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WAPD-T-1802

CONF-650602-88

SEVEN-POINT DIFFERENCE EQUATIONS WITH ARBITRARY MESH SPACING
FOR NEUTRON DIFFUSION IN 60° GEOMETRY BY THE INTEGRATION TECHNIQUE

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UNIT PRICES

CONTRACT AT-11-1-GEN-14

H.C. \$ 1.00; MN .50

January 20, 1965

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

American Nuclear Society Meeting

June 1965

Gatlinburg, Tennessee

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SEVEN-POINT DIFFERENCE EQUATIONS WITH ARBITRARY MESH SPACING
FOR NEUTRON DIFFUSION IN 60° GEOMETRY BY THE INTEGRATION TECHNIQUE

R. C. Gast

I. INTRODUCTION

The purpose of this paper is to present two new derivations of two-dimensional, seven-point difference equations for neutron diffusion in 60° geometry with arbitrary mesh by the well known "integration technique" (this technique was used by R. S. Varga (Reference 1) to derive the present five point, x-y, PDQ equation). The first derivation defines a class of difference equations which contains a simpler seven-point difference equation for arbitrary mesh than presently in use. It also contains, as special cases, the seven-point difference equations for 60° geometry presently in use at Bettis-KAPL. The second derivation results in a simple seven-point difference equation which is of special interest since a positive guarantee against negative flux can be given.

The history of development of Bettis-KAPL programs using seven-point difference equations for 60° geometry may be summarized as follows. The first program was KAPL's "FLEER" as described by J. P. Jewett (Reference 2). The integration technique was used to arrive at a seven-point difference equation for uniform mesh. The area of integration used by Jewett to obtain the leakage term in the diffusion equation, through the use of Green's theorem, is the hexagon formed by the perpendicular bisectors of the radii from a given mesh point to its neighbors. The first Bettis program to use a seven-point difference equation for 60° geometry was PDQ5P. This program used a difference equation for non-uniform mesh derived by R. B. Kellogg through the use of the integration technique. The area of integration was that used by Jewett. However, a restriction had to be placed on mesh variation to avoid a degenerate hexagon. Since the ability to specify an arbitrary mesh was required, R. B. Kellogg derived another seven-point difference equation by a variational technique (Reference 3) which would meet this requirement. This equation is currently in PDQ-06.

The two new derivations may be briefly characterized as follows. In the first derivation one assumes, in obtaining the leakage term of the diffusion equation, that the neutron flux is given by a regionwise-linear continuous surface through the flux values at the mesh points; i.e., each set of three neighboring flux values defines a plane over the triangular region formed by the corresponding three mesh points. The treatment of the removal and source terms is permitted a degree of freedom designated by a parameter C . For one value of C one obtains Kellogg's variational difference equation. For another value one obtains a difference equation completely consistent with the above assumption of regionwise-linear continuous flux, and for a third value of C one obtains a simpler difference equation that reduces to that in FLEER for uniform mesh.

In the second derivation one assumes in obtaining the leakage term of the diffusion equation that the neutron flux over the triangular region formed by three neighboring mesh points is given by three discontinuous planes defined completely by the flux values at these three mesh points. This derivation is an exact analog of that used by R. S. Varga for x - y geometry. The resulting difference equation reduces to that in FLEER for uniform mesh.

For both derivations the form of the one-dimensional reduction will be given for blackness theory use.

II. DIFFERENCE EQUATION ASSUMING REGIONWISE-LINEAR CONTINUOUS FLUX

The mesh points used in the derivation of the difference equation will be taken as the intersections of lines parallel to the x -axis with a set of parallel lines making an angle of 120° with the x -axis. The spacing of these lines is arbitrary as shown in Figure (1). The equation for the more general case of completely arbitrary mesh (i.e., mesh points not defined by the intersections of two sets of parallel lines) may be easily derived following the methods given here.

The difference equation for point P shown in Figure (1) will be obtained in terms of the six neighboring points $P_1, \dots, P_6$. Figure (1) also shows the six neighboring points associated with P_2 and P_1 .

FIGURE (1) - MESH DESCRIPTION

- - 6 neighboring points associated with point P
- - 6 neighboring points associated with point P_1
- - 6 neighboring points associated with point P_2
- - mesh lines
- - triangulation lines to form composition triangles
- - lines connecting midpoints of adjacent radii

The triangular regions between mesh points shown in Figure (1) define compositions; i.e., the diffusion theory parameters (D , Σ_a , Σ_R , $\vee\Sigma_f$) are constant in each such region but in general have different values in the different triangular regions. The area of integration associated with point P to be used in deriving a difference equation by the integration technique will be formed in the following way (one should keep in mind the requirement that such areas taken for all mesh points should completely fill two-dimensional space). First connect adjacent midpoints of the six radii from point P as shown in Figure (1); same is also done for P_1 and P_2 . This process will form a set of hexagons that will fill the x - y plane except for a triangular region omitted in each composition triangle; see Figure (1). It is clear that these hexagons and triangles can never be degenerate even for completely arbitrary mesh (degeneracy here means an area defined by a boundary that crosses itself).

Figure (2) shows an enlarged view of triangle $P P_1 P_2$. The omitted triangular region is triangle ADB . Triangles PAB , AP_1D , BDP_2 and ADB are congruent and equal. Symmetry of derivation for the difference equation demands that the omitted area, ADB , be equally divided into three parts; each part to be associated with one of the three points P , P_1 , or P_2 . This is conveniently done by the boundaries formed by the medians shown in Figure (2).

FIGURE (2)

COMPOSITION TRIANGLE I FOR CONTINUOUS FLUX

For example, the portion of the omitted triangular region to be associated with point P is triangle ABC. Thus one of the sides of the hexagon about point P, i.e., BA shown in Figure (2), is replaced by the two line segments BC and AC. Repeating this procedure for each side of the hexagon about P will result in an area defined by a twelve sided figure which can never be degenerate. This dodecagon defines the area of integration to be used in the difference equation derivation.

The flux for obtaining the leakage term in the difference equation is assumed to be linear over each composition triangle such as PP_1P_2 . Thus across the boundaries of the dodecagon will exist continuity of flux and current, furthermore, strict neutron conservation in the difference equation will result in each dodecagon or any collection of dodecagons.

The flux assumption used to obtain the removal and source terms in the diffusion equation does not have to be consistent with the linear assumption made to obtain the leakage term. For example, one could assume that the flux at P is constant over the area defined by its dodecagon; i.e., an analog of what was done in the x-y PDQ derivation. On the other hand the consistent treatment of the removal and source terms involves, for example in composition triangle I shown in Figure 2, the product of the flux at the centroid of the quadrilateral PACB and the area of PACB. The centroid of the quadrilateral PACB lies on the median PD. A degree of freedom will be permitted in the treatment of the removal and source terms through the introductions of a parameter C. This parameter for composition triangle I is the fraction of the length of the median PD, of a line segment measured from P, to the point on PD where the flux is computed in the removal and source terms. Thus C = 0 will give the analog of the x-y PDQ treatment of these terms. The value of C for a consistent treatment is obtained as follows. Let the length of PD be ℓ . From Figure (2) it is seen that the centroid of PAB is at a distance, C_1 , from P on PD given by $C_1 = 1/3 \ell$. The centroid of ACB is at a distance, C_2 , from E on PD given by $C_2 = 1/18 \ell$. The area of ACB is $1/3$ the area of PAB. Thus, C for consistent treatment is given by

$$C = \frac{C_1 + \frac{1}{3} \left(\frac{1}{2} \ell + C_2 \right)}{\frac{1}{3} \ell} = \frac{7}{18} .$$

We will leave C an undetermined parameter for the remainder of the difference equation derivation. For any value of C, the flux at a point $C\ell$, say $\phi(C\ell)$, along the median from P in composition triangle I is:

$$\begin{aligned} \phi(C\ell) &= \phi_P + \left[\frac{\phi_{P_2} + \phi_{P_1}}{2} - \phi_P \right] C , \\ &= \phi_P (1 - C) + \frac{C}{2} (\phi_{P_2} + \phi_{P_1}) . \end{aligned}$$

A similar expression holds for each composition triangle.

Following the above discussion, the difference equation is readily obtained by the integration technique as follows. The diffusion equation for a given group in composition triangle I may be taken as

$$-D_1 \nabla^2 \phi(x,y) + \sigma_1 \phi(x,y) = S(x,y) ,$$

where D_1 and σ_1 are constants. The source term $S(x,y)$ is treated the same as $\sigma_1 \phi(x,y)$ and hence will be omitted. To derive the difference equation at mesh point P, we will integrate the above diffusion equation over the area of each composition triangle, A_1 , contained in the dodecagon about P and add results. Thus,

$$\sum_{i=1}^6 \left(-D_1 \int_{A_1} \nabla^2 \phi(x,y) dA + \sigma_1 \int_{A_1} \phi(x,y) dA \right) = 0 .$$

By Green's theorem, each of the terms, $\int_{A_1} \nabla^2 \phi(x,y) dA$, reduces to a line integral of the outward directed normal derivative over the boundary that defines A_1 . For example, in composition triangle I, this boundary is given by the line segments PA, AC, CB, and BP. Since the line integral over PA and BP vanish in the sum over all composition triangles, one may write the above equation as

$$\sum_{i=1}^6 \left(-D_1 \int_{L_i} \frac{\partial \phi}{\partial n}(x,y) dL + \sigma_1 \int_{A_i} \phi(x,y) dA \right) = 0 ,$$

where L_i is the part of the dodecagon boundary that lies in composition triangle i.

Each of the above terms are defined for the difference equation by assumptions already made. The normal derivatives on L_1 needed in $\int_{L_1} \frac{\partial \phi}{\partial n} dL$ are defined by the assumption of linear flux over each composition triangle. Also $\int_{A_1} \phi dA$ is computed as the A_1 times ϕ , where ϕ is computed at some point on the median from P in A_1 . The form that $D_1 \int_{L_1} \frac{\partial \phi}{\partial n} dL$ and $\sigma_1 \int_{A_1} \phi dA$ take for composition triangle I, shown in Figure (2), are L_1 given below to illustrate. L_1 is given by the two segments AC and CB. Since $\nabla^2 \phi = 0$ on a plane, then

$$\int_{AC+CB} \frac{\partial \phi}{\partial n} dL = - \int_{AB} \frac{\partial \phi}{\partial n} dL .$$

Let N be the point of intersection with line P_1P_2 of the perpendicular dropped from P. Then

$$\phi_N = \phi_{P_1} + \frac{(\phi_{P_2} - \phi_{P_1})}{k_1} \frac{h_2}{2} .$$

Thus we have:

$$- \left. \frac{\partial \phi}{\partial n} \right|_{AB} = \frac{\phi_N - \phi_P}{h_2 \frac{\sqrt{3}}{2}} ,$$

and

$$D_1 \int_{L_1} \frac{\partial \phi}{\partial n} dL = -D_1 \phi_P \frac{k_1}{h_2 \sqrt{3}} + D_1 \frac{\phi_{P_2}}{2 \sqrt{3}} + D_1 \frac{\phi_{P_1} (2k_1 - h_2)}{2 \sqrt{3} h_2} .$$

We also have,

$$\int_{A_1} dA = \frac{k_1 h_2}{4 \sqrt{3}} .$$

Thus

$$\sigma_1 \int_{A_1} \phi \, dA = \sigma_1 \frac{k_1 h_2}{4\sqrt{3}} \left[\phi_P (1 - C) + \frac{C}{2} (\phi_{P_2} + \phi_{P_1}) \right] .$$

Performing similar calculations for each of the six composition triangles and collecting terms, we have the desired difference equation given below:

$$\begin{aligned} & \phi_P \left\{ \frac{1}{4\sqrt{3}} \left[D_1 \frac{k_1}{h_2} + D_2 \frac{h_2}{k_1} + \frac{D_3}{k_1 h_1} (k_1^2 + h_1^2 - k_1 h_1) + \frac{D_4 k_2}{h_1} + \frac{D_5 h_1}{k_2} + D_6 \frac{(h_2^2 + k_2^2 - h_2 k_2)}{h_2 k_2} \right] \right. \\ & \left. + \frac{(1-C)}{4\sqrt{3}} \left[\sigma_1 k_1 h_2 + \sigma_2 k_1 h_2 + \sigma_3 k_1 h_1 + \sigma_4 h_1 k_2 + \sigma_5 k_2 h_1 + \sigma_6 h_2 k_2 \right] \right\} \\ & + \phi_{P_1} \left\{ -\frac{1}{2\sqrt{3}} \left[D_1 \frac{(2k_1 - h_2)}{h_2} + D_6 \frac{(2k_2 - h_2)}{h_2} \right] + \frac{C}{8\sqrt{3}} \left[\sigma_1 k_1 h_2 + \sigma_6 k_2 h_2 \right] \right\} \\ & + \phi_{P_2} \left\{ -\frac{1}{2\sqrt{3}} \left[D_1 + D_2 \right] + \frac{C}{2\sqrt{3}} \left[\sigma_1 k_1 h_2 + \sigma_2 k_1 h_2 \right] \right\} \\ & + \phi_{P_3} \left\{ -\frac{1}{2\sqrt{3}} \left[D_2 \frac{(2h_2 - k_1)}{k_1} + D_3 \frac{(2h_1 - k_1)}{k_1} \right] + \frac{C}{8\sqrt{3}} \left[\sigma_2 k_1 h_2 + \sigma_3 k_1 h_1 \right] \right\} \\ & + \phi_{P_4} \left\{ -\frac{1}{2\sqrt{3}} \left[D_3 \frac{(2k_1 - h_1)}{h_1} + D_4 \frac{(2k_2 - h_1)}{h_1} \right] + \frac{C}{8\sqrt{3}} \left[\sigma_3 k_1 h_1 + \sigma_4 h_1 k_2 \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \phi_{P_5} \left\{ -\frac{1}{2\sqrt{3}} \left[D_4 + D_5 \right] + \frac{C}{8\sqrt{3}} \left[\sigma_4 h_1 k_2 + \sigma_5 k_2 h_1 \right] \right\} \\
& + \phi_{P_6} \left\{ -\frac{1}{2\sqrt{3}} \left[D_5 \frac{(2h_1 - k_2)}{k_2} + D_6 \frac{(2h_2 - k_2)}{k_2} \right] + \frac{C}{8\sqrt{3}} \left[\sigma_5 k_2 h_1 + \sigma_6 h_2 k_2 \right] \right\} = 0 .
\end{aligned}$$

Upon comparing this equation with that obtained by Kellogg through the assumption of regionwise-linear continuous flux and use of a variational technique, we see that they are identical if $C = 1/2$. That is, $C = 1/2$ yields a difference equation that has the property of minimizing the functional (Reference 3).

$$I(\phi) = \int_A \left(D \left(\frac{\partial \phi}{\partial x} \right)^2 + D \left(\frac{\partial \phi}{\partial y} \right)^2 - 2S\phi \right) dA .$$

The consistent treatment of the assumption of regionwise-linear continuous flux by the above integration technique leads to $C = 7/18$. Thus, the consistent difference equations are nearly the same as those obtained through the above variational method.

For $C = 0$, i.e., the analog of x-y PDQ, the difference equation simplifies. For example, source and removal terms now involve the flux only at the point P and not the six neighboring points as is the case if $C \neq 0$. For $C = 0$ and uniform mesh, the above difference equation is identical to that in FLEER.

As mentioned in the introduction negative fluxes can occur in any of these difference equations (with the exclusion of the special case where $C = 0$ and mesh is uniform). However, as $C \rightarrow 0$, the likelihood of this decreases since the coefficients of $\phi_{p_1}, \dots, \phi_{p_6}$ decrease as $C \rightarrow 0$.

We will now obtain the one dimensional reduction of the above difference equation in order to obtain the difference equation required for blackness theory. The one dimensional reduction will be made by removing the η dependence of the flux (see Figure 1). Of course, the same result would be obtained if ξ dependence were removed. It must be anticipated that the removal of the η dependence of the flux will not be possible for arbitrary mesh. Thus our approach will be to make the flux uniform in the η direction and then find the restrictions on the k mesh in order to remove the mesh variable k completely. Then upon rotating the ξ axis 30° , we will obtain the one dimensional reduction desired. In the above difference equation we set

$$\begin{aligned}
 \phi_{p_2} &= \phi_{p_1} & \sigma_1 &= \sigma_2 = \sigma_6 = \sigma^+ \\
 \phi_{p_3} &= \phi_{p_6} = \phi_p & \sigma_3 &= \sigma_4 = \sigma_5 = \sigma^- \\
 & & D_1 &= D_2 = D_6 = D^+ \\
 \phi_{p_5} &= \phi_{p_4} & D_3 &= D_4 = D_5 = D^- .
 \end{aligned}$$

After collecting terms, we have:

$$\begin{aligned}
 & \phi_P \left\{ \frac{1}{\sqrt{3}} \left[D^+ \frac{(k_1 + k_2)}{h_2} + D^- \frac{(k_1 + k_2)}{h_2} \right] \right\} \\
 & + \phi_P \left\{ \frac{1}{4\sqrt{3}} \sigma^+ \left[2k_1 h_2 (1 - \frac{3}{4}C) + h_2 k_2 (1 - \frac{C}{2}) \right] \right. \\
 & \left. + \frac{1}{4\sqrt{3}} \sigma^- \left[2h_1 k_2 (1 - \frac{3}{4}C) + h_1 k_1 (1 - \frac{C}{2}) \right] \right\} \\
 & + \phi_{P_1} \left\{ -\frac{D^+}{\sqrt{3}} \frac{(k_1 + k_2)}{h_2} \right\} + \phi_{P_1} \left\{ \frac{\sigma^+}{8\sqrt{3}} [\sqrt{3}k_1 h_2^C + h_2 k_2^C] \right\} \\
 & + \phi_{P_4} \left\{ -\frac{D^-}{\sqrt{3}} \frac{(k_1 + k_2)}{h_1} \right\} + \phi_{P_4} \left\{ \frac{\sigma^-}{8\sqrt{3}} [\sqrt{3}h_1 k_2^C + h_1 k_1^C] \right\} = 0 .
 \end{aligned}$$

We see that k dependence can be removed only if $k_1 = k_2$, i.e., uniform mesh in the η direction. Setting $k_1 = k_2$, and performing a rotation of 30° of the ξ axis by setting $h_1' = h_1 \frac{\sqrt{3}}{2}$ and $h_2' = h_2 \frac{\sqrt{3}}{2}$, we obtain:

$$\begin{aligned}
 & \phi_P \left\{ \frac{D^+}{h_2'} + \frac{D^-}{h_1'} \right\} - \phi_{P_1} \left\{ \frac{D^+}{h_2'} \right\} - \phi_{P_4} \left\{ \frac{D^-}{h_1'} \right\} \\
 & + \phi_P \left\{ \frac{\sigma^+}{6} h_2' (3-2C) + \frac{\sigma^-}{6} h_1' (3-2C) \right\} \\
 & + \phi_{P_1} \left\{ \frac{\sigma^+}{3} h_2'^C \right\} + \phi_{P_4} \left\{ \frac{\sigma^-}{3} h_1'^C \right\} = 0 .
 \end{aligned}$$

If $C = 0$, this difference equation is identical to WANDA (for this one dimensional reduction negative fluxes cannot occur although they can still occur for $C \neq 0$). Thus the formula for lumped absorber D and Σ_a as a function of the blackness parameters (α, β) and interval mesh size h' is that given by Gelbard (Reference 4). For $C = 1/2$, i.e. the value of C that yields Kellogg's variational difference equation, the formula for the lumped absorber D and Σ_a has been derived by A. A. Gonzalez (private communications) and is currently being used in connection with PDQ06.

III. DIFFERENCE EQUATIONS ASSUMING REGIONWISE-LINEAR DISCONTINUOUS FLUX

The derivation, by the integration technique, of a seven point difference equation with arbitrary mesh for 60° geometry assuming a regionwise-linear discontinuous flux for the leakage term is very similar to the derivation given in the proceeding section. The area of integration is still the dodecagon previously described. Compositions are defined over the triangular regions previously described. The treatment of the source and removal terms may be given the same degree of freedom through the parameter C as before, although this section will give the difference equation only for $C = 0$; i.e., the x-y PDQ analog for the treatment of these terms. The only major difference, then, in this section is the assumption of regionwise-linear discontinuous flux rather than regionwise-linear continuous flux to describe the neutron leakage term in the difference equation.

Since the treatment of the leakage term will now be a close analog of the treatment of this term for the 5 point x-y PDQ difference equation, we will first review the flux assumptions made there. In Figure 3 is shown some x-y mesh lines and the four mesh points, $P_1, \dots, P_4$, associated with the point P .

FIGURE (3)

REGIONS OF LINEAR FLUX FOR x-y PDQ

Regions that define compositions are the rectangles formed by the mesh lines. The area of integration is formed by constructing perpendicular bisectors to the radii from point P to its four neighbors as shown in Figure (3). Points A, B, C , and D are the intersections of these perpendicular bisectors. Rectangle $ABCD$ defines the area of integration. The flux assumptions are described as follows. The flux is required to be linear over each of the four quadrilaterals given by PDP_1A , PAP_2B , PBP_3C , and PCP_4D . For example, this is done for quadrilateral PDP_1A by requiring that the flux over this quadrilateral be a plane passing through ϕ_P and ϕ_{P_1} and oriented such that a line in this flux plane perpendicular to the line connecting ϕ_P and ϕ_{P_1} is parallel to the x - y plane. A similar requirement is made for each quadrilateral. Thus, in general, the flux is discontinuous on all such quadrilateral boundaries. This flux assumption and area of integration uniquely define the leakage term in the PDQ x - y difference equation (they could also be used to define consistent removal and source terms).

The analog of the above flux assumptions for the case of 60° geometry, seven-point difference equations are easily made by referring to Figure (4). Figure (4) is a reproduction of Figure (1) used in the previous section except that the six centroids, i.e., $C_1, \dots, C_6$, of the six composition triangles associated with point P, as well as the mid-points, A, B, C, D, E and F, of the radii from point P are shown.

FIGURE (4)
REGIONS OF LINEAR FLUX FOR 60° GEOMETRY

As previously stated, the area of integration is defined by the dodecagon of the previous section, i.e., $AC_1 BC_2 CC_3 DC_4 EC_5 FC_6$.

The flux assumptions are described as follows. First form the six quadrilaterals defined by lines connecting the centroid of each composition triangle with its vertices as shown in Figure (4). Now, in an analog of x-y coordinates, require the flux to be linear over each of the six quadrilaterals given by $PC_1P_1C_1$, $PC_1P_2C_2$, $PC_2P_3C_3$, $PC_3P_4C_4$, $PC_4P_5C_5$, and $PC_5P_6C_6$. For example, this is done for quadrilateral $PC_6P_1C_1$ by requiring that the flux over this quadrilateral be a plane passing through ϕ_P and ϕ_{P_1} and oriented such that a line in this flux plane perpendicular to the line connecting ϕ_P and ϕ_{P_1} is parallel to the x-y plane. A similar requirement is made for each quadrilateral. Thus, in general, the flux is discontinuous on all such quadrilateral boundaries. The above flux assumption and area of integration uniquely define the leakage term for the seven-point difference equation by the integration technique. The form of the leakage term in composition Triangle I, i.e.,

$$D_1 \int_{AC_1+C_1B} \frac{\partial \phi}{\partial n} dL ,$$

is given below to illustrate the manner of derivation. Figure (5) shows an enlarged view of composition triangle I, i.e., PP_1P_2 in Figure (4).

COMPOSITION TRIANGLE I FOR DISCONTINUOUS FLUX

The portions of the six discontinuous flux planes associated with point P, that lie over composition Triangle I, lie over the triangular regions, PC_1P_2 and PC_1P_1 . Let N_1 and N_2 be the points of intersection on PP_2 and PP_1 respectively with the perpendiculars dropped from the centroid C_1 . By the flux assumptions made above the normal neutron current on PP_2 and PP_1 is zero. Thus we may write

$$D_1 \int_{AC_1+C_1B} \frac{\partial \phi}{\partial n} dL = D_1 \int_{N_2C_1} \frac{\partial \phi}{\partial n} dL + D_1 \int_{C_1N_1} \frac{\partial \phi}{\partial n} dL ,$$

where n is directed away from P. The length of N_2C_1 is given by

$$N_2C_1 = \frac{k_2}{2\sqrt{3}} .$$

The length of C_1N_1 is given by

$$C_1N_1 = \frac{k_1 h_2}{2\sqrt{3} \sqrt{h_2^2 + k_1^2 - h_2 k_1}} .$$

We also have:

$$\left. \frac{\partial \phi}{\partial n} \right|_{N_2C_1} = \frac{\phi_{P_1} - \phi_P}{h_2} ,$$

$$\left. \frac{\partial \phi}{\partial n} \right|_{C_1N_1} = \frac{\phi_{P_2} - \phi_P}{\sqrt{h_2^2 + k_1^2 - h_2 k_1}} .$$

Thus,

$$D_1 \int_{N_2 C_1} \frac{\partial \phi}{\partial n} dL = \frac{D_1}{2\sqrt{3}} \frac{k_1}{h_2} (\phi_{P_1} - \phi_P) ,$$

$$D_1 \int_{CN_1} \frac{\partial \phi}{\partial n} dL = \frac{D_1 k_1 h_2 (\phi_{P_2} - \phi_P)}{2\sqrt{3} (h_2^2 + k_1^2 - h_2 k_1)} .$$

To obtain the difference equation similar calculations are performed for each of the six composition triangles and terms collected in the form

$$\sum_{i=1}^6 \left(-D_1 \int_{L_i} \frac{\partial \phi}{\partial n} (x,y) dL + \sigma_i \int_{A_i} \phi(x,y) dA \right) = 0$$

Here $\sigma_i \int_{A_i} \phi(x,y) dA$ has the same value as in the previous derivation for $C = 0$ since it involves only the flux at P and the area of the dodecagon. This difference equation may be written as follows:

$$\begin{aligned} \frac{\phi_P}{2\sqrt{3}} \left\{ \left[D_1 \left(\frac{k_1 h_2}{h_2^2 + k_1^2 - h_2 k_1} + \frac{k_1}{h_2} \right) + D_2 \left(\frac{h_2 k_1}{k_1^2 + h_2^2 - k_1 h_2} + \frac{h_2}{k_1} \right) \right. \right. \\ \left. \left. + D_3 \left(\frac{h_1}{k_1} + \frac{k_1}{h_1} \right) + D_4 \left(\frac{h_1 k_2}{h_1^2 + k_2^2 - h_1 k_2} + \frac{k_2}{h_1} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + D_5 \left(\frac{h_2 h_1}{h_1^2 + k_2^2 - h_1 k_2} + \frac{h_1}{k_2} \right) + D_6 \left(\frac{k_2}{h_2} + \frac{h_2}{k_2} \right) \Big] \\
& + \frac{1}{2} \left[\sigma_1 k_1 h_2 + \sigma_2 k_1 h_2 + \sigma_3 k_1 h_1 + \sigma_4 h_1 k_2 + \sigma_5 k_2 h_1 + \sigma_6 h_2 k_2 \right] \Big\} \\
& + \frac{\phi_{P_1}}{2\sqrt{3}} \left\{ -\frac{1}{h_2} (D_1 k_1 + D_6 k_2) \right\} \\
& + \frac{\phi_{P_2}}{2\sqrt{3}} \left\{ -\frac{k_1 h_2}{h_2^2 + k_1^2 - h_2 k_1} (D_1 + D_2) \right\} \\
& + \frac{\phi_{P_3}}{2\sqrt{3}} \left\{ -\frac{1}{k_1} (D_2 h_2 + D_3 h_1) \right\} \\
& + \frac{\phi_{P_4}}{2\sqrt{3}} \left\{ -\frac{1}{h_1} (D_3 k_1 + D_4 k_2) \right\} \\
& + \frac{\phi_{P_5}}{2\sqrt{3}} \left\{ -\frac{h_1 k_2}{h_1^2 + k_2^2 - h_1 k_2} (D_4 + D_5) \right\} \\
& + \frac{\phi_{P_6}}{2\sqrt{3}} \left\{ -\frac{1}{k_2} (D_5 h_1 + D_6 h_2) \right\} = 0 \quad .
\end{aligned}$$

Upon comparing this difference equation with that in the previous section, it is observed that they are identical if $C = 0$ and the mesh is uniform (i.e., $k_1 = h_2 = h_1 = h_2$). Thus, for uniform mesh these equations are identical to those in FLEER.

For the difference equation in the previous section negative fluxes could occur, here however, they cannot.

Upon performing a one dimensional reduction as before, one finds that the mesh variable k can be removed only if $k_1 = k_2 = h_1 = h_2$, i.e., uniform mesh. The form of the one dimensional reduction is, of course, WANDA.

IV. SUMMARY AND CONCLUSIONS

Various difference equations for neutron diffusion in 60° geometry with arbitrary mesh can be readily obtained by the integration technique. This is an appealing method because of simplicity and ease of physical interpretation.

The relative merits of these different difference equations can be established, from the viewpoint of a nuclear designer, only by having them inserted in a PDQ type program. This is because the designer is concerned with errors in power peaks, eigenvalue, capture rates, etc. as a function of mesh size. This type of error analysis is not possible without a computer program. However, in advance of such comparisons, one can state that it is generally desirable to avoid negative fluxes and to have a one dimensional reduction to WANDA which is the one dimensional reduction of the present x-y PDQ. These requirements point to the use of the latter difference equation derived with the assumption of regionwise-linear discontinuous flux. In addition, there may be computer time savings by the use of this equation due to the simpler source and removal terms. All of the above statements are also true for the former difference equations derived with the assumption of regionwise-linear continuous flux and $C = 0$, except that a positive guarantee against negative fluxes cannot be given. (subsequent to the derivations in this paper, "point error analysis" by R. P. Kellogg has indicated that the latter difference equation can have poor accuracy in the leakage term, hence the former one is to be preferred.)

ACKNOWLEDGMENT

Special thanks are due to Dr. R. B. Kellogg, Dr. L. A. Hageman, and to Mr. B. H. Ruth for several helpful discussions concerning properties of difference equations for neutron diffusion in 60° geometry.

REFERENCES

- (1) R. S. Varga, "Numerical Solution of the Two-Group Diffusion Equations in x-y Geometry, "WAPD-159, August, 1956.
- (2) J. P. Jewett, "Numerical Solution of the Diffusion Equation in Triangular and Hexagonal Geometries," KAPL-2085, September 1960
- (3) R. P. Kellogg, "A Ritz Finite Difference Approximation to the Neutron Diffusion Equation", WAPD-BT-31, 1964.
- (4) A. F. Henry, "A Theoretical Method for Determining the Work of Control Rods," WAPD-218, pp. 1, August 1959.