At a point (R, θ, ϕ) in a spherical fluid shell, the particle velocity has a component $U \cos \theta$ in the direction of the axis of the cone. The components of fluid velocity in mutually orthogonal directions at right angles to the axis of the cone are $U \sin \theta \sin \phi$ and $U \sin \theta \cos \phi$. For that segment of the spherical shell lying within the cone, the component of momentum in the radial direction is:

$$M_R = \int_{R_1}^{R_2} \int_0^{\theta'} \int_0^{2\pi} \rho U \cos \theta R^2 \sin \theta dR d\theta d\phi \\ = \pi \sin^2 \theta' \int_{R_1}^{R_2} \rho R^2 U dR, \quad (EA)$$

where R_1 and R_2 are the inner and outer radii of the spherical shell. The other two components of momentum are zero by symmetry. The kinetic

energy of the fluid element is:

$$\begin{aligned}
 T &= \frac{1}{2} \int_{R_1}^{R_2} \int_0^{\theta'} \int_0^{2\pi} \rho \left[(U \cos \theta)^2 + (U \sin \theta \cos \phi)^2 \right. \\
 &\quad \left. + (U \sin \theta \sin \phi)^2 \right] R^2 \sin \theta \, dR \, d\theta \, d\phi \\
 &= \pi (1 - \cos \theta') \int_{R_1}^{R_2} \rho R^2 U^2 \, dR.
 \end{aligned}$$

The contribution to the kinetic energy from the component of velocity in the direction of the axis of the cone is:

$$\begin{aligned}
 T_R &= \frac{1}{2} \int_{R_1}^{R_2} \int_0^{\theta'} \int_0^{2\pi} \rho (U \cos \theta)^2 R^2 \sin \theta \, dR \, d\theta \, d\phi \\
 &= \frac{\pi}{3} (1 - \cos^3 \theta') \int_{R_1}^{R_2} \rho R^2 U^2 \, dR. \quad (\text{EB})
 \end{aligned}$$

Obviously, $T > T_R$ if $0 < \theta' < \pi$. Also,

$$T/T_R = \frac{1}{3} (1 + \cos \theta' + \cos^2 \theta'),$$

and

$$T/T_R \rightarrow 1 \text{ as } \theta' \rightarrow 0.$$

Hence, if θ' is small enough, a good analog of T_R is an equally good analog of the total kinetic energy T . Next, if $\underline{n}_R$ is a unit vector in the direction of the axis of the cone, then the component of force in this direction is:

$$\mathcal{F}_R = - \underline{n}_R \cdot \int_S (P + Q) \underline{n} \, d\sigma$$

$$\begin{aligned}
 &= -R_2^2 \int_0^{\theta'} \int_0^{2\pi} \left[P(R_2) + Q(R_2) \right] \cos \theta \sin \theta \, d\theta \, d\phi \\
 &\quad + R_1^2 \int_0^{\theta'} \int_0^{2\pi} \left[P(R_1) + Q(R_1) \right] \cos \theta \sin \theta \, d\theta \, d\phi \\
 &\quad + \int_{R_1}^{R_2} \int_0^{2\pi} (P+Q) \sin \theta' R \sin \theta' \, dR \, d\phi \\
 &= -\pi \sin^2 \theta' \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] - 2 \int_{R_1}^{R_2} (P+Q)R \, dR \right\}.
 \end{aligned}$$

Thus, for the component of momentum in the radial direction $\underline{n}_R$, the momentum conservation Eq. (II) implies:

$$\frac{d}{dt} \int_{R_1}^{R_2} \rho R^2 U \, dR + \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] - \int_{R_1}^{R_2} (P+Q) dR^2 \right\} = 0. \quad (EC)$$

The mass of fluid in the segment of the spherical shell lying within the cone is:

$$m = 2\pi(1-\cos \theta') \int_{R_1}^{R_2} \rho R^2 \, dR = 4\pi \sin^2 \theta/2 \int_{R_1}^{R_2} \rho R^2 \, dR. \quad (ED)$$

Let $m_{RAD} = m/2\pi(1-\cos \theta') =$ mass of spherical shell per unit solid angle.

Then Eq. (I) for conservation of mass gives:

$$\frac{dm_{RAD}}{dt} = \frac{d}{dt} \int_{R_1}^{R_2} \rho R^2 \, dR = 0. \quad (EE)$$

Now, since $\rho R^2 \geq 0$, we have

$$\int_{R_1}^{R_2} \rho R^2 U dR = U(\bar{R}) \int_{R_1}^{R_2} \rho R^2 dR ; R_1 < \bar{R} < R_2 ,$$

by a basic mean-value theorem.¹⁰ Hence:

$$\frac{d}{dt} \left[m_{\text{RAD}} U(\bar{R}) \right] + \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] - \int_{R_1}^{R_2} (P+Q) dR^2 \right\} = 0 . \quad (\text{EF})$$

In view of Eq. (EE):

$$\frac{d}{dt} U(\bar{R}) + \frac{1}{m_{\text{RAD}}} \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] - \int_{R_1}^{R_2} (P+Q) dR^2 \right\} = 0 . \quad (\text{EG})$$

Again using mean-value theorems, we get:

$$\lim_{R_2 - R_1 \rightarrow 0} \frac{1}{m_{\text{RAD}}} \left\{ R_2^2 \left[P(R_2) + Q(R_2) \right] - R_1^2 \left[P(R_1) + Q(R_1) \right] - \int_{R_1}^{R_2} (P+Q) dR^2 \right\} = \frac{1}{\rho} (P+Q)_R ,$$

and

$$U_t + \frac{1}{\rho} (P+Q)_R = 0 . \quad (\text{EH})$$

If the radial position of a particle at time (t_0) is used as a Lagrangian coordinate (a) in place of the Eulerian coordinate (R), then:

$$R = R(a, t) ; a = R(a, t_0) .$$

The Jacobian for the transformation from Cartesian coordinates to the coordinates (a, θ, ϕ) is:

$$J = R^2 (\partial R / \partial a) \sin \theta = \frac{1}{3} (R^3)_a \sin \theta \equiv J_{\text{RAD}} \sin \theta. \quad (\text{EJ})$$

Thus

$$m_{\text{RAD}} = \int_{a_1}^{a_2} \rho J_{\text{RAD}} da,$$

where $R_1 = R(a_1, t)$; $R_2 = R(a_2, t)$.

Furthermore, since $dm/dt = 0$, we have $dm_{\text{RAD}}/dt = 0$. Thus, applying the mean-value theorem, we obtain:

$$\left(\rho J_{\text{RAD}} \right)_t = 0. \quad (\text{EK})$$

If we now divide Eq. (EK) by $(a_2 - a_1)$ and let $(a_2 - a_1) \rightarrow 0$, we get:

$$\left(\rho J_{\text{RAD}} U \right)_t + r^2 P_a = 0, \quad (\text{EL})$$

or:

$$\left(\rho J_{\text{RAD}} U \right)_t + J_{\text{RAD}} P_R = 0. \quad (\text{EM})$$

Equations (EK), (EM) imply:

$$\left(\frac{1}{2} \rho J_{\text{RAD}} U^2 \right)_t + J_{\text{RAD}} U (P + Q)_R = 0. \quad (\text{EN})$$

Equations (EK) and (EM) could have been gotten from Eqs. (9a), (10a) by writing $\underline{U}$, $\underline{\nabla}$, and $\underline{J}$ in spherical coordinates. An alternate procedure was followed to exhibit more directly the connection between the integral and differential forms.

For the total energy (H), we have:

$$H = \int_{R_1}^{R_2} \int_0^{\theta'} \int_0^{2\pi} \rho \left(\frac{1}{2} U^2 + E \right) R^2 \sin \theta dR d\theta d\phi = 2\pi(1 - \cos \theta') \int_{R_1}^{R_2} \rho R^2 \times \left(\frac{1}{2} U^2 + E \right) dR,$$

or

$$H = 2\pi(1-\cos\theta')H_{\text{RAD}},$$

where

$$H_{\text{RAD}} = \int_{a_1}^{a_2} \rho J_{\text{RAD}} \left(\frac{1}{2} U^2 + E \right) da.$$

The rate at which work is done by the segment of the spherical shell on its surroundings is

$$\begin{aligned} W &= \int_S (P+Q) \underline{U} \cdot \underline{n} d\sigma \\ &= 2\pi(1-\cos\theta') \left\{ R_2^2 U(R_2) \left[P(R_2) + Q(R_2) \right] - R_1^2 U(R_1) \left[P(R_1) + Q(R_1) \right] \right\}. \end{aligned}$$

Hence, the energy conservation Eq. (III) is equivalent to:

$$dH_{\text{RAD}}/dt + R_2^2 U(R_2) \left[P(R_2) + Q(R_2) \right] - R_1^2 U(R_1) \left[P(R_1) + Q(R_1) \right] = 0. \quad (\text{EP})$$

Finally, Eq. (IV) for conservation of volume is equivalent to:

$$dv_{\text{RAD}}/dt = R_2^2 U(R_2) - R_1^2 U(R_1), \quad (\text{EQ})$$

where

$$v_{\text{RAD}} = \frac{1}{3} \left(R_2^3 - R_1^3 \right) = \int_{a_1}^{a_2} J_{\text{RAD}} da.$$

APPENDIX F

Deduction of Analogs of Eqs. (9a) - (15a) in the

Case of Spherical Symmetry

The mass conservation equation, Eq. (U), together with the definitions

of $\left(\rho_{j+\frac{1}{2}}^n, J_{j+\frac{1}{2}}^n, v_{j+\frac{1}{2}}^n\right)$, implies:

$$\left(\rho_{j+\frac{1}{2}}^{n+1} J_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n\right) / \Delta^{n+\frac{1}{2}}_t = 0, \quad (9h)$$

which is a good difference analog of Eq. (9a). From Eq. (AG) and the definitions of $\left(\rho_j^n, J_j^n, v_j^n\right)$, we conclude that

$$\begin{aligned} & \left(\rho_j^{n+1} J_j^{n+1} U_j^{n+1} - \rho_j^n J_j^n U_j^n\right) / \Delta^{n+\frac{1}{2}}_t \\ & + \left[\left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}}\right) \right] (R^2)_j^{n+\frac{1}{2}} \left[\left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \right. \\ & \quad \left. / \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \right] = 0, \quad (10h) \end{aligned}$$

where

$$R_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{4} \left(R_{j+1}^{n+1} + R_{j+1}^n + R_j^{n+1} + R_j^n \right).$$

Equation (10h) is our difference analog of Eq. (M), the spherical counterpart of the momentum equation (10a). If we compare Eq. (10h) with Eq. (10a), it is evident that

$$J_j^{n+\frac{1}{2}} = (R^2)_j^{n+\frac{1}{2}} \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \right) \quad (FA)$$

must be taken as a difference analog of the radial Jacobian

$$\left[J_{\text{RAD}} \left(a_j, t^{n+\frac{1}{2}} \right) = \frac{1}{3} \left(\partial R^3 / \partial a \right)_{a_j, t^{n+1/2}} \right]. \text{ If } (R^3)_a \text{ is written as } 3R^2 R_a,$$

then $J_j^{n+\frac{1}{2}}$ not only preserves the form of $J_{\text{RAD}} \left(a_j, t^{n+\frac{1}{2}} \right)$, but is even a

good difference analog of $J_{\text{RAD}} \left(a_j, t^{n+\frac{1}{2}} \right)$. Equation (10h) is then a good

difference analog of Eq. (10a). On the other hand, if J_{RAD} is written as

$$\frac{1}{3} (R^3)_a, \text{ then } J_j^{n+\frac{1}{2}} \text{ does not even preserve the form of } J_{\text{RAD}} \left(a_j, t^{n+\frac{1}{2}} \right),$$

and Eq. (10h) is not an FPA of Eq. (10a). The difficulty is that we have been led to a difference analog of $R^2 R_a$ which does not satisfy an FPA of the equation $R^2 R_a = \frac{1}{3} (R^3)_a$. We have no solution for this problem, which arises in many of the analogs of Eqs. (9a) - (21a). As a result, the spherical difference equations are not as satisfactory as the linear equations. It might

be argued that the definition of $R_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ in Eq. (10h) could be abandoned in

favor of some such definition as

$$3 \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) (R^2)_j^{n+\frac{1}{2}} = \frac{1}{4} \left[\left(R_{j+1}^{n+1} \right)^3 + \left(R_{j+1}^n \right)^3 - \left(R_{j-1}^{n+1} \right)^3 - \left(R_{j-1}^n \right)^3 \right],$$

which is a good difference analog of $R^2 R_a = \frac{1}{3} (R^3)_a$. However, with this

definition there is no guarantee that the sum of the radial distances

$\left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right)$ will equal the radial distance between the boundaries of a

given material. This, in fact, appears to be the nub of the problem:

volumes must be strictly additive and distances must also be strictly additive. In the linear case there is no difference between these two requirements, but for spherical and cylindrical systems they are quite independent.

It is interesting that in the cylindrical case both conditions can be met rather simply. For cylindrical systems we have $\frac{1}{2} (R^2)_a$ in place of $\frac{1}{3} (R^3)_a$, and $RR_a = \frac{1}{2} (R^2)_a$ replaces the equation $R^2 R_a = \frac{1}{3} (R^3)_a$. We again obtain Eq.

(10h), except that $(R^2)_j^{n+\frac{1}{2}}$ is replaced by $R_j^{n+\frac{1}{2}} = \frac{1}{2} (R_j^{n+1} + R_j^n)$. Then

$$\begin{aligned} \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \left(R_j^{n+\frac{1}{2}} \right) &= \frac{1}{4} \left(R_{j+1}^{n+1} + R_{j+1}^n - R_{j-1}^{n+1} - R_{j-1}^n \right) \frac{1}{2} (R_j^{n+1} + R_j^n) \\ &= \frac{1}{2} \left[\frac{1}{4} (R_{j+1}^{n+1} + R_{j+1}^n) (R_j^{n+1} + R_j^n) - \frac{1}{4} (R_j^{n+1} + R_j^n) (R_{j-1}^{n+1} + R_{j-1}^n) \right], \end{aligned}$$

which is a good analog of $RR_a = \frac{1}{2} (R^2)_a$. The cylindrical equations are therefore superior to the spherical equations by our standards. Their properties will not be gone into in detail, however, since the development of the various difference analogs proceeds just as in the linear and spherical cases.

Turning now to energy conservation, it can be shown that the spherical equivalent of Eq. (11a) is:

$$\left[\rho J_{\text{RAD}} \left(\frac{1}{2} U^2 + E \right) \right]_t + (1/R^2) J \left[R^2 (P + Q) U \right]_R = 0. \quad (\text{FB})$$

Just as in the linear case (see Eq. (11g)), the equation for total energy conservation (AF) yields no FPA of Eq. (11a) (or Eq. (FB)) if the mass varies from zone to zone. If the mass is constant for all zones, we get a good

difference analog of Eq. (AF) if we write $J_{\text{RAD}} \left(a_j, t^{n+\frac{1}{2}} \right)$ as $R^2 R_a$.

This can be seen by putting Eq. (AF) in the form

$$\left(\rho_j^{n+1} J_j^{n+1} h_j^{n+1} - \rho_j^n J_j^n h_j^n \right) / \Delta^{n+\frac{1}{2}}_t$$

$$+ J_j^{n+\frac{1}{2}} \left[1/(R^2)_j^{n+\frac{1}{2}} \right] \left(W_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0,$$

where

$$h_j^n = \frac{1}{2} \left(U_j^n \right)^2 + \frac{1}{2} \left(E_{j+\frac{1}{2}}^n + E_{j-\frac{1}{2}}^n \right).$$

Next, the spherical counterpart of Eq. (12a) is:

$$(1/R^2) (R^2 U)_R = (J_{RAD})_t / J_{RAD} = \lim_{v_{RAD} \rightarrow 0} \left[(v_{RAD})_t / v_{RAD} \right]$$

$$= \lim_{R_2 \rightarrow R_1} \left(1/v_{RAD} \right) \left(R_2^2 U_2 - R_1^2 U_1 \right). \quad (FC)$$

Equation (AC) implies:

$$\left(v_{j+\frac{1}{2}}^{n+1} - v_{j+\frac{1}{2}}^n \right) / v_{j+\frac{1}{2}}^n \Delta^{n+\frac{1}{2}}_t$$

$$= \left\{ 1 / \left[\left(R_{j+1}^n \right)^2 + R_{j+1}^n R_j^n + \left(R_j^n \right)^2 \right] \right\} \left[\left(R^2 \right)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - \left(R^2 \right)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right]$$

$$/ \left(R_{j+1}^n - R_j^n \right), \quad (12h)$$

which is a fair analog of Eq. (FC). Similarly, difference analogs can be found from Eq. (AC) for the other parts of Eq. (FC).

From Eqs. (W), (AD) we find:

$$\left[\frac{1}{2} \rho_j^{n+1} J_j^{n+1} \left(U_j^{n+1} \right)^2 - \frac{1}{2} \rho_j^n J_j^n \left(U_j^n \right)^2 \right] / \Delta^{n+\frac{1}{2}}_t$$

$$+ (R^2)_j^{n+\frac{1}{2}} \left[\left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \right) \right] U_j^{n+\frac{1}{2}} \left[\left(PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} - PQ_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) / \left(R_{j+\frac{1}{2}}^{n+\frac{1}{2}} - R_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \right] = 0. \quad (14h)$$

This is to be compared with Eq. (R), the spherical form of the kinetic energy equation (14a). Just as in the case of the momentum equations (10h), Eq. (14h)

is a good difference analog of Eq. (Y) if $J_{RAD} \left(a_{j,t}^{n+\frac{1}{2}} \right)$ is written as

$R^2 R_a$. Otherwise Eq. (14h) is not an FPA of Eq. (14a). Thus, while one of our major guides in developing Eqs. (U), (AG), (AF), (AC) was the attempt to generate a good difference analog of Eq. (14a), this goal has not been entirely realized. The search for this analog has nevertheless led to a set of equations which includes both a kinetic energy equation and an internal energy equation, and yields a good analog of the energy conservation equation (III).

The internal energy equation (X) can be combined with Eq. (AC) to give:

$$\left(\rho_{j+\frac{1}{2}}^{n+1} v_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n v_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t$$

$$+ PQ_{j+\frac{1}{2}}^{n+\frac{1}{2}} \left[(R^2)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - (R^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] = 0,$$

or

$$\begin{aligned}
& \left(\rho_{j+\frac{1}{2}}^{n+1} J_{j+\frac{1}{2}}^{n+1} E_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n J_{j+\frac{1}{2}}^n E_{j+\frac{1}{2}}^n \right) / \Delta^{n+\frac{1}{2}}_t \\
& + \left\{ \frac{1}{3} \left[\left(R_{j+1}^n \right)^3 - \left(R_j^n \right)^3 \right] / \left(a_{j+1} - a_j \right) \right\} P Q_j^{n+\frac{1}{2}} \left\{ 1 / \left[\left(R_{j+1}^n \right)^2 + R_{j+1}^n R_j^n + \left(R_j^n \right)^2 \right] \right\} \\
& \times \left[\left(R^2 \right)_{j+1}^{n+\frac{1}{2}} U_{j+1}^{n+\frac{1}{2}} - \left(R^2 \right)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}} \right] / \left(R_{j+1}^n - R_j^n \right) = 0. \quad (15h)
\end{aligned}$$

Equation (15h) is a fair difference analog of the equation

$$(\rho JE)_t + J(P+Q)(1/R^2)(R^2 U)_R = 0,$$

which is the spherical equivalent of Eq. (15a). The analogs of Eqs. (16a) - (21a) could now be found. However, this will not be done here since the procedure is obvious and the important properties of Eqs. (U), (AG), (AF), (AC) have now been established.

FOOTNOTES

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

¹ For one-dimensional linear motion, the general equations of continuum mechanics are formally identical with the hydrodynamic equations, for materials whose stress is uniquely determined by the internal energy and the state of strain. In this case, our discussion applies to the motion of any continuous medium.

² J. von Neumann and R. Richtmyer, J. Appl. Phys. 21, 232 (1950).

³ Various difference analogs of the equations of motion in characteristic form are superior to the von Neumann-Richtmyer equations for many problems. However, a numerical solution of the characteristic equations

becomes very difficult for systems having many permanent and transient discontinuities (interfaces and shocks).

⁴ We feel that the von Neumann-Richtmyer equations are successful largely because they possess these properties.

⁵ For example, given Eq. (2b), we could not have reached the conservation form exhibited by Eq. (8b) if we had defined $1/V^n_{j+\frac{1}{2}} = \left(\rho^n_{j+\frac{3}{2}} + 2\rho^n_{j+\frac{1}{2}} + \rho^n_{j-\frac{1}{2}} \right) / 4$.

⁶ Since the discussion is limited to one-dimensional linear motion, we can think of the system as a cylinder of unit cross section, whose axis lies along the direction of flow.

⁷ J. von Neumann and R. Richtmyer, J. Appl. Phys. 21, 236 (1950). We have not written a specific Q -equation because it has no effect on the transformation properties that concern us. For the same reason, no equation of state is included in our equations. For the present, we neglect boundary problems and the special equations needed to meet them.

⁸ For one-dimensional motion, the Lagrangian coordinate (a) can be so defined that (P_a) becomes continuous across a density discontinuity (see R. D. Richtmyer, Difference Methods for Initial Value Problems (Interscience Publishers, N. Y., 1957), pp. 202, 203).

⁹ We take $m \geq M$ to allow for the possibility that the coordinates of some mesh points appear explicitly in the difference equation but do not appear implicitly as arguments of the functions $Y_1(\underline{y}), \dots, Y_N(\underline{y})$.

¹⁰ See, for example, D. V. Widder, Advanced Calculus, (Prentice-Hall, Inc., N. Y., 1947), p. 34.