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UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

AEC Contract No. W-7405-eng-48

COMPTON SCATTERING FROM RELATIVISTIC ELECTRONS*

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MAY 27, 1966

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COMPTON SCATTERING FROM RELATIVISTIC ELECTRONS

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I. INTRODUCTION

We present here a short numerical method for analyzing Compton scattering from a relativistic distribution of electrons. It is comparatively quick and is capable of any desired accuracy. The method is derived within quite a general framework, but for present purposes only a specialized scattering is considered.

The special scattering considered is that of a photon from a moving free electron in lowest order perturbation theory, called single Compton scattering. The electron is taken to be one of a distribution in phase space, and any degeneracy is explicitly neglected by taking the distribution to be relativistic Maxwell-Boltzmann in the non-configuration part of phase space. Finally the distribution (referred to occasionally subsequently as RMB for relativistic Maxwell-Boltzmann) is considered unpolarized.

II. FORMALISM

A. Interaction Rate

Particles of a given type are observed in some convenient frame of reference to be characterized by a range of dynamical states. A dynamical state is described by a particle energy and a direction of travel.

Quantitatively a distribution function f is defined so that the density of type #1 particles, having energy E and direction $\vec{\Omega}$ within infinitesimals dE and $d^2\vec{\Omega}$ of a chosen dynamical state labeled (I), is

$$f_1 d^3(I) = \text{density (particles per volume) of \#1-types within } d^3(I) \text{ of (I)}$$

where (I) = dynamical state described by energy E , direction $\vec{\Omega}$

$$d^3(I) = dE d^2\Omega$$

Then the density ρ_1 of #1-types is the total number per unit volume

$$\rho_1 = \int_{(I)} d^3(I) f_1$$

The case to be considered is that of two types of particles each with its own distribution function in the same region of space. Further the two types interact in some way to produce a third type having also its own distribution. The interaction is given as a rate R_3 such that

$$R_3 d^3(I) d^3(II) d^3(III) = \text{density of \#3-types per unit time produced with } d^3(III) \text{ of state (III) from the action of \#1's in state (I) on \#2's in state (II).}$$

Qualitatively the interaction rate R_3 is larger as the density of either type of particle is larger. Also the rate is larger if the types encounter each other more often. That is, R_3 should depend on the apparent relative velocity $|\vec{v}_I - \vec{v}_{II}|$ with which #1's and #2's travel:

$$R_3 \sim |\vec{v}_I - \vec{v}_{II}| f_1 f_2$$

B. Cross Section

Such an argument may be made quantitative by defining a function D_3 for any given $|\vec{v}_I - \vec{v}_{II}|$, f_1 , f_2 , and R_3 by

$$R_3 \equiv |\vec{v}_I - \vec{v}_{II}| f_1 f_2 D_3, \quad \underline{\text{definition of } D_3}.$$

Then by this definition

D_3 = density of 3-types created within $d^3(\text{III})$ of state (III) per unit apparent relative velocity, per unit 1-density, per unit 2-density, per unit III-state interval, per unit time,

and $D_3 d^3(\text{III})$ has units of area.

In the special frame of reference where one of the initial particles, say 2-type, is at rest the definition of D_3 takes on special meaning.

Denoting quantities measured in the rest frame by an asterisk,

$$R_3^* = |\vec{v}_1^*| f_1^* f_2^* D_3^*$$

and $D_3^* \equiv \frac{d^3\sigma}{d^3(\text{III})^*} = \underline{\text{differential cross section}}.$

When an explicit expression is derived for a cross section it is usually given in the rest frame. Anticipating this fact for Compton scattering,

the task now is to obtain a useable expression for D_3 in terms of $D_3^* \equiv \frac{d^3\sigma}{d^3(\text{III})^*}.$

C. Cross Section Transformations

A suitable transformation for D_3 may most easily be obtained by the use of Lorentz invariants. A pure number is such an invariant. So then "proper" quantities (those calculated in the rest frame) are invariant:

$$\gamma_{II}(d^3\vec{r}) = (d^3\vec{r})^* \quad \text{proper volume element}$$

$$\frac{1}{\gamma_{II}}(dt) = (dt)^* \quad \text{proper time interval}$$

where

$$\gamma_{II} = \left[1 - \left| \frac{\vec{v}_{II}}{c} \right|^2 \right]^{-1/2}$$

In the same sense that $\gamma_{II}(d^3\vec{r})$ is invariant, so also are

$$f_2 d^3(II) d^3\vec{r} = \text{number of 2-types in an element of phase space;}$$

$$R_3 d^3(I) d^3(II) d^3(III) d^3\vec{r} dt = \text{number of 3-types in an element of phase space.}$$

Landau and Lifshitz¹ have pointed out that another useful invariant is the scalar product of the 4-velocities,

$$u_I \cdot u_{II} = \pm \gamma_I \gamma_{II} (1 - \vec{\beta}_I \cdot \vec{\beta}_{II}) \quad (\text{sign depends on metric}).$$

The ratio

$$\frac{u_I \cdot u_{II}}{\gamma_I \gamma_{II}} = 1 - \vec{\beta}_I \cdot \vec{\beta}_{II}$$

-
1. Landau and Lifshitz: The Classical Theory of Fields, Addison-Wesley, revised 2nd ed. (1962), p. 38.

is defined even in the limiting case where one mass goes to zero (photon).

The definition of D_3 may now be rewritten as

$$[R_3 d^3(I) d^3(II) d^3(III) d^3 \vec{r} dt]$$

$$= \frac{[f_1 d^3(I) d^3 \vec{r}]}{[\gamma_I d^3 \vec{r}]} \frac{[f_2 d^3(II) d^3 \vec{r}]}{[\gamma_{II} d^3 \vec{r}]} \frac{[u_1 \cdot u_{II}]}{(1 - \vec{\beta}_I \cdot \vec{\beta}_{II})} |\vec{v}_I - \vec{v}_{II}| D_3 d^3(III) [d^3 \vec{r} dt]$$

where each quantity in square brackets is Lorentz-invariant.

Therefore,

$$\frac{|\vec{v}_I - \vec{v}_{II}|}{1 - \vec{\beta}_I \cdot \vec{\beta}_{II}} D_3 d^3(III) \text{ is Lorentz-invariant.}$$

Again Landau and Lifshitz² argue cleverly that $d^3 \vec{p}$, which is the volume element in (ordinary) 3-momentum space, is the fourth component of an element of the mass shell in 4-momentum space. As such it is parallel in 4-space to the mass shell normal, which is just the 4-momentum. The fourth component of the 4-momentum is $\frac{1}{c}E$. Since the ratio of corresponding components of parallel 4-vectors is invariant, then

$$\frac{d^3 \vec{p}}{E} = \frac{1}{c^2} |\vec{p}_{III}| d^3(III) \text{ is invariant,}$$

and finally,

$$\frac{|\vec{v}_I - \vec{v}_{II}|}{(1 - \vec{\beta}_I \cdot \vec{\beta}_{II})} \frac{D_3}{|\vec{p}_{III}|} \text{ is Lorentz-invariant.}$$

2. Ibid., p. 31.

When the quantity of interest is the total density of 3-type particles formed per unit time, then the definition of D_3 is integrated over all possible (III) states.

$$\int_{(III)} d^3(III) R_3 = |\vec{v}_I - \vec{v}_{II}| f_1 f_2 \int_{(III)} d^3(III) D_3$$

In the rest frame the integrated differential cross section is the total cross section σ .

$$\int_{(III)}^* d^3(III)^* D_3^* \equiv \sigma = \text{total cross section.}$$

Then the useful invariant to relate to σ is

$$\frac{|\vec{v}_I - \vec{v}_{II}|}{(1 - \vec{\beta}_I \cdot \vec{\beta}_{II})} \int_{(III)} d^3(III) D_3 \quad (\text{Lorentz-invariant}).$$

In the rest frame definitions of D_3^* and $\int d^3(III)^* D_3^*$ take the forms

$$R_3^* \equiv |\vec{v}_I^*| f_1^* f_2^* \frac{d^3\sigma}{d^3(III)^*}$$

$$\int_{(III)}^* d^3(III)^* R_3^* \equiv |\vec{v}_I^*| f_1^* f_2^* \sigma.$$

The usefulness of these forms is that the production rate of 3-type density is the mathematical product of the relative 1-type flux density, the 2-type density, and the cross section function. A useful thing to have would be a similar cross-section-like function for use in a laboratory frame of reference.

D. Distribution Averages

A distribution-averaged cross section function can be defined in several ways. Most directly one could take

$$\frac{\overline{d^3\sigma}}{d^3(\text{III})} \equiv \frac{\int d^3(\text{I}) \int d^3(\text{II}) R_3}{\int d^3(\text{I}) \int d^3(\text{II}) |\vec{v}_\text{I} - \vec{v}_\text{II}| f_1 f_2}$$

and

$$\overline{\sigma} \equiv \frac{\int d^3(\text{I}) \int d^3(\text{II}) \int d^3(\text{III}) R_3}{\int d^3(\text{I}) \int d^3(\text{II}) |\vec{v}_\text{I} - \vec{v}_\text{II}| f_1 f_2}$$

However in the usual case for a lab frame of reference what one knows are densities, ρ_1 and ρ_2 , and mean velocities,

$$\langle \vec{v}_\text{I} \rangle \equiv \frac{1}{\rho_1} \int d^3(\text{I}) \vec{v}_\text{I} f_1$$

$$\langle \vec{v}_\text{II} \rangle \equiv \frac{1}{\rho_2} \int d^3(\text{II}) \vec{v}_\text{II} f_2$$

In fact it may be very difficult to evaluate

$$\overline{Q} \equiv \int d^3(\text{I}) \int d^3(\text{II}) |\vec{v}_\text{I} - \vec{v}_\text{II}| f_1 f_2$$

It seems appropriate therefore to take somewhat more useable definitions for distribution-averaged cross sections. Define

$$\langle Q \rangle \equiv \rho_1 \rho_2 |\langle \vec{v}_\text{I} \rangle - \langle \vec{v}_\text{II} \rangle|$$

Then take the definitions

$$\left\langle \frac{d^3\sigma}{d^3(\text{III})} \right\rangle \equiv \frac{1}{\langle Q \rangle} \int d^3(\text{I}) \int d^3(\text{II}) R_3$$

$$\langle \sigma \rangle \equiv \frac{1}{\langle Q \rangle} \int d^3(I) \int d^3(II) \int d^3(III) R_3$$

These definitions differ from the former, $\frac{\overline{d^3\sigma}}{d^3(III)}$ and $\bar{\sigma}$, only in that $\bar{Q}$ differs from $\langle Q \rangle$.

$$\begin{aligned} \langle Q \rangle &\equiv \rho_1 \rho_2 |\langle \vec{v}_I \rangle - \langle \vec{v}_{II} \rangle| \\ &= |\rho_2(\rho_1 \langle \vec{v}_I \rangle) - \rho_1(\rho_2 \langle \vec{v}_{II} \rangle)| \\ &= |\int d^3(I) \int d^3(II) (\vec{v}_I - \vec{v}_{II}) f_1 f_2| \end{aligned}$$

Thus $\langle Q \rangle$ is not necessarily equal to

$$\bar{Q} \equiv \int d^3(I) \int d^3(II) |\vec{v}_I - \vec{v}_{II}| f_1 f_2$$

used in defining $\frac{\overline{d^3\sigma}}{d^3(III)}$ and $\bar{\sigma}$.

In either case, of course, the correct rate, R_3 or $\int d^3(III) R_3$, may be obtained, since by construction,

$$\langle Q \rangle \left\langle \frac{d^3\sigma}{d^3(III)} \right\rangle = \bar{Q} \frac{\overline{d^3\sigma}}{d^3(III)}$$

$$\langle Q \rangle \langle \sigma \rangle = \bar{Q} \bar{\sigma}.$$

III. COMPTON SCATTERING

A. The Problem

The specialized case to be considered is that of scattering of photons from a relativistic distribution of electrons. In contrast to the more usual treatments of the photon-electron system, that of elegant covariant dynamics from weak interaction perturbation theory, this paper simply accepts the lowest-order perturbation approximation performed by Klein and Nishina³ and concentrates on the non-dynamical details of this one problem. Other forms of interaction and correction terms may similarly be treated by the formalism given in part II of this paper.

Scattering of photons is further complicated by induced effects. The probability of scattering into a final state is enhanced by the density of photons already in that final state. For this reason it has sometimes been assumed⁴ that photons and electrons coexist before the scattering in respective distributions characteristic of local thermodynamic equilibrium. More generally some other photon distribution is present. To the extent that induced effects are to be treated one must have prior knowledge of the photon distribution. The present paper therefore assumes no photon density in the final state before scattering. Data so obtained may be used for a general prior distribution by integrating the data multiplied by⁵

$$[1 + \text{term proportional to prior distribution}]$$

The distribution to take then is unit for the photons. In this way only one photon state is studied at a time. By ignoring any electron degeneracy the appropriate distribution there is relativistic Maxwell-Boltzmann (RMB). Polarizations are assumed random.

3. Klein, O., and Y. Nishina, Zeits. fur Physik, 29, 853 (1929).

4. Sampson, D. H., Astrophysical Jour., 129, 734 (1959).

5. Chandrasekhar: An Introduction to the Study of Stellar Structure, Univ. of Chicago Press (1939)

B. Units and Variables

As is customary it is convenient to use units in which

$$c = 1, \quad h = 1, \quad k = 1$$

respectively for light velocity, Planck's constant, and Boltzmann's constant.

Then mass, momentum, frequency, and temperature are measured in units of energy.

Define energies and polar angle cosines as follows:

<u>particle</u>	<u>energy</u>	<u>cosine of polar angle</u>
incident photon	ν	μ
incident electron	E	α
scattered photon	ν'	μ'
scattered electron	E'	α'

Describe the azimuthal angle of state (i) as ϕ_i .

Another angle of importance is that between scattered photon and incident electron. Define

$$\begin{aligned} \eta &\equiv \text{cosine (angle between scattered photon, incident electron)} \\ &= \alpha\mu' + \cos\phi_{III} \sqrt{(1-\alpha^2)(1-\mu'^2)} \end{aligned}$$

Constants used are

$$m \equiv \text{electron mass}$$

$$= 511.006 \text{ kev}$$

$$r_0 \equiv \text{"classical radius of electron"}$$

$$= 2.81776 \times 10^{-13} \text{ cm.}$$

For electron energy E define

$$\gamma \equiv \frac{E}{m}$$

$$\beta \equiv \sqrt{1 - \frac{1}{\gamma^2}}$$

Then

$$\begin{aligned} p &\equiv \text{magnitude of electron 3-momentum} \\ &= \beta E \end{aligned}$$

C. The Distribution Functions

Unit photon distribution is proportional to delta functions. The particular choice

$$f_p d^3(I) \sim \frac{1}{2\pi} \delta(v-v) \delta(\mu-1) dv d\mu d\phi_I$$

effectively orients the lab frame of reference. In this way

$$\int d^3(I) f_p \sim 1$$

and the unit photon density is incident with energy v along the polar axis of the lab frame.

The lab frame is further chosen so that the RMB for electrons is isotropic.

$$f_e d^3(III) \sim \exp \left[- \frac{(E-m)}{T} \right] p^2 dp d\alpha d\phi_{II}$$

where the distribution is characterized as shown by

$$T \equiv \text{"temperature" of electron distribution.}$$

For convenience define

$$x \equiv \frac{E-m}{T}$$

Then

$$f_e d^3(II) \sim e^{-x} \beta E^2 dE d\alpha d\phi_{II}.$$

Again for later convenience define

$$N_e(T) \equiv \frac{4\pi}{T} \int_0^\infty dE e^{-x_{\beta} E^2}$$

$$= \int_0^\infty dx \int_{-1}^{+1} d\alpha \int_0^{2\pi} d\phi_{II} e^{-x_{\beta} E^2}.$$

Then

$$\rho_e \sim T N_e(T).$$

D. Transformation from the Rest Frame

In part II. C. it was shown that

$$\left. \begin{aligned} & \frac{|\vec{v}_I - \vec{v}_{II}|}{(1 - \vec{\beta}_I \cdot \vec{\beta}_{II})} \frac{D_3}{|\vec{p}_{III}|} \\ & \frac{|\vec{v}_I - \vec{v}_{II}|}{(1 - \vec{\beta}_I \cdot \vec{\beta}_{II})} \int d^3(III) D_3 \end{aligned} \right\} \text{are Lorentz invariants.}$$

For the present case state (III), the scattered photon state, has

$$|\vec{p}_{III}| = v'.$$

Further, $\beta_I = 1, \quad \beta_{II} = \beta, \quad \vec{\beta}_I \cdot \vec{\beta}_{II} = \beta\alpha.$

There obtains

$$|\vec{v}_I - \vec{v}_{II}| D_3 = c(1 - \beta\alpha) \frac{v'}{v'^*} \frac{d^3\sigma}{d^3(III)^*}$$

$$|\vec{v}_I - \vec{v}_{II}| \int d^3(III) D_3 = c(1 - \beta\alpha) \sigma^*.$$

In order to evaluate σ^* , $\frac{d\sigma}{d^3(\text{III})}^*$, and $\frac{v'}{v'^*}$ it is necessary to have the following energy and angle transformations to the rest frame:

$$v^* = v\gamma(1-\beta\alpha)$$

$$v'^* = v'\gamma(1-\beta\eta)$$

$$\mu'^* = 1 - \frac{(1-\mu')}{\gamma^2(1-\beta\alpha)(1-\beta\eta)}$$

E. The Klein-Nishina Formulae

For random polarizations single-Compton scattering in the electron's rest frame is given by⁶

$$\frac{d^3\sigma}{d^3(\text{III})}^* = g^*(v^*, v'^*, \mu'^*) \delta \left[v'^* - \frac{v^*}{1 + \frac{v^*}{m}(1-\mu'^*)} \right]$$

where

$$g \equiv \frac{1}{2} (r_0)^2 \left(\frac{v'^*}{v^*} \right)^2 \left[\frac{v'^*}{v^*} + \frac{v^*}{v'^*} - 1 + (\mu'^*)^2 \right]$$

The delta function is the result of conservation of 4-momentum.

Total cross section is

$$\sigma^* = 2\pi (r_0)^2 \left\{ \frac{\left(1 + \frac{v^*}{m}\right)}{\left(\frac{v^*}{m}\right)^3} \left[\frac{2\left(\frac{v^*}{m}\right)\left(1 + \frac{v^*}{m}\right)}{\left(1 + 2\frac{v^*}{m}\right)} - \ln \left(1 + 2\frac{v^*}{m}\right) \right] + \frac{\ln \left(1 + 2\frac{v^*}{m}\right)}{2\frac{v^*}{m}} - \frac{\left(1 + 3\frac{v^*}{m}\right)}{\left(1 + 2\frac{v^*}{m}\right)^2} \right\}.$$

6. See, for example, Heitler, W.: The Quantum Theory of Radiation, Oxford, 3rd ed. (1954), pp. 211 - 224.

IV. MATHEMATICAL METHOD

A. Distribution-Averaged Differential Cross Section

The formalism of part II applied to the special case of part III yields an expression for distribution-averaged differential cross section:

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{1}{\langle Q \rangle} \int d^3(I) \int d^3(II) f_p f_e \frac{c(1-\beta\alpha)}{\gamma(1-\beta\eta)} g^* \delta[h(v^*, v'^*, \mu'^*)]$$

where

$$h(v^*, v'^*, \mu'^*) \equiv v'^* - \frac{v^*}{1 + \frac{v^*}{m}(1 - v'^*)}$$

and

$$\begin{aligned} \langle Q \rangle &= \left| \rho_e \left(\rho_p \langle \vec{v}_I \rangle \right) - \rho_p \left(\rho_e \langle \vec{v}_{II} \rangle \right) \right| \\ &\sim \left| T N_e(T) [c \hat{n}_p] - [O] \right| \end{aligned}$$

This result obtains because the unit photon distribution is oriented along the polar axis, $\hat{n}_p$, and the lab frame is chosen by specifying f_e to be isotropic so that

$$\langle \vec{v}_e \rangle = 0$$

Integration over $d^3(I)$ simply picks out the photon variables.

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{1}{N_e(T)} \int_0^\infty dx \int_{-1}^{+1} d\alpha \int_0^{2\pi} d\phi_{II} e^{-x\beta E^2} \frac{(1-\beta\alpha)}{\gamma(1-\beta\eta)} g^* \delta[h]$$

The integrand of the $d^3(II)$ integral represents the contribution to

$\left\langle \frac{d^3\sigma}{d^3(III)} \right\rangle$ of a single electron (unit density per II-state interval) in state II having variables (E, α, ϕ_{II}) . But the full integral is independent of azimuth,

being symmetric about the polar axis. Therefore the angle ϕ_{II} of incident electrons and the angle ϕ_{III} of scattered photons are not absolute but relative, reflecting a freedom to choose lab frame axes with any zero azimuth. The integrand, a function of ϕ_{III} through

$$\eta = \alpha\mu' + \cos \phi_{III} \sqrt{(1-\alpha^2)(1-\mu'^2)}$$

may then be considered exactly the same function of the relative azimuth ϕ

$$\phi \equiv \phi_{III} - \phi_{II}$$

simply by appealing to azimuthal symmetry and rotation of axes to where $\phi_{II} = 0$. Moreover the integrand depends only on $\cos \phi$ (through η), so first two quadrants in ϕ are symmetric to the last two.

$$\left\langle \frac{d^3\sigma}{d\nu' d^2\Omega'} \right\rangle = \frac{2}{N_e(T)} \int_0^\infty dx \int_{-1}^{+1} d\alpha \int_0^\pi d\phi e^{-x\beta E^2} \frac{(1-\beta\alpha)}{\gamma(1-\beta\eta)} g^* \delta(h)$$

To help evaluate the delta function it is convenient to introduce a new variable

$$z \equiv \gamma(1-\beta\eta)$$

Since α , μ' , and $\cos \phi$ are real and in $(-1, +1)$, then so is η . Moreover η is monotonic decreasing in ϕ (α, μ' not zero). For z this means a one-to-one correspondence with ϕ , and

$$\frac{d\phi}{dz} = \frac{1}{\gamma \sqrt{\beta^2(1-\alpha^2)(1-\mu'^2) - [1-\beta\alpha\mu' - z/\gamma]^2}}$$

which is always positive.

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{2}{N_e(T)} \int dx e^{-x} \beta E^2 \int d\alpha (1-\beta\alpha) \int dz \frac{g^* \delta[h]}{z\gamma \sqrt{\beta^2(1-\alpha^2)(1-\mu'^2) - [1-\beta\alpha\mu' - z/\gamma]^2}}$$

The delta function may be transformed:

$$\begin{aligned} \delta[h] &\equiv \delta \left[v'^* - \frac{v^*}{1 + \frac{v^*}{m}(1-\mu'^*)} \right] \\ &= \frac{\delta(z-z_0)}{\left| \frac{d}{dz} h(z_0) \right|} \end{aligned}$$

where

$$z_0 \equiv \frac{v}{v'} \gamma (1-\beta\alpha) \left[1 - \frac{v'(1-\mu')}{E(1-\beta\alpha)} \right]$$

and

$$\left| \frac{d}{dz} h(z_0) \right| = v' \left[1 - \frac{v'(1-\mu')}{E(1-\beta\alpha)} \right]$$

Then integration over z of the delta function evaluates at z_0 . It is helpful to define

$$\rho \equiv \frac{v'(1-\mu')}{E(1-\beta\alpha)}$$

Then in terms of $\rho(E, \alpha, v', \mu')$, and with $\gamma \equiv \frac{E}{m}$,

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{2m^2}{vN_e(T)} \int_0^\infty dx e^{-x} \beta \int_{-1}^{+1} d\alpha \frac{g^*}{(1-\rho)^2 \sqrt{\beta^2(1-\alpha^2)(1-\mu'^2) - [1-\beta\alpha\mu' - \frac{v}{v'}(1-\beta\alpha)(1-\rho)]^2}}$$

The differential cross section term g^* may also be simplified. With the definitions

$$z \equiv \gamma(1-\beta\eta)$$

$$z_0 \equiv \frac{v}{v'} \gamma (1-\beta\alpha)(1-\rho)$$

$$\rho = \frac{v'(1-\mu')}{E(1-\beta\alpha)}$$

then the relativistic energy and angle transformations to the rest frame become

$$v^* = v\gamma(1-\beta\alpha)$$

$$v'^* = v\gamma(1-\beta\alpha)(1-\rho)$$

$$\mu'^* = 1 - \frac{\rho^2}{(1-\rho)} \frac{m^2}{vv'(1-\mu')}$$

Define

$$\omega = \frac{m^2}{vv'(1-\mu')}$$

and

$$\Delta = 2 + \frac{\rho^2}{(1-\rho)} \left[\omega^2 \frac{\rho^2}{(1-\rho)} - 2\omega + 1 \right].$$

Then

$$g^* = \frac{1}{2} (r_0)^2 (1-\rho)^2 \Delta.$$

The integrands are sometimes zero over a part of the range of α and x . This is true because $\frac{d\phi}{dz} > 0$ means that the radicand in the denominator must always be non-negative. To be explicit define

$$y_1 = \left(\frac{v}{v'}\right)^2 + 1 - 2\left(\frac{v}{v'}\right)\mu'$$

$$y_2 = \left(\frac{v}{v'}\right) + 1 + \frac{v}{E}\left(\frac{v}{v'} - \mu'\right)$$

$$y_3 = \left(\frac{v}{v'}\right) - 1 - \frac{v}{E}(1-\mu')$$

$$y_4 = 1 - \frac{y_2}{y_1}(1-\mu')$$

$$y_5 = \sqrt{y_4^2 + \frac{1}{y_1} [\beta^2(1-\mu'^2) - y_3^2]}$$

In terms of the y's the denominator square root becomes

$$\sqrt{\beta^2(1-\alpha^2)(1-\mu'^2) - [1-\beta\alpha\mu' - \frac{\nu}{\nu'}(1-\beta\alpha)(1-\rho)]^2}$$

$$= \sqrt{\nu_1} \sqrt{y_5^2 - (\beta\alpha - y_4)^2}$$

Since the radicand must be positive or zero, and since y_1 is always positive,

$$\alpha_{\min} \equiv \frac{1}{\beta} (y_4 - y_5); \quad \alpha_{\max} \equiv \frac{1}{\beta} (y_4 + y_5)$$

where

$$-1 \leq \alpha_{\min} \leq \alpha_{\max} \leq +1.$$

It is therefore convenient to transform the entire α -integral. Define

$$\xi \equiv \frac{1}{y_5} (\beta\alpha - y_4); \quad -1 \leq \xi \leq +1$$

Then

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{(r_0)^2 m^2}{\nu\sqrt{y_1} N_e(T)} \int_{x_{\min}}^{\infty} dx e^{-x} \int_{-1}^{+1} d\xi \frac{\Delta}{\sqrt{1-\xi^2}}$$

Just as non-zero radicand limits the range of α integration to $(\alpha_{\min}, \alpha_{\max})$, so also

$$\alpha \text{ real} \implies y_5^2 \geq 0$$

This means that the range of x integration is $(x_{\min}, \infty)$, where

$$x_{\min} \equiv \frac{1}{2\pi} \left\{ \nu' \left[1 + \sqrt{y_1(2\omega+1)} \right] - 2m - \nu \right\} \geq 0$$

and

$$\omega \equiv \frac{m^2}{\nu\nu'(1-\mu')}$$

Again it is convenient to transform the entire integral. Define

$$\zeta \equiv x - x_{\min}$$

Then

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{(r_0)^2 m^2 e^{-x_{\min}}}{v\sqrt{y_1} N_e(T)} \int_0^\infty d\zeta e^{-\zeta} \int_{-1}^{+1} d\xi \frac{\Delta}{\sqrt{1-\xi^2}}$$

B. Limit of Forward Scattering

The expression for $\left\langle \frac{d^3\sigma}{d^3(\text{III})} \right\rangle$ tends to infinite values only as $\mu' \rightarrow +1$ (given $v, T > 0$). In particular the divergence as $|\xi| \rightarrow 1$ is integrated over, and the integral converges. How handle the divergence at $\mu' = 1$?

The limit of forward scattering is the limit

$$\begin{aligned} \lim_{\mu' \rightarrow +1} \mu'^* &= \lim_{\mu' \rightarrow +1} \left[1 - \frac{(1-\mu')}{\gamma^2(1-\beta\alpha)(1-\beta\eta)} \right] \\ &= +1 \end{aligned}$$

which also is forward scattering in the rest frame. The 4-momentum-conserving delta function requires $v'^* \rightarrow v^*$, and transforming back this means

$$\lim_{\mu' \rightarrow +1} v' = v$$

$$\lim_{\mu' \rightarrow +1} y_1 = 0$$

Such a result simply means that

$$\lim_{\mu' \rightarrow +1} \left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle \sim \delta(v-v')$$

What is needed is the factor of proportionality or, more meaningfully,

$$\left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}} \equiv \lim_{\mu' \rightarrow +1} \int_0^\infty dv' \left\langle \frac{d^3\sigma}{d^3(\text{III})} \right\rangle$$

To evaluate $\left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}}$ it is convenient to go back to the form of definition that contains the 4-momentum-conserving delta function.

$$\left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}} = \frac{2}{N_e(T)} \lim_{u' \rightarrow +1} \int_0^\infty dv' \int_0^\infty dx e^{-x\beta E^2} \int_{-1}^{+1} d\alpha (1-\beta\alpha) \int_0^\pi d\phi \frac{g^*\delta[h]}{\gamma(1-\beta\eta)}$$

The integrand of each integral vanishes outside of a particular sub-range. Each sub-range is less than or equal to the range of integration. Each integrand is a "testing function"⁷ except for $\delta[h]$, which is not manipulated for present purposes as a testing function. Therefore the order of integration may be changed, and the limit may be taken before integrating.

$$\left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}} = \frac{2}{N_e(T)} \int_0^\infty dx e^{-x\beta E^2} \int_{-1}^{+1} d\alpha (1-\beta\alpha) \int_0^\pi d\phi \int_0^\infty dv' \lim_{u' \rightarrow +1} \frac{g^*\delta[h]}{\gamma(1-\beta\eta)}$$

Then convert the delta function to

$$\delta[h] = \frac{1}{\gamma(1-\beta\eta)} \delta \left[v' - \frac{v(1-\beta\alpha)}{(1-\beta\alpha) + \frac{v}{E}(1-u')} \right]$$

But

$$\begin{aligned} \lim_{u' \rightarrow +1} \eta &= \alpha \\ \lim_{u' \rightarrow +1} g^* &= \frac{(r_0)^2}{2} \lim_{u' \rightarrow +1} (1-\rho)^2 \Delta \\ &= \frac{(r_0)^2}{2} (1)^2 (2) \\ &= (r_0)^2 \end{aligned}$$

7. Friedman, B.: Principles and Techniques of Applied Mathematics, Wiley & Sons (1956), p. 137.

So

$$\begin{aligned}
 \left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}} &= \frac{2}{N_e(T)} r_0^2 \int_0^\infty dx e^{-x} \beta \frac{E^2}{\gamma^2} \int_{-1}^{+1} d\alpha (1-\beta\alpha) \int_0^\pi d\phi \int_0^\infty dv' \frac{\delta(v'-v)}{(1-\beta\alpha)^2} \\
 &= (r_0)^2 \frac{2\pi m^2}{N_e(T)} \int_0^\infty dx e^{-x} \beta \int_{-1}^{+1} \frac{d\alpha}{(1-\beta\alpha)} \\
 &= (r_0)^2 \frac{2\pi m^2}{N_e(T)} \int_0^\infty dx e^{-x} \ln\left(\frac{1+\beta}{1-\beta}\right)
 \end{aligned}$$

C. Distribution-Averaged Total Cross Section

For the case of total scattering

$$\begin{aligned}
 \langle \sigma \rangle &= \frac{2}{N_e(T)} \int_0^\infty dx \int_{-1}^{+1} d\alpha \int_0^\pi d\phi e^{-x} \beta E^2 (1-\beta\alpha) \sigma \\
 &= \frac{4\pi}{N_e(T)} \pi (r_0)^2 \int_0^\infty dx e^{-x} \beta E^2 \int_{-1}^{+1} d\alpha (1-\beta\alpha) S
 \end{aligned}$$

where

$$\begin{aligned}
 S(v, x, \alpha) &\equiv \frac{\left(1 + \frac{v^*}{m}\right)}{\left(\frac{v^*}{m}\right)^3} \left[\frac{2\frac{v^*}{m} \left(1 + \frac{v^*}{m}\right)}{\left(1 + 2\frac{v^*}{m}\right)} - \ln \left(1 + 2\frac{v^*}{m}\right) \right] \\
 &\quad + \frac{\ln \left(1 + 2\frac{v^*}{m}\right)}{2\frac{v^*}{m}} - \frac{\left(1 + 3\frac{v^*}{m}\right)}{\left(1 + 2\frac{v^*}{m}\right)^2}
 \end{aligned}$$

and

$$v^* = v\gamma(1-\beta\alpha)$$

This form for S is valid for $\frac{v^*}{m}$ in the full range $(0, \infty)$. However for very small $\frac{v^*}{m}$ the form approaches apparent indeterminacy.

$$S \xrightarrow{\frac{v^*}{m} \rightarrow 0} \frac{0}{0} + \frac{0}{0} - \frac{0}{0}$$

For this reason it is more accurate for small $\frac{v^*}{m}$ to use

$$S = \frac{4}{3} \frac{1}{\left(1 + 2\frac{v^*}{m}\right)^2} \left[1 + 2\frac{v^*}{m} + \frac{6}{5} \left(\frac{v^*}{m}\right)^2 - \frac{1}{2} \left(\frac{v^*}{m}\right)^3 + \frac{2}{7} \left(\frac{v^*}{m}\right)^4 - \frac{6}{35} \left(\frac{v^*}{m}\right)^5 + \frac{8}{105} \left(\frac{v^*}{m}\right)^6 + \frac{4}{105} \left(\frac{v^*}{m}\right)^7 \right],$$

which is accurate to 9th order in $\left\{\frac{v^*}{m}\right\}$ because the first neglected term (8th order) has zero coefficient. To first order only in $\frac{v^*}{m}$,

$$S \longrightarrow \frac{4}{3} \left(1 - 2\frac{v^*}{m}\right)$$

D. Quadrature Forms

Three integral forms have been encountered:

$$(a) \quad \int_0^{\infty} dx e^{-x} F_x(x)$$

$$(b) \quad \int_{-1}^{+1} d\xi \frac{F_{\xi}(\xi)}{\sqrt{1-\xi^2}}$$

$$(c) \quad \int_{-1}^{+1} d\alpha F_{\alpha}(\alpha)$$

In each case the form F does not integrate analytically to a convenient form. However in each case F is well-behaved, being continuous and having continuous

derivatives in the interval. For such F's Gauss-type quadrature is a powerful integrating technique capable of high accuracy.⁸

Briefly the method is this. An integration is written as

$$\int dy w(y) F(y) = \sum_{j=1}^J W_j F(y_j)$$

Here $w(y)$ is a characteristic weight function. F is the complicated function that is approximated as being well-behaved. Corresponding to $w(y)$ is a set of polynomials $P(y)$. The y_j , J in number, are the zeros of $P_J(y)$. The W_j are quadrature weights; they are given in terms of the J values of $P_{(J+1)}(y_j)$. The error is proportional to $\frac{d^{(2J)}}{dy^{(2J)}} F(y)$

The appropriate polynomials are

$$(a) \quad w(y) = e^{-y} \text{ in } (0, \infty) \longrightarrow \text{Laguerre polynomials}$$

$$(b) \quad w(y) = \frac{1}{\sqrt{1-y^2}} \text{ in } (-1, +1) \longrightarrow \text{Chebyshev polynomials}$$

$$(c) \quad w(y) = 1 \text{ in } (-1, +1) \longrightarrow \text{Legendre polynomials}$$

For the present application an error of the order of

$$\text{error} \lesssim \frac{1}{1000}$$

was obtained by using

$$J_{(a)} = 15; \quad J_{(b)} = 5, \quad J_{(c)} = 10$$

8. Hildebrand, F. B.: Introduction to Numerical Analysis, McGraw-Hill (1956)

V. RESULTS

A. Formulae for the Distribution Averages

The techniques that have been developed reduce the Compton problem to pure numerical computation. Given

ν = energy of incident photon

T = temperature of the target RMB electron distribution

ν' = energy of scattered photon

μ' = polar angle cosine of scattered photon

m = electron mass in units of ν , T , ν'

r_0 = "classical electron radius"

then a set of distribution-averaged functions can be found by the following prescription.

Choose the number of points to be used for each quadrature depending upon the accuracy desired. Take the appropriate quadrature quantities⁹ to be represented by the following table of symbols:

kind of quadrature	Laguerre	Chebyshev	Legendre
indexing symbol	j	k	ℓ
no. points in quadrature	J	K	L
quadrature abscissa (indexed)	ζ_j	ξ_k	α_ℓ
quadrature weight (indexed)	W_j	π/K	H_ℓ

Define the following functions of the given (ν, T, ν', μ') :

$$\omega = \frac{m^2}{\nu\nu'(1-\mu')}$$

9. For tabulated values see reference 8, or for more detail see for example Abramowitz & Stegun: Handbook of Mathematical Functions, U.S. Gov. Printing Office (1965).

$$y_1 = \left(\frac{v}{v'}\right)^2 - 2\left(\frac{v}{v'}\right)u' + 1$$

$$x_{\min} = \frac{1}{2T} \left\{ v' \left[1 + \sqrt{y_1(2\omega+1)} \right] - 2\omega - v \right\}$$

For each of the indexed quadrature abscissas define

$$E_j = m + T(\zeta_j + x_{\min})$$

$$\beta_j = \sqrt{1 - \left(\frac{m}{E_j}\right)^2}$$

$$(y_2)_j = \frac{v}{v'} + 1 + \frac{v}{E_j} \left(\frac{v}{v'} - u' \right)$$

$$(y_3)_j = \frac{v}{v'} - 1 - \frac{v}{E_j} (1-u')$$

$$(y_4)_j = 1 - \frac{(y_2)_j}{y_1} (1-u')$$

$$(y_5)_j = \sqrt{(y_4)_j^2 + \frac{1}{y_1} \left[\beta_j^2 (1-u')^2 - (y_3)_j^2 \right]}$$

$$\rho_{jk} = \frac{v'}{E_j} \frac{(1-u')}{\left[1 - (y_4)_j - (y_5)_j \xi_j \right]}$$

$$x_{j\ell} = \frac{vE_j}{m^2} (1 - \beta_j \alpha_\ell)$$

Using the two functions

$$\Delta(\rho) \equiv 2 + \frac{\rho^2}{(1-\rho)} \left[\omega^2 \frac{\rho^2}{(1-\rho)} - 2\omega + 1 \right]$$

$$S(x) \equiv \frac{1+x}{x^3} \left[\frac{2x(1+x)}{(1+2x)} - \ln(1+2x) \right] \\ + \frac{\ln(1+2x)}{2x} - \frac{(1+3x)}{(1+2x)^2},$$

form

$$\Delta_{jk} \equiv \Delta(\rho_{jk}) ; \quad S_{j\ell} \equiv S(x_{j\ell})$$

Define four sums:

$$\sum_0 \equiv \sum_{j=1}^J w_j \beta_j E_j^2$$

$$\sum_1 \equiv \sum_{j=1}^J w_j \sum_{k=1}^K \frac{\pi}{K} \Delta_{jk}$$

$$\sum_2 \equiv \sum_{j=1}^J w_j \ln \left(\frac{1+\beta_j}{1-\beta_j} \right)$$

$$\sum_3 \equiv \sum_{j=1}^J w_j \beta_j E_j^2 \sum_{\ell=1}^L H_{\ell} (1-\beta_j \alpha_{\ell}) S_{j\ell}$$

Then

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle = \frac{(r_0)^2}{4\pi} \frac{m^2 e^{-x_{\min}}}{\sqrt{y_1}} \frac{\sum_1}{\sum_0}$$

$$\left\langle \frac{d^2\sigma}{d^2\Omega'} \right\rangle_{\text{forward}} = \frac{(r_0)^2}{2} m^2 \frac{\sum_2}{\sum_0}$$

$$\langle \sigma \rangle = \pi (r_0)^2 \frac{\sum_3}{\sum_0}$$

B. Scattered Densities and Energy Transfer

As pointed out in part III.C, the cross section functions are derived for unit incident photon density. Thus with lab frame electron density ρ_e ,

$$\rho_e \left\langle \frac{d^3\sigma}{dv' d^2\Omega'} \right\rangle = \text{probability per incident photon for scattering into particular final state (III), per unit time per unit scattered state.}$$

$$\rho_e \langle \sigma \rangle = \text{probability per unit time per incident photon for scattering.}$$

The way in which photon energy is redistributed by scattering is contained in an energy transfer function.

$$\left\langle \frac{d^3\epsilon_t}{dv' d^2\Omega'} \right\rangle \equiv (v-v') \left\langle \frac{d^3\sigma}{dv' d^2\Omega'} \right\rangle, \text{ units of } \frac{\text{energy-area}}{\text{unit state (III)}}$$

Then

$$\rho_e \left\langle \frac{d^3\epsilon_t}{dv' d^2\Omega'} \right\rangle = \text{probable energy transfer per photon if scattering into state (III), per unit time per unit scattered state.}$$

Just as

$$\langle \sigma \rangle = \int d^3(\text{III}) \left\langle \frac{d^3\sigma}{d^3(\text{III})} \right\rangle,$$

so also

$$\langle \epsilon_t \rangle = \int d^3(\text{III}) \left\langle \frac{d^3\epsilon_t}{d^3(\text{III})} \right\rangle.$$

Note that

$\nu c p_e \langle \sigma \rangle$ = total probable energy scattered from the
initial photon, per photon per unit time;

$c p_e \langle et \rangle$ = total probable energy transferred to the
RMB electrons, per incident photon per unit
time;

$c p_e [\nu \langle \sigma \rangle - \langle et \rangle]$ = total probable energy carried away by scattered
photon, per initial photon per unit time.

Other useful functions can be formed. For example,

$$\left\langle \frac{d^2 \sigma}{d^2 \Omega'} \right\rangle \equiv \int_0^\infty dv' \left\langle \frac{d^3 \sigma}{dv' d^2 \Omega'} \right\rangle$$

= differential cross section function for
scattering with any energy to within $d^2 \Omega'$ of
scattered direction $\hat{\Omega}'$, in units of millibarns
per steradian.

where

$$\lim_{\mu' \rightarrow 1} \left\langle \frac{d^2 \sigma}{d^2 \Omega'} \right\rangle = \left\langle \frac{d^2 \sigma}{d^2 \Omega'} \right\rangle_{\text{forward}}$$

specially treated because $\left\langle \frac{d^3 \sigma}{dv' d^2 \Omega'} \right\rangle$ becomes a delta function in v' as $\mu' \rightarrow 1$.
Similarly

$$\left\langle \frac{d et}{dv'} \right\rangle \equiv \int_{4\pi} d^2 \Omega' \left\langle \frac{d^3 et}{dv' d^2 \Omega'} \right\rangle = 2\pi(\nu - \nu') \int_{-1}^{+1} du' \left\langle \frac{d^3 \sigma}{dv' d^2 \Omega'} \right\rangle$$

= differential energy transfer function for a photon
scattered to within dv' of $(\nu - \nu')$ regardless
of angle, in units of millibarns.

Finally it is useful to define a cumulative cross section and a cumulative energy transfer as

$$QX(\bar{\nu}, \bar{\mu}) \equiv 2\pi \int_0^{\bar{\nu}} d\nu' \int_{-1}^{\bar{\mu}} d\mu' \left\langle \frac{d^3\sigma}{d\nu' d^2\Omega'} \right\rangle$$

$$QE(\bar{\nu}, \bar{\mu}) \equiv 2\pi \int_0^{\bar{\nu}} d\nu' \int_{-1}^{\bar{\mu}} d\mu' \left\langle \frac{d^3 et}{d\nu' d^2\Omega'} \right\rangle$$

with

$$0 \leq \bar{\nu} < \infty \quad \text{and} \quad -1 \leq \bar{\mu} \leq +1$$

The cumulative functions depend on scattered photon energy and direction, now taken to be $\bar{\nu}$ and $\bar{\mu}$ to distinguish from dependence by $\left\langle \frac{d^3\sigma}{d\nu' d^2\Omega'} \right\rangle$ and $\left\langle \frac{d^3 et}{d\nu' d^2\Omega'} \right\rangle$.

Then

$$c p_e \left[QX(\bar{\nu}_2, \bar{\mu}_2) - QX(\bar{\nu}_2, \bar{\mu}_1) - QX(\bar{\nu}_1, \bar{\mu}_2) + QX(\bar{\nu}_1, \bar{\mu}_1) \right]$$

= probability per incident photon for scattering with ν' in $(\bar{\nu}_1, \bar{\nu}_2)$ and μ' in $(\bar{\mu}_1, \bar{\mu}_2)$, per unit time,

and

$$c p_e \left[QE(\bar{\nu}_2, \bar{\mu}_2) - QE(\bar{\nu}_2, \bar{\mu}_1) - QE(\bar{\nu}_1, \bar{\mu}_2) + QE(\bar{\nu}_1, \bar{\mu}_1) \right]$$

= probable energy transferred to RMB electrons by incident photon scattering with ν' in $(\bar{\nu}_1, \bar{\nu}_2)$ and μ' in $(\bar{\mu}_1, \bar{\mu}_2)$, per unit time.

C. Output of Computer Code

A computer code was written in Fortran II and used on the IBM 7094.

For each value of ν and T it calculated $\langle \sigma \rangle$,

$$\left\langle \frac{d^2 \sigma}{d^2 \Omega'} \right\rangle_{\text{forward}}$$

$$\left\langle \frac{d^3 \sigma}{d\nu' d^2 \Omega'} \right\rangle(\nu', \mu'),$$

$$\left\langle \frac{d^3 \text{et}}{d\nu' d^2 \Omega'} \right\rangle(\nu', \mu'),$$

$$QX(\bar{\nu}, \bar{\mu}),$$

$$QE(\bar{\nu}, \bar{\mu}),$$

$$\left[\int d^2 \Omega' \left\langle \frac{d^3 \sigma}{d\nu' d^2 \Omega'} \right\rangle \right](\nu'),$$

$$\left[\int d^2 \Omega' \left\langle \frac{d^3 \text{et}}{d\nu' d^2 \Omega'} \right\rangle \right](\nu'),$$

$$\left[\int d\nu' \left\langle \frac{d^3 \sigma}{d\nu' d^2 \Omega'} \right\rangle \right](\mu'),$$

and

$$\left[\int d\nu' \left\langle \frac{d^3 \text{et}}{d\nu' d^2 \Omega'} \right\rangle \right](\mu')$$

using 100 values of ν' and $\bar{\nu}$ and 41 values of μ' and $\bar{\mu}$. In the form used the code computed these 18,324 data, printed them and plotted 16 curves in about 75 seconds. Accuracy was approximately 1 or less in 1000.

The tables and graphs which follow are representative examples of the code's output. Note that Tables A and B are of $\left\langle \frac{d^3 \sigma}{d\nu' d^2 \Omega'} \right\rangle$ and QE respectively, both for the one case of $T = 50$ kev, $\nu = 10$ kev. Figures 1 through 6 are typical plots at three sets of (T, ν) :

- (a) $T = 1$ kev, $\nu = 100$ kev (fairly nearly Klein-Nishina from rest electrons)
- (b) $T = 50$ kev, $\nu = 10$ kev
- (c) $T = 100$ kev, $\nu = 100$ kev.

Finally Figures 7 through 10 depict functions of T and ν in the ranges

$$1 \text{ kev} \leq T \leq 120 \text{ kev}$$

$$1 \text{ kev} \leq \nu \leq 1000 \text{ kev}.$$

Table A:

DIFFERENTIAL CROSS-SECTION,

$$\left\langle \frac{d^3\sigma}{dv'd^2\Omega'} \right\rangle$$

(T=50, v=10)

ENERGY	-1.0	-0.9	-0.8	COSINE -0.7	-0.6	-0.5
0.8197	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
2.4214	0.0485	0.0851	0.0730	0.0618	0.0514	0.0420
3.7750	0.5845	0.5215	0.4794	0.4382	0.3977	0.3582
4.9189	1.1463	1.1357	1.0777	1.0199	0.9633	0.9077
5.8856	1.7598	1.7044	1.6425	1.5829	1.5265	1.4733
6.7025	2.2163	2.1518	2.0916	2.0363	1.9866	1.9434
7.3929	2.5431	2.4802	2.4232	2.3729	2.3307	2.2975
7.9764	2.7761	2.7146	2.6601	2.6139	2.5777	2.5531
8.4695	2.9405	2.8798	2.8270	2.7836	2.7514	2.7327
8.8862	3.0565	2.9951	2.9442	2.9024	2.8728	2.8577
9.2383	3.1385	3.0780	3.0264	2.9855	2.9573	2.9444
9.5360	3.1967	3.1360	3.0844	3.0437	3.0162	3.0045
9.7875	3.2384	3.1773	3.1254	3.0847	3.0573	3.0461
10.0000	3.2684	3.2068	3.1546	3.1136	3.0862	3.0750
10.2125	3.2939	3.2317	3.1790	3.1376	3.1097	3.0983
10.4500	3.3171	3.2542	3.2007	3.1586	3.1301	3.1181
10.7153	3.3368	3.2729	3.2185	3.1754	3.1459	3.1327
11.0117	3.3516	3.2864	3.2306	3.1861	3.1551	3.1403
11.3428	3.3595	3.2927	3.2353	3.1889	3.1567	3.1385
11.7127	3.3585	3.2898	3.2301	3.1813	3.1454	3.1249
12.1260	3.3462	3.2751	3.2128	3.1609	3.1214	3.0968
12.5877	3.3200	3.2461	3.1806	3.1250	3.0811	3.0512
13.1036	3.2772	3.2000	3.1307	3.0707	3.0216	2.9854
13.6799	3.2152	3.1343	3.0607	2.9956	2.9404	2.8967
14.3237	3.1313	3.0464	2.9690	2.8972	2.8352	2.7834
15.0430	3.0236	2.9344	2.8510	2.7741	2.7048	2.6441
15.8466	2.8906	2.7971	2.7086	2.6256	2.5488	2.4790
16.7444	2.7320	2.6345	2.5411	2.4522	2.3683	2.2897
17.7475	2.5486	2.4477	2.3501	2.2560	2.1657	2.0794
18.8681	2.3427	2.2394	2.1386	2.0406	1.9454	1.8529
20.1200	2.1182	2.0138	1.9115	1.8112	1.7131	1.6168
21.5187	1.8803	1.7767	1.6747	1.5744	1.4757	1.3785
23.0813	1.6357	1.5347	1.4353	1.3374	1.2410	1.1460
24.8270	1.3915	1.2935	1.2009	1.1030	1.0166	0.9269
26.7774	1.1554	1.0663	0.9789	0.8933	0.8095	0.7276
28.9563	0.9344	0.8540	0.7756	0.6992	0.6251	0.5532
31.3907	0.7344	0.6642	0.5961	0.5304	0.4671	0.4065
34.1103	0.5593	0.5004	0.4434	0.3889	0.3370	0.2880
37.1487	0.4129	0.3645	0.3185	0.2750	0.2345	0.1963
40.5433	0.2941	0.2561	0.2204	0.1872	0.1565	0.1284
44.3357	0.2017	0.1731	0.1466	0.1223	0.1002	0.0805
48.5725	0.1329	0.1123	0.0939	0.0765	0.0614	0.0481
53.3060	0.0840	0.0698	0.0570	0.0457	0.0359	0.0274
58.5943	0.0507	0.0414	0.0332	0.0260	0.0200	0.0149
64.5024	0.0292	0.0234	0.0184	0.0141	0.0105	0.0076
71.1029	0.0166	0.0125	0.0096	0.0072	0.0053	0.0037
78.4771	0.0083	0.0064	0.0048	0.0035	0.0025	0.0017
86.7156	0.0041	0.0031	0.0022	0.0016	0.0011	0.0007
95.8196	0.0019	0.0014	0.0010	0.0007	0.0005	0.0003
106.2025	0.0008	0.0006	0.0004	0.0003	0.0002	0.0001

Table A (continued)

ENERGY	COSINE					
	-0.4	-0.3	-0.2	-0.1	0.0	0.1
0.8197	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.4214	0.0335	0.0261	0.0196	0.0141	0.0097	0.0062
3.7750	0.3195	0.2317	0.2449	0.2089	0.1740	0.1403
4.9189	0.8530	0.7468	0.7449	0.6901	0.6339	0.5748
5.8856	1.4244	1.3770	1.3337	1.2929	1.2534	1.2131
6.7025	1.9073	1.8794	1.8602	1.8506	1.8507	1.8602
7.3929	2.2760	2.2669	2.2726	2.2954	2.3380	2.4031
7.9764	2.5425	2.5484	2.5742	2.6236	2.7017	2.8141
8.4695	2.7300	2.7467	2.7969	2.8557	2.9597	3.1074
9.8862	2.8602	2.8941	2.9339	3.0157	3.1374	3.3094
9.2383	2.9501	2.9784	3.0343	3.1245	3.2575	3.4450
9.5360	3.0119	3.0428	3.1023	3.1974	3.3373	3.5343
9.7875	3.0544	3.0866	3.1481	3.2460	3.3897	3.5919
10.0000	3.0836	3.1163	3.1797	3.2779	3.4234	3.6283
10.2125	3.1068	3.1395	3.2021	3.3017	3.4479	3.6537
10.4500	3.1260	3.1582	3.2202	3.3192	3.4648	3.6696
10.7153	3.1395	3.1703	3.2308	3.3280	3.4712	3.6729
11.0117	3.1451	3.1737	3.2314	3.3252	3.4639	3.6595
11.3428	3.1403	3.1657	3.2192	3.3076	3.4393	3.6254
11.7127	3.1230	3.1435	3.1911	3.2719	3.3936	3.5661
12.1260	3.0898	3.1040	3.1437	3.2143	3.3227	3.4774
12.5877	3.0378	3.0441	3.0738	3.1316	3.2232	3.3554
13.1036	2.9642	2.9609	2.9789	3.0208	3.0922	3.1973
13.6799	2.8666	2.8520	2.8566	2.8799	2.9280	3.0027
14.3237	2.7431	2.7160	2.7039	2.7082	2.7307	2.7723
15.0430	2.5929	2.5524	2.5234	2.5068	2.5027	2.5103
15.8466	2.4167	2.3625	2.3166	2.2790	2.2487	2.2235
16.7444	2.2167	2.1494	2.0876	2.0303	1.9760	1.9215
17.7475	1.9969	1.9181	1.8423	1.7682	1.6933	1.6158
18.8681	1.7630	1.6752	1.5886	1.5018	1.4129	1.3186
20.1200	1.5222	1.4287	1.3354	1.2410	1.1439	1.0416
21.5187	1.2825	1.1871	1.0917	0.9954	0.8969	0.7945
23.0813	1.0520	0.9589	0.8661	0.7731	0.6792	0.5837
24.8270	0.8384	0.7512	0.6652	0.5800	0.4957	0.4122
26.7774	0.6476	0.5695	0.4935	0.4195	0.3479	0.2791
28.9563	0.4838	0.4169	0.3528	0.2918	0.2343	0.1809
31.3907	0.3487	0.2940	0.2426	0.1948	0.1511	0.1120
34.1103	0.2421	0.1993	0.1601	0.1246	0.0931	0.0661
37.1487	0.1614	0.1296	0.1011	0.0761	0.0547	0.0371
40.5433	0.1031	0.0806	0.0610	0.0443	0.0306	0.0198
44.3357	0.0630	0.0479	0.0351	0.0246	0.0162	0.0100
48.5725	0.0367	0.0271	0.0192	0.0129	0.0082	0.0048
53.3060	0.0203	0.0145	0.0099	0.0064	0.0039	0.0021
58.5443	0.0107	0.0074	0.0049	0.0030	0.0017	0.0009
64.5024	0.0053	0.0036	0.0023	0.0013	0.0007	0.0004
71.1029	0.0023	0.0016	0.0010	0.0006	0.0003	0.0001
78.4771	0.0011	0.0007	0.0004	0.0002	0.0001	0.0000
86.7156	0.0005	0.0003	0.0002	0.0001	0.0000	0.0000
95.9196	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
106.2025	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

Table A (continued)

ENERGY	0.8	0.9	1.0	COSINE
0.8197	0.0000	0.0000	0.	
2.4214	0.0000	0.0000	0.	
3.7750	0.0003	0.0000	0.	
4.9189	0.0219	0.0005	0.	
5.8856	0.2508	0.0250	0.	
6.7025	1.1056	0.2920	0.	
7.3929	2.7673	1.4265	0.	
7.9764	4.8862	3.9386	0.	
8.4695	6.9337	7.4568	0.	
8.8862	8.5728	11.0163	0.	
9.2383	9.7170	13.8542	0.	
9.5360	10.4276	15.7185	0.	
9.7875	10.8139	16.7163	0.	
10.0000	10.9803	17.0841	0.	
10.2125	11.0023	17.0122	0.	
10.4500	10.8654	16.4535	0.	
10.7153	10.5359	15.3365	0.	
11.0117	9.9826	13.6591	0.	
11.3428	9.1948	11.5158	0.	
11.7127	8.1861	9.1019	0.	
12.1260	7.0014	6.6791	0.	
12.5877	5.7157	4.5085	0.	
13.1036	4.4249	2.7759	0.	
13.6799	3.2279	1.5477	0.	
14.3237	2.2055	0.7768	0.	
15.0430	1.4037	0.3494	0.	
15.8466	0.8282	0.1404	0.	
16.7444	0.4513	0.0503	0.	
17.7475	0.2263	0.0160	0.	
18.8681	0.1042	0.0046	0.	
20.1200	0.0440	0.0012	0.	
21.5137	0.0170	0.0003	0.	
23.0813	0.0060	0.0001	0.	
24.8270	0.0019	0.0000	0.	
26.7774	0.0006	0.0000	0.	
28.9563	0.0002	0.0000	0.	
31.3907	0.0000	0.0000	0.	
34.1103	0.0000	0.0000	0.	
37.1487	0.0000	0.0000	0.	
40.5433	0.0000	0.0000	0.	
44.3357	0.0000	0.0000	0.	
48.5725	0.0000	0.0000	0.	
53.3060	0.0000	0.0000	0.	
58.5943	0.0000	0.0000	0.	
64.5024	0.0000	0.0000	0.	
71.1029	0.0000	0.0000	0.	
78.4771	0.0000	0.0000	0.	
86.7156	0.0000	0.0000	0.	
95.9196	0.0000	0.0000	0.	
106.2025	0.0000	0.0000	0.	

Table B: CUMULATIVE ENERGY TRANSFER, QE(T=50, v=10)

ENERGY	-1.0	-0.9	-0.8	COSINE -0.7	-0.6	-0.5
0.8197	0.	0.	0.	0.	0.	0.
2.4214	0.	0.17	0.32	0.44	0.54	0.62
3.7750	0.	1.79	3.40	4.83	6.09	7.19
4.9189	0.	5.18	9.95	14.35	18.33	22.05
5.8856	0.	9.21	17.83	25.93	33.48	40.51
6.7025	0.	12.93	25.17	38.74	47.68	57.99
7.3929	0.	15.93	31.10	45.53	59.25	72.30
7.9764	0.	19.16	35.91	52.07	67.89	83.01
8.4695	0.	19.72	38.59	56.65	73.94	90.52
8.8862	0.	20.75	40.62	59.67	77.95	95.49
9.2393	0.	21.38	41.89	61.55	80.43	98.58
9.5360	0.	21.74	42.60	62.61	81.83	100.32
9.7875	0.	21.91	42.93	63.11	82.50	101.15
10.0000	0.	21.96	43.02	63.24	82.67	101.37
10.2125	0.	21.91	42.93	63.11	82.49	101.15
10.4300	0.	21.75	42.61	62.63	81.86	100.36
10.7153	0.	21.43	41.98	61.69	80.61	99.91
11.0117	0.	20.90	40.92	60.12	78.53	98.22
11.3428	0.	20.08	39.31	57.72	75.36	92.28
11.7127	0.	18.90	36.97	54.24	70.75	86.56
12.1260	0.	17.25	33.69	49.37	64.31	79.56
12.5877	0.	14.99	29.23	42.74	55.53	67.69
13.1036	0.	11.99	23.29	33.91	43.89	53.23
13.6799	0.	8.06	15.50	22.37	28.65	34.35
14.3237	0.	2.98	5.49	7.53	9.09	10.17
15.0430	0.	-3.44	-7.17	-11.21	-15.58	-20.30
15.8466	0.	-11.44	-22.93	-34.49	-45.13	-58.03
16.7444	0.	-21.25	-42.21	-62.94	-93.49	-103.94
17.7475	0.	-33.07	-65.41	-97.08	-128.17	-158.77
18.8681	0.	-47.06	-92.80	-137.30	-180.67	-223.00
20.1200	0.	-63.30	-124.52	-183.75	-241.10	-296.72
21.5187	0.	-81.79	-160.49	-236.25	-309.19	-379.46
23.0813	0.	-102.33	-200.37	-294.25	-384.12	-470.14
24.8270	0.	-124.67	-243.54	-356.77	-464.56	-567.06
26.7774	0.	-149.32	-289.06	-422.43	-548.63	-667.87
28.9563	0.	-172.70	-335.77	-489.46	-634.03	-769.73
31.3907	0.	-197.10	-382.28	-555.87	-718.17	-869.54
34.1103	0.	-220.77	-427.17	-619.59	-798.43	-964.17
37.1487	0.	-243.00	-469.05	-678.67	-872.41	-1050.88
40.5433	0.	-263.13	-506.75	-731.51	-938.14	-1127.38
44.3357	0.	-280.70	-539.43	-777.01	-994.34	-1192.35
48.5725	0.	-295.44	-566.63	-814.61	-1040.47	-1245.33
53.3060	0.	-307.30	-588.35	-844.40	-1076.73	-1286.67
58.5943	0.	-316.41	-604.91	-866.94	-1103.96	-1317.51
64.5024	0.	-323.10	-616.97	-883.20	-1123.46	-1339.43
71.1029	0.	-327.77	-625.30	-894.36	-1136.74	-1354.24
78.4771	0.	-330.86	-630.78	-901.62	-1145.30	-1363.72
86.7156	0.	-332.80	-634.17	-906.08	-1150.52	-1369.46
95.9195	0.	-333.95	-635.16	-908.67	-1153.52	-1372.74
106.2025	0.	-334.58	-637.25	-910.07	-1155.14	-1374.49

Table B (continued)

ENERGY	COSINE					
	-0.4	-0.3	-0.2	-0.1	0.0	0.1
0.8197	0.	0.	0.	0.	0.	0.
2.4214	0.68	0.73	0.77	0.79	0.81	0.82
3.7750	9.14	8.96	9.62	10.18	10.62	10.96
4.9189	25.36	28.34	30.99	33.33	35.36	37.10
5.8856	47.04	53.09	58.66	63.78	68.45	72.62
6.7025	67.72	76.88	85.51	93.62	101.24	108.58
7.3929	84.72	96.54	107.92	118.57	128.85	138.69
7.9764	97.48	111.35	124.66	137.49	149.88	161.89
8.4645	106.44	121.76	136.59	150.86	164.78	178.38
8.8662	112.32	128.67	144.44	159.75	174.71	189.39
9.2893	116.07	132.96	149.34	165.29	180.89	196.25
9.5360	118.15	135.39	152.12	168.42	184.39	200.14
9.7375	119.14	136.54	153.43	169.90	186.05	201.99
10.0000	119.41	136.85	153.79	170.30	186.50	202.48
10.2125	119.14	136.54	153.43	169.90	186.05	201.98
10.4500	118.20	135.44	152.19	168.49	184.47	200.22
10.7153	116.34	133.28	149.70	165.69	181.35	196.76
11.0117	113.25	129.68	145.59	161.06	176.16	191.00
11.3428	108.54	124.20	139.32	155.99	168.26	182.23
11.7127	101.71	116.25	130.25	148.76	156.85	169.57
12.1260	92.16	106.16	117.59	129.50	140.95	151.96
12.5877	79.19	90.09	100.41	110.18	119.42	128.15
13.1036	61.95	70.08	77.62	84.58	90.95	96.72
13.6799	39.49	44.04	48.01	51.37	54.08	56.10
14.3237	10.73	10.77	10.23	9.08	7.25	4.65
15.0430	-25.43	-31.00	-37.09	-43.75	-51.10	-59.26
15.8466	-70.12	-82.50	-95.28	-108.56	-122.45	-137.12
16.7444	-124.35	-144.85	-165.53	-186.53	-207.99	-230.06
17.7475	-188.97	-218.91	-249.70	-279.50	-308.46	-338.74
18.8681	-264.44	-305.10	-345.13	-384.70	-423.95	-463.03
20.1200	-350.72	-403.28	-454.53	-504.63	-553.73	-601.94
21.5197	-447.20	-512.58	-575.77	-636.90	-696.10	-753.46
23.0813	-552.49	-631.33	-708.83	-779.13	-848.36	-914.55
24.8270	-664.48	-757.01	-844.82	-928.06	-1006.86	-1081.26
26.7774	-780.58	-886.38	-986.07	-1079.65	-1167.25	-1248.96
28.9563	-896.95	-1015.66	-1126.43	-1229.39	-1324.78	-1412.74
31.3907	-1010.32	-1140.87	-1261.56	-1372.73	-1474.71	-1567.77
34.1103	-1117.26	-1258.18	-1387.42	-1505.45	-1612.77	-1709.82
37.1487	-1214.61	-1364.31	-1500.59	-1624.13	-1735.58	-1835.59
40.5433	-1300.00	-1456.82	-1598.68	-1726.42	-1840.92	-1943.02
44.3357	-1372.04	-1534.39	-1680.44	-1811.26	-1927.89	-2031.40
48.5725	-1430.37	-1596.81	-1745.88	-1878.82	-1996.88	-2101.27
53.3060	-1475.61	-1644.93	-1796.03	-1930.38	-2049.34	-2154.27
58.5943	-1509.12	-1680.37	-1832.81	-1968.02	-2087.51	-2192.74
64.5024	-1532.79	-1705.25	-1858.51	-1994.22	-2114.01	-2219.40
71.1029	-1548.68	-1721.86	-1875.59	-2011.59	-2131.53	-2237.00
78.4771	-1558.79	-1732.39	-1886.36	-2022.51	-2142.53	-2248.04
86.7156	-1564.87	-1738.69	-1892.79	-2029.01	-2149.06	-2254.59
95.9196	-1568.33	-1742.25	-1896.41	-2032.66	-2152.73	-2258.26
106.2025	-1570.17	-1744.13	-1898.32	-2034.58	-2154.66	-2260.20

Table B (continued)

ENERGY	COSINE						
	0.2	0.3	0.4	0.5	0.6	0.7	
0.8197	0.	0.	0.	0.	0.	0.	0.
2.4214	0.83	0.83	0.83	0.83	0.84	0.84	0.84
3.7750	11.21	11.38	11.49	11.55	11.58	11.59	11.59
4.9189	38.54	39.71	40.60	41.23	41.62	41.82	41.82
5.8856	76.46	79.78	82.62	84.93	86.68	87.81	87.81
6.7025	115.04	121.20	126.82	131.81	136.05	139.32	139.32
7.3929	148.10	157.09	165.62	173.62	180.91	187.19	187.19
7.9764	173.56	184.92	195.99	206.70	216.93	226.33	226.33
8.4695	191.73	204.88	217.98	230.74	243.35	255.48	255.48
8.8862	203.88	218.27	232.63	246.99	261.34	275.49	275.49
9.2383	211.47	226.64	241.86	257.21	272.68	288.18	288.18
9.5360	215.77	231.39	247.11	263.02	279.15	295.45	295.45
9.7875	217.81	233.65	249.61	265.79	282.24	298.92	298.92
10.0000	218.36	234.26	250.28	266.53	283.07	299.85	299.85
10.2125	217.81	233.64	249.60	265.78	282.23	298.91	298.91
10.4500	215.86	231.50	247.23	263.15	279.30	295.61	295.61
10.7153	212.03	227.26	242.65	257.96	273.52	289.12	289.12
11.0117	205.66	220.23	234.78	249.37	263.97	278.42	278.42
11.3428	195.97	209.64	223.00	236.35	249.51	262.27	262.27
11.7127	181.99	194.14	206.04	217.65	228.82	239.25	239.25
12.1260	162.57	172.78	182.67	191.83	200.37	207.76	207.76
12.5877	136.36	144.02	151.04	157.28	162.46	166.09	166.09
13.1036	101.83	106.23	109.76	112.23	113.31	112.49	112.49
13.6799	57.34	57.68	56.94	54.89	51.16	45.32	45.32
14.3237	1.16	-3.39	-9.18	-16.50	-25.63	-36.63	-36.63
15.0430	-68.38	-78.63	-90.23	-103.41	-118.37	-135.08	-135.08
15.8466	-132.72	-169.44	-187.47	-206.95	-227.89	-249.69	-249.69
16.7444	-252.93	-276.75	-301.63	-327.59	-354.39	-381.23	-381.23
17.7475	-369.48	-400.80	-432.72	-465.06	-497.31	-528.29	-528.29
18.8681	-502.04	-541.01	-579.94	-618.18	-655.25	-689.66	-689.66
20.1200	-649.31	-695.78	-741.13	-784.83	-825.91	-862.74	-862.74
21.5187	-808.95	-862.49	-913.70	-961.95	-1006.19	-1044.81	-1044.81
23.0813	-977.67	-1037.54	-1093.77	-1145.69	-1192.28	-1232.08	-1232.08
24.8270	-1151.21	-1216.54	-1276.37	-1331.59	-1379.82	-1420.34	-1420.34
26.7774	-1324.78	-1394.59	-1459.10	-1514.84	-1564.14	-1606.08	-1606.08
28.9563	-1493.37	-1566.68	-1632.62	-1690.64	-1740.60	-1781.77	-1781.77
31.3907	-1652.18	-1728.08	-1795.65	-1854.55	-1904.90	-1946.18	-1946.18
34.1103	-1797.03	-1874.76	-1943.31	-2002.85	-2053.40	-2094.74	-2094.74
37.1487	-1924.79	-2003.74	-2072.96	-2132.60	-2183.46	-2224.82	-2224.82
40.5433	-2033.55	-2113.28	-2182.89	-2242.90	-2293.60	-2334.97	-2334.97
44.3357	-2122.78	-2202.97	-2272.79	-2332.38	-2383.61	-2424.99	-2424.99
48.5725	-2193.16	-2273.62	-2343.65	-2403.68	-2454.42	-2496.79	-2496.79
53.3060	-2246.45	-2327.04	-2397.03	-2457.12	-2507.92	-2549.29	-2549.29
58.5943	-2285.07	-2365.73	-2435.75	-2495.90	-2546.64	-2588.02	-2588.02
64.5024	-2311.81	-2392.51	-2462.53	-2522.69	-2573.43	-2614.81	-2614.81
71.1029	-2329.45	-2410.16	-2480.19	-2540.34	-2591.08	-2632.46	-2632.46
78.4771	-2340.50	-2421.21	-2491.24	-2551.40	-2602.14	-2643.52	-2643.52
86.7156	-2347.06	-2427.77	-2497.80	-2557.96	-2608.70	-2650.08	-2650.08
95.9196	-2350.73	-2431.43	-2501.48	-2561.63	-2612.38	-2653.75	-2653.75
106.2025	-2352.67	-2433.38	-2503.41	-2563.57	-2614.31	-2655.69	-2655.69

Table B (continued)

ENERGY	0.8	0.9	1.0	COSINE
0.8197	0.	0.	0.	
2.4214	0.84	0.84	0.84	
3.7750	11.59	11.59	11.59	
4.9189	41.89	41.89	41.89	
5.8855	88.35	88.49	88.49	
6.7025	141.39	142.15	142.20	
7.3929	191.93	194.40	194.69	
7.9764	234.33	239.59	240.69	
8.4645	266.52	275.02	277.83	
8.8862	288.96	300.39	305.60	
9.2383	303.33	316.97	324.55	
9.5360	311.60	326.63	335.29	
9.7875	315.55	331.28	341.61	
10.0000	316.63	332.53	343.14	
10.2125	315.55	331.26	341.59	
10.4500	311.78	326.84	336.29	
10.7153	304.39	318.19	325.93	
11.0117	292.24	304.08	309.53	
11.3428	274.00	283.16	285.91	
11.7127	248.18	253.95	253.96	
12.1260	213.17	215.02	212.59	
12.5877	167.33	164.96	160.63	
13.1036	109.06	102.52	96.89	
13.6799	36.96	26.59	20.08	
14.3237	-50.10	-63.70	-70.69	
15.0430	-152.87	-158.98	-176.23	
15.8466	-271.64	-289.54	-296.91	
16.7444	-406.17	-425.24	-432.68	
17.7475	-555.65	-575.43	-582.88	
18.8681	-718.61	-738.77	-746.24	
20.1200	-892.89	-913.25	-920.71	
21.5187	-1075.62	-1095.07	-1103.63	
23.0813	-1283.25	-1283.74	-1291.21	
24.8270	-1451.71	-1472.21	-1479.68	
26.7774	-1636.54	-1657.05	-1664.52	
28.9563	-1813.27	-1833.78	-1841.25	
31.3907	-1977.69	-1998.20	-2005.67	
34.1103	-2126.25	-2146.76	-2154.23	
37.1487	-2256.34	-2276.85	-2284.32	
40.5433	-2366.50	-2387.01	-2394.48	
44.3357	-2456.51	-2477.02	-2484.49	
48.5725	-2527.32	-2547.83	-2555.30	
53.3060	-2580.82	-2601.33	-2608.80	
58.5943	-2619.54	-2640.05	-2647.62	
64.5024	-2646.33	-2666.84	-2674.31	
71.1029	-2663.98	-2684.49	-2691.96	
78.4771	-2675.04	-2695.58	-2703.02	
86.7156	-2681.60	-2702.11	-2709.58	
95.9196	-2685.28	-2705.79	-2713.25	
106.2025	-2687.21	-2707.72	-2715.19	

Figure 1(a)

Figure 1(b)

Figure 1(c)

Figure 2(a)

Figure 2(b)

Figure 2(c)

Figure 3(a)

Figure 3(b)

TEMP

50.0

SCATTERING ANGLE COSINE

$h\nu$ 10.0

Figure 3(c)

Figure 4(a)

Figure 4(b)

Figure 4(c)

Figure 5(a)

Figure 5(b)

Figure 5(c)

TEMP

100.0

hν 100.0

Figure 6(a)

Figure 6(b)

Figure 6(c)

Figure 7

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Figure 8

Figure 9

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Figure 10