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SCATTERING OF PARTICLES WITH SPIN;MACKEY STATE FORMULATION

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Abstract

The conventional states used to describe the scattering of particles with spin are the covariant states and the Wigner states. In this paper we describe the scattering of massive particles in terms of Mackey states, which are analogous to the states used by Mackey in his induced theory of group representations, and whose advantage is that their Lorentz transformation law does not involve the spin indices explicitly.

The description with Mackey states follows the pattern of the description in the well-known papers of Jacob and Wick and of Trueman and Wick and covers in particular partial-wave analyses, discrete symmetries and crossing. We hope that the usefulness and elegance of the Mackey states will become evident in the course of the discussion.

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Contents

1. Introduction
2. Two-Particle States. Natural Basis
3. Reduced Two-Particle States
4. Reduced T-matrix
5. Parity
6. Time-Reversal
7. Identical Particles
8. Crossing for Covariant and Wigner States
9. Crossing for Mackey States

Appendix

References

1. Introduction

It is customary to describe relativistic particles of momentum p ($p^2 = m^2 > 0$, $p_0 > 0$) either by Wigner states $|p, \zeta\rangle$ or by covariant states $|p, \alpha\rangle$ which have the Lorentz transformation laws and inner products^(1,2,3,4,5,6,7,8)

$$U(\lambda) |p, \zeta\rangle = |\lambda p, \zeta'\rangle D_{\zeta\zeta'}^j(\ell'(\lambda) \ell(p)) \quad (1.1)$$

$$\langle p', \zeta' | p, \zeta \rangle = 2p_0 \delta^3(p' - p) \delta_{\zeta\zeta'}$$

and

$$U(\lambda) |p, \alpha\rangle = |\lambda p, \alpha'\rangle \mathcal{D}_{\alpha\alpha'}(\lambda) \quad (1.2)$$

$$\langle p', \alpha' | p, \alpha \rangle = 2p_0 \delta^3(p' - p) [\mathcal{D}^\dagger(\ell(p)) \mathcal{D}(\ell(p))]_{\alpha\alpha'}$$

respectively. Here λ is the Lorentz transformation in question, $\ell(p)$ is a standard boost from the rest momentum $k = (m, 0, 0, 0)$ to momentum $p = \ell(p)k$, D^j is the spin j representation of the rotation group and $\mathcal{D}(\lambda)$ is a representation of the Lorentz group $\{\lambda\}$ whose restriction to rotations contains the representation D^j . To avoid redundant components α , $\mathcal{D}(\lambda)$ may be taken to be the $(j, 0)$ or $(0, j)$ representation of $\{\lambda\}$.

In general, $\ell(p)$ can be written as

$$\ell(p) = r(\phi, \theta, -\phi) z(p) r(p) \quad (1.3)$$

where $(\phi, \theta, -\phi)$ is the Euler angle parametrization of the rotation group, $z(p)$ is an acceleration in the positive z -direction which takes k to $(p_0, 0, 0, |\vec{p}|)$, (θ, ϕ) are the polar angles of $\vec{p}$, and $r(p)$ is a rotation whose dependence on p fixes the choice of $\ell(p)$. Two standard $r(p)$ are

$$r(p) = 1 \quad (\text{Jacob and Wick})^{(7)} \quad (1.4)$$

$$r(p) = r(\phi, \theta, -\phi)^{-1} \quad (\text{Foldy})^{(8)} \quad (1.5)$$

If $W_\mu = \epsilon_{\mu\nu\lambda\rho} M_{\nu\lambda} p_\rho$ is the Pauli-Lubanski operator, the Jacob-Wick states $|p, \zeta\rangle_{\text{JW}}$ are eigenstates of the helicity operator proportional to $\vec{W} \cdot \vec{p}$.

Each kind of state just described has its own advantages. For example, because helicity commutes with the total angular momentum operator, the Jacob-Wick states are very suitable for partial-wave expansions, while for reasons of locality, the covariant states are useful for the study of crossing symmetry.

In the present paper, we wish to propose the use of a third kind of state, namely the Mackey states⁽⁹⁾ $|\lambda, \zeta\rangle$ defined by

$$|\lambda, \zeta\rangle = |p, \alpha\rangle \mathcal{D}_{\alpha\zeta}(\lambda) = |p, \zeta'\rangle D_{\zeta'\zeta}^j(l(p)\lambda), \quad p = \lambda k \quad (1.6)$$

Special cases of (1.6) are

$$|l(p), \zeta\rangle = |p, \zeta\rangle, \quad (1.7)$$

and these special cases of Mackey states have been discussed previously by Moussa and Stora⁽⁵⁾. From (1.1) or (1.2), we see that the Lorentz transformation property of the Mackey states is, simply,

$$U(\lambda')|\lambda, \zeta\rangle = |\lambda'\lambda, \zeta\rangle \quad (1.8)$$

while the inner product is

$$\langle \lambda, \zeta | \lambda', \zeta' \rangle = 2p_0 \delta^3(p - p') D_{\zeta'\zeta}^j(\lambda^{-1}\lambda'), \quad p = \lambda k, p' = \lambda' k \quad (1.9)$$

Note that the δ -function in the inner product guarantees that $\lambda^{-1}\lambda'$ is a rotation. The simplicity of the transformation law (1.8) and the independence of the Mackey states of the boost $l(p)$ are the principal advantages of the Mackey states and are the reason for their introduction. The price paid for these advantages is that the states $|\lambda, \zeta\rangle$ are functions of the six variables λ instead of the three variables $\vec{p}$. Hence they are not linearly independent, but, as can be readily verified from the definition (1.6), they satisfy the subsidiary condition

$$|\lambda k, \zeta\rangle = |\lambda, \zeta\rangle D_{\zeta\zeta}^j(h), \quad h \in SO(3)_k \quad (1.10)$$

where $SO(3)_k$ is the little group of $k = (m, 0, 0, 0)$. Note that the subsidiary condition is a condition on the functional dependence on λ of $|\lambda, \zeta\rangle$ and not a condition to remove redundant components.

Equations (1.8), (1.9) and (1.10) are the basic equations satisfied by Mackey states, and it is easy and instructive to check that they are self-consistent. In the following sections, we shall describe the reduction of the two-particle Mackey states, using methods which are fairly standard (2,3,5,6,7,10,11). We hope that the elegance and usefulness of Mackey states will become evident during the discussion.

It might be worth mentioning at this point a natural affinity between the Mackey and Jacob-Wick states which arises as follows: Since every Lorentz transformation λ can be written in the canonical form⁽¹⁾ $\lambda = rzh$, where r and h are rotations in the little group $SO(3)_k$, r is of the form $r(\phi, \theta, -\phi)$ and z is an acceleration in the positive z -direction, we can write

$$|\lambda, \zeta\rangle = |rz, h, \zeta\rangle = |rz, \zeta'\rangle D_{\zeta, \zeta'}^j(h) \quad (1.11)$$

by using the subsidiary condition. Thus the states $|rz, \zeta\rangle$ form a 'natural basis' for the general Mackey states and we shall henceforth refer to them as one-particle base states.

On the other hand, from (1.3), (1.4) and (1.7) we see that

$$|rz, \zeta\rangle = |p, \zeta\rangle_{JW} \quad (1.12)$$

where the subscript JW emphasizes that the Wigner states in question are defined by the Jacob-Wick choice of boosts $l(p)$, namely

$$l(p) = r(\phi, \theta, -\phi) \Xi(p) \quad (1.13)$$

Thus the one-particle Mackey base states and the one-particle Jacob-Wick states are identical.

Finally, it should be mentioned that although we have used here the parameters λ of the Lorentz group $L_+^\uparrow$ as the variables in $|\lambda, \zeta\rangle$, this is for intuitive reasons. Strictly speaking, to allow for two-valued ray representations D^j and $\mathbb{D}$, we should use as variables the parameters u of the covering group $SL(2, \mathbb{C})$, with $\lambda(u)$ as the representative of u in $L_+^\uparrow$. However, the distinction between $\lambda(u)$ and u becomes significant only in the section on crossing and so we use λ rather than u in all other sections.

2. Two-Particle States. Natural Basis

We shall assume until the end of section 6 that the two particles we are considering are not identical. In section 7 we shall indicate the modification that is necessary if they are identical.

The most general non-identical two-particle Mackey state is clearly

$$|\lambda_1, \lambda_2, \xi_1, \xi_2\rangle = |\lambda_1, \xi_1\rangle \otimes |\lambda_2, \xi_2\rangle \quad (2.1)$$

On account of the subsidiary conditions the two-particle states (2.1) are not linearly independent. We now prove a lemma which selects out a natural set of base states.

Lemma. For any pair of Lorentz transformations λ_1, λ_2 , there exists a single Lorentz transformation λ such that

$$(\lambda_1, \lambda_2) = (\lambda \mathfrak{z}_1 h_1, \lambda \mathfrak{z}_2^{-1} h_2) \quad (2.2)$$

where

$$h_1 = \mathfrak{z}_1^{-1} \lambda_1^{-1} \lambda_1 \quad \text{and} \quad h_2 = \mathfrak{z}_2 \lambda_2^{-1} \lambda_2 \quad (2.3)$$

are rotations around the k -axis (Wigner rotations), and the $\mathfrak{z}_a$ are pure boosts in the positive z -direction to energies

$$\omega_a(s) = (s + m_a^2 - m_b^2) / 2\sqrt{s}, \quad a \neq b \quad (2.4)$$

where

$$s = (p_1 + p_2)^2, \quad p_a = \lambda_a k_a, \quad k_a = (m_a, 0, 0, 0)$$

Proof: Let λ^{-1} be the Lorentz transformation that brings $p_1 + p_2$ to rest and aligns $\vec{p}_1, \vec{p}_2$ along the positive z -axis. Then

$$\lambda^{-1} \lambda_1 k_1 = \mathfrak{z}_1 k_1, \quad \lambda^{-1} \lambda_2 k_2 = \mathfrak{z}_2 k_2 \quad (2.5)$$

where

$$(\omega_1 + \omega_2)^2 = (\mathfrak{z}_1 k_1 + \mathfrak{z}_2^{-1} k_2)^2 = s \quad (2.6)$$

From these equations the results follow at once. Note that λ is unique up to a rotation around the z-axis.

Using the lemma we see clearly that for any general Mackey state

$|\lambda_1, \lambda_2, \zeta_1, \zeta_2\rangle$ we have

$$|\lambda_1, \lambda_2, \zeta_1, \zeta_2\rangle = |\lambda z_1(s), \lambda z_2^{-1}(s), \zeta_1, \zeta_2\rangle D_{\zeta_1 \zeta_1}^{j_1}(h_1) D_{\zeta_2 \zeta_2}^{j_2}(h_2) \quad (2.7)$$

and hence that the states $|\lambda z_1(s), \lambda z_2^{-1}(s), \zeta_1, \zeta_2\rangle$ form a natural basis.

(The basis is slightly overcomplete since a rotation around the z-axis in λ can be pulled through the z's and absorbed in the $D(h)$ matrices, but this is a trivial point which we shall ignore).

For the centre-of-mass states in particular, the two-particle base states $|\lambda z_1(s), \lambda z_2^{-1}(s), \zeta_1, \zeta_2\rangle$ can be reduced to $|r z_1(s), r z_2^{-1}(s), \zeta_1, \zeta_2\rangle$ where $r = r(\phi, \theta, -\phi)$ and it is interesting to compare these states with the conventional Jacob-Wick states. Using (1.11) we have

$$\begin{aligned} |r z_1(s), r z_2^{-1}(s), \zeta_1, \zeta_2\rangle &= |p_1, \zeta_1\rangle_{\text{JW}} \otimes |p_2, \zeta_2\rangle_{\text{JW}} \\ &= |p_1 p_2, \zeta_1, -\zeta_2\rangle_{\text{JW}}, \end{aligned} \quad (2.8)$$

where the minus sign before ζ_2 comes from the Jacob-Wick convention, and we have $\vec{p}_1 + \vec{p}_2 = 0$. Note that there is no phase-factor in (2.8).

3. Reduced Two-Particle States

To reduce the two-particle states with respect to the Poincaré group, we take the centre-of-mass states $|rz_1, rz_2^{-1}, \zeta_1, \zeta_2\rangle$ and follow the conventional procedure of averaging them over the rotation group $SO(3)_k$ to obtain a state

$$\delta_{\zeta_1, \zeta_2, n} |1, m(j, s), \zeta_1, \zeta_2\rangle = \int d\mu(r) D_{mn}^{j*}(r) |rz_1(s), rz_2^{-1}(s), \zeta_1, \zeta_2\rangle, \quad (3.1)$$

$r \in SO(3)_k$

Here we normalize the $SO(3)_k$ invariant measure $d\mu(r)$ so that the volume of the covering group $SU(2)$ is unity [see Appendix], and $s = (p_1 + p_2)^2 = (z_1 k_1 + z_2^{-1} k_2)^2$ as in (2.6). In contrast to the conventional case, if we now apply a Lorentz transformation $U(\lambda)$ to each side of (3.1), we pick up no D-matrices (thanks to the simple transformation law for Mackey state vectors) and we obtain the analogous formula

$$\delta_{\zeta_1, \zeta_2, n} |\lambda, m(j, s), \zeta_1, \zeta_2\rangle = \int d\mu(r) D_{mn}^{j*}(r) |\lambda rz_1, \lambda rz_2^{-1}, \zeta_1, \zeta_2\rangle \quad (3.2)$$

We call the states $|\lambda, m(j, s), \zeta_1, \zeta_2\rangle$ the reduced two-particle states. What we now wish to show is that the nomenclature is justified, i.e. that (3.2) does indeed effect a reduction with respect to the Poincaré group. To show this, we must show that the states $|\lambda, m(j, s), \zeta_1, \zeta_2\rangle$ satisfy the basic relations (1.8), (1.9) and (1.10) of the Mackey states.

The first property, the transformation law

$$U(\lambda') |\lambda, m(j, s), \zeta_1, \zeta_2\rangle = |\lambda' \lambda, m(j, s), \zeta_1, \zeta_2\rangle, \quad (3.3)$$

follows immediately from (3.2). The third relation, the subsidiary condi-

tion, is established as follows: from (3.2), we have for $g \in SO(3)_k$

$$\begin{aligned}
 |\lambda g, m(j_s), \zeta_1, \zeta_2\rangle &= \int d\mu(r) D_{mn}^{j*}(r) |\lambda g r \zeta_1, \lambda g r \zeta_2, \zeta_1, \zeta_2\rangle, \quad n = \zeta_1 + \zeta_2 \\
 &= \int d\mu(r) D_{mn}^{j*}(g^{-1}r) |\lambda r \zeta_1, \lambda r \zeta_2, \zeta_1, \zeta_2\rangle \\
 &= D_{mn}^{j*}(g^{-1}) \int d\mu(r) D_{mn}^{j*}(r) |\lambda r \zeta_1, \lambda r \zeta_2, \zeta_1, \zeta_2\rangle \\
 &= D_{mn}^{j\dagger}(g^{-1}) |\lambda, m'(j_s), \zeta_1, \zeta_2\rangle = |\lambda, m'(j_s), \zeta_1, \zeta_2\rangle D_{nm}^j(g)
 \end{aligned} \tag{3.4}$$

where we have used the unitarity of D^j . This is the required relation.

It should be emphasized, however, that this result depends critically

on our use in equation (3.1) of the two-particle base states

$|r z_1, r z_2^{-1}, \zeta_1, \zeta_2\rangle$ corresponding to the Jacob-Wick states of the conventional theory. Had we used, for example, the states $|r z_1 r^{-1}, r z_2^{-1} r^{-1}, \zeta_1, \zeta_2\rangle$ which correspond to the Foldy states, the kets on the right hand side of (3.4) would have been replaced by $|\lambda g r z_1 r^{-1}, \lambda g r z_2^{-1} r^{-1}, \zeta_1, \zeta_2\rangle$ and not $|\lambda g r z_1 r^{-1} g^{-1}, \lambda g r z_2^{-1} r^{-1} g^{-1}, \zeta_1, \zeta_2\rangle$, and so the g could not have been transferred to $D_{mn}^{j*}(r)$.

The third condition, namely the existence of an invariant positive definite inner product for the states $|\lambda, m(j_s), \zeta_1, \zeta_2\rangle$ is actually guaranteed by the first two conditions (3.3) and (3.4), since these conditions tell us that we have a reduction of a representation which is already unitary. However we wish to show that not only does the positive definite invariant inner product exist, but that it has the same explicit form as the inner product (1.9) for the one-particle states. To show this we note that equation (3.2) defines in principle the Clebsch-Gordon coefficients $\langle \lambda z_1, \lambda z_2^{-1}, \zeta_1, \zeta_2 | \lambda', m(j_s), \eta_1, \eta_2 \rangle$, and a (nontrivial) calculation in the

appendix shows them to be

$$\langle \lambda \zeta_1, \lambda \zeta_2 | \lambda', m(j, s), \eta_1, \eta_2 \rangle = 2P_0 \delta^3(P-P') D_{mm'}^j(\lambda^{-1} \lambda') \delta_{\zeta_1 \eta_1} \delta_{\zeta_2 \eta_2} \sigma(s) \delta(s-s') \quad (3.5)$$

where

$$P = \lambda \zeta_1 k_1 + \lambda \zeta_2 k_2, \quad k_a = (m_a, 0, 0, 0) \\ P' = \lambda' \zeta_1' k_1 + \lambda' \zeta_2'^{-1} k_2 \quad (3.6)$$

$$s = P, \quad s' = P'^2, \quad n = \zeta_1 + \zeta_2, \quad \sigma(s) = s^{1/2} / \pi q(s)$$

$$q(s)^2 = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2] / 4s, \quad q(s) = |q(s)|$$

Using the expansion (3.2) and the orthogonality relation

$$\int d\mu(r) D_{mm}^{j*}(r) D_{m'n'}^{j'}(r) = \frac{1}{2j+1} \delta_{jj'} \delta_{mm'} \delta_{nn'} \quad (3.7)$$

we obtain at once the inner product relation

$$\langle \lambda, m(j, s), \zeta_1, \zeta_2 | \lambda', m'(j', s'), \zeta_1', \zeta_2' \rangle = 2P_0 \delta^3(P-P') D_{mm'}^j(\lambda^{-1} \lambda') N \quad (3.8)$$

where

$$N = \frac{s^{1/2}}{\pi q(s)} \cdot \delta(s-s') \delta_{\zeta_1 \zeta_1'} \delta_{\zeta_2 \zeta_2'} \cdot \frac{\delta_{jj'}}{2j+1} \quad (3.9)$$

Apart from the relativistically invariant "normalization factor" N , the inner product (3.8) is of exactly the required form.

We see therefore that the ansatz (3.2) not only effects a reduction into irreducible representations of the Poincaré group, but effects it in such a way that the reduced states $|\lambda, m(j, s), \zeta_1, \zeta_2\rangle$ have exactly the same properties as one-particle states of mass $s^{1/2}$, spin j and "internal" quantum numbers ζ_1 and ζ_2 .

The inverse formula to (3.2) is obtained by using the completeness relation

$$\sum_{j, m, n} (2j+1) D_{mn}^{j*}(r) D_{mn}^j(h) = \delta(r^{-1}h), \quad \int d\mu(r) \delta(r) = 1, \quad (3.10)$$

Multiplying both sides of (3.2) by $(2j+1) D_{mn}^j(h)$ and summing over j, m, n we obtain

$$|\lambda \hbar \mathbb{Z}_1, \lambda \hbar \mathbb{Z}_2^{-1}, \zeta_1, \zeta_2\rangle = \sum_{j,m} (2j+1) |\lambda, m(js), \zeta_1, \zeta_2\rangle D_{m, \zeta_1+\zeta_2}^j(h), \quad (3.11)$$

and in particular

$$|\lambda \mathbb{Z}_1, \lambda \mathbb{Z}_2^{-1}, \zeta_1, \zeta_2\rangle = \sum_j (2j+1) |\lambda, \zeta_1+\zeta_2(js), \zeta_1, \zeta_2\rangle \quad (3.12)$$

We have already seen that the states $|\lambda \mathbb{Z}_1, \lambda \mathbb{Z}_2^{-1}, \zeta_1, \zeta_2\rangle$ form a complete basis for the two-particle states. From (3.12) we now see that the same is true for the reduced states $|\lambda, m(js), \zeta_1, \zeta_2\rangle$.

4. Reduced T-Matrix

We turn now to scattering, and except when there is danger of confusion, we shall not distinguish between ingoing and outgoing particles, denoting both pairs with indices 1 and 2. We do not assume that any of the masses are equal.

Let T be the conventional T-matrix normalized by the unitarity condition

$$i(T^\dagger - T) = T^\dagger T \quad (4.1)$$

and let Π be the projection operator on the two-particle sector. Since T is Poincaré invariant, it commutes with linear and angular momenta.

Hence if $|1, m(js), \zeta_1, \zeta_2\rangle$ is a rest-frame state with quantum numbers $(m j s)$ so is $\Pi T |1, m(js), \zeta_1, \zeta_2\rangle$. It follows that $\Pi T |1, m(js), \zeta_1, \zeta_2\rangle$ is a linear combination of the states $|r, m(js), \zeta_1, \zeta_2\rangle$, and since the latter in turn are linear combinations of the states $|1, m(js), \zeta_1, \zeta_2\rangle$ on account of the subsidiary condition, it follows that we have the expansion

$$\Pi T |1, m(js), \zeta_1, \zeta_2\rangle = \sum_{\mu_1, \mu_2} T_{\mu_1 \mu_2; \zeta_1 \zeta_2}^j(s) |1, m(js), \mu_1, \mu_2\rangle \quad (4.2)$$

This expansion defines the reduced T-matrix $T_{\mu_1 \mu_2; \zeta_1 \zeta_2}^j(s)$. In the presence of only elastic scattering the reduced T-matrix satisfies the same formal unitarity condition as T itself, for if we sandwich equation (4.1) between states of the form $|1, m(js), \zeta_1, \zeta_2\rangle$ and there are only two-body intermediate states in $T^\dagger T$, we obtain

$$i(T_{\zeta_1 \zeta_2; \mu_1 \mu_2}^{j*}(s) - T_{\mu_1 \mu_2; \zeta_1 \zeta_2}^j(s)) = \sum_{\sigma_1, \sigma_2} T_{\mu_1 \mu_2; \sigma_1 \sigma_2}^{j*}(s) T_{\zeta_1 \zeta_2; \sigma_1 \sigma_2}^j(s), \quad (4.3)$$

which is formally the same as (4.1).

The expansion (4.2) defines the two-particle matrix elements of the full T -matrix (4.1) in terms of the elements of the reduced T -matrix,

and they can be extracted as follows: First, we have

$$\begin{aligned} \langle \lambda, m(j, s), \zeta_1, \zeta_2 | T | 1, m'(j', s'), \zeta'_1, \zeta'_2 \rangle &= \sum_{\mu_1 \mu_2 \zeta'_1 \zeta'_2} T_{\mu_1 \mu_2 \zeta'_1 \zeta'_2}^j(s) \langle \lambda, m(j, s), \zeta_1, \zeta_2 | 1, m'(j', s'), \mu_1, \mu_2 \rangle \\ &= \frac{s^{1/2}}{\pi q(s)} \delta(s-s') 2P_0 \delta^3(P-P') \frac{\delta_{jj'}}{2j+1} D_{mm'}^j(\lambda^{-1}) T_{\zeta_1 \zeta_2 \zeta'_1 \zeta'_2}^j(s) \end{aligned} \quad (4.4)$$

using the value for the inner product found in the appendix. Then from (3.12) we have

$$\begin{aligned} \langle \lambda \zeta_1(s), \lambda \zeta_2(s), \zeta_1, \zeta_2 | T | \zeta_1(s'), \zeta_2(s'), \zeta'_1, \zeta'_2 \rangle &= \\ &= \sum_{j, j'} (2j+1)(2j'+1) \langle \lambda, \zeta_1 + \zeta_2(j, s), \zeta_1, \zeta_2 | T | 1, \zeta'_1 + \zeta'_2(j', s'), \zeta'_1, \zeta'_2 \rangle \\ &= \frac{s^{1/2}}{\pi q(s)} \delta(s-s') 2P_0 \delta^3(P-P') \sum_j (2j+1) D_{\zeta_1 \zeta_2 \zeta'_1 + \zeta'_2}^j(\lambda^{-1}) T_{\zeta'_1 \zeta'_2 \zeta_1 \zeta_2}^j(s) \end{aligned} \quad (4.5)$$

In equations (4.4) and (4.5) the δ -functions ensure that λ is a rotation, so that $D^j(\lambda^{-1})$ makes sense. It is necessary to put λ in only one of the kets in each equation, because T is a Lorentz invariant, and hence the more general matrix elements $\langle \lambda \dots | T | \lambda' \dots \rangle$ can be changed immediately to $\langle \lambda'^{-1} \lambda \dots | T | 1 \dots \rangle$. The only change in (4.4) and (4.5) which is necessary to allow for the change is $D^j(\lambda^{-1}) \rightarrow D^j(\lambda'^{-1} \lambda')$, with the same subscripts.

To identify the reduced matrix $T^j(s)$ in conventional terms we compare (4.5) with the corresponding formula for Jacob and Wick states (equation (30) of their paper) and allowing for the different normalizations of the product states, we obtain

$$T_{\zeta_1 \zeta_2 \mu_1 \mu_2}^j(s) = \langle \zeta_1, -\zeta_2 | T^j(E) | \mu_1, -\mu_2 \rangle \quad (4.6)$$

From the comparison we see that the two reduced T -matrices are identical except for the signs of the second indices ζ_2, μ_2 .

5. Parity

The incorporation of the discrete space-time symmetries (parity and time-reversal) is one of the most elegant parts of the Mackey state formalism. In particular it avoids the use of phase conventions.

We consider first the parity operator $\mathcal{P}$ and assume in the conventional manner that it is linear. For the one-particle Mackey states this means that

$$\mathcal{P}|\lambda, \zeta\rangle = |\lambda^\pi, \zeta'\rangle \eta_{\zeta'\zeta} \quad (5.1)$$

where λ^π is the parity transform of λ , sometimes written as $\lambda^\pi = g\lambda g^{-1}$ where g is the metric tensor, and η is a matrix to be determined. Its matrix elements $\eta_{\zeta'\zeta}$ are determined by the compatibility of (5.1) with the subsidiary condition

$$|\lambda h, \zeta\rangle = |\lambda, \zeta'\rangle D_{\zeta'\zeta}^j(h), \quad h \in SO(3)_{\mathbb{R}} \quad (5.2)$$

as follows: Parity means space-inversion in the rest-frame k . Noting that $(\lambda h)^\pi = \lambda^\pi h$ we obtain from (5.2) after applying the parity operator $\mathcal{P}$ to both sides

$$\eta_{\zeta''\zeta'} D_{\zeta''\zeta}^j(h) = D_{\zeta''\zeta'}^j(h) \eta_{\zeta'\zeta} \quad (5.3)$$

Since $D(h)$ is irreducible, (5.3) implies that η must be a multiple of the identity, and hence (5.1) reduces to

$$\mathcal{P}|\lambda, \zeta\rangle = \eta |\lambda^\pi, \zeta\rangle, \quad (5.4)$$

where η is a number which we call the intrinsic parity. The involutive nature of $\mathcal{P}$ then implies that

$$\eta^2 = 1 \quad (5.5)$$

Equations (5.4) and (5.5) incorporate the operation of parity on the one-particle states.

Let us turn now to the reduced two-particle states. From the definition of the reduced states $|\lambda, m(j_s), \zeta_1, \zeta_2\rangle$, the commutativity of $\mathcal{P}$ with rotations r and $z^{\hat{\pi}} = z^{-1}$, we have

$$\mathcal{P}|\lambda, m(j_s), \zeta_1, \zeta_2\rangle = \int d\mu(r) D_{mn}^{j*}(r) \mathcal{P}|\lambda r z_1, \lambda r z_2^{-1}, \zeta_1, \zeta_2\rangle \quad (5.6)$$

$$= \int d\mu(r) D_{mn}^{j*}(r) |\lambda^{\pi} r z_1^{-1}, \lambda^{\pi} r z_2, \zeta_1, \zeta_2\rangle \eta_1 \eta_2, \quad (5.7)$$

$$\eta = \zeta_1 + \zeta_2$$

To proceed further with this expression it is convenient to introduce at this point the charge-conjugation matrix $C = \exp(i\pi J_2)$ which corresponds to rotations around the y-axis through an angle π in $SO(3)_k$, and has the properties

$$C_{\lambda\mu}^j = (-1)^{j-\mu} \delta_{\lambda, -\mu}, \quad C^j D^j(k) C^{j-1} = D^{j*}(k), \quad C = C^1 \quad (5.8)$$

$$C^2 C = \mathbb{I}^{-1}$$

with respect to $SO(3)$. We then have

$$\mathcal{P}|\lambda, m(j_s), \zeta_1, \zeta_2\rangle = \int d\mu(r) D_{mn}^{j*}(r) |\lambda^{\pi} r C^1 z_1 C, \lambda^{\pi} r C^1 z_2^{-1} C, \zeta_1, \zeta_2\rangle \eta_1 \eta_2 \quad (5.9)$$

$$= \int d\mu(r) D_{mn}^{j*}(r) |\lambda^{\pi} r z_1 C, \lambda^{\pi} r z_2^{-1} C, \zeta_1, \zeta_2\rangle \eta_1 \eta_2$$

$$= \int d\mu(r) D_{mn}^{j*}(r) |\lambda^{\pi} r z_1, \lambda^{\pi} r z_2^{-1}, \zeta_1, \zeta_2\rangle \eta_1 \eta_2 C_{\zeta_1 \zeta_1}^{j*} C_{\zeta_1 \zeta_1}^{j_1} C_{\zeta_2 \zeta_2}^{j_2}$$

where we have used the subsidiary condition to extract the last two C 's. Hence

$$\mathcal{P}|\lambda, m(j_s), \zeta_1, \zeta_2\rangle = |\lambda^{\pi}, m(j_s), \zeta_1', \zeta_2'\rangle \eta_1 \eta_2 C_{\zeta_1' \zeta_1' \zeta_1 + \zeta_2}^{j*} C_{\zeta_1' \zeta_1}^{j_1} C_{\zeta_2' \zeta_2}^{j_2} \quad (5.10)$$

$$= \eta_{12} |\lambda^{\pi}, m(j_s), -\zeta_1, -\zeta_2\rangle$$

 

where

$$\eta_{12} = \eta_1 \eta_2 (-1)^{j-j_1-j_2} \quad (5.11)$$

Thus the effect of parity on the reduced states is the same as on the one-particle states, except that the sign of the 'internal' quantum numbers ζ_1 and ζ_2 are reversed, and that the intrinsic parity is the compound expression η_{12} .

The selection rules for the reduced T -matrix when parity is conserved are now easy to obtain. On the two-particle sector we have, from the definition of the reduced T -matrix and using the summation convention

$$T_{\zeta_1 \zeta_2; \mu_3 \mu_4}^j(s) |1, m(j s), \zeta_1 \zeta_2\rangle = \pi T |1, m(j s), \mu_3 \mu_4\rangle \quad (5.12)$$

Applying the parity operator $\mathcal{P}$ to both sides of this equation, assuming that $\mathcal{P}$ commutes with T , and using (5.10) we obtain

$$\begin{aligned} T_{\zeta_1 \zeta_2; \mu_3 \mu_4}^j(s) |1, m(j s), -\zeta_1, -\zeta_2\rangle \eta_{12} &= \pi T |1, m(j s), -\mu_3, -\mu_4\rangle \eta_{34} \\ &= T_{-\zeta_1 -\zeta_2; -\mu_3 -\mu_4}^j(s) |1, m(j s), -\zeta_1, -\zeta_2\rangle \eta_{34} \end{aligned} \quad (5.13)$$

from which on comparing coefficients, and using (5.11), we have

$$T_{\zeta_1 \zeta_2; \mu_3 \mu_4}^j(s) = T_{-\zeta_1 -\zeta_2; -\mu_3 -\mu_4}^j(s) \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{j_1+j_2-j_3-j_4} \quad (5.14)$$

The relation (5.14) is identical to that obtained by Jacob and Wick.

6. Time-Reversal

We assume as usual that the time reversal operator $\mathcal{T}$ is anti-linear.

We then have

$$\mathcal{T}|\lambda, \mathfrak{z}\rangle = |\lambda^\tau, \mathfrak{z}'\rangle \varepsilon_{\mathfrak{z}'\mathfrak{z}} \quad (6.1)$$

where $\varepsilon_{\mathfrak{z}'\mathfrak{z}}$ is a matrix to be determined, λ^τ is the time-reversal of λ and the transformation is understood to be anti-linear. λ^τ is actually the same as the parity transform λ^π because, like parity, time-reversal leaves the rotations invariant and reverses the boosts. However, the anti-linearity of $\mathcal{T}$ means that the condition for the compatibility of (6.1) and the subsidiary condition (5.2) is not the analogue of (5.3) but

$$D_{\mathfrak{z}'\mathfrak{z}}^j(h) \varepsilon_{\mathfrak{z}'\mathfrak{z}} = \varepsilon_{\mathfrak{z}'\mathfrak{z}} D_{\mathfrak{z}'\mathfrak{z}}^{j*}(h) \quad (6.2)$$

where star denotes as usual complex conjugation. To solve (6.2) we note that if C is the charge-conjugation matrix defined in the last section, then from (5.8) C^j has exactly the property (6.2) required of ε . Hence from the irreducibility of $D^j(h)$ we have

$$\varepsilon_{\mathfrak{z}'\mathfrak{z}} = \varepsilon C_{\mathfrak{z}'\mathfrak{z}}^j \quad (6.3)$$

where ε is a number. The involutive nature of $\mathcal{T}$ together with the anti-linearity, imply that

$$\varepsilon^* \varepsilon = 1 \quad (6.4)$$

Hence the action of $\mathcal{T}$ on the one-particle states is

$$\mathcal{T}|\lambda, \mathfrak{z}\rangle = \varepsilon |\lambda^\tau, \mathfrak{z}'\rangle C_{\mathfrak{z}'\mathfrak{z}}^j \quad (6.5)$$

where the number ε is unitary and the transformation is understood to be anti-linear. Using the subsidiary condition we can actually reduce (6.5)

to a more compact form, namely,

$$\mathcal{J}|\lambda, \mathfrak{z}\rangle = \varepsilon |\lambda^T c, \mathfrak{z}\rangle \quad (6.6)$$

a simplification that is clearly peculiar to Mackey states.

To determine the operation of $\mathcal{J}$ on the reduced two-particle states we first note that since $r^T = r$, $\mathfrak{z}^T = c\mathfrak{z}c^{-1}$,

$$\mathcal{J}|\lambda r \mathfrak{z}, \mathfrak{z}\rangle = \varepsilon |\lambda^T r \mathfrak{z}^T c, \mathfrak{z}\rangle = \varepsilon |\lambda^T r c \mathfrak{z}, \mathfrak{z}\rangle \quad (6.7)$$

and hence, using also the anti-linearity of $\mathcal{J}$,

$$\begin{aligned} \mathcal{J}|\lambda, m(j_s), \mathfrak{z}_1, \mathfrak{z}_2\rangle &= \mathcal{J} \int d\mu(r) D_{mn}^{j*}(r) |\lambda r \mathfrak{z}_1, \lambda r \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2\rangle \\ &= \int d\mu(r) D_{mn}^{j*}(r) |\lambda^T r c \mathfrak{z}_1, \lambda^T r c \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2\rangle \varepsilon_1 \varepsilon_2 \\ &= \int d\mu(r) D_{mn}^{j*}(rc^{-1}) |\lambda^T r \mathfrak{z}_1, \lambda^T r \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2\rangle \varepsilon_1 \varepsilon_2 \end{aligned} \quad (6.8)$$

On the other hand from (5.8) we have

$$D^j(rc^{-1}) = D^j(r) (c^j)^{-1} = (c^j)^{-1} D^{j*}(r) = \tilde{c}^j D^{j*}(r) \quad (6.9)$$

where tilde denotes transpose (c^j may be taken to be real). Hence

$$\begin{aligned} \mathcal{J}|\lambda, m(j_s), \mathfrak{z}_1, \mathfrak{z}_2\rangle &= \tilde{c}^j_{mm'} \int d\mu(r) D_{mn}^j(r) |\lambda^T r \mathfrak{z}_1, \lambda^T r \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2\rangle \varepsilon_1 \varepsilon_2 \\ &= \varepsilon_1 \varepsilon_2 |\lambda^T, m'(j_s), \mathfrak{z}_1, \mathfrak{z}_2\rangle c^j_{m'm} \\ &= \varepsilon_1 \varepsilon_2 |\lambda^T c, m(j_s), \mathfrak{z}_1, \mathfrak{z}_2\rangle \end{aligned} \quad (6.10)$$

Thus the operation of $\mathcal{J}$ on the reduced states is exactly the same as on the one-particle states, with $\varepsilon = \varepsilon_1 \varepsilon_2$.

To obtain the selection rules imposed on the reduced T -matrix by time-reversal invariance we use the general relation

$$(a, Tb) = (T^\dagger a, b) = (\mathcal{J}b, \mathcal{J}T^\dagger a) = (\mathcal{J}b, T\mathcal{J}a) \quad (6.11)$$

where the second step expresses the anti-linearity of $\mathcal{T}$, and the third step T -invariance. Inserting the reduced states (3.1) into (6.11) we obtain in particular

$$\langle \Lambda, M(j_s), \zeta_3, \zeta_4 | T | 1, m(j_s), \zeta_1, \zeta_2 \rangle = \langle 1, m(j_s), \zeta_1, \zeta_2 | T | \Lambda, M(j_s), \zeta_3, \zeta_4 \rangle \times \epsilon_1^* \epsilon_2^* \epsilon_3 \epsilon_4 \quad (6.12)$$

and hence

$$T_{\zeta_3 \zeta_4; \zeta_1 \zeta_2}^j(s) = \epsilon_1^* \epsilon_2^* \epsilon_3 \epsilon_4 T_{\zeta_1 \zeta_2; \zeta_3 \zeta_4}^j(s) \quad (6.13)$$

a result which again agrees with that of Jacob and Wick. Note that for elastic scattering $T^j(s)$ is properly symmetric.

7. Identical Particles

The physically realizable two-particle states of identical particles are the symmetrized and anti-symmetrized combinations for bosons and fermions respectively. Hence they can be obtained from the two-particle states which we have hitherto discussed by applying to them the projection operator

$$Q = [1 + (-1)^{2j_1} P_{12}] / \sqrt{2} \quad (7.1)$$

where P_{12} is the exchange operator defined by

$$P_{12} |\lambda_1, \lambda_2, \zeta_1, \zeta_2\rangle = |\lambda_2, \lambda_1, \zeta_2, \zeta_1\rangle \quad (7.2)$$

In this section we wish to derive the explicit form of Q for the reduced states $|\lambda, m(j_s), \zeta_1, \zeta_2\rangle$. From (3.2) and (7.2) we have

$$P_{12} |\lambda, m(j_s), \zeta_1, \zeta_2\rangle = \int d\mu(r) D_{mn}^{j_s}(r) |\lambda r \zeta_2^{-1}, \lambda r \zeta_1, \zeta_1, \zeta_2\rangle \quad (7.3)$$

Comparing this equation with (5.7) we see that

$$P_{12} |\lambda, m(j_s), \zeta_1, \zeta_2\rangle = \mathcal{P} |\lambda^*, m(j_s), \zeta_1, \zeta_2\rangle \eta_1 \eta_2, \quad (7.4)$$

where $\mathcal{P}$ is the parity operator. Hence from (5.10) and (5.11) we have

$$P_{12} |\lambda, m(j_s), \zeta_1, \zeta_2\rangle = |\lambda, m(j_s), -\zeta_2, -\zeta_1\rangle (-1)^{j-2j_1} \quad (7.5)$$

From (7.1) we then have

$$Q = [1 + (-1)^j R_{12}] / \sqrt{2} \quad (7.6)$$

where R_{12} is the exchange operator

$$R_{12} |\lambda, m(j_s), \zeta_1, \zeta_2\rangle = |\lambda, m(j_s), -\zeta_2, -\zeta_1\rangle \quad (7.7)$$

8. Crossing for Covariant and Wigner States

Before proceeding to the crossing relations for Mackey states, it is convenient to recall first the conventional crossing relations, using as a starting point the crossing relations for the covariant states derived rigorously by Bros, Epstein and Glaser⁽¹⁰⁾. The relations are

$$(p_1 p_2, \alpha_1 \alpha_2 | \tilde{T} | p_3 p_4, \alpha_3 \alpha_4 \rangle = (-1)^{\sigma} (p_1 -p_3, \alpha_1 \alpha_3 | \tilde{T} | -p_2 p_4, \alpha_2 \alpha_4 \rangle, \quad (8.1)$$

where σ is unity if 2 and 3 are fermions and is zero otherwise⁽¹¹⁾. $\tilde{T}$ is the T -matrix with the energy-momentum conservation δ -function removed. The right hand side is obtained by the substitution $(p_2, p_3) \rightarrow (-p_2, -p_3)$ from the alternative channel amplitude $(p_1 p_3, \alpha_1 \alpha_3 | \tilde{T} | p_2 p_4, \alpha_2 \alpha_4 \rangle$ transformation law

$$(p_1 p_3, \alpha_1 \alpha_3 | \tilde{T} | p_2 p_4, \alpha_2 \alpha_4 \rangle = (\lambda_{p_1} \lambda_{p_3, \alpha'_1 \alpha'_3} | \tilde{T} | \lambda_{p_2} \lambda_{p_4, \alpha'_2 \alpha'_4} \rangle \times \\ \times \mathcal{D}_{\alpha'_1 \alpha_1}^{(1)}(\lambda) \dots \mathcal{D}_{\alpha'_4 \alpha_4}^{(4)}(\lambda) \quad (8.2)$$

We distinguish the covariant and Wigner states by the use of $\alpha, \alpha', \dots$ and $\zeta, \zeta', \dots$ for the spin indices. It is crucial in this equation that both the incoming and outgoing states corresponding to particle 2 and 3 transform according to the same representation $\mathcal{D}^{(a)}(\lambda)$ of the Lorentz group. Otherwise the positions of α_2 and α_3 could not be interchanged in (8.1). This means, however, that the outgoing states $(p, \alpha |$ are not the natural adjoints $\langle p, \alpha |$ of the incoming states, since the latter would transform according to the conjugate representations $\mathcal{D}^{(a)*}(\lambda)$. The relationship between the two sets of states is

$$(p, \alpha) = (p, \alpha' | M_{\alpha' \alpha}(p) \quad (8.3)$$

where the matrix $M(p)$ is given by

$$M(p) = \mathcal{D}(\ell) \mathcal{D}^\dagger(\ell) C \quad (8.4)$$

ℓ being any Lorentz transformation that accelerates the momentum from rest to p , and C being the charge conjugation matrix of section 5, which with respect to the complexification $\mathcal{D}(\lambda)$ of $\mathcal{D}(r)$ has the property

$$C \mathcal{D}^* C^{-1} = (\mathcal{D}^\dagger)^{-1} \quad (8.5)$$

Equations (8.1) - (8.4) summarize the crossing relations for covariant states.

The transformation law (8.2) is actually important not only for defining the transformation properties of the states in the manner just discussed, but also for relating centre-of-mass configurations. For $(p_1 p_2)$ and $(p_1 p_3)$ cannot both be centre-of-mass configurations except in the trivial case of forward scattering. But $(p_1 p_2)$ and $(\lambda p_1, \lambda p_3)$ can both be centre-of-mass, provided that λ is suitably chosen. Indeed if $(p_1 p_2)$ and $(\lambda p_1, \lambda p_3)$ are both z -directional centre-of-mass configurations (i.e. incoming particles moving along the z -axis in both cases) then λ is actually unique up to a rotation around the z -axis, and we can write $\lambda = \lambda(p_1 p_2 p_3 p_4)$.

Finally, we note that in order to insert (8.2) into (8.1) and relate centre-of-mass configurations we must continue the $\mathcal{D}(\lambda)$, $\lambda = \lambda(p_1 p_2 p_3 p_4)$ to $\mathcal{D}(\lambda)$, $\lambda = \lambda(p_1, -p_2, -p_3, p_4)$. This requires a specification of paths in the $SL(2, C) \otimes SL(2, C)$ space corresponding to the path $p \rightarrow -p$. This complication is discussed in detail in refs. (10) and (11) and we shall return to it briefly in the discussion of the Mackey states.

To obtain the crossing relations corresponding to (8.1), (8.2) and (8.3) for Wigner states one uses the definition of the Wigner states in

terms of covariant states from (1.6) and (1.7), namely

$$|p, \zeta\rangle = |p, \alpha\rangle \mathcal{D}_{\alpha\zeta}(\ell(p)),$$

where $\ell(p)$ is the standard boost. One obtains at once from (8.1), (8.2) and (8.3) the relations

$$(p_1 p_2, \zeta_1 \zeta_2 | \tilde{T} | p_3 p_4, \zeta_3 \zeta_4) = (-1)^{\sigma} (p_1 - p_3, \zeta_1 \zeta_3 | \tilde{T} | -p_2 p_4, \zeta_2' \zeta_4) \mathcal{D}_{\zeta_3 \zeta_3}^{(3)}(-1) \mathcal{D}_{\zeta_2' \zeta_2}^{(2)}(-1) \quad (8.6)$$

where the amplitude on the right hand side is obtained by the substitution $(p_2, p_3) \rightarrow (-p_3, -p_2)$ from the alternative channel amplitude with the transformation law

$$(p_1 p_3, \zeta_1 \zeta_3 | \tilde{T} | p_2 p_4, \zeta_2 \zeta_4) = (\lambda_{p_1} \lambda_{p_3}, \zeta_1' \zeta_3' | \tilde{T} | \lambda_{p_2} \lambda_{p_4}, \zeta_2' \zeta_4') \times \quad (8.7)$$

$$\times D_{\zeta_1' \zeta_1}^{(1)}(h_1) \dots D_{\zeta_4' \zeta_4}^{(4)}(h_4)$$

the h_a being the Wigner rotations

$$h_a = \ell^{-1}(\lambda_{p_a}) \lambda \ell(p_a)$$

The values of $\mathcal{D}^{(a)}(-1)$ in (8.6) are to be specified by the path of analytic continuation. The relationship between the outgoing kets $(p, \zeta|$ and the adjoints $\langle p, \zeta|$ of the ingoing kets is much simpler for the Wigner case, namely

$$(p, \zeta| = \langle p, \zeta' | C_{\zeta' \zeta} \quad (8.8)$$

where C is the charge-conjugation matrix. The reason is that the $D(h)$ are rotation matrices and hence (8.5) reduces to $D(C) D^x D(C^{-1}) = D$ from (5.8).

9. Crossing for Mackey States

To obtain the crossing relations for Mackey states, we proceed in analogy to the Wigner case, using the covariant crossing relations (8.1), (8.2) and (8.3) and the definition (1.6) of the Mackey states in terms of the covariant states, namely,

$$|\lambda, \zeta\rangle = |p, \alpha\rangle \mathcal{D}_{\alpha\zeta}(\lambda), \quad p = \lambda k \quad (9.1)$$

In this manner we obtain for the Mackey states the crossing relations

$$\begin{aligned} (\lambda_1, \lambda_2, \zeta_1, \zeta_2 | \tilde{T} | \lambda_3, \lambda_4, \zeta_3, \zeta_4) &= (-1)^{\sigma} (\lambda_1, -\lambda_3, \zeta_1, \zeta_3 | \tilde{T} | -\lambda_2, \lambda_4, \zeta_2, \zeta_4) \mathcal{D}_{\zeta_1 \zeta_3}^{(3)}(-1) \mathcal{D}_{\zeta_2 \zeta_4}^{(2)}(-1) \\ &= (-1)^{\sigma} (\lambda_1, -\lambda_3, \zeta_1, \zeta_3 | \tilde{T} | -\lambda_2, \lambda_4, \zeta_2, \zeta_4) \mathcal{D}_{\zeta_1 \zeta_3}^{(3)}(-1) \mathcal{D}_{\zeta_2 \zeta_4}^{(2)}(-1) \end{aligned} \quad (9.2)$$

where

$$(\lambda, \zeta | = \langle \lambda, \zeta' | \mathcal{C}_{\zeta' \zeta} \quad (9.3)$$

and the values of $\mathcal{D}^{(a)}(-1)$ depends on the path of analytic continuation.

The last equation is the same as in the Wigner case and is equivalent to the condition that all the states in (9.2) have the same Mackey subsidiary condition i.e.

$$|\lambda h, \zeta\rangle = |\lambda, \zeta'\rangle D_{\zeta \zeta'}^*(h), \quad \langle \lambda h, \zeta | = \langle \lambda, \zeta' | D_{\zeta \zeta'}^*(h), \quad (\lambda h, \zeta | = (\lambda, \zeta' | D_{\zeta \zeta'}^*(h) \quad (9.4)$$

Equation (9.3) is the same as in the Wigner case and is equivalent to the condition that all the states occurring in (9.2) have a D^j and not a D^{j*} matrix in the Mackey subsidiary condition.

In practice, however, what is needed for some purposes such as partial-wave

expansions is not so much the crossing relations (9.2) and (9.3) for general Mackey states, but the crossing relation for Mackey base states. The latter relation is not displayed in (9.2) because, as can easily be checked, there is no value of λ for which incoming states $|\lambda_1, \lambda_2, \zeta_1, \zeta_2\rangle$ and $|\lambda\lambda_1, \lambda\lambda_3, \zeta_1, \zeta_3\rangle$ are both base states. (Similarly the outgoing states cannot both be base states). To obtain a crossing relation for base-states we let the Mackey states in one channel, $(\lambda_1, \lambda_2, \zeta_1, \zeta_2 | \tilde{T} | \lambda_3, \lambda_4, \zeta_3, \zeta_4\rangle$ say, be base states, and expand the states in the other channel into linear combinations of base states according to the lemma of section 2. We then obtain as a supplement to (9.2) the relation

$$(\lambda_1, \lambda_3, \zeta_1, \zeta_3 | \tilde{T} | \lambda_2, \lambda_4, \zeta_2, \zeta_4\rangle = (\lambda\tilde{\zeta}_1(s)h_1, \lambda_3\tilde{\zeta}_3(s)h_3, \zeta_3 | \tilde{T} | \lambda\tilde{\zeta}_2(s)h_2, \lambda\tilde{\zeta}_4(s)h_4, \zeta_4\rangle \quad (9.5)$$

where $\lambda, \tilde{\zeta}_i(s)$ and h_i have been defined in section 2. Equations (9.2), (9.3) and (9.5) are the complete set of crossing relations for Mackey states, including base states. It will be seen that (9.5) rather than (9.3) leads to the real analogue of the Trueman-Wick relation, since it contains the Wigner rotations h_a . However, (9.5) is formally simpler since the Wigner rotations appear in the arguments of the states instead of D -matrices. The D -matrices can, of course, be extracted from (9.5) at any time by using the Mackey subsidiary condition.

A further slight advantage of the Mackey states is the possibility of displaying explicitly in the argument the choice of paths in $SL(2, C) \otimes SL(2, C)$ -space corresponding to $p \rightarrow -p$, mentioned earlier. The ambiguity in paths originates in the fact that for $p \rightarrow -p$, we have $\lambda(u) \rightarrow -\lambda(u)$, $u \in SL(2, C) \otimes SL(2, C)$, and the equation $\lambda(v) = -1$ has two solutions for v in $SL(2, C) \otimes SL(2, C)$. But for half-integral spin the proper definition of the Mackey states is

$$|u, \zeta\rangle = |p, \alpha\rangle \mathcal{D}_{\alpha\zeta}(u), \quad u \in SL(2, C) \quad (9.6)$$

as mentioned in the introduction, and if we use this definition, then in equations (9.1), (9.3) and (9.5) λ is replaced by u throughout, while for equation (9.2) we obtain

$$\begin{aligned}
 (u_1, u_2, \zeta_1, \zeta_2 | \tilde{T} | u_3, u_4, \zeta_3, \zeta_4) &= (-1)^{\sigma} (u_1, v_3 u_3, \zeta_1, \zeta'_3 | \tilde{T} | v_2 u_2, u_4, \zeta'_2, \zeta_4) \times \\
 &\quad \times g_{\zeta'_3 \zeta_3}^{(3)}(-1) g_{\zeta'_2 \zeta_2}^{(2)}(-1)
 \end{aligned}
 \tag{9.7}$$

where v_2 and v_3 are solutions of $\lambda(v) = -1$. In this way the choice of path, namely the choice of v , is displayed explicitly in the states.

Appendix

From the definition of $|\lambda, m(j_s), \zeta_1, \zeta_2\rangle$ and the one-particle inner products we have

$$\Delta \equiv \langle \zeta_1, \zeta_2^{-1}, \zeta_1, \zeta_2 | \lambda, m(j_s), \mu_1, \mu_2 \rangle = \\ = \int d\mu(r) D_{mn}^{j*}(r) D_{\zeta_1 \mu_1}^{j_1}(\zeta_1^{-1} \lambda r \zeta_1) D_{\zeta_2 \mu_2}^{j_2}(\zeta_2 \lambda r \zeta_2^{-1}) 4\omega_1 \omega_2 \delta^3(p_1 - p'_1) \delta^3(p_2 - p'_2)$$

where

$$p_1 = \zeta_1 k_1 = (\omega_1, 0, 0, q) \quad p'_1 = \lambda r \zeta_1^{-1} k_1 = \lambda r (\omega'_1, 0, 0, q') \quad k_a = (m_a, 0, 0, 0)$$

$$p_2 = \zeta_2^{-1} k_2 = (\omega_2, 0, 0, -q) \quad p'_2 = \lambda r \zeta_2^{-1} k_2 = \lambda r (\omega'_2, 0, 0, -q') \quad \omega_a = (m_a^2 + q^2)^{1/2}$$

$$P \equiv P(q) = (p_1 + p_2) = (\omega_1 + \omega_2, 0, 0, 0) \quad P' = (p'_1 + p'_2) = \lambda r P(q') = \lambda P(q') \quad s^{\frac{1}{2}} = P_0 = \omega_1 + \omega_2$$

$$2Q \equiv 2Q(q) = (p_1 - p_2) = (\omega_1 - \omega_2, 0, 0, 2q) \quad 2Q' = p'_1 - p'_2 = 2\lambda r Q(q') \quad 4sq^2 = [s - (m_1 + m_2)^2] \times [s - (m_1 - m_2)^2]$$

We then have

$$\begin{aligned} 4\omega_1 \omega_2 \delta^3(p_1 - p'_1) \delta^3(p_2 - p'_2) &= 4\omega_1 \omega_2 \delta^3(P - P') \delta^3(Q - Q') \\ &= 4\omega_1 \omega_2 \delta^3(P - P') \delta(|\vec{Q}| - |\vec{Q}'|) |\vec{Q}|^{-2} \delta(\Omega(Q, Q')) \\ &= 4\omega_1 \omega_2 \delta^3(P - P') \delta(q - q') q^{-2} \delta(\Omega(n, \lambda r n)), \quad n = (0, 0, 0, 1) \\ &\quad q = |\vec{Q}| \\ &= 4\omega_1 \omega_2 \delta^3(P - P') \frac{dP_0}{dq} \delta(P_0 - P'_0) q^{-2} \delta(\Omega(n, \lambda r n)) \\ &= 4P_0 q^{-1} \delta^4(P - P') \delta(\Omega(n, \lambda r n)) \end{aligned}$$

where $\Omega(a, b)$ denotes the angle between $\vec{a}$ and $\vec{b}$, and we have $P'_0 = \omega'_1 + \omega'_2$ because of the δ -function $\delta^3(P - P') = \delta^3(P')$.

Since $4P_0 q^{-1} \delta^4(P - P')$ is independent of r we may then write

$$\Delta = 4P_0 q^{-1} \delta^4(P - P') \int d\mu(r) D_{mn}^{j*}(r) D_{\zeta_1 \mu_1}^{j_1}(\zeta_1^{-1} \lambda r \zeta_1) D_{\zeta_2 \mu_2}^{j_2}(\zeta_2 \lambda r \zeta_2^{-1}) \delta(\Omega(n, \lambda r n)) =$$

$$\begin{aligned}
&= 4P_0 q^{-1} \delta^4(P-P') \int d\mu(r) D_{mn}^{j*}(\lambda^{-1}r) D_{\zeta_1 \mu_1}^{j_1}(\mathfrak{F}_1^{-1} r \mathfrak{F}_1) D_{\zeta_2 \mu_2}^{j_2}(\mathfrak{F}_2^{-1} r \mathfrak{F}_2) \delta(\Omega(n, \lambda r n)) \\
&= 4P_0 q^{-1} \delta^4(P-P') D_{mn'}^{j*}(\lambda^{-1}) \int d\mu(r) D_{nn'}^{j*}(r) D_{\zeta_1 \mu_1}^{j_1}(\mathfrak{F}_1^{-1} r \mathfrak{F}_1) D_{\zeta_2 \mu_2}^{j_2}(\mathfrak{F}_2^{-1} r \mathfrak{F}_2) \delta(\Omega(n, \lambda r n))
\end{aligned}$$

where we have set $z'_a = z_a$, because of the δ -function $\delta(q'-q)$, and have used the fact that λ is a rotation because of the δ -function $\delta^4(P-P')$.

To evaluate the integral we parametrize r by means of Euler angles

$$r = (\alpha, \beta, \gamma), \quad 0 \leq \beta < \pi, \quad 0 \leq \alpha < 2\pi, \quad 0 \leq \gamma < 4\pi$$

$$16\pi^2 d\mu(r) = d\alpha d(\cos\beta) d\gamma, \quad \delta(\Omega(n, r n)) = \delta(\cos\beta) \delta(\alpha)$$

and using the fact that $(\alpha \circ \gamma)$ commutes with z_a , we obtain at once

$$\begin{aligned}
\Delta &= 4P_0 q^{-1} \delta^4(P-P') D_{mn}^{j*}(\lambda^{-1}) \frac{4\pi}{16\pi^2} \delta_{n, \mu_1 + \mu_2} \delta_{\zeta_1 \mu_1} \delta_{\zeta_2 \mu_2} \\
&= \frac{s^{1/2}}{\pi q(s)} \delta^4(P-P') D_{nm}^j(\lambda) \delta_{\zeta_1 \mu_1} \delta_{\zeta_2 \mu_2}, \quad n = \zeta_1 + \zeta_2
\end{aligned}$$

Using the Lorentz transformation property of the Mackey states, we then have

$$\begin{aligned}
\langle \lambda \mathfrak{F}_1, \lambda \mathfrak{F}_2^{-1}, \zeta_1, \zeta_2 | \lambda', m(j's), \mu_1, \mu_2 \rangle &= \langle \mathfrak{F}_1, \mathfrak{F}_2^{-1}, \zeta_1, \zeta_2 | \lambda' \lambda', m(j's), \mu_1, \mu_2 \rangle \\
&= \frac{s^{1/2}}{\pi q(s)} \delta^4(P-P') D_{\zeta_1 + \zeta_2, m}^j(\lambda' \lambda') \delta_{\zeta_1 \mu_1} \delta_{\zeta_2 \mu_2} \\
&= \left[\frac{s^{1/2}}{\pi q(s)} \delta(s-s') \delta_{\zeta_1 \mu_1} \delta_{\zeta_2 \mu_2} \right] 2P_0 \delta^3(P-P') D_{\zeta_1 + \zeta_2, m}^j(\lambda' \lambda')
\end{aligned}$$

References

1. E. Wigner, Ann. of Math. 40, 39 (1939).
2. H. Joos, Fortschr. Physik, 10, 65 (1962).
3. A.D. Martin and T.D. Spearman, Elementary Particle Theory
(North-Holland, Amsterdam 1970).
4. Y. Shirokov, JETP 6, 919 (1958).
J.C. Guillot and J.L. Petit, Helv. Phys. Acta 39, 281, 300 (1966).
5. P. Moussa and R. Stora, Lectures at Hercegnovi International
School (September 1966).
6. J. Stratthdee, J. Boyce, R. Delbourgo and A. Salam, 'Partial
Wave Analyses Part I', Trieste preprint IC/67/9 (1967)(unpublished).
A.P. Balachandran, D.N. Goswami and M. Modjtahedzadeh, 'The Poincaré
Group', Syracuse preprint SU-1206-212 (1969) (unpublished)
P. Winternitz, 'The Poincaré Group, Its Little Groups and Their
Applications in Physics', preprint RPF/T/3, Rutherford Laboratory,
England (1968) (unpublished)
and references therein.
7. M. Jacob and G.C. Wick, Ann. Phys. 7, 404 (1959).
8. L. Foldy, Phys. Rev. 102, 568 (1956).
9. G. Mackey, Induced Representations of Groups and Quantum Mechanics,
(Benjamin, New York 1968).
L. O'Raiheartaigh and U.H. Niederer, Nato Istanbul Summer School
Lectures 1970 (to be published).
10. J. Bros, H. Epstein and V. Glaser, Comm. Math. Phys. 1, 240 (1965).
G. Cohen-Tannoudji, A. Morel and H. Navelet, Ann. Phys. 46, 239 (1968).
11. T.L. Trueman and G.C. Wick, Ann. Phys. 26, 322 (1964).