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NEUTRON RESONANCE ABSORPTION

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IDAHO NUCLEAR CORPORATION
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U. S. ATOMIC ENERGY COMMISSION

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NEUTRON RESONANCE ABSORPTION

by

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IDAHO NUCLEAR CORPORATION

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ABSTRACT

An investigation of the solution to the slowing down equations for a homogeneous mixture of substances is presented. The slowing down equation can be reduced to a system of first order differential equations under rather mild assumptions. The assumptions do not limit the physical phenomena associated with resonance absorption. Flux oscillations and resonance shadowing are taken into account by the form of the solution. Furthermore, the solution is amenable to numerical calculation and has the same form as the exact solution for hydrogen.

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I. Introduction

The theory of resonance absorption has progressed in the past decade from little more than an empirical art to a well formulated science. The marriage between experiment and theory is more than satisfactory. However, there remains much to be completed and many loose ends brought together before the subject becomes academic.

E. P. WIGNER et al. (1955) in a pioneering paper laid the supporting framework for the physical theory while on the other hand H. BETHE (1937) and G. PLACZEK (1946) formulated and solved what have now become the classical mathematical solutions to the slowing down equation.

Today, many mathematical solutions, exact within the limits of their assumptions, exist to the slowing down equation. The works of R. MARSHAK (1947) and BOFFI (1960, 1961, 1965) give a concise review of the current mathematical techniques. Also, an excellent exposition of the physical and mathematical formulation of resonance absorption is given by L. DRESNER (1960).

Due to the nature of the slowing down equation, the homogeneous mixture and the pure substance of infinite extent seem to be the most tractable from the mathematical point of view. Hence, many investigators have found various solutions for these cases. However, few, if any, of the forms of the general solution to the slowing down equation are suitable for numerical evaluation. Inherently interesting in themselves, these solutions prove cumbersome when applied specifically to absorption by resonances.

WEINBERG and WIGNER (1958) have placed much emphasis on the physical aspect of resonance absorption, and in a paper by the same authors (1957) they showed that for a pure substance second order corrections to Wigner's approximate solution involve flux oscillations occurring in the resonance. Concurrently, CORNGOLD (1957) showed that Weinberg and Wigner's heuristic derivation of the governing equation for the collision density could be obtained straightforwardly from the general Boltzman equation. In the same paper Corngold derived an equation governing the homogeneous mixture of absorbing elements. Correspondingly, HASEGAWA (1959) followed through the derivation for the heterogeneous system. The work of the above authors points out the importance of the Placzek oscillations which occur as a result of anomalous scattering or absorption.

SPINNEY (1957) and CORNGOLD (1957) have presented a solution for the case of a homogeneous mixture of hydrogen and a heavy absorber. Originally, a solution was presented to the same problem by PLACZEK (1946). We shall attempt to carry this formulation one step further. By two rather mild restrictions, a solution will be presented for a homogeneous mixture of arbitrary masses. In addition, a solution will be given for a pure substance which bears an interesting comparison with the solution for hydrogen given by WEINBERG and WIGNER (1958).

Our main attention will be placed on the actual physical process of absorption in resonances. To this end we shall design a solution which incorporates all the physical aspects of the process as well as being easily handled for numerical evaluation.

It is our purpose here to develop a method by which absorption by resonances can be studied accurately. Furthermore, the method shows

hope of providing an investigation into four phases of resonance absorption: 1) oscillations produced in the flux as a result of the resonance, 2) the shadowing of one resonance by an adjacent higher energy resonance, 3) studies of neutron memory produced by absorptions at a higher energy, 4) interference effects produced by overlapping resonances. It would seem appropriate now that such detail structure be studied and carried to completion.

The work reported here was initiated several years ago while the author was a summer employee with Phillips Petroleum Co. The work was completed in July of 1965. Since that time a computer code, GAROL, was issued which allows a more detailed study as suggested above. In addition, it treats heterogeneous systems.

II. General Slowing Down Equations

A. A Homogeneous Mixture

The integral equation governing the distribution of neutrons for a homogeneous substance composed of n constituents is

$$\Sigma_t(E)\phi(E) = \sum_{i=1}^n \int_E^{\frac{E}{1-\alpha_i}} \Sigma_{si}(E')\phi(E') \frac{dE'}{\alpha_i E'} \quad (1)$$

where $\Sigma_t(E)$ is the total cross section of the mixture;

$$\Sigma_t(E) = \sum_{j=1}^n \Sigma_{tj}(E) \quad (2)$$

For a homogeneous mixture of two substances

$$\Sigma_t(E)\phi(E) = \int_E^{\frac{E}{1-\alpha_1}} \Sigma_{s1}(E')\phi(E') \frac{dE'}{\alpha_1 E'} + \int_E^{\frac{E}{1-\alpha_2}} \Sigma_{s2}(E')\phi(E') \frac{dE'}{\alpha_1 E'} \quad (3)$$

where $\Sigma_t = \Sigma_{t1}(E) + \Sigma_{t2}(E)$. With the slowing down equation expressed in this form, the ZUT program effectively calculates the neutron flux, $\phi(E)$, by numerical integration. However, using this scheme the oscillations which are present in $\phi(E)$ across the resonance cannot be calculated. These oscillations appear, mathematically speaking, as discontinuities in the derivatives of the neutron flux. Only in the case of hydrogen is the neutron flux and its derivatives continuous over the entire energy range.

It is well known that the absorption cross section interspaced between resonances is inversely proportional to the neutron velocity.

We may therefore expect that the neutron flux will not have a true $1/E$ distribution, but may deviate slightly from its asymptotic form. The solution to the slowing down equation for a pure substance for this case has been given by MARSHAK (1947) and PLACZEK (1946). For a mixture of substances with a $1/v$ absorption cross section the reader is referred to BOFFI (1965). Therefore, we shall incorporate a completely arbitrary source function into the following formulation.

It is well known that differentiation of Eq. (1) yields a functional equation which relates the flux at E to the flux at some higher energy E' . The resulting functional equation is not directly amenable to numerical techniques; however, BOFFI (1965) has shown that a solution is representable in terms of iterated kernels.

It is our purpose here to reduce Eq. (1) to a form that can be easily handled by a computer and to present an exact solution to the extent that our assumptions apply. To facilitate the discussion, we shall do so in two cases: 1) for a pure substance, and 2) for a mixture of two constituents.

B. Pure Substance Slowing Down Equation

Let us assume that there is an energy E_1 such that for $E > E_1$, $\Sigma_a(E) = 0$. Furthermore, let $\phi_0(E)$ be the neutron flux for $E > E_1$, which has a known functional form.

We shall now divide the resonance region, $E_2 < E < E_1$, into collision intervals, the first of which is $(1-\alpha)E_1$, the second $(1-\alpha)^2E_1$, the N th $(1-\alpha)^N E_1 = E_2$.

Denote by $F_1(E) = \Sigma_t(E)\phi_1(E) \dots$, $F_N(E) = \Sigma_t\phi_N(E)$ the collision density in the 1st, $\dots$, Nth collision interval, and $F_0(E) = \Sigma_t(E)\phi_0(E)$.

The integral equation (1) for a pure substance (i.e., $n = 1$) can now be separated into N differential equations corresponding to the N collision intervals. However, instead of mathematically carrying out the conversion process, let us derive the governing equations from the first principles. We shall do so for the 1st and Nth collision interval. To accomplish this, let us define the following functions:

$F(E)dE$ - the total number of neutron-nuclei interactions in the interval E to $E + dE$.

$F_N(E)dE$ - the total number of neutron-nuclei interactions in the interval E to $E + dE$, $(1-\alpha)^N E_1 < E < (1-\alpha)^{N-1} E_1$,
 $N = 1, 2, \dots, N$.

$K_\alpha(E' \rightarrow E)dE$ - the probability of a neutron-nuclei interaction at E' will produce in one collision a neutron in the energy interval E to $E + dE$. For isotropic scattering, this function has the well known form $dE/\alpha E'$.

If $\Sigma_t(E)$ denotes the cumulative probability of a neutron-nuclei interaction at E , then $\Sigma_s(E)/\Sigma_t$ denotes the probability that the interaction will be a scattering collision. Therefore, the product $F(E')dE' \cdot \Sigma_s(E')/\Sigma_t(E') \cdot K_m(E' \rightarrow E)dE$ represents the scattering collisions at E' which result in neutrons in the energy interval E to $E + dE$. By the conservation of neutrons, the number of neutrons entering the interval E to $E + dE$ must exactly equal those absorbed in E to $E + dE$ in addition to those scattered out of E to $E + dE$.

For the first collision interval $(1-\alpha)E_1 < E < E_1$, the contribution from scattering collisions at $E' > E$ can arise from two sources, scatterings from energies above E to E_1 , and scattering from energies above E_1 but less than $E/(1-\alpha)$.

Therefore, equating the number of neutrons entering E to $E + dE$ by scattering collision above E , say at E' , to the number of neutrons absorbed in, and scattered out of, E to $E + dE$ we have

$$F_1(E)dE = \int_E^{E_1} F_1(E') \frac{\Sigma_s(E')}{\Sigma_t(E')} K_\alpha(E' \rightarrow E) dE dE' + \int_E^{\frac{E}{1-\alpha}} F_0(E') \frac{\Sigma_s(E')}{\Sigma_t(E')} K_\alpha(E' \rightarrow E) dE dE', \quad (1-\alpha)E_1 < E < E_1 \quad (4)$$

In a similar manner for the N_{th} collision interval, $(1-\alpha)^N E_1 < E < (1-\alpha)^{N-1} E_1$, scattering collisions from the $(N-1)^{th}$ collision interval and the N^{th} collision interval may contribute to the number of neutrons entering the interval E to $E + dE$. However, scattering collisions above $E/(1-\alpha)$ cannot, by first collisions, contribute to the number of neutrons entering the interval E to $E + dE$. We shall see that scattering collisions in $E/(1-\alpha)$ to $(1-\alpha)^{N-1} E_1$ enter as a source term as intuitively they should.

Therefore,

$$F_N(E)dE = \int_E^{(1-\alpha)^{N-1} E_1} F_N(E') \frac{\Sigma_s(E')}{\Sigma_t(E')} K_\alpha(E' \rightarrow E) dE dE'$$

$$+ \int_{(1-\alpha)^{N-1}E_1}^{\frac{E}{1-\alpha}} F_{N-1}(E') \frac{\Sigma_s(E')}{\Sigma_t(E')} K_\alpha(E' \rightarrow E) dE dE', E_1(1-\alpha)^N < E < (1-\alpha)^{N-1}E_1 \quad (5)$$

The reader will note that the limits of integration are not both variable in E . Thus, differentiation of Eq. (5) does not give a functional equation for $F_N(E)$, but yields a linear first order differential equation in $F_N(E)$ where $F_{N-1}(\frac{E}{1-\alpha})$ appears as a source function.

The result of differentiation with respect to E is

$$F_N'(E) + \frac{\Sigma_s(E)}{\alpha \Sigma_t(E)E} F_N(E) = \frac{\Sigma_s(\frac{E}{1-\alpha})}{\Sigma_t(\frac{E}{1-\alpha})(1-\alpha)E} F_{N-1}(\frac{E}{1-\alpha}) \quad (6)$$

where the isotropic form of $K_\alpha(E' \rightarrow E) dE$ has been used for the scattering kernel.

Solving and mutatis mutandis

$$F_N(E) = \frac{1}{(1-\alpha)^{\frac{1}{\alpha}} \frac{\alpha-1}{E^{\frac{1}{\alpha}}}} \int_{\frac{E}{1-\alpha}}^{\frac{E}{1-\alpha}} \frac{d\xi \Sigma_s(\xi) F_{N-1}(\xi)}{\Sigma_t(\xi) \xi^{\frac{1}{\alpha}}} \exp \left(- \int_E^{\xi(1-\alpha)} \frac{\Sigma_a(E'') dE''}{\Sigma_t(E'') E''} \right)^{\frac{1}{\alpha}} \\ + \frac{C_N}{E^{\frac{1}{\alpha}}} \exp \left(\int^E \frac{\Sigma_a(E'') dE''}{\Sigma_t(E'') E''} \right)^{\frac{1}{\alpha}} \quad E_1(1-\alpha)^N < E < E_1(1-\alpha)^{N-1} \quad (7)$$

where $\xi = \frac{E'}{1-\alpha}$, and where the boundary condition which appears implicitly in the integral equation is

$$\lim_{\epsilon \rightarrow 0} F_{N-1} [(1-\alpha)^{N-1}E_1 - \epsilon] = \lim_{\epsilon \rightarrow 0} F_N [(1-\alpha)^{N-1}E_1 + \epsilon]. \quad (8)$$

An interesting comparison is seen between Eq. (7) and that given by WEINBERG and WIGNER (1958) for hydrogen. One may readily observe that the function $F_{N-1}(\frac{E}{1-\alpha})$ does indeed correspond to a source function for the solution of $F_N(E)$. Also the physical reasoning applied to slowing down by hydrogen by WEINBERG and WIGNER (1958) is seen to apply here to within certain limits.

A formal solution to Eq. (7) is easily obtained. It has been assumed that $F_0(E)$ is known; we may therefore generate any other solution, $F_m(E)$, by solving successively Eq. (7) starting with $F_0(E)$. In this way we proceed across the resonance, or for that matter, to any energy.

C. Two Constituent Homogeneous Mixture

Let us now proceed to a situation which is of some practical interest. We shall confine ourselves here to a homogeneous mixture composed of two elements. A generalization to the case of more than two constituents follows directly.

Assume that there exists an energy E_1 such that for $E > E_1$, $\Sigma_a(E) = 0$, and further let us make the not too strenuous assumption that there is an integral number of collision integrals of the heavier element per collision interval of the lighter element. Let N be that number. This assumption may be removed; however, for simplicity we shall let it remain.

Under these two assumptions, it will be shown that the governing integral equation is no longer a functional equation. It will be seen that the solution to the problem is of a form which is amenable to numerical techniques.

Eq. (3) is the governing equation. We shall now modify that equation according to the previous assumptions. Rather than give a rederivation similar to that given in Section I, Part B, we shall merely list total contributions to the N^{th} collision interval by scatterings from energies above by each of the two constituents.

Consider the energy interval E to $E + dE$ such that $(1-\alpha_o)^N E_1 < E < E_1(1-\alpha_o)^{N-1}$ whereby the second of the previous assumptions $(1-\alpha_o)^N E_1$ corresponds to $(1-\alpha_1)E_1$.

The total contribution of neutrons of energy $E' > E$ which are scattered into E to $E + dE$ arise from scattering collisions above E with both the lighter and heavier substance. Therefore, we may write:

$$\begin{aligned} \text{Total contribution by scatterings} &= \int_E^{E_1(1-\alpha_o)^{N-1}} F_N(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} \\ \text{with the light element} & \\ & \times \int_{E_1(1-\alpha_o)^{N-1}}^{E_1(1-\alpha_o)^{N-2}} F_{N-1}(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} + (N-2 \text{ integrals of the same form}) \\ & + \int_{E_1}^{\frac{E}{1-\alpha_1}} F_o(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'}. \end{aligned} \quad (9)$$

In a similar manner we may write for the heavier element;

$$\begin{aligned} \text{Total contribution by scatterings} &= \int_E^{E_1(1-\alpha_o)^{N-1}} F_N(E') \frac{\Sigma_{so}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'} \\ \text{with the heavy element} & \\ & + \int_{E_1(1-\alpha_o)^{N-1}}^{\frac{E}{1-\alpha_o}} F_{N-1}(E') \frac{\Sigma_{so}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'}. \end{aligned} \quad (10)$$

Now the sum of the two contributions above must be equal to the total number of neutrons removed from E to $E + dE$ by scattering collisions in dE in addition to those neutrons removed in dE by absorptions.

Therefore, the integral equation for the N^{th} collision interval of the heavier substance contained in the first collision interval of the lighter substance becomes:

$$\begin{aligned}
 F_N(E) = & \int_E^{E_1(1-\alpha_o)^{N-1}} F_N(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} + \left\{ \begin{array}{l} N-1 \text{ integrals with constant} \\ \text{limits of integration} \end{array} \right\} \\
 & + \int_{E_1}^{\frac{E}{1-\alpha_1}} F_o(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} + \int_E^{E_1(1-\alpha_o)^{N-1}} F_N(E') \frac{\Sigma_{so}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'} \\
 & \quad \int_{E_1(1-\alpha_o)^{N-1}}^{\frac{E}{1-\alpha_o}} F_{N-1}(E') \frac{\Sigma_{so}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'} \quad (11)
 \end{aligned}$$

The number of collision intervals of the heavier element per collision interval of the lighter element, N , is readily found to be

$$N = \frac{\ln(1-\alpha_1)}{\ln(1-\alpha_o)} \quad (12)$$

Let us now write the equation for the second collision interval of the lighter constituent. We shall use " M " to denote the M^{th} collision interval of the heavier element counting from the end of the first collision interval of the lighter element, $M = 1, 2, \dots = N + 1, N + 2, \dots$

In this case

$$\Sigma_t(E) F_{N+M}(E) = \int_E^{E_1(1-\alpha_o)^{N+M-1}} F_{N+M}(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} + \left\{ \begin{array}{l} N-1 \text{ integrals with constant} \\ \text{limits of integration} \end{array} \right\}$$

$$\begin{aligned}
& + \int_{E_1(1-\alpha_o)^j}^{\frac{E}{1-\alpha_1}} F_j(E') \frac{\Sigma_{s1}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_1 E'} + \int_E^{E_1(1-\alpha_o)^{N+M-1}} F_{N+M}(E') \frac{\Sigma_{s0}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'} \\
& + \int_{E_1(1-\alpha_o)^{N+M-1}}^{\frac{E}{1-\alpha_o}} F_{N+M-1}(E') \frac{\Sigma_{s0}(E')}{\Sigma_t(E')} \frac{dE'}{\alpha_o E'} \quad (13)
\end{aligned}$$

If we define

$$\Delta(M) = \frac{(1-\alpha_o)^{N+M}}{(1-\alpha_1)} E_1 \quad N < M \leq 2N \quad (14)$$

then $\Delta(M)$ is a measure of maximum integer number of collision intervals by which scatterings in the light element can contribute to E to $E + dE$ in the $(N + M)^{th}$ collision interval of the heavier substance. To find this integer number, which is denoted by "j", we must find the least integer value of "j" such that

$$\Delta(M) > (1-\alpha_o)^j E_1 \quad (15)$$

Hence the remaining contribution to E to $E + dE$ comes from an energy E' in the interval $\frac{E}{1-\alpha_1} < E' < E_1(1-\alpha_o)^j$.

Eq. (10) may be immediately generalized to include any collision interval of the lighter element by counting "M" starting at the beginning of that interval.

The solution to the above equations can be in principle easily found. If we differentiate Eq. (9), noting that the integrals with constant limits make no contribution to the rate of change of the collision density, we obtain

$$F_N(E) = - \frac{F_N(E) \Sigma_{s1}(E)}{\Sigma_t(E) \alpha_1 E} + \frac{F_0\left(\frac{E}{1-\alpha_1}\right) \Sigma_{s1}\left(\frac{E}{1-\alpha_1}\right) (1-\alpha_1)}{\Sigma_t\left(\frac{E}{1-\alpha_1}\right) \alpha_1} - \frac{F_N(E) \Sigma_{s0}(E)}{\Sigma_t(E) \alpha_0 E} + \frac{F_{N-1}\left(\frac{E}{1-\alpha_0}\right) \Sigma_{s0}\left(\frac{E}{1-\alpha_0}\right) (1-\alpha_0)}{\Sigma_t\left(\frac{E}{1-\alpha_0}\right) \alpha_0 E} \quad (16)$$

or

$$F_N(E) + \psi(E) F_N(E) = S(0, 1-\alpha_1; N-1, 1-\alpha_0; E) \quad (17)$$

where

$$\psi(E) = \frac{\Sigma_{s0}(E)}{E \Sigma_t(E) \alpha_0} + \frac{\Sigma_{s1}(E)}{E \Sigma_t(E) \alpha_1} \quad (18)$$

and

$$S(0, 1-\alpha_1; N-1, 1-\alpha_0; E) = \frac{F_0\left(\frac{E}{1-\alpha_1}\right) \Sigma_{s1}\left(\frac{E}{1-\alpha_1}\right) (1-\alpha_1)}{\Sigma_t\left(\frac{E}{1-\alpha_1}\right) \alpha_1} + \frac{F_{N-1}\left(\frac{E}{1-\alpha_0}\right) \Sigma_{s0}\left(\frac{E}{1-\alpha_0}\right) (1-\alpha_0)}{E \Sigma_t\left(\frac{E}{1-\alpha_0}\right) \alpha_0} \quad (19)$$

An exactly analogous procedure may be used to obtain the corresponding differential equations for the 2nd, 3rd, etc. collision intervals in the lighter substance (see Appendix I).

To within a constant value we may solve for the functional form of $F_N(E)$

$$F_N(E) = \int_E^E S(0, 1-\alpha_1; N-1, 1-\alpha_0; E') \exp \left(- \int_E^{E'} \psi(E'') dE'' \right) dE' + C_N \exp \left(- \int^E \psi(E') dE' \right). \quad (20)$$

The constant C_N may be evaluated by the boundary conditions implied in the original integral equation

$$\lim_{\epsilon \rightarrow 0} [F_N E_1 (1-\alpha_0)^{N-1} + \epsilon] = \lim_{\epsilon \rightarrow 0} [F_{N-1} E_1 (1-\alpha_0)^{N-1} - \epsilon] \quad (21)$$

That the integrals with constant limits should not contribute to the solution can be seen from physical considerations. The rate of change of the function $F_N(E)$ is physically equivalent to considering how the function $F_N(E)$ changes when E changes by ΔE . Now, scattering collision in those energy intervals with constant limits of integration may produce neutrons of energy below E and $E + \Delta E$. In other words a constant supply of neutrons is produced to the energy $E + \Delta E$ by neutrons suffering a scattering collision in those energy intervals. However, this is not the case for the interval E_1 to $\frac{E}{1-\alpha_1}$ since if it is changed by ΔE so must be the upper limit $\frac{E}{1-\alpha_1}$ to $\frac{E + \Delta E}{1-\alpha_1}$ since neutrons above $\frac{E + \Delta E}{1-\alpha_1}$ will now not be able to scatter to $E + \Delta E$. By similar considerations we may conclude that in general we should expect two integrals for each constituent to contribute to terms in the differential equation.

One may also note in passing that the phenomena of "shadowing" is also shown, that is the neutron flux will experience a diminution below a given resonance due to the absorption process taking place within the resonance. If one examines Eq. (20) and notes that the source function S for an energy interval below the resonance involves the collision density in an energy interval within the resonance then one readily sees that an attenuation of the collision density below the resonance will take place due to absorptions within the

resonance. In other words, neutrons absorbed in the resonance which would have otherwise reached some energy E below the resonance cause a dip in the collision density at E . If a resonance appears at this energy this dip will be even more pronounced. If the dip occurs in a region just ahead of a resonance the source distribution for that resonance will not be $1/E$ and would yield a significant effect in the collision density calculation within the lower resonances.

Since we have assumed that $F_0(E')$ is a known function, the solution to Eq. (20) can be determined exactly for any $F_N(E)$, $(1-\alpha_1) E_1 < E < E_1$. Furthermore, we may determine in an exactly analogous manner the collision densities for any "M", by the corresponding solution to Eq. (13).

Hydrogen forms a special case for the above formulation. Eq. (11) reduces to SPINNEY's (1957) mutatis mutandis.

The general form for an arbitrary number of elements can be obtained from the above equations by analogy. Only the $\psi(E)$ and the $S(0, 1-\alpha_1; N-1, 1-\alpha_1; E)$ change (see Appendix I).

III. Discussion of Assumptions

The solution for the flux distribution in a resonance of a single substance was based on the assumption that an energy E_1 could be found such that for E above E_1 , $\Sigma_a(E) = 0$, and for $E > E_1$ the collision density, $F_0(E)$ is known. These assumptions do not place a severe restriction on the solution.

In the case of a homogeneous mixture we incorporated the above two assumptions in addition to requiring that the number of collision intervals of the absorber per moderation collision interval have an integral value. For practical cases where the absorber mass is very much greater than the moderator mass this requirement can easily meet to a good approximation by choosing the absorber mass slightly smaller so that N is an integer. This accentuates the scattering by the absorber and increases the resonance escape probability. However, the approximation can be met exactly in a number of important cases.

IV. Approximate Solution

If one assumes that the neutron flux is continuous, it is possible to approximately solve the slowing down equation. The assumption of continuity physically has the connotation that the flux oscillations are neglected and hence the actual neutron flux is replaced by a continuous flux distribution. The approximation is reasonably good within a resonance and increases in accuracy as one proceeds away from regions of strongly varying absorption. For the purpose of solving this problem, it is expedient to express the slowing down equation in terms of the lethargy variable, μ . In addition, to illustrate the method we shall deal only with a homogeneous material composed of a single absorber. The slowing down equation is then

$$F(\mu) = \int_{\mu-\gamma}^{\mu} d\mu' \frac{\Sigma_s(\mu')}{\Sigma_t(\mu')} F(\mu') \frac{e^{-(\mu-\mu')}}{1-\alpha} + S(\mu)$$

Differentiating the above with respect to μ one obtains the functional equation

$$F' + \left(1 - \frac{\Sigma_s(\mu)}{\Sigma_t(\mu)(1-\alpha)} \right) F(\mu) + \frac{\Sigma_s(\mu-\gamma)}{\Sigma_t(\mu-\gamma)} \frac{e^{-\gamma}}{1-\alpha} F(\mu-\gamma) = \frac{dS}{d\mu} + S$$

We now make the assumption that the collision density is continuous and expand in a Taylor series about μ

$$F(\mu-\gamma) = F(\mu) - \gamma \frac{dF}{d\mu} + \frac{\gamma^2}{2!} \frac{d^2F}{d\mu^2} + \dots$$

We shall retain, for ease of computation, only the first two terms of the above expansion. Then the functional equation for $F(\mu)$ becomes

$$\left(1 - \frac{\Sigma_s(\mu-\gamma)}{\Sigma_t(\mu-\gamma)} \frac{\bar{e}^\gamma}{1-\alpha} \gamma\right) F' + \left(1 - \frac{\Sigma_s(\mu)}{\Sigma_t(\mu)(1-\alpha)} + \frac{\Sigma_s(\mu-\gamma)}{\Sigma_t(\mu-\gamma)} \frac{\bar{e}^\gamma}{(1-\alpha)}\right) F = \frac{dS}{d\mu} + S$$

The above equation can be easily solved for $F(\mu)$. For the case of a point source at lethargy μ_1 , $S(\mu) = S\delta(\mu-\mu_1)$; and a constant scattering cross section with $1/v$ absorption, the solution is

$$F(\mu) = \Sigma_t S \left\{ \frac{\xi \Sigma_s [\xi \Sigma_s + \Sigma_a(\mu_1)] [\xi \Sigma_s + \Sigma_a(\mu)] + \delta(\mu-\mu_1) [\xi \Sigma_s + \Sigma_a(\mu)]^3}{[\xi \Sigma_s + \Sigma_a(\mu)]^4} \right\}$$

Higher order terms can naturally be found by retaining more terms in the Taylor series expansion. If more than one substance is present a similar set of equations can be obtained by using a Taylor series expansion for the collision density.

V. Summary

Solutions have been obtained to the slowing down equation in a homogeneous media under rather mild assumptions. The assumptions do not limit the physical phenomena of resonance absorption. In this respect it is seen that the Placzek oscillations which occur at a point source also occur within a resonance. The oscillations become more pronounced the more abrupt the change in the absorption properties of the media. Furthermore, the frequency of these oscillations is greater the number of elements present.

The dip in the collision density at an energy below the resonance due to absorption in the resonance is also observed from the form of the solution. Therefore in a mixture of elements or for a single element with a suitable resonance spacing this dip could cause sufficient error in the calculation procedure usually used to warrant a complete calculation of the collision density through all energies. This dip would appear as a fictitious absorption process.

The solution presented here can also be used to calculate the effect of resonance overlap. Overlap would increase the effect of flux oscillations within the overlap energy region.

The basic form of the solution contained herein for an arbitrary element has the same form as that given by WEINBERG and WIGNER (1958).

Finally, an approximate solution neglecting flux oscillations is indicated.

VI. References

- BETHE, H. (1937), Rev. Mod. Phys. 9, 69.
- BOFFI, V. (1960), Ann. of Phys. 9, 435.
- BOFFI, V. (1961), Nuovo Cimento, 21 No. 6.
- BOFFI, V. (1965), Nuovo Cimento, 35, No. 4.
- CORNGOLD, N. (1957), Proc. Phys. Soc. A70, 793.
- DRESNER, L. (1960), Resonance Absorption in Nuclear Reactors, Pergamon Press.
- HASEGAWA, K. (1959) ORNL-2705.
- MARSHAK, R. (1947), Rev. Mod. Phys. 19, 185.
- PLACZEK, G. (1946), Phys. Rev. 69, 423.
- SPINNEY, K. (1957), BNL-433, 103. Also see J. Nucl. Energy 6, 53 (1957).
- WEINBERG and WIGNER (1958), The Physical Theory of Neutron Chain Reactors, University of Chicago Press. See also BNL-433, 125 (1957).
- WIGNER, E. P., CREUTZ, E., JUPNIK, H., and SNYDER, T. (1955), J. Appl. Phys. 26.

Appendix I

The form of the functions " ψ " and " S " will now be given for 1) two constituent mixture of the k^{th} collision interval of the lighter substance; 2) for a mixture of ℓ elements.

1) The function $\psi(E)$ for two constituent mixture has already been given. The function " S ", however, describes the number of neutrons scattered into E to $E + dE$ from the energy interval immediately above the interval in which E appears in addition to the neutrons scattered into E to $E + dE$ from $(1-\alpha_0)^j E_1$ to $\frac{E}{1-\alpha_1}$. Therefore, the form for " S " for an arbitrary collision interval, say the M^{th}

$$S(j, 1-\alpha_1; M-1, 1-\alpha_0; E) \quad M = 1, 2, \dots \quad (1-A)$$

The notation for " S " is; the first place denotes the subscript on the collision density of the lightest substance contributing by scattering collisions to E to $E + dE$, the second place denotes the energy at which the above collision density is evaluated, i.e., $\frac{E}{1-\alpha_1}$, etc. The last place is to denote that S is a function of energy.

2) For a mixture of ℓ elements when there is an integral number of collision intervals of the heaviest per collision interval of all substances present. Let the subscript 0 denote the heaviest substance, 1 the next heaviest, etc.

Let us define

$$\Delta_1(M) = \frac{(1-\alpha_0)^{N+M} E_1}{(1-\alpha_1)}$$

$$\Delta_2^{(M)} = \frac{(1-\alpha_0)^{N+M} E_1}{(1-\alpha_2)} \quad (2-A)$$

$$\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix}$$

$$\Delta_{\ell-1}^{(M)} = \frac{(1-\alpha_0)^{N+M} E_1}{(1-\alpha_{\ell-1})}$$

Then we determine for each $\ell-1$ the least integer values $j_1, j_2, \dots$
 $j_{\ell-1}$ such that

$$\Delta_{\ell-1}^{(M)} > (1-\alpha_0)^{j_{\ell-1}} E_1. \quad (3-A)$$

We then have contributions from $(1-\alpha_0)^{j_1}$ to $\frac{E}{1-\alpha_1}$; $(1-\alpha_0)^{j_2}$ to $\frac{E}{1-\alpha_2}$; $\dots$ $(1-\alpha_0)^{j_{\ell-1}}$ to $\frac{E}{1-\alpha_{\ell-1}}$ in addition to the contribution arising from E to $(1-\alpha_0)^{N+M-1} E_1$ for each element.

Then the form of S will be

$$S(j_{\ell-1}, 1-\alpha_{\ell-1}; j_{\ell-2}, 1-\alpha_{\ell-2}; \dots, \dots; M-1, 1-\alpha_0; E) \quad (4-A)$$

The function ψ will be

$$\psi(E) = \frac{1}{E \sum_t (E)} \sum_{i=0}^{i=\ell} \frac{\sum_{si} (E)}{\alpha_1} \quad (5-A)$$