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THE ANALYTIC REPRESENTATION OF
ROD CLUSTER CONTROL ELEMENTS
IN NUCLEAR DESIGN CALCULATIONS

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ABSTRACT

Analytic schemes for the representation of Rod Cluster Control elements in reactor core design calculations have been developed and compared with experiment. Theoretical bases and limitations of these methods are discussed in this report. Finally, procedures have been provided to expedite the application of these methods to design.

In the course of this comparison between analysis and experiment, a fundamental improvement has been made in the analytic form used in the "worth" comparison. This improvement has relevance to all analysis compared with results derived from any form of "worth" substitution experiments.

I. INTRODUCTION

Heretofore, control elements in light water reactors designed by WAPD have been cruciform arrangements either of poison slabs or of poison rods in linear arrays so closely packed as to be equivalent to slabs. The functions of regulation and safety control in large chemically shimmed PWR reactors presently under design have now been assigned to so-called rod cluster control elements. Elements of this type consist of a cluster of 12 to 20 cylindrical Ag-In-Cd absorber rods having approximately the same dimensions as fuel rods which move as a unit within thimbles that are an integral part of every fuel assembly. The absorber rods are distributed within an approximately annular region with two fuel rods between absorbers.

A series of reactivity measurements related to this cluster design was initiated at the Westinghouse Reactor Evaluation Center. Reference 1 summarizes the experimental effort; it should be considered as a companion report to the analytic effort summarized in the following pages. The absolute worths in a variety of critical configurations were derived using partial moderator height techniques. The consequent experimental values provided criteria against which methods of cluster representation could be tested.

Experiments were conducted with various configurations rectilinear in plan view which encompassed cluster arrays of solid cylindrical absorbers; uniform arrays of absorbers were also measured. The absorbers were composed of Ag-In-Cd and three different absorber diameters - not in combination - were involved in these experiments - 0.260 in., 0.330 in., and 0.405 in. In addition to bare absorber measurements, experiments were conducted with combinations of stainless steel clad and thimbles of various thicknesses surrounding the absorbers; furthermore absorber-out critical heights were also measured for the various assemblies. Four methods were developed for the analytic representation of the clustered and the uniform arrays of these cylindrical absorbers. Since mutual interference between

absorber elements was virtually nil in the uniform arrays, that facet of analysis which accounted for this interference could be evaluated from results derived by applying the same method of representation to both types of arrays. Furthermore, all locations within the lattice corresponding to the positions for these elements of the uniform arrays were predetermined and the holes in the lattice spacer plates at these locations were enlarged. Thus these positions could be occupied by the next-to-largest poison rods in combination with a 60 mil radial thickness of stainless steel that simulates clad and thimble. All methods of analysis employed a four group representation ultimately based on surface boundary conditions for the individual cluster element derived from nuclear data⁽²⁾ which were then appropriately transformed for application to the representation of an individual cylindrical absorber. All methods of analysis were derived from this basic representation. Each method involved different transformations of this basic representation to achieve a final approximate representation for either the cluster or the uniform array. Also common to all methods was a reliance on the Wigner-Wilkins or conventional spectrum for the calculation of the homogenized fuel region thermal group constants used in the determination of the elementary capture rates. In all cases, however, one of the intermediate transformations involved a change from this conventional thermal model to the Mixed Number Density Model⁽³⁾. Use of the latter model (MND) in design analysis results in greater accuracy in the determination of peaking factors and over-all power distributions. Finally, those methods recommended for use in actual design are compatible with the coarse two-dimensional diffusion code mesh description normally used in analyzing large reactor configurations.

II. SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS

Analytic procedures based on the development of surface boundary conditions for cylindrical absorbers and the utilization of such boundary conditions in diffusion theory representations for the fuel-moderator lattice have been found to be adequate for the determination of the reactivity effects of Rod Cluster Control elements. The correlation between calculations and critical experiments is such that these methods are recommended for use in design of UO_2 fueled water moderated reactors utilizing cylindrical absorbers interspersed in the fuel rod lattice for control.

Three methods have been developed for the representation in design of Rod Cluster Control elements. The best of these, the Hybrid method, has been shown to produce calculated worths which, relative to experiment, fall within a band the magnitude of whose half-width is 2.5% in relative error about an average value which corresponds to a worth overestimate of 1% only; this method is also most versatile in that the choice of the final geometry approximating the cluster is at the discretion of the designer. The Shell method of representation has been proven to be almost as accurate as the Hybrid relative to experiment; this method requires only hand calculations for the development of a shell geometry approximation to the cluster. The third method is a representation in which the cluster absorption is homogenized into an appropriate volume of the fuel.

III. NEUTRON ABSORPTION OF CYLINDRICAL RODS

The contents of this section are abstracted from a report by H. Aisu dealing with the representation of a closely-packed cruciform array of cylindrical absorbers. The first part of this section describes the representation of a linear array of rods based on integration of the diffusion kernel. This is the analytic approach which, by means of a straight-forward hand calculation, provides absorption rates required for both the homogeneous and the shell cluster representations; the various facets of cluster representation in general, however, are nicely illuminated in the development of this approach. The second part of the section describes the calculation of the current-to-flux ratio parameter associated with individual cylindrical absorbers. This surface boundary condition is the starting point for all methods of cluster representation described in this report.

A. ABSORPTION OF A LINEAR ARRAY OF RODS

Usually in control rods, the neutron absorptions in the first and second groups (> 5530 ev in the normal 4 group scheme) are so small that the neutron source to the epithermal group can be taken to be spatially uniform, with fairly good accuracy. However, we can no longer expect a flat source to the thermal group, because of the strong neutron absorption in the epithermal group. At first, we will calculate the neutron absorption in an array of cylindrical rods in the flat source approximation and extend it to the non-flat source problem.

1. FLAT SOURCE

Let us consider the absorption of an array of cylindrical rods immersed in an infinite homogeneous medium.

We take the origin of cylindrical coordinate at the center of one of the cylinders, and define the notation as follows. (Fig. 3-1)

r_n : distance between the center of the cylinder n and point (r, ϕ)

a : pitch of the array

R : radius of the cylinder

Fig. 3-1

If we assume that the neutron source is zero in the cylinders and flat in the external medium, the neutron flux in the external medium can be expressed by

the superposition of the two contributions; from the flat source and from the neutron sink at the cylinder surfaces.

At this point we make the assumption that the diffusion kernel in the infinite medium can be used at any place in the external medium. This assumption corresponds to neglecting the perturbation in the diffusion kernel which is caused by the absence of neutrons that would have come through the space occupied by the cylinders. This assumption leads us to the same results which can be obtained from the zero order approximation in the Bessel expansion method.*

If we take J as the net current per unit length of the cylinder in which the origin of coordinates has been located, equation (3-1) follows.

$$\psi(r, \phi) = S - J \sum_n P(r_n) \quad (3-1)$$

$$S = Q/\Sigma_a$$

Q : neutron source density in the external medium

Σ_a : macroscopic absorption cross section of the external medium

$P(r_n)$ is a diffusion kernel from the surface of the cylinder n and is normalized by eq. (3-2).

$$\int P(r) dV = 1/\Sigma_a = L^2/D \quad (3-2)$$

D : diffusion coefficient of the external medium

L : diffusion length of the external medium

The integration covers the entire external medium.

If we define α as the ratio of net current to scalar flux at the cylinder surface,**

* The criterion for neglecting higher order terms in the Bessel expansion is given by Neumann (4) as,

$$\frac{0.13 \left(\frac{2R}{a}\right)^4}{M \left[1 + 0.4 \left(\frac{2R}{a}\right)^4\right]} \ll 1$$

although the effect of this approximation is not clear quantitatively. This criterion is satisfied in our problems. (M is given in eq. (3-13).

** Although α is dependent on the angle, we can take it as the ratio of the average current to the average flux at the cylinder surface.

$$\begin{aligned}\frac{J}{\alpha R} &= \int_0^{2\pi} \psi(R, \phi) d\phi \\ &= 2\pi S - J \int_0^{2\pi} \sum_n P(R_n) d\phi\end{aligned}$$

$$J = S \frac{D}{L} \frac{2\pi R}{B + \frac{D}{\alpha L}} \quad (3-3)$$

$$B = \frac{DR}{L} \sum_n \int_0^{2\pi} P(R_n) d\phi \quad (3-4)$$

$$R_n^2 = (na)^2 + R^2 - 2naR \cos \phi$$

The diffusion kernel from a cylindrical shell in an infinite medium is⁽⁵⁾,

$$P'(r) = \frac{1}{2\pi D} I_0\left(\frac{r'}{L}\right) K_0\left(\frac{r}{L}\right) \quad \text{for } r \geq r'$$

where r' is a radius of the source shell, I_n and K_n are n order Bessel functions in imaginary argument. The diffusion kernel $P'(r)$ is normalized to L^2/D in whole space, but the kernel $P(r)$ should be normalized to L^2/D in the external medium (moderating region) only.

$$P(r) = \frac{L^2}{DE} K_0\left(\frac{r}{L}\right) \quad (3-5)$$

$$E = \int_{\text{mod}} K_0\left(\frac{r}{L}\right) r dr d\phi \quad (3-6)$$

So that

$$B = \sum_n B_n \quad (3-7)$$

$$B_n = \frac{LR}{E} \int_0^{2\pi} K_0\left(\frac{R_n}{L}\right) d\phi \quad (3-8)$$

Using the addition theorem of Bessel function, for $n \neq 0$,

$$\begin{aligned}B_n &= \frac{LR}{E} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} K_m\left(\frac{na}{L}\right) I_m\left(\frac{R}{L}\right) \cos m\phi d\phi \\ &= 2\pi \frac{LR}{E} K_0\left(\frac{na}{L}\right) I_0\left(\frac{R}{L}\right)\end{aligned} \quad (3-9)$$

For $n = 0$,

$$B_0 = 2\pi \frac{LR}{E} K_0 \left(\frac{R}{L} \right) \quad (3-10)$$

where $n = 0$ correspond to the cylinder at whose center the origin of coordinates has been located.

The integration in eq. (3-6) is also obtained by the addition theorem after separating it into two parts.

$$\begin{aligned} E &= \int_{\text{mod}} K_0 \left(\frac{r}{L} \right) r \, dr \, d\phi \\ &= \int_{V_1} K_0 \left(\frac{r}{L} \right) r \, dr \, d\phi - \int_{V_2} K_0 \left(\frac{r}{L} \right) r \, dr \, d\phi \end{aligned}$$

where V_1 and V_2 denote the whole space except $n = 0$ rod region and rod region except $n = 0$ rod, respectively.

$$\begin{aligned} E &= \int_R^\infty dr \int_0^{2\pi} d\phi K_0 \left(\frac{r}{L} \right) r - \sum_{n \neq 0} \int_{\text{rod } n} K_0 \left(\frac{r}{L} \right) r \, dr \, d\phi \\ &= 2\pi LR K_1 \left(\frac{R}{L} \right) - \sum_{n \neq 0} \int_0^R dx \int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} K_m \left(\frac{na}{L} \right) I_m \left(\frac{x}{L} \right) \cos m\theta \\ &= 2\pi LR K_1 \left(\frac{R}{L} \right) - 2\pi LR \sum_{n \neq 0} I_1 \left(\frac{R}{L} \right) K_0 \left(\frac{na}{L} \right) \quad (3-11) \end{aligned}$$

From eq. (3-7), (3-9), (3-10) and (3-11),

$$B = \frac{K_0 \left(\frac{R}{L} \right) + I_0 \left(\frac{R}{L} \right) M}{K_1 \left(\frac{R}{L} \right) - I_1 \left(\frac{R}{L} \right) M} \quad (3-12)$$

$$M = 2 \sum_{n=1}^{\infty} K_0 \left(\frac{na}{L} \right) \quad (3-13)$$

Eq. (3-3), (3-12) and (3-13) give the total current into a cylindrical rod which is located in a symmetric array of the same type of rods. This result is exactly the same as the one that is obtainable from the Bessel expansion

method⁽⁶⁾ assuming the angular independence of the flux around the cylinder.

The M term expresses the interference effect between the rods and becomes zero for the isolated rod.

2. NON-FLAT SOURCE

The epithermal flux depression in the vicinity of control rods is important in the calculation of the thermal neutron absorption. Since the neutron source in the epithermal energy range can be taken as spatially flat, the epithermal flux in the external medium is expressed by eq. (3-1) and the thermal neutron source is obtained by multiplying this by the removal cross section, Σ_r .

$$\psi_4(r, \phi) = \int_{\text{mod}} \Sigma_{r3} \psi_3(r') Q_4(r' \rightarrow r) dr' - J_4 \sum_n P_4(r_n) \quad (3-14)$$

$$\psi_3(r, \phi) = S - J_3 \sum_n P_3(r_n) \quad (3-15)$$

where suffix 3 and 4 denote the epithermal and thermal group in the normal 4 group scheme, respectively; $Q(r' \rightarrow r)$ is the diffusion kernel from a line source and is normalized to L^2/D .*

$$\int_{\text{mod}} Q(r' \rightarrow r) dr = \frac{L^2}{D} \quad (3-16)$$

From the same argument in the derivation of eq. (3-3), eq. (3-17) follows.

$$J_4 = S' \frac{D_4}{L_4} \frac{2\pi R}{B_4 + \frac{D_4}{\alpha_4 L_4}} \quad (3-17)$$

$$S' = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{mod}} \Sigma_{r3} \psi_3(r') Q_4(r' \rightarrow R) dr' \quad (3-18)$$

S' can be separated into two parts.

$$S' = S_1 - S_2 \quad (3-19)$$

* Strictly speaking, it is impossible to set the normalization condition (3-16) independent of position, but we can easily show that the position dependence is small. (Appendix B).

$$\begin{aligned}
S_1 &= \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{mod}} \Sigma_{r3} S Q_4 (r \rightarrow R) dr \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{mod}} \Sigma_{r3} S \Sigma_4 (R \rightarrow r) dr = \Sigma_{r3} S \frac{L_4^2}{D_4}
\end{aligned} \tag{3-20}$$

$$S_2 = \Sigma_{r3} J_3 \Sigma_n \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{mod}} P_3(r_n) Q_4(r \rightarrow R) dr \tag{3-21}$$

The diffusion kernel from a line source is,

$$Q_4(r) = \frac{1}{2\pi D_4} K_0\left(\frac{r}{L_4}\right) \tag{3-22}$$

and $P_3(r_n)$ is expressed as,

$$P_3(r_n) = N_3 K_0\left(\frac{r_n}{L_3}\right) \tag{3-23}$$

$$N_3 = \frac{1}{2\pi D_3} \frac{1}{\frac{R}{L_3} \left[K_1\left(\frac{R}{L_3}\right) - I_1\left(\frac{R}{L_3}\right) M_3 \right]} \tag{3-24}$$

So that, S_2 is described as follows.

$$S_2 = \Sigma_{r3} J_3 \frac{N_3}{D_4} \Sigma_n S_{2n} \tag{3-25}$$

$$S_{2n} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{mod}} K_0\left(\frac{r_n}{L_3}\right) K_0\left(\frac{x}{L_4}\right) r dr \frac{d\phi}{2\pi} \tag{3-26}$$

$$r_n^2 = (na)^2 + r^2 - 2nar \cos \theta$$

$$x^2 = R^2 + r^2 - 2Rr \cos(\theta - \phi)$$

Since the moderator region is expressed by subtracting the rod region from the whole space,

$$S_{2n} = S_{2n}^{(1)} - S_n^{(2)} \tag{3-27}$$

$$s_{2n}^{(1)} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{V_1} K_0\left(\frac{r}{L_3}\right) K_0\left(\frac{x}{L_4}\right) r \, dr \, \frac{d\theta}{2\pi} \quad (3-28)$$

$$s_{2n}^{(2)} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{V_2} K_0\left(\frac{r}{L_3}\right) K_0\left(\frac{x}{L_4}\right) r \, dr \, \frac{d\theta}{2\pi} \quad (3-29)$$

V_1 : whole space except $n=0$ rod region

V_2 : all rod regions except $n=0$ rod

$s_{2n}^{(1)}$ can be obtained again using the addition theorem for the Bessel function.
For $n \neq 0$,

$$\begin{aligned} s_{2n}^{(1)} &= \int_0^{2\pi} \frac{d\phi}{2\pi} \int_R^{na} r \, dr \int_0^{2\pi} \sum_{\ell=-\infty}^{\infty} K_{\ell}\left(\frac{na}{L_3}\right) I_{\ell}\left(\frac{r}{L_3}\right) \cos \ell\theta \sum_{m=-\infty}^{\infty} K_m\left(\frac{r}{L_4}\right) I_m\left(\frac{R}{L_4}\right) \cos m(\theta-\phi) \frac{d\theta}{2\pi} \\ &+ \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{na}^{\infty} r \, dr \int_0^{2\pi} \sum_{\ell=-\infty}^{\infty} K_{\ell}\left(\frac{r}{L_3}\right) I_{\ell}\left(\frac{na}{L_3}\right) \cos \ell\theta \sum_{m=-\infty}^{\infty} K_m\left(\frac{r}{L_4}\right) I_m\left(\frac{R}{L_4}\right) \cos m(\theta-\phi) \frac{d\theta}{2\pi} \\ &= K_0\left(\frac{na}{L_3}\right) I_0\left(\frac{R}{L_4}\right) \int_R^{na} I_0\left(\frac{r}{L_3}\right) K_0\left(\frac{r}{L_4}\right) r \, dr + I_0\left(\frac{na}{L_3}\right) I_0\left(\frac{R}{L_4}\right) \int_{na}^{\infty} K_0\left(\frac{r}{L_3}\right) K_0\left(\frac{r}{L_4}\right) r \, dr \end{aligned} \quad (3-30)$$

After performing the integration and a little arrangement, noting the relation in Bessel functions,

$$I_0(x) K_1(x) + I_1(x) K_0(x) = 1/x$$

the following equation for $s_{2n}^{(1)}$ is obtained.

$$s_{2n}^{(1)} = \frac{L_4^2 I_0\left(\frac{R}{L_4}\right)}{1 - \left(\frac{L_4}{L_3}\right)^2} \left[W_1 K_0\left(\frac{na}{L_3}\right) - K_0\left(\frac{na}{L_4}\right) \right] \quad (3-31)$$

$$W_1 = \frac{R}{L_3} I_1\left(\frac{R}{L_3}\right) K_0\left(\frac{R}{L_4}\right) + \frac{R}{L_4} I_0\left(\frac{R}{L_3}\right) K_1\left(\frac{R}{L_4}\right) \quad (3-32)$$

For $n=0$, by the similar method,

$$S_{20}^{(1)} = \frac{L_4^2 I_0 \left(\frac{R}{L_4}\right)}{1 - \left(\frac{L_4}{L_3}\right)^2} W_2 \quad (3-33)$$

$$W_2 = \frac{R}{L_4} \left[K_1\left(\frac{R}{L_4}\right) K_0\left(\frac{R}{L_3}\right) \right] - \frac{R}{L_3} \left[K_1\left(\frac{R}{L_3}\right) K_0\left(\frac{R}{L_4}\right) \right] \quad (3-34)$$

The calculation of $S_{2n}^{(2)}$ is more difficult because of its complexity in the integral region and it is impossible to get a simple expression for this quantity, but we can show that $S_{2n}^{(2)}$ is negligible compared to $S_{2n}^{(1)}$ by an approximate method (Appendix A). So, from now on we neglect this term.

$$S_{2n}^{(2)} \ll S_{2n}^{(1)} \quad (3-35)$$

From eq. (3-19),

$$S' = \Sigma_{r3} S \frac{L_4^2}{D_4} Y \quad (3-36)$$

$$Y = 1 - \frac{N_3 J_3}{S} \frac{I_0\left(\frac{R}{L_4}\right)}{1 - \left(\frac{L_4}{L_3}\right)^2} \left[W_2 + W_1 M_3 - M_4 \right] \quad (3-37)$$

$$M_i = 2 \sum_{n=1}^{\infty} K_0\left(\frac{na}{L_i}\right), \quad i = 3, 4$$

The Y factor is related to the epithermal flux depression caused by the neutron absorption of the cylinder in that energy range, and naturally, it becomes unity by eq. (3-37) if there is no absorption at this energy.

If the radius of the cylinder becomes large compared to L_3 and L_4 , Y for the isolated rod should approach $Y(\text{slab})$ which is the corresponding source depression factor in slab geometry.

$$Y(\text{slab}) = 1 - \frac{1}{\left(1 + \frac{L_4}{L_3}\right) \left(1 + \frac{D_3}{\alpha_3 L_3}\right)} \quad (3-38)$$

In this case, using the expansion form of the Bessel function,

$$M_3 = M_4 = 0$$

$$B_3 \longrightarrow 1$$

$$\frac{I_0\left(\frac{R}{L_4}\right) W_2}{K_1\left(\frac{R}{L_3}\right)} \longrightarrow \frac{1}{2} \left(1 - \frac{L_4}{L_3}\right)$$

So that,

$$Y \longrightarrow 1 - \frac{1}{2} \frac{1}{\left(1 + \frac{L_4}{L_3}\right) \left(1 + \frac{D_3}{\alpha_3 L_3}\right)} \quad (3-39)$$

This discrepancy in a factor of two comes from the uncertainty in the normalization of the diffusion kernel for a line source, but this uncertainty is negligible if $R/L \lesssim 0.5$, as we show in Appendix B and, therefore, the discrepancy is not a hindrance in the practical situation.

B. CURRENT TO FLUX RATIO (α) AT CYLINDRICAL ROD SURFACE

1. FORMULA OF α FOR MONOENERGETIC NEUTRONS

In the calculation of the neutron absorption of control rods, the current to flux ratio at the surface, α , is as shown above, a required quantity.

$$\alpha = j / \phi_s \quad (3-40)$$

j : net current to the absorber

ϕ_s : scalar flux at the absorber surface

Since the net current into the absorber is equal to the total absorption in the absorber, divided by the surface,

$$\alpha = \frac{\Sigma_c \bar{\phi} V / S_o}{\phi_s} = \frac{\Sigma_c \bar{\ell}}{4G} \quad (3-41)$$

$$G = \phi_s / \bar{\phi}$$

Σ_c : macroscopic absorption cross section of the absorber

$\bar{\phi}$: average flux in the absorber

$\bar{\ell} := 4 V / S_o$, where V and S_o are volume and surface area of the absorber, respectively.

If we assume the isotropy of the neutron flux at the absorber surface, surface to average flux ratio, G , can be calculated by the arguments identical to those of references⁽⁷⁾ and ⁽⁸⁾. The effect of this assumption on α value is very small and is not essential in this problem.* Under this assumption, G can be related to P_{ae} , the capture probability of the entering neutrons in the absorber.

* The effect of the introduction of linear angular dependence on the flux can easily be examined, in cylindrical case, using the collision probability which is calculated by the successive generations method⁽⁹⁾; in slab geometry, using the blackness coefficient⁽¹⁰⁾ and for pure absorption case, comparing

$$1 + 2E_3 \frac{(\Sigma_c \bar{\ell})}{2} \quad \text{and} \quad 1 + 3E_4 \frac{(\Sigma_c \bar{\ell})}{2} \quad (11)$$

$$G = \frac{\Sigma_c \bar{l}}{2} \left[\frac{2}{P_{ae}} - 1 \right] \quad (3-42)$$

If we assume a uniform distribution of scattered neutrons in the absorber,*

$$P_{ae} = P_e \frac{\Sigma_c}{\Sigma_T} + P_e \frac{\Sigma_s}{\Sigma_T} P_a \quad (3-43)$$

P_e : first collision probability in the absorber for the entering neutrons

P_a : absorption probability in the absorber for the uniformly distributed neutron source in the absorber

Σ_T, Σ_s : macroscopic total and scattering cross section in the absorber, respectively.

P_e and P_a are related to P_c , the first collision probability in the absorber of the uniformly distributed neutron source in the absorber⁽¹²⁾.

$$P_e = \Sigma_T \bar{l} (1 - P_c) \quad (3-44)$$

$$1 - P_a = \frac{1 - P_c}{1 - \frac{\Sigma_s}{\Sigma_T} P_c} \quad (3-45)$$

From eq. (3-42) - (3-45), G follows.

$$G = 1 - \frac{\Sigma_c \bar{l}}{2} + \frac{\Sigma_c}{\Sigma_T} \frac{P_c \left(\frac{\Sigma_T \bar{l}}{2} \right)}{1 - P_c \left(\frac{\Sigma_T \bar{l}}{2} \right)} \quad (3-46)$$

For slab geometry and the zero scattering case,

$$2x \left[1 - P_c(x) \right] = 1 - 2E_3(x)$$

* We can eliminate this assumption by a more complicated treatment but the effect of this assumption on P_{ae} is also found to be small.⁽⁹⁾

$$\alpha_{\text{slab}} = \frac{1}{2} \frac{1 - 2E_3 \left(\frac{\Sigma_c \bar{l}}{2} \right)}{1 + 2E_3 \left(\frac{\Sigma_c \bar{l}}{2} \right)} \quad (3-47)$$

The difference in the denominator from the more exact formula in this special case⁽²⁾ comes from the neglecting of linear angular dependence in surface flux; and this difference produces a maximum error in the α value of 3.7% at $\Sigma_c \bar{l}/2 \approx 0.6$.

The difference between slab and cylindrical geometry but of the same $\bar{l}$, for the zero scattering case, is less than 10% in α and the maximum error occurs at $\Sigma_c \bar{l}/2 \approx 0.5$, as is shown in fig. 3-2. Since the difference shows a fairly sharp peak and goes down to about 3% at $\Sigma_c \bar{l}/2 = 1.5$, we can expect fairly good agreement in the α of Ag-In-Cd for both slab and cylindrical geometry since absorption of this alloy is mainly composed of sharp resonances at intermediate energies and strong absorption in the lower energies.

2. EPITHERMAL AVERAGING PROCESS

In the epithermal region, the α values change greatly in magnitude and the averaging process is important.

From the definition of α , it is reasonable to take the surface flux as the weighting factor.

$$\bar{\alpha} = \int \alpha(u) \phi_s(u) du / \int \phi_s(u) du \quad (3-48)$$

For slab geometry and narrow resonances $\phi_s(u)$ can be taken as⁽²⁾,

$$\phi_s(u) = \frac{1}{1 + \sqrt{3} \alpha(u)} \quad (3-49)$$

This representation of $\phi_s(u)$ is derived from the flat source assumption and is valid for widely separated resonances. But even for Ag-In-Cd absorber, the comparison of few group results with the multigroup calculation shows

Fig.3-2 Comparison of α for Slab and Cylindrical Geometry

that this assumption is not valid at the lower energies because of the strong depression of the flux at the higher energies in the vicinity of the absorber⁽¹³⁾. If we take $\phi_s(u)$ as eq. (3-49), the current into the Ag-In-Cd in the epithermal region is, in a typical case, larger in a few group calculation; also the thermal current is even more of an overestimate; the latter is caused by the misrepresentation of the slowing down density to the thermal group in the vicinity of the absorber.

In spite of these errors, the eigenvalue calculated in few group scheme shows very good agreement with the multigroup results⁽²⁾⁽¹³⁾. This excellent agreement comes from the cancellation of this error with the error caused by the misrepresentation of the slowing down phenomena in the vicinity of the absorber.

Since we do not have a better averaging process at present, at least in the normal four group scheme, we adopt the same assumption in cylindrical geometry in expectation of the same cancellation in eigenvalue. Therefore, the uncorrected α_3 , $\alpha_3(1)$, is taken from fig. 3-3 and $\alpha_4(1)$ can be taken as 0.5 (black).

3. MODIFICATION OF α FOR FINITE RADIUS

In the above section, the external medium is not taken into consideration in the calculation of α except in the averaging process in the epithermal group in which the effect of the external medium on the weighting factor is considered in a simple manner through its mean free path.

If we use the exact equation of the neutron transport in the external medium, we can describe the flux behavior correctly even in the vicinity of an absorber surface. However, if we use the diffusion equation in the external medium, some modification on α is necessary to represent the correct flux in the asymptotic region.

The correction factor on α for a black absorber in cylindrical geometry - let us denote it by C_B - has been calculated by many authors⁽¹⁴⁾ and the results are approximated in a simple equation⁽¹⁵⁾.

$$\alpha_{cy}(\text{Black}) = \frac{1}{2.13} \frac{1}{C_B} \quad (3-50)$$

Fig.3-3 α_3 for Ag-In-Cd

$$C_B = \left(1 + \frac{0.7604 \lambda_{tr}}{R} \right) / \left(1 + \frac{0.4052 \lambda_{tr}}{R} \right) \quad (3-51)$$

λ_{tr} : transport mean free path in the external medium

The factor of 2.13 in eq. (3-50) is calculated by transport theory. If we remain consistent with the assumption of isotropy in the surface flux, this factor should be taken as 2.

α value for a cylindrical gray absorber is expressed by eq. (3-52)^{(16)*}.

$$\frac{1}{\alpha_{cy}} = 4 \left(\frac{1}{g(a)} - 1 \right) + \frac{1}{\alpha_{cy}(\text{Black})} \quad (3-52)$$

$$g(a) = 2 \left[1 - P_c(a) \right] a$$

$$a = \Sigma_c R$$

If we denote $\alpha(C_B=1)$ as the α value which is calculated by eq. (3-41) and (3-46), α for the gray cylinder is obtained from eq. (3-50) and (3-52).

$$\alpha_{cy} = \frac{\alpha(C_B=1)}{1 + 2(C_B-1)\alpha(C_B=1)} \quad (3-53)$$

$\alpha(C_B=1)$: defined by eq. (3-41) and (3-46)

The correction factor for the grey cylinder is dependent on $\alpha(C_B=1)$ and approaches to 1, as it should, when the absorption is very weak.

* Eq. (3-52) can be derived from eq. (3-41) and (3-46) except for the C_B factor.

4. CORRECTION FOR CLAD EFFECT

The current to flux ratio α calculated in the section 3 is the value for a bare absorber. The actual control rods are clad in stainless steel and the steel reduces the neutron current into the rod.

If we assume the clad thickness is small compared to both the radius of the absorber and the mean free path in the clad, the conservation of neutrons leads us to the following equations.

$$R_o J_{in o} = R J_{in} (1 - 2\Sigma\Delta) F_2 + R J_{in} 2\Sigma_s \Delta \bar{F}_1 + R_o J_{out o} 2\Sigma_s \Delta \bar{F}_1 \quad (3-54)$$

$$\begin{aligned} R J_{out} &= R_o J_{out o} (1 - 2\Sigma\Delta) + R_o J_{out o} 2\Sigma_s \Delta (1 - \bar{F}_1) \\ &+ R J_{in} (1 - F_2) (1 - 2\Sigma\Delta) + R J_{in} 2\Sigma_s \Delta (1 - \bar{F}_1) \end{aligned} \quad (3-55)$$

J_{in}, J_{out} : entering and outgoing neutron current at the clad surface, respectively.

R : outer radius of the clad

$J_{in o}, J_{out o}, R_o$: quantities corresponding to J_{in}, J_{out} and R at the absorber surface, respectively.

Σ_s, Σ : scattering and total macroscopic cross section of clad material, respectively.

Δ : clad thickness

$\bar{F}_1$: The average probability that neutrons enter the absorber region from the uniformly distributed source in the clad

F_2 : The probability that isotropic incoming neutrons enter the absorber region

The albedo is defined by the ratio of the outgoing current to the incoming current and is related to α by eq. (3-57).

$$\beta = J_{\text{out}}/J_{\text{in}} \quad (3-56)$$

$$\alpha = \frac{1}{2} \frac{1-\beta}{1+\beta} \quad (3-57)$$

From eq. (3-54)-(3-56) and noting that $\Sigma\Delta \ll 1$, the blackness at the outer surface is,

$$\beta' = \beta + (1-F_2)(1-\beta) + \Sigma_a \Delta \left[(-2+2F_2-4F_2\beta)k + (2F_2-2\bar{F}_1-2F_2\bar{F}_1\beta)(1-\beta) \right] \quad (3-58)$$

where k is the ratio of the absorption cross section to the scattering cross section in the clad. For stainless steel k is negligibly small in the epithermal region and is about 0.25 in the thermal energy range. However, β is zero in the thermal energy range and,

$$1-F_2 \ll 1 \quad \text{and} \quad 1-2\bar{F}_1 \ll 1$$

we can obtain a simpler relation from (3-58).

$$\beta' = \beta + (1-F_2)(1-\beta) + \Sigma_a \Delta (1-\beta)^2 \quad (3-59)$$

The calculation of F_2 is as follows. By the assumption of isotropic surface flux, the angular dependence of the current is given by eq. (3-60).

$$J(\theta)d\theta = \cos \theta \sin \theta d\theta \quad (3-60)$$

θ : angle between the normal direction at the outer surface and the neutron direction

If we define $P(\theta)$ as the probability that neutrons of direction θ enter into the absorber region,

$$P(\theta) = 1 \quad \text{for } \theta \leq \theta_0 \quad (3-61)$$

$$P(\theta) = \frac{2\gamma}{\pi} \quad \text{for } \theta \geq \theta_0 \quad (3-62)$$

$$\theta_o = \cos^{-1} \frac{\sqrt{2R_o \Delta + \Delta^2}}{R_o + \Delta} \quad (3-63)$$

where γ is indicated in Fig. 3-4 and is calculated by simple algebra to be,

$$\gamma = \sin^{-1} \left(\frac{R_o}{\sqrt{2R_o \Delta + \Delta^2}} \cot \theta \right) \quad (3-64)$$

PQ, PT : Tangents to the cylinder from the point P on the surface

(Fig. 3-4)

From these equations $P(\theta)$ in Eq. (3-62) approaches 1 as θ approaches θ_o , as it should.

$$\begin{aligned} F_2 &= \int_0^{\pi/2} J(\theta) \cdot P(\theta) d\theta / \int_0^{\pi/2} J(\theta) d\theta \\ &= 1 - \frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2} + \frac{2\epsilon + \epsilon^2}{\pi} 4 \int_0^1 \frac{y \sin^{-1} y}{[1 + (2\epsilon + \epsilon^2)y^2]^2} dy \end{aligned} \quad (3-65)$$

$$\epsilon = \Delta / R_o \quad (3-66)$$

The integration of Eq. (3-65) can be carried out by expanding the denominator in series noting $\epsilon \ll 1$ and using,

$$\int_0^1 y \sin^{-1} y dy = \frac{\pi}{8} \quad \int_0^1 y^3 \sin^{-1} y dy = \frac{5\pi}{64} ;$$

$$\therefore F_2 = 1 - \epsilon + \epsilon^2 \quad (3-67)$$

If we calculate the higher order terms of ϵ , the next equation follows, which is also obtainable by a different approach.

$$F_2 = 1/(1+\epsilon) \quad (3-68)$$

The equations (3-57), (3-59) and (3-67) give us the value of α corrected for the clad effect.*

If the clad becomes thicker, we have to modify eq. (3-59) to make it more rigorous; one such modification is the application of the first collision probability for tubular geometry.

* We can show that the combination of the equations (3-53), (3-57), (3-59) and (3-67), taking $\Sigma_s=0$, agrees with the correction formula for a small air gap derived by Davison and Kushnerick⁽¹⁶⁾ in the case of small absorber radius.

IV. AVERAGE NEUTRON ABSORPTION FOR A CYLINDRICAL ELEMENT IN A CLUSTER

A. FORMULA FOR THE NEUTRON CURRENT

The formulas for neutron absorption of a cylindrical element in a linear array are given in Section III. Since the absorption of a cluster can be calculated by considering the elements of a cluster to be elements of superimposed linear arrays with aperiodic gaps, the pertinent results of Section III are repeated below with modifications where necessary in either the equations or their context.

Assuming a spatially uniform source to group 3, the neutron absorption in this group is expressed as follows:

$$J_3 = 2\pi RL_3 Q_3 / (B_3 + (D_3 / \alpha_{cy3} L_3)) \quad (4-1)$$

$$B_1 = [K_0(R/L_1) + I_0(R/L_1)M_1] / [K_1(R/L_1) - I_1(R/L_1)M_1] \quad (4-2)$$

$$M_1 = \frac{1}{N} \sum_{j=1}^N \left[\sum_{\substack{n=1 \\ n \neq j}}^N K_0(d_n/L_1) \right] \quad (4-3)$$

where J is the net neutron current into a rod; Q_3 , D and L are the neutron source (normalized to unity), diffusion coefficient and migration length of the external medium, respectively; R is the radial distance equal to the sum of the absorber radius plus total radial thickness of clad and thimble SS; (gaps are neglected) d_n is the distance between element j and the n th element of the cluster. The cluster is composed of N cylindrical absorbers. α_{cy1} is the current to flux ratio at the surface delimited by R .

For group 4, eq. (4-1) is modified to eq. (4-4) taking the neutron source depression into consideration.

$$J_4 = 2\pi RL_4 Q_4 Y_4 / (B_4 + (D_4 / \alpha_{cy4} L_4)) \quad (4-4)$$

Q_4 is $Q_3 \Sigma_{r3}/(\Sigma_{r3} + \Sigma_{a3})$ where Σ_{r3} and Σ_{a3} are the removal and absorption cross section of the external medium in group 3. Y_4 is the correction factor due to the non-flat neutron source and is expressed by,

$$Y_4 = 1 - \frac{I_0(R/L_4)(W_2 + W_1 M_3 - M_4)}{(B_3 + (D_3/\alpha_{cy3} L_3)) [K_1(R/L_3) - I_1(R/L_3) M_3] [1 - (L_4/L_3)^2]} \quad (4-5)$$

$$W_1 = R I_1(R/L_3) K_0(R/L_4)/L_3 + R I_0(R/L_3) K_1(R/L_4)/L_4 \quad (4-6)$$

$$W_2 = R K_1(R/L_4) K_0(R/L_3)/L_4 - R K_1(R/L_3) K_0(R/L_4)/L_3 \quad (4-7)$$

In the numerator in eq. (4-5), W_2 accounts for the source depression effect for a single rod whereas the quantity $(W_1 M_3 - M_4)$ accounts for the effect of macroscopic source depression in the vicinity of the cluster of absorbers. Normally, the source depression effect for a single rod is approximately a few per cent of the current and is not important. Since the effect for a cluster of absorbers, however, is quite large and may range from 10% to 20% of the current depending on the tightness of the cluster and the size of each absorber, this factor is important if the homogenization method is adopted; this is discussed later.

The fundamental assumptions on which the above equations depend are: (1) that diffusion equation is valid, (2) that the external medium is uniform, (3) that the neutron source for group 3 is spatially uniform. The first assumption is taken for granted in this paper except as regards the transport modification for the current to flux ratio at the cylindrical absorber surface which has been discussed. The third assumption is not important in the practical problem since the local depression in thermal flux is recovered in group 2. The second assumption, however, may introduce error for the interference term, M_3 , for group 4, in the case of a very tight cluster since the diffusion character is completely different in water

than in fuel rods. The diffusion length for water in group 4 is much larger than that for homogenized fuel so that the error, if any, might be an underestimate of the interference term. This will also be discussed in section VI.

If the array of absorbers is systematic, the M term can be simply determined and it has been examined for an infinite linear array, square lattice, rectangular lattice and hexagonal lattice by Neumann⁽⁴⁾.

M is mainly determined by the pitch between elements of a cluster although it depends slightly on the number of elements. This is explained by the fact that $K_0(x)$ is a rapidly decreasing function of its argument. The calculation of the direct sum in eq. (4-3), however, is recommended for an actual cluster which might be composed of double circular or square rings.

B. THERMAL GROUP MODEL

Normally, the Mixed Number Density (MND) model is used in the thermal group in the critical calculation because this model gives a better power distribution⁽³⁾⁽¹⁷⁾. However, as has been shown by Buslik⁽¹⁸⁾, the MND model has no theoretical justification in the vicinity of a strong absorber. Even in the absence of a strong absorber, the flux distribution calculated by the MND model changes too rapidly around the interface of two compositions, compared to multi-group results, because the diffusion length expressed by the MND model is too small. The reason that the MND model gives a better power distribution than the conventional model is explained by the compensation of errors in the flux distribution and in the effective fission cross section distribution. Since the diffusion length is an important quantity in the determination of neutron current into a strong absorber, the conventional model is superior in such a case. The numerical confirmation of this is given below comparing the neutron current by one and multi-thermal group calculations.

The lattice chosen was Yankee Core I and the study was done in slab geometry. A slab absorber (0.266 in. thick Ag-In-Cd), which can be

considered thermally black, was put at the left boundary of the lattice and a current equal zero condition was imposed at the right boundary. The cell thickness was 7.2 and 8 cm for case A and B, respectively, which is thick enough to represent the infinite core medium. In case A, the thermal neutron source was set as spatially uniform. In case B the source distribution was input from the few group calculation and represented a realistic distribution. In both cases, SLOP-1⁽¹⁹⁾ was run using the double P-1 option. Two types of boundary conditions at the left boundary were applied in the MND model calculation; that is $\alpha/(1/v)_{\max}$ and $\alpha/(1/v)_{\text{ww}}$, where $\alpha/(1/v)_{\max}$ and $(1/v)_{\text{ww}}$ are the current to flux ratio in conventional scheme, $(1/v)$ average over Maxwellian spectrum and $(1/v)$ average over Wigner-Wilkins spectrum, respectively. Neither $(1/v)$ average has any theoretical basis but they were used to show the inability of reproducing the correct neutron current within the MND model by merely adjusting current to flux ratio on some reasonable basis. The results are shown in Table 4-1.

TABLE 4-1 COMPARISON OF THERMAL NEUTRON CURRENT

Case	Temp. (°C)	Conv.	R* MND with $\alpha/(1/v)_{\max}$	MND with $\alpha/(1/v)_{\text{ww}}$
A(Flat Source)	260	1.05	0.88	-
B(Realistic Source)	260	1.05	0.81	0.85
	20	1.04	0.66	0.73

$$^*R = (\text{Neutron current by 1 group})/(\text{Neutron current by SLOP-1})$$

The superiority of the conventional model in the calculation of neutron current is clear from Table 4-1. The MND model would not only underestimate the thermal neutron absorption by an absorber but would also introduce a temperature dependent error even if it is adjusted by a factor at room

temperature, if the factor were assumed to be independent of temperature. This error would be, from Table 4-1, about 12 to 15% overestimate in the thermal neutron current at operating temperature, if it is adjusted at room temperature, which would introduce 5 to 7% overestimate in rod worth for the practical configuration.

C. WORTH ESTIMATE OF WATER OCCUPYING BALANCE OF CYLINDRICAL ELEMENT UNIT CELL

The current to flux ratio at the cylindrical absorber surface appears in the formula for neutron current and is considered to be the most fundamental quantity determining the absorber's character. The derivation and modifications of this quantity are discussed in section III. It is derived, at first for a bare cylindrical absorber, by Amouyal and Benoist method⁽⁸⁾ assuming isotropic flux at the cylindrical absorber surface. The current to flux ratio for a cylindrical absorber was compared with that for a slab and it was concluded that the ratio in group 3 for a cylinder can be approximated by the one for a slab with the same chord length. For the thermal group, the normal Ag-In-Cd absorber can be taken as black. After knowing the current to flux ratio for a bare cylinder, it is modified for the clad (and thimble) effect considering neutron balance in order to make it applicable to the outside clad (or thimble) surface. And again the modification for the transport effect is applied to the current to flux ratio, which enables us to apply it to an absorber of an arbitrary radius with the diffusion equation applying outside the absorber region.

The current to flux ratio at the cylinder surface is obtainable by the above process and can be applied to the equations (4-1) and (4-4) if the external medium is uniform. In the practical problem, however, there is a water region immediately adjacent to the cylindrical absorber and the effect of this should be examined.

The main effects of this water in the balance of the unit cell are two: the increase of the slowing down power for group 3 in the core and the increase of thermal neutron absorption by the absorber. The increase

of slowing down power increases the reactivity because the light water moderator core is normally undermoderated. The effect on reactivity can be roughly estimated from the experimental measurement of the reactivity worth of water holes. Experimental results show that this reactivity worth is around 5 to 7% of the rod worth depending on the radius of the absorber. Since the water volume is reduced when the absorber is inserted, the effect becomes smaller. This effect on reactivity might be around 2% of rod worth for a larger absorber and around 5% of rod worth for a smaller absorber in the 2.9 W/U lattice if the absorber has no neutron absorption. The increase of thermal neutron absorption by an absorber increases the rod worth whereas the increase of slowing down power decreases the worth. The effect of the increase of thermal neutron absorption on the reactivity can be estimated analytically. However, the integral result of the two effects has been examined below.

Initially, one dimensional cylindrical criticality calculations were performed in cylindrical geometry putting an absorber at the center, a water slot next to the absorber and a core region next to the water slot; 0.3302 and 0.5118 cm radius Ag-In-Cd were chosen as the absorber. The core region contained 2.7% enriched, 2.9/1 W/U or 3.7% enriched 2.9/1 W/U lattices. The outside radii of the water slot were taken as 0.6236 and 0.8118 cm. The former corresponds to the unit cell size of the 2.9/1 W/U lattice. Infinite medium group constants were used both for the water slot and the core region. A current equal zero condition was imposed at the outside boundary, located at a radial position of 6 cm.

Another set of calculations was performed in which fuel displaced the water slot. The worth differences between the two sets of calculations are shown in Table 4-2. As can be seen from the table, the error caused by neglecting the water slot is less than 2% of the absorber worth. The error shows no dependence on enrichment and no significant dependence on water slot size. The homogeneous and shell methods of representation approximate the true situation by assuming that the (clad) absorber is intimately surrounded by homogenized fuel. The uncertainty in worth associated with one aspect of this assumption has thus been shown to be small. However, the effect of

this assumption on mutual interference between absorbers in close proximity will be examined in Section VI. The discrete and hybrid methods of representation do not involve this approximation on the water slot under either aspect.

TABLE 4-2
Error Due to Neglecting Water Slot

Enrichment (%)	Absorber Radius (cm)	Water Slot Outside Radius (cm)	Error Due to Neglecting Water Slot (Relative to the Rod Worth)
2.7	0.3302	0.6236	1.4×10^{-2}
"	0.5118	"	0.6×10^{-2}
"	"	0.8118	1.7×10^{-2}
3.7	0.3302	0.6236	1.4×10^{-2}
"	0.5118	"	0.6×10^{-2}
"	"	0.8118	1.7×10^{-2}

V. REPRESENTATIONS FOR ABSORBER ARRAYS

The mesh number limitation in a two dimensional diffusion code is usually one of the most serious problems in the analysis of a power reactor. Although it is not impossible to represent each absorber discretely in some simple configurations, there are several problems which would be caused by use of this representation in an actual design. One such problem is inability to concentrate the mesh about the hot spot where a detailed description of the actual heterogeneity is desirable. Another problem is the error in rod worth due to coarse mesh around the absorbers, the amount of which is very difficult to evaluate.

One of the geometries which is not only simple but also fairly close to the real cluster arrangement is a square shell. Only 1 mesh space is needed across the width of the annulus to represent a cluster in this case. A simpler representation, discussed below, is a homogenization of absorbers into a fuel region such as an assembly. This will not give a realistic power distribution in the locale of a cluster and is considered to be useful only as a supplement to the shell representation whenever necessary. The transformation to a shell discussed below consists of a straightforward hand calculation based on the approaches outlined in the preceding two sections. A correction for the shell representation which preserves worth despite internal and external mesh coarseness is also described in this section.

A discrete method of representation, as used in the analysis of criticals, is also discussed. This representation as such is necessarily so extravagant of mesh that it is limited to representation of clusters only in simple criticals or hypothetical cells. The method, however, does not involve certain of the approximations associated with the direct shell transformation. Accordingly, upon the availability of the results of the application to criticals of both the shell and also the discrete methods of representation, the most accurate method of representation clearly required a synthesis of these two methods - thus the hybrid representation. The latter combines the accuracy of a type of discrete representation with the practicality of the shell geometry.

A. HOMOGENIZATION

The reactivity worth of an absorber is determined by the neutron absorption balance between the absorber and the fuel region if the position of an absorber is fixed in the core. Since the importance of neutron absorption is a function of the position of such an absorption, the conservation of neutron absorption balance only does not always give the correct rod worth. If the neutron absorption is homogenized in the vicinity of the actual absorber whilst conserving the neutron balance, however, control rod worth might be approximately preserved on the assumption that the importance of neutron absorption does not, at least locally, change much. Since this assumption is essential for the simple homogenization method in which only the neutron absorption balance between the absorber and the fuel region is conserved, this method should be checked at different positions in the core; this is discussed later.

The neutron absorption by an absorber in groups 3 and 4 can be obtained by eqs. (4-1) and (4-4). If this absorption is homogenized into some area (V), homogenized absorption cross section for absorbers (Σ_p) are given by

$$NJ_1/Q_1V = \Sigma_{p1} / [(\Sigma_r + \Sigma_a)_1 + \Sigma_{p1}]$$

or

$$\Sigma_{p1} = (\Sigma_r + \Sigma_a)_1 / [Q_1V/NJ_1 - 1] \quad (5-1)$$

where N is the number of absorbers in a cluster. In a uniform array of absorbers it is possible to choose the homogenized area (V) rather uniquely by dividing the core into as many equal areas as there are absorbers. There is, however, an arbitrariness in choosing the homogenized area for a cluster arrangement. The effect of this arbitrariness should be checked, especially for off-center positions, since the importance of neutron absorption varies.

The presence of a cluster in (V) means considerably fewer fuel rods in ~~that~~ homogenized area. Accordingly, $v\Sigma_{f1}$ and Σ_{a1} of the fuel should be reduced by

the volume ratio of the fuel region of the total homogenized area. Since the total unit cell area in which the absorber is inserted is considered as an absorber region (cf. section 4-C), the fraction which should multiply $v\Sigma_{fi}$ and Σ_{ai} is the ratio of the number of fuel rods in V to the total number of unit cells in the homogenized area. $v\Sigma'_{fi}$ for all groups can be obtained directly by this process and Σ'_{ai} for groups 3 and 4 is obtained by simply adding Σ_{pi} to the reduced Σ_{ai} , which is the contribution from the fuel rod, where primes indicate the cross sections in the homogenized region. This process, however, has not been examined further because the homogenization itself is not considered a powerful tool to represent a cluster and is only supplemental to the shell representation.

For groups 1 and 2, Σ'_{ai} can be taken as Σ_{ai} assuming that the neutron absorption by an absorber is the same as that of a fuel rod. This approximation might introduce a slight underestimate of the rod worth. However, it is not worth a more rigorous treatment because the effect is very small compared to the approximation in groups 3 and 4.

B. TRANSFORMATION TO A CIRCULAR OR A SQUARE SHELL

Let us first calculate the neutron absorption by a circular shell absorber. Notation is shown in Fig. 5-1. The diffusion equation is assumed to be valid in the absorber region. The neutron source is assumed to be spatially uniform except in the shell region where the source is zero. Regions 1 and 3 are filled by homogeneous fuel and region 2 extends to infinity.

The neutron fluxes in each region, ϕ_i , are expressed by,

$$\phi_1 = A I_0(r/L) + Q/\Sigma_a \quad (5-2)$$

$$\phi_2 = C I_0(r/L_a) + E K_0(r/L_a) \quad (5-3)$$

$$\phi_3 = B K_0(r/L) + Q/\Sigma_a \quad (5-4)$$

Fig. 5-1

where the suffix a to R, L and D indicates an absorber region. I_n and K_n are the first and second kind modified Bessel functions, respectively. After imposing flux and current continuity at each boundary and arranging the equation, the following relations are obtained.

$$a_1 A + b_1 B = Q/\Sigma_a M \quad (5-5)$$

$$a_2 A + b_2 B = Q/\Sigma_a M \quad (5-6)$$

$$a_1 = \left[I_0(R/L_a) K_1(R_a/L_a) + K_0(R/L_a) I_1(R_a/L_a) \right] \left[I_1(R/L)/\Delta \right] - I_0(R/L)/M \quad (5-7)$$

$$a_2 = L_a I_1(R/L)/R_a \Delta \quad (5-8)$$

$$b_1 = L_a K_1(R_a/L)/R \Delta \quad (5-9)$$

$$b_2 = \left[I_0(R_a/L_a) K_1(R/L_a) + K_0(R_a/L_a) I_1(R/L_a) \right] \left[K_1(R_a/L)/\Delta \right] - K_0(R_a/L)/M \quad (5-10)$$

$$\Delta = K_1(R_a/L_a) I_1(R/L_a) - K_1(R/L_a) I_1(R_a/L_a) \quad (5-11)$$

$$M = (D/L)/(D_a/L_a) \quad (5-12)$$

When the condition, $R_a - R \ll L_a$ or $R \gg L_a$, is satisfied, a_1 , b_2 and Δ can be approximated by,

$$a_1 = L_a \cosh(m) I_1(R/L)/\sqrt{RR_a} \Delta - I_0(R/L)/M \quad (5-13)$$

$$b_2 = L_a \cosh(m) K_1(R_a/L)/\sqrt{RR_a} \Delta - K_0(R_a/L)/M \quad (5-14)$$

$$\Delta = -L_a \sinh(m) \sqrt{RR_a} \quad (5-15)$$

$$m = (R_a - R)/L_a \quad (5-16)$$

From these A and B are obtained in eqs. (5-17) and (5-18).

$$A = \frac{KM(\sqrt{R_a/R} / \sinh(m) - \coth(m)) - 1}{\left[1 + M(I+K) \coth(m) + IKM^2 \right] I_0(R/L)} \cdot Q/\Sigma_a \quad (5-17)$$

$$B = \frac{IM(\sqrt{R_a/R} / \sinh(m) - \coth(m)) - 1}{\left[1 + M(I+K) \coth(m) + IKM^2 \right] K_0(R_a/L)} \cdot Q/\Sigma_a \quad (5-18)$$

$$I = I_1(R/L)/I_0(R/L) \quad (5-19)$$

$$K = K_1(R_a/L)/K_0(R_a/L) \quad (5-20)$$

The total neutron current into a shell, J, is given by,

$$\begin{aligned} J &= -2\pi RD I_1(R/L)A/L - 2\pi R_a D K_1(R_a/L)B/L \\ &= 2\pi(R + R_a) QT \end{aligned} \quad (5-21)$$

Noting that $(R_a - R) \ll R$, T is presented in eq. (5-22)

$$T = Q \frac{(I+K)/2 + IKM \tanh(m/2)}{I + (I+K)M \coth(m) + IKM^2} L \quad (5-22)$$

The eqs. (5-21) and (5-22) give the neutron current into a circular shell. Deviation of I and K from unity shows the effect of flux depression inside the circular shell. When the radius becomes very large compared to the diffusion length, I and K become 1 and T gives the neutron current into a unit area of slab absorber. This flux depression effect inside the shell is naturally more important for a strong absorber and becomes less important for a weak absorber if the dimension of the shell and character of the core medium are fixed. Since the inequality $R/L > 4$ normally holds for the thermal group, I and K can be replaced by 1 in this group without any errors. For the epithermal group, however, R/L can be as small as 1.5

in some cases and a numerical check is necessary before approximating I and K by 1 in this group. Fortunately, neutron absorption in the epithermal group is not so strong. The m value is normally 0.2 and hardly exceeds 0.4 in practical problems. Since M is around 0.4 in normal lattice, the error due to the replacement of I and K by 1 can be numerically checked and can be shown to be below 3% relative in this extreme case: $R/L = 1.5$ and $m = 0.4$. Thus the replacement is allowed in epithermal group also and the problem reduces to slab geometry.

In slab geometry the neutron current into a unit area of an absorber is expressed by,

$$j_s = QL / (1 + D/\alpha_s L) \quad (5-23)$$

where α_s is the current to flux ratio at a slab absorber surface. Equating this expression to T with $I=K=1$ leads to the following relation which is given elsewhere (2).

$$\alpha_s = D_a / L_a \cdot \tanh (m/2) \quad (5-24)$$

The total current into a circular shell, J , is now given by,

$$J = 2\pi (R + R_a) j_s \quad (5-25)$$

α_s can be used as an interim parameter between total neutron current into a circular shell and the group constants in the absorber. Knowing the total neutron current, α_s can be determined by setting T equal to j_s . And the group constants in the absorber again are obtained by the relation (5-24) and one more supplemental relation. The supplemental relation is given by,

$$\Sigma_{aa} D_a = 1/p \quad (5-26)$$

where Σ_{aa} is the macroscopic absorption cross section in the shell region. It should be noted that normal relation $L_a^2 = D_a / \Sigma_{aa}$ is retained in the shell region too. The p value approaches 3 for a very weak absorber and 4 for a very strong absorber⁽²⁾. The choice of a p value is discussed in section 5C.

In square shell geometry, the edge effect should be examined. The edge effect is caused, qualitatively, by the same flux depression as discussed above for circular geometry. However, the amount of the effect is expected to be larger because of the sharper angle in the square shell. The edge effect is estimated by the "absorption area" method⁽²⁰⁾. Since the zero flux boundary condition was applied in this analysis, the absolute value for neutron current is not correct. The correction for the edge effect, however, is considered to be usable in estimating the per cent contribution of this effect to the thermal neutron current. In the absorption area method, the ratio of the edge effect to the total neutron absorption in a square shell, ϵ , is given by,

$$\epsilon = 4(N_1 + N_2) \cdot L/S \quad (5-27)$$

where S is total surface area of a square shell (per unit length); N_1 and N_2 are the correction factors of the angles $3\pi/2$ and $\pi/2$, respectively. The values of N_1 and N_2 are given by Hurwitz and Roe⁽²⁰⁾ as,

$$N_1 = 4(4\pi/3 \sqrt{3} - 1)/3\pi = 0.602$$

$$N_2 = -4/\pi = -1.273$$

thus the fraction is about $-2.7 L/S$, which is around -3% to -4% in a practical problem. The edge effect is neglected in this report although the modification entailed is straightforward.

The process of transforming a cluster into a circular or square shell is now obvious from the above discussion. At first, the total neutron

current into a cluster of absorbers is calculated by the method described in Section IV and a shell is constructed preserving the area of the total fuel region and the fuel region inside the cluster; j_s is obtained next by dividing the total current by the surface area of the shell. j_s determines the current to flux ratio at the shell surface, which defines the effective group constants inside the shell. One thing, however, should be made clear in this process. As is discussed in Section IV, the conventional model is superior to the MND model in the calculation of thermal neutron current into an absorber. Since the MND model, however, gives a better power distribution in a critical calculation, which is a very important quantity, it is preferable to adopt the MND model in a critical calculation. In the derivation of J_s and α_s , total neutron current of a shell is being forced to the value precalculated in conjunction with conventional model. Thus it is possible to switch to the MND model in this process by changing L and D in the core region into MND values. This combination gives a better over-all power distribution while conserving correct neutron balance between the absorber and the core region.

In the practical problem, the calculation of the Y_4 factor is troublesome and an approximation for this factor is desirable. For slab geometry, the correction factor (γ_4) on the thermal neutron current due to source depression is given by,

$$\gamma_4 = 1 - \frac{1}{(1 + D_3/\alpha_{s3}L_3)(1 + L_4/L_3)} \quad (5-28)$$

where α_{s3} is an effective current to flux ratio derived by the above method. If Y_4 and γ_4 are the same or very close to each other in magnitude, the replacement of Y_4 and γ_4 by 1 is allowed; that is, the correction factors for the source depression in both geometries can be neglected without error. Numerical examples are given in Table 5-1 for the experimental and practical cluster arrangement.

TABLE 5-1

Comparison of the Correction Factors Y_4 and γ_4

Radius of Absorber (in.)	Number of Abs. in Cluster	Lattice	Y_4^*	γ_4^*
0.285	12	3.7%, 2.9/1	0.8996	0.9007
"	16	"	0.9099	0.9103
0.330	12	"	0.8667	0.8670
"	16	"	0.8799	0.8791
0.285	20	2.7%, 2.9/1	0.9239	0.9231
"	24	"	0.9129	0.9089
0.428	16	"	0.8860	0.8868
"	20	"	0.8663	0.8615
"	24	"	0.8468	0.8372

* Y_4 is calculated using L_4 by conventional model
 whereas γ_4 is by MND model.

The maximum difference is about 1% relative in these cases and the replacement of Y_4 and γ_4 by 1 is allowed.

C. FINITE MESH SIZE CORRECTION FOR THE SHELL REPRESENTATION

The analysis in Section VB has been performed by the diffusion equation which corresponds to adoption of an extremely fine mesh in the difference approximation. In a practical problem, however, especially in the calculation of a large power reactor, it is impossible to adopt a very fine mesh inside and outside the shell region due to the mesh number limitation in a diffusion code. Coarse mesh inside the shell region changes the relation

between the neutron current into a unit area of a slab and the current to flux ratio. Thus eqs. (5-24) and (5-23) should be modified if a coarse mesh is adopted inside and outside the shell region.

If uniform neutron source (Q) and mesh size (h) are assumed in a region, the solution of the three point difference approximation to the diffusion equation is given by⁽²⁾.

$$\phi_n = \left[A \sinh(knh) + B \sinh(k(b-nh)) \right] / \sinh(kb) + S \quad (5-29)$$

where ϕ_n , b and S are the neutron flux at the mesh point number n , the region thickness and Q/Σ_a , respectively. A and B are arbitrary constants. k is defined by,

$$\cosh(kh) = 1 + h^2/2L^2 \quad (5-30)$$

The origin of the coordinate in eq. (5-29) is taken at the left boundary of the region and this will be kept throughout this sub-section.

The neutron current at the left boundary is given in a diffusion code by,

$$J_L = -D(\phi_1 - \phi_{-1})/2h$$

where ϕ_{-1} is defined by setting $n = -1$ in eq. (5-29). Direct calculation leads to,

$$J_L = -F \left[A \operatorname{csch}(kb) - B \coth(kb) \right] \quad (5-31)$$

$$F = D \sinh(kh)/h = \left(D/L \right) \sqrt{1 + (h/2L)^2} = fD/L \quad (5-32)$$

In the same way, the neutron current at the right boundary is given by,

$$J_R = -F \left[A \coth(kb) - B \operatorname{csch}(kb) \right] \quad (5-33)$$

We further define E for later use:

$$E = F \tanh(kb/2) \quad (5-34)$$

E is the current to flux ratio at the surface of the region in which no neutron source exists.

Now let us study the most fundamental arrangement shown in Fig. 5-2. The medium 1 is an absorber and has no neutron source inside. From now on, the suffix shows a medium number.

If we denote the neutron flux at the interface of two regions by ϕ , the relations $A_1 = \phi$ and $B_1 = \phi \operatorname{sech}(k_1 b_1/2)$ follow from eqs. (5-29) and (5-31). So that,

Fig. 5-2

$$J_{1R} = -E_1 \phi \quad (5-35)$$

$$J_{2L} = -E_2(S_2 - \phi) \quad (5-36)$$

Eliminating ϕ in eq. (5-35) using the current continuity, $J_{1R} = J_{2L}$, the neutron current into an absorber slab is expressed by,

$$J = |J_{1R}| = S_2 E_2 / (1 + E_2/E_1) \quad (5-37)$$

The current to flux ratio at the absorber surface is given by E_1 . This result has already been obtained in reference (2). The relation between E_1 and $\Sigma_{aa} t_a$, where t_a is the thickness of the absorber, can be expressed in a convenient form when the number of meshes inside the absorber is small. These expressions are,

$$E_1 = \frac{z(1 + pz^2/32)(2 + pz^2/32)}{2[(2 + pz^2/16)^2 - 2]} \quad \text{for 4 meshes} \quad (5-38)$$

$$E_1 = \frac{z(1 + pz^2/16)}{2(1 + pz^2/8)} \quad \text{for 2 meshes} \quad (5-39)$$

$$E_1 = z/2 \quad \text{for 1 mesh} \quad (5-40)$$

$$z = \Sigma_{aa} t_a, \quad \Sigma_{aa} D_a = 1/p$$

These relations are shown in Fig. 5-3 and 5-4 with $p = 3$ and 4 .

E_1 for all meshes is expressed by the same form, $z/2$, for a very weak absorber ($z \ll 1$). For a very strong absorber, E_1 for 4 and 2 meshes approaches $z/8$ and $z/4$, respectively, which is physically consistent if the flux average is defined as a simple algebraic average in a diffusion code. In these two limits, the value of the diffusion coefficient has no effect on the neutron current for all mesh numbers, so far as it is finite. The neutron current is also determined only by z in 1 mesh case. The neutron current in the 4 and 2 mesh cases is dependent slightly on the value of the diffusion coefficient, that is the p value in eqs. (5-38) and (5-39), for an intermediate z and the best p value is a function of z , which lies between 3 and 4⁽²⁾. In practical problems, however, it can be shown that the choice of the p value is not important because of two facts: that E_1 for epithermal group is normally below 0.1 and that E_1 for thermal group is above 0.3 which means p can be taken as 4. Although 4 is recommended for p value of thermal group, the difference in rod worth calculated by two extreme p values has been checked, using the above equations, and hardly exceed 1% of the rod worth in the practical problem.

Since a diffusion code calculates the neutron current into an absorber by eq. (5-37), some pre-adjustment on the group constants in region 1 or 2 is necessary to have the correct current. The easier way to do this is to adjust the group constants in region 1, that is, to adjust E_1 . From eqs. (5-32), (5-34) and (5-37), this adjustment can be performed as,

$$E'_1 = E_1 / \left[1 + (1 - 1/f_2) E_1 L_2 / D_2 \right] \quad (5-41)$$

Fig. 5-3 Current to Flux Ratio vs. $\Sigma_{aa} t_a$ for Various Mesh Descriptions Across Width of Absorber (Epithermal Range)

Fig. 5-4 Current to Flux Ratio vs. $\Sigma_{aa} t_a$ for Various Mesh Descriptions Across Width of Absorber (Thermal Range)

where E_1' is the corrected current to flux ratio. Although the neutron source distribution in the practical problem is not spatially uniform, the uniform source assumption, on which eq. (5-41) is based, is not considered to be a hindrance to the application of eq. (5-41) because the neutron current is mainly determined by the source within a few migration lengths from the absorber and the migration length in the epithermal group is much larger than the one in thermal group. Correction to the epithermal group is not important as can be seen from eq. (5-41), since E_1 is small and L_2 is large in this group. However, numerical confirmation is considered to be desirable before adopting this correction formula and is given below.

A 3.7% enrichment, 2.9/1 W/U lattice (at room temperature) was chosen as typical to show the numerical examples. In this lattice the migration length in the epithermal and thermal group are 2.59 and 0.965 cm (MND), respectively. Case a-1 and a-2 correspond to a one dimensional study whereas case b corresponds to a two dimensional study. The thickness of the fuel region was taken as 16 cm and that of the absorber, as 1 cm, in case a. A zero current condition was set at the center of each region corresponding to a cyclic array in slab geometry. Four meshes were used inside the absorber. In case b, cell geometry was used as shown in Fig. 5-5. Only one internal mesh point was taken for the absorber. The absorption cross sections in the absorber region were:

$$\text{Case a-1 } \Sigma_{a4} = 1.0 \quad \Sigma_{a3} = 0.15$$

$$\text{Case a-2 } \Sigma_{a4} = 1.0 \quad \Sigma_{a3} = 0$$

$$\text{Case b } \Sigma_{a4} = 2.0 \quad \Sigma_{a3} = 0.2$$

Σ_a for groups 1 and 2 was set equal to zero.

Case b corresponds approximately to the shell representation of the actual rod cluster.

Case a-2 is a fictitious one and any difference between case a-1 and a-2 gives us information

about the effect of the uniform source assumption.

Fig. 5-5

Eigenvalues for cases a and b were calculated by AIM-5⁽¹⁵⁾ and PDQ-3⁽²¹⁾, respectively, changing the mesh size outside the absorber. The results are shown in Table 5-2. The error in eigenvalue is shown by $\Delta^2\lambda/\Delta\lambda$, where $\Delta^2\lambda$ is a deviation of eigenvalue from the correct one and $\Delta\lambda$ is a difference between the eigenvalues of the clean core and the absorber-in core. This quantity is considered to be one measure of the error due to the coarse mesh.

TABLE 5-2
Errors in Eigenvalue Due to Coarse Mesh
(Uncorrected and Corrected)

Mesh (cm)	Error (Relative to the rod worth), $\Delta^2\lambda/\Delta\lambda \times 10^2$					
	Case a-1		Case a-2		Case b	
	Uncorr.	Corr. by Eq. (5-41)	Uncorr.	Corr. by Eq. (5-41)	Uncorr.	Corr. by Eq. (5-41)
0.5	0.8	- 0.3	2.0	- 0.4	--	--
0.875	--	--	--	--	2.9	0.2
1.0	2.9	- 0.4	7.6	- 0.7	--	--
1.75	--	--	--	--	9.9	0.6
2.0	9.6	1.4	23.8	- 0.4	--	--
3.5	--	--	--	--	24.4	4.0
4.0	23.7	2.9	54.6	- 0.7	--	--

From this table the formula to correct error due to coarse mesh, Eq. (5-41), can be considered quite satisfactory. Deviation for the corrected cases for a very coarse mesh in case a-1 and case b is considered to be due to the inadequacy of the uniform source assumption because this deviation does not appear in case a-2. The fact that the correction by Eq. (5-41) is quite accurate in case a-2 is explained by the lack of the epithermal contribution to the worth in this case, in addition to the above fact. Epithermal neutron absorption is overestimated only by a few per cent even in the coarsest mesh size and correction is unnecessary in this group.

It is now possible to apply a correction to account for a uniformly coarse mesh outside an absorber. It is necessary in some cases, however, to adopt a transient mesh size region next to an absorber. This is examined below.

At first, the neutron current into an absorber in the arrangement shown in Fig. 5-6 will be calculated.

Using eqs. (5-29), (5-31) and (5-33) and imposing flux and current continuity at two interfaces, the neutron current into the absorber can be expressed, after some algebra and re-arrangements,

Fig. 5-6

$$J = \frac{E_1 [S_3(E_2 E_3 + F_3^2) + S_2 E_2 F_3 \operatorname{csch}(k_3 b_3)]}{E_1 E_2 + F_3(E_1 + E_2) \coth(k_3 b_3) + F_3^2} \quad (5-42)$$

If media 2 and 3 are assumed the same,

$$J = S_2 \frac{E_1 F_3 [F_3 + E_2 \coth(k_3 b_3)]}{E_1 E_2 + F_3(E_1 + E_2) \coth(k_3 b_3) + F_3^2} \quad (5-43)$$

If the thickness of region 3 is thin enough, eq. (5-43) approaches eq. (5-37). If the thickness of region 3 is thick enough, J approaches $S_2 E_1 F_3 / (E_1 + F_3)$, which is the special case of eq. (5-37), $b_2 \rightarrow \infty$, and the mesh size of region 2, naturally, does not affect the neutron current. In the intermediate thickness the neutron current is a function of the transient region thickness as well as mesh size. We denote this by $J(G)$, where G is $\coth(k_3 b_3)$.

$$\begin{aligned} J(G)/J(1) &= \frac{1 + [E_2(G-1)/(F_3 + E_2)]}{1 + [F_3(E_1 + E_2)(G-1)/(F_3 + E_1)(F_3 + E_2)]} \\ &= (1 + \delta_1)/(1 + \delta_2) \end{aligned} \quad (5-44)$$

Since both δ_1 and δ_2 are smaller than $(G-1)$, $J(G)/J(1)$ can be expanded to $1 + \delta_1 - \delta_2$, if $(G-1) \ll 1$. This expansion gives the following expression for the relative error in the neutron current.

$$\Delta J/J = (G-1)(E_2/F_3 - 1)/(1 + F_3/E_1)(1 + E_2/F_3) \quad (5-45)$$

If the thickness of region 2 is thick enough, E_2 in eq. (5-45) can be replaced by F_2 . The corresponding quantity without a transient region is derived from eq. (5-33).

$$\Delta J/J = (f_2 - 1)/(1 + F_2/E_1) \quad (5-46)$$

Comparing these equations, the error is found to be reduced approximately by the factor of β .

$$\beta = \frac{(G-1)(f_2/f_3 - 1)}{(1 + f_2/f_3)(f_2 - 1)} \lesssim (G-1)/(1 + f_2) = \beta' \quad (5-47)$$

β approaches β' when the mesh size in the transient region is very fine.

The error reduction factor by transient region, β , is mainly determined by G which is given in Table 5-3 for various mesh sizes and numbers.

TABLE 5-3
Table of G-Factors

Number of meshes	h/L	0.25	0.50	0.75	1.0
1		4.093	2.183	1.600	1.342
2		2.169	1.320	1.112	1.044
3		1.577	1.108	1.025	1.006
4		1.315	1.039	1.006	1.001
5		1.180	1.014	1.001	1.000

The G factor is mainly determined by region thickness although it depends slightly on the mesh size. From these results, it can be concluded that the error is reduced by the factor of ~ 6 by the use of a fine mesh region 1 migration length thick and by the factor of ~ 20 by use of a fine mesh region 1.5 migration lengths thick. If the thickness of a transient region is fixed, the reduction of error due to coarse mesh in the region 2 becomes more effective when the mesh size in the transient region becomes coarser. In the practical problem it is recommended to adopt, if it is necessary, at least 1 migration thickness for a transient region next to the absorber and to apply the correction corresponding to the mesh size in this region. The above process helps to simplify the method in the practical problem without much error although it is possible to apply a more exact correction using eq. (5-43), when it is necessary.

D. DISCRETE REPRESENTATION

A Discrete representation, so-called, was developed and applied to the analysis of the rod cluster critical experiments. The method derives its title from the fact that the individual lattice unit cells which contain the absorber elements are discretely represented in the PDQ radial plant description of the critical assembly.

The starting point for this method of representation is the same as for all methods described in this report: the four group current-to-flux boundary conditions described in Section III. A set of group diffusion constants, appropriate to these discretely represented unit cells and corresponding to a properly weighted homogenization of the absorber, water and steel, is developed according to the following procedure. A cylindrical AIM-5 few group diffusion calculation is set up in which the origin of the difference equation mesh is specified to be at the radial position of the (clad) absorber surface. Recall that actual thimble thickness is added to clad, and void is ignored. The appropriate boundary conditions in all groups are applied at this position. The water occupying the absorber unit cell is described as an annular region about the origin. The balance, of the mesh, approximately 6 cm., is filled with appropriate homogenized fuel constants; a zero current boundary is applied at the end of the mesh. The mesh is quite fine and conventional thermal

group constants are used throughout.

The transformations involving homogenization, conversion to rectilinear geometry and thermal model are accomplished by means of a PDQ set-up which is made analogous to that of the AIM-5 by means of a simple device. The zero current outer boundary of the AIM is approximated by the declaration of a properly indented "rod region" in one corner of the square PDQ mesh; these "rod regions" in PDQ are specified only by current-to-flux ratio boundary conditions in each group; by specifying such ratios to be very small, a zero current boundary is established in the PDQ mesh the average position of which is identical to that of the AIM - see Fig. 5-7. This figure also shows the absorber unit cell representation at the center of physical symmetry of the PDQ mesh. Simple volume averaged fluxes from the AIM are used to generate the group constants for the three fast groups in the PDQ.

$$\Sigma_{a1} = \frac{\begin{bmatrix} \text{rod} \\ \alpha_1 & \phi_1^{\text{surface}} & 2\pi R \end{bmatrix} + \begin{bmatrix} \text{H}_2\text{O} & \text{H}_2\text{O} & \text{H}_2\text{O} \\ \phi_1 & V & \Sigma_{a1} \end{bmatrix}}{\begin{bmatrix} \phi_1 & \text{H}_2\text{O} \\ \text{H}_2\text{O} & (V^{\text{H}_2\text{O}} + V^{\text{rod}}) \end{bmatrix}}$$

$$\Sigma_{r1} = \frac{\begin{bmatrix} \text{H}_2\text{O} \\ V^{\text{H}_2\text{O}} \end{bmatrix}}{\begin{bmatrix} \text{H}_2\text{O} & \text{rod} \\ V^{\text{H}_2\text{O}} & + V \end{bmatrix}} \left(\begin{bmatrix} \text{H}_2\text{O} \\ \Sigma_{r1} \end{bmatrix} \right)$$

The fourth group macroscopic absorption is adjusted until the eigenvalue of the PDQ matches that of the AIM. Mixed Number Density thermal group constants are used for the fuel representation in the PDQ. The PDQ mesh is tapered toward the absorber unit cell, that is, the mesh is progressively finer with increasing proximity to the unit cell.

Similar pains are taken with the PDQ mesh description of the actual absorber configurations so as to avoid the characteristic and unpredictable worth overestimate associated with coarseness of mesh in the vicinity of a strong neutron sink. If a given mesh is to correspond to a critical

Fig.5-7 Comparison of AIM-5 and PDQ Cell Outer Boundaries

assembly containing a uniform array of absorbers, the tapered mesh used in the vicinity of each absorber is the same as that used in the PDQ cell described above; if the mesh is to describe a cluster, a constant mesh spacing equal to one-fourth of the side of a unit cell is used within the area circumscribed by the cluster elements whereas the usual mesh tapering is described around the cluster as a whole.

E. HYBRID REPRESENTATION

The results of the comparison of the methods of representation described above with experimental determinations of worth are described in the next section. The analysis of these results led to the conclusion that the most accurate method of representation for use in actual design would be based on a synthesis of the Shell and a type of Discrete method - thus the designation Hybrid.

As in the Discrete method, homogenized absorber unit cell group constants are developed through a correspondence between AIM and PDQ cells. A modification is that use of the conventional thermal model is retained in the PDQ. The next stage of the calculation involves a representation in a PDQ cell of the particular cluster configuration. The fine mesh described above in the discussion of the discrete representation of a cluster is used. At least 6 cm. of fuel is placed between the cluster and the zero current mesh boundaries. Conventional fuel region constants are utilized. A cell of identical area with a shell description for the cluster is also set up; the fuel constants in this cell reflect the Mixed Number Density model; there is one mesh across the width of the shell and the mesh in the fuel is coarse enough so that an actual core description based on cyclic use of this cell mesh would not exceed the mesh capacity of the particular version of PDQ to be used in the actual design problem. Unlike the straight shell representation described above, removal cross sections are included within this shell. If for any reason the mesh in the actual design run has to be altered, the methods outlined in part C above may be used to adjust the thermal shell representation. This method is also adaptable to use of a discrete coarse-mesh description for the cluster in the overall reactor PDQ. In this case, a coarse mesh cell with

MND constants is normalized to the usual fine-mesh cell by adjustment of the absorber region cross sections. These adjusted parameters are then utilized in the PDQ for use in describing the reactor as a whole.

* User may choose any convenient geometry for approximating cluster

Fig. 5-8 "Flow Chart" Showing Development of Discrete and Hybrid Representations

VI. WORTH COMPARISONS BETWEEN ANALYSIS AND EXPERIMENT

This section will include a discussion of the results of applying the various methods of absorber representation described in the preceding section to the analysis of a variety of critical experiments. Part A of this section, however, consists of an explanation to the reader of what is measured and the specific forms derived from analysis which correspond to these experimentally measured quantities. The discussion in this first part should be of interest to those engaged in the analysis of worth substitution experiments of any kind.

A. DERIVATIONS OF EXPRESSIONS FOR RELATIVE ERROR

The particular experimental measurements discussed in this report consist basically of partial moderator height determinations, i.e., over-all leakage is traded off against insertion of volumetrically discrete parasitic absorption. The discussion which follows, however, could be applied just as validly to any sort of experimental worth determination involving substitution techniques such as moderator poison insertion with control rod withdrawal, poison withdrawal of any sort with increase of moderator temperature, etc.

1. ANALYTIC COUNTERPARTS OF THE INTEGRAL EXPERIMENTAL WORTHS

The experimentally determined quantity, absorber worth, is expressed henceforth as $\Delta\rho_{\text{EXP}}$. Between the core heights, H_1 and H_2 , it is defined by the following equation:

$$\Delta\rho_{\text{EXP}} = \int_{H_1}^{H_2} \frac{\partial\rho}{\partial H} dH \quad (6-1)$$

The values of $\frac{\partial\rho}{\partial H}$ at different water heights are approximated by values of $\frac{\Delta\rho}{\Delta H}$ obtained with the aid of an analog computer as described in reference (1), the report companion to this which summarizes the experimental work. This variation of $\frac{\Delta\rho}{\Delta H}$ with water height, H , has been well approximated by the function $(aH+b)^{-3}$ where a and b are fitted constants. The experimental expressions are:

$$\frac{\partial\rho}{\partial H} = (0.112H + 1.80)^{-3} \frac{\Delta k}{k} / \text{cm for } 3.7\% \text{ } 2.9/1 \text{ W/U}$$

SS Clad

$$\frac{\partial \rho}{\partial H} = (0.1064H + 2.132)^{-3} \frac{\Delta k}{k} / \text{cm for } 2.7\% \text{ } 2.9/1 \text{ W/U}$$

SS Clad

These expressions for $\frac{\partial \rho}{\partial H}$ are derived maintaining the reactor close to critical at all heights. An excessively supercritical condition is avoided by adjusting poison content and/or leakage of the core. It is possible to simulate this experimental procedure with analysis. For instance, a poison cross section could be introduced into a calculation at some core height forcing criticality; the core height could then be slightly increased; the resulting eigenvalue difference could be translated to a $\Delta \rho$; the core height could then be increased, poison added, etc.

We shall relate experiment to the eigenvalues derived from analysis by discussing in detail the analytic analog of the step-by-step experimental procedure; the process will then be reduced to the differential limit and simple logarithmic eigenvalue relationships will be derived with which and only with which $\Delta \rho_{\text{EXP}}$ can be validly compared.

Reactivity is defined by the following equation:

$$\rho = (k_{\text{eff}} - 1) / k_{\text{eff}} \quad (6-2)$$

where k_{eff} is the criticality factor. We shall denote the k_{eff} of the clean reactor as $\lambda(H)$ where λ is the eigenvalue corresponding to H ; further, let $\lambda(H_1)$ be denoted by λ_1 and $\lambda(H_2)$ by λ_2 . Therefore, the reactivity difference between clean cores with different water heights is

$$\Delta \rho = \frac{\lambda_2 - 1}{\lambda_2} - \frac{\lambda_1 - 1}{\lambda_1} = \frac{\lambda_2 - \lambda_1}{\lambda_2 \lambda_1} \quad (6-3)$$

For the sake of emphasis, we shall repeat that the experimental determination of a set of $\frac{\partial \rho}{\partial H}$ values is always carried out near criticality. This is achieved by poisoning the core sufficiently at each successively greater water height, at which a "bump", ΔH , in the water height is to occur.

The effect of this poison on the criticality factors involved in the "bump" is a multiplicative one, i.e., if λ_n is reduced by the factor $\frac{1}{\lambda_n}$, then λ_{n+1} is also reduced by the same factor. Thus, in general, the experimental quantity measured, $\Delta\phi_n$, is

$$\Delta\phi_n = \frac{\lambda_{n+1} \times \frac{1}{\lambda_n} - \lambda_n \times \frac{1}{\lambda_n}}{\left[\lambda_{n+1} \times \frac{1}{\lambda_n} \right] \left[\lambda_n \times \frac{1}{\lambda_n} \right]} = \frac{(\lambda_{n+1}/\lambda_n) - 1}{(\lambda_{n+1}/\lambda_n)} = \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \quad (6-4)$$

where λ_{n+1} and λ_n are the multiplication factors for the clean core at the two water heights corresponding to the measurement. Table 6-1 compares $\Delta\phi_n$, the quantity actually measured, with $\Delta\phi$, the quantity that would be measured if there were no step-wise return to criticality. λ_1 is set equal to unity.

TABLE 6-1

Eigenvalue Analogs of Step-Wise Experimental Measurements

Step n	Lower Height of Step	$\Delta\phi_n$	$\Delta\phi$
1	H	$\frac{\lambda_2 - 1}{\lambda_2}$	$\frac{\lambda_2 - 1}{\lambda_2}$
2	H + ΔH	$\frac{\lambda_3 - \lambda_2}{\lambda_3}$	$\frac{\lambda_3 - \lambda_2}{\lambda_3 \lambda_2}$
3	H + 2 ΔH	$\frac{\lambda_4 - \lambda_3}{\lambda_4}$	$\frac{\lambda_4 - \lambda_3}{\lambda_4 \lambda_3}$

Now if a sequence of measurements is made,

$$\sum_n (\Delta\phi)_n = \sum_n \left(\frac{\Delta\lambda}{\lambda} \right)_n \quad (6-5)$$

where λ is understood to be the higher value of the pair in the interval n . In the limit, however, this distinction is not essential and

$$\lim_{\Delta p \rightarrow 0} \sum (\Delta p)_n = \int dp = \Delta p_{cal} = \int_{\lambda_A}^{\lambda_B} \frac{d\lambda}{\lambda} = \ln (\lambda_B / \lambda_A) \quad (6-6)$$

where λ_B is the eigenvalue corresponding to the clean core description with the absorber-in critical axial buckling as input and λ_A is the eigenvalue corresponding to the clean core description with the clean critical axial buckling as input. Thus, $\ln \frac{\lambda_B}{\lambda_A}$ is an analytic form with which Δp_{EXP} can be compared.

In this same differential limit, however, one can invert the sequential process of bumping, and poisoning back to criticality. One could consider that reactivity differences are taken between configurations which differ only by the addition to one of them of differential amounts of poison; criticality is then restored by a change in leakage which is multiplicative on the pair of eigenvalues involved in the next differential sequence, etc. Thus, instead of the parameter being the clean core description and the variable being leakage, the parameter is leakage and the variable is the poison representation in the core, so that

$$\Delta p_{cal} = - \int_{\lambda_D}^{\lambda_C} \frac{d\lambda}{\lambda} = \int_{\lambda_C}^{\lambda_D} \frac{d\lambda}{\lambda} = \ln \left(\frac{\lambda_D}{\lambda_C} \right) \quad (6-7)$$

where λ_D is the calculated eigenvalue for the clean just critical core and λ_C is the eigenvalue for the absorber-in core, the axial buckling input to both cases being identically that for the clean core.

With "perfect" analysis, of course, the RHS of eq. 6-6 would always equal the RHS of eq. 6-7. In this report, however, the analytic form constituted by the RHS of eq. 6-7 has been chosen for the comparison with Δp_{EXP} . There are two reasons for this choice, one of which should be

quite obvious:

- (1) The comparison of the eigenvalue ratio in eq. 6-7 with the corresponding experimental quantity affords a direct test of the accuracy of the analytic absorber representation.
- (2) The RHS of eq. 6-6 specifically consists of the ratio of eigenvalues derived from analysis which are identical in that they correspond to the same configuration in the radial plane - that of a clean core. The eigenvalues differ because of the differing axial bucklings imposed on these two problems: one buckling reflects the height measured for the absorber-in core, the other corresponds to the actual clean core height. Since the latter height is rather small, the eigenvalue ultimately calculated is extraordinarily sensitive to the reflector savings associated with this height in the determination of the axial buckling. It is more appropriate to reserve a discussion of this aspect of criticality analysis involving the determination of axial reflector savings as a function of moderator height to a future report which will be concerned only with the analysis of clean cores; the introduction of such a necessarily lengthy discussion into this report would be irrelevant.

Simply from a qualitative point of view, one could note two important properties shared by our definition and $\Delta\rho_{\text{EXP}}$; these are additiveness* and uniqueness. The quantities $\ln(\lambda_3/\lambda_2)$ and $\ln(\lambda_2/\lambda_1)$ add perfectly to $\ln(\lambda_3/\lambda_1)$ just as $\Delta\rho_{\text{EXP}}$, in the corresponding height ranges, no matter what their

* An explicit illustration of the additive property: If these heights are measured, H_1 , H_2 and H_3 , corresponding to the clean configuration, a poisoned configuration and the accreted poison configuration, then

$$\Delta\rho_{\text{EXP}}(H_1 \rightarrow H_3) = \Delta\rho_{\text{EXP}}(H_1 \rightarrow H_2) + \Delta\rho_{\text{EXP}}(H_2 \rightarrow H_3)$$

Definition 6-7 is additive:

$$\ln \frac{\lambda_3}{\lambda_1} = \ln \frac{\lambda_3}{\lambda_2} + \ln \frac{\lambda_2}{\lambda_1}$$

Definition 6-11 is additive to an accuracy of a term in $(\frac{\Delta\lambda}{\lambda})^2$:

$$\begin{aligned} \frac{\lambda_2 - \lambda_1}{(\lambda_2 + \lambda_1)/2} + \frac{\lambda_3 - \lambda_2}{(\lambda_3 + \lambda_2)/2} &= \frac{2(\lambda_3 - \lambda_1)}{\lambda_3 + \lambda_1 - (\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)/2\lambda_2} \\ &= \frac{\lambda_3 - \lambda_1}{(\lambda_3 + \lambda_1)/2} \left[1 + \frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{2\lambda_2(\lambda_3 + \lambda_1)} \right] \cong \frac{\lambda_3 - \lambda_1}{(\lambda_3 + \lambda_1)/2} \end{aligned}$$

magnitude, is simply additive. The logarithmic ratio is as unique as the experimental quantity, a property notable by its absence in other forms which one might adopt for comparison of analysis with experiment. This negative aspect of certain other definitions for analytic equivalents to $\Delta\rho_{\text{EXP}}$ will be discussed below. For the present, we shall simply point out that any analytic form resembling eq. 6-7 is in error if it is dependent on the single axial buckling input identically to the clean and absorber-in cases. There is no physical reason for choosing the axial buckling associated either with the clean or the absorber-in case as the common input for the pair of calculations; numerical calculations verify this independence: a worth determination equivalent to more than 6% in $\Delta\rho_{\text{cal}}$ changes by only 2 parts in 600 with the substitution of absorber-in buckling for clean axial buckling. This result could also be deduced from the similarity in treatment afforded buckling and poison by the diffusion equations and the multiplicative nature of such simultaneous modifications.

One analytic form with which $\Delta\rho_{\text{EXP}}$ might be compared is simply the expression for $\Delta\rho$ applied to the endpoints of the stepwise process:

$$\frac{\lambda_D - \lambda_C}{\lambda_D \lambda_C} \quad (6-8)$$

This is equivalent to:

$$\frac{1}{\lambda_D} \left[\frac{1 - (\lambda_C / \lambda_D)}{\lambda_C / \lambda_D} \right] \quad (6-9)$$

Equation 6-8 or its equivalent in 6-9 is perfectly additive; these forms, however, are not unique. As shown above both theoretically and by means of the results of a numerical calculation, $\frac{\lambda_C}{\lambda_D}$ is quite constant independent of the single buckling value input to both calculations. But the eigenvalues themselves certainly are not; therefore, as shown in 6-9, this analytic form can arbitrarily differ in magnitude by as much as the eigenvalue would differ with input either of the clean core critical axial buckling or the absorber-in core critical axial buckling.

Other analytic forms that might be used to compare with $\Delta\rho_{\text{EXP}}$ are

$$(\lambda_D - \lambda_C)/\lambda_C \quad \text{or} \quad (\lambda_D - \lambda_C)/\lambda_D \quad (6-10)$$

Since these forms are a function of the eigenvalue ratios only, the uniqueness requirement is fulfilled. They are additive, however, only if terms in $\frac{\Delta\lambda}{\lambda}$ are ignored.

One final analytic form is

$$\frac{\lambda_D - \lambda_C}{(\lambda_D + \lambda_C)/2} \quad (6-11)$$

This also is unique; furthermore it is additive if terms in $(\frac{\Delta\lambda}{\lambda})^2$ are ignored. This form can be shown to be an excellent approximation to 6-7; the latter, however, has a more exact theoretical basis and is used to compare with the bulk of the experimental measurements in the next part of this section.

Table 6-2A in part B of this section contains a listing of $\Delta\rho_{\text{EXP}}$ for configurations analyzed by means of integral worth comparisons. The significant quantity in this tabulation is the Relative Error:

$$\text{Relative Error} = \text{RE} = (\Delta\rho_{\text{cal}} - \Delta\rho_{\text{EXP}}) / \Delta\rho_{\text{EXP}} \quad (6-12)$$

Where $\Delta\rho_{\text{cal}}$ is the analytic form defined by eq. 6-7.

2. THE ERROR DERIVED USING EXPERIMENTALLY MEASURED CRITICAL WATER HEIGHTS

Corresponding to the measurements on a given configuration of absorbers, there are output to the analysts two items of information which can be used in virtually distinct evaluations of the accuracy of a given analytic absorber representation; one item is $\Delta\rho_{\text{EXP}}$ and the analytic correlation with this has been described above; the other item is the critical water height measured for the absorber-in configuration. This height, together with the eigenvalue calculated for a critical clean loading which is fully inundated, are the only essentials required for a direct evaluation of the Relative Error as it can be shown that

$$\text{RE} = \ln(\lambda_A/\lambda_E) / \Delta\rho_{\text{EXP}} \quad (6-13)$$

where λ_E is the eigenvalue corresponding to the absorber-in core description with the absorber-in critical axial buckling as input.

For if $\Delta\rho_{EXP}$ is approximated by eq. 6-6, then from 6-12,

$$RE = \frac{\ln(\lambda_D/\lambda_C) - \ln(\lambda_B/\lambda_A)}{\ln(\lambda_B/\lambda_A)} = \frac{\ln(\lambda_D/\lambda_C \cdot \lambda_A/\lambda_B)}{\ln(\lambda_B/\lambda_A)} \quad (6-14)$$

Recall that λ_A is the eigenvalue corresponding to the clean core description with the clean core critical axial buckling as input and λ_B is the eigenvalue corresponding to the clean core description with the absorber-in core critical axial buckling as input.

The ratio λ_D/λ_C is insensitive to axial buckling, so that $\lambda_D/\lambda_C = \lambda_B/\lambda_E$. Therefore equation 6-14 reduces to equation 6-13. Inasmuch as we are concerned with the relative representation of control rods rather than with calculation of low water height cores in which the axial reflector savings employed is an unusually critical parameter, the eigenvalue obtained for a fully inundated clean critical configuration is used in place of λ_A . The critical height with absorbers in the core is sufficiently great that the selection of reflector savings is likewise not crucial.

Tables 6-2B in part B of this section contains a listing of RE values for configurations analyzed in the basis of such absorber-in critical eigenvalue correlations.

We would ask the reader to note, in this expression for Relative Error, the insensitivity to the denominator, $\Delta\rho_{EXP}$; were $\Delta\rho_{EXP}$ to have a 20% uncertainty and if the RE calculated with the nominal value were to equal 5% (or 1%), a recalculation of RE with the $\Delta\rho_{EXP}$ reduced by 20% would result in a 6.3% (or 1.3%) value. This insensitivity to the accuracy of $\Delta\rho_{EXP}$ indicates an approach whereby analysis can evaluate the accuracy of the expressions used to obtain $\Delta\rho_{EXP}$. This evaluation is made later in the report.

At the time this report is being written, an experimental expression for the variation of $\frac{\partial\rho}{\partial H}$ in units of $\frac{\Delta k}{k}$ /cm as a function of just critical water

height in the lattice of zirconium clad fuel is not available. Noting the insensitivity, however, of 6-13 to $\Delta\rho_{\text{EXP}}$, it can be seen that it is quite justifiable to substitute for this quantity the $\ln(\lambda_B/\lambda_A)$ which is the RHS of Eq. 6-6. The uncertainties in either $\Delta\rho_{\text{EXP}}$ or $\ln(\lambda_B/\lambda_A)$ (due to reflector savings uncertainty) are, of course, far less than the 20% in $\Delta k/k$ considered above.

B. COMPARISON BETWEEN ANALYTIC AND EXPERIMENTAL WORTHS

Experiments analyzed in this report were conducted in three lattice types. Two involved steel clad fuel elements, approximately 300 mils in diameter which differed only in enrichment. One enrichment was 2.7% and the other was 3.7%. Appendix II of reference⁽¹⁷⁾ provides a detailed physical description of these fuel rods. The lattice in which these rods were used was square with a 0.493" pitch. The water to equivalent-uranium metal ratio, (W/U), in this lattice was 2.9/1. The third lattice type was composed of zircaloy clad fuel rods which were approximately 400 mils in diameter and 2.7% enriched; Appendix C provides a detailed physical description of these elements. This fuel was employed in a square lattice with a 0.6" pitch; W/U was also 2.9/1.

Table 6-2 summarizes the comparison between analysis and experiment in the determination of worth. The symbols in the column marked "cluster positions in the core" are schematically illustrated in Fig. C-1 of Appendix C. Any slight differences in worths and heights between values quoted in Table 6-2 for identical configurations are to be attributed to the fact that the critical heights were not quite identical when some configurations were re-measured.

1. Shell and Homogeneous Representations

The eigenvalues for the clean cores and the absorber-in cores using the Shell or Homogeneous representations were obtained by either one or two dimensional critical calculations fixing the axial buckling to the value for the clean core. The calculated control worths are then derived by Eq. (6-7). These analytic worths and their associated Relative Errors are shown in columns (A), (B), (C) and (D) of Table 6-2A. Two points, however, should be considered before discussing the worth comparison. These are the clad correction and the interference effect between cylindrical absorbers in a cluster.

Table 6-2A Identification of Experiments and Integral Worth Comparisons

Total Fuel Rods	Absorber Diameter (in)	Clad Thickness (in)	Number of Absorbers	Cluster Positions in the Core	Critical Height (cm)	$\Delta \rho_{exp} \times 10^2$	(A) $\Delta \rho_{cal} \times 10^2$ Shell	(B) $RE \times 10^2$ Shell	(C) $\Delta \rho_{cal} \times 10^2$ Homogenization	(D) $RE \times 10^2$ Homogenization	(E) $\Delta \rho_{cal} \times 10^2$ Discrete $\delta \text{ clad} = 0.7$	(F) $RE \times 10^2$ Discrete $\delta \text{ clad} = 0.7$	(G) Definitive % RE Uniform RE from Discrete Cluster RE from Hybrid $\delta \text{ clad} = 0.84$
2.7 w/o Uniform Array													
3920	.260	0.0	49	--	72.39 ⁺	4.27	--	--	4.05	-5.2	--	--	--
3920	.260	0.0	49	--	71.33	4.16	--	--	--	--	4.19	0.7	0.7
3944	.330	0.0	25	--	74.26	4.46	--	--	4.19	-6.1	--	--	--
3920	.330	0.0	49	--	88.77	5.62	--	--	5.29	-5.9	--	--	--
3920	.330	0.0	49	--	87.70	5.55	--	--	--	--	5.64	1.6	1.6
3920	.330	0.04	49	--	90.06	5.70	--	--	5.41	-5.1	--	--	--
3920	.330	0.04	49	--	89.03	5.64	--	--	--	--	5.95	5.5	1.5
3920	.330	0.06	49	--	90.87	5.76	--	--	5.44	-5.6	6.02	4.5	-1.5
3944	.403	0.0	25	--	87.76	5.56	--	--	5.20	-6.5	--	--	--
2.7 w/o Cluster Array													
3701	.260	0.01	20	--	71.62	3.67	3.62	-1.4	--	--	3.85	4.9	0.2
3697	.260	0.01	24	--	77.46	4.25	4.17	-1.9	--	--	4.50	5.9	1.2
3703	.403	0.01	16	--	79.54	4.43	4.27	-3.6	--	--	4.69	5.9	1.2
3701	.403	0.01	20	--	91.87	5.30	5.01	-5.5	--	--	5.48	3.4	-1.3
3697	.403	0.01	24	--	105.15	5.96	5.65	-5.2	--	--	6.10	2.4	-1.3
3237	.330	0.0	12	A1	75.45	2.90	2.78	-4.1	--	--	3.01	3.8	0.1
3233	.330	0.0	16	A1	88.19	3.91	3.71	-5.1	--	--	4.01	2.6	-1.1
3693	.330	0.0	28	A1	76.58	4.17	3.94	-5.5	--	--	4.27	2.4	-1.3
3.7 w/o Uniform Array													
1872	.330	0.0	23	--	97.61	4.77	--	--	4.58	-4.0	--	--	--
1872	.330	0.0	23	--	98.01	4.79	--	--	--	--	4.84	1.0	1.0
1872	.330	0.02	23	--	99.35	4.85	--	--	4.65	-4.1	--	--	--
1872	.330	0.04	23	--	100.09	4.88	--	--	4.69	-3.9	--	--	--
1872	.330	0.06	23	--	101.43	4.95	--	--	4.72	-4.7	5.18	4.7	-1.4
2032	.260	0.0	25	--	61.21	3.67	--	--	3.51	-4.4	--	--	--
2032	.403	0.0	25	--	84.89	6.14	--	--	5.78	-5.9	--	--	--
2203	.260	0.0	25	--	55.61	3.68	--	--	3.43	-6.8	3.64	-1.1	-1.1
2203	.330	0.0	25	--	62.42	4.83	--	--	4.52	-6.4	--	--	--
2203	.403	0.0	25	--	72.50 ⁺	6.07	--	--	5.66	-6.8	6.29	3.6	-3.6
3.7 w/o Cluster Array													
2221	.260	0.01	12	A1	52.56	3.05	3.06	0.3	2.94	-3.6	3.15	3.3	1.4
2221	.260	0.01	12	A2	47.45	1.78	1.79	0.6	--	--	--	--	--
2221	.260	0.01	12	A3	43.90	0.71	0.71	0.0	--	--	--	--	--
2221	.330	0.0	12	A1	55.41	3.64	3.58	-1.7	3.48	-4.4	3.70	1.7	0.8
2221	.330	0.0	12	A2	48.55	2.08	2.02	-2.9	--	--	--	--	--
2221	.330	0.0	12	A3	44.27	0.83	0.80	-3.6	--	--	--	--	--
2217	.330	0.0	16	A1	62.71	4.87	4.83	-0.8	4.54	-6.8	5.03	3.3	2.4
2209	.330	0.0	24 ^x	B1	68.47	5.62	5.53	-1.6	5.54	-1.4	--	--	--
2209	.330	0.0	24 ^x	B2	64.56	5.13	5.04	-1.8	5.04	-1.7	--	--	--
2209	.330	0.0	24 ^x	B3	60.90	4.60	4.53	-1.5	4.54	-1.3	--	--	--
2209	.330	0.0	24 ^x	C1	77.77	6.56	6.34	-3.4	--	--	--	--	--
2209	.330	0.0	24 ^x	C2	75.49	6.36	6.13	-3.6	--	--	--	--	--
2209	.330	0.0	24 ^x	C3	56.23	3.80	3.65	-4.0	--	--	--	--	--
2185	.330	0.0	48 ^a	D2	53.61	3.27	3.26	-0.3	--	--	--	--	--
2185	.330	0.0	48 ^a	D2	57.00	3.94	3.74	-5.1	--	--	--	--	--

^xTwo 12 Rod Clusters^aFour 12 Rod Clusters⁺This height is coincident with the top of lucite spacer plate; small error thus introduced into measurement

Table 6-2B Identification of Experiments and Relative Errors Based on Critical Eigenvalues

Total Fuel Rods	Absorber Diameter (in)	Clad Thickness (in)	Number of Absorbers	Cluster Positions in the Core	Critical Height (cm)	λ_E	λ_A	$\ln(\frac{\lambda_A}{\lambda_E})$	$\Delta\rho^*_{exp} \times 10^2$	(A)	(B)	(C)
										Definitive % RE Uniform RE from Discrete Cluster RE from Hybrid 6 clad = 0.84	% RE Homogenization	% RE Shell
2.7 w/o Uniform Array												
3920	.260	0.0	49	--	71.33	1.0034	1.0037	.0003	4.16	+0.7	--	--
3920	.330	0.0	49	--	87.70	1.0021	1.0037	.0015	5.55	+2.7	--	--
3920	.330	0.04	49	--	89.03	.9999	1.0037	.0038	5.64	+2.7	--	--
3920	.330	0.06	49	--	90.87	1.0003	1.0037	.0034	5.76	-0.1	-4.2	--
2.7 w/o Cluster Array												
3701	.260	0.01	20	--	71.62	1.0015	1.0037	.0022	3.67	+1.3	--	-0.3
3697	.260	0.01	24	--	77.46	1.0004	1.0037	.0033	4.25	+3.1	--	0.0
3705	.403	0.01	16	--	79.54	1.0001	1.0037	.0035	4.43	+3.2	--	-1.6
3701	.403	0.01	20	--	91.87	1.0003	1.0037	.0034	5.30	+1.7	--	-2.5
3697	.403	0.01	24	--	105.15	1.0000	1.0037	.0036	5.96	+1.4	--	-2.5
3237	.330	0.0	12	A1	75.45	1.0019	1.0037	.0017	2.90	+2.3	--	-1.9
3233	.330	0.0	16	A1	88.19	1.0010	1.0037	.0027	3.91	+3.2	--	-0.8
3693	.330	0.0	28	A1	76.58	1.0019	1.0037	.0018	4.17	+0.6	--	-3.6
3.7 w/o Uniform Array												
1872	.330	0.0	23	--	98.01	.9994	1.0019	.0025	4.79	+5.2	--	--
1872	.330	0.06	23	--	101.43	.9975	1.0019	.0043	4.95	+2.7	-6	--
2208	.260	0.0	25	--	55.61	1.0019	1.0019	0.0	3.68	+0.0	-5.7	--
2208	.403	0.0	25	--	72.50	.9986	1.0019	.0033	6.07	+5.4	-5.0	--
3.7 w/o Cluster Array												
2221	.260	0.01	12	A1	52.56	1.0004	1.0019	.0015	3.05	+3.0	-2.0	+1.9
2221	.330	0.0	12	A1	55.41	1.0009	1.0019	.0010	3.64	+1.8	-3.4	-0.7
2217	.330	0.0	16	A1	62.71	1.0000	1.0019	.0018	4.87	+2.8	-6.4	-0.4
Zr Clad 2.7 w/o Uniform Array												
936	.403	0.0	25	--	92.15	.9918	.9914	-.0004	9.20	-0.4	-6.0	--
Zr Clad 2.7 w/o Cluster Array												
945	.403	0.0	16	--	89.75	.9889	.9914	.0024	9.10	+2.6	--	--

* Eq. 6-6 used in place of $\Delta\rho_{exp}$ for 2 Zr Cases

a) Clad Correction

An empirical factor, δ , has been included in the correction formula for the clad effect of eqs. (3-59) and (3-68):

$$\beta_1^* = \beta_1 + \left[\frac{\epsilon}{1+\epsilon} (1-\beta_1) + \Sigma_{SS} \Delta (1-\beta_1)^2 \right] \delta$$

The adoption of an empirical factor in this formula is due to the fact that the clad correction without this factor overpredicted the clad effect in previous analyses of cruciform arrays of cylindrical absorbers. This is also true in the analysis of the present cluster experiments. Examples of the effect of the empirical factor are shown in Table 6-3 where the calculated rod worths with and without the empirical factor, $\delta = 0.7$, are compared with the experimental values.

TABLE 6-3
Comparison of Calculated and Experimental Rod Worths
(Uniform arrays, 2.7%, 2.9/1 W/U)

Abs. Dia. (in)	Clad Thick. (in)	No. of Abs.	No. of Total Rods	Critical Height (cm)	$\Delta \rho_{EXP}$ $\times 10^2$	$\delta = 0.7$		$\delta = 1.0$	
						$\Delta \rho_{cal}$ $\times 10^2$	RE $\times 10^2$	$\Delta \rho_{cal}$ $\times 10^2$	RE $\times 10^2$
0.330	0	49	3969	88.77	5.62	5.29	-5.9	5.29	-5.9
"	0.04	"	"	90.06	5.70	5.41	-5.1	4.95	-13.2
"	0.06	"	"	90.87	5.76	5.44	-5.6	4.70	-18.4

The effect of the empirical factor is clear from Table 6-3, although the reason why this factor is necessary is not understood at present. However, it is the only empirical factor introduced into the analysis. A different numerical value is used for δ in the other methods of representation; this is discussed later.

b) Interference Effect

In the representation of a cluster by either the Shell or Homogeneous methods of representation, the thermal group interference term, M_4 , is underestimated due to the assumption, described in Section III, in which the homogenized fuel is assumed to occupy all space between (clad) absorbers. M_4 involved zero order modified Bessel functions of the second kind whose several arguments consist of thermal mean free paths between elements in an array, i.e., $K_0(\frac{d_n}{L_4})$ where d_n corresponds to the spacing between elements of a cluster and L_4 is the thermal diffusion length. This function decreases rapidly with increasing argument. Accordingly, erroneous determination of the thermal diffusion length has little effect on the worth calculations for uniform arrays in which the physical spacing is quite large. In clusters, however, the spacing is much reduced; the d_n corresponding to nearest neighbors largely determines M_4 . Accordingly, use of the thermal diffusion length corresponding to the homogenized fuel in the argument of the Bessel function is erroneous because a varying but large fraction of the cluster element spacing, d , corresponds to the water which occupies the balance of the unit cell containing the absorber. The table below compares magnitudes of the thermal diffusion length of the three fuel types and water.

Table 6-4 Conventional Thermal Model Diffusion Lengths (cm.)

<u>Water</u>	<u>2.7% SS Clad</u>	<u>3.79% SS Clad</u>	<u>2.7% Zr Clad</u>
3.65	1.46	1.33	1.59

Experimental worths were requested and obtained for clusters, which have different amounts of thermal interference, in order to check this point. The clusters measured were 12, 16 and 28 element clusters in a 2.7%, SS clad lattice. The comparison of experimental results with the calculated, in which M_4 and $1.7 M_4$ (arbitrarily increased) are used for the thermal interference terms, is shown in Table 6-5.

Table 6-5 Comparison of Calculated and Experimental Worths

Abs. Dia. (in)	Clad Thickness (in)	No. of Abs.	No. of Total Rods	Critical Height (cm)	M_4			$1.7 M_4$	
					$\Delta \rho_{exp}$ $\times 10^2$	$\Delta \rho_{cal}$ $\times 10^2$	RE $\times 10^2$	$\Delta \rho_{cal}$ $\times 10^2$	RE $\times 10^2$
0.330	0	12	3249	75.45	2.90	2.78	-4.1	2.61	-10.0
"	"	16	"	88.19	3.91	3.71	-5.1	3.59	-8.2
"	"	28	3721	76.58	4.17	3.94	-5.5	3.61	-13.4

It is obvious that any increase of the interference term would increase the deviation of calculated from experimental worth. Therefore, it must be assumed that some cancellation of this effect occurs in the over-all calculation.

c) Worth Comparison

The worth summary on Table 6-2 shows, for the Homogeneous representation, worth underestimates of 0.6% to 6.8%. The method is quite consistent in the analysis of uniform arrays and single clusters; analysis of configurations involving two clusters shows markedly different results. This result is not unexpected in view both of the grossness of the representation when applied to clusters and also the importance of interference between clusters.

For the shell representation, the summary shows a range from + 1.9% to -5.5% in variation of calculated from experimental worth. If the edge correction mentioned previously had been applied, the (negative) magnitude of the bias would have been corrected. More important, however, is the fact that the worth overestimate due to underestimate of thermal interference between cluster elements is evidently cancelled out, more or less, by the worth underestimate due to the original assumption (in Section III) concerning the diffusion kernel in the infinite medium. This assumption amounted to neglecting the perturbation in the diffusion kernel which is

caused by absence of neutrons that would have come through the volume occupied by the other cylinders in the linear array.

2. Discrete and Hybrid Representations

Two dimensional critical calculations only were used in the application of the Discrete and Hybrid representations. The calculated worths, based on eq. (6-7), and associated Relative Errors are shown in columns (E) through (G) of Table 6-2A. Before discussing the worth comparison, the clad correction and the interference effect between cluster elements will again be considered.

a) Clad Correction

Column (F) of Table 6-2A shows the Relative Error associated with the worths initially calculated by the Discrete Method of representation. The worth results for the uniform arrays show marked dependence on the thickness of steel around the Ag-In-Cd absorber. Implicit in these worth results is the 0.7 value for the δ factor which had been developed on the basis of Shell and Homogeneous worth comparisons with experiment.

The development of the Discrete method of representation commences with a one dimensional representation and homogenization of the unit cell containing an isolated absorber. In the Shell and Homogeneous methods, the absorber unit cell, per se, is not considered. Within this fundamental difference in approach might be found the reason for the necessity of the δ factor in the first place; a one dimensional cell study and the results shown in column (F), however, provided the basis for simply choosing a δ factor suitable for use with the Discrete (and also the related Hybrid) representation. This cell study provided the variation in worth for a .330" diameter absorber surrounded by 60 mils of stainless steel with variation in δ factor. The variational relationship proved to be identical for both the 2.7% and 3.7% SS clad fuel. Table 6-3 demonstrates the linearity of the correction for other thicknesses. Figure 6-1 summarizes this worth variation relative

Fig. 6-1 % Worth Difference Relative to Use of 0.7 for δ Factor

to the use of 0.7 for δ ; the best value was estimated to be 0.84; column (G) of Table 6-2A shows Relative Errors based on the Discrete and Hybrid representations and the 0.84 δ factor. The difference in this empirical factor between .7 and .84 amounts to a reduction of 1% in worth for each additional 10 mils of SS around the Ag-In-Cd.

b) Interference Effect and Worth Comparisons

Column (F) of Table 6-2A shows that the Discrete representation slightly overestimates the worth of clusters. The results for uniform arrays are shown in column (G) and, with one exception - see footnote on Table 6-2A, are within 1.6% of the experimental results. The cluster worth overestimate clearly, then, is related to the interference between elements in a cluster. The specific cause of the difficulty is clear-cut and is not related to any water slot approximation. It is due to the erroneous homogenized fuel region diffusion lengths implicit in the MND thermal model.

Recall that the geometrically rectilinear representation of the unit cell containing an individual element in an array is generated from an original reference representation which utilizes, as it should, the conventional thermal model for the media surrounding the absorber. An analogous PDQ cell is used to transform the thermal model (and geometry). In this PDQ and in the final Discrete runs, the thermal group constants used are derived from the MND model. (The compelling reasons for the use of this model, particularly in design, have been previously stated.) The arguments presented above in connection with the Shell method which illustrated the importance of cluster element separation in terms of thermal mean free paths have application in this instance in that only thermal diffusion lengths derived from the conventional thermal model are valid in the vicinity of an absorber. In the two steps beyond the reference AIM, the Discrete method of representation used the MND model; thus, when this representation is applied to clusters, the interference is erroneously calculated.

This error is illustrated by means of Table 6-6 which compares the results of applying the two thermal models in the calculation of the

modified Bessel function which corresponds to part of the interference term. The quantity d in the table is the modal distance between next nearest neighbors in the 16 element clusters; it is equivalent to three lattice pitches.

Table 6-6
Effect of Thermal Model on Cluster Interaction

	<u>2.7% SS</u>	<u>3.7% SS</u>	<u>2.7% Zr</u>
$d(\text{cm})$	3.3147	3.3147	4.5720
$L_{\text{Conv.}} \text{ cm}^{-1}$	1.465	1.347	1.587
$L_{\text{MND}} \text{ cm}^{-1}$	1.113	0.9806	1.237
$(\frac{d}{L_{\text{Conv.}}})$	2.263	2.461	2.881
$K_o(\frac{d}{L_{\text{Conv.}}})$	0.08278	0.06528	0.03987
$(\frac{d}{L_{\text{MND}}})$	2.9776	3.3802	3.6952
$K_o(\frac{d}{L_{\text{MND}}})$	0.03565	0.02246	0.01572

Table 6-7 provides the modal next nearest neighbor separation in mean free paths for 16 element clusters in a design for the Southern California Edison core; the information is provided for two enrichments at two boron concentrations. It can be seen from the table that the interference values in these critical experiment cluster configurations bracket the interference in a practical design.

Table 6-7

Cluster Separation in an SCE Cluster

<u>Boron Concentration</u>	<u>Enrichment</u>	<u>L_{Conv}</u>	<u>d/L[*]</u>
1500	3.01	1.8052	2.3117
	4.30	1.6338	2.554
2500	3.01	1.7452	2.3911
	4.30	1.5892	2.6258

* d = 4.173 cm

One purpose for developing the Hybrid method of representation was the elimination of this defect in the analysis of critical experiments. The other purpose (simultaneously achieved) was the incorporation in the Hybrid method of a feature which allowed the user to choose both any geometry as an approximation for the cluster and also any mesh in conjunction with this geometry. The Discrete method culminates in a (discrete) representation (of the unit cell containing the absorber) which, for a cluster, necessarily involves a fine mesh in the cluster vicinity. The Hybrid method, on the other hand, retains the conventional thermal model through an auxiliary cluster cell representation involving a discrete fine mesh description; use is then made of a second auxiliary cell calculation for simultaneous transformation of thermal model, mesh description and approximating geometry while preserving capture (and removal) fractions. This latter representation is then simply "planted" in the mesh describing the actual configuration.

The definitive cluster results in column (G) of Table 6-2A and column (A) of Table 6-2B were obtained by applying a correction to Discrete results for clusters. The corrections were obtained from the Hybrid runs for a 16 rod cluster in each of the three lattices. As one might anticipate, the magnitudes of the corrections in worth which are constituted by the

differences in Relative Error between the Hybrid and Discrete results are proportional to the magnitudes of the Bessel functions shown in Table 6-6. For the two SS clad lattices, the corrections in RE, Hybrid method relative to Discrete method, are -3.7% for the 2.7% fuel and -.9% for the 3.7% fuel; for the zircaloy clad fuel, the correction was only -.5% in RE. Thus, in terms of worth only, the Hybrid method is seen as a small correction on the Discrete which places the cluster results in better agreement with uniform arrays.

The Hybrid cluster results for the Zr-clad fuel shown in Table 6-2 correspond to an approximating representation consisting of a shell with about a .5 cm mesh spacing in the shell vicinity. For this same case, two other Hybrid representations were developed and two other worth calculations were made: in one representation, a shell was developed in which the local mesh external to the shell was about 2 cm; in the other, a coarse mesh discrete approximation was used (one mesh per lattice pitch in the cluster vicinity). The Relative Errors were 4.6% and 5.3%, respectively.

3. Evaluation of the Accuracy of the Worth Comparison

An evaluation has been made of the accuracy of the explicit comparisons made for the configurations constructed in the two steel clad lattices between $\Delta\rho_{\text{CAL}}$ and $\Delta\rho_{\text{EXP}}$. This evaluation indicates that the various values for $\Delta\rho_{\text{EXP}}$ are quite accurate; an overestimate in $\Delta\rho_{\text{EXP}}$ of 2% is suggested by this evaluation but other facets of the evaluating method might be responsible for such a small discrepancy.

Table 6-8 compares, for configurations in the two SS lattices, the average values of Relative Error as calculated by eqs. 6-12 and 6-13; these averages are derived from the tabulation in Tables 6-2A and 6-2B, respectively. Recall that each type of RE is based on virtually distinct experimental information: in one case, the measured critical water heights; in the other the $\Delta\rho_{\text{EXP}}$. Note also the insensitivity of the RE derived from eq. 6-13 to $\Delta\rho_{\text{EXP}}$. The last column in Table 6-8 provides the information which could

Table 6-8

Estimate of Uncertainty in the Definitive Worth Comparisons

	Average % RE from Eq. 6-12	Average % RE from Eq. 6-13	Difference in Average % RE
2.7 w/o Uniform Array	.58	1.5	.9
2.7 w/o Cluster Array	-.29	2.1	2.3
3.7 w/o Uniform Array	.53	3.3	2.8
3.7 w/o Cluster Array	1.53	2.5	1.0
Grand Average Difference			2

suggest a 2% worth overestimate in $\Delta\rho_{\text{EXP}}$. By the same token, however, an error greater than 2% or a worth underestimate would be ruled out. This is because the sign and magnitude of the difference in Relative Errors can be taken as a measure of the signed uncertainty in $\Delta\rho_{\text{EXP}}$. The reasoning follows: postulate a revision upwards of $\Delta\rho_{\text{EXP}}$ by 20%. If the relative error calculated according to eq. 6-12 were to be maintained at the values quoted in the table, the analytic absorber worth would have to be enhanced in some manner such that $\Delta\rho_{\text{CAL}}$, i.e. $\ln(\lambda_D/\lambda_C)$, would also increase by 20% and this change would correspond to a change in λ_C only. But any change in absorber representation reflected in λ_C would necessarily be reflected in λ_E . Now the 20% change in $\Delta\rho_{\text{EXP}}$ only would cause a negligible change in the RE values in Table 6-8 calculated according to eq. 6-13 but the (approximate) 20% decrease in λ_E would be seen in the final revised value of % RE as an addition of 20. As shown in Table 6-2B, the critical heights for absorber-in configurations are, in almost all cases, at least 70 cm. Accordingly, uncertainties in axial reflector savings are equivalent to little over 1% in RE in this height range. Thus the grand average RE difference noted in the last column of Table 6-8 is a good measure of the accuracy of $\Delta\rho_{\text{EXP}}$.

A qualitative way to summarize this process follows: if a given method of absorber representation does not enable accurate correlation with measured critical water heights for absorber-in configurations and if this same method produces $\Delta\rho_{\text{CAL}}$ in excellent agreement with $\Delta\rho_{\text{EXP}}$ then the representation is in error and so also is $\Delta\rho_{\text{EXP}}$. Upon correction both of the latter and also the analytic method, analysis should correlate with experiment on water heights and $\Delta\rho_{\text{CAL}}$ will still agree with $\Delta\rho_{\text{EXP}}$. The Relative Errors corresponding to eq. 6-13 are precisely the measure of the accuracy of an absorber-in configuration critical height correlation; the Relative Errors corresponding to eq. 6-12 are precisely the measure of the accuracy of $\Delta\rho_{\text{CAL}}$ with respect to $\Delta\rho_{\text{EXP}}$. In this connection, it might be mentioned that the Discrete method outlined in this report predicted the critical height of the 25 rod uniform array in the Zr-clad configuration

as 91 cm. even before this fuel had been assembled into any sort of critical configuration.

VII. CONCLUSIONS AND RECOMMENDATIONS

The purpose of the effort described in this report was the generation of methods useful in design for the representation of Rod Cluster Control elements. Three such methods have been developed. The Hybrid method of representation has proven to be most accurate in comparison with experiment; a conservative estimate of the error in the method is $\pm 2.5\%$ about an average value corresponding to a worth overestimate of 1% only; this figure includes the uncertainty in the experimental worths. The Hybrid method has the soundest theoretical bases and should be the most accurate in application to the representation of clusters at temperature and under conditions of 2,000 psi. This method is also most versatile in that the designer can choose the particular geometry with which the cluster is approximated; a shell geometric representation is recommended, however, as being the best approximation compatible with the mesh limitations encountered in design. The Hybrid method involves iterative use of the computer for simple problems intrinsic to the development of the final representation. The Shell method of representation, on the other hand, requires only hand calculations for the development of a cluster approximation which utilizes the shell geometry representation. The Shell method worth comparisons with experiment show very little more scatter than the Hybrid; the worth, on the average, calculated using this method appears to be underestimated by only 2%. The designer may well prefer the Shell method for survey work provided the results of Hybrid calculations for a typical design are available as a worth standard. The Homogeneous method of representation also requires only hand calculations for the development of a homogeneous approximation to the cluster. This representation is the least sound theoretically of the three but it could be useful in certain design applications, e.g., stuck cluster analyses which require a full core representation instead of the more usual quarter core description and therefore correspondingly coarser mesh for representation of the heterogeneities within the core.

All methods are based on an identical analytical four group representation for the individual cylindrical Ag-In-Cd absorber. The only empiricism

involved in any methods discussed in this report involve a correction for the stainless steel in clad and thimble which will surround the poison rods in practical designs.

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REFERENCES

1. R. D. Leamer et al., "Critical Experiments Performed with Cluster and Uniform Arrays of Rodded Absorbers", WCAP-3269-39, (To be published).
2. A. F. Henry, "A Theoretical Method for Determining the Worth of Control Rods", WAPD-218, (1959).
3. R. J. Breen, "A One Group Model for Thermal Activation Calculations", N.S.E. 2, 91, (1961).
4. H. Neumann, "Thermal Utilization, Thermal Flux and Lattice Cell Shape", HW-52048, (1957).
5. A. M. Weinberg and E. P. Wigner, "The Physical Theory of Neutron Chain Reactors", The University of Chicago Press.
6. G. C. Pomraning, "Control Worth of B_4C Rods", GEAP-3800, (1961).
7. M. H. Theys, "Integral Transport Theory of Thermal Utilization Factor in Infinite Slab Geometry", N.S.E. Vol. 7, 58, (1960).
8. A. Amouyal and P. Benoist, "Nouvelle Methode de Determination du Facteur d'Utilisation Thermique d'une Cellule", CEA-571, (1956).
9. G. W. Stuart and R. W. Woodruff, "Method of Successive Generations", N.S.E. Vol. 3, 339, (1958).
10. C. W. Maynard, "Blackness Theory and Coefficients for Slab Geometry", WAPD-TM-168, (1958); N.S.E. Vol. 6, 174 (1959).
11. K. M. Case et al., "Introduction to the Theory of Neutron Diffusion".
12. G. W. Stuart, "Multiple Scattering of Neutrons", N.S.E. Vol. 2, 617, (1957).
13. W. J. Eich and H. T. Williams, Jr., "The Heterogeneous Representation of Control Rods in Light Water Lattices; A Comparison of Analysis and Experiment", WCAP-3711 (1963).
14. G. A. Bat and D. F. Zaretskii, "Effective Boundary Conditions in the Theory of Neutron Diffusion", J.N.E. Vol. 9, 252, (1959).

REFERENCES (CONT'D)

15. H. P. Flatt and D. C. Baller, "AIM-5: A Multigroup, One Dimensional Diffusion Equation Code", NAA-SR-4694, (1960).
16. "Reactor Physics Constants", ANL-5800, (1958).
17. W. J. Eich and W. P. Kovacic, "Reactivity and Neutron Flux Distribution Studies in Multi-region Reactor Cores", WCAP-1433, (1961).
18. A. J. Buslik, "The Description of the Thermal Neutron Spatially Dependent Spectrum by Means of Variational Principles", WAPD-BT-25, (1962).
19. H. Bohl et al., "SLOP-1, A Thermal Multigroup Program for the IBM-704", WAPD-TM-188, (October 1960).
20. H. Hurwitz and G. M. Roe, "Absorption of Neutrons by Black Control Rods", J.N.E. 2, 85, (1955).
21. W. R. Cadwell et al., "PDQ-3 - A Program for the Neutron-Diffusion Equations in Two Dimensions on the IBM-704", WAPD-TM-179, (March 1960).

APPENDIX A

CONTRIBUTION TO THE INTEGRAL FROM THE $S_2^{(2)}$ TERM

$$S_{2n}^{(2)} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\text{rods}} \text{except } n=0 K_0\left(\frac{r_n}{L_3}\right) K_0\left(\frac{x}{L_4}\right) r dr \frac{d\theta}{2\pi}$$

$$= \sum_{m \neq 0} S_{2n m}^{(2)} \quad (A-1)$$

$$S_{2n m}^{(2)} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^R \int_0^{2\pi} K_0\left(\frac{r_{nm}}{L_3}\right) K_0\left(\frac{x_m}{L_4}\right) r_m dr_m \frac{d\gamma}{2\pi}$$

If we approximate the distance x by x_1 , (cf. Fig. A-1)

$$S_{2n m}^{(2)} \approx \int_0^{2\pi} K_0\left(\frac{x_{1m}}{L_4}\right) \frac{d\phi}{2\pi} \int_0^R r_m dr_m \int_0^{2\pi} K_0\left(\frac{r_{nm}}{L_3}\right) \frac{d\gamma}{2\pi}$$

$$x_{1m}^2 = (ma)^2 + R^2 - 2 ma R \cos \phi$$

$$r_{nm}^2 = y^2 + r_m^2 - 2y r_m \cos \gamma$$

$$y = |m-n|a$$

for $m \neq n$

$$S_{2n m}^{(2)} = L_3^2 K_0\left(\frac{ma}{L_4}\right) I_0\left(\frac{R}{L_4}\right) \frac{R}{L_3} I_1\left(\frac{R}{L_3}\right) K_0\left(\frac{y}{L_3}\right) \quad (A-3)$$

for $m = n$

$$S_{2n m}^{(2)} = L_3^2 K_0\left(\frac{ma}{L_4}\right) I_0\left(\frac{R}{L_4}\right) \left[1 - \frac{R}{L_3} K_1\left(\frac{R}{L_3}\right) \right] \quad (A-4)$$

Fig. A-1

Noting that the main contribution to $S_2^{(2)}$ comes from the nearest neighbors,

$$S_2^{(2)} \lesssim \sum_{m \neq 0} K_0 \left(\frac{ma}{L_4} \right) L_3^2 I_0 \left(\frac{R}{L_4} \right) \left[1 - \frac{R}{L_3} K_1 \left(\frac{R}{L_3} \right) + \frac{R}{L_3} I_1 \left(\frac{R}{L_3} \right) M_3 \right]$$

$$= L_3^2 I_0 \left(\frac{R}{L_4} \right) \left[1 - \frac{R}{L_3} K_1 \left(\frac{R}{L_3} \right) + \frac{R}{L_3} I_1 \left(\frac{R}{L_3} \right) M_3 \right] M_4 \quad (A-5)$$

The percentage of $S_2^{(2)}$ to $S_2^{(1)}$ is obtained from eq. (A-5) and (A-6) and is shown on Table A-1 for the typical cases.*

$$S_2^{(1)} = \frac{L_4^2 I_0 \left(\frac{R}{L_4} \right)}{1 - \left(\frac{L_4}{L_3} \right)^2} \left[W_2 + W_1 M_3 - M_4 \right] \quad (A-6)$$

The error from neglecting the $S_2^{(2)}$ term is thought to be about 10% in (1-Y) and within 4% in Y for a practical case.

Table A-1 $S_2^{(2)} / S_2^{(1)}$ *

Pitch (cm)	$S_2^{(2)}$	$S_2^{(1)}$	$S_2^{(2)} / S_2^{(1)} \times 10^2$
0.80	1.31	7.67	17
1.60	0.19	3.87	5

* rod radius : 0.362 cm
core : 2.7% enrichment, W/U = 2.9 UO_2 fuel

APPENDIX B

UNCERTAINTY IN THE NORMALIZATION OF THE DIFFUSION KERNEL FROM A LINE SOURCE

The diffusion kernel from a line source can be taken as,

$$Q(r' \rightarrow r) = \frac{1}{2\pi DH} K_0 \left(\frac{r'-r}{L} \right) \quad (B-1)$$

$$\int_{\text{mod}} Q(r' \rightarrow r) dr = \frac{L^2}{D} \quad (B-2)$$

where H is the normalization factor which is calculated from Eq. (B-1) and (B-2).

$$\begin{aligned} H &= \frac{1}{2\pi L^2} \int_{\text{mod}} K_0 \left(\frac{r'-r}{L} \right) dr \\ &= 1 - \frac{1}{2\pi L^2} \int_{\text{rods}} K_0 \left(\frac{r'-r}{L} \right) dr \\ &= 1 - \frac{R}{L} I_1 \left(\frac{R}{L} \right) \sum_n K_0 \left(\frac{r'_n}{L} \right) \end{aligned} \quad (B-3)$$

r'_n : distance between the point r' and the center of rod n .

For an isolated rod,

$$H = 1 - \frac{R}{L} I_1 \left(\frac{R}{L} \right) K_0 \left(\frac{R}{L} \right) \approx 1 - \frac{R}{L} I_1 \left(\frac{R}{L} \right) K_0 \left(\frac{R}{L} \right) \equiv H_{\min} \quad (B-4)$$

For $R/L \lesssim 0.5$, which condition is satisfied in the practical problem,

$$1 - H_{\min} \lesssim 0.1 \quad (B-5)$$

and if we extend this to $R \rightarrow \infty$ mechanically, a discrepancy of a factor of 2 appears.

$$H_{\min} (R \rightarrow \infty) \rightarrow \frac{1}{2} \quad (B-6)$$

APPENDIX C
Summary of Nuclear Analysis Data

TABLE C1
Few Group Diffusion Constants Derived from FORM and THERMOS

.435" Lattice W/U = 2.9: 1 SS Clad Fuel

		2.7 w/o	3.7 w/o	Slot H ₂ O	Reflector
Group 1	Σ_r	0.081767	0.082047	0.114860	0.107118
(8.21 Kev to 10 Mev)	Σ_a	0.005153	0.005220	0.0011818	0.001440
	$\nu\Sigma_f$	0.009647	0.009819	0.0	0.0
	D	1.80439	1.78294	1.82804	2.04729
Group 2	Σ_r	0.077272	0.076819	0.153910	0.14907
(5.53 Kev to 8.21 Kev)	Σ_a	0.003002	0.003153	0.000013	0.000013
	$\nu\Sigma_f$	0.000944	0.001292	0.0	0.0
	D	0.965130	0.96104	1.08180	1.06006
Group 3	Σ_r	0.067047	0.065504	0.095284	0.15119
(0.625 ev to 5.53 Kev)	Σ_a	0.026094	0.028521	0.0006953	0.000949
	$\nu\Sigma_f$	0.013281	0.017807	0.0	0.0
	D	0.64369	0.63937	0.440869	0.59769
Thermal (Conventional)	Σ_a	0.12529	0.14793	0.014991	--
(0.0 ev to 0.625 ev)	$\nu\Sigma_f$	0.18516	0.23462	0.0	--
	D	0.26890	0.26830	0.20283	--
Thermal MND	Σ_a	0.18885	0.23221	0.022146	0.021680
(0.0 ev to 0.625 ev)	$\nu\Sigma_f$	0.27909	0.36828	0.0	0.0
	D	0.23400	0.22330	0.18523	0.17157

TABLE C2

Few Group Diffusion Constants Derived from FORM and THERMOS

.600" Lattice W/U = 3.0:1 Zr Clad Fuel

		2.72 w/o
Group 1 (8.21 Kev to 10 Mev)	Σ_r	0.079626
	Σ_a	0.005248
	$\nu\Sigma_f$	0.009205
	D	1.79758
Group 2 (5.53 Kev to 8.21 Kev)	Σ_r	0.075725
	Σ_a	0.002964
	$\nu\Sigma_f$	0.000904
	D	0.96037
Group 3 (0.625 ev to 5.53 Kev)	Σ_r	0.066870
	Σ_a	0.023238
	$\nu\Sigma_f$	0.012415
	D	0.71218
Thermal (Conventional) (0.0 ev to 0.625 ev)	Σ_a	0.107500
	$\nu\Sigma_f$	0.17734
	D	0.27070
Thermal MND (0.0 ev to 0.625 ev)	Σ_a	0.15586
	$\nu\Sigma_f$	0.25711
	D	0.23860

TABLE C3

Cylindrical Boundary Condition α_{cy}
for Unclad Absorber Rods

Absorber O.D. (inches)	$\alpha_{cy\ 1}$	$\alpha_{cy\ 2}$	$\alpha_{cy\ 3}$	$\alpha_{cy\ 4}$
0.260	0.000886	0.004077	0.0832	0.3623
0.330	0.001125	0.005175	0.0963	0.3759
0.403	0.001374	0.006320	0.1072	0.3876

TABLE C4

Discrete Method of Representation:Few Group Constants for Homogenized Unit Cell Containing Unclad Absorber Rods

Absorber O.D. (inches)	Group	2.7 w/o SS Clad Fuel Σ_a	3.7 w/o SS Clad Fuel Σ_a	2.7 w/o & 3.7 w/o SS Clad Fuel Σ_r	2.72 w/o Zr Clad Fuel Σ_a	2.72 w/o Zr Clad Fuel Σ_r
0.260	1	0.002354	0.002355	0.082632	---	---
	2	0.006928	0.006928	0.110735	---	---
	3	0.13907	0.13909	0.068548	---	---
	4	0.7300	0.7500	---	---	---
0.330	1	0.003072	0.003073	0.062942	---	---
	2	0.011159	0.011159	0.084341	---	---
	3	0.20427	0.204280	0.052215	---	---
	4	1.100	1.1370	---	---	---
0.403	1	0.004004	0.004004	0.037432	0.002662	0.074163
	2	0.016645	0.016645	0.050158	0.008739	0.099377
	3	0.27912	0.27913	0.031053	0.144170	0.061523
	4	1.7420	1.8300	---	0.59850	---

D = 1.0 in all groups

TABLE C-5

Zircaloy Clad and UO₂ Fuel DescriptionDIMENSIONS

Pitch 0.600 inch

Pellet (Fuel)

Material	UO ₂
Enrichment w/o U-235	2.72%
Pellet Diameter	0.400 inches
Active Fuel Length	48.0 inches
% Theoretical Density	95%

Tubes (Cladding)

Material	Zircaloy-4
OD	0.4683 inches
ID	0.4054 inches
Length	49.47 inches
Density	6.55 gm/cc

CHEMICAL ANALYSISFuel Analysis

U	88.15	w/o
C	< 10	ppm
F	< 10	"
Al	40	"
B	< .5	"
Bi	< 1	"
Cd	< .3	"
Co	< 4	"
Ca	9.5	"
Cr	34	"
Cu	2.0	"
Fe	266	"
In	< 3	"
Mg	4.4	"
Mn	2.4	"
Mo	6.2	"
Ni	24.3	"
Pb	< 8	"
Si	21	"
Sn	< 2	"
Ti	3.9	"
V	< 1	"
W	< 50	"
Zn	< 8	"
N	< 18	"

Clad Analysis

Zr	98.20%
Fe	0.21%
Cr	0.10%
Ni	< .004%
Sn	1.4%
Hf	< 100 ppm
Al	< 20 "
B	0.2 "
C	95 "
Cd	0.2 "
Co	10 "
Cu	33 "
Mg	< 10 "
Mn	< 20 "
Mo	< 20 "
N	45 "
Pb	< 20 "
Si	58 "
Ti	< 20 "
V	< 20 "
W	< 50 "

A Cluster
1 Moving Cluster
(3 positions)

B Cluster
1 Fixed Cluster
1 Moving Cluster
(3 positions)

C Cluster
2 Moving Clusters
(3 positions)

D Cluster
1. 2 Fixed Clusters
2. 4 Fixed Clusters

Fig. C-1 12 Element Cluster Positions

Fig. C-2 16 and 28 Rod Cluster Configurations
(Each Square Represents a Lattice Unit Cell)

16 Rod

28 Rod

APPENDIX D

POWER DISTRIBUTION COMPARISON FOR VARIOUS CLUSTER REPRESENTATIONS

The first two figures in this appendix display good agreement in power shape between the shell representation and the more exact fine mesh discrete representation. The homogeneous approximation does not represent the power in the center of the cluster very well but, outside the discontinuity at the homogenized region interface, the power distribution is more accurate.

In general, if two clusters are close and the interference between them is important, the shell method is preferred even if the homogeneous method is as accurate in the worth representation of a single cluster. The final two figures in this appendix display comparisons in power shape between a fine mesh Discrete cluster representation and a coarse mesh discrete representation for this cluster derived according to the Hybrid method. The worth difference between these two representations was only 2%; at the least, the comparison suggests that depletion of a core in which clusters are represented by a coarse discrete mesh should be carefully evaluated; errors in depletion accumulating for an absorber-in configuration could distort the peaking calculated for the same location upon absorber withdrawal.

Fig. D-1 Comparison of Source Distribution in 2.7%, 2.9/l Lattice
(16, 0.403" dia. Ag-In-Cd Rods Clad with 0.01" S.S.)

Fig. D-2 Comparison of Source Distribution in 2.7%, 2.9/l Lattice
(20, 0.260" dia. Ag-In-Cd Rods Clad with 0.01" S.S.)

Fig. D-3 Power Comparison for Discrete Representation - Diagonal Scan

Fig. D-4 Power Comparison for Discrete Representation - Transverse Scan

APPENDIX E

PROCEDURE SHEETS

This appendix contains procedures which should aid the designer in the practical application of the shell, homogeneous and hybrid representations. With each procedure are included sample calculations, each of which constitutes a numerical illustration of the particular method in representing typical configurations.

Because of their concise step-wise arrangement, these procedures also serve the purpose of clarifying the descriptions and discussions in the body of the report.

1. SHELL REPRESENTATION

A. SQUARE SHELL GEOMETRIC REPRESENTATION FOR CLUSTER ARRAYS

The process of transforming a cluster into a square shell is based on preserving the area of the total fuel region, outside and inside the shell. The procedure is described with the aid of Figure E-1.

1. Determine the outside dimension of the shell, B.
2. Calculate total area of unit cells that contain absorber rods:

$$\text{Shell Area} = N \times \text{Unit Cell Area}$$

where N = Number of Absorbers in the Cluster.

3. Calculate the fuel area interior to the shell.

$$\text{Interior fuel area} = B^2 - \text{shell area.}$$

Figure E-1: Cluster to Shell Conversion

4. Dimension A is given by

$$A = (\text{Interior fuel area})^{1/2}$$

5. Finally, the shell thickness is

$$t = \frac{B-A}{2}$$

B. Equivalent Diffusion Constants for the Shell in Groups 1 and 2

The equivalent macroscopic absorption cross section for groups 1 and 2 of Ag-In-Cd are:

$$\Sigma_1 = .00536 \text{ cm}^{-1}$$

$$\Sigma_2 = .02470 \text{ cm}^{-1}$$

The geometric shell representation, presented in Section A, homogenizes the absorber rods over the unit cell. Therefore, we must homogenize Σ_i by the ratio of absorber area (excluding the cladding if any) to the shell area.

$$\Sigma_{sh i} = \Sigma_i \times \frac{N \times \text{Absorber Area}}{\text{Shell Area}} \quad i=1,2$$

$$D_{sh i} = 2.0 \text{ cm}^{-1*}, \quad i=1,2$$

C. Calculation of the Boundary Conditions α in Groups 3 and 4 for an Individual Cylindrical Ag-In-Cd Absorber^{cy}

The symbols used in this section are defined below:

$$\alpha_i, \quad i = 3,4 \quad \text{Slab Boundary Condition}$$

* macroscopic removal of the shell is assigned zero value for all 3 fast groups

$\alpha_{cy\ 1}$, $i=3,4$ Cylindrical Boundary Condition

R_o = Absorber Radius

Δ = Radial SS thickness, ie., in determination of Δ , thimble included if any.

$R = R_o + \Delta$; because the definition of Δ ignores the thin water region between clad and thimble, R is not quite equal to the outer radius of the thimble.

Σ_{s1} , $i=3,4$ Macroscopic scattering cross section of the stainless steel.

$\lambda_{tr\ 3}$, The total mean free path in Group 3

$= \frac{1}{\Sigma_{s\ 3}}$ for the homogenized fuel region, cf procedure Sheet 3.

$$\bar{\lambda}_{tr\ 4} = \frac{1}{2} \left[\frac{1}{\Sigma_{tr\ 4\ \text{Homg. fuel}}} + \frac{1}{\Sigma_{tr\ 4\ \text{water slot}}} \right]$$

$\frac{1}{\Sigma_{tr\ 4\ \text{region}}}$ = 3 x Conventional thermal group diffusion coefficient for the region.

1. Read value of α_3 corresponding to an absorber radius $R_0 \equiv$ Slab thickness t from graph of Figure E-3 N.B. abscissa of Figure E-3 is t in inches.
2. $\alpha_4 = .5$
3. $\beta_1 = \frac{1 - 2\alpha_1}{1 + 2\alpha_1}, \quad i=3,4$
4. $\epsilon = \Delta/R_0$
5. $\beta'_1 = \beta_1 + \left[(\epsilon - \epsilon^2)(1 - \beta_1) + \Sigma_{s1} \Delta (1 - \beta_1)^2 \right] \times 0.7 \quad i=3,4$
6. $\alpha'_1 = \frac{1}{2} \frac{1 - \beta'_1}{1 + \beta'_1} \quad i=3,4$
7. $C_B = (1 + \frac{0.7604 \lambda_{tr \ 1}}{R}) / (1 + \frac{0.4052 \lambda_{tr \ 1}}{R}) \quad i=3,4$
8. $\alpha_{cy \ 1} = \frac{\alpha'_1}{1 + 2(C_B - 1)\alpha'_1} \quad i=3,4$

D. Equivalent Diffusion Constants for the Shell in Groups 3 and 4

Additional symbols introduced in this section are defined below:

$$J_1 = 2 \pi R L_1 / (B_1 + (D_1 / \alpha_{cy \ 1} L_1)) \quad i=3,4$$

$$D_1^+ = \text{homogenized fuel region diffusion coefficients (cm}^{-1}\text{)}$$

$$L_1^+ = \text{homogenized fuel region diffusion length (cm)}$$

* For $i=4$, conventional constants are to be used unless the symbols have an asterisk, in which case use MND values.

Figure E-3 α_3 Vs. Total Slab Thickness (Ag-In-Cd)

$$B_i = [K_0(R/L_i) + I_0(R/L_i)M_i] / [K_1(R/L_i) - I_1(R/L_i)M_i]$$

K_0, I_0, K_1, I_1 are ordinary Bessel Functions of zero and first order for imaginary arguments.

$$M_i = \frac{1}{N} \sum_{j=1}^N \left[\sum_{\substack{n=1 \\ n \neq j}}^{N-1} K_0(d_{nj}/L_i) \right]$$

d_{nj} is the distance between the center of j th absorber and the center of absorber n in the cluster containing N total absorbers. The inner summation is computed for $(N-1)$ absorbers interacting with the remaining or j th absorber. This interaction is summed and averaged over all N absorbers. Symmetry of a cluster leads to obvious simplifications in the summation. Figure E-4 schematically depicts the three families of d_{nj} corresponding to the three basic rod interaction positions in a 16 rod cluster. The effective boundary condition appropriate to the surface of the shell is given by

$$1. \quad \alpha_{si} = (D_i^*/L_i^*) / ((L_i^* \ell / N J_i) - 1), i=3,4$$

ℓ is the total surface area of the shell:

$$\ell = 4B + 4A$$

2. Group 3⁺:

$$\Sigma_{sh 3} = \frac{2 \alpha_{s 3}}{t}$$

$$D_{sh 3} = \frac{1}{3 \Sigma_{sh 3}}$$

*One is referred to Section V for a graphical generalization of the procedure to cover the less likely situation wherein more than one mesh space is used across the shell width.

Figure E-4: Rod Interaction Positions

Type 1

Type 2

Type 3

3. Group 4⁺:

$$\alpha_{s4}^* = \alpha_{s4} / \left[1 + (1-f) \cdot \alpha_{s4} \cdot L_4^* / D_4^* \right]$$

where

$$f = \sqrt{1 + (h/2L_4^*)^2}$$

and h = mesh size in fuel region adjacent to both inner and outer surfaces of shell.

$$\Sigma_{sh 4} = \frac{2 \alpha_{s4}^*}{t}$$

$$D_{sh 4} = \frac{1}{4 \Sigma_{sh 4}}$$

SAMPLE CALCULATIONS

The following sample calculation is carried out for a 16 rod cluster in a quondam SCE design. The cluster considered is for the inner fuel region, 3.01 w/o, having a moderator boron concentration of 2000 ppm. Figure E-5 schematically depicts the mesh description used in conjunction with the shell representation.

Table E-1 presents the fuel region constants for groups 3 and 4 as well as other pertinent parameters used in the calculation.

TABLE E-1

Group	D	$\Sigma_a + \Sigma_r$	L
3	0.74854	0.076688	3.1242
4 Conv.	0.37320	0.11854	1.7743
4 MND	0.50073	0.22858	1.4800

⁺One is referred to Section V for a graphical generalization of the procedure to cover the less likely situation wherein more than one mesh space is used across the shell width.

Figure E-5 SCE PDQ MESH DESCRIPTION

$$R_o = 0.5493 \text{ cm}$$

$$\Delta = 0.0813 \text{ cm}$$

$$R = 0.6306 \text{ cm}$$

$$\text{Pitch} = 1.391 \text{ cm.}$$

$$\text{Unit cell area} = 1.9349 \text{ cm}^2$$

A.

$$B = 10 \times 1.391 = 13.91 \text{ cm.}$$

$$\text{Shell Area} = 16 \times 1.9349 \text{ cm}^2 = 30.958 \text{ cm}^2$$

$$\text{Fuel Area interior to the shell} = (13.91 \text{ cm})^2 - (30.958 \text{ cm}) = 162.53 \text{ cm}^2$$

$$A = (162.53 \text{ cm}^2)^{1/2} = 12.749 \text{ cm.}$$

$$t = \frac{13.91 \text{ cm} - 12.749 \text{ cm.}}{2} = 0.5805 \text{ cm.}$$

B.

$$\text{Absorber Area} = \pi (0.5493 \text{ cm})^2 = 0.9479 \text{ cm}^2$$

$$\text{Shell Area} = 30.958 \text{ cm}^2$$

$$\frac{V_{\text{abs. Area}}}{V_{\text{Shell Area}}} = \frac{16 \times 0.9479}{30.958} = 0.4899$$

$$\Sigma_{\text{sh } 1} = 0.00536 \text{ cm}^{-1} \times 0.4899 = 0.0026 \text{ cm}^{-1}$$

$$\Sigma_{\text{sh } 2} = 0.02470 \text{ cm}^{-1} \times 0.4899 = 0.0121 \text{ cm}^{-1}$$

C.

$$\Sigma_{s-3} = 0.08807 \times 9.6 = 0.846$$

$$\Sigma_{s-4} = 0.08807 \times 10.1 = 0.890$$

$$\text{For } t = 0.5493 \text{ cm} = 0.2163 \text{ inches}$$

$$\alpha_3 = 0.1218$$

$$\alpha_4 = 0.50$$

$$\beta_3 = \frac{1 - 2(0.1218)}{1 + 2(0.1218)} = 0.6082$$

$$\beta_4 = 0$$

$$\epsilon = 0.0813 \text{ cm} / 0.5493 \text{ cm} = 0.1480$$

$$\begin{aligned} \beta_3' &= 0.6082 + \left[(0.1261)(0.3918) + (0.846)(0.0813)(0.1535) \right] 0.7 \\ &= 0.6502 \end{aligned}$$

$$\begin{aligned} \beta_4' &= 0.0 + \left[(0.1261)(1.0) + (0.890)(0.0813)(1.0) \right] 0.7 \\ &= 0.1389 \end{aligned}$$

$$\alpha_3' = 0.5 \left[\frac{1 - 0.6502}{1 + 0.6502} \right] = 0.1060$$

$$\alpha_4' = 0.5 \left[\frac{1 - 0.1389}{1 + 0.1389} \right] = 0.3780$$

$$\lambda_{tr 3} = 1.25$$

$$\bar{\lambda}_{tr\ 4} = 0.983$$

$$C_{B_3} = 2.5073/1.8032 = 1.3904$$

$$C_{B_4} = 2.1853/1.6316 = 1.3394$$

$$\alpha_{cy\ 3} = \frac{0.1060}{1 + [(2)(0.3904)(0.1060)]} = 0.09790$$

$$\alpha_{cy\ 4} = \frac{0.3780}{1 + [(2)(0.3394)(0.3780)]} = 0.30081$$

D.

$$\frac{R}{L_3} = 0.2018$$

$$\frac{R}{L_4} = 0.3554$$

Bessel Function	Group 3	Group 4
$K_0(R/L_1)$	1.7439	1.2189
$K_1(R/L_1)$	4.7296	2.5134
$I_0(R/L_1)$	1.0102	1.0319
$I_1(R/L_1)$	0.1015	0.1806

See Table E-2 and Figure E-4 for the details of the $\bar{M}_1$ calculation.

$$\bar{M}_3 = \frac{(4)(1.03638) + (8)(1.41599) + (4)(1.72052)}{16} = 1.3972$$

$$\bar{M}_4 = \frac{(4)(0.25155) + (8)(0.36206) + (4)(0.47066)}{16} = 0.3616$$

$$B_3 = \frac{1.7439 + (1.0102)(1.3972)}{4.7296 - (0.1015)(1.3972)} = 0.6878$$

$$B_4 = \frac{1.2189 + (1.0319)(0.3616)}{2.5134 - (0.1806)(0.3616)} = 0.6503$$

$$J_3 = \frac{2 \pi (0.6306)(3.1242)}{0.6878 + \left[\frac{0.74854}{(0.0979)(3.1242)} \right]} = 3.9484$$

$$J_4 = \frac{2 \pi (0.6306)(1.7743)}{0.6503 + \left[\frac{0.37320}{(0.30081)(1.7743)} \right]} = 5.2094$$

$$l = 4(13.91 \text{ cm}) + 4(12.749 \text{ cm}) = 106.636 \text{ cm.}$$

$$\alpha_{s,3} = \frac{0.74854/3.1242}{\left[\frac{(3.1242)(106.636)}{(16)(3.9484)} \right] - 1} = 0.23959/4.2735 = 0.05606$$

$$\Sigma_{a3} = \frac{2(0.05606)}{0.5805} = 0.1931$$

TABLE E-2

Type 1

No. of times d_n used	d_n	$\frac{d_n}{L_3}$	$K_o(\frac{d_n}{L_3})$	$\frac{d_n}{L_4}$	$K_o(\frac{d_n}{L_4})$
2	4.173	1.336	0.53040	2.352	0.14872
2	8.346	2.671	0.10190	4.704	0.01022
2	12.519	4.007	0.02218	7.056	0.00079
2	13.196	4.224	0.01738	7.437	0.00053
2	15.046	4.816	0.00902	8.480	0.00019
2	10.127	3.241	0.05264	5.708	0.00342
1	3.934	1.259	0.29401	2.217	0.08746
1	13.770	4.408	0.00708	7.761	0.00020
1	17.705	5.667	0.00177	9.978	0.00002
			$\sum_{i=1}^{15} = 1.03638$		$\sum_{i=1}^{15} = 0.25155$

Type 2

2	4.173	1.336	0.53040	2.352	0.14872
1	8.346	2.671	0.05095	4.704	0.00511
1	5.902	1.889	0.13061	3.326	0.02389
1	3.110	0.995	0.42405	1.753	0.15480
1	6.221	1.991	0.11516	3.506	0.01947
2	9.331	2.987	0.07052	5.259	0.00556
1	11.803	3.778	0.01431	6.652	0.00077
1	9.836	3.148	0.02929	5.544	0.00201
1	11.214	3.589	0.01772	6.320	0.00088
2	13.196	4.224	0.01738	7.437	0.00054
1	12.519	4.007	0.01109	7.056	0.00025
1	15.046	4.816	0.00451	8.480	0.00006
			$\sum_{i=1}^{15} = 1.41599$		$\sum_{i=1}^{15} = 0.36206$

Type 3

1	3.934	1.259	0.29401	2.217	0.08746
2	3.110	0.995	0.84810	1.753	0.30960
2	6.221	1.991	0.23032	3.506	0.03894
2	10.127	3.241	0.05264	5.708	0.00342
2	6.995	2.239	0.17032	3.942	0.02380
1	9.836	3.148	0.02929	5.544	0.00201
2	11.214	3.589	0.03544	6.320	0.00176
1	10.376	3.321	0.02403	5.848	0.00146
1	9.836	3.148	0.02929	5.544	0.00201
1	13.770	4.408	0.00708	7.761	0.00020
			$\sum_{i=1}^{15} = 1.72052$		$\sum_{i=1}^{15} = 0.47066$

$$D_3 = \frac{1}{3\Sigma_a} = 1.726$$

$$\alpha_{s4} = \frac{0.50073/1.4800}{\left[\frac{(1.4800)(106.636)}{(16)(5.2094)} \right] - 1}$$

$$\alpha_{s4} = \frac{0.33833}{0.89347} = 0.37867$$

$$h = 1.40 \text{ cm.}$$

$$f = \sqrt{1 + (1.40 \text{ cm}/2 \times 1.48)^2} = 1.1062$$

$$\alpha'_{s4} = 0.37867 / \left[1 + \left(1 - \frac{1}{1.1062} \right) (0.37867) \left(\frac{1.480}{0.50073} \right) \right] =$$

$$= 0.37867/1.10745$$

$$= 0.34193$$

$$\Sigma_{sh4} = \frac{2(0.34193)}{0.5805} = 1.178$$

$$D_{sh4} = \frac{1}{4 \Sigma_{sh4}} = 0.2122$$

2. HOMOGENEOUS REPRESENTATION

A. Geometric Representation

1. Determination of the Total Area, A_T , over which the Array is to be Homogenized

The total area, A_T , over which a square lattice of N absorbers should be homogenized is equal to $N \times (S)^2$ where S is the distance between absorber centers. Extension to non-square uniform lattice is obvious. The total area, A_T , over which a cluster of N absorbers should be homogenized is somewhat arbitrary and might be determined by the particular circumstances of the over-all core configuration.

2. Determination of the Reduction Factor, R :

$$R = A_F / A_T .$$

where A_F is the area of the homogenized fuel region in A_T .

B. Diffusion Constants for the Homogenized Region, A_T .

1. D_i , Σ_{ri} in the three fast groups is the same for A_T as for the fuel region surrounding the absorber.
2. Σ_{a1} and Σ_{a2} for A_T are also the same as for the fuel.
3. $v\Sigma_{fi}$ in all groups is equal to

$$R \times v\Sigma_{fi}^F$$

where $v\Sigma_{fi}^F$ corresponds to that for the ambient fuel.

4. Macroscopic absorption in groups 3 and 4 and the fourth group diffusion coefficients. See part C and D below.

C. Calculation of the Boundary Conditions α_{cy} in Groups 3 and 4 for an Individual Cylindrical Ag-In-Cd Absorber

See part C of the shell representation procedure in this appendix.

D. Σ_{a3} , Σ_{a4} and D_4 for the Homogenized Region A_T

The symbols used in this section are defined below:

$$J_1 = 2\pi R L_1 Y_1 / (\beta_1 + (D_1 / \alpha_{cy1} L_1))_{i=3,4}$$

where

$$D_1 = \text{conventional fuel region diffusion coefficients (cm}^{-1}\text{)}$$

$$L_1 = \text{conventional fuel region diffusion length (cm)}$$

$$Y_3 = 1.0$$

$$Y_4^* = 1 - \frac{I_0\left(\frac{R}{L_4}\right) W_2}{\left(B_3 + \frac{D_3}{\alpha_{cy3} L_3}\right) \left[K_1\left(\frac{R}{L_3}\right) \left[1 - \left(\frac{L_4}{L_3}\right)^2\right]\right]}$$

$$W_2 = \left[\frac{R}{L_4} K_1\left(\frac{R}{L_4}\right) K_0\left(\frac{R}{L_3}\right) + \frac{R}{L_3} K_1\left(\frac{R}{L_3}\right) K_0\left(\frac{R}{L_4}\right) \right]$$

$$B_1^* = \frac{K_0\left(\frac{R}{L_1}\right)}{K_1\left(\frac{R}{L_1}\right)}$$

I_0 , K_0 , K_1 are ordinary Bessel Functions of zero and first order for imaginary arguments.

The macroscopic absorption cross section, Σ_{p1} , representing the absorbers homogenized over A_T :

* These quantities for an isolated rod, i.e., a rod in a uniform array with negligible interaction between elements of the array.

$$i = 3,4 \quad \Sigma p_i = (\Sigma r_i + R \Sigma a_i^F) / \left[(A_T / N J_i) - 1 \right]$$

Σa_i^F = Macroscopic absorption cross section of fuel; in the thermal group use MND value.

N = Total number of absorbers in the cluster.

$$i = 3,4 \quad \Sigma a_i = \Sigma p_i + R \Sigma a_i^F$$

D_4 is the MND fuel diffusion coefficient

SAMPLE CALCULATION

The following sample calculation corresponds to a uniform array of twenty five 0.403 inch diameter unclad Ag-In-Cd rods.

Table E-3 presents the fuel region constants for groups 3 and 4 as well as other pertinent parameters used in the calculation.

TABLE E-3

Group	D	$\Sigma_a + \Sigma_r$	L	$R \Sigma_a$	$(R \Sigma_a + \Sigma_r)$
3	0.71218	0.090102	2.811	0.22587	0.089457
4 Conv.	0.27070	0.10750	1.587	0.10451	0.10451
4 MND	0.23860	0.15586	--	0.15153	--

$$R_o = 0.5118 \text{ cm}$$

$$\text{Pitch} = 1.5240 \text{ cm}$$

$$\text{Unit Cell Area} = 2.3226 \text{ cm}^2$$

A.

$$1. \quad A_T = 2090.322 \text{ cm}^2$$

$$\text{Absorber area } (25 \times 2.3226 \text{ cm}^2) = 58.065 \text{ cm}^2$$

$$A_F = 2032.257 \text{ cm}^2$$

$$2. \quad R = A_F/A_T = \frac{2032.257}{2090.322} = .97222$$

B. Table E-4 presents all but the Σ_{a3} and Σ_{a4} few group diffusion constants for the homogenized region.

TABLE E-4

Group	D	Σ_a	Σ_r	$v\Sigma_f = [R \times v\Sigma_f^F]$
1	1.79758	0.005248	0.079626	0.0089493
2	0.96037	0.002964	0.075725	0.0008789
3	0.71218	-	0.066870	0.012070
4 MND	0.23860	-	-	0.24997

$$C. \quad \Sigma_{s3} = 0.08807 \times 9.6 = 0.846$$

$$\Sigma_{s4} = 0.08807 \times 10.1 = 0.890$$

for $t = 0.5118 \text{ cm} = 0.403 \text{ inches}$

$$\alpha_3 = 0.117$$

$$\alpha_4 = 0.50$$

$$\beta_3 = \frac{1 - 2(0.117)}{1 + 2(0.117)} = 0.6207$$

$$\beta_4 = 0$$

$$\alpha_3^* = 0.5 \left[\frac{1-0.6207}{1+0.6207} \right] = 0.11702$$

$$\alpha_4^* = 0.50$$

$$\lambda_{tr3} = 1.01$$

$$\bar{\lambda}_{tr4} = 0.625$$

$$c_{B3} = \frac{2.5006}{1.7996} = 1.3895$$

$$c_{B4} = \frac{1.9286}{1.4948} = 1.2902$$

$$\alpha_{cy3} = \frac{0.117}{1 + [(2)(0.3895)(0.11702)]}$$

$$\alpha_{cy3} = 0.1072$$

$$\alpha_{cy4} = \frac{0.50}{1 + [(2)(0.2902)(0.50)]}$$

$$\alpha_{cy4} = 0.3875$$

D.

$$\frac{R}{L_3} = 0.1821$$

$$\frac{R}{L_4} = 0.3225$$

Bessel Function	Group 3	Group 4
$K_0(R/L_1)$	1.843	1.307
$K_1(R/L_1)$	5.281	2.814
$I_0(R/L_1)$	-	1.0262

$$B_3 = \frac{1.843}{5.281} = 0.3490$$

$$B_4 = \frac{1.307}{2.814} = 0.4645$$

$$J_3 = \frac{2\pi(0.5118)(2.811)(1.0)}{0.3490 + \left[\frac{0.71218}{(0.1072)(2.811)} \right]}$$

$$J_3 = 3.333$$

$$\Sigma_{p3} = \frac{0.089457}{\left[\frac{2090.322}{(25)(3.333)} - 1 \right]}$$

$$= 0.0037140$$

$$\Sigma_{a3} = 0.22587 + 0.003714$$

$$\Sigma_{a3} = 0.026301$$

$$W_2 = [(0.3225)(2.814)(1.843) - (0.1821)(5.281)(1.307)]$$

$$W_2 = 0.41565$$

$$Y_4 = 1 - \frac{(1.0262)(0.41565)}{\left[0.3490 + \frac{0.71218}{(0.1072)(2.811)} \right] (5.281)(0.6813)}$$

$$Y_4 = 0.9563$$

$$J_4 = \frac{2\pi (0.5118)(1.587)(0.9563)}{0.4645 + \left[\frac{0.27070}{(0.3875)(1.587)} \right]}$$

$$J_4 = 5.394$$

$$\Sigma_{p4} = \frac{0.15153}{\left[\frac{2090.322}{(25)(5.394)} - 1 \right]}$$

$$\Sigma_{p4} = 0.010449$$

$$\Sigma_{a4} = 0.15153 + 0.010449$$

$$\Sigma_{a4} = 0.16198$$

3. HYBRID REPRESENTATION

A. Calculation of the Boundary Conditions α_{cy} in Four Groups for an Individual Ag-In-Cd Cluster Element

1. α_{cy} in Groups 1 and 2

$$\alpha_{cy\ 1} = (.002684) \frac{R_o^2}{R}$$

$$\alpha_{cy\ 2} = (.012348) \frac{R_o^2}{R}$$

where R_o , R have been defined in part 1-C of this appendix.

2. α_{cy} in Groups 3 and 4

With one exception, these two boundary conditions are to be calculated according to the procedure outlined in Section 1-C of this appendix. The exception is that the 0.7, the last numerical factor in step 5 of that section, should be replaced by 0.84. (This is a substitution applying to this method only; it is not a revision of that section.)

B. Geometric Transformation of the Isolated Absorber

1. Set up a four group cylindrical AIM-5 according to the general set-up of figure E-6.

Figure E-6

- (a) Establish a zero current boundary condition at R_F .
- (b) Establish the rod boundary conditions at R . AIM-5 has the capability of specifying such boundary conditions at an off-origin position.
- (c) The area delimited by R_w is the same as the area of the unit cell containing the absorber. Thus the water is that occupying the balance of this unit cell.
- (d) The magnitude of R_F is somewhat arbitrary; 6 cm is suggested.
- (e) Conventional Thermal group constants should be used in both the fuel and water regions.
- (f) An extremely fine mesh is used throughout.

2. Set up a four group PDQ-03 equivalent to the AIM-5 cell in terms of outer boundary (See Figure E-7).

The purpose of this run is the proper homogenization of the absorber and associated water into a geometry suitable for use in the next step in the method.

- (a) Region 3 is specified as a "rod region" in PDQ, that is, it is described in all four groups simply by current-to-flux boundary conditions having magnitudes of 0.0001. This device effectively transfers the zero current boundaries C and D to the (stepped) equivalent of the AIM-5 outer boundary.
- (b) The input for region 2 is identical to the input for the fuel region of the AIM-5; the conventional thermal model is retained.
- (c) Note in Figure E-7 that the mesh in the fuel region is coarse relative to the AIM mesh but that it is tapered, that is, the mesh spacing is finer where the flux gradients are greater. Only 2 meshes are required across the quarter unit cell represented by region 1.

Region Description

- [1] Homogenized absorber and slot water (area = 1 unit cell)
- [2] fuel (Area = $\frac{1}{4}$ fuel area of AIM)
- [3] Boundary region

PDQ Fuel Region Mesh Spacing

Mesh Spacing	Mesh Thickness Over Interval (cm)
0 → 2	0.3810
2 → 6	0.2076
6 → 11	0.2640
11 → 17	0.5000
17 → 18	0.2500

Figure E-7 PDQ Cell Required for Unit Cell Transformation of AIM-5 Reference Description

- (d) The constants for the first three groups in region 1 are derived from the AIM-5 according to the following equations.

$$\Sigma_{a_i}^{Abs} = \frac{\alpha_i \phi_i^{surface} 2\pi R + \bar{\phi}_i \frac{H_2O}{V} \Sigma_{a_i}^{H_2O}}{\phi_i \frac{H_2O}{V} (V^{H_2O} + V^{rod})}$$

$$\Sigma_{r_i}^{Abs} = \frac{V \frac{H_2O}{V}}{V \frac{H_2O}{V} + V^{rod}} (\Sigma_{r_i}^{H_2O})$$

$$D_i^{Abs} = 1.0$$

- (e) For group 4 in region 1, use a D of 1.0. Vary Σ_a until the PDQ eigenvalue matches that of the AIM.

To guide the user, the following guesses are recommended for Σ_{a_4} ; they are based, however, on experience gained over a limited range of fuel enrichment and for room temperature criticals only. Two PDQ runs with these guesses have enabled the authors to extrapolate or interpolate to the correct value without running a third case.

$$\text{1st Guess } \Sigma_{a_4}^{abs} = 13 \left(\frac{\alpha_{cy 4} \Sigma_{a_4}^{fuel conv.}}{\text{Pitch (cm)}} \right)$$

$$\text{2nd Guess } \Sigma_{a_4}^{abs} = 0.85 \text{ 1st Guess}$$

In these and all PDQ cells used in the method, computer time can be saved by not requesting rectangular region integrals and pointwise edits, i.e., series 7000 and 8000 in PDQ-03, respectively. The required quantities, namely the integrals of the fluxes and sources, are automatically calculated for each composition.

C. Simultaneous Transformation to the Ultimate Geometry and MND Thermal Model

1. Reference Cell Problem with Discrete Representation of the Absorber Unit Cells

- (a) Set up a PDQ cell similar to that of Figure E-8 in which one quarter of a 16 rod cluster is depicted.
- (b) Use a constant mesh equivalent in magnitude to one-fourth of the lattice pitch from the center of symmetry to point A; from point A to point B use a tapered mesh similar to that shown in Figure E-8.
- (c) The group constants used for the representation of the unit cells containing cluster elements are those derived in part B.
- (d) Run the problem for 1 iteration only.

Zero current conditions on all boundaries

Figure E-8

2. Cell Problem with Shell Representation:

- (a) Set up another PDQ-03 with total area identical to that of the reference problem but with the cluster geometry approximated by a shell - see Figure E-9. Specify the shell dimensions in the manner described in part 1A of this appendix.
- (b) Use Mixed Number Density constants for the thermal group representation of the fuel.

Figure E-9

- (c) The mesh used in the cell may be quite coarse. The sole determination of the mesh is the actual core design which is to be represented by cyclic use of the cell mesh.
- (d) Running one iteration problems only, vary Σa_i in the shell until capture fractions for the shell approximation match capture fractions for the cluster representation in the reference discrete problem:

$$\text{Capture Fraction in Group } i = \frac{(\Sigma a_i \int \phi_i dV)_{\text{cluster representation}}}{(\Sigma a_i \int \phi_i dV)_{\text{cluster}} + (\Sigma a_i \int \phi_i dV)_{\text{fuel representation}}}$$

- (e) Shell capture fractions are matched to the reference values by adjusting the shell absorption cross sections, the removal cross section in each group is adjusted by the same factor as the absorption cross section. The following guide to the input for running two simultaneous shell problems is provided below. For the first problem, use

$$\Sigma a_i^{\text{shell}} = 0.994 \times \Sigma a_i^{\text{reference}}$$

$$\Sigma a_2^{\text{shell}} = 0.995 \times \Sigma a_2^{\text{reference}}$$

$$\Sigma a_3^{\text{shell}} = 0.99 \times \Sigma a_3^{\text{reference}}$$

$$\Sigma a_4^{\text{shell}} = 1.85 \times \Sigma a_4^{\text{reference}}$$

For the second shell problem, the input should be identical in groups 1 and 2. Let

$$\Sigma a_3^{\text{shell}} = .98 \times \Sigma a_3^{\text{reference}}$$

and

$$\Sigma a_4^{\text{shell}} = 1.75 \times \Sigma a_4^{\text{reference}}$$

The final values for groups 1 and 2 can be obtained by simply scaling capture fractions. Final values for groups 3 and 4 can be linearly interpolated from the results of these two problems making a third problem usually unnecessary. The factors used for thermal input to the two shell problems may be temperature dependent. They have been found by the authors to bracket the final correct values at room temperature over a range of external mesh spacing from relatively fine to coarse (2 cm).

Let

$$D_i = 1.0 \quad \text{for } i = 1, 2$$

In group 3,

$$D = \frac{1}{3(\Sigma a_3 + \Sigma r_3)}$$

In group 4,

$$D = \frac{1}{4 \Sigma a_4}$$

SAMPLE CALCULATION

The following sample calculation illustrates the method using 16 0.403" unclad Ag-In-Cd absorber rods in a square cluster array in 2.72 w/o Zr clad fuel.

A.

$$1. \quad \alpha_{cy 1} = \frac{(.002684)(.51181)^2}{(.51181)}$$

$$= .001374$$

$$\alpha_{cy\ 2} = \frac{(.012348)(.51181)^2}{(.51181)}$$

$$= .006320$$

$$2. \quad \Sigma_s\ 3 = 0.08807 \times 9.6 = 0.846$$

$$\Sigma_s\ 4 = 0.08807 \times 10.1 = 0.890$$

$$\text{for } t = 0.5118 \text{ cm} = 0.403 \text{ inches}$$

$$\alpha_3 = 0.117$$

$$\alpha_4 = 0.50$$

$$\beta_3 = \frac{1 - [2(0.117)]}{1 + [2(0.117)]} = 0.6207$$

$$\beta_4 = 0$$

$$\alpha'_3 = 0.5 \left[\frac{1 - 0.6207}{1 + 0.6207} \right] = 0.11702$$

$$\alpha'_4 = 0.50$$

$$\lambda_{tr\ 3} = 1.01$$

$$\bar{\lambda}_{tr\ 4} = 0.625$$

$$C_{B\ 3} = \frac{2.5006}{1.7996} = 1.3895$$

$$C_{B\ 4} = \frac{1.9286}{1.4948} = 1.2902$$

$$\alpha_{cy\ 3} = \frac{0.117}{1 + [(2)(0.3895)(0.11702)]}$$

$$\alpha_{cy\ 3} = 0.1072$$

$$\alpha_{cy\ 4} = \frac{0.50}{1 + [(2)(0.2902)(0.50)]}$$

$$\alpha_{cy\ 4} = 0.3875$$

B.

$$1. \quad \lambda_{\text{conv.}}(\text{ATM-5}) = 1.24496$$

where

$$R = .51181 \text{ cm}$$

$$R_W = .85983 \text{ cm}$$

$$R_F = 5.9048 \text{ cm}$$

2. Region 1 Homogenized Constants

Group	Σa	Σr	D
1	0.002662	0.074163	1.0
2	0.008739	0.099377	1.0
3	0.144166	0.061523	1.0

Iterative PDQ Σa_h vs. λ

$$(\text{1st guess}) \quad \Sigma a_h = 13 \left(\frac{0.3875 \times 0.1075}{1.524} \right)$$

$$= 0.355$$

$$(2\text{nd Guess}) \quad \Sigma a_4 = 0.3553 \times 0.85 = 0.302$$

	Σa_4	λ
1st Guess	0.355	1.2442
2nd Guess	0.302	1.2506
Interpolated	0.348	1.2450

C.

1. Reference PDQ Σa_1^{abs} and capture fractions

Group	Σa^{abs}	Capture fractions
1	0.002662	0.0252
2	0.008739	0.1299
3	0.144166	0.2321
4	0.3480	0.1186

2. Iterative Shell Group Constants and Capture Fractions

Group	1st Guess		2nd Guess		Final Values	
	Σa^{abs}	Capture Fraction	Σa^{abs}	Capture Fraction	Σa^{abs}	Capture Fractions
1	0.00265	0.0253	0.00265	0.0253	0.00264	0.0252
2	0.00870	0.1300	0.00870	0.1300	0.00869	0.1299
3	0.1427	0.2333	0.1412	0.2317	0.1416	0.2321
4	0.6440	0.1195	0.6090	0.1164	0.6335	0.1186

Shell Group Constants

Group	Σ_a^{shell}	Σ_r^{shell}	D^{shell}
1	0.00264	0.07361	1.0
2	0.00869	0.09879	1.0
3	0.1416	0.05981	1.655
4	0.6335	--	0.3946