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University of California Ernest O. Lawrence Radiation Laboratory

- TWO TOPICS IN PION PHYSICS
- I. PHOTOPRODUCTION OF NEUTRAL PIONS FROM
COMPLEX NUCLEI AND THE PAIR-CORRELATION
FUNCTION
 - II. THE RELATIVE RATE OF ABSORPTION OF
NEGATIVE PIONS IN HYDROGEN AND DEUTERIUM

Berkeley, California

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Robert Henry Traxler

(Ph. D. Thesis)

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TWO TOPICS IN PION PHYSICS

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ABSTRACT

I. A simple theory of the photoproduction of π^0 mesons from complex nuclei based on the impulse and closure approximation is studied. The theory is compared with an experiment of Goodwin, Anderson, and Kenney in which, with a single counter, they observed the dependence of the number of decay gammas from π^0 mesons photo-produced from various nuclei as a function of the atomic number at 45, 90, and 135 degrees for incident photon energies from threshold to 350 MeV. The dependence they measure near threshold and at 45 degrees does not agree with the simple theory nor is it consistent with partial elastic production as they claim. By a method of Placzek and Wick the corrections to the simple theory are estimated, and found to be large near threshold. It is suggested that these corrections might explain the discrepancy between the theory and the experimental results. No information about the nuclear-pair correlation function has been obtained.

II. The impulse approximation is used to calculate the relative ratio of the radiative absorption rate of π^- mesons absorbed in hydrogen and deuterium from an atomic s state. The calculated value of the ratio $T = R(\pi^- + d \rightarrow 2n + \gamma) / R(\pi^- + p \rightarrow n + \gamma)$ is 0.83 ± 0.08 . The experimental value is 0.41 ± 0.06 . The closure approximation is used to evaluate T. By a method due to Placzek the error due to the closure approximation is shown to be small.

I. PHOTOPRODUCTION OF NEUTRAL PIONS FROM COMPLEX NUCLEI AND THE PAIR-CORRELATION FUNCTION

1. INTRODUCTION

The initial motivation of this research was to use the formula

$$\sigma = \sigma_f \left\{ A^2 |c(\theta)|^2 + A \left[1 - |c(\theta)|^2 + \rho C(\theta) \right] \right\}, \quad (1)$$

familiar from x-ray scattering from liquids,¹ to analyze an experiment of Goodwin, Anderson, and Kenney to obtain information about correlations between nucleons in the nucleus. In their experiment, Goodwin, Anderson, and Kenney looked at the A dependence of π^0 photoproduction at various angles for photon energies from 350 MeV down to threshold.² In Eq. (1) the symbols have the following meaning:

σ_f : cross section for photoproduction from a free nucleon,

A : number of nucleons in the nucleus,

$\rho = A/V$: density of the nucleus,

$c(\theta)$: $\int P(\underline{x}) e^{i\underline{q} \cdot \underline{x}} d^3x$,

$C(\theta)$: $\int G(\underline{x}) e^{i\underline{q} \cdot \underline{x}} d^3x$,

$\underline{q}$: momentum transfer,

$\underline{\nu}$: photon's momentum,

$\underline{k}$: meson's momentum,

θ : angle between $\underline{\nu}$ and $\underline{k}$,

$P(\underline{x})$: probability of finding a nucleon at $\underline{x}$ in the nucleus,

$P(\underline{x}, \underline{x}') = P(\underline{x}) P(\underline{x}') [1 + G(\underline{x} - \underline{x}')]$: joint probability of finding one nucleon at $\underline{x}$ and another at $\underline{x}'$, and

$G(\underline{x} - \underline{x}')$: nuclear-pair correlation function.

See Figs. I-1, I-2, I-3, and I-4.

Watson has pointed out³ that if Serber's direct-interaction model⁴ adequately describes nuclear reactions, Eq. (1) can be derived for high-energy reactions (such as photomeson production) by using multiple-scattering theory, the impulse approximation, and the closure approximation. Further, the applicability of Eq. (1) to a particular reaction at a particular energy can be determined by estimating the

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Fig. I-1. Expected form of various form factors in Eq. (1);
 (a) $A^2 |c(\theta)|^2$, (b) $A[1 - |c(\theta)|^2]$: no correlations,
 (c) $A[1 - |c(\theta)|^2 + \rho C(\theta)]$: with correlations; $q_R \sim R^{-1}$,
 $q_c \sim r_c^{-1}$.

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Fig. 1-2. The nuclear density is $NP(x)$; the number of nucleons in the nucleus is N ; and $R = r_0 A^{1/3}$.

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Fig. I-3. The expected form of the nuclear pair correlation function. The radius of the hard core is r_c .

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Fig. I-4. The real and imaginary part of the optical potential as a function of the meson's kinetic energy outside the nucleus (Ref. 15).

error in the impulse approximation from a formula given by Chew and Wick, and by estimating the error in the closure approximation by a method due to Placzek.

In photomeson production, the interaction of the meson with the nucleus is one of the important factors that determine the number and the angular distribution of photomesons produced from a nucleus. The direct-interaction-model description of photoproduction follows.⁵

The meson is produced from a single nucleon. Before the meson escapes the nucleus, it can interact with the nucleus in three ways:

(a) the meson can be absorbed by a pair of nucleons and not get out of the nucleus;

(b) the meson can scatter inelastically (inelastic scattering is due to inelastic collisions with individual nucleons in the nucleus); and

(c) the meson can interact with the nucleus elastically. Elastic interaction is described by the optical-model potential.⁶

A form frequently taken for the optical-model potential is

$$\mathcal{V}(\underline{x}) = V_0(\epsilon_k) P(\underline{x}), \quad (2)$$

where $\underline{x}$ and ϵ_k are the position and energy of the meson, $V_0(\epsilon_k)$ is the well depth of the optical-model potential, and $P(\underline{x})$ is defined in the list of symbols following Eq. (1).

2. THE EXPERIMENT

π^0 mesons decay rapidly into two gamma rays. One way of detecting the π^0 mesons produced from a nucleus is to observe one of the decay gammas with a single scintillation counter. The distribution of the decay gammas is isotropic in the meson's center-of-mass frame. For low π^0 energy, one would expect that the detection of a single gamma would give information only about the total π^0 photoproduction cross section and would not give information about the angular distribution. If the energy of the π^0 is high enough, the decay gammas tend to go in the same direction as the π^0 , and one can observe the angular distribution of the photoproduced π^0 mesons with a single counter.

In the experiment that motivated this paper, Goodwin, Anderson, and Kenney used a single scintillation counter to observe the intensity of the decay gammas, I , from π^0 mesons produced from targets made of Li, Be, C, N, Al, Ti, Cu, and Pb. They made observations at 45, 90, and 135 degrees and at maximum beam energies of 165, 190, 200, 230, and 340 MeV. They plotted $\log(I)$ vs $\log(A^n)$, and determined n . A is the atomic number. The various values of n obtained are shown in Table I. This table is taken from reference 2.

Table I. Values of n for angles of 45°, 90°, and 135°.

Lab angle (deg)	165 (MeV)	190 (MeV)	200 (MeV)	230 (MeV)	340 (MeV)
45	1.77 ± 0.10	1.40 ± 0.11	1.34 ± 0.22		0.80 ± 0.03
90	1.04 ± 0.04	0.96 ± 0.07		0.84 ± 0.08	0.64 ± 0.04
135	1.30 ± 0.14	0.93 ± 0.11			0.77 ± 0.01

(Their results are further illustrated in Fig. I-6.) It is clear from Table I that n increases at small angles and energies. Goodwin, Anderson, and Kenney have asserted that this increase is consistent with partial elastic production of the π^0 mesons. This is, however, not the case.

A. Comments on the Results of the Experiment

I would like, at this point, to make several remarks based on Eq. (1) about π^0 photoproduction, notably about why the results of Goodwin, Anderson, and Kenney do not seem to be consistent with elastic production. The scattering amplitude for the elastic photoproduction of a π^0 from a spinless nucleus must be of the form

$$f_e = \hat{\epsilon} \times \underline{v} \cdot \underline{k} F(\theta), \quad (3)$$

where $\underline{v}$ and $\hat{\epsilon}$ are the momentum and the polarization vector of the photon, $\underline{k}$ is the momentum of the meson, and $F(\theta)$ is a form factor that depends on energy and angle. The π^0 is a pseudoscalar and $\hat{\epsilon} \times \underline{v} \cdot \underline{k}$ is a pseudoscalar linear in $\hat{\epsilon}$. In the cross section $\hat{\epsilon} \times \underline{v} \cdot \underline{k}$ leads to a $\sin^2 \theta$ dependence,

$$\sigma_e(\theta) \sim \sin^2 \theta |F(\theta)|^2. \quad (4)$$

For $|F(\theta)|^2$ it is reasonable to take $N^2 \exp(-q^2 R^2) = N^2 \exp(-q^2 r_0^2 N^{2/3})$. The differential production cross section for elastic production, therefore, goes as

$$\sigma_e(\theta) \sim \sin^2 \theta A^2 \exp(-q^2 r_0^2 A^{2/3}), \quad (5)$$

which goes to zero for $\theta = 0$.

Near threshold, k is small; therefore $q^2 \approx v^2$, or $q^2 = v^2$ and the total elastic cross section goes as

$$\sigma_e^T \sim A^2 \exp(-v^2 r_0^2 A^{2/3}). \quad (6)$$

If one is working sufficiently above threshold that $|\underline{v}| \approx |\underline{k}|$, one must use $q^2 = v^2 + k^2 - 2kv \cos \theta$ and integrate $\sigma_e(\theta)$, Eq. (5), over all angles to get the total cross section, and

$$\sigma_e^T \propto A^{2/3}. \quad (7)$$

It is hard to see that the results of Goodwin, Anderson, and Kenney are consistent with Eq. (5), Eq. (6), or Eq. (7). Leiss and Schrack have published results of experiments in which they observed both decay photons from the π^0 meson for production energies near 170 MeV.⁷ They stated that production below 200 MeV is elastic, and

they fitted their data apparently quite well with a cross section of the form given in Eq. (5). However, they did not first obtain a cross section from their data and then fit that cross section with the assumed form of the cross section, but analyzed their raw data in terms of the assumed form of the cross section. Kenney has stated that previous experiments at MIT have shown a peaking of the cross section in the forward direction.⁸ If one assumed $1 - pG(\theta) \approx \frac{1}{2}$, one would expect that production from a large nucleus such as lead would be largely inelastic for 200-MeV incident photons.

3. THE CROSS SECTION

In this section, I review the mathematical formulation of the direct-interaction model of photomeson production, discuss the multiple-scattering formalism, and obtain expressions for the transition matrix, T , and for the cross section.

A. Preliminary Definitions

and the Formal Expression for the Cross Section

The Hamiltonian that governs photomeson production is

$$\begin{aligned} H &\equiv H_N + H_\pi + H_\gamma + V \\ &\equiv H_0 + V, \end{aligned} \quad (8)$$

where

$$H_0 = H_N + H_\pi + H_\gamma. \quad (9)$$

Here H_N , H_γ , and H_π are the Hamiltonians of the nucleus, the photon, and the meson; V is the interaction between the photon, the meson, and the nucleons in the nucleus. I assume that

$$V = \sum_{i=1}^N v_i, \quad (10)$$

where v_i is the interaction potential between the i th nucleon and the photon and the meson. The sum in Eq. (10) is over the N nucleons in the nucleus.

For the form of the nuclear Hamiltonian, I take

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2M} + \frac{1}{2} \sum_{i,j} V_{ij}, \quad (11)$$

where $p_i^2/2M$ is the kinetic energy of the i th nucleon and V_{ij} is the potential between the i th and the j th nucleon, i. e., I assume that the forces between the nucleons in the nucleus are two-body forces. Only in Section 4 is it necessary to have a definite form for the nuclear Hamiltonian. The eigenstates of the nucleus, denoted by $|f\rangle$, are complete,⁹

$$\sum_f |f\rangle \langle f| = 1, \quad (12)$$

and they satisfy the eigenvalue equation

$$H_N |f\rangle = E_f |f\rangle \quad (13)$$

The ground state of the nucleus is denoted by the special symbol $|0\rangle$. Equation (7) expresses the closure property of the nuclear wave function. For use in the latter parts of this section it is convenient to state the usual definition of a function of an operator:⁹

$$g(H_N) \equiv \sum_f |f\rangle g(E_f) \langle f|. \quad (14)$$

Single-photon states, denoted by $|\hat{e}, \underline{\nu}\rangle$, where $\hat{e}$ and $\underline{\nu}$ are the polarization vector and the momentum of the photon respectively, or denoted by $|\underline{\nu}\rangle$ if an average over the photon's polarization is implied, satisfy the eigenvalue equation

$$H_Y |\underline{\nu}\rangle = \nu |\underline{\nu}\rangle. \quad (15)$$

Single-meson states, denoted by $|\underline{k}\rangle$, where $\underline{k}$ is the momentum of the meson, satisfy the eigenvalue equation

$$H_\pi |\underline{k}\rangle = \epsilon_k |\underline{k}\rangle, \quad (16)$$

where

$$\epsilon_k = (k^2 + m^2)^{\frac{1}{2}}.$$

In photoproduction, the initial state contains the nucleus in its ground state and a photon with momentum $\underline{\nu}$. The final state contains the nucleus in some excited state and a meson with momentum $\underline{k}$. The initial state is denoted by

$$|0, \underline{\nu}\rangle = |0\rangle |\underline{\nu}\rangle, \quad (17)$$

and the final state is denoted by

$$|f, \underline{k}\rangle = |f\rangle |\underline{k}\rangle. \quad (18)$$

The formal expression for the photoproduction cross section is

$$\sigma = (2\pi)^4 \int d^3k \sum_f \delta(\nu + E_0 - \epsilon_k - E_f) |\langle \underline{k}, f | T | 0, \underline{\nu} \rangle|^2 \quad (19)$$

The transition matrix, T , satisfies

$$T = V \Omega = \bar{\Phi} V. \quad (20)$$

The Møller matrices Ω and $\bar{\Phi}$ satisfy the integral equations

$$\Omega = 1 + \frac{1}{d} V \Omega, \quad (21)$$

and

$$\bar{\Phi} = 1 + \bar{\Phi} V \frac{1}{d}, \quad (22)$$

where

$$d = E_0 + \nu + i\eta - H_0.$$

Here η is the usual small number used to define the Green's function d^{-1} .

B. The Multiple-Scattering Formalism

Formally the transition matrix T is given by the solution of Eqs. (20), (21), and (22). The multiple-scattering formalism¹⁰ allows one to write Eqs. (20), (21), and (22) in such a way that it is clear what approximations to make to obtain an expression for T appropriate to the direct-interaction model.

The operators Φ_C and F are defined by

$$\Phi = \Phi_C F, \quad (23)$$

$$\Phi_C = 1 + \Phi_C \mathcal{V} \frac{1}{a}, \quad (24)$$

and

$$F = 1 + F (V - \mathcal{V}) \frac{1}{a}, \quad (25)$$

where $a = d - \mathcal{V}$. The operator Φ_C and the optical-model potential $\mathcal{V}$, which is defined by Eq. (20) together with

$$F = 1 + FVP^\dagger \frac{1}{a}, \quad (26)$$

are diagonal in the states of the nucleus:

$$\langle n | \mathcal{V} | m \rangle = \mathcal{V}(n) \delta_{nm}, \quad (27)$$

and

$$\langle n | \Phi_C | m \rangle = \Phi_C(n) \delta_{nm}, \quad (28)$$

where n and m are two eigenstates of the nucleus, and $\Phi_C(n)$ is the wave function of the meson appropriate to the optical potential $\mathcal{V}(n)$, which is the optical potential the meson sees if the nucleus is in its n th eigenstate. The operator $P^\dagger$ is an instruction operator that forbids the repetition of an intermediate state that it multiplies to the left of that state. For example,

$$\begin{aligned} \langle n | O_1 O_2 P^\dagger O_3 | m \rangle \\ = \sum_a \sum_{b \neq a, n} \langle n | O_1 | a \rangle \langle a | O_2 | b \rangle \langle b | O_3 | m \rangle, \end{aligned}$$

where O_1 , O_2 , and O_3 are operators.

The multiple-scattering equations, which are equivalent to Eqs. (20), (21), and (22), are

$$F = 1 + \sum_a F_a t_a \frac{P^\dagger}{a}, \quad (29a)$$

$$F_a = 1 + \sum_{\beta (\neq a)} F_\beta t_\beta \frac{P^\dagger}{a}, \quad (29b)$$

and

$$t_\beta = v_\beta + t_\beta \frac{P^\dagger}{a} v_\beta, \quad (29c)$$

where the subscripts refer to the a th and β th nucleons. That Eqs. (29a), (29b), and (29c) are equivalent to Eqs. (15), (16), and (17) is seen as follows:

$$\begin{aligned} \sum_a F_a t_a &= \sum_a F_a t_a \frac{P^\dagger}{a} v_a + \sum_a F_a v_a \\ &= \sum_a F_a t_a \frac{P^\dagger}{a} v_a + \sum_{\beta \neq a} F_\beta t_\beta \frac{P^\dagger}{a} v_a + \sum_a v_a \\ &= \left(\sum_\beta F_\beta t_\beta \frac{P^\dagger}{a} + 1 \right) \sum_a v_a. \end{aligned}$$

Therefore,

$$\sum_a F_a t_a = FV. \quad (30)$$

By using Eqs. (20), (23), and (30), one obtains the important equation

$$T = \Phi_C \sum F_a t_a. \quad (31)$$

The connection between Eq. (31) and the direct-interaction model can be seen by using Eq. (29b) to iterate F_a in Eq. (26):

$$T = \Phi_C \left(\sum_a t_a + \sum_{\beta \neq a} \sum_a t_\beta \frac{P^\dagger}{a} t_a + \sum_{\gamma \neq \beta} \sum_{\beta \neq a} \sum_a t_\gamma \frac{P^\dagger}{a} t_\beta \frac{P^\dagger}{a} t_a + \dots \right).$$

For photomeson production, the photon is absorbed and the meson is emitted by the first t matrix on the right. The subsequent t matrices to the left correspond to inelastic scatterings of the meson. Because of $P^\dagger$ and the restrictions on the sums, each scattering is inelastic and is from a different nucleon. Between inelastic scatterings, the meson interacts with the nucleus elastically. This interaction is described by the optical potential $\mathcal{V}$ in the propagator a^{-1} .

C. Approximations

To reduce Eq. (31) to a simple form that is the mathematical expression of the direct-interaction model of photoproduction, three approximations are made.

a. The amplitude t_a is replaced by t_a^0 , the amplitude for photoproduction from a free nucleon. This is the impulse approximation.¹¹

Because in photoproduction from a free nucleon momentum is conserved, the matrix elements of t_a^0 in a momentum representation contain a delta function:

$$\langle \underline{k}, \underline{P}'_a | t_a^0 | \underline{P}_a, \underline{v} \rangle = \delta(\underline{v} + \underline{P}_a - \underline{k} - \underline{P}'_a) t(\underline{k}, \underline{v}, \underline{\sigma}_a). \quad (32)$$

Here $\underline{P}_a$, $\underline{P}'_a$, and $\underline{\sigma}_a$ are the initial momentum, the final momentum, and the Pauli spin matrix of the nucleon.

Now, I make a slight change in notation. From this point on in the paper, I denote $t(\underline{k}, \underline{v}, \underline{\sigma}_a)$ by t_a . The dependence of t_a on $\underline{\sigma}_a$ is explicitly displayed by

$$t_a = a + \underline{b} \cdot \underline{\sigma}_a, \quad (33)$$

where a and $\underline{b}$ depend on $\underline{k}$ and $\underline{v}$. To calculate the cross section for photomeson production, only $|\underline{a}|^2$ and $|\underline{b}|^2$ are needed. The quantity $|\underline{a}|^2 + |\underline{b}|^2$ can be obtained directly from the differential cross section for photoproduction from a free nucleon. Obtaining $|\underline{a}|^2$ and $|\underline{b}|^2$ separately is not easy at all energies. At energies of interest in this paper--i. e., photon energies below 300 MeV--a theory of photomeson production from free nucleons, which gives $|\underline{a}|^2 \approx |\underline{b}|^2$, is available.¹²

Besides $\underline{k}$, $\underline{v}$, and $\underline{\sigma}_a$, the amplitude t_a can also depend on $\hat{\epsilon}$, $\underline{P}_a$, and $\underline{\tau}_a$, where $\underline{\tau}_a$ is the isotopic Pauli spin matrix of the nucleon. The dependence on $\hat{\epsilon}$ is simply not stated explicitly. The amplitude t_a depends on initial and final relative momenta, which depend on $\underline{P}_a$; however, because the lever arm of $\underline{P}_a$ is small, the dependence of t_a on $\underline{P}_a$ is not critical. I simply ignore this dependence. For photon energies below 300 MeV, the cross section

for photoproduction from neutrons and protons is nearly the same, and it is a good approximation to take t_a independent of isotopic spin.

It follows from Eq. (32) that in a mixed representation, the matrix elements of t_a^0 are

$$\langle \underline{k}, \underline{x}'_a | t_a^0 | \underline{v}, \underline{x}_a \rangle = e^{i(\underline{v}-\underline{k}) \cdot \underline{x}_a} t(\underline{k}, \underline{v}, \sigma_a) \delta(\underline{x}_a - \underline{x}'_a), \quad (34)$$

where $\underline{x}_a$ and $\underline{x}'_a$ are the initial and final positions of the nucleon.

b. The operator F is replaced by the number Q.¹³

$|Q|^2$ is shown in Fig. I-5. The difference between $|Q|^2$ and unity represents the effect upon the photoproduction cross section due to the inelastic scattering of the meson from nucleons that are correlated with the meson's parent nucleon. If the meson scatters inelastically from a nucleon uncorrelated with its parent nucleon, it must excite that nucleon above the nuclear Fermi sea. To do so, the meson must lose considerable momentum; it will be slowed down and probably absorbed. I assume such mesons do not escape the nucleus.

c. The matrix elements of Φ_C are calculated from the optical potential appropriate to the nuclear ground state.⁵

This approximation is made on the assumption that the meson outruns the disturbances that are set up in the nucleus by the production of the meson and the inelastic scatterings of the meson. Therefore, the part of the nucleus the meson sees always looks undisturbed, and the optical potential the meson sees is always that of the ground state of the nucleus.

From Eq. (24), one sees that the matrix elements $\langle \underline{k} | \Phi_C | \underline{x} \rangle$ satisfy the integral equation

$$\langle \underline{k} | \Phi_C | \underline{x} \rangle = \langle \underline{k} | \underline{x} \rangle + \int d^3 x' \frac{e^{i\mathbf{k} \cdot (\underline{x} - \underline{x}')}}{|\underline{x} - \underline{x}'|} \mathcal{V}'(\underline{x}') \langle \underline{k} | \Phi_C | \underline{x}' \rangle. \quad (35)$$

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Fig. I-5. The factor $|Q|^2$ corrects the cross section for inelastic scattering of the meson from a nucleon correlated with the meson's parent nucleon. The quantity $|Q|^2$ is shown as a function of the meson's kinetic energy outside the nucleus (Ref. 13).

If one remembers that $\langle \underline{k} | \underline{x} \rangle = (2\pi)^{-3/2} e^{-i\underline{k} \cdot \underline{x}}$ and denotes $\langle \underline{k} | \bar{\Phi}_C | \underline{x} \rangle$ by $\psi_{-\underline{k}}(\underline{x})$, the meaning of Eq. (35) becomes clear. Equation (35) is the integral form of the Schroedinger equation for the wave function of a particle that scatters from a potential and initially has a momentum $-\underline{k}$ rather than $+\underline{k}$.

Taken together with Eq. (31), the three approximations just described yield

$$T = Q \sum_a \psi_{-\underline{k}}(\underline{x}_a) e^{i\underline{v} \cdot \underline{x}_a} t_a. \quad (36)$$

The amplitude t_a depends on the momentum of the pion, $\underline{k}$. If the optical potential $\mathcal{V}$ were zero, $\psi_{-\underline{k}}(\underline{x})$ would be a plain wave with a definite momentum $-\underline{k}$. But the optical potential is not zero, and $\psi_{-\underline{k}}(\underline{x})$ does have a definite momentum. It is natural to replace $\underline{k}$ in t_a by $i\underline{\nabla}$ acting on $\psi_{-\underline{k}}(\underline{x})$ only. For a large nucleus, the W. K. B. approximation may be applicable, in which case $i\underline{\nabla}$ can be replaced by $\underline{k}$, the momentum of the pion inside the nucleus.

Equations (19) and (36) yield

$$\sigma = (2\pi)^4 \int d^3k |Q|^2 \sum_f \delta(\nu - \epsilon_k + E_0 - E_f) \cdot \left| \langle f | \sum_a \psi_{-\underline{k}}(\underline{x}_a) e^{i\underline{v} \cdot \underline{x}_a} t_a | 0 \rangle \right|^2, \quad (37)$$

which can be written

$$\sigma = (2\pi)^4 \int d^3k |Q|^2 \sum_f \delta(\nu - \epsilon_k + E_0 - E_f) \sum_{\alpha, \beta} \langle 0 | t_\alpha^\dagger e^{-i\underline{v} \cdot \underline{x}_\alpha} \psi_{-\underline{k}}^*(\underline{x}_\alpha) | f \rangle \langle f | \psi_{-\underline{k}}(\underline{x}_\beta) e^{i\underline{v} \cdot \underline{x}_\beta} t_\beta | 0 \rangle. \quad (37')$$

4. EVALUATION OF THE CROSS SECTION

A. The Closure Approximation

If the cross section is represented schematically by $\sigma = \sum_f \rho_f |\langle f | T | i \rangle|^2$, where ρ_f is in phase space, $\langle f | T | i \rangle$ is the matrix element, and f refers to final states of the nucleus, the closure approximation consists in replacing ρ_f by an average value $\bar{\rho}$, extending the sum over nuclear states to all nuclear states and not just those consistent with energy conservation, and using Eq. (12) to obtain

$$\sigma = \bar{\rho} \langle i | T^\dagger T | i \rangle. \quad (38)$$

One expects the closure approximation to be applicable if, as a function of the variables of the final nuclear state, ρ_f is slowly varying and $\langle f | T | i \rangle$ is peaked.

The closure approximation is applied to Eq. (37') by replacing the nuclear excitation energy, $E_f - E_0$, in the delta function by an average excitation Δ . The terms in Eq. (37') for which $\alpha = \beta$ correspond to direct inelastic photoproduction from a single nucleon. For those terms, I choose

$$\Delta(\alpha = \beta) = (q^2 + M^2)^{1/2} - M, \quad (39)$$

where $\underline{q} = \underline{v} - \underline{k}$; $\underline{k}$ is the momentum of the meson inside the nucleus. In Eq. (37'), the terms for which $\alpha \neq \beta$ are nonzero, because of nuclear structure; they are small except for small nuclear excitation. For those terms, I choose

$$\Delta(\alpha \neq \beta) = 0. \quad (40)$$

In the closure approximation the cross section is

$$\begin{aligned} \sigma = & (2\pi)^4 \sum_{\alpha} \int d^3k |Q|^2 \delta(v - \epsilon_k + M - \sqrt{q^2 + M^2}) \cdot \langle 0 | |\psi_{-\underline{k}}(\underline{x}_{\alpha})|^2 t_{\alpha}^{\dagger} t_{\alpha} | 0 \rangle \\ & + (2\pi)^4 \sum_{\alpha \neq \beta} \int d^3k |Q|^2 \delta(v - \epsilon_k) \cdot \langle 0 | \psi_{-\underline{k}}^*(\underline{x}_{\alpha}) \psi_{-\underline{k}}(\underline{x}_{\beta}) e^{i v \cdot (\underline{x}_{\beta} - \underline{x}_{\alpha})} t_{\alpha}^{\dagger} t_{\beta} | 0 \rangle. \end{aligned} \quad (41)$$

To evaluate the matrix elements in Eq. (41), I assume that the only vector the spin of a nucleon, $\underline{\sigma}_a$, is correlated with is the spin of another nucleon; therefore if $\underline{\sigma}_a$ is multiplied by a vector other than the spin of another nucleon, I average over the direction of that vector and $\underline{\sigma}_a$ independently. If $\underline{t}_a = \underline{a} + \underline{b} \cdot \underline{\sigma}_a$, then

$$\sum_a \langle 0 | |\psi_{-\underline{k}}(\underline{x}_a)|^2 \underline{t}_a^\dagger \underline{t}_a | 0 \rangle = A \left[|\underline{a}|^2 + |\underline{b}|^2 \right] \int P(\underline{x}) |\psi_{-\underline{k}}(\underline{x})|^2 d^3x. \quad (42)$$

To evaluate this term, I have simply averaged over the spins of the nucleons and replaced the integration over $\underline{x}_a$ and the sum over a by an integration over the nuclear density $NP(\underline{x})$.

The evaluation of the matrix element of the second term in Eq. (41) is a bit more difficult. Consider

$$\langle 0 | \sum_{a \neq \beta} \sigma_{a\beta} | 0 \rangle = \int d^3x_1 d^3x_2 \text{Tr} \left[\rho(\underline{x}_1 \underline{x}_2 \sigma_1 \sigma_2) \sigma(\underline{x}_1 \underline{x}_2 \sigma_1 \sigma_2) \right], \quad (43)$$

where $\sigma_{a,\beta} \equiv \sigma(\underline{x}_a \underline{x}_\beta \sigma_a \sigma_\beta)$ is an operator that depends on the positions and spins of the a th and β th nucleon, and $\rho(\underline{x}_1 \underline{x}_2 \sigma_1 \sigma_2)$ is a density matrix. To evaluate the cross term, I simply take

$$\rho = \rho_F + \rho_C, \quad (44)$$

where ρ_F is the density matrix for an antisymmetrized product wave function, i. e., a degenerate Fermi gas, and ρ_C is added to take care of short-ranged correlations between the nucleons. For an anti-symmetrized product wave function, one obtains

$$\begin{aligned} \rho_F(\underline{x}_1 \underline{x}_2 \sigma_1 \sigma_2 \tau_1 \tau_2) = \\ (A/4)^2 \rho(\underline{x}_1) \rho(\underline{x}_2) - (A/4^2) \rho(\underline{X}) \rho G_F(\underline{x}) \frac{1}{2} (1 + \sigma_1 \cdot \sigma_2) \cdot \frac{1}{2} (1 + \tau_1 \cdot \tau_2). \end{aligned} \quad (45)$$

where

$$\underline{X} = (\underline{x}_1 + \underline{x}_2)/2, \quad \underline{x} = \underline{x}_1 - \underline{x}_2, \quad (46)$$

and

$$G_F(x) = [3 j_1(k_F x)/k_F x]^2. \quad (47)$$

We note that $G_F(x)$ is the usual correlation function for a degenerate Fermi gas, k_F is the Fermi momentum, and j_1 is the spherical Bessel function of order one. Thus,

$$j_1(x) = \frac{\sin x}{x} - \frac{\cos x}{x}. \quad (48)$$

If one averages over the isotopic spin--that is, takes the trace with respect to τ_1 and τ_2 --Eq. (45) becomes

$$\rho_F(x_1 x_2 \sigma_1 \sigma_2) = (A^2/4) \rho(x_1) \rho(x_2) - (A/4^2) \rho(X) \rho G_F(x) (1 + \sigma_1 \cdot \sigma_2). \quad (45')$$

The choice of the form of ρ_C is not very critical. To have variety and yet simplicity, I choose

$$\rho_C(x_1 x_2 \sigma_1 \sigma_2) = A \rho(X) \rho G_N(x) (1 + a \sigma_1 \cdot \sigma_2), \quad (49)$$

where a is simply a number. To further simplify the evaluation of the second term, I make the approximation that $\psi_{-k}^*(x_a) \psi_{-k}(x_\beta)$ may be replaced by $e^{i\mathbf{k} \cdot (\mathbf{x}_a - \mathbf{x}_\beta)} |\psi_{-k}(X)|^2$ for the terms that contain the correlation functions G_F and G_N . This assumption is not too unreasonable if one remembers that G_F and G_N are short-ranged functions. The evaluation of the cross term is now straightforward:

$$\begin{aligned} & \sum_{a \neq \beta} \langle 0 | \psi_{-k}^*(\underline{x}_a) \psi_{-k}(\underline{x}_\beta) e^{i\mathbf{k} \cdot (\underline{x}_\beta - \underline{x}_a)} t_a^\dagger t_\beta | 0 \rangle \\ &= A^2 |a|^2 |c(\theta)|^2 + A\Lambda \left\{ -\frac{\rho}{4} C_F(\theta) \left[|a|^2 + |b|^2 \right] + \rho C_N(\theta) \left[|a|^2 + a|b|^2 \right] \right\}, \end{aligned} \quad (50)$$

where

$$C(\theta) \equiv \int \psi_{-\mathbf{k}}(\mathbf{x}) e^{i\mathbf{v} \cdot \mathbf{x}} \rho(\mathbf{x}) d^3\mathbf{x}, \quad (51a)$$

$$\Lambda \equiv \int |\psi_{-\mathbf{k}}(\mathbf{x})|^2 \rho(\mathbf{x}) d^3\mathbf{x}, \quad (51b)$$

and

$$C_{F,N}(\theta) \equiv \int e^{i(\mathbf{v}-\mathbf{k}) \cdot \mathbf{x}} G_{F,N}(\mathbf{x}) d^3\mathbf{x}. \quad (51c)$$

Further, I define

$$\sigma_i(\theta) \equiv (2\pi)^4 \int k^2 dk \delta[\mathbf{v} - \epsilon_{\mathbf{k}} + (q^2 + M^2)^{1/2} - M] [|\mathbf{a}|^2 + |\mathbf{b}|^2], \quad (52a)$$

$$\sigma_e(\theta) \equiv (2\pi)^4 \int k^2 dk \delta(\mathbf{v} - \epsilon_{\mathbf{k}}) |\mathbf{a}|^2, \quad (52b)$$

and

$$r(\theta) \sigma_i(\theta) \equiv (2\pi)^4 \int k^2 dk \delta(\mathbf{v} - \epsilon_{\mathbf{k}}) [|\mathbf{a}|^2 + |\mathbf{b}|^2]. \quad (52c)$$

The closure approximation for the cross section is

$$\begin{aligned} \sigma(\theta) = A^2 \sigma_e(\theta) |C(\theta)|^2 |Q|^2 + A\Lambda \sigma_i(\theta) |Q|^2 & \left\{ 1 - \frac{\rho}{4} C_F(\theta) r(\theta) \right. \\ & \left. + \rho C_N(\theta) \left[(1 - \alpha) \sigma_e(\theta) / \sigma_i(\theta) + \alpha r(\theta) \right] \right\}. \end{aligned} \quad (53)$$

Equation (53) is quite similar to Eq. (1).

B. Corrections to the Closure Approximation

To evaluate the corrections to the closure approximation, I use a method due to Placzek.¹⁴ I follow Fowler and Watson's exposition of the method.¹³

To evaluate the corrections, I set $|Q|^2 = 1$ and $\psi_{-\underline{k}}(\underline{x}) = e^{-i\underline{k} \cdot \underline{x}}$ where $\underline{k}$ is the momentum of the meson inside the nucleus. Then the cross section becomes

$$\sigma = (2\pi)^4 \int k^2 dk \sum_f \delta(\nu - \epsilon_k + E_0 - E_f) \left| \left\langle f \left| \sum_a t_a e^{i\underline{q} \cdot \underline{x}_a} \right| 0 \right\rangle \right|^2, \quad (54)$$

where $\underline{q} = \underline{\nu} - \underline{k}$. I use

$$\sum_f |f\rangle \delta(\nu - \epsilon_k + E_0 - E_f) \langle f| = \delta(\nu - \epsilon_k + E_0 - H_N) \quad (55)$$

to express the cross section [cf. Eq. (14)] in the elegant form

$$\sigma = (2\pi)^4 \int k^2 dk \sum_{a, \beta} \left\langle 0 \left| t_a^\dagger e^{-i\underline{q} \cdot \underline{x}_a} \delta(\nu - \epsilon_k + E_0 - H) e^{i\underline{q} \cdot \underline{x}_\beta} t_\beta \right| 0 \right\rangle. \quad (54')$$

I expand the delta function in a Taylor series about the most probable value of the nuclear excitation. For the $a = \beta$ term,

$$\delta(\nu - \epsilon_k + E_0 - H) = \sum_{n=0}^{\infty} \frac{1}{n!} (E_0 - H + \frac{q^2}{2M})^n \delta^n(\nu - \epsilon_k - \frac{q^2}{2M}); \quad (56)$$

I have replaced $(q^2 + M^2)^{1/2} - M$ by $q^2/2M$ because the effects of correlations on the cross section are small unless $q^2 \ll M^2$. For the $a \neq \beta$ term,

$$\delta(\nu - \epsilon_k + E_0 - H) = \sum_{n=0}^{\infty} \frac{1}{n!} (E_0 - H)^n \delta^n(\nu - \epsilon_k), \quad (57)$$

where δ^n is the n th derivative of the delta function and $\delta^0 = \delta$.

By using $e^{-iq \cdot x} p_a e^{iq \cdot x} = p_a + q$ and assuming the only momentum-dependent term in H is the kinetic energy, which is $\sum_i p_i^2/2M$, one obtains

$$e^{-iq \cdot x_\beta} (E_0 - H)^n e^{iq \cdot x_\beta} = \left(E_0 - H - \frac{q^2}{2M} - \frac{q \cdot p_\beta}{2M} \right)^n \quad (58)$$

and

$$e^{iq \cdot x_\beta} \left(E_0 - H + \frac{q^2}{2M} \right)^n e^{iq \cdot x_\beta} = \left(E_0 - H - \frac{q \cdot p_\beta}{M} \right)^n \quad (59)$$

With the help of Eqs. (53), (54), (55), and (56) the cross section, Eq. (54), is

$$\sigma = \sigma_d + \sigma_{nd}, \quad (60)$$

where

$$\sigma_d = \int k^2 dk \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n \left(\nu - \epsilon_k - \frac{q^2}{2M} \right) \sum_a \left\langle 0 \left| t_a^\dagger \left(E_0 - H - \frac{q \cdot p_a}{M} \right)^n t_a \right| 0 \right\rangle \quad (60a)$$

and

$$\sigma_{nd} = \int k^2 dk \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n (\nu - \epsilon_k) \quad (60b)$$

$$\sum_{a \neq \beta} \left\langle 0 \left| e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger \left(E_0 - H - \frac{q^2}{2M} - \frac{q \cdot p_\beta}{M} \right)^n t_\beta \right| 0 \right\rangle.$$

The zero-order terms in Eqs. (60a) and (60b) are included in the closure-approximation expression for the cross section, Eq. (41).

The other terms in Eqs. (60a) and (60b) are corrections to the closure approximation. I evaluate the first few terms to estimate the error in the closure approximation.

In Eq. (60a) the $n = 0$ term is

$$\langle 0 | t_a^\dagger t_a | 0 \rangle = \left[|a|^2 + |b|^2 \right];$$

this term contributes to the closure approximation. The $n = 1$ term is

$$\langle 0 | t_a^\dagger (E_0 - H - \frac{q \cdot p_a}{M}) t_a | 0 \rangle.$$

If the dependence of t_a on p_a is ignored, an average over the direction of p_a makes

$$\langle 0 | t_a^\dagger \frac{q \cdot p_a}{M} t_a | 0 \rangle = 0.$$

The amplitude t_a is not independent of q , and therefore, one can not simply average over the direction of q in $q \cdot p$. The neglect of the dependence of t_a on p_a corresponds to assuming t_a to be a slowly varying function of p_a . Thus, we have

$$\begin{aligned} \langle 0 | t_a^\dagger (E_0 - H) t_a | 0 \rangle &= \frac{1}{2} \langle 0 | [H, t_a], t_a | 0 \rangle \\ &= \frac{8}{3} |b|^2 \langle 0 | V_{S_a} | 0 \rangle. \end{aligned}$$

The commutator $[H, t_a] \neq 0$, because of the spin-dependent part of the potential V_S . To evaluate this term I use

$$V_S = \frac{1}{2} \sum_a \sum_{\beta \neq a} u_{a\beta} \sigma_a \cdot \sigma_\beta, \quad (61)$$

where $u_{a\beta}$ depends on the separation between the a th and β th particles only, and

$$V_{S_a} = \sum_{\beta (\neq a)} u_{a\beta} \sigma_a \cdot \sigma_\beta. \quad (61')$$

A similar value of the $n = 1$ term is obtained with other types of spin-dependent potentials. The $n = 2$ term is

$$\langle 0 | t_a^\dagger (E_0 - H - \frac{q \cdot p_a}{M})^2 t_a | 0 \rangle .$$

Let us ignore the terms that are nonzero because $[H, t_a] \neq 0$ and consider only

$$\langle 0 | t_a^\dagger t_a \left(\frac{p_a \cdot q}{M} \right)^2 | 0 \rangle = \frac{1}{3} q^2 \left[|a|^2 + |b|^2 \right] \langle 0 | \left(\frac{p_a}{M} \right)^2 | 0 \rangle .$$

To evaluate this term I have averaged over the direction of $\underline{p_a}$ and σ_a .

Next, terms in Eq. (60b) up to $n = 1$ are evaluated. In a sense the evaluation of these terms is consistent with the evaluation of the terms in Eq. (60a) up to $n = 2$. All the terms in Eq. (60b) are nonzero because of the structure of the nucleus, which is a consequence of the forces between the nucleons. In other words, all the terms in Eq. (60b) are nonzero because of the forces between the nucleons and are, in a sense, binding-energy corrections. In Eq. (60b), the $n = 0$ term

$$\langle 0 | e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger t_\beta | 0 \rangle$$

is included in the closure-approximation expression for the cross section. The $n = 1$ term is

$$\langle 0 | e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger (E_0 - H - \frac{p_a \cdot q}{M} - \frac{q^2}{2M}) t_\beta | 0 \rangle .$$

From Eq. (61) I obtain

$$\langle 0 | e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger (E_0 - H) t_\beta | 0 \rangle = -\frac{8}{3} |b|^2 \langle 0 | e^{-iq \cdot (x_a - x_\beta)} \sigma_a \cdot \sigma_\beta | 0 \rangle .$$

From integration by parts, it follows

$$\langle 0 | e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger t_\beta \left(\frac{q \cdot p_\beta}{M} \right) | 0 \rangle = - \frac{q^2}{2M} \langle 0 | e^{-iq \cdot (x_a - x_\beta)} t_a^\dagger t_\beta | 0 \rangle ;$$

to see this, consider the matrix element written schematically:

$$\int d^3 x_a d^3 x_\beta e^{-iq \cdot (x_a - x_\beta)} \bar{\psi}(x_a) t_a^\dagger t_\beta q \cdot \frac{1}{i} \nabla_\beta \psi(x_\beta).$$

The equality follows if

$$\bar{\psi}(x_\beta) t_a^\dagger t_\beta q \cdot \nabla_\beta \psi(x_\beta) = \frac{1}{2} q \cdot \nabla_\beta [\bar{\psi}(x_\beta) t_a^\dagger t_\beta \psi(x_\beta)] \quad --$$

that is to say, if $\bar{\psi}(x_\beta)$ differs only from $\psi(x_\beta)$ by a phase factor that is independent of x_β . If such is the case, the last two terms of the $n = 1$ term cancel.

To summarize, the cross section including the first-order corrections to the closure approximation is

$$\sigma = \sigma_0 + \sigma_1 + \sigma_2, \quad (62)$$

where

$$\begin{aligned} \sigma_1 = (2\pi)^4 \int k^2 dk \left\{ \delta^1(v - \epsilon_k - \frac{q^2}{2M}) \frac{8}{3} N |b|^2 \langle 0 | \frac{1}{A} V_S | 0 \rangle \right. \\ \left. - \delta^1(v - \epsilon_k) \frac{8}{3} |b|^2 \langle 0 | \sum_{a \neq \beta} e^{-iq \cdot (x_a - x_\beta)} u_{a\beta} \sigma_a \cdot \sigma_\beta | 0 \rangle \right\} \end{aligned} \quad (62a)$$

and

$$\sigma_2 = (2\pi)^4 \int k^2 dk \delta^2(v - \epsilon_k - \frac{q^2}{2M}) \frac{1}{6} q^2 \langle 0 | (\frac{p}{M})^2 | 0 \rangle A [|a|^2 + |b|^2]. \quad (62b)$$

To estimate the value of σ_1/σ_c and σ_2/σ_c , I ignore the factor $q^2/2M$ in the delta functions. To estimate σ_1/σ_c , I set

$$\langle 0 | \sum_{(a \neq \beta)} \left[1 - e^{iq \cdot (x_a - x_\beta)} \right] u_{a\beta} \sigma_a \cdot \sigma_\beta | 0 \rangle \approx \frac{1}{5} \langle 0 | V_S | 0 \rangle. \quad (63)$$

The spin-flip potential V_S is a sum of short-ranged potentials between pairs of nucleons. I obtain Eq. (63) by setting $q \approx m$ and by using a Hulthén wave function. Using

$$\int k^2 dk \delta(v - \epsilon_k) = d(k\epsilon)/d\epsilon = k(1 + (\epsilon/k)^2),$$

$$|b|^2 \approx \frac{1}{2} |t|^2, \text{ and } \sigma_c \approx (2\pi)^4 \epsilon k A |t|^2, \text{ I obtain}$$

$$\sigma_1/\sigma_c \approx \frac{4}{3} \cdot \frac{1}{5} \cdot \langle 0 | A^{-1} V_S | 0 \rangle \frac{1}{\epsilon} (1 + \frac{\epsilon^2}{k^2}). \quad (64)$$

Similarly,

$$\sigma_2/\sigma_c \approx \frac{1}{6} \langle 0 | (\frac{p_a}{M})^2 | 0 \rangle \frac{1}{\epsilon k} \frac{d^2(\epsilon k q^2)}{d\epsilon^2}. \quad (65)$$

Since $q^2 = v^2 + k^2 - 2vk \cos \theta$, it follows

$$\frac{1}{\epsilon k} \frac{d^2(k\epsilon q^2)}{d\epsilon^2} = \left[2 - \frac{m_\pi^2}{k^2} \right] \frac{q^2}{k^2} + 2 \left(\frac{dk}{d\epsilon} \right)^2$$

$$+ 2(k-v \cos \theta) \left[2(1 + \frac{\epsilon^2}{k^2}) \cdot \frac{1}{\epsilon} \frac{dk}{d\epsilon} + \frac{d^2 k}{d\epsilon^2} \right]. \quad (66)$$

I take $\langle 0 | A^{-1} V_S | 0 \rangle \approx 20$ MeV, $\langle 0 | (p_a/M)^2 | 0 \rangle \approx 0.1$, and $q \approx m_\pi$, and estimate $dk/d\epsilon$ and $d^2 k/d\epsilon^2$ from Fig. I-6.¹⁵ I obtain

$$\text{for } v \approx 200 \text{ MeV, } \sigma_1/\sigma_c \approx 0.1 \text{ and } \sigma_2/\sigma_c \approx \frac{1}{2};$$

$$\text{for } v < 200 \text{ MeV, } \sigma_2/\sigma_c \approx 1;$$

for $v > 200$ MeV, σ_2/σ_c is even less than without the optical potential.

It is surprising that the optical potential should decrease the binding-energy corrections in the region of 140 MeV, i.e., near the 3-3 resonance. It must be remembered that to calculate the effect of the optical potential a crude approximation has been made, namely: $\psi_{-k}(x)$ was replaced by $e^{i k \cdot x}$, which is, in a sense, the WKB method. But the WKB method is certainly of marginal applicability, firstly

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Fig. I-6. The momentum of the meson inside and outside the nucleus as a function of the kinetic energy of the meson outside the nucleus. The momentum of the meson outside the nucleus is k ; the momentum of the meson inside the nucleus is K (Ref. 15).

because--below 100 MeV at least--the meson wavelength is not small compared with the thickness of the nuclear surface; and secondly because, just above 100 MeV the mean free path of the mesons changes rapidly and becomes quite small; this certainly must affect the binding-energy corrections, but in a way that is not clear. The crudeness of the WKB method probably precludes one from saying more than that the effect of the optical potential upon the cross section in the region of the 3-3 resonance and below may be considerable. Glassgold¹⁶ has stressed the unreliability of the WKB method in optical-model calculations and the desirability of exact numerical calculations. A feasible, exact calculation could be based upon some simple model of the nucleus--such as the Fermi gas model or shell model--and an exact meson wave function, $\psi_{-k}(x)$, obtained numerically from a reasonable optical potential. Such a calculation could afford a better insight into the effect of the optical potential upon the photoproduction cross section near and below the 3-3 resonance.

C. The Impulse Approximation

According to Chew and Wick, the impulse approximation¹¹ is applicable if the parameter α is small, where

$$\alpha = \frac{f}{\lambda} \frac{B}{\epsilon} \quad (67)$$

That is to say, they assert that if f_B is the scattering amplitude for scattering from a bound nucleon and f_F is the scattering amplitude for scattering from a free nucleon, then

$$f_B = f_F [1 + \mathcal{O}(\alpha)] \quad (68)$$

where $\mathcal{O}(\alpha)$ means of the order of, e. g., $\mathcal{O}(\alpha) < c |\alpha|$, where c is of order unity. They do not, however, calculate the constant c .

In Eq. (67), B is the average kinetic energy or binding energy of the bound nucleon. For photomeson production, ϵ and λ are the kinetic energy and de Broglie wavelength of the meson. Here, f is the meson-nucleon (not the photomeson-nucleon) scattering amplitude. The magnitude of α depends on what we take for B . If B is the average kinetic energy, it is of the order of 40 MeV; if B is the average binding energy, it is of the order of only 10 MeV. Depending on which value we take for B , α will be of the order of either 1 or 1/4 for meson energies from 50 MeV to 250 MeV, i. e., above the 3-3 resonance. This follows if one takes $f = (\sigma/4\pi)^{1/2}$. Because one does not know the parameter c or the function $\mathcal{O}(\alpha)$, it is not exactly clear how small α has to be for one to safely assume--on the basis of the criterion of Chew and Wick--that the impulse approximation is valid below the 3-3 resonance.

Serber has implied that the impulse approximation is valid when the collision time t_C for photomeson production is short compared with the collision time t_N between the nucleons in the nucleus.⁴ A reasonable value for t_C is $(v_\pi n_\pi)^{-1}$ --if we assume the meson has to travel a distance equal to its Compton wave length, m_π^{-1} , at a velocity v_π before it gets away from the nucleon from which it is produced. A reasonable value of t_N is $6/m_\pi$ --if we assume that the average

velocity of the nucleons in a nucleus is one-third the velocity of light, and their mean free path is $2/m_\pi$. Therefore, we get

$$\frac{t_c}{t_N} = \frac{1}{6v_\pi}$$

It is, therefore, intuitively reasonable that the impulse approximation is satisfied by mesons with energies above 50 MeV. Near the 3-3 resonance, probably $t_c \approx (m_\pi v_\pi)^{-1} + m_\pi^{-1}$. The second m_π^{-1} is the lifetime of the resonance.

The physical interpretation of the parameter α , Eq. (67), should perhaps be pointed out. Note that ϵ^{-1} and B^{-1} correspond to t_c and t_N , respectively. The interpretation that Chew and Wick have given to f/λ is that it is a factor that says α is small if the Born approximation is applicable.

5. CONCLUSION

Goodwin, Anderson, and Kenney observed the decay photons from π^0 mesons photoproduced from nuclei of various atomic numbers, A . They determined the dependence of the number of photons counted upon the atomic number. For this dependence, they assumed the form A^n . Their results are that n increases with decreasing angle and incident photon energy as shown in Table I.

I have explored the possibility of explaining their results upon the basis of a simple theory which yields an expression for the cross section similar to that of Eq. (1). In the simple model one assumes that the production amplitude is the sum of the production amplitudes from the individual nucleons, that the production amplitude of one of the individual nucleons is the same as that of a free nucleon, that the interaction of the nucleus with those mesons that escape the nucleus can be taken into account by changing the meson's wave function from a plain wave to one appropriate to an optical potential, and that one can use the closure approximation to evaluate the cross section.

It does not seem possible to explain the results of Goodwin, Anderson, and Kenney on the basis of this simple theory. Furthermore, their results do not seem to be consistent with partial elastic production--as they have asserted. Near threshold one would expect that the detection of a single decay photon would give information mostly about the total production cross section rather than about the differential production cross section. One would expect the dependence of the total cross section near threshold to be of the form $A^2 \exp(-a A^{2/3})$, where a is of order unity, rather than of the form A^2 .

I have looked at the binding-energy corrections to the closure approximation. From an evaluation of these corrections based on the WKB method (which is admittedly crude and unreliable), it appears that because of the magnitude and rather rapid variation of the optical potential, the corrections may be rather large below and in the neighborhood of the 3-3 pion-nucleon resonance. For several reasons one might expect the effect of the optical potential upon the cross section and upon the corrections to the cross section to be larger for large

nuclei than for small. For example, the optical potential is usually taken to be proportional to the nuclear density. Because the ratio of surface--in which the density is lower--to volume in smaller nuclei is larger, there is a smaller portion of a small nucleus over which the optical potential has its maximum value than there is of a large nucleus. If the corrections to the closure approximation are larger for larger nuclei, they might account for the difference between the simple theory and the experimental results. One might see whether such is the case by using a simple model for the nucleus and by taking the effect of the optical potential upon the meson wave function into account exactly. Such a calculation, though feasible, would have to be done numerically.

Another possible explanation of the experimental result is that the elementary targets in the nucleus for π^0 photoproduction are not single nucleons but pairs of nucleons and that these pairs act like bosons. It would then seem that one might get a different dependence upon atomic number with bosons than he could get with fermions.

APPENDIX I-1

In this appendix, I point out several things of interest about Eqs. (60a) and (60b). Consider a simplified version of the matrix elements

$$\begin{aligned} & \langle 0 | e^{-iq \cdot x_a} (E_0 - H - \frac{q^2}{2M})^n e^{iq \cdot x_a} | 0 \rangle \\ &= \langle 0 | (E_0 - H - \frac{p_a \cdot q}{M})^n | 0 \rangle . \end{aligned} \quad (A. 1)$$

If $|\langle p_a | 0 \rangle|^2$, the momentum distribution of the ath nucleon, goes to zero for large p_a as some power p_a^{-1} rather than as $\exp(-p_a^2)$ --as it does for a particle bound by a harmonic oscillator potential;--then for n sufficiently large, $\langle 0 | (p_a \cdot q)^n | 0 \rangle$ will be infinite. The matrix elements (A. 1) may also be infinite because for arbitrary operators the moments of the nuclear excitation $E_0 - E_f$ for the distribution $\langle f | \theta | 0 \rangle \cdot \langle 0 | \theta' | f \rangle$, i. e.,

$$\sum_f (E_0 - E_f)^n \langle 0 | \theta' | f \rangle \langle f | \theta | 0 \rangle = \langle 0 | \theta' (E_0 - H) \theta | 0 \rangle , \quad (A. 2)$$

are in general not all finite. Therefore, Eqs. (60a) and (60b) as they stand contain infinite matrix elements. These infinities may be removed by remembering that the sum over final nuclear states [cf Eq. (37)] can include only states with energy less than $E_0 + \nu$. In operator notation, this condition may be expressed by inserting $H(E_0 + \nu - H)$ into the matrix element, where $H(x)$ is the Heaviside function; the Heaviside function is unity if its argument is positive, and zero otherwise. The matrix elements in Eq. (60) with $H(E_0 + \nu - H)$ inserted become

$$\langle 0 | t_a^+ e^{-iq \cdot x_a} H(E_0 + \nu - H) (E_0 - H - \frac{q^2}{2M})^n e^{iq \cdot x_a} t_a | 0 \rangle . \quad (A. 3)$$

For the lower-order matrix elements, i. e., n small, the inclusion of $H(E_0 + \nu - H)$ in the matrix element leads to only a small correction. But for high-order matrix elements $H(E_0 + \nu - H)$ is very important.

To gain a bit of insight into the structure of the Placzek-Wick method, consider the cross section for the scattering of a neutron, for example, from a system of particles,

$$\begin{aligned} \sigma &= (2\pi)^4 |a|^2 v_I^{-1} \int d^3k \sum_f \delta\left(\frac{k_0^2}{2m} - \frac{k^2}{2m} + E_0 - E_f\right) |\langle f | \sum_a e^{iq \cdot x_a} | 0 \rangle|^2 \\ &= (2\pi)^4 v_I^{-1} |a|^2 \int d^3k \sum_{\alpha, \beta} \langle 0 | e^{-iq \cdot x_\alpha} \delta\left(\frac{k_0^2}{2m} - \frac{k^2}{2m} + E_0 - H\right) e^{iq \cdot x_\beta} | 0 \rangle. \end{aligned} \quad (A. 4)$$

If one commutes $e^{iq \cdot x_\beta}$ through the delta function, the matrix element in Eq. (A. 4) takes the form

$$\langle 0 | e^{-iq \cdot (x_\alpha - x_\beta)} \delta\left(\frac{k_0^2}{2m} - \frac{k^2}{2m} - \frac{q^2}{2M} - \frac{q \cdot p_\beta}{M} + E_0 - H\right) | 0 \rangle. \quad (A. 5)$$

In the delta function in Eq. (A. 5) the first three terms go as k_0^2 , the fourth term goes as k_0 , and the last term goes as a constant. If one uses $\delta(ax) = a^{-1} \delta(x)$ to take a factor k_0^{-2} out of the delta function, the first three terms then go as a constant, the fourth term as k_0^{-1} , and the last two terms as k_0^{-2} . An expansion of the delta function in the last three terms is therefore an expansion in powers of k_0^{-1} .

Placzek evaluated the terms in the expansion up to k_0^{-6} . The inclusion of

$H\left(\frac{k_0^2}{2m} + E_0 - H\right)$ in the matrix elements, however, may lead to correction of a lower power in k_0^{-1} , which Placzek did not calculate.

For example, if one assumes that the target is a single particle bound by a finite-ranged potential and that the ground state of the target is an s state, the inclusion of $H\left(\frac{k_0^2}{2m} + E_0 - H\right)$ leads to corrections that go as k_0^{-5} .

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II. THE RELATIVE RATE OF THE ABSORPTION OF NEGATIVE PIONS IN HYDROGEN AND DEUTERIUM

1. INTRODUCTION

A. Background

Low-energy pion physics, or the interaction of low-energy π mesons with neutrons, protons, and deuterons is almost a classic subject.¹ Low-energy pion physics, however, is not understood on the basis of a fundamental theory that has a definite starting point from which all quantities of experimental interest can be calculated systematically. Rather, it is organized and interpreted: (a) by means of general physical principles that hold for almost any theory--such as the reciprocity theorem, which comes from time-reversal invariance; (b) by means of some simple conservation laws that seem to hold for strongly interacting particles--such as charge independence or the conservation of isotopic spin; and (c) by means of some simple models that seem plausible for relating several particular reactions.² Of the important things that have been learned about the π mesons by such means are that the pion has odd intrinsic parity and zero spin. A particle with odd intrinsic parity is referred to as a pseudoscalar.

That the pion has odd intrinsic parity was learned from studies of the absorption of π^- mesons in hydrogen and in deuterium.³ This paper is about one aspect of the absorption of π^- mesons in hydrogen and in deuterium, namely the ratio of the absorption rate for the reaction

denoted by R_D to the absorption rate for the reaction

denoted by R_H . The quantity that is calculated in this paper is

$$T = R_D/R_H. \quad (1-3)$$

Figure II-1 shows the connection between the reaction rates of reactions that occur when π^- mesons are absorbed in hydrogen and deuterium, and other low-energy cross sections for reactions that involve mesons, photons, protons, neutrons, and deuterons. By detailed balancing the cross section for a reaction can be determined from the cross section for the inverse reaction. By charge symmetry, the cross section for a reaction can be found from the cross section of the reaction with the charges of all the particles reversed. By charge independence, cross sections containing similar particles are related. By extrapolation to zero energy, the absorption rate is obtained from the indicated cross sections.

The meaning of Fig. II-1 is as follows: (a) By knowing the cross section for the reaction $p + p \rightarrow \pi^+ + d$, one can find the absorption rate for the reaction

(b) By knowing⁴ the cross sections for the scattering of π^+ and π^- mesons from hydrogen, one can calculate the absorption rate for the reaction

(c) By knowing the ratio of π^- mesons to π^+ mesons photo-produced from deuterons, and the cross section for the photoproduction of π^+ mesons from hydrogen, one can multiply them together--or even try some more sophisticated approach--and hope to get the cross section for the photoproduction of π^- mesons from neutrons, from which one can calculate the absorption rate for the reaction

T, P, and S are simply the ratios of the absorption rates:

$$P = R(\pi^+ + p \rightarrow \pi^0 + n) / R(\pi^- + p \rightarrow n + \gamma), \quad (1-7)$$

$$S = R(\pi^- + d \rightarrow 2n) / R(\pi^- + d \rightarrow 2n + \gamma), \quad (1-8)$$

and

$$T = R(\pi^- + d \rightarrow 2n + \gamma) / R(\pi^- + p \rightarrow n + \gamma). \quad (1-9)$$

Legend:

R - absorption rate

σ - cross section

P, S, T - various ratios between the absorption rates as defined by Eqs. (1-7), (1-8), (1-9)

C I - charge independence

C S - charge symmetry

D B - detailed balancing

E Z E - extrapolation to zero energy

X - multiply together

Pion production in nucleon-nucleon collision

$$\sigma(p + p \rightarrow \pi^+ + d)$$

C I

$$\sigma(n + n \rightarrow \pi^- + d)$$

D B

$$\sigma(\pi^- + d \rightarrow 2n)$$

E Z E

$$R(\pi^- + d \rightarrow \gamma + 2n)$$

S

$$R(\pi^- + d \rightarrow 2n)$$

T

$$R(\pi^- + p \rightarrow \gamma + n)$$

P

$$R(\pi^- + p \rightarrow \pi^0 + n)$$

E Z E

$$\sigma(\pi^- + p \rightarrow \gamma + n)$$

E Z E

$$\sigma(\pi^- + p \rightarrow \pi^0 + n)$$

D B

$$\sigma(\gamma + n \rightarrow \pi^- + p)$$

C I

$$\sigma(\pi^- + p \rightarrow \pi^- + p) \quad \sigma(\pi^+ + p \rightarrow \pi^+ + p)$$

X

$$\frac{\sigma(\gamma + d \rightarrow \pi^+ + 2n) \quad \sigma(\gamma + p \rightarrow \pi^+ + n)}{\sigma(\gamma + d \rightarrow \pi^- + 2p)}$$

Pion-nucleon scattering

Pion photoproduction

Fig. II-1. ¹⁹ Connections between low-energy scattering, production, and absorption reaction for π mesons.

P is the famous Panofsky ratio. P and S can not be calculated at present; they must be measured.

J. Ryan has recently measured both P and S.⁵ Taken together with $R(\pi^- + d \rightarrow 2n)$ and $R(\pi^- + p \rightarrow \pi^0 + n)$, determined in the way just discussed, a knowledge of P and S allows one to determine T. J. Ryan has also determined T.

B. The Model

The basic and well-known formula for the calculation of an absorption rate is⁶

$$R = 2\pi \sum_f \rho_f |\langle f|M|i \rangle|^2, \quad (1-10)$$

where $|i\rangle$ and $|f\rangle$ are the wave functions of the initial and final states, respectively, M is the absorption matrix, ρ_f is the phase space available for a particular final state, and $\sum_f$ is a sum over all final states allowed by energy conservation and experimental resolution.

The assumptions of the model⁷ used to calculate T are as follows.

1. The meson is captured from an atomic s state--though not necessarily from the s state of lowest energy. For a discussion of this assumption, see Appendix II.

As will be seen in Sec. 2. A, it turns out that the dependence on the atomic wave function of the various absorption processes separates out and one may write

$$R = \bar{R} C, \quad (1-11)$$

where $\bar{R}$ is independent of (and C depends only on) the atomic state from which the meson is captured. To obtain S to interpret his experimental data, J. Ryan has assumed that C is the same for $R(\pi^- + d \rightarrow 2n)$ and $R(\pi^- + d \rightarrow 2n + \gamma)$, and that C is the same for $R(\pi^- + p \rightarrow n + \gamma)$ and $R(\pi^- + p \rightarrow n + \pi^0)$. The values of C are not necessarily the same for hydrogen and deuterium. It follows that Ryan's values of S, P, and T contain no atomic dependence. The value of T that I have calculated is without atomic dependence.

2. The second assumption, which is known as the impulse approximation,⁸ is that the basic absorption process and the matrix that describes it are the same in hydrogen and deuterium. The process is

$$\pi^- + p \rightarrow n + \gamma, \quad (1-12)$$

and the matrix that describes it is

$$a \underline{\sigma} \cdot \underline{\hat{e}} \delta(\underline{q} + \underline{p} - \underline{k} - \underline{p}'). \quad (1-13)$$

Between the time the meson is absorbed and the photon is emitted, the neutron is merely a spectator. It does not interact with the proton nor does it interact with the meson to change its density at the proton. It only interacts with the proton before the meson is absorbed--this is the interaction that holds the deuteron together--and after the photon is emitted when the proton has changed into a neutron.

In the absorption matrix, Eq. (1-13), the symbols have the following meaning: a is a constant. For deuterium, taking a to be a constant corresponds to neglecting p-wave absorption due to the motion of the proton in the deuteron. Here $\underline{\sigma}$ is the spin of the nucleon and $\underline{\hat{e}}$ is the polarization vector of the photon; $\underline{k}$ and $\underline{q}$ are the momenta of the meson and the photon, and $\underline{p}'$ and $\underline{p}$ are the momenta of the initial proton and the final neutron it changes into. Under space inversion, $\underline{\sigma} \cdot \underline{\hat{e}}$ is a pseudoscalar, which it must be because the meson has odd intrinsic parity. The only coordinate of the photon that $\underline{\sigma} \cdot \underline{\hat{e}}$ contains is $\underline{\hat{e}}$, which means the photon emitted is an electric dipole photon (cf. Sec. 2, B). The delta function indicates that momentum is conserved.

The final state of the reaction,

$$\pi^- + d \rightarrow 2n + \gamma, \quad (1-14)$$

is a three-body system--two neutrons and a gamma ray. Experimentally,⁹ it is observed that the energy spectrum of the γ ray is not monochromatic but has the form shown in Fig. II-2. As is well known, for a three-body final state the conditions of total energy and total momentum conservation do not uniquely fix the distribution of energy

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Fig. II-2. Energy spectrum of gamma rays from $\pi^- + d \rightarrow 2n + \gamma$: (a). Without experimental resolution. (b). With experimental resolution.

among the particles. The probability of a given energy distribution, which manifests itself in the energy spectra of the particles, depends upon the interaction that produces the particles, the interactions between the particles, and the available phase space for that energy distribution. The sharp peak in the γ -ray energy spectrum (Fig. II-2) is due to the attractive interaction between the two neutrons. Because of this interaction the neutrons like to act like a single particle, i. e., they like to be in a nearly bound state and have a low relative energy. This tendency causes the γ -ray spectrum to peak and be more monochromatic. If the two neutrons were a single particle, the spectrum would be monochromatic. Although the force between the neutrons causes a striking change in the γ -ray spectrum, it does not appreciably change the total absorption rate or the ratio T .¹⁰

To evaluate the ratio T , I shall use the closure approximation. The closure property¹¹ of a complete set of functions, denoted by $|f\rangle$, is

$$\sum_f |f\rangle \langle f| = 1. \quad (1-15)$$

By multiplying both sides of Eq. (1-15) by any function ψ , say one gets immediately the formal expression for the expansion of the function

$$\psi = \sum_f |f\rangle \langle f|\psi\rangle \quad (1-16)$$

The $\langle f|\psi\rangle$ are just the expansion coefficients.

Equation (1-15) is used to evaluate the absorption rate. Let R be the absorption rate, given by Eq. (1-10). To use Eq. (1-15), one replaces ρ_f by an average phase-space factor $\bar{\rho}_f$ that is independent of f , and extends the sum over the final state to all final states and not just to those allowed by energy conservation to obtain

$$R = \bar{\rho}_f \sum_f \langle i|M^\dagger|f\rangle \langle f|M|i\rangle = \bar{\rho}_f \langle i|M^\dagger M|i\rangle, \quad (1-17)$$

where, to go from Eq. (1-10) to Eq. (1-17), the identity

$\langle f|M|i \rangle^* = \langle i|M^\dagger|f \rangle$ has been used. One can expect the closure approximation to be a good approximation if ρ_f is a slowly varying function of f , if $\langle f|M|i \rangle$ is a peaked function of f , and if the extra states included in the sum do not contribute much. Actually, one would expect the closure approximation to be a particularly good way to evaluate T because of the sharp peak in the γ -ray energy spectrum that is due just to $\langle f|M|i \rangle$ and not to ρ_f .

I shall obtain the corrections to the closure approximation by a method due to Placzek.¹² There are a number of possible corrections to the model used to calculate T , all of which one would expect to be most important if the π^- is absorbed when the nucleons in the deuteron are close together: (a) one nucleon may interact with the other between the time the meson is absorbed and the photon is emitted; (b) the meson may be focused on the proton by scattering back and forth between the nucleons;¹³ (c) because the nucleons are close together, by the uncertainty principle the proton's momentum may be large enough to make p-wave absorption important;⁶ and (d) the meson may be absorbed by one nucleon and the photon emitted by the other. Let us estimate the importance of all of these processes together. Experimentally, it is observed that $\pi^- + d \rightarrow 2n$ occurs roughly 60% of the time and $\pi^- + d \rightarrow 2n + \gamma$ occurs 30% of the time.³ The absorption of the meson to give no photon must occur when the nucleons are close together; one may guess that the rate of the four processes listed above, which involve the emission of a photon and occur when the nucleons are close together, is of the order of e^2 times the rate of $\pi^+ + d \rightarrow 2n$, which means, since $e^2 \approx 10^{-2}$, 5% of the γ rays observed are not accounted for by the model. Most of these γ rays probably go into the tail of the γ -ray energy spectrum. Because the nucleons are close together, they probably have a high relative momentum and will have a high relative energy after the absorption. A high relative energy for the nucleons means a low energy for the γ ray.

2. FORMULATION

A. Absorption in Hydrogen, $\pi^- + p \rightarrow n + \gamma$

In the initial state, the π^- and the proton form an atom. They are held together by the Coulomb potential. In the final state, the neutron and the γ ray have equal and opposite momentum and a kinetic energy of about 70 MeV, i. e., $(1/2) m_\pi$.

The π^- reaches an atomic s state--though not necessarily the lowest s state, i. e., a K state--before it is absorbed.¹⁴ Only absorption from an atomic s state is considered.

To take into account the change of the proton into a neutron, I shall use the isotopic-spin formalism.¹ In this familiar formalism, the neutron and proton are merely different states of the same particle, the nucleon. Let us label these states $|n\rangle$ for the neutron and $|p\rangle$ for the proton. Further, let us introduce the Pauli isotopic-spin operator τ . Of particular interest is τ^- :

$$\tau^- |p\rangle = |n\rangle; \tau^- |n\rangle = 0. \quad (2-1)$$

The change of the proton to a neutron is described by writing the absorption operator in such a way that it changes the nucleon from one state, $|p\rangle$, to another state, $|n\rangle$. To do this, one simply includes τ^- in the absorption operator, Eq. (1-13).

The initial state is

$$|i\rangle = \phi_n(\underline{x}) \Phi_{cm}(\underline{X}) \chi_i |p\rangle, \quad (2-2)$$

where $\underline{x}$ is the relative distance between the π^- meson and the proton, and $\underline{X}$ is the position of their center of mass. Thus, we have

$$\underline{x} = \underline{r}_\pi - \underline{r}; \quad \underline{X} = \frac{m_\pi \underline{r}_\pi + M \underline{r}}{m_\pi + M}, \quad (2-3)$$

where $\underline{r}_\pi$ and m_π are the position vector and mass of the π^- meson, and $\underline{r}$ and M are the position vector and mass of the nucleon, i. e., the proton; $\phi_n(\underline{x})$ is the atomic s state labeled with the principal quantum number n ; χ_i is the Pauli spinor of the proton, and $|p\rangle$

is the isotopic spinor that indicates there is a proton in the initial state; and $\Phi_{\text{cm}}(\mathbf{X})$ is the center-of-mass wave function of the atom. It is normalized to unity, which means we are going to calculate the absorption rate for a single atom. Other details about $\Phi_{\text{cm}}(\mathbf{X})$ are not very important. The electrostatic forces between the atom and its environment, which determine $\Phi_{\text{cm}}(\mathbf{X})$, simply serve to contain the atom in the target initially; their effect upon the absorption rate is negligible.

The final state is

$$|f\rangle = (2\pi)^{-3/2} e^{i\mathbf{P} \cdot \mathbf{r}} \chi_f |n\rangle |\mathbf{v}, \hat{\mathbf{e}}\rangle, \quad (2-4)$$

where $\hat{\mathbf{e}}, \mathbf{v}$, and $|\mathbf{v}, \hat{\mathbf{e}}\rangle$ are the polarization vector, the momentum, and the state vector of the photon, respectively. Also, $\mathbf{P}, \chi_f$, and $|n\rangle$ are, respectively, the momentum, Pauli spinor, and isotopic spinor of the nucleon, i. e., the neutron.

Let us define the absorption operator, M , by specifying its matrix elements in momentum space and its operator form in spin and isospin space:

$$\langle \mathbf{v}, \hat{\mathbf{e}}, \mathbf{P} | M | \mathbf{k}, \mathbf{P}' \rangle = a \boldsymbol{\sigma} \cdot \hat{\mathbf{e}} \tau^- \delta(\mathbf{v} + \mathbf{P} - \mathbf{k} - \mathbf{P}'), \quad (2-5)$$

where $\mathbf{P}'$ and $\mathbf{P}$ are the initial and final momenta of the nucleon, and $\mathbf{k}$ is the momentum of the meson. The magnitude of the absorption matrix is characterized by the constant a , and $\boldsymbol{\sigma}$ is the Pauli spinor of the nucleon. The delta function indicates that total momentum is conserved in the absorption. Under space inversion, $\boldsymbol{\sigma} \cdot \hat{\mathbf{e}}$ is a pseudoscalar, which it must be because the pion is a pseudoscalar. To see that $\boldsymbol{\sigma} \cdot \hat{\mathbf{e}}$ is a pseudoscalar, remember that under space inversion $\hat{\mathbf{e}}$ goes into $-\hat{\mathbf{e}}$ ($\hat{\mathbf{e}}$ behaves like an electric field), and $\boldsymbol{\sigma}$ goes into $\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ behaves like an angular momentum). Besides $\mathbf{v}$ in the delta function, the only dependence on the variables of the photon is $\hat{\mathbf{e}}$ in $\boldsymbol{\sigma} \cdot \hat{\mathbf{e}}$. Here $\hat{\mathbf{e}}$ has odd parity and unit angular momentum; the photon emitted also has odd parity and unit angular momentum. It is an electric dipole photon.

It is now straightforward to find the matrix element of the absorption operator with respect to the initial and final state:

$$\langle f | M | c \rangle = a \chi_f^\dagger \sigma \cdot \hat{e} \chi_i \phi_n(0) (2\pi)^{3/2} \langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle, \quad (2-6)$$

where $\phi_n(0)$ is $\phi_n(\underline{x})$ evaluated at $\underline{x} = 0$ and

$$\langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle = \frac{1}{(2\pi)^{3/2}} \int e^{-i \underline{P} \cdot \underline{X}} e^{-i \underline{v} \cdot \underline{X}} \underline{\Phi}_{cm}(\underline{X}) d^3 X. \quad (2-7)$$

The absorption rate is

$$\begin{aligned} R &= \sum_f \rho_f |M_{fi}|^2 \\ &= \sum_{pol} \sum_{\chi_f} \int d^3 v \int d^3 P \delta(m_\pi - v - \frac{P^2}{2M}) |\langle f | M | i \rangle|^2 \\ &= \sum_{pol} \sum_{\chi_f} (2\pi)^3 |a|^2 \int d^3 v \int d^3 P \delta(m_\pi - v - \frac{P^2}{2M}) \\ &\quad \cdot |\chi_f^\dagger \sigma \cdot \hat{e} \chi_f|^2 |\langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle|^2. \end{aligned} \quad (2-8)$$

The simplification of Eq. (2-8) yields two examples of the use of the closure approximation:

$$\begin{aligned} \sum_f |\chi_f^\dagger \sigma \cdot \hat{e} \chi_i|^2 &= \sum_f \chi_i^\dagger \sigma \cdot \hat{e} \chi_f \chi_f^\dagger \sigma \cdot \hat{e} \chi_i \\ &= \chi_i^\dagger (\sigma \cdot \hat{e})^2 \chi_i = 1. \end{aligned} \quad (2-9)$$

In Eq. (2-4) I have used the following equalities:

$$(\chi_f^\dagger \underline{\sigma} \cdot \underline{\hat{e}} \chi_i)^* = \chi_i^\dagger \underline{\sigma} \cdot \underline{\hat{e}} \chi_f;$$

$\sum_f \chi_f \chi_f^\dagger = 1$, the closure property of the nucleon Pauli spinors;

$\chi_i^\dagger \chi_i = 1$, χ_i is normalized to unity; and the identity

$$\underline{\sigma} \cdot \underline{a} \underline{\sigma} \cdot \underline{b} = \underline{a} \cdot \underline{b} + i \underline{\sigma} \cdot (\underline{a} \times \underline{b}). \quad (2-10)$$

The matrix element $\langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle$, [see Eq. (2-8)], is almost a delta function--remember that the distance over which $\underline{\Phi}_{cm}$ varies is determined by electrostatic forces. Thus,

$$\begin{aligned} \int d^3 \underline{P} | \langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle |^2 &= \int \langle \underline{\Phi}_{cm} | e^{i \underline{v} \cdot \underline{X}} | \underline{P} \rangle d^3 \underline{P} \langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle \\ &= \langle \underline{\Phi}_{cm} | e^{i \underline{v} \cdot \underline{X}} e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle = \langle \underline{\Phi}_{cm} | \underline{\Phi}_{cm} \rangle \\ &= 1. \end{aligned} \quad (2-11)$$

One sees from Eq. (2-11) that $| \langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle |^2$ is normalized like a delta function. Therefore, we may take

$$| \langle \underline{P} | e^{-i \underline{v} \cdot \underline{X}} | \underline{\Phi}_{cm} \rangle |^2 = \delta(\underline{v} + \underline{P}). \quad (2-12)$$

Equation (2-12) expresses total-momentum conservation.

Let us use Eq. (2-12) and the result of Eq. (2-9) to simplify the absorption rate, Eq. (2-8). Let us also display explicitly the electrostatic dependence of the absorption rate,

$$R = C R_H, \quad (2-13a)$$

with

$$R_H = \sum_{\text{pol}} |a|^2 \int d^3v \delta(m_\pi - v - \frac{v^2}{2M}) \quad (2-13b)$$

and

$$C = (2\pi)^3 |\Phi_n(0)|^2, \quad (2-13c)$$

where C is the electrostatic dependence of the absorption rate. The inclusion of $(2\pi)^3$ in C rather than in R_H has no intrinsic significance. I have done this to avoid writing the $(2\pi)^3$ in some subsequent equations.

For deuterium, a quantity R_D , which is similar to R_H , will be defined. The quantity T , which was discussed in the introduction and which it is the business of this paper to calculate, is $R_D R_H^{-1}$.

B. Absorption in Deuterium, $\pi^- + d \rightarrow 2n + \gamma$

In the initial state, the π^- and the deuteron form an atom. As for hydrogen, only capture from an s state is considered. The separation of the dependence of the absorption rate on the atomic wave function, ϕ_n , is the same as for hydrogen and gives the same result. It will not be repeated. In this section, it is the equivalent of R_H , which I have called R_D , that is calculated. The absorption rate in deuterium from a particular atomic state, ϕ_n , is $R = C R_D$, where C is given by Eq. (2-13c). Also, in the same way as for hydrogen, the center-of-mass dependence of the absorption rate gives total-momentum conservation [see Eq. (2-12)].

The final state is a three-body system--two neutrons and a γ ray. The two neutrons are fermions and obey Pauli's exclusion principle. The force between the two neutrons causes the energy spectrum of the γ ray to peak as shown in Fig. II-1 and discussed in Sec. I-B.

Again it will be convenient to use the isotopic-spin formalism. To formulate an expression for the absorption rate, three points of particular interest must be discussed: (a) the form of the absorption matrix; (b) the force between the two neutrons; and (c) the Pauli principle.

First let us consider the Pauli principle. The expression for the absorption rate is

$$R = \sum_f \rho_f |\langle \Psi_f | M | \Psi_i \rangle|^2. \quad (2-14)$$

The sum over final states must be over only physically distinct states. According to the exclusion principle, Ψ_f and Ψ_i must be antisymmetric in fermion quantum numbers. Physically, two final states that differ only by the interchange of the quantum numbers of two fermions are the same state, which must be counted only once in the sum over final states. If there are N fermions in the final state, Ψ_f , there are $N!$ possible permutations of their quantum numbers. In the usual way, the sum over all physically distinct final states, $\sum_f$, may be replaced by an unrestricted sum over all quantum numbers divided by $N!$, the number of times a physically distinct state is counted. Let us denote the sum over all quantum numbers by $\sum_{\text{all qm}}$. Then

$$R = (N!)^{-1} \sum_{\text{all qm}} \rho_f |\langle \Psi_f | M | \Psi_i \rangle|^2. \quad (2-15)$$

Let us parameterize the final states. First, we introduce the usual antisymmetric projection operator

$$A = (N!)^{-1} \sum_P \delta^P P, \quad (2-16)$$

where P is a permutation operator that acts on fermion quantum numbers, and δ^P is -1 if P is an odd permutation and $+1$ if P is an even permutation. A is Hermitian and a projection operator, i. e.,

$$A = A^\dagger \quad (2-17a)$$

and

$$A^2 = A. \quad (2-17b)$$

Next, let us introduce the complete set of unsymmetrized, normalized states, Φ_f , which satisfy the same Schroedinger equation as Ψ_f and Ψ_i . In Φ_f , each individual fermion has a definite set of quantum numbers, which may, for example, be its spin and asymptotic momentum.

We may write

$$\Psi_f = (N!)^{1/2} A \Phi_f, \quad (2-18)$$

where Ψ_f is normalized (because Φ_f is) and labeled by the quantum numbers of Φ_f .

If M is symmetric,

$$[M, A] = 0. \quad (2-19)$$

A symmetric and an antisymmetric operation commute.

By using Eqs. (2-17a), (2-17b), and (2-19) one obtains

$$\begin{aligned} \langle \Psi_f | M | \Psi_i \rangle &= \langle \sqrt{N!} A \Phi_f | M | \sqrt{N!} A \Phi_i \rangle \\ &= \sqrt{N!} \langle \Phi_f | M | \sqrt{N!} A \Phi_i \rangle \\ &= \sqrt{N!} \langle \Phi_f | M | \Psi_i \rangle. \end{aligned} \quad (2-20)$$

By inserting Eq. (2-20) into Eq. (2-15), one obtains

$$R = \sum_{\text{all } qn} \rho_f |\langle \Phi_f | M | \Psi_i \rangle|^2. \quad (2-21)$$

Therefore, to take care of the Pauli principle it is necessary to antisymmetrize only the initial state, Ψ_i , and use an absorption matrix, M , that is symmetric. $\sum_{\text{all } qn}$ is a sum over all final states, Φ_f , and the Φ_f are simply a complete set of normalized unsymmetrized states that are solutions of the same Schroedinger equation as Ψ_f .

For the initial state, let us take

$$\Psi_f = 2^{-1/2} [n(1) p(2) - n(2) p(1)] \phi_D(\underline{r}). \quad (2-22)$$

The dependence of the initial state upon the π^- meson has already been taken out in C [see Eq. (2-13c)]; $\phi_D(\underline{r})$ is the space and spin part of the deuteron's wave function. Let us take it to be of the usual form

$$\phi_D(\underline{r}) = (4\pi r)^{-1} [u(r) + 8^{-1/2} w(r) S_{12}] \chi_T, \quad (2-23)$$

where $\underline{r}$ is the distance between the neutron and the proton, i. e., the relative position vector of the deuteron; χ_T is simply a triplet spinor. The first term in the brackets, $u(r)$, is the s-state part of the deuteron wave function, and the second term, $w(r) S_{12}$, is the d-state part; S_{12} is the usual tensor operator. Hence,

$$S_{12} = 3 \underline{\sigma}_1 \cdot \hat{\underline{r}} \underline{\sigma}_2 \cdot \hat{\underline{r}} - \underline{\sigma}_1 \cdot \underline{\sigma}_2, \quad (2-24)$$

where $\hat{\underline{r}} = \underline{r}/r$. Now S_{12} satisfies the identity

$$(S_{12})^2 = -2S_{12} + 8 P_T, \quad (2-25)$$

where $P_T = \frac{1}{4} (3 + \underline{\sigma}_1 \cdot \underline{\sigma}_2)$ is the triplet projection operator. Also $\phi_D(\underline{r})$ is normalized, so that

$$\int_0^\infty \left\{ [u(r)]^2 + [w(r)]^2 \right\} dr = 1. \quad (2-26)$$

Thus, $\phi_D(\underline{r})$ satisfies the eigenvalue equation

$$h_0 \phi_D = \epsilon_0 \phi_D, \quad (2-27a)$$

where

$$h_0 = \frac{p^2}{M} + v_0 \quad (2-27b)$$

and

$$\epsilon_0 = -\frac{a^2}{M}. \quad (2-27c)$$

Asymptotically $u(r)$ and $w(r)$ go as e^{-ar} . Now, v_0 is the potential between the neutron and the proton in the deuteron. So, v_0 must, of course, have a noncentral part for the deuteron wave function to have a d-state component, but this noncentral part does not lead to any large corrections to the absorption rate that is to be calculated--even the corrections due to the d-state component of the wave function are small. Therefore, since it does not lead to any confusion, I shall simply talk about and use v_0 as though it were just a central potential.

As can be seen from Eq. (2-22), the notation for isotopic-spin spinors has been changed in an obvious way. It is necessary to use an isotopic singlet spinor to make $\bar{\Psi}_1$ antisymmetric. And, $\phi_D(\underline{r})$ is symmetric in both space and spin.

The final state is

$$\bar{\Phi}_f = |\underline{y}, \hat{\epsilon}\rangle n(1)n(2) \phi_f(\underline{r}), \quad (2-28)$$

where $\phi_f(\underline{r})$ is the space and spin part of the wave function of the two final neutrons. It is unsymmetrized and satisfies the eigenvalue equation

$$h_f \phi_f = \epsilon_f \phi_f, \quad (2-29)$$

where

$$h_f = \frac{p^2}{M} + V. \quad (2-30)$$

The choice of V is not critical; to have a definite form for V , let us choose

$$V = v_0 P_S, \quad (2-31)$$

where v_0 is the same potential as the one between the neutron and the proton in a deuteron, and $P_S = \frac{1}{4} (1 - \underline{\sigma}_1 \cdot \underline{\sigma}_2)$ is the singlet projection operator. The spectrum of h_0 consists of a single bound state and a continuum. The spectrum of h_f consists of a continuum and at most a single bound state. The continuum part of the spectra of both operators may be labeled in the usual way, i. e., by the asymptotic value of the momentum or energy. Actually the details of the spectra do not come up in the subsequent discussion.

For the absorption operator M let us take the sum of the absorption operators of the individual nucleons. The matrix elements of M in a momentum representation are

$$\begin{aligned} \langle \underline{P}_1 \underline{P}_2 \underline{y} \hat{\epsilon} | M | \underline{P}'_1 \underline{P}'_2 \rangle &= a \underline{\sigma}_1 \cdot \hat{\epsilon} \tau_1^- \delta(\underline{y} + \underline{P}_1 - \underline{P}'_1) \delta(\underline{P}_2 - \underline{P}'_2) \\ &+ a \underline{\sigma}_2 \cdot \hat{\epsilon} \tau_2^- \delta(\underline{y} + \underline{P}_2 - \underline{P}'_2) \delta(\underline{P}_1 - \underline{P}'_1), \end{aligned} \quad (2-32)$$

where the capital $\underline{P}'$'s are the momenta of the two nucleons. I have omitted the momentum of the meson on the right-hand side of the matrix element, Eq. (2-32), because in the atom it is effectively zero. As for hydrogen, a is taken to be a constant. This corresponds to neglecting p-wave absorption due to the motion of the proton in the deuteron. As can be seen from Eq. (2-32), the nucleon that absorbs the meson gets all the recoil from the photon. By using the identity

$$\begin{aligned}\delta(\underline{x}) \delta(\underline{y}) &= \delta(\underline{x} + \underline{y}) \delta[(\underline{x} - \underline{y})/2] \\ &= \delta[(\underline{x} + \underline{y})/2] \delta(\underline{x} - \underline{y}),\end{aligned}$$

we can explicitly take the dependence on the center-of-mass momenta of the nucleons out of the matrix elements of the absorption operator. Thus,

$$\begin{aligned}M &= \delta(\underline{y} + \underline{P} - \underline{P}') \left[a \underline{\sigma}_1 \cdot \hat{\underline{e}} \tau_1^- \delta(\underline{y}/2 + \underline{p} - \underline{p}') \right. \\ &\quad \left. + a \underline{\sigma}_2 \cdot \hat{\underline{e}} \tau_2^- \delta(-\underline{y}/2 + \underline{p} - \underline{p}') \right],\end{aligned}\quad (2-33)$$

where the large $\underline{P}'$'s are the center-of-mass momenta of the nucleons and the small $\underline{p}'$'s are their relative momenta, i. e.,

$$\underline{P} = \underline{P}_1 + \underline{P}_2 \quad \text{and} \quad \underline{p} = (\underline{P}_1 - \underline{P}_2)/2.$$

The delta function $\delta(\underline{y} + \underline{P} - \underline{P}')$ in Eq. (2-33) expresses total momentum conservation. If the center-of-mass dependence of the absorption rate is taken out consistently--as it was for hydrogen by putting Φ_{cm} in--the result is that only $(2\pi)^3 \delta(\underline{y} + \underline{P} - \underline{P}')$ comes out of the matrix element squared, $|M_{fi}|^2$ (in the formula for the absorption rate), rather than in $\delta(\underline{y} + \underline{P} - \underline{P}')$ squared. The factor of $(2\pi)^3$ is included in C [see Eq. (2-13c)]. It is convenient to express the part of the absorption matrix that is independent of the center-of-mass momentum in a coordinate representation rather than in a momentum representation, i. e., in terms of $\underline{r}$ rather than of $\underline{p}$.

If we use

$$\delta(\underline{y}/2 + \underline{p} - \underline{p}') = \langle \underline{p} | e^{-i\underline{y} \cdot \underline{x}/2} | \underline{p}' \rangle \quad (2-34)$$

and

$$\int \langle \underline{x} | \underline{p} \rangle d^3 \underline{p} \langle \underline{p} | e^{-i \underline{v} \cdot \underline{x} / 2} | \underline{p}' \rangle d^3 \underline{p}' \langle \underline{p}' | \underline{x}' \rangle$$

$$= \delta(\underline{x}' - \underline{x}) e^{-i \underline{v} \cdot \underline{x} / 2}, \quad (2-35)$$

the part of the absorption matrix that is independent of the center of mass takes the form

$$M = a \left(\underline{\sigma}_1 \cdot \hat{\underline{e}} \tau_1^- e^{-i \underline{v} \cdot \underline{x} / 2} + \underline{\sigma}_2 \cdot \hat{\underline{e}} \tau_2^- e^{i \underline{v} \cdot \underline{x} / 2} \right).$$

(2-36)

By using Eq. (2-1), Eq. (2-22), Eq. (2-28), and Eq. (2-26), one easily obtains

$$\langle \Phi_f | M | \Psi_i \rangle = 2^{-1/2} \int \phi_f^\dagger(\underline{r}) (\underline{\sigma}_1 \cdot \hat{\underline{e}} e^{-i \underline{v} \cdot \underline{x} / 2} - \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{i \underline{v} \cdot \underline{x} / 2} \phi_D(\underline{r}) d^3 \underline{r}$$

$$= 2^{-1/2} \langle f | (\underline{\sigma}_1 \cdot \hat{\underline{e}} e^{-i \underline{v} \cdot \underline{x} / 2} - \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{i \underline{v} \cdot \underline{x} / 2} | 0 \rangle.$$

(2-37)

Equation (2-37) may be inserted into Eq. (8) to give

$$R_D = \sum_{\text{pol}} \sum_f |a|^2 \int d^3 \underline{v}_0 (m_\pi + \epsilon_0 - v - \epsilon_f - \frac{v^2}{4M})$$

$$\cdot \frac{1}{2} |\langle f | (\underline{\sigma}_1 \cdot \hat{\underline{e}} e^{-i \underline{v} \cdot \underline{x} / 2} - \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{i \underline{v} \cdot \underline{x} / 2} | 0 \rangle|^2.$$

(2-38)

In the delta function, $v^2/4M$ is the center-of-mass energy of the two final neutrons. By using the identity

$$\sum_f |f\rangle g(\epsilon_f) \langle f| \equiv g(h_f),$$

where $g(\epsilon_f)$ is a function of ϵ_f , and the identity

$$\langle n | \theta | m \rangle^* = \langle m | \theta^\dagger | n \rangle,$$

where θ is any operator, we may write Eq. (2-38) in the form

$$R_D = \sum_{\text{pol}} |a|^2 \int d^3 \nu \cdot \frac{1}{2} \langle 0 | (\underline{\sigma}_1 \cdot \hat{\underline{e}} e^{i \underline{\nu} \cdot \underline{x}/2} - \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{-i \underline{\nu} \cdot \underline{x}/2})$$

$$\cdot \delta(m_\pi - \nu - \nu^2/4M + \epsilon_0 - h_f) (\underline{\sigma}_1 \cdot \hat{\underline{e}} e^{-i \underline{\nu} \cdot \underline{x}/2} - \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{i \underline{\nu} \cdot \underline{x}/2}) |0\rangle,$$

which can be reduced to

$$R_D = \sum_{\text{pol}} |a|^2 \int d^3 \nu \cdot \left[\langle 0 | \underline{\sigma}_1 \cdot \hat{\underline{e}} e^{i \underline{\nu} \cdot \underline{x}/2} \delta(m_\pi - \nu - \frac{\nu^2}{4M} + \epsilon_0 - h_f) \underline{\sigma}_1 \cdot \hat{\underline{e}} e^{-i \underline{\nu} \cdot \underline{x}/2} |0\rangle \right. \\ \left. - \langle 0 | \underline{\sigma}_1 \cdot \hat{\underline{e}} e^{i \underline{\nu} \cdot \underline{x}/2} \delta(m_\pi - \nu - \frac{\nu^2}{4M} + \epsilon_0 - h_f) \underline{\sigma}_2 \cdot \hat{\underline{e}} e^{i \underline{\nu} \cdot \underline{x}/2} |0\rangle \right]. \quad (2-39)$$

3. EVALUATION OF THE RELATIVE ABSORPTION RATE

A. The Closure Approximation

In this section $T = R_D/R_H$ is evaluated by the closure approximation. The R_D and R_H are given by Eq. (2-39) and Eq. (2-13). As mentioned in the introduction, because the γ -ray energy spectrum is sharply peaked (see Fig. II-2), owing to the force between the two final neutrons, one might expect the use of the closure approximation to be a good way to evaluate T . To apply the closure approximation, let us simply replace $\epsilon_0 - h_f$ by $-v^2/(4M)$ in the argument of the delta function in R_D of Eq. (2-39). Then the arguments of the delta functions of R_D and R_H are the same and, in T , the phase-space factors of R_D and R_H cancel. In T , $|a|^2$ cancels also. The sum over the polarization of the photon, $\sum_{\text{pol}}$, in R_H is redundant and gives just 2. Therefore, one gets simply

$$T = R_D/R_H$$

$$= \frac{1}{2} \sum_{\text{pol}} \left[\langle 0 | \underline{\sigma}_1 \cdot \underline{\hat{e}} \underline{\sigma}_1 \cdot \underline{\hat{e}} | 0 \rangle - \langle 0 | \underline{\sigma}_1 \cdot \underline{\hat{e}} \underline{\sigma}_2 \cdot \underline{\hat{e}} e^{i \underline{v} \cdot \underline{x}} | 0 \rangle \right]. \quad (3-1)$$

The second term must be evaluated at the zero of the argument of the delta function, which is

$$v = (M^2 + 2M m_\pi)^{1/2} - M. \quad (3-2)$$

The two terms in Eq. (3-1) simplify to

$$(\underline{\sigma}_1 \cdot \underline{\hat{e}})^2 = 1;$$

therefore,

$$\frac{1}{2} \sum_{\text{pol}} \langle 0 | (\underline{\sigma}_1 \cdot \underline{\hat{e}})^2 | 0 \rangle = 1, \quad (3-3)$$

$$\begin{aligned} \frac{1}{2} \sum_{\text{pol}} \underline{\sigma}_1 \cdot \hat{\mathbf{e}} \underline{\sigma}_2 \cdot \hat{\mathbf{e}} &= \frac{1}{2} \left[\underline{\sigma}_1 \cdot \underline{\sigma}_2 - \underline{\sigma}_1 \cdot \hat{\mathbf{v}} \underline{\sigma}_2 \cdot \hat{\mathbf{v}} \right] \\ &= \frac{1}{2} \left[\frac{2}{3} \underline{\sigma}_1 \cdot \underline{\sigma}_2 + \frac{1}{3} (\underline{\sigma}_1 \cdot \underline{\sigma}_2 - 3 \underline{\sigma}_1 \cdot \hat{\mathbf{v}} \underline{\sigma}_2 \cdot \hat{\mathbf{v}}) \right] \end{aligned} \quad (3-4)$$

where, as a function of $\hat{\mathbf{v}}$, the first term has angular momentum zero and the second term has angular momentum two. Because experimentally the deuterons in the target are unpolarized and the γ rays are emitted isotropically, it is consistent to average over the direction of $\hat{\mathbf{v}}$. By means of the usual expansion of a plane wave

$$e^{i \underline{v} \cdot \underline{x}} = 4\pi \sum_{lm} i^l j_l(vx) Y_{lm}(\hat{\mathbf{v}}) Y_{lm}(\hat{\mathbf{x}}), \quad (3-5)$$

it is easily seen that if $f_1(\hat{\mathbf{v}})$ has angular momentum 1, it follows that

$$\int f_1(\hat{\mathbf{v}}) e^{i \underline{v} \cdot \underline{x}} \frac{d\Omega_{\hat{\mathbf{v}}}}{4\pi} = f_1(\hat{\mathbf{x}}) \cdot i^1 j_1(vx). \quad (3-6)$$

The left-hand side of Eq. (3-6) is just $f_1(\hat{\mathbf{v}})$ times $e^{i \underline{v} \cdot \underline{x}}$ averaged over the direction of $\hat{\mathbf{v}}$. Comparing Eqs. (3-1), (3-4), and (3-6), one sees

$$\frac{1}{2} \sum_{\text{pol}} \langle 0 | \underline{\sigma}_1 \cdot \hat{\mathbf{e}} \underline{\sigma}_2 \cdot \hat{\mathbf{e}} e^{i \underline{v} \cdot \underline{x}} | 0 \rangle = \frac{1}{3} \langle 0 | j_0(vx) | 0 \rangle + \frac{1}{6} \langle 0 | j_2(vx) S_{12} | 0 \rangle, \quad (3-7)$$

where $S_{12} = 3 \underline{\sigma}_1 \cdot \underline{x} \underline{\sigma}_2 \cdot \underline{x} - \underline{\sigma}_1 \cdot \underline{\sigma}_2$. To obtain Eq. (3-7), I have used also $\underline{\sigma}_1 \cdot \underline{\sigma}_2 | 0 \rangle = | 0 \rangle$, which follows because $\underline{\sigma}_1 \cdot \underline{\sigma}_2$ has the eigenvalue one for a triplet state.

Combining Eqs. (3-1), (3-3), and (3-7), one obtains the closure approximation expression for T:

$$T = 1 - \frac{1}{3} \langle 0 | j_0(vx) | 0 \rangle - \frac{1}{6} \langle 0 | j_2(vx) S_{12} | 0 \rangle. \quad (3-8)$$

In terms of $u(r)$ and $w(r)$ [see Eq. (2-23)], note the last two terms of the following equation:

$$\begin{aligned} \langle 0 | j_0(\nu x) | 0 \rangle &= \int_0^\infty j_0(\nu x) \left\{ [u(r)]^2 + [w(r)]^2 \right\} dr \\ &= G_a(\nu); \end{aligned} \quad (3-9)$$

then to simplify this expression, one uses Eq. (2-25) to obtain

$$\begin{aligned} \langle 0 | j_2(\nu x) S_{12} | 0 \rangle &= \int_0^\infty j_2(\nu x) 2w \left[\sqrt{8}u - w \right] dr \\ &= \sqrt{8} G_b(\nu). \end{aligned} \quad (3-10)$$

J. A. McIntyre has tabulated G_a and G_b for several models of the deuteron.¹⁴ Using his values, one gets a value for T between 0.85 and 0.81, where the first value is for a deuteron held together by a potential with a hard core and the second is for a Yukawa potential. The last two terms in Eq. (3-8) are of the order of 15% and 2% of the first term, respectively. Therefore, the inclusion of the d state in the deuteron wave function leads to a correction only of the order of 2%.

B. Corrections to the Closure Approximation

In this section, corrections to the closure-approximation value of T are obtained by a method due to Placzek. I shall follow an exposition of Placzek's method¹² according to Fowler and Watson.¹⁵

To begin, let us note the following operator identity:

$$e^{i\mathbf{v} \cdot \mathbf{x}/2} \mathbf{p} e^{-i\mathbf{v} \cdot \mathbf{x}/2} = \mathbf{p} - \mathbf{v}/2. \quad (3-11)$$

Equation (3-11) may be interpreted by saying that $e^{i\mathbf{v} \cdot \mathbf{x}/2}$ is a translation operator in momentum space. By noting the kinetic energy, p^2/M is the only momentum-dependent term in h_f and further, by noting

$$e^{i\mathbf{v} \cdot \mathbf{x}/2} p^2 e^{-i\mathbf{v} \cdot \mathbf{x}/2} = (\mathbf{p} - \mathbf{v}/2)^2 = p^2 - \mathbf{v} \cdot \mathbf{p} + v^2/4, \quad (3-12)$$

one obtains:

$$e^{i\mathbf{v} \cdot \mathbf{x}/2} h_f e^{-i\mathbf{v} \cdot \mathbf{x}/2} = h_f - \mathbf{v} \cdot \mathbf{p} + v^2/4 \quad (3-13)$$

and

$$\begin{aligned} e^{i\mathbf{v} \cdot \mathbf{x}/2} \delta(m_\pi - v - v^2/4M + \epsilon_0 - h_f) e^{-i\mathbf{v} \cdot \mathbf{x}/2} \\ = \delta(m_\pi - v - v^2/(2M) + \epsilon_0 - h_f + \mathbf{v} \cdot \mathbf{p}/M). \end{aligned} \quad (3-14)$$

To obtain the closure approximation and corrections to it, one simply uses the Taylor-series expansion of a delta function

$$\delta(x + y) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \cdot \delta^n(y), \quad (3-15)$$

where $\delta^n(y) = d^n \delta(y)/d^n y$ and $\delta^0(y) = \delta(y)$, to expand the left-hand side of Eq. (3-14), and inserts the result into Eq. (2-39). The resulting expansion for R_D (from which we omit $|a|^2$ because it cancels in $T = R_D/R_H$) is

$$R_D = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\text{pol}} \int d^3 v \delta^n(m_{\pi} - v - v^2/2M) \cdot \left[M_{11}(n) - M_{12}(n) \right], \quad (3-16)$$

where

$$M_{11}(n) \equiv \langle 0 | \sigma_1 \cdot \hat{e} (\epsilon_0 - h_f + \underline{v} \cdot \underline{p}/M)^n \sigma_1 \cdot \hat{e} | 0 \rangle \quad (3-17)$$

and

$$M_{12}(n) \equiv \langle 0 | \sigma_1 \cdot \hat{e} e^{i \underline{v} \cdot \underline{x}} (\epsilon_0 - h_f - \underline{v} \cdot \underline{p}/M)^n \sigma_2 \cdot \hat{e} | 0 \rangle. \quad (3-18)$$

The first term of the expansion Eq. (3-7) is just the closure approximation. Some of the lower-order matrix elements are shown in Table I.

Some of the matrix elements in Eq. (3-16) are large or infinite because of their bad behavior at short distances or for high momenta. As can be seen from Table I, some terms that are large in $M_{11}(n)$ tend to be canceled by large terms in $M_{12}(n)$. For example, in $M_{11}(2)$ there is $\frac{2}{3} \langle 0 | v_0^2 | 0 \rangle$, which is large, but this term tends to be canceled by $\frac{2}{3} \langle 0 | v_0^2 e^{i \underline{v} \cdot \underline{x}} | 0 \rangle$ in $M_{12}(2)$. Because the major contribution to these two matrix elements comes from short distances, to estimate the second matrix element, one can set $e^{i \underline{v} \cdot \underline{x}}$ equal to 1. But there are other matrix elements, for which large terms in $M_{11}(n)$ are not canceled by terms in $M_{12}(n)$, that would be zero if $e^{i \underline{v} \cdot \underline{x}}$ were 1. Such terms are not infinite but, nevertheless, they may be large owing to their behavior at short distances. To determine whether they are or are not large, they must be examined in detail.

To see what to do about large and infinite matrix elements in Eq. (3-16), let us consider the following simple example:

Table I. The matrix elements M_{11} and M_{12}
[See Eqs. (3-17) and (3-18).]

n	$M_{11}(n)$	$M_{12}(n)$
0	1	$\langle 0 e^{i \underline{v} \cdot \underline{x}} \sigma_1 \cdot e \sigma_2 \cdot e 0 \rangle$
1	$\frac{2}{3} \langle 0 v_0 0 \rangle$	$\frac{v^2}{2M} \langle 0 e^{i \underline{v} \cdot \underline{x}} \sigma_1 \cdot e \sigma_2 \cdot e 0 \rangle + \frac{2}{3} \langle 0 e^{i \underline{v} \cdot \underline{x}} v_0 0 \rangle$
2	$\frac{1}{3} v^2 \langle 0 \left(\frac{\underline{p}}{M} \right)^2 0 \rangle + \frac{2}{3} \langle 0 v_0^2 0 \rangle$	$-\frac{1}{3} \langle 0 \frac{v \cdot \underline{p}}{M} e^{i \underline{v} \cdot \underline{x}} \frac{v \cdot \underline{p}}{M} 0 \rangle + \frac{2}{3} \langle 0 v_0^2 e^{i \underline{v} \cdot \underline{x}} 0 \rangle$ $+ \frac{2}{3} \frac{v^2}{M} \langle 0 v_0 e^{i \underline{v} \cdot \underline{x}} 0 \rangle + \frac{2}{3} \langle 0 e^{i \underline{v} \cdot \underline{x}} \left[\frac{v \cdot \underline{p}}{M}, v_0 \right] 0 \rangle$
3	$\langle 0 \frac{1}{2} \left[\left[h_0, \frac{v \cdot \underline{p}}{M} \right], \frac{v \cdot \underline{p}}{M} \right] 0 \rangle$	$-\frac{1}{3} \langle 0 \frac{1}{2} \left[\left[h_0, \frac{v \cdot \underline{p}}{M} \right], \frac{v \cdot \underline{p}}{M} \right] e^{i \underline{v} \cdot \underline{x}} 0 \rangle$ $-\frac{v^2}{2M} \cdot \frac{1}{3} \cdot \langle 0 \frac{v \cdot \underline{p}}{M} e^{i \underline{v} \cdot \underline{x}} \frac{v \cdot \underline{p}}{M} 0 \rangle$ $+ \frac{v^2}{2M} \cdot \frac{1}{3} \cdot \langle 0 \left[h_0, \frac{v \cdot \underline{p}}{M} \right] e^{i \underline{v} \cdot \underline{x}} 0 \rangle$
4	$\langle 0 \left(\frac{v \cdot \underline{p}}{M} \right)^4 0 \rangle$ $- \langle 0 \left(\left[h_0, \frac{v \cdot \underline{p}}{M} \right] \right)^2 0 \rangle$	$\frac{1}{3} \langle 0 \left(\frac{v \cdot \underline{p}}{M} \right)^2 e^{i \underline{v} \cdot \underline{x}} \left(\frac{v \cdot \underline{p}}{M} \right)^2 0 \rangle$ $+ \frac{1}{3} \langle 0 \left(\left[h_0, \frac{v \cdot \underline{p}}{M} \right] \right)^2 e^{i \underline{v} \cdot \underline{x}} 0 \rangle$

$$\begin{aligned}
 & \int_0^{\infty} v^2 dv \left\langle 0 \left| \delta\left(v + \frac{p^2}{M} - m_{\pi}\right) \right| 0 \right\rangle \\
 &= \left\langle 0 \left| \left(m_{\pi} - \frac{p^2}{M}\right)^2 H\left(m_{\pi} - \frac{p^2}{M}\right) \right| 0 \right\rangle \\
 &= m_{\pi}^2 \left\langle 0 \left| \left(1 - \frac{2}{m_{\pi}} \frac{p^2}{M} + \frac{1}{m_{\pi}^2} \frac{p^2}{M^2}\right) \cdot H\left(m_{\pi} - \frac{p^2}{M}\right) \right| 0 \right\rangle,
 \end{aligned}
 \tag{3-19}$$

where H is the Heaviside function. The value of H is 1 if its argument is positive, and zero if its argument is negative. It is usual that, if a delta function is a function of two variables and if the delta function is used to integrate over one of the variables, the delta function sets limits on the other variable. The inclusion of H in Eq. (3-19) is just an expression, in operator form, of the limit placed on the integration over momentum in the matrix element.

In Eq. (2-38), the delta function, when used to perform the integration over v , sets a limit on the sum over final states. Only final states with an energy less than $m_{\pi} + \epsilon_0$ are permitted. As is discussed in the introduction (Sec. 1. B), one of the assumptions made in the closure approximation is that the phase-space factor, ρ_f , in the expression for the absorption rate, Eq. (1-10), can be replaced by an average value, $\bar{\rho}_f$, and the other is that one can extend the sum over final states, $\sum_f$, to all states and not to just those states that are consistent with energy conservation. This last assumption is equivalent to setting the Heaviside function, which should be in M_{11} and M_{12} , Eq. (3-17) and Eq. (3-18), identically equal to 1. While setting $H \equiv 1$ is justified for the first few matrix elements, it is not justified in the higher-order ones. It is just the omission of H from M_{11} and M_{12} that allows them to be large or infinite. The Heaviside

function would cut them off at short distances and high momenta. However, the explicit inclusion of the Heaviside function in the matrix elements makes them very difficult to evaluate accurately.

If one evaluates the terms in Eq. (3-19) with a Hulthén wave function, for example, one finds that the inclusion of the Heaviside function in the first two terms does not affect their value much, but that its inclusion in the third term reduces its value markedly, i. e., $\langle 0|H|0\rangle \approx \langle 0|0\rangle = 1$ and $\langle 0|p^2 H|0\rangle \approx \langle 0|p^2|0\rangle = 1$, but $\langle 0|p^4|0\rangle \gg \langle 0|p^4 H|0\rangle \approx [\langle 0|p^2|0\rangle]^2$. Therefore, if the second term in Eq. (3-10) is of the order of 20% of the first term, the third term will be of the order of 1% of the first term. To evaluate Eq. (3-19) to 5% accuracy, it is necessary to keep only the first two terms and not the third term, which is difficult to evaluate because it must contain the Heaviside function.

The conclusion I would like to draw is that, if Eq. (3-16) is treated properly either by including a Heaviside function in the matrix elements initially or by summing singular matrix elements and reinterpreting them to get the Heaviside function, then (as in the simple example just discussed) terms containing operators of order p^4 will be of the order of the square of terms that contain operators of the order of p^2 ; and that, if terms containing operators of the order of p^2 are of the order of 20% of the first terms in Eq. (3-16), one need keep only terms containing operators of order p^2 (and lower) to evaluate Eq. (3-16) with 5% accuracy. To tell what the order of an operator is (i. e., how singular the operator is) in one of the matrices in Eq. (3-16), it is necessary to look at the term in detail, but it is not difficult to tell.

The absorption rate, R_D , including corrections that include operators of order p^2 , is

$$\begin{aligned}
 R_D = \sum_{\text{pol}} \int d^3 v \left\{ \delta(a) \left[1 - \langle 0 | e^{i \underline{v} \cdot \underline{x}} \underline{\sigma}_1 \cdot \underline{\hat{e}} \underline{\sigma}_2 \cdot \underline{\hat{e}} | 0 \rangle \right] \right. \\
 + \delta^1(a) \frac{2}{3} \left[\langle 0 | v_0 | 0 \rangle - \langle 0 | v_0 e^{i \underline{v} \cdot \underline{x}} | 0 \rangle \right] \\
 \left. + \frac{1}{2!} \delta^2(a) \frac{v^2}{3} \left[\langle 0 | \left(\frac{\underline{p}}{m} \right)^2 | 0 \rangle - \frac{1}{3} \langle 0 | e^{i \underline{v} \cdot \underline{x}} \left(\frac{\underline{p}}{m} \right)^2 | 0 \rangle \right] \right\}, \quad (3-20)
 \end{aligned}$$

where

$$a = m_\pi - v - v^2/(2m). \quad (3-21)$$

By examining Table I, one sees that there are several other terms that contain operators of the same order as p^2 . These terms give only a small correction to Eq. (3-20) and, for the sake of simplicity, are omitted.

The first term in Eq. (3-20) is the closure approximation expression for R_D and is evaluated in Sec. 3. A. The other two terms are corrections to the closure approximation. I have evaluated the corrections by using a Hulthén wave function for the deuteron; thus, $u(r) = N[e^{-\alpha r} - e^{-\beta r}]$, and $w(r) = 0$ [for notation see Eq. (2-23)]. Here, N is simply a normalization constant. To evaluate the matrix element of the first correction term I set $v_0 = p^2/m - \epsilon_0$. Then

$$\langle 0 | v_0 | 0 \rangle - \langle 0 | v_0 e^{i \underline{v} \cdot \underline{x}} | 0 \rangle \approx \epsilon_0. \quad (3-22)$$

For β between 5α and 7α , the result is about the same. The expectation value of v_0 is about -10 MeV or $5\epsilon_0$. The second matrix element in Eq. (3-22) is therefore about four-fifths of the first. The evaluation of the matrix elements of the second correction term in Eq. (3-20) is similar to that of the first. The result is

$$\langle 0 | \left(\frac{\underline{p}}{M} \right)^2 | 0 \rangle - \frac{1}{3} \langle 0 | e^{i \underline{v} \cdot \underline{x}} \left(\frac{\underline{p}}{M} \right)^2 | 0 \rangle \approx 3 \frac{\epsilon_0}{M}. \quad (3-23)$$

To evaluate the delta functions in Eq. (3-20), let us simply ignore $v^2/(2m)$, which is not a bad approximation. Then,

$$\int v^2 \delta(m_\pi - v) dv = m_\pi^2, \quad (3-24a)$$

$$\frac{2}{3} \int v^2 \delta'(m_\pi - v) dv = \frac{4}{3} m_\pi, \quad (3-24b)$$

and

$$\frac{1}{6} \int v^4 \delta''(m_\pi - v) dv = 2 m_\pi^2. \quad (3-24c)$$

From Eqs. (3-20), (3-22), (3-23), and (3-24), one sees that relative to the first term the first correction term is

$$(m_\pi)^{-2} (m_\pi \frac{4}{3} \epsilon_0) = \frac{4}{3} \frac{\epsilon_0}{m_\pi} \approx -0.02, \quad (3-25)$$

and the second is

$$(m_\pi)^{-2} (2m_\pi^2 \frac{1}{3} \frac{\epsilon_0}{m}) = 6 \frac{\epsilon_0}{m} \approx 0.02. \quad (3-26)$$

Therefore, the corrections to the closure approximation are quite small and tend to cancel. Actually the correction due to the interaction between the two final neutrons, if one chooses $V = (P_S - P_T)v_0$ or $V = v_0$ rather than $V = P_S v_0$, is -0.04, or zero, rather than -0.02. The correction to the closure approximation from matrix elements that contain operators of the order of p^2 is about 2%. The correction from the rest of the matrix elements is probably also about 2%. Therefore the closure approximation is good to about 5%.

4. RESULT AND SUMMARY OF EXPECTED ERRORS

The error due to the model, as discussed in Sec. I.B., is probably about 5%. The error due to the closure approximation as estimated by the method of Placzek is about 5%. The error due to the uncertainty in the deuteron wave function is about 5%. Therefore the error in the calculation is about 10%.

The result of the calculation is

$$T = 0.83 \pm 0.08.$$

The result is in substantial disagreement with experiment. The experimental result obtained by J. Ryan is 0.41 ± 0.08 . J. Ryan has also drawn my attention to an experiment by Kuehner, Merrison, and Tornabene, which gives a value for T of 0.51 ± 0.08 .¹⁶

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APPENDIX II-1

In this Appendix I would like to show how I have evaluated Eqs. (3-17) and (3-18) to obtain Table I.

For the usual unpolarized deuterium target, the angular distribution of decay photons for the reaction $\pi^- + d \rightarrow 2n + \gamma$ is isotropic and the photons are unpolarized, i. e., randomly polarized. This means that in evaluating Eqs. (3-17) and (3-18), one can average over the variables of the decay photon, $\underline{\nu}$ and $\hat{\epsilon}$, independently of the variables of the deuteron. But one cannot necessarily average over $\underline{\nu}$ and $\hat{\epsilon}$ separately since they satisfy the condition $\underline{\nu} \cdot \hat{\epsilon} = 0$. The dependence on $\hat{\epsilon}$ of the matrix elements is of the form $\underline{A} \cdot \hat{\epsilon} \underline{B} \cdot \hat{\epsilon}$, where $\underline{A}$ and $\underline{B}$ are two operators that are vectors. To average over $\hat{\epsilon}$ and $\underline{\nu}$, one can first use the identity

$$\frac{1}{2} \sum_{\text{pol}} \underline{A} \cdot \hat{\epsilon} \underline{B} \cdot \hat{\epsilon} = \frac{1}{2} [\underline{A} \cdot \underline{B} - \underline{\nu} \cdot \underline{A} \underline{\nu} \cdot \underline{B}] \quad (\text{A-1})$$

to eliminate $\hat{\epsilon}$ and to satisfy the condition $\underline{\nu} \cdot \hat{\epsilon} = 0$, and then one can average over the direction of $\underline{\nu}$. Note that if $\underline{A}$ and $\underline{B}$ contain the vector $\underline{\nu}$ an odd number of times, the matrix element becomes zero when averaged over the direction of $\underline{\nu}$. If the matrix element contains $e^{i\underline{\nu} \cdot \underline{x}}$, Eq. (3-5) is useful. To average over $\underline{\nu}$ and $\hat{\epsilon}$ in the way just mentioned usually gives only a few extra terms not obtained if one averages over $\hat{\epsilon}$ and $\underline{\nu}$ separately. These extra terms, although they have a complicated structure and are difficult to evaluate exactly, have a negligible value. Therefore, for the evaluation of the matrix elements, I shall simply average over the directions of $\underline{\nu}$ and $\hat{\epsilon}$ separately.

For the sake of definiteness and simplicity, I have evaluated Eqs. (3-17) and (3-18) by using a specific, simple, and reasonable form for the potential between the two final neutrons, namely $V = v_0 P_S$ [see Eq. (2-31)]. The results for other forms are similar; and the procedure to obtain them is also similar and no more difficult. Note that Placzek's method is not very useful for the phenomenological

spin-orbit force which has sometimes been used to fit nucleon-nucleon scattering experiments. It is too singular at short distances. If an operator in one of the matrix elements is singular at short distances, then to obtain the correct value of the matrix element, remember that one must include a Heaviside function, which suppresses the singular behavior of the operator at short distances and makes the matrix element small, but also makes it difficult to evaluate. For the phenomenological spin-orbit force, one must include a Heaviside function even to evaluate matrix elements that contain the potential only once. Placzek's method may be generalized to a potential containing a hard core by means of the notion of a symbolic operator. To lowest order the correction terms are the same as without a hard core.

To save on bookkeeping, it is convenient to write $h_f = h_0 - \Delta$, where $\Delta = v_0 - V$, and to treat v_0 as a central, spin-independent potential. Observe that Δ does not commute with $\underline{\sigma}_1 \cdot \hat{e}$ or $\underline{\sigma}_2 \cdot \hat{e}$ because V does not. Remember $V = v_0 P_S$, where P_S is the singlet projection operator. From the identity

$$P_S \underline{\sigma}_1 \cdot \hat{e} |0\rangle = \frac{1}{2} (\underline{\sigma}_1 \cdot \hat{e} - \underline{\sigma}_2 \cdot \hat{e}) |0\rangle, \quad (A-2)$$

which one gets by remembering that a triplet-spin state is even, a singlet-spin state is odd, and $|0\rangle$ is a triplet state, it follows that

$$\langle 0 | \underline{\sigma}_1 \cdot \hat{e} \Delta^n \underline{\sigma}_1 \cdot \hat{e} | 0 \rangle_{\hat{e}} = \frac{2}{3} v_0^n \quad (A-3)$$

and

$$\langle 0 | \underline{\sigma}_1 \cdot \hat{e} \Delta^n \underline{\sigma}_2 \cdot \hat{e} | 0 \rangle_{\hat{e}} = \frac{2}{3} v_0^n, \quad (A-4)$$

where the subscript $\hat{e}$ indicates an average over the direction of $\hat{e}$.

Before evaluating the matrix elements, let us notice several useful identities:

$$\underline{\sigma} \cdot \underline{a} = \underline{a} \cdot \underline{\sigma} + i \underline{\sigma} \cdot \underline{a} \times \underline{\sigma} \quad (A-5)$$

from which it follows $(\underline{\sigma} \cdot \underline{e})^2 = 1$;

$$\langle \underline{x} \cdot \underline{A} \underline{x} \cdot \underline{B} \rangle_{\hat{x}} = \frac{1}{3} \underline{x}^2 \underline{A} \cdot \underline{B}, \quad (\text{A-6})$$

where the subscript $\hat{x}$ indicates an average over the direction of $\underline{x}$.

We note that

$$[h_0, \sigma_1, 2 \cdot \hat{e}] = 0 \quad (\text{A-7})$$

by assumption, and

$$(\epsilon_0 - h_0) |0\rangle = \langle 0 | (h_0 - \epsilon_0) = 0, \quad (\text{A-8})$$

from which it follows that for an arbitrary operator, θ ,

$$(h_0 - \epsilon_0) \theta |0\rangle = [h_0, \theta] |0\rangle. \quad (\text{A-9})$$

Let us start with the $M_{11}(n)$ terms.

For $n = 1$,

$$\langle 0 | \sigma_1 \cdot e (\epsilon_0 - h_0 + \Delta + \frac{\underline{v} \cdot \underline{p}}{M}) \sigma_1 \cdot e | 0 \rangle$$

$$= \frac{2}{3} \langle 0 | v_0 | 0 \rangle.$$

The first two terms in the parenthesis are zero by Eqs. (A-7) and (A-8); the third is equal to the second line by Eq. (A-3); and the last becomes zero when one averages over the direction of $\underline{v}$.

For $n = 2$,

$$\langle 0 | \sigma_1 \cdot e (\epsilon_0 - h_0 + \Delta + \frac{\underline{v} \cdot \underline{p}}{M})^2 \sigma_1 \cdot e | 0 \rangle$$

$$= \frac{2}{3} \langle 0 | v_0^2 | 0 \rangle + \langle 0 | (\sigma_1 \cdot e)^2 (\frac{\underline{v} \cdot \underline{p}}{M})^2 | 0 \rangle. \quad (\text{A-8})$$

Again I have used Eqs. (A-3), (A-7), and (A-8). An average over the direction of $\underline{v}$ yields the result shown in Table I.

For $n > 2$, I have not evaluated any terms that contain Δ . They all contain operators that are more singular than the first power of the kinetic energy, i.e., p^2 , and I have argued (Sec. 3. B) that I can get an estimate of the error in the closure approximation by keeping terms to order p^2 only. That there are terms of higher order in the table is merely for the sake of illustrating the discussion of Placzek's method made in Sec. 3, B. To ignore terms that contain Δ is equivalent for computation to setting it equal to zero.

For $n = 3$, let us omit $\sigma_1 \cdot \hat{e}$, since $(\sigma_1 \cdot \hat{e})^2 = 1$:

$$\begin{aligned} & \langle 0 | (\epsilon_0 - h_0 + \frac{\mathbf{v} \cdot \mathbf{p}}{M})^3 | 0 \rangle \\ &= \langle 0 | \frac{\mathbf{v} \cdot \mathbf{p}}{M} (\epsilon_0 - h_0) \frac{\mathbf{v} \cdot \mathbf{p}}{M} | 0 \rangle \\ &= \frac{1}{2} \langle 0 | \left[[h_0, \frac{\mathbf{v} \cdot \mathbf{p}}{M}], \frac{\mathbf{v} \cdot \mathbf{p}}{M} \right] | 0 \rangle . \end{aligned}$$

For the last step, Eq. (A-9) is used.

For $n = 4$, the evaluation is straightforward.

Let us now turn to the evaluation of $M_{12}(n)$. First notice the identity

$$e^{i\mathbf{v} \cdot \mathbf{x}} (\epsilon_0 - h_f - \frac{\mathbf{v} \cdot \mathbf{p}}{M}) = (\epsilon_0 - h_f + \frac{\mathbf{v} \cdot \mathbf{p}}{M}) e^{i\mathbf{v} \cdot \mathbf{x}} .$$

For $n = 1$,

$$\begin{aligned} & \langle 0 | \sigma_1 \cdot e^{i\mathbf{v} \cdot \mathbf{x}} (\epsilon_0 - h_f - \frac{\mathbf{v} \cdot \mathbf{p}}{M}) \sigma_2 \cdot e | 0 \rangle \\ &= \frac{1}{2} \langle 0 | \sigma_1 \cdot e \left[\frac{\mathbf{v} \cdot \mathbf{p}}{M}, e^{i\mathbf{v} \cdot \mathbf{x}} \right] \sigma_2 \cdot e | 0 \rangle \\ &= \frac{\mathbf{v}^2}{2M} \langle 0 | \sigma_1 \cdot \hat{e} \sigma_2 \cdot e e^{i\mathbf{v} \cdot \mathbf{x}} | 0 \rangle . \end{aligned}$$

The second line is obtained by using Eqs. (A-10), (A-7), and (A-8).

For $n = 2$, the evaluation is the same as for $M_{12}(1)$:

$$\begin{aligned} & \langle 0 | \underline{\sigma}_1 \cdot \hat{e} e^{i \underline{v} \cdot \underline{x}} (\epsilon_0 - h_f - \frac{\underline{v} \cdot \underline{p}}{M})^2 \sigma_2 \cdot e | 0 \rangle \\ &= \langle 0 | \sigma_1 \cdot \hat{e} (\epsilon_0 - h_f + \frac{\underline{v} \cdot \underline{p}}{M}) e^{i \underline{v} \cdot \underline{x}} (\epsilon_0 - h_f - \frac{\underline{v} \cdot \underline{p}}{M}) \sigma_2 \cdot e | 0 \rangle \\ &= \frac{2}{3} \langle 0 | v_0^2 e^{i \underline{v} \cdot \underline{x}} | 0 \rangle - \langle 0 | \sigma_1 \cdot e \sigma_2 \cdot e \frac{\underline{v} \cdot \underline{p}}{M} e^{i \underline{v} \cdot \underline{x}} \frac{\underline{v} \cdot \underline{p}}{M} | 0 \rangle . \end{aligned}$$

If one averages over the direction of $\hat{e}$ and uses $\underline{\sigma}_1 \cdot \underline{\sigma}_2 | 0 \rangle = | 0 \rangle$, which follows because $| 0 \rangle$ is a triplet-spin state, one obtains the result seen in Table I. The evaluation of $M_{12}(3)$ and $M_{12}(4)$ is straightforward.

$M_{11}(0)$ and $M_{12}(0)$ are evaluated in Sec. 3. A.

APPENDIX II-2

I think that there is no question but that absorption takes place from an atomic s state. Even the mechanism for how it gets to an s state is fairly well understood. For the meson to be captured from an s state it must not decay before it gets to the s state--the lifetime of a π^- is 2.5×10^{-8} second.

By the usual mechanism of ionization the meson is slowed from several MeV to a few eV in $\approx 10^{-12}$ second.¹⁸ Of absorption processes, the one that has the most chance of taking place from a p state is $\pi^- + 0 \rightarrow n + n$. The rate for this process can be inferred by detailed balancing from the total cross section for $p + p \rightarrow d + \pi^+$, which is of the form $\sigma = \eta[a + b\eta^2]$, where $\eta = k/m_\pi =$ (momentum of meson)/(mass of meson), $a \approx 14 \times 10^{-27} \text{ cm}^2$, and $b \approx 1 \times 10^{-27} \text{ cm}^2$.² The ratio of p-wave absorption to s-wave absorption for $\pi^- + d \rightarrow n + n$ is $b\eta^2/a = 14(k^2/12m_\pi)/m_\pi$, which for a few eV is small.

The time it would take a meson of a few MeV to be absorbed rather than slowed down, I estimate by the formula $t = (v_\pi \sigma \rho)^{-1}$, where v_π is the velocity, σ is the cross section for absorption, and $\rho \approx 10^{22} \text{ cm}^{-3}$ is the density of hydrogen or deuterium. I take $\sigma \approx (a/\eta)(e^2/\eta)$, where $a \approx 10^{-27}$ or 10^{-26} cm^2 ; the second factor represents the effect of the attractive Coulomb potential. Now $e^2 \approx 10^{-2}$ and $v_\pi = \frac{c}{c\eta} = \eta \times 3 \times 10^{10} \text{ cm/sec}$; therefore, $t \approx 10^{-4} v_\pi$, where v_π must be given in cm/sec. Now $v_\pi \approx 10^6$ (energy of meson in electron volts)^{1/2}; therefore, slowing down of the meson takes place much faster than absorption.

A meson might be absorbed from a p state if it got stuck in one and could not get out except by absorptions. There are ample mechanisms to get a meson from a p state to an s state. For example, the rate for radiative transition from a p state to an s state is twenty times as fast as the rate for absorption in a p state.²⁰ For principal quantum numbers $n > 4$, other mechanisms are even faster than radioactive transitions.

The time to stop a meson in hydrogen has been measured to be 5×10^{-12} sec, which is about half the previously calculated time.²¹ This disagreement has led to some confusion. People have asked if the meson is really captured from an s state. I should like to point out emphatically that there is no question about the π^- meson's being captured from an s state. The disagreement raises questions only about the mechanism for the meson's getting to an s state. That is, the meson gets to an s state even faster than allowed by previous mechanisms.

Now, after some controversy, there seems to be a consistent scheme for the slowing down of a meson in hydrogen (and deuterium).²² From several MeV to a few eV, in a time 10^{-12} sec, the meson loses energy by ionizing hydrogen atoms and molecules. With a few eV it is captured to form (π^-, H_2^+) , which quickly dissociates ($< 10^{-13}$ sec) to form (π^-, p) in a state with molecular dimensions, i. e., principal quantum number $n \approx 30$. From $n \approx 30$ to $n \approx 15$ (2×10^{-12} sec), the (π^-, p) ionizes H_2 molecules by collisions. From $n \approx 15$ to $n \approx 4$ (10^{-12} sec), the (π^-, p) ionizes molecules by the Auger effect coupled with the Stark effect, i. e., collisions. Finally the meson is absorbed. For $n \approx 15$ to $n \approx 4$, there has been some controversy about the importance of the Stark effect. In his initial calculation, Wightman did not use the Stark effect. Mandansky evidently commented that the Stark effect might account for the difference between Wightman's time for the meson to slow down and the measured time. Day, Sucher, and Snow²³ did a calculation which they asserted showed that the Stark effect removes the discrepancy. Adair²⁴ objected to the validity of the calculation by Day et al. but not to the possibility that the Stark effect might remove the discrepancy. Subsequently, more careful calculations have apparently shown agreement between experiment and theory.²²

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