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LA-5003-MS

INFORMAL REPORT

Self-Focusing in Passive Media I

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UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

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Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.95

LA-5003-MS
Informal Report
UC-34

ISSUED: March 1973

Self-Focusing in Passive Media I

by

B. R. Suydam

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ABSTRACT

We study the propagation of light in a transparent, passive medium whose anomalous index of refraction is cubic in the electric field strength. For finite beams, we find that self-focusing occurs when a critical power (not power density) is surpassed and give general formulas from which critical power and self-focusing distances can be easily computed. Infinite beams, i.e. plane waves are always unstable and will break up into cells, each of which contains roughly twice critical power. Examples are computed from the formulas developed, and we show how high critical power and long self-focusing length may be attained. The attainment of very high laser power will surely be difficult, but self-focusing seems not to render it impossible.

I. INTRODUCTION

In an earlier report⁽¹⁾ we initiated the study of the nonlinear propagation of light pulses. The present report continues this program by studying the self-focusing problem. We here limit our attention to transparent passive media and shall see that, by accepting such a limitation, a very nearly complete picture can be drawn. As we are particularly interested in glass lasers working in the 10-100 picosecond range and gas lasers in the nanosecond range, we can ignore processes with slow response times, such as electrostriction and Kerr effect in solids, and thermal effects in gases. This introduces the further simplification that we may consider the nonlinearity in the index of refraction to develop instantaneously. Thus the quasi-optical equation which describes the nonlinear propagation takes the simple form

$$\frac{\partial E_\alpha}{\partial z} = \frac{v}{2i\omega} [\nabla^2 E_\alpha + \gamma_{\alpha\beta\gamma\delta}^* E_\beta^* E_\gamma E_\delta]; \quad v \equiv c\sqrt{\epsilon_0}. \quad (1.1)$$

Here Greek indices indicate Cartesian vector (or generally tensor) components, and we use, here and throughout the report, the usual convention that repeated indices are summed. Light propagation is

in the z-direction, thus the electric field has only x and y components; thus our indices run over 1, 2 (or x,y) and the Laplacian operator ∇^2 acts in the plane perpendicular to z. The reader is reminded that here, as in Ref. (1), we are working in variables (x,y,x;t*) with t* the proper time,

$$t^* \stackrel{\text{def}}{=} t - z\sqrt{\epsilon_0}/c, \quad (1.2)$$

rather than (x,y,z;t), and that $\partial/\partial z$ is calculated with $t^* = \text{const.}$ not $t = \text{const.}$ The E_α are the Cartesian components of the envelope, represented in the usual complex manner, i.e. the full electric field is given by

$$\vec{E}_{\text{full}} = 1/2[\vec{E} e^{i\omega t^*} + \vec{E}^* e^{-i\omega t^*}]. \quad (1.3)$$

The quantities $\gamma_{\alpha\beta\gamma\delta}$ are constants and it is clear from Eq. (1.1) that they can be taken symmetric in the last two indices without loss of generality. For a passive transparent medium energy is conserved and this leads to

$$\gamma_{\alpha\beta\gamma\delta}^* = \gamma_{\gamma\delta\alpha\beta}. \quad (1.4)$$

Here and elsewhere in this report the star * represents the complex conjugate of a quantity (other

than time). Equation (1.4), together with symmetry in the last two indices, implies symmetry also in the first two. Thus

$$\gamma_{\alpha\beta\gamma\delta} = \gamma_{\beta\alpha\gamma\delta} = \gamma_{\alpha\beta\delta\gamma} = \gamma_{\beta\alpha\delta\gamma} \quad (1.5)$$

We write dots over some of the indices of the γ -quantities to indicate that these indices are to be contracted with components of $\vec{E}^*$.

For a general anisotropic medium, we must use the full array of γ 's. In two important cases, however, the nonlinear term greatly simplifies so that Eq. (1.1) reduces to the simpler form

$$\frac{\partial \vec{E}}{\partial z} = \frac{v}{2i\omega} [\nabla^2 \vec{E} + p(\vec{E} \cdot \vec{E}^*) \vec{E} + q(\vec{E} \cdot \vec{E}) \vec{E}^*], \quad (1.6)$$

with p, q positive constants. As we have seen,⁽¹⁾ for an isotropic medium $p:q$ is in the ratio 2:1, and this same ratio is a quite accurate description also for a cubic crystal. For a medium dominated by an instantaneously responding Kerr effect, such as a gas (see Appendix A), $p:q$ is in the ratio 1:3. In either case, for plane polarized light, Eq. (1.6) further simplifies to

$$\frac{\partial E}{\partial z} = \frac{v}{2i\omega} [\nabla^2 E + \gamma |E|^2 E]; \quad \gamma \stackrel{\text{def}}{=} p + q, \quad (1.7)$$

and E is now a (complex) scalar. It is sometimes convenient to express the electric field in terms of amplitude A , phase φ , and the two polarization parameters η, χ , writing⁽¹⁾

$$\vec{E} = A e^{i\varphi} \left[\vec{e}_x (\cos \eta \cos \chi - i \sin \eta \sin \chi) + \vec{e}_y (\cos \eta \sin \chi + i \sin \eta \cos \chi) \right]. \quad (1.8)$$

Here η measures the degree of ellipticity of the polarization, so that $\eta = \pm\pi/4$ is circular polarization; χ measures the angle the polarization ellipse makes to the x -axis. In terms of these parameters, Eq. (1.6) takes the form⁽¹⁾

$$\frac{2\omega}{v} A \frac{\partial A}{\partial z} = \nabla \cdot [A^2 (\nabla \varphi - \sin 2\eta \nabla \chi)] ,$$

$$- \frac{2\omega}{v} A \frac{\partial \varphi}{\partial z} = \nabla^2 A - A[(\nabla \varphi)^2 + (\nabla \eta)^2 + (\nabla \chi)^2]$$

$$- \frac{1}{A} \tan 2\eta \nabla \cdot [A^2 \nabla \eta] + (p + q) A^3 ,$$

$$\frac{2\omega}{v} A^2 \frac{\partial \eta}{\partial z} = 2A^2 (\nabla \varphi) \cdot (\nabla \eta) - \nabla \cdot [A^2 \cos 2\eta \nabla \chi] ,$$

$$\frac{2\omega}{v} A^2 \cos 2\eta \frac{\partial \chi}{\partial z} = \nabla \cdot [A^2 \nabla \eta] +$$

$$2A^2 \cos 2\eta (\nabla \varphi) \cdot (\nabla \chi) - q A^4 \sin 2\eta \cos 2\eta. \quad (1.9)$$

For plane polarized light $\eta = 0$ or $\pi/2$; η and χ remain constant and these equations reduce to

$$\frac{2\omega}{v} A \frac{\partial A}{\partial z} = \nabla \cdot (A^2 \nabla \varphi) , \quad (1.10)$$

$$- \frac{2\omega}{v} A \frac{\partial \varphi}{\partial z} = \nabla^2 A - A(\nabla \varphi)^2 + \gamma A^3 ,$$

which are equivalent to Eq. (1.7).

II. BEHAVIOR OF PLANE WAVES

In this section we wish to study in some detail the propagation of a plane wave in a nonlinear medium. For this purpose, it is more convenient to use the propagation equations in the form of Eq. (1.9) rather than Eq. (1.6). For a plane wave, everything is independent of x and y and the solution to Eq. (1.9) is simply

$$A = A_0 = \text{const}, \quad \eta = \eta_0 = \text{constant},$$

$$\varphi = \varphi_0 \equiv \text{const} - \frac{vz}{2\omega} (p + q) A_0^2, \quad (2.1)$$

$$\chi = \chi_0 \equiv \text{const} - \frac{vz}{2\omega} q A_0^2 \sin^2 2\eta_0.$$

Thus the amplitude A and the ellipticity of the polarization η remain constant as the wave propagates. Owing to the nonlinear refractive index there is a slow phase shift, as viewed by an observer moving with the pulse. Moreover, the axes of the polarization ellipse rotate as the wave propagates. In the special case of plane polarization $\sin 2\eta_0 = 0$ and χ remains constant as the wave propagates. Note, finally, that $\partial/\partial z$ is computed with the retarded time t^* held constant. This means that our "constant" of integration A_0 is indeed independent of z , but it may depend on t^* . If it does, Eq. (2.1) says that the phase shift varies with t^* . Thus a plane wave which is initially amplitude modulated (A_0 depends on t^*) becomes phase modulated as well as it passes through the medium.

The solution discussed above is an exact one, but whether it is observable in practice is another question. In actual fact such a solution will be

perturbed, by imperfections of the medium, or even by thermal noise, and the question now is, do these perturbations grow? To answer this question, let us set

$$\begin{cases} A = A_0 + \tilde{A}, & \varphi = \varphi_0 + \tilde{\varphi}, \\ \eta = \eta_0 + \tilde{\eta}, & \chi = \chi_0 + \tilde{\chi}, \end{cases} \quad (2.2)$$

where $(A_0, \varphi_0, \eta_0, \chi_0)$ are the plane wave solution given by Eq. (2.1) and $(\tilde{A}, \tilde{\varphi}, \tilde{\eta}, \tilde{\chi})$ are small perturbations. If we substitute Eq. (2.2) into Eq. (1.9), and if we assume that the perturbation is so small that we may neglect its square, we obtain the following linearized equations for the perturbed quantities:

$$\begin{aligned} \frac{2\omega}{v} \frac{\partial \tilde{A}}{\partial z} &= A_0 [\nabla^2 \tilde{\varphi} - \sin 2\eta_0 \nabla^2 \tilde{\chi}], \\ -\frac{2\omega}{v} A_0 \frac{\partial \tilde{\varphi}}{\partial z} &= \nabla^2 \tilde{A} - A_0 \tan 2\eta_0 \nabla^2 \tilde{\eta} + 2(p+q)A_0^2 \tilde{A}, \\ \frac{2\omega}{v} \frac{\partial \tilde{\eta}}{\partial z} &= -\cos 2\eta_0 \nabla^2 \tilde{\chi}, \end{aligned} \quad (2.3)$$

$$\frac{2\omega}{v} \cos 2\eta_0 \frac{\partial \tilde{\chi}}{\partial z} = \nabla^2 \tilde{\eta} - 2qA_0^2 \tilde{A} \sin 2\eta_0 \cos 2\eta_0$$

$$-2qA_0^2 \tilde{\eta} \cos^2 2\eta_0.$$

These equations are linear and have constant coefficients; therefore, we can Fourier analyze, writing

$$\begin{pmatrix} \tilde{A} \\ \tilde{\varphi} \\ \tilde{\eta} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \tilde{A}_0 \\ \tilde{\varphi}_0 \\ \tilde{\eta}_0 \\ \tilde{\chi}_0 \end{pmatrix} e^{i\omega z + i\vec{k} \cdot \vec{x}}, \quad (2.4)$$

where $\tilde{A}_0, \tilde{\varphi}_0, \tilde{\eta}_0, \tilde{\chi}_0$ are constants. We obtain the system of equations

$$\begin{pmatrix} X & , & A_0 k^2 & , & 0 & , & -k^2 A_0 \sin 2\eta_0 \\ 2(p+q)A_0^2 - k^2 & , & A_0 X & , & k^2 A_0 \tan 2\eta_0 & , & 0 \\ 0 & , & 0 & , & X & , & -k^2 \cos 2\eta_0 \\ 2qA_0 \sin 2\eta_0 \cos 2\eta_0 & , & 0 & , & k^2 + 2qA_0^2 \cos^2 2\eta_0 & , & X \cos 2\eta_0 \end{pmatrix} \begin{pmatrix} \tilde{A}_0 \\ \tilde{\varphi}_0 \\ \tilde{\eta}_0 \\ \tilde{\chi}_0 \end{pmatrix} = 0, \quad (2.5)$$

where we have written

$$X \stackrel{\text{def}}{=} 2\omega\alpha/v$$

for short. The only way Eq. (2.5) can have a nontrivial solution is for the determinant to vanish, and this yields the dispersion relation

$$X^4 - 2k^2(pA_0^2 - k^2)X^2 + k^8 - 2pk^6 A_0^2 - 4q(p+q)k^4 A_0^4 \cos 2\eta_0 = 0. \quad (2.6)$$

which solves to yield

$$X^2 = k^2(pA_0^2 - k^2) \pm k^2 A_0^2 \sqrt{p^2 + 4q(p+q) \cos^2 2\eta_0}. \quad (2.7)$$

We see that for perturbations of wave number vector $\vec{k}$ such that

$$0 < k^2 < 2\gamma A_0^2.$$

$$\gamma \stackrel{\text{def}}{=} \frac{1}{2}p + \frac{1}{2}\sqrt{p^2 + 4q(p+q) \cos^2 2\eta_0}, \quad (2.8)$$

one of the roots of X^2 is positive and gives rise to exponentially growing waves. The greatest growth rate is attained when

$$k^2 = k_m^2 \equiv \gamma A_0^2, \quad (2.9)$$

and this maximum growth rate is

$$\alpha_m = \frac{v}{2\omega} \gamma A_0^2, \quad (2.10)$$

where γ is a constant, defined by Eq. (2.8). We see that a plane wave is unstable against perturbations with wave lengths in the range defined by Eq. (2.8) and that the growth rate of the instability is maximum for plane polarized light ($\cos^2 2\eta_0 = 1$) and minimum for circularly polarized light ($\cos^2 2\eta_0 = 0$); for elliptically polarized light the growth rate lies between the two extremes.

In a general case, as we have seen, the quantities $\tilde{A}$, $\tilde{\varphi}$, $\tilde{\eta}$, $\tilde{\chi}$ all grow exponentially in an instability. The two limiting cases of linear polarization and of circular polarization are, however, somewhat special. First consider linear polarization. We put $\sin 2\eta_0 = 0$ in Eq. (2.5) and notice at once that they break up into two decoupled sets, namely

$$\begin{pmatrix} X & , & k^2 \\ 2(p+q)A_0^2 - k^2 & , & X \end{pmatrix} \begin{pmatrix} \tilde{A}_0 \\ \tilde{\varphi}_0 \end{pmatrix} = 0 \quad (2.11)$$

and

$$\begin{pmatrix} X & , & -k^2 \\ k^2 + 2qA_0^2 & , & X \end{pmatrix} \begin{pmatrix} \tilde{\eta}_0 \\ \tilde{\chi}_0 \end{pmatrix} = 0 \quad (2.12)$$

Equation (2.12) yields the dispersion relation

$$X^2 = -k^2(k^2 + 2qA_0^2), \quad (2.13)$$

whence α is always purely imaginary, whatever k^2 may be. Thus the polarization state is stable. On the other hand Eq. (2.11) yields

$$(2.14) \quad X^2 = k^2[2(p+q)A_0^2 - k^2], \quad (2.14)$$

which implies instability of amplitude and phase, with

$$\begin{cases} k_m^2 = (p+q)A_0^2, \\ \alpha_m = \frac{v}{2\omega} (p+q)A_0^2. \end{cases} \quad (2.15)$$

Equations (2.14) and (2.15) are precisely Eqs. (2.7), (2.9) and (2.10) evaluated for $\sin 2\eta_0 = 0$ ($\cos^2 2\eta_0 = 1$). For circularly polarized light there is no such quantity as χ (no orientation of a circle)

and n is permanently $\pm \pi/4$. Thus in these two limiting cases the state of polarization is stable against small perturbations whereas amplitude and phase are unstable.

In the important case where the perturbation is some sort of random noise, all wave lengths will be present in roughly equal amounts and therefore the modes of fastest growth rate will be those observed, namely those characterized by Eqs. (2.9) and (2.10). In a random process, there is no reason to prefer the x over the y direction, so we may suppose $k_x = k_y$. Thus the perturbation breaks up the wave front into square cells of side length a , given by

$$k^2 = 2k_x^2 = 2\left(\frac{2\pi}{a}\right)^2 = \gamma A_c^2, \quad (2.16)$$

whence the cell area is

$$a^2 = 8\pi^2/\gamma A_c^2. \quad (2.17)$$

In a plane wave $\vec{B}$ is orthogonal to $\vec{E}$ and $|\vec{B}| = \sqrt{\epsilon_0} |\vec{E}|$, thus the energy flux is

$$W = \frac{c\sqrt{\epsilon_0} A_0^2}{8\pi} \quad (2.18)$$

in our plane wave. The power per cell is

$$W a^2 = \pi\sqrt{\epsilon_0} (c/\gamma). \quad (2.19)$$

Suppose next that we somehow arrange perturbations with $k_x \neq k_y$, say with

$$k_x = \frac{2\pi}{a}, \quad k_y = \frac{2\pi}{b} \quad (2.20)$$

Now we have

$$4\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \gamma A_c^2. \quad (2.21)$$

If we define the aspect ratio of the cell ρ by

$$\rho \stackrel{\text{def}}{=} b/a, \quad (2.22)$$

we can write the area as

$$ab = \frac{8\pi^2}{\gamma A_c^2} \cdot \left(\frac{1 + \rho^2}{2\rho}\right), \quad (2.23)$$

whence power per cell becomes

$$Wab = (c/\gamma)\pi\sqrt{\epsilon_0} \left(\frac{1 + \rho^2}{2\rho}\right) \quad (2.24)$$

A plane wave thus will break up into cells inside of each of which the perturbation grows. At first it grows exponentially, but later on, when the perturbation has become large enough to produce nonlinear effects on itself, the growth becomes faster than exponential until a point focus is formed. For each focus so formed a power given by Eq. (2.24) is required and it clearly depends on the cell shape through the factor $(1 + \rho^2)/2\rho$. We shall meet this factor again when we discuss critical self-focusing power for finite beams.

We assumed a special form for the normal modes of our perturbation, namely that given by Eq. (2.4). We could instead have set

$$\begin{pmatrix} \tilde{A} \\ \tilde{\varphi} \\ \tilde{\eta} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \tilde{A}_0 \\ \tilde{\varphi}_0 \\ \tilde{\eta}_0 \\ \tilde{\chi}_0 \end{pmatrix} e^{i\alpha z} U_k(x,y), \quad (2.25)$$

where $U_k(x,y)$ represents any eigenfunction of the two dimensional Laplacian operator, with eigenvalue $-k^2$. Had we done so we would still have obtained Eq. (2.5). From this point on the analysis would be identical to that given above. Thus, for example we could choose the normal modes to be of the form

$$U_k = J_0(kr), \text{ or } U_k = J_m(kr) e^{im\varphi}, \quad (2.26)$$

using now polar coordinates (r, φ) in the x, y -plane. If for example, we chose the first of Eq. (2.26) for our normal modes, as would be natural in cases of cylindrical symmetry, the central cell is a circle of radius a given by

$$ka = \rho_{11} = 3.8317, \quad (2.27)$$

where ρ_{11} denotes the first zero of the J_1 function, i.e. the first minimum of J_0 . The total power in this cell is

$$\pi a^2 W = (c/\gamma) \frac{\rho_{11}^2}{8} \sqrt{\epsilon_0} = 1.835 \sqrt{\epsilon_0} (c/\gamma). \quad (2.28)$$

Comparing Eqs. (2.19), (2.24) and (2.28) we can see that the more compact the elementary cell, the lower is the critical energy per cell for instability.

So far, we have considered only the fastest growing normal mode. If instead we ask for the smallest possible cell size for instability, and hence the minimum power per cell, we consider the stability limit $\vec{k}_s$ given by

$$k_s^2 = 2k_m^2 = 2\gamma A_0^2. \quad (2.29)$$

Thus, if for some reason more rapidly growing shorter wave lengths are suppressed and only waves of wave number vector $\vec{k}_s$ need be considered, we arrive at power per cell figures precisely one half of those given by Eqs. (2.19), (2.24) and (2.28), which correspond with maximum growth rate.

We have seen that a plane wave propagated in a medium with nonlinear refractive index is unstable and will break up into cells. The energy per cell depends on the cell shape, so it is natural to inquire what this shape will be. If every attempt has been made to obtain a plane wave front, so that only random noise and random irregularities of the refractive index remain to perturb the beam, there will be no preferred direction in the x,y-plane. Thus, in this case, the cells will be as symmetric as is possible, that is they will be (very nearly) regular hexagons. To compute precisely the power per unit cell in this case one would have to solve, probably numerically, the equation

$$\nabla^2 u + k^2 u = 0 \quad (2.30)$$

on the hexagon. We have not done this but rather content ourselves with the observation that the hexagon is nearly as compact a figure as is the circle and therefore Eq. (2.28) will underestimate the power per cell but little.

III. SOME EXACT SOLUTIONS

In this section we consider the simplest case of a plane or circularly polarized beam. In this case the quasi-optical equation is a scalar equation, namely

$$\frac{2i\omega}{v} \frac{\partial E}{\partial z} = \nabla^2 E + \gamma |E|^2 E, \quad (3.1)$$

where $\gamma = p + q$ for plane polarized light, or $\gamma = p$ for circularly polarized light. We further restrict ourselves to axial symmetry so that

$$\nabla^2 E = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E}{\partial r} \right). \quad (3.2)$$

It is easy to see that Eq. (3.1) possesses a very simple scaling law, namely if

$$E = u(x,y,z) \quad (3.3)$$

is a solution to Eq. (3.1), then so also is

$$E = au(ax, ay; a^2 z). \quad (3.4)$$

This suggests the possibility of finding similarity solutions of the form $E = au(ar)$, where a is now allowed to depend on z , but not on r . In fact, if we substitute this expression into Eq. (3.1), we find that the variables separate and thus we can solve the equation. The solutions thus found are, however, not very interesting.

Using the above idea we can find interesting solutions of a similarity type by relaxing somewhat the similarity condition. Thus, let us represent the field in terms of amplitude and phase by

$$E = Ae^{i\varphi}. \quad (3.5)$$

Now, Eq. (3.1) takes the form

$$\frac{2i\omega}{v} A \frac{\partial A}{\partial z} = \nabla \cdot [A^2 \nabla \varphi], \quad (3.6)$$

$$- \frac{2\omega}{c} A \frac{\partial \varphi}{\partial z} = \nabla^2 A - A(\nabla \varphi)^2 + \gamma A^3,$$

and we assume a similarity form only for the amplitude A , writing

$$A(r,z) = \frac{1}{\zeta(z)} X(\xi); \quad \xi \stackrel{\text{def}}{=} r/\zeta(z). \quad (3.7)$$

If now we choose φ to be of the form

$$\varphi(r,z) = \varphi_0(z) - \frac{\omega r^2}{2v} \frac{\zeta'(z)}{\zeta(z)}, \quad (3.8)$$

we find the first of Eq. (3.6) identically satisfied. If we set Eqs. (3.7) and (3.8) into the second of Eq. (3.6) we readily find

$$\frac{\partial^2 X}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial X}{\partial \xi} + \gamma X^3 + \left[\frac{2\omega}{v} \zeta^2 \varphi_0' - \frac{\omega^2}{v^2} \zeta^2 \zeta'' \right] X = 0. \quad (3.9)$$

So far φ_0 and ζ have been arbitrary functions of z .

We must now choose them so that the z -dependence of the coefficient of X in Eq. (3.9) disappears, for by hypothesis X depends on ξ only. Thus we set

$$\varphi_0 = -\frac{v\beta}{2\omega} \int \frac{dz}{[\zeta(z)]^2} \quad (3.10)$$

and

$$\zeta^3 \zeta''' = 4n^2 v^2 / \omega^2, \quad (3.11)$$

where n^2 , β are arbitrary constants. Equation (3.11) solves to give

$$\zeta = \sqrt{(az+b)^2 + 4n^2 v^2 / a^2 \omega^2}, \quad (3.12)$$

whence

$$\varphi = \frac{-\beta}{4n} \tan^{-1} \frac{a\omega(az+b)}{2nv} - \frac{\omega}{2v} \frac{a(az+b)r^2}{(az+b)^2 + 4n^2 v^2 / a^2 \omega^2}, \quad (3.13)$$

where a , b are new integration constants. With this determination of ζ and φ , Eq. (3.9) becomes

$$X'' + \frac{1}{\xi} X' + \gamma X^3 - [\beta + 4n^2 \xi^2] X = 0. \quad (3.14)$$

Of particular interest are those solutions for which the total beam power is finite, finite beams for short, i.e., solutions for which

$$\int_0^\infty [X(\xi)]^2 \xi d\xi < \infty. \quad (3.15)$$

For such a beam it is clearly necessary that $X \rightarrow 0$ as $\xi \rightarrow \infty$. Unless $n = 0$, such solutions to Eq. (3.14), if they exist, have the asymptotic form

$$X \sim C J_0(in\xi^2) + D Y_0(in\xi^2), \quad (3.16)$$

where C , D are constants and J_0 , Y_0 are the zero order Bessel functions. For purely imaginary n (negative n^2) these form a two parameter set and therefore can be matched to the solution of Eq. (3.14) which is finite at the origin. Thus for negative n^2 , the asymptotic solution to Eq. (3.14) is surely of the form of Eq. (3.16) and therefore the integral of Eq. (3.15) diverges. We therefore restrict ourselves to real n (positive n^2). For real n , only one of the two Bessel functions of Eq. (3.16) is small as $\xi \rightarrow \infty$, and Eq. (3.16) reduces to

$$X \sim C K_0(n\xi^2); \quad (3.17)$$

the other asymptotic solution is not small as $\xi \rightarrow \infty$. In general, the solution of Eq. (3.14) which is well behaved at the origin will attain a mixture of the two asymptotic solutions as $\xi \rightarrow \infty$ and thus not be small. In order that the asymptotic solution to Eq. (3.17) be attained, the constant β must be properly chosen, i.e. we are faced with an eigenvalue problem.

The case $n = 0$ is both special and especially interesting. The first thing we notice is that there is no solution for $n = 0$ and $\gamma = 0$. For then Eq. (3.14) reduces to Bessel's equation whose solutions are the I_0 and K_0 functions, one of which blows up at the origin and the other at infinity. As $\gamma \neq 0$, we can scale it to unity. Thus if we set

$$X(\xi) = \alpha u(\alpha \xi \sqrt{\gamma}) \equiv \alpha u(x) \quad (3.18)$$

say, Eq. (3.14) becomes

$$u'' + \frac{1}{x} u' + u^3 - \lambda^2 u = 0 \quad (3.19)$$

where we have defined

$$\lambda^2 \stackrel{\text{def}}{=} \beta / \alpha^2 \gamma \quad (3.20)$$

and α is an arbitrary scale factor. It can be chosen so that the condition at the origin is

$$u(0) = 1. \quad (3.21)$$

Equations (3.12) and (3.13) now reduce to

$$-\zeta = az+b, \quad \varphi = \frac{v\alpha^2 \lambda^2 \gamma}{2a\omega(az+b)} - \frac{\omega a r^2}{2v(az+b)}. \quad (3.22)$$

A special case of this solution has been previously found,⁽²⁾ namely the case $a = 0$. In this case $\zeta = \text{const.}$ and the beam intensity profile is independent of z ; we are dealing with the steady state self-constricted beam of Chiao et al.⁽²⁾

In any case the intensity profile is governed by Eq. (3.19). This equation clearly has three possible asymptotic solutions, namely

$$u \sim \pm \lambda \quad \text{and} \quad u \sim C K_0(\lambda x), \quad (3.23)$$

only the third of which is finite. In order to

match this solution which is small at infinity with the initial condition Eq. (3.12), λ^2 must assume a one of a discrete set of eigen values, say $\lambda_0^2, \lambda_1^2, \dots$. These have the property that, if arranged in order* so that

$$\lambda_0^2 > \lambda_1^2 > \lambda_2^2 > \dots > 0, \quad (3.24)$$

then the corresponding eigen functions $u_0, u_1, u_2, \dots$ have respectively 0, 1, 2, ... zeros and ultimately decay exponentially to zero as $x \rightarrow \infty$. Chiao et al. (2) found the first of these eigen values and eigen functions. They make no mention of the rest, but their existence and properties as given above are very easily demonstrated. We can multiply Eq. (3.19) through by u' and thus re-write it in the form

$$\frac{d}{dx} \left\{ \frac{1}{2}(u')^2 + \frac{1}{4}u^4 - \frac{1}{2}\lambda^2 u^2 \right\} = -\frac{1}{x}(u')^2. \quad (3.25)$$

If now we interpret u as displacement and x as time we see that the left hand side of Eq. (3.25) simply describes a particle oscillating in the potential well

$$V = \frac{1}{4}u^4 - \frac{1}{2}\lambda^2 u^2, \quad (3.26)$$

and the right hand side states that the motion is damped.

Fig. 1. Potential Well of Eq. (3.25)

If we sketch the potential V , it appears as in Fig. 1. The two minima are at $u = \pm \lambda$ and at these points $V(\lambda) = -\lambda^4/4$. A particle starts at point P and executes damped oscillations. In general it will come to rest at one of the two minima; this corresponds with one of the first two asymptotic solutions given by Eq. (3.23). Exceptionally it may come to rest in the center, corresponding with the third of Eq. (3.23), but clearly this can only happen for certain discrete heights of the central hump, i.e. values of λ^2 . As the motion is damped, it can end at the center on the first pass, after one passage over the hump, and so on. Thus the general character of the finite solutions of Eq. (3.19) is clear. Knowing these properties, it is a straightforward matter to compute the various u_n numerically. In Fig. 2 we show the first, u_0 , and compare it with a gaussian.

We can give this special class of solutions a very simple physical interpretation. In the first place, setting $a = 0$ in Eq. (3.22) reduces our solution to the steady state solution of Chiao et al. (2) which describes a beam of light with plane constant phase surfaces, which propagates neither spreading by diffraction nor contracting by self-focusing. Now suppose $a \neq 0$. In the plane $z = 0$ we have exactly the same radial profile as though

Fig. 2. Comparison of Minimum Critical Power Profile (Solid) with Gaussian (Dotted). The Gaussian is of Critical Power.

*Zero is not an eigen value. See Ref. (1), Appendix C.

a were zero, but the phase is

$$\varphi(0) = \text{const} - \frac{\omega}{2v} \frac{r^2}{(b/a)}, \quad (3.27)$$

that is to say we have a constant phase front with radius curvature $R = -b/a$ such as would have been produced by a lens of focal length $-b/a$. At $z = R = -b/a$, ζ vanishes and hence the amplitude A becomes infinite; we have produced a point focus. Our solution with $a \neq 0$ is simply the result of putting the beam of Chiao et al. through an aberration free lens, converging or diverging according to the sign of b/a . When the lens is converging, a point focus is formed rather than a diffraction limited spot.

The total power represented by the solutions found above is readily found. As $\vec{E}$, $\vec{B}$ are orthogonal for a quasi-plane wave, and as the magnitude of B is $\sqrt{\epsilon_0}|E|$, we have for the total power

$$P = \frac{1}{8\pi} \int_{\Sigma} c\sqrt{\epsilon_0} |E|^2 ds \quad (3.28)$$

in general, where Σ is the cross-sectional area of the beam. In our case this becomes

$$P_m = \frac{c\sqrt{\epsilon_0}}{4} \int_0^\infty [A_m(r)]^2 r dr \quad (3.29)$$

$$= \frac{c\sqrt{\epsilon_0}}{4} \int_0^\infty [X_m(\xi)]^2 \xi d\xi = \frac{c\sqrt{\epsilon_0}}{4\gamma} \int_0^\infty [u_m(x)]^2 x dx$$

where $u_m(x)$ is the eigen solution of Eqs. (3.19), (3.21) with eigen value λ_m^2 . For the lowest mode we have found $u_0(x)$ and the above integral numerically and have

$$P_0 = 0.4656 c\sqrt{\epsilon_0}/\gamma \quad (3.30)$$

As we shall see later, this represents the minimum critical self-focusing power.

So far we have discussed only the case $n = 0$. When $n \neq 0$, Eq. (3.14) is more difficult to analyze, but we may see qualitatively what happens by examining the special case $\gamma = 0$. In this case Eq. (3.14) reduces to a recognizable linear equation. If we impose the finiteness condition, Eq. (3.15), β is restricted to a discrete set of eigen values given by

$$\beta_m = 4n(2m+1), \quad m = 0, 1, 2, \dots \quad (3.31)$$

and the corresponding eigenfunctions are

$$X_m(\xi) = C L_m^{(0)}(2n\xi^2) e^{-n\xi^2} \quad (3.32)$$

where C is an arbitrary constant and the $L_m^{(0)}$ are Laguerre polynomials which are, in general, defined by

$$e^{-x} X_m^{(v)} L_m^{(v)}(x) = \frac{1}{m!} \left(\frac{d}{dx} \right)^m \left(e^{-x} x^{m+v} \right). \quad (3.33)$$

Note that $X_0(\xi)$, $\beta_0 = 4n$ simply describes a Gaussian profile.

When $a \neq 0$ and $\gamma = 0$ we have a set of similarity solutions corresponding with the general choice of ζ , φ given by Eqs. (3.12) and (3.13). Clearly ζ is minimum where

$$z = z_f \equiv -b/a \quad (3.34)$$

which is, therefore, the waist of the beam. At $z = z_f$ the phase front is a plane, whereas for $z < z_f$ it is spherical and converging. These solutions, then, describe a diffraction limited focal spot produced by a lens in the region $z \leq z_f$; conversely considering the region $z \geq z_f$ we may view the solution as describing the divergence of an initially collimated beam owing to diffraction. The nonlinear case $\gamma \neq 0$ leaves Eqs. (3.12) and (3.13) unaltered, so again we have a diffraction limited focal spot, now somewhat smaller than had γ been zero. (The focal spot size does not change with γ , but the initial beam radius does.) These solutions with $n \neq 0$, $\gamma \neq 0$ represent therefore the nonlinear behavior of focused beams, or of initially collimated sources, when total beam power is less than that required for self focusing.

IV. FINITE BEAMS - THE METHOD OF MOMENTS

In a short but very important paper⁽³⁾ Vlasov, Petrichchev and Talanov have shown how the method of moments can be applied to the quasi-optical equation. They considered this equation in the form of our Eq. (1.7) and obtained a very simple rule for calculating critical self-focusing power and, when this power is exceeded, the self-focusing distance. The initial beam may be parallel, con-

verging or diverging, and the profile arbitrary except that it contains finite total power. As this paper does not seem to exist in translation, we shall give in this section an exposition of the method.

To illustrate the general method of moments, let us suppose that we have a scalar quantity w , a pair of vector quantities $\vec{S}$ and $\vec{Q}$, and a tensor quantity $\vec{T}$ related by the following conservation equations:

$$\begin{aligned}\frac{\partial w}{\partial t} &= -\nabla \cdot \vec{S}, \\ \frac{\partial \vec{S}}{\partial t} &= \nabla \cdot \vec{T},\end{aligned}\quad (4.1)$$

$$\frac{\partial}{\partial t} (T_{\alpha}^{\alpha}) = -\nabla \cdot \vec{Q},$$

where T_{α}^{α} represents the trace of the tensor $\vec{T}$. An example of such a system is Maxwell's vacuum equations, with the identification

$$w = \frac{1}{4\pi} (E^2 + B^2), \quad \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B},$$

$$T_{\alpha\beta} = \frac{c^2}{4\pi} [E_{\alpha}E_{\beta} + B_{\alpha}B_{\beta} - \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)]. \quad (4.2)$$

In This case $T_{\alpha}^{\alpha} = c^2 w$ whence $\vec{Q} = -c^2 \vec{S}$.

Let us now form the moments

$$\begin{aligned}W &\stackrel{\text{def}}{=} \int_V w \, dv, \\ \vec{R} &\stackrel{\text{def}}{=} \frac{1}{W} \int_V \vec{r} w \, dv, \\ a_{\text{eff}}^2 &\stackrel{\text{def}}{=} \frac{1}{W} \int_V r^2 w \, dv,\end{aligned}\quad (4.3)$$

where $\vec{r}$ is the position vector and the volume of integration V is the whole region of space over which the fields are non-zero. Clearly we have, by Gauss' theorem

$$\frac{dW}{dt} = \int_V \frac{\partial w}{\partial t} \, dv = - \int_{\Sigma} \vec{n} \cdot \vec{S} \, d\sigma = 0, \quad (4.4)$$

where Σ is the surface bounding $\vec{V}$ and $\vec{n}$ is the outward unit normal to Σ . Similarly we have

$$\frac{d^2 \vec{R}}{dt^2} = \frac{1}{W} \frac{d}{dt} \int_V \vec{r} \frac{\partial w}{\partial t} \, dv = \frac{1}{W} \frac{d}{dt} \int_V [\vec{S} \cdot \nabla \vec{r} - \nabla \cdot (\vec{S} \vec{r})] \, dv \quad (4.5)$$

$$= \frac{1}{W} \int_V \frac{\partial \vec{S}}{\partial t} \, dv = \frac{1}{W} \int_{\Sigma} \vec{n} \cdot \vec{T} \, d\sigma = 0,$$

and likewise

$$\frac{d^3 (a_{\text{eff}}^2)}{dt^3} = \frac{1}{W} \frac{d^2}{dt^2} \int_V [\vec{S} \cdot \nabla (r^2) - \nabla \cdot (\vec{S} r^2)] \, dv \quad (4.6)$$

$$= \frac{1}{W} \frac{d}{dt} \int_V 2(\vec{r} \cdot \frac{\partial \vec{S}}{\partial t}) \, dv = \frac{2}{W} \frac{d}{dt} \int_V [\nabla \cdot (\vec{T} \cdot \vec{r}) - T_{\alpha}^{\alpha}] \, dv$$

$$= \frac{1}{W} \int_{\Sigma} \vec{n} \cdot \vec{Q} \, d\sigma = 0.$$

In deriving these results we have assumed that

$$\vec{n} \cdot \vec{S} = \vec{n} \cdot \vec{Q} = 0 = \vec{n} \cdot \vec{T} \quad (4.7)$$

on the surface Σ and that, if Σ is at infinity, all integrals converge. We see that W , $d\vec{R}/dt$, and $d^2(a_{\text{eff}}^2)/dt^2$ are all constants, whence we have

$$\begin{aligned}W &= W_0, \\ \vec{R} &= \vec{R}_0 + \vec{v} t,\end{aligned}\quad (4.8)$$

$$a_{\text{eff}}^2 = a_0^2 + 2Bt + Ct^2,$$

and in the process of our derivation we have seen

$$\text{that } \vec{v} = \frac{1}{W_0} \int_V \vec{S} \, dv \Big|_{t=0}, \quad C = -\frac{1}{W_0} \int_V (T_{\alpha}^{\alpha}) \, dv \Big|_{t=0}, \quad (4.9)$$

$$B = \frac{1}{W_0} \int_V \vec{r} \cdot \vec{S} \, dv \Big|_{t=0},$$

and $\vec{R}_0$, a_0^2 , are simply Eq. (4.3) evaluated at $t = 0$. In fact, as W , $\vec{v}$ and C are clearly constants of the motion, these three integrals may be evaluated wherever we like.

In Ref. (3) these results were applied to Eq. (1.7) as follows. First we note upon multiplying Eq. (1.7) by E^* and its complex conjugate by E and adding, that

$$\frac{\partial |E|^2}{\partial z} = -\nabla \cdot \left\{ \frac{\vec{v}}{2i\omega} (E \nabla E^* - E^* \nabla E) \right\}. \quad (4.10)$$

Thus we start by identifying

$$w = |E|^2, \quad \vec{S} = -\frac{\vec{v}}{2i\omega} (E \nabla E^* - E^* \nabla E). \quad (4.11)$$

If now we form $\partial \vec{S} / \partial z$ according to the second of

Eqs. (4.11) and then use Eq. (1.7) to eliminate the z-derivatives, we find that the result may be arranged as

$$\frac{\partial \vec{S}}{\partial z} = \nabla \cdot \vec{T}, \quad (4.12)$$

where $\vec{T}$ has the components

$$T_{\alpha\beta} = -\frac{v^2}{2\omega^2} [(\nabla_\alpha E)(\nabla_\beta E^*) + (\nabla_\alpha E^*)(\nabla_\beta E)] + \frac{v^2}{4\omega^2} \delta_{\alpha\beta} [\nabla^2(|E|^2) + \gamma|E|^4] \quad (4.13)$$

From this, we form the trace

$$T_\alpha^\alpha = -\frac{v^2}{2\omega^2} [E\nabla^2 E^* + E^*\nabla^2 E + \gamma|E|^4]. \quad (4.14)$$

Finally, differentiating this by z and using Eq. (1.7) we find

$$\frac{\partial T_\alpha^\alpha}{\partial z} = \nabla \cdot \vec{Q}, \quad (4.15)$$

with $\vec{Q}$ given by

$$\vec{Q} = \frac{c^3}{4i\omega^3} \{E^*\nabla(\nabla^2 E) + (\nabla E)(\nabla^2 E^*) - E\nabla(\nabla^2 E^*) - (\nabla E^*)(\nabla^2 E) + 2\gamma|E|^2(E^*\nabla E - E\nabla E^*)\}. \quad (4.16)$$

The third of our equations (4.8) is, now

$$(a_{\text{eff}}^2)^2 = Cz^2 + 2Bz + A_0^2, \quad (4.17)$$

where

$$C = \frac{v^2}{\omega^2 W_0} \int_\Sigma \{|\nabla E|^2 - \frac{\gamma}{2}|E|^4\} d\sigma, \quad B = \frac{v}{2i W_0} \int_\Sigma \vec{r} \cdot [E\nabla E^* - E^*\nabla E] d\sigma \Big|_{z=0}, \quad (4.18)$$

$$a_0^2 = \frac{1}{W_0} \int_\Sigma |E|^2 d\sigma \Big|_{z=0},$$

$$W_0 = \int_\Sigma |E|^2 d\sigma,$$

and the integrals are taken over the full cross-section of the beam. Although the integrals W_0 and C are constants of the motion, it will normally be most convenient to evaluate them at $z = 0$.

Equations (4.17) and (4.18) are the real meat of this development. Thus, according to the definitions a_0^2 must be positive and a_{eff}^2 is non-negative. But if C should be negative there must be some value of z where the right-hand side of Eq. (4.17) vanishes. At this point $a_{\text{eff}}^2 = 0$ which can only happen if the field is singular at this

point. In fact, as

$$a_{\text{eff}}^2 \stackrel{\text{def}}{=} \frac{\int_\Sigma r^2 |E|^2 d\sigma}{\int_\Sigma |E|^2 d\sigma}, \quad (4.19)$$

this quantity is simply the mean square radius of the beam. When it vanishes the beam has collapsed to a point, the self-focus. Thus the condition for a self focus is that the equation

$$Cz^2 + 2Bz + a_0^2 = 0 \quad (4.20)$$

have real roots, and these roots are the self-focus points. In general the roots will be distinct, say z_1 and z_2 . If we shift the origin of z to the point $(z_1 + z_2)/2$, the roots will be symmetrically disposed about $z = 0$ and B will vanish. In passing we might note that B is most conveniently written in terms of amplitude and phase. Thus if

$$E = A e^{i\varphi}, \quad (4.21)$$

we have

$$B = -\frac{v}{\omega W_0} \int_\Sigma A^2 \vec{r} \cdot \nabla \varphi d\sigma \Big|_{z=0}. \quad (4.22)$$

The vanishing of B , as above, for $z = 0$ normally occurs because the phase front is here flat. Having made $B = 0$, Eq. (4.20) has real roots if and only if $C < 0$. Exceptionally, for our solution of Section III, for example, there can be a single focal point. In this case Eq. (4.20) has a double root and B is negative. The phase front is everywhere convergent in this exceptional case; we are dealing with a beam which has passed through a convergent lense. Excluding this exceptional case, the condition $C < 0$ is necessary and sufficient for self focusing. Accordingly it is natural to define critical power by the requirement

$$\int_\Sigma \{|\nabla E|^2 - \frac{\gamma}{2}|E|^4\} d\sigma = 0. \quad (4.23)$$

As we shall see, critical self focusing power depends strongly on the cross sectional shape of the beam and on the illumination profile.

The above considerations can be extended to

include the general anisotropic case described by Eq. (1.1). The calculations are the same as given above but more lengthy. The result is that we can make the identification with Eq. (4.1) thus:

$$w = |\vec{E}|^2, \quad \vec{S} = \frac{v}{2i\omega} [E_\alpha \nabla E_\alpha^* - E_\alpha^* \nabla E_\alpha],$$

$$T_{\alpha\beta} = -\frac{v^2}{2\omega^2} \{ (\nabla_\alpha E_\pi) (\nabla_\beta E_\pi^*) + (\nabla_\alpha E_\pi^*) (\nabla_\beta E_\pi) \}$$

$$+ \frac{v^2}{4\omega^2} \delta_{\alpha\beta} \{ \nabla^2 (|\vec{E}|^2) + \gamma_{\pi\rho\sigma\tau} E_\pi^* E_\rho^* E_\sigma E_\tau \}.$$

From the last we have

$$T_\alpha^\alpha = \frac{v^2}{2\omega^2} [E_\alpha \nabla^2 E_\alpha^* + E_\alpha^* \nabla^2 E_\alpha + \gamma_{\alpha\beta\gamma\delta} E_\alpha^* E_\beta^* E_\gamma E_\delta], \quad (4.25)$$

and finally

$$\frac{\partial T_\alpha^\alpha}{\partial z} = \nabla \cdot \vec{Q}, \quad (4.26)$$

with

$$\vec{Q} = \frac{v^3}{4i\omega^3} \{ E_\alpha^* \nabla (\nabla^2 E_\alpha) + (\nabla E_\alpha) (\nabla^2 E_\alpha^*) - cc \}$$

$$+ \frac{v^3}{2i\omega^3} \gamma_{\alpha\beta\gamma\delta} E_\beta^* E_\gamma (E_\alpha^* \nabla E_\delta - E_\delta \nabla E_\alpha^*).$$

Here "cc" means the complex conjugate of the preceding quantity, the E_α are cartesian vector components, Greek indices range over 1,2 (or x,y) and repeated indices are summed.

As a particular sub-case of the general expressions above, we may consider the specialization of the $\gamma_{\alpha\beta\gamma\delta}$ which lead to Eq. (1.6). In this case our identification of the quantities $w, \vec{S}$ remains as in Eq. (4.24), but the tensor $\vec{T}$ and vector $\vec{Q}$ simplify to

$$T_{\alpha\beta} = -\frac{v^2}{2\omega^2} \{ (\nabla_\alpha E_\pi) (\nabla_\beta E_\pi^*) + (\nabla_\alpha E_\pi^*) (\nabla_\beta E_\pi) \}$$

$$+ \frac{c^2}{4\omega^2} \delta_{\alpha\beta} \{ \nabla^2 (|\vec{E}|^2) + (p+q) |\vec{E}|^4 + q (E_x^* E_y^* - E_y^* E_x^*)^2 \},$$

$$\vec{Q} = \frac{v^3}{4i\omega^3} \{ [E_\alpha^* \nabla (\nabla^2 E_\alpha) + (\nabla E_\alpha) (\nabla^2 E_\alpha^*) - cc]$$

$$+ 2(p+q) |\vec{E}|^2 [E_\alpha^* \nabla E_\alpha - cc]$$

$$- 2q (E_x^* E_y^* - cc) [E_x^* \nabla E_y - E_y^* \nabla E_x + cc] \}.$$

As $E_x^* E_y^* - E_y^* E_x^*$ is purely imaginary, its square is simply $-|\vec{E} \times \vec{E}^*|^2$ and we can write the simplified version of Eq. (4.25) as

$$T_\alpha^\alpha = \frac{v^2}{2\omega^2} \{ \vec{E} \cdot \nabla^2 \vec{E}^* + \vec{E}^* \cdot \nabla^2 \vec{E} + (p+q) |\vec{E}|^4$$

$$- q |\vec{E} \times \vec{E}^*|^2 \}$$

For plane polarized light $\vec{E}$ and $\vec{E}^*$ are parallel so that $\vec{E} \times \vec{E}^* = 0$ and we obtain Eq. (4.23) for critical power with $\gamma = p + q$. For circularly

polarized light $\vec{E}$ and $\vec{E}^*$ are orthogonal so that $|\vec{E} \times \vec{E}^*|^2 = |\vec{E}|^4$. Now Eq. (4.23) gives critical power when we set $\gamma = p$. Thus critical self focusing power is higher for circularly polarized light than for linearly polarized light by the factor $(p+q)/p$.

We may try generalizing our results in another direction. Let us consider the simple scalar case of plane or circularly polarized light,

$$\frac{\partial E}{\partial z} = \frac{v}{2i\omega} [\nabla^2 E + \chi E], \quad (4.30)$$

but now let us suppose that χ is a more general function of $|\vec{E}|^2$ than purely linear. Again we recover the equations

$$\frac{\partial w}{\partial z} = -\nabla \cdot \vec{S}, \quad \frac{\partial \vec{S}}{\partial z} = \nabla \cdot \vec{T}, \quad (4.31)$$

with

$$w = |\vec{E}|^2, \quad \vec{S} = \frac{v}{2i\omega} [E \nabla E^* - E^* \nabla E], \quad (4.32)$$

as before. Now $T_{\alpha\beta}$ becomes

$$T_{\alpha\beta} = -\frac{v^2}{2\omega^2} [(\nabla_\alpha E) (\nabla_\beta E^*) + (\nabla_\alpha E^*) (\nabla_\beta E)]$$

$$+ \frac{v^2}{4\omega^2} [\nabla^2 (|\vec{E}|^2) + 2\chi |\vec{E}|^2 - 2X] \delta_{\alpha\beta},$$

where X is defined by

$$\chi(u) \stackrel{\text{def}}{=} \int_u^u \chi(u) du,$$

$$u \stackrel{\text{def}}{=} |\vec{E}|^2 \quad (4.35)$$

When now we form T_α^α and differentiate it, the nonlinear contribution becomes

$$\frac{\partial (T_\alpha^\alpha)^{NL}}{\partial z} = \frac{v^3}{4\omega^3} \{ \nabla \cdot [\chi (E^* \nabla E - cc)]$$

$$+ [|\vec{E}|^2 \chi' - \chi] [E^* \nabla^2 E - cc] \},$$

where χ' means $d\chi/d|\vec{E}|^2$. This reduces to a gradient only if $|\vec{E}|^2 \chi' - \chi$ is a constant, i.e. if

$$\chi = \alpha + \gamma |\vec{E}|^2, \quad (4.37)$$

with α, γ constants; α can of course be transformed to zero by redefining the velocity v . We cannot extend Eqs. (4.17), (4.18) to a nonlinear term of more general form than the cubic. This is not serious, for when higher powers of $|\vec{E}|^2$ become important in the susceptibility, the fields have

already reached catastrophic proportions. We note, however, that although the integral of T_{α}^{α} is not a constant of the motion for a noncubic medium, we nevertheless have the four constants of the motion

$$W = \int_{\Sigma} |E|^2 d\sigma, \quad \vec{V} = \int_{\Sigma} \frac{V}{2i\omega} [E^* \nabla E - E \nabla E^*] d\sigma, \quad (4.37)$$

$$I \equiv \int_{\Sigma} [|VE|^2 - X] d\sigma,$$

where X is defined by Eq. (4.35) above. The first two of these follows from Eqs. (4.31), (4.32) above, and the last by direct calculation. In fact we find

$$\frac{\partial}{\partial z} \{ |VE|^2 - X^2 \} = -\nabla \cdot \left\{ \frac{V}{2i\omega} [\nabla E^2 E^* + X E^*] - c\vec{c} \right\}. \quad (4.38)$$

In closing this section we should point out that all the results here derived are strictly true only for unconfined beams (or pulses) of finite power. For examination of our tensor $T_{\alpha\beta}$ makes it clear that the "surface" integrals, here contour integrals in the xy -plane around the outside boundary of the beam cross-section, can vanish only if both $\vec{E}$ and $\text{grad } \vec{E}$ vanish. But this is impossible on a finite boundary unless $\vec{E}$ vanishes everywhere. It is possible in the limit, for a finite beam, as the boundary recedes to infinity. In our numerical solutions, it is necessary to introduce a perfectly conducting wall. We normally make the field very small as this wall is approached so that $\text{grad } \vec{E}$ becomes very small. The "surface" integrals thus become quite small, in principle as small as one would like, but not dead zero. For this reason it is possible for numerical solutions to be correct, in that they solve the problem posed, and yet disagree with Eq. (4.17).

V. APPLICATIONS OF METHOD OF MOMENTS

In this section we shall apply the method of moments to a set of particular problems selected primarily to give insight into the phenomenon of self-focusing. The basic equations we shall use are Eqs. (4.17) and (4.18), which means that we confine our attention to plane or circularly polarized light, propagating in a medium of at least cubic symmetry, or in a medium dominated by

Kerr effect. As the coefficients of Eq. (4.17) involve integrals over the beam cross-section, critical power will depend, among other things, on the beam profile. Accordingly, it seems very natural to ask: "Of all possible profiles, which will give minimum critical power?" This we now discuss.

A. Minimum Critical Self-Focusing Power

Vlasov, Petrishev and Talaov⁽³⁾ solve this problem as follows. Let us suppose that at $z = 0$ the electric field has the distribution

$$E(z=0) = E_0 f(x, y), \quad (5.1A)$$

when E_0 is a constant which determines the scale factor; then critical power is determined by Eq. (4.25), which may be written in the form

$$E_c^2 = \frac{2}{\gamma} \frac{\int_{\Sigma} (\nabla f)^2 d\sigma}{\int_{\Sigma} f^4 d\sigma}, \quad (5.2A)$$

where E_c is the critical value of the constant E_0 of Eq. (5.1A). Thus the critical power is given by

$$P_c = \frac{c\sqrt{E_0}}{4\pi\gamma} \frac{[\int_{\Sigma} (\nabla f)^2 d\sigma] [\int_{\Sigma} f^2 d\sigma]}{\int_{\Sigma} f^4 d\sigma}. \quad (5.3A)$$

If now we ask what function f makes P_c a minimum, we are faced with an ordinary calculus of variations problem which yields the Euler-Lagrange equation

$$\nabla^2 f - \frac{\int_{\Sigma} (\nabla f)^2 d\sigma}{\int_{\Sigma} f^2 d\sigma} f + \frac{\int_{\Sigma} (\nabla f)^2 d\sigma}{\int_{\Sigma} f^4 d\sigma} f^3 = 0. \quad (5.4A)$$

All the integrals entering into Eq. (5.4A) are constants, thus if we scale, setting

$$f = au, \quad (\bar{x}, \bar{y}) = (bx, by), \quad (5.5A)$$

we obtain the equation

$$\nabla^2 u + u^3 - \lambda^2 u = 0, \quad (5.6A)$$

provided we choose the scale factors so that

$$b = \lambda \left\{ \frac{\int_{\Sigma} (\nabla f)^2 d\sigma}{\int_{\Sigma} f^2 d\sigma} \right\}^{-\frac{1}{2}} a^2 b^2 = \left\{ \frac{\int_{\Sigma} (\nabla f)^2 d\sigma}{\int_{\Sigma} f^4 d\sigma} \right\}^{-1} \quad (5.7A)$$

In the axisymmetric case Eq. (5.6A) is identical

with Eq. (3.19), thus the solutions found in Section III corresponding with $n = 0$ are all (local) minima of the critical power. The particular one u_0 , with no zeros gives smallest power of all these minima, hence Eq. (3.30), here repeated

$$(P_c)_{\min} = 0.4656 (c/\gamma) \sqrt{E_0} \quad (5.8A)$$

gives the minimum critical power.

It is interesting to ask whether Eq. (5.6A) possesses any solutions which are not axisymmetric. Marburger has shown that Eq. (5.6A) in fact possesses no solution corresponding with an elliptical cross-section and speculates that indeed solutions exist only for cylinder symmetry.⁽⁴⁾ I know of no proof of this speculation, but it certainly seems intuitively reasonable.

B. Effect of a Lense on Self-Focusing

In this section we shall suppose that the electric field at $z = 0$ is given by

$$\begin{cases} E(z=0) = E_0 f(x,y) e^{i\omega r^2/2\nu R} \\ r^2 \equiv x^2 + y^2 \end{cases} \quad (5.1B)$$

Again E_0 is a constant scale factor, f describes the profile and the complex factor describes a spherical isophase front of radius of curvature R ; R is positive (negative) for a beam which converges (diverges) as z increases.

Substituting Eq. (5.1B) into Eq. (4.18) we readily obtain

$$a_0^2 = \frac{E_0^2}{W_0} \tilde{I}, \quad B = -\frac{E_0^2}{RW_0} \tilde{I}, \quad C = C_0 + \frac{E_0^2}{R^2 W_0} \tilde{I}, \quad (5.2B)$$

where

$$\tilde{I} \stackrel{\text{def}}{=} \int_{\Sigma} r^2 f^2 d\sigma, \quad C_0 \stackrel{\text{def}}{=} \int_{\Sigma} \left\{ (\nabla f)^2 - \frac{\gamma}{2} E_0^2 f^4 \right\} d\sigma. \quad (5.3B)$$

Thus Eq. (4.17) becomes simply

$$(a_{\text{eff}})^2 = C_0 z^2 + \frac{E_0^2 \tilde{I}}{R^2} [z-R]^2 \quad (5.4B)$$

Let us confine our attention to the region $z \geq 0$. For positive R (convergent lense) the condition that $(a_{\text{eff}})^2$ should vanish for some positive z is simply that $C_0 \leq 0$. Thus critical power is given by $C_0 = 0$ and self focusing occurs at $z = R$ for critical power, just at the focal point of the lense. But $C_0 = 0$ also defines critical power for

an initially parallel beam of light ($R = \infty$). Thus we see that: If we start with a parallel beam, the insertion of a convergent lens or lenses into this beam does not alter the critical power.

Next, suppose R is negative and ask the condition for a self-focus for positive z . The condition is clearly that

$$C_0 + \frac{E_0^2}{R^2 W_0} \tilde{I} \leq 0, \quad (5.5B)$$

with equality defining critical power. As $E_0^2 \tilde{I}/R^2 W_0$ is clearly positive, initial divergence raises the critical power. The effect is in fact quite large, and can be estimated quite easily. The quantity a_0 , defined by Eq. (4.18) clearly measures the size of the beam, so that, in some average sense, we have $(\nabla f)^2 \sim f^2/a_0^2$ and can therefore set

$$\int_{\Sigma} (\nabla f)^2 d\sigma = \frac{1}{\alpha a_0^2} \int_{\Sigma} f^2 d\sigma = \frac{1}{\alpha a_0^2}, \quad (5.6B)$$

if we normalize so that f^2 integrates to unity. Then clearly

$$\tilde{I} = a_0^2 \quad (5.7B)$$

by the definition of a_0 . Using Eqs. (5.6B) and (5.7B) in Eq. (5.5B) written as an equality, we find the relationship

$$P_R = \left(1 + \frac{4\pi^2 a_0^4}{\alpha \lambda^2 R^2} \right) P_{||} \quad (5.8B)$$

between critical power ($P_{||}$) for a parallel beam and critical power (P_R) for a divergent beam. The quantity α is a constant of order unity and depends on the details of the profile function f ; for a gaussian distribution we shall find $\alpha = 4$. As λ , the wavelength of the light measured in the medium, is normally $\ll a_0$ we see that a modest beam divergence can raise the critical power by quite a large factor.

C. Circular Beam of Gaussian Profile

As a particularly simple, but practically important case, let us suppose that

$$E(z=0) = E_0 \exp \left\{ -\frac{r^2}{a^2} + \frac{i\omega r^2}{2\nu R} \right\}. \quad (5.1C)$$

With such an initial distribution all of the integrals become elementary and we find

$$W_0 = \frac{\pi a^2}{2} E_0^2, \quad (5.2C)$$

and the other integrals yield

$$\begin{cases} a_0^2 = a^2/2, & B = -a^2/2R, \\ C = \frac{2v^2}{a^2\omega^2} \left[1 - \frac{\gamma}{8} E_0^2 + \frac{a^4\omega^2}{4v^2R^2} \right], \end{cases} \quad (5.3C)$$

whence

$$(a_{\text{eff}})^2 = \frac{2v^2z^2}{a^2\omega^2} \left[1 - \frac{\gamma}{8} E_0^2 \right] + \frac{a^2}{2R^2} [z - R]^2. \quad (5.4C)$$

For $R \geq 0$, the condition for criticality is clearly $\gamma a^2 E_0^2 = 8$ whence the critical power is

$$P_c = \frac{1}{2} (c/\gamma) \sqrt{\epsilon_0}. \quad (5.5C)$$

Comparing with Eq. (5.8A), we see that the critical power for a gaussian is only 7 percent larger than minimum critical power. If $R < 0$, then critical power for a self-focus with $z > 0$ is simply Eq. (5.8B) with $\alpha = 4$.

D. Elliptical Gaussian Beam

In Section II we saw that the power per cell in the break-up of a plane wave depends on the aspect ratio (length to width ratio) of the cell. The same is true of a finite beam as we shall see by considering the initial configuration

$$E(z=0) = E_0 \exp \left\{ -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) + \frac{i\omega}{2v} \left(\frac{x^2}{R_1} + \frac{y^2}{R_2} \right) \right\}. \quad (5.1D)$$

This describes a beam of elliptical cross-section which has been passed through a pair of crossed cylindrical lenses of focal lengths R_1 and R_2 . The distribution is gaussian in intensity. All of the integrals are elementary in the sense that they can be found in all standard tables. We find

$$W_0 = \iint_{-\infty}^{\infty} |E|^2 dx dy = \frac{1}{2} \pi a b E_0^2, \quad (5.2D)$$

and Eq. (4.18) then gives

$$\begin{aligned} a' &= \frac{1}{4} (a^2 + b^2), \\ B &= -\frac{1}{4} \left(\frac{a^2}{R_1} + \frac{b^2}{R_2} \right), \end{aligned} \quad (5.3D)$$

$$C = \frac{v^2}{\omega^2} \left[\frac{a^2+b^2}{a^2b^2} - \frac{\gamma}{4} E_0^2 \right] + \frac{1}{4} \left[\frac{a^2}{R_1^2} + \frac{b^2}{R_2^2} \right],$$

whence we find for the mean square radius

$$\begin{aligned} (a_{\text{eff}})^2 &= \frac{v^2 z^2}{\omega^2} \left[\frac{a^2+b^2}{a^2b^2} - \frac{\gamma}{4} E_0^2 \right] \\ &+ \frac{a^2}{4R_1^2} [z-R_1]^2 + \frac{b^2}{4R_2^2} [z-R_2]^2. \end{aligned} \quad (5.4D)$$

Written in this form it is clear that if $R_1 = R_2$, so that the lense is anastigmatic, critical self-focusing power is determined by the condition

$$\frac{1}{4} \gamma E_c^2 = \frac{a^2+b^2}{a^2b^2} \quad (5.5D)$$

whence it follows that critical self-focusing power is, in any of these cases

$$P_c = \frac{1}{2} (c/\gamma) \sqrt{\epsilon_0} \cdot \left[\frac{1+\rho^2}{2\rho} \right]; \quad \rho \stackrel{\text{def}}{=} b/a \quad (5.6D)$$

Written in this form, we can readily compare with Eq. (5.5C) and thus note the factor $(1+\rho^2)/2\rho$ advantage for a cross-sectional aspect ratio ρ as compared with a completely symmetric cross-section. We met this factor earlier in our study of the break-up of a plane wave [see Eq. (2.24)]. In the general convergent case we may rewrite Eq. (5.4D) as

$$\begin{aligned} (a_{\text{eff}})^2 &= C \left[z - \frac{1}{4C} \left(\frac{a^2}{R_1} + \frac{b^2}{R_2} \right) \right]^2 + \frac{1}{4} (a^2 + b^2) \\ &- \frac{1}{16C} \left(\frac{a^2}{R_1} + \frac{b^2}{R_2} \right)^2 \end{aligned} \quad (5.7D)$$

whence it is clear that the waist of the beam is at

$$z = z_m \equiv \frac{1}{4C} \left(\frac{a^2}{R_1} + \frac{b^2}{R_2} \right); \quad (5.8D)$$

the condition that this be positive defines what we mean by a convergent beam. At this point, the minimum mean square radius is

$$(a_{\text{eff}})_m^2 = \frac{1}{4} (a^2 + b^2) - \frac{1}{16C} \left(\frac{a^2}{R_1} + \frac{b^2}{R_2} \right). \quad (5.9D)$$

Critical self-focusing power is achieved when this vanishes, i.e. for

$$\gamma \left(\frac{a^2+b^2}{a^2b^2} \right) E_c^2 = 1 + \frac{\omega^2}{4v^2} \frac{a^4b^4}{a^2+b^2} \left[\frac{1}{R_1} - \frac{1}{R_2} \right]^2 \quad (5.10D)$$

For the general divergent case criticality is determined by the condition $C = 0$, which specifically gives

$$\frac{\gamma}{4} \left(\frac{a^2 b^2}{a^2 + b^2} \right) E_c^2 = 1 + \frac{\omega^2}{4v^2} \frac{a^2 b^2}{a^2 + b^2} \left(\frac{a^2}{R_1^2} + \frac{b^2}{R_2^2} \right). \quad (5.11D)$$

As the beam dimensions a, b are normally quite large compared with the wavelength of the light involved, it is seen that a modest amount of beam divergence can raise the critical self-focusing power quite dramatically. It is interesting to enquire where the beam will self-focus when critical power is exceeded. To study this, we first define the following short-hand symbols

$$\begin{cases} Y \stackrel{\text{def}}{=} \frac{\omega^2}{4v^2} \frac{a^2 b^2}{a^2 + b^2} \left(\frac{a^2}{R_1^2} + \frac{b^2}{R_2^2} \right), \\ R_0 \stackrel{\text{def}}{=} \frac{a^2 + b^2}{(a^2/R_1^2) + (b^2/R_2^2)}, \\ N \stackrel{\text{def}}{=} \frac{Y}{4} \left(\frac{a^2 b^2}{a^2 + b^2} \right) E_0^2. \end{cases} \quad (5.12D)$$

Thus R_0 is a mean focal length and N is the ratio of actual power to that power which would be critical in a parallel beam ($R_1 = R_2 = \infty$). Equation (5.4D) becomes

$$(N-1-Y)z^2 + \frac{\omega^2}{2v^2} \frac{a^2 b^2}{R_0} z - \frac{\omega^2}{4v^2} a^2 b^2 = -\frac{\omega^2}{v^2} \left(\frac{a^2 b^2}{a^2 + b^2} \right) (a_{\text{eff}})^2, \quad (5.13D)$$

and the focal points are at the vanishing of the left hand side, that is at

$$z_{\pm} = \frac{\omega^2 a^2 b^2}{4v^2 R_0 (N-1-Y)} \left\{ -1 \pm \sqrt{1 + \frac{4v^2}{\omega^2} \frac{R_0^2}{a^2 b^2} (N-1-Y)} \right\}. \quad (5.14D)$$

As we suppose that the power exceeds critical, $N > 1 + Y$. If $N \gg (1+Y)$, then the second term under the radical dominates and we have

$$z_{\pm} \approx \pm \frac{\omega a b}{2v\sqrt{N}} \quad (5.15D)$$

for very large N . At very large factor over critical power, initial divergence of the beam no longer helps.

E. Apertured Beam of Flat Profile

As an extreme example of the effect beam profile has on critical power, let us consider a profile at $z = 0$ given by

$$E(z=0) = E_0 \times \begin{cases} 1 & \text{if } r < a - \delta \\ 0 & \text{if } r > a \end{cases}, \quad E_0 = \text{const.}, \quad (5.1E)$$

and in between $a-\delta$ and a $E(z=0)$ varies linearly. Then, if $\delta \ll a$ we have

$$\begin{cases} \int_{\Sigma} |\nabla E|^2 d\sigma = \frac{2\pi a}{\delta} E_0^2, & \int_{\Sigma} |E|^2 d\sigma = \pi a^2 E_0^2, \\ \int_{\Sigma} r^2 |E|^2 d\sigma = \frac{\pi}{2} a^4 E_0^2, & \int_{\Sigma} |E|^4 d\sigma = \pi a^2 E_0^4, \end{cases} \quad (5.2E)$$

when evaluated at $z = 0$. The critical value of E_0 is given by

$$a^2 E_c^2 = 4a/\delta\gamma, \quad (5.3E)$$

and critical power by

$$P_c = \frac{ac\sqrt{E_0}}{2\gamma\delta}. \quad (5.4E)$$

In principal, it should be possible so to aperture the beam that in the aperture plane ($z = 0$) δ is of the order of the light wavelength λ , thus P_c can in principal be very high for such a beam. Comparing Eqs. (5.4E) and (5.5C), we see that an advantage over the gaussian distribution of several thousand is readily conceivable. This seems really too easy and one thinks immediately that there must be some catch; indeed there is.

One obvious candidate for a phenomenon which will spoil our nice result is the inevitable diffraction pattern produced by our aperture. This will put a ripple on the intensity profile which might drastically reduce critical power. In fact this does not happen as we can very easily demonstrate. Suppose the region ($0 < z < \ell$) to be vacuum and our non-linear medium to start at $z = \ell$. In the region $0 < z < \ell$, $\gamma = 0$ and therefore

$$\int_{\Sigma} |\nabla E|^2 d\sigma \Big|_{z=\ell} = \int_{\Sigma} |\nabla E|^2 d\sigma \Big|_{z=0}. \quad (5.5E)$$

At $z = \ell$, the beam will have diffracted some, therefore we will have

$$\int_{\Sigma} |E|^4 d\sigma \Big|_{z=\ell} < \int_{\Sigma} |E|^4 d\sigma \Big|_{z=0} \quad (5.6E)$$

Furthermore diffraction will have rendered the wave front somewhat divergent. Thus the effect of diffraction is to raise, not to lower, the critical power.

What spoils everything if we try to take advantage of Eq. (5.4E) is not diffraction then, but rather it is instability. Let us choose $a/2\delta$ to give us an advantage of, say, a factor 2000. Other than at the edges, our initial beam is essentially a plane wave, such as studied in Section II. Comparing Eqs. (2.28) and (5.5C), and remembering we are assuming a power 2000 times greater than the latter, we see that we have sufficient power to break up the beam into 500 or more cells. Most of these will not feel edge effects and hence will grow according to the formulas developed in Section II. Thus, unless we can avoid initial perturbations in the critical wavenumber range [see Eq. (2.8)] instability will foil any attempts to buy really large factors of improvement over Eq. (5.5C) by profile shaping.

VI. CONCLUSIONS

We have studied self-focusing in transparent, passive cubic media (i.e. the nonlinear term is cubic in the field strength) and found it to be essentially an instability phenomenon. Thus, when critical power is exceeded, an initial perturbation will grow, concentrating the energy more and more until a point focus, or a set of point foci, is produced. Both our study of plane waves and of finite beams agree in predicting that the critical quantity is total power and not power density. In Eq. (5.8A) we give an expression for the minimum critical power. It is interesting to evaluate it for two cases. If we write Maxwell's displacement vector as

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_2 |\vec{E}|^2 \vec{E} \quad (6.1)$$

then our parameter γ works out to be

$$\gamma = \frac{3}{4} \frac{\omega^2}{c^2} \epsilon_2 \quad (6.2)$$

for plane polarized light. In glass $\epsilon_0 = 2.418$, $\epsilon_2 = 2 \times 10^{-13}$ in e.s.u.* and, for the neodymium glass laser $\omega = 1.78 \times 10^{15}$ rad/sec. Using these numbers Eq. (5.8A) gives

$$(P_c)_{\min} = 4.1 \times 10^6 \text{ watts (Nd - glass)} \quad (6.3)$$

For carbon dioxide at STP we estimate $\epsilon_2 = 0.84 \times 10^{-15}$. The laser frequency is 1.78×10^{14} rad/sec whence

$$(P_c)_{\min} = 6.3 \times 10^{10} \text{ watts (CO}_2\text{)} \quad (6.4)$$

For air at STP minimum critical power will be about 4.5 times higher. These numbers indicate that heating of pellets to the burn point will require exceeding minimum critical power by a very large factor.

In Section V we saw essentially three ways of increasing critical power above the minimum, namely using a very eccentric elliptical beam, introducing divergence into the beam, and using a very flat profile, namely an apertured plane wave. We further saw that this last ruse is likely to be foiled by instabilities. Thus we are left really with only two devices. Going to an elliptical cross section buys the factor $(1 + \rho^2/2\rho)$, where ρ is the aspect ratio of the beam cross-section. A value $\rho = 10$ is not ridiculous; this buys a factor 5 in critical power. Introducing divergence into the beam buys a factor of about $\pi^2 a^4 \rho^2 / \lambda^2 R^2$, where a is the minor semiaxis, ρ the aspect ratio, λ the light wavelength and R the divergence of the beam expressed as the radius of curvature of an isophase front (focal length of a divergent lens, for example). This can easily buy a factor of 1000, say with $\rho = 10$ and $a^2/\lambda R = 1$. By choosing a larger value of a , a couple more powers of 10 are available, so this combination of devices looks quite promising. Finally, we can always operate above critical power provided we manage to keep the self-focal point well outside the nonlinear medium (glass, CO₂, air, etc.). What the above says is what everyone knew before, namely, for high power make the cross-sectional area large, preferably long and narrow in shape, and put in a divergent lens. But now we can say how large an area, how large an aspect ratio, and how strong a lens.

As mentioned above, a cubic nonlinearity of the medium leads inevitably to a point focus, with the attendant infinite intensity, when

*See Appendix B

critical power is exceeded. Physically this is an unacceptable answer. Actually, of course, the physics changes as the singularity is approached, usually owing to ionization of the medium. Thus complete collapse of the beam is averted by the fact that the nonlinearity ultimately departs from cubic, but this does not happen before the field has reached catastrophic proportions.

It will be noticed that we have left out of account the possibility of a time dependent nonlinearity, such as might be produced, for example, by electrostriction. The reason we did this is simple. In the case of electrostriction, the response time is of the order of sound time across the smallest transverse dimension of the beam. If we are interested in truly high powers, such dimensions must be of the order of several centimeters, whence the response time is in the order of a millisecond. For nanosecond pulses, this is truly forever. A faster time dependent mechanism in solids is the Kerr effect. This has a time constant of the order of a tenth of a microsecond. This is still very long compared with pulse lengths, but could conceivably be important in solids with a large Kerr constant (say like that of CS_2) in conjunction with pulses tens of nanoseconds long. In gases the Kerr effect is essentially instantaneous, as are also electronic effects, whereas thermal effects are essentially infinitely slow. Only in liquids is there an important, in fact dominant, effect which is time dependent, namely the Kerr effect. In a liquid composed of small, simple molecules the response time to the Kerr effect is of the order of 10 picoseconds; it becomes longer for large molecules or for a liquid with strong clustering. This suggests that liquids be kept away from the high energy end of a laser chain.

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Appendix A - Appendix A was to have treated the Kerr effect in gases. In the time interval between the original calculations and the typing, a better treatment of this subject has been found, and especially we have found better numerical values for the Kerr effect ϵ_2 of a number of important gases. Accordingly, our original Appendix A is now deleted. The material will be more carefully treated in our next report. We confine ourselves here to the statement that we calculate $\epsilon_2 = 0.84 \times 10^{-15}$ in CO_2 at S.T.P.

APPENDIX B

ON THE VALUE OF ϵ_2

I. ϵ_2 IN SOLIDS

In an earlier report¹ we interpreted the non-linearity in terms of bound electrons moving in an anharmonic potential well, of potential φ per unit mass of the form

$$\varphi = \frac{1}{2}a|\vec{x}|^2 = \frac{1}{4}b|\vec{x}|^4 + , \quad (B.1)$$

where $\vec{x}$ is the electron displacement from its equilibrium position. From the solution of the equations of motion, Ref. 1, Eq. (3.10) and from the fact that the polarization of the medium is simply $4\pi n e \vec{x}$, we readily find Maxwell's displacement vector becomes

$$\vec{D} = \vec{E} + 4\pi n e \left(\frac{e}{m\Delta_1} \right) \left\{ \vec{E} + \frac{b}{4\Delta_1} \left(\frac{e}{m\Delta_1} \right)^2 [2|\vec{E}|^2 \vec{E} + (\vec{E} \cdot \vec{E}) \vec{E}^*] \right\} \quad (B.2)$$

In this equation m, e, n are the electronic mass, charge, and number density, $\Delta_1 = a\omega^2$, and a, b , are as defined in Eq. (B.1). Clearly, for a plane wave, we have simply

$$D = \epsilon_0 E + \epsilon_2 |E|^2 E \quad (B.3)$$

whence

$$\epsilon_0 = 1 + 4\pi n e \left(\frac{e}{m\Delta_1} \right) \text{ or } \left(\frac{e}{m\Delta_1} \right) = \frac{\epsilon_0 - 1}{4\pi n e}, \quad (B.4)$$

$$\epsilon_2 = \frac{b}{a} 4 n e \left(\frac{e}{m\Delta_1} \right)^3 = \frac{b}{a} \frac{(\epsilon_0 - 1)^3}{16\pi^2 n e^2},$$

so long as we are well below resonance, so that

$\Delta_1 \sim a$. Now in a solid φ varies more or less periodically from atom to atom. As a rough model, then, we set

$$\varphi \sim A[1 - \cos(\frac{2\pi x}{\ell})] = \frac{1}{2}A(\frac{2\pi x}{\ell})^2 - \frac{1}{4!}A(\frac{2\pi x}{\ell})^4 + \dots \quad (B.5)$$

Upon identification of this with Eq. (B.1) we find

$$b/a \sim 2\pi^2/3\ell^2 \quad (B.6)$$

If now we suppose that there is essentially one electron per atom (i.e. the most loosely bound electron dominates) we have

$$n = 1/\ell^3 \quad (B.7)$$

and finally

$$\epsilon_2 \sim (\epsilon_0 - 1)^3 \ell^4 / 24e^2 \quad (B.8)$$

As an example we consider glass, which for simplicity we take to be SiO_2 . From the molecular weight, 60.06 and the density 2.3 we readily calculate $n = 6.87 \times 10^{22}$ atoms/(cm)³, whence $\ell = 2.44\text{\AA}$ and $\ell^4/4e^2 = 0.064 \times 10^{-12}$. With a typical ϵ_0 of 2.418 this gives

$$\epsilon_2 \sim 1.8 \times 10^{-13} \text{ e.s.u.} \quad (B.9)$$

We have actually been using 2×10^{-13} in our numerical studies. Measured values⁷ for a number of different laser glasses, using pulses in the 30 picosecond-1 nanosecond range all cluster around 2.2×10^{-13} , with the second figure probably not significant. That our crude model should agree with this in order of magnitude tends to lend confidence in the model. That the agreement should be within experimental error is purely fortuitous.