First and Second Factorization in a Dual
Multiparticle Theory with Nonlinear Trajectories*

8684000

S. Yu and M. Baker

MASTER

Physics Department, University of Washington, Seattle, Washington 98195

and

D. D. Coon

Physics Department, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

ABSTRACT

The N-point tree graphs of a dual resonance theory with nonlinear trajectories are factorized. The factorization procedure applies equally well with the external lines on or off the mass shell. The particle spectrum of the theory is found to have a degeneracy of approximately 6^l for the l^{th} level. Since the trajectory is logarithmic, the asymptotic density of levels grows only as some power of the mass. The techniques used in first factorization are applied to second factorization to arrive at an amplitude for processes involving two external particles with nonzero spins. Two sets of linear dependence relations among the amplitudes obtained by factorization are proven.

*Work supported in part by the U. S. Atomic Energy Commission under Contracts # A.T. (45-1)-1388, A.T. (11-1)-1764 and A.T. (30-1)-3829.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

INTRODUCTION

In this paper and a related one,¹ we derive the following properties for a dual resonance theory with nonlinear trajectories:²

- (i) first and second factorization,
- (ii) generalized Ward-like identities,
- (iii) the twist relation,
- (iv) third factorization and the general three-point vertex.

Some of these results have been obtained using an operator formalism.³ In these papers, direct c-number techniques will be developed, which aside from giving new understanding of the nature of factorization, allow us to obtain results which have not yet been obtained with the operator method.

From these and previous results,² it is clear that the nonlinear theory closely parallels the Veneziano model.⁴ Thus, we have found a larger class of theories with many of the interesting and perhaps desirable properties of the Veneziano model. In addition to the intercept and slope parameters $\alpha(0)$ and $\alpha'(0)$, which characterize the Veneziano model, the nonlinear theory has another parameter q (where $0 < q < 1$). The particles lie on a logarithmic trajectory and q determines the degree of nonlinearity of the trajectory. The $q \rightarrow 1$ limit reduces to the Veneziano model² while the $q \rightarrow 0$ limit gives a nondual ϕ^3 theory.⁵ From a purely mathematical viewpoint, the nonlinear theory is a natural extension of the

Veneziano model because the functions of the theory appear in the mathematical literature⁶ as the natural one-parameter generalizations of the trigonometric, gamma, beta, and hypergeometric functions.

In the present paper, we prove that the N-Point Born terms of the nonlinear theory factorize. The factorization procedure applies equally well with external lines on- or off-the-mass shell.⁷

The particle spectrum of the theory is found to have a degeneracy of approximately 6^L for the L^{th} level. Because the trajectories of the theory are logarithmic, the asymptotic density of states grows only as some power of the mass. The same techniques which are used in first factorization are applied to second factorization to arrive at an amplitude for processes involving two external particles with nonzero spins. We will also give two sets of linear dependence relations among the amplitudes obtained by factorization. The linear dependence relations presented here hold only if a particular one of the external particles is on-the-mass shell and confined to the same trajectory as the internal particles.

FIRST FACTORIZATION

The N-point Born term of the nonlinear theory is given by

$$B_N = \sum_{\text{all } n_{ij}=0}^{\infty} \prod_{i=1}^{N-2} \prod_{j=i+1}^{N-1} \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{k=i+1}^j \prod_{l=j+1}^{N-1} g^{n_{ij} n_{kl}} \quad (1)$$

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

where

$$\sigma_{ij} = a(p_i + p_{i+1} + \dots + p_j)^2 + b \quad (2)$$

and a and b are constants. The quantity $\sigma_{1, N-1}$, which is included in Eq. (1) for notational convenience, is defined to be zero. The coefficients $f(n_{ij})$ of the power series are defined by

$$\begin{aligned} f(0) &= 1 \\ f(n) &= (1-q)(1-q^2)\dots(1-q^n) \quad n \geq 1 \end{aligned} \quad (3)$$

To exhibit the poles in σ_{ij} , we use the summation formula²

$$\sum_{n=0}^{\infty} \frac{\sigma^n}{f(n)} = \prod_{n=0}^{\infty} \frac{1}{(1-\sigma q^n)} = \frac{1}{G(\sigma)} \quad (4)$$

$1/G(\sigma)$ which has poles at $\sigma = q^{-n}$ $n=0, 1, 2, \dots$ is the generalization of the gamma function. B_N is defined so that there exists one factor $q^{n_{ij} n_{kl}}$ for each pair of dual variables σ_{ij} and σ_{kl} . The presence of a factor $q^{n_{ij} n_{kl}}$ prevents simultaneous poles in the variables σ_{ij} and σ_{kl} .

We will prove that the amplitude B_N factorizes at a pole of σ_{1k} (see Fig. 1). First, by means of Eq. (4), we perform the n_{1k} sum in Eq. (1) and obtain a factor

$$1/G(\sigma_{1k} q^{\sum_{i=2}^k \sum_{j=k+1}^{N-1} n_{ij}})$$

The extra q -factors in the argument of G come from the q^{nm} "duality factors" appearing in B_N . Next, we will rearrange

the remaining sums in B_N by defining a new summation index

$$l = \sum_{i=2}^k \sum_{j=k+1}^{N-1} n_{ij} \quad (5)$$

Instead of summing each n_{ij} in Eq. (5) independently from 0 to ∞ , we will sum over the n_{ij} subject to the condition of Eq. (5), with l being some fixed integer. We then sum over l from 0 to ∞ . This rearrangement of summations allows us to write the N-point amplitude in the form

$$B_N = \sum_{l=0}^{\infty} \frac{K_l}{\prod_{i=0}^{\infty} (1 - \sigma_{1k} q^{l+i})} \quad (6)$$

where K_l consists of all remaining sums in B_N .

Factorization is the statement that at any pole of σ_{1k} , the residue may be written in the form

$$\sum_{n=1}^d F_n(p_l) G_n(p_r) \quad (7)$$

In Eq. (7), $F_n(p_l)$ is a function only of momenta p_l of lines which lie to the left of σ_{1k} in Fig. 1, and $G_n(p_r)$ is a function only of momenta p_r of lines lying to the right of σ_{1k} . Moreover, the number of terms d in the sum in Eq. (7) is independent of the number of external lines. From Eq. (6) we see that the residue of the pole at

$\sigma_{1k} = q^{-j}$ will have contributions only from terms with $l \leq j$. Thus, if K_l can be written in the form

of Eq. (7), the residue of the poles can also be written in this way.

We now proceed to prove that K_ℓ does have the form of Eq. (7), which means that B_N is factorizable. In the multiple power series representation of B_N in Eq. (1), there is one summation index n_{ij} associated with each variable σ_{ij} . K_ℓ is a function of all σ_{ij} variables except σ_{1k} . We now separate these σ_{ij} variables and their associated indices into three different sets:

- (i) The left set of variables and the left set of indices are those variables σ_{ij} and indices n_{ij} for which $1 \leq i < j \leq k$ except σ_{1k} and n_{1k} .
- (ii) The right set of variables σ_{ij} and the right set of indices n_{ij} have $i=1, j > k$ or $k < i < j \leq N-1$.
- (iii) The crossed set of variable σ_{ij} and the corresponding indices n_{ij} have $2 \leq i \leq k < j$.

The left and right variables are associated with the left and right "blobs" in Fig. 1. None of the left variables are dual to any of the right variables, and none of the left or right variables are dual to σ_{1k} . Therefore, there are no q^{nm} duality factors involving a left index and a right index in B_N . Consequently, we can separately perform the summations associated with these two sets of variables. Summation over the left set of indices yields a factor

$$B_{k+1}(\sigma_{ij} g^{\sum_{r=i+1}^j n_r}) \quad (8)$$

where

$$n_r = \sum_{s=k+1}^{N-1} n_{rs} \quad (9)$$

The expression (8) means the $(k+1)$ -point tree graph, as defined in Eq. (1), with each σ_{ij} replaced by $\sigma_{ij} g^{\sum_{r=i+1}^j n_r}$. The σ variables depend only on the "left" momenta. The extra q factors are functions of the crossed indices. They came from the q^{nm} factors between the left and crossed indices.

The amplitude given by Eq. (8) is called a satellite Born term.²

Similarly, by summing over the right set of indices, we obtain

$$B_{N-k+1}(\sigma_{ij} g^{\sum_{s=i}^{j-1} m_s}) \quad (10)$$

where

$$m_s = \sum_{r=2}^k n_{rs} \quad (11)$$

Expression (10) is defined analogously to Eq. (8). The variables are those belonging to the right blob. In order for the notation to be consistent, we must be careful to write σ_{ij} , $j > k$, which is a right variable, as $\sigma_{j+1, N}$. σ_{ij} and $\sigma_{j+1, N}$ are equivalent by energy momentum conservation.

Having performed the sums over the left and right indices, we are left with only the crossed indices to sum. We recall that K_l was defined such that the sum over the crossed indices is finite, subject to the condition imposed by Eq. (5). Therefore, K_l is a polynomial in the crossed variables. Equations (8) and (9) indicate that in the satellite term for the left variables, the crossed indices occur only in small groups characterized by $\{n_r\}$. Similarly, in the satellite term for the right variables, we have the crossed indices occurring in groups characterized by $\{m_s\}$. This natural grouping of the crossed indices suggest that we should make a corresponding rearrangement of the sums over the crossed indices. Our new arrangement is as follows. Let n_r , $r=2,3,\dots,k$, and m_s , $s=k+1,k+2,\dots,N-1$ be any set of non-negative integer parameters subject only to the conditions

$$\sum_{r=2}^k n_r = \sum_{s=k+1}^{N-1} m_s = l \quad (12)$$

The condition of Eq. (12) necessarily follows from the definitions of l , n_r , and m_s in Eqs. (5), (9), and (11). Now, we sum over all n_{ij} in the crossed set of indices subject to the constraints

$$\sum_{s=k+1}^{N-1} n_{rs} = n_r, \quad r=2,3,\dots,k, \quad (9)$$

$$\sum_{r=2}^k n_{rs} = m_s, \quad s = k+1, \dots, N-1. \quad (11)$$

We will denote this sum by the symbol

$$\sum_{n_i | n_i, m_j}.$$

Next, we will sum over all n_i and m_j independently subject only to the conditions of Eq. (12). We will denote these two sums by the symbols $\sum_{n_i | \ell}$ and $\sum_{m_j | \ell}$ respectively.

This rearrangement allows us to write K_ℓ in Eq. (6) in the form

$$K_\ell = \sum_{n_i | \ell} \sum_{m_j | \ell} B_{k+1}(\sigma_{ij} g^{\sum_{r=i+1}^j n_r}) B_{N-k+1}(\sigma_{ij} g^{\sum_{s=j+1}^{N-1} m_s}) C, \quad (13)$$

where C is just the expression which remains unsummed. It is a polynomial in the crossed variables, given by

$$C = \sum_{n_i | n_i, m_j} \prod_{i=2}^k \prod_{j=k+1}^{N-1} \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{r=i+1}^k \prod_{s=j+1}^{N-1} g^{n_{ij} n_{rs}} \quad (14)$$

Equation (13) expresses K_ℓ as a sum of terms each of which depends on a set of integers $\{n_r\}$ and $\{m_s\}$. The sum goes over all possible values of the integers n_r and m_s subject to the conditions of Eq. (12). Each term in the sum in turn is a product of the following three factors: a factor B_{k+1} , which is a function only of the left kinematic variables and of the left indices n_r , a factor B_{N-k+1} which is a

function only of the right variables and of the indices m_s , and C , which is a function of both the left and right variables and of both the n_r , and m_s indices.

Since C is a function of the crossed variables, we first write each crossed variable in a factorized form. We define a "6-vector" $P_{i\alpha}$ by

$$P_{i\mu} = \sqrt{2a} (p_1 + p_2 + \dots + p_i)_\mu, \quad 0 \leq \mu \leq 3 \quad (15)$$

$$P_{i4} = 1,$$

$$P_{i5} = a (p_1 + p_2 + \dots + p_i)^2 + \frac{b}{2}$$

and another 6 vector $P'_{i\alpha}$ by

$$P'_{i\mu} = -P_{i\mu}, \quad 0 \leq \mu \leq 3 \quad (16)$$

$$P'_{i4} = P_{i4},$$

$$P'_{i5} = P_{i5}.$$

If we now introduce a six-dimensional metric $g_{\alpha\beta}$ by

$$g_{\alpha\beta} = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}, \quad (17)$$

then

$$\sigma_{ij} = P_{i-1\alpha} P_j'^{\alpha}, \quad (18)$$

where the repeated index implies summation over the 6-vector indices.

C of Eq.(14) is a polynomial in the crossed variables. Each σ_{ij} in C may be written in the form of Eq.(18). $P_{i-1\alpha}$ is a "left" variable and $P_j'^\alpha$ is a "right" variable. Therefore, Eq.(18) allows us to write each term in C in a factorized form. We now insert the expression (18) for σ_{ij} into Eq.(14) for C. This yields a factorized form for the polynomial C in terms of left variables P_i^α and right variables $P_j'^\alpha$ etc. However, the number of terms in the polynomial increases as the number of external lines increases. We will show that these terms in the polynomial can be regrouped nicely, so that C can always be written in a factorized form independent of the number of external lines. First, let us write down our result.

We define a 6^l by 6^l matrix which depends only on the parameter q,

$$N_{\beta_1\beta_2\cdots\beta_l}^{\alpha_1\alpha_2\cdots\alpha_l} = (1-q)^l \sum_P q^{T_P} \delta_{\beta_{P(1)}}^{\alpha_1} \delta_{\beta_{P(2)}}^{\alpha_2} \cdots \delta_{\beta_{P(l)}}^{\alpha_l} \quad (19)$$

where the indices α, β run from 0 to 5. δ_β^α is the usual kronecka delta symbol. The sum $\sum_P$ is taken over all

$l!$ permutations of the integers 1 to l and T_P is the number of transpositions associated with the permutation P.

Then C has the factorized form

$$C = L_{\alpha_1\alpha_2\cdots\alpha_l} (P_{k-1} n_k; P_{k-2} n_{k-1}; \cdots; P_1 n_2) \quad (20)$$

$$\times N_{\beta_1\beta_2\cdots\beta_l}^{\alpha_1\alpha_2\cdots\alpha_l} L_{\beta_1\beta_2\cdots\beta_l} (P'_{k+1} m_{k+1}; P'_{k+2} m_{k+2}; \cdots; P'_{N-1} m_{N-1})$$

where

$$L_{\alpha_1 \alpha_2 \dots \alpha_\ell} (P_{k-1} n_k; P_{k-2} n_{k-1}; \dots; P_1 n_2) \quad (21)$$

$$= \frac{(P_{k-1} \alpha_1 P_{k-1} \alpha_2 \dots P_{k-1} \alpha_{n_k})}{f(n_k)} \frac{(P_{k-2} \alpha_{n_k+1} \dots P_{k-2} \alpha_{n_k+n_{k-1}})}{f(n_{k-1})} \dots \frac{(P_1 \alpha_{\ell-n_2+1} \dots P_1 \alpha_\ell)}{f(n_2)}$$

and $L_{\beta_1 \beta_2 \dots \beta_\ell} (P'_{k+1} m_{k+1}; P'_{k+2} m_{k+2}; \dots; P'_{N-1} m_{N-1})$ is

the same function of its arguments and indices as in Eq. (21).

The repeated indices α and β are summed over.

Eq. (20) has the following content: The Left-hand side, as defined by Eq. (13), is a function of both left and right kinematic variables, and of both the left indices n_i and the right indices m_j . In the right-hand side, we have a sum over $6^\ell \times 6^\ell$ terms (corresponding to the sum over α and β indices) each of which is the product of the following three factors:

$$N_{\beta_1 \beta_2 \dots \beta_\ell}^{\alpha_1 \alpha_2 \dots \alpha_\ell}, \quad \text{which}$$

is independent of kinematic variables and indices n_i and m_j ,

$$L_{\alpha_1 \alpha_2 \dots \alpha_\ell} (P_{k-1} n_k; \dots; P_1 n_2), \quad \text{which depends only on the left 6 vectors } P_i \alpha \quad \text{and indices } n_i, \text{ and } L_{\beta_1 \beta_2 \dots \beta_\ell} (P'_{k+1} m_{k+1}; \dots; P'_{N-1} m_{N-1})$$

which depends only on the right 6 vectors $P'_j \beta$ and indices m_j . If we substitute the expression C in Eq. (20)

into Eq. (13), we can rewrite each term in the sum as the

product of five objects, B_{k+1} , B_{N-k+1} , $L_{\alpha_1 \alpha_2 \dots \alpha_\ell}$,

$$L_{\beta_1 \beta_2 \dots \beta_\ell}, \quad \text{and} \quad N_{\beta_1 \beta_2 \dots \beta_\ell}^{\alpha_1 \alpha_2 \dots \alpha_\ell}. \quad \text{Now in Eq. (13),}$$

we have a sum over n_i and m_j . Since B_{k+1} and $L_{\alpha_1 \alpha_2 \dots \alpha_\ell}$

depend only on n_i and B_{N-k+1} and $L^{\beta_1 \beta_2 \dots \beta_l}$ depend only on m_j , we can carry out the sum over the two sets of indices separately. This yields

$$K_l = B_{k+1 \alpha_1 \alpha_2 \dots \alpha_l} N_{\beta_1 \beta_2 \dots \beta_l}^{\alpha_1 \alpha_2 \dots \alpha_l} B_{N-k+1}^{\beta_1 \beta_2 \dots \beta_l} \quad (22)$$

where

$$B_{k+1 \alpha_1 \alpha_2 \dots \alpha_l} = \sum_{n_i | l} B_{k+1}(\sigma_{ij} g^{\sum_{r=i+1}^j n_r}) L_{\alpha_1 \dots \alpha_l}(\bar{p}_{k-1 n_k}; \bar{p}_{n_2}) \quad (23)$$

and

$$B_{N-k+1}^{\beta_1 \beta_2 \dots \beta_l} = \sum_{m_j | l} B_{N-k+1}(\sigma_{ij} g^{\sum_{s=i}^{j-1} m_s}) L^{\beta_1 \dots \beta_l}(\bar{p}'_{k+1} m_{k+1}; \dots; \bar{p}'_{N-1} m_{N-1}) \quad (23')$$

where $L_{\alpha_1 \alpha_2 \dots \alpha_l}$ is defined in Eq. (21). Repeated indices are summed over. As the number of external lines increase, the expressions $B_{k+1 \alpha_1 \alpha_2 \dots \alpha_l}$ and $B_{N-k+1}^{\beta_1 \beta_2 \dots \beta_l}$ in Eq. (23) change accordingly. However, the number of terms in the sum of Eq. (22) remains unchanged. Thus K_l factorizes independently of the number of external lines. K_l of Eq. (22) when substituted into Eq. (6), gives the factorized form for B_N . $B_{k+1}^{\alpha_1 \alpha_2 \dots \alpha_l}$ cf Eq. (23) is the scattering amplitude for $k+1$ particles where one particle is in the " l^{th} excited state," and $N_{\beta_1 \beta_2 \dots \beta_l}^{\alpha_1 \alpha_2 \dots \alpha_l}$ may be regarded as the numerator of the propagator for a particle in the " l^{th} excited state."

Since α_i can take on the values 0 to 5, we have 6^l amplitudes $B_{k+1} \alpha_1 \alpha_2 \dots \alpha_l$. From Eqs. (23) and (21), we see that the amplitudes possess no obvious symmetry properties under permutations of their indices. Thus, most of the 6^l components could be linearly independent. From the discussion following Eq. (6), we argued that the residue of the pole at $\sigma_{jk} = g^{-j}$ will come only from the terms K_l with $l \leq j$. Thus, the degeneracy at the pole $\sigma = g^{-j}$ is at most

$$\sum_{m=0}^j 6^m \sim \left(\frac{6}{5}\right) 6^j$$

We have found two sets of linear dependence relations connecting these states which we will prove in the last section of this paper. However, the total number of linear dependence relations is small compared to the number of states.

To get the degeneracy as a function of the energy, we note that $l = -\frac{\ln \sigma}{\ln g}$. Therefore, $N(s) \sim 6^l = (as+b)^{-\frac{\ln 6}{\ln g}}$.

Thus, the density of states grows like a power of s .

Equation (20) gives C in a factorized form. For a direct proof of the factorization of C , it is most convenient to give the following alternative form for C .

$$C = L_{\alpha_1 \alpha_2 \dots \alpha_l} R^{\alpha_1 \alpha_2 \dots \alpha_l}$$

(24)

where $L \alpha_1 \alpha_2 \dots \alpha_\ell$ is given in Eq. (21) and

$$R^{\alpha_1 \alpha_2 \dots \alpha_\ell} = \sum_{dP} g^T P P'^{\alpha_1}_{P(k+1)} P'^{\alpha_2}_{P(k+1)} \dots P'^{\alpha_{m_{k+1}}}_{P(k+1)} \quad (25)$$

$$\times P'^{\alpha_{m_{k+1}+1}}_{P(k+2)} \dots P'^{\alpha_{m_{k+1}+m_{k+2}}}_{P(k+2)} \dots P'^{\alpha_{\ell-m_{N-1}+1}}_{P(N-1)} \dots P'^{\alpha_\ell}_{P(N-1)}$$

The sum in (25) is performed over all distinct permutations of the ℓ 6-vectors. Among the ℓ 6-vectors, we have m_{k+1} of the type P'_{k+1} , m_{k+2} of the type P'_{k+2} etc.⁸ and T_P , the exponent of the q coefficient, is again the number of transpositions associated with the permutation P . In the rest of this section, we will concentrate on the direct proof of Eq. (24). The equivalence of the two factorized forms Eqs. (20) and (24) will be given at the final step of this proof.

We will first prove Eq. (24) for the special case where $N-k > \ell$, and for which the parameters n_i and m_j take on the following special values:

$$n_2 = \ell, \quad n_i = 0 \quad \text{for } i = 3, 4, \dots, k \quad \text{and}$$

$$m_j = 1 \quad \text{for } j = k+1, k+2, \dots, k+\ell$$

$$m_j = 0 \quad \text{for } j = k+\ell+1, \dots, N-1$$

Then by Eqs. (14), (21) and (25), we can write Eq. (24) in the form

$$\frac{\sigma_{2k+1} \sigma_{2k+2} \cdots \sigma_{2k+l}}{[f(1)]^l} = \frac{P_{1\alpha_1} P_{1\alpha_2} \cdots P_{1\alpha_l}}{f(l)} \left\{ \sum_P g^T P_{P(k+1)}^{\alpha_1} \cdots P_{P(k+l)}^{\alpha_l} \right\} \quad (26)$$

By Eq. (18), we can write the right-hand side of Eq. (26) as

$$\left\{ \sigma_{2k+1} \sigma_{2k+2} \cdots \sigma_{2k+l} \right\} \frac{1}{f(l)} \sum_P g^T P$$

Thus Eq. (26) follows if we can prove the following Lemma:

Lemma 1:

$$\sum_P g^T P = \frac{f(l)}{[f(1)]^l}$$

(all primed vectors distinct)

Proof:

If $l = 1$, we have only one term in R^α . The coefficient is $g^T = 1 = f(1)/f(1)$.

Suppose the formula is true for $l-1$. We wish to prove it for l .

We can generate all possible permutations of the primed vectors in the following way. We start with the term corresponding to the identity permutation $P_{k+1}^{\alpha_1} P_{k+2}^{\alpha_2} \cdots P_{k+l}^{\alpha_l}$, which, for notational convenience, we will denote by

$$k+1 \ k+2 \ \cdots \ k+l-1 \ k+l.$$

We first generate $l-1$ permutations by interchanging each of the primed vectors with the last vector. Thus, we have the

following permutations with the associate weight factors :

<u>Permutation</u>	<u>Weight Factor</u>
$k+1 \ k+2 \ \dots \ k+l-1 \ k+l$	1
$k+1 \dots k+j-1 \ k+j+1 \dots k+l-1 \ k+l \ k+j$	q^{l-j}
$k+2 \ k+3 \ \dots \ k+l-1 \ k+l \ k+1$	q^{l-1}

Starting with each of these l permutations we now take all possible permutations of the first $l-1$ vectors. By assumption, the total weight factor associated with the permutation of the first $l-1$ vectors is $\frac{f(l-1)}{[f(1)]^{l-1}}$. Therefore, the total weight factor associated with the permutation of l vectors is

$$\frac{f(l-1)}{[f(1)]^{l-1}} (1 + q + \dots + q^{l-1}) = \frac{f(l-1)}{[f(1)]^{l-1}} \frac{(1-q^l)}{(1-q)} \quad (27)$$

From the definition of $f(l)$ in Eq. (3), we see that the right-hand side of Eq. (27) is exactly $f(l)/[f(1)]^l$. This proves Lemma 1.

Next, we wish to prove Eq. (24) for the case $n_2 = l$, $n_i = 0$ for $i \neq 2$, and m_j arbitrary (subject only to the condition $\sum m_j = l$). In this case, Eq. (24) reduces to

$$\frac{\sigma_{2k+1}^{m_{k+1}} \sigma_{2k+2}^{m_{k+2}} \dots \sigma_{2N-1}^{m_{N-1}}}{f(m_{k+1}) f(m_{k+2}) \dots f(m_{N-1})} = \frac{P_{1\alpha_1} P_{1\alpha_2} \dots P_{1\alpha_l}}{f(l)} R^{\alpha_1 \alpha_2 \dots \alpha_l}, \quad (28)$$

Eq.(28) then follows from Lemma 1.

Lemma 1

$$\sum_{\mathbf{dP}} \frac{q^{T_P}}{\delta} = \frac{f(l)}{\prod_{j=k+1}^{N-1} f(m_j)}$$

To prove Lemma 1, we first treat the δ -vectors in $R^{\alpha_1, \alpha_2, \dots, \alpha_l}$ as though they are all distinct, and take all possible permutations. The sum of the coefficients is $\frac{f(l)}{[f(1)]^l}$.

However, in treating m_j identical primed vectors as distinct, we will have overcounted each permutation $m_j!$ times. The associated weight factor is $f(m_j)/[f(1)]^{m_j}$. Thus to count correctly, we have to make up for the overcounting by multiplying by $[f(1)]^{m_j}/f(m_j)$. Doing this for every j , we have as our final answer

$$\frac{f(l)}{[f(1)]^l} \prod_j \frac{[f(1)]^{m_j}}{f(m_j)} = \frac{f(l)}{\prod_{j=k+1}^{N-1} f(m_j)}.$$

We note that Lemma 1 is just the derivation of a "q-generalized" multinomial distribution. Indeed, as $q \rightarrow 1$, Lemma 1 gives exactly the usual multinomial coefficient $\frac{l!}{\prod_j m_j!}$.

To prove Eq. (24) in general, we first note that C , as defined in Eq. (14), is a sum of terms, each of which is characterized by a fixed set of indices $\{n_{ij}\}$. From the definition of $L_{\alpha_1, \alpha_2, \dots, \alpha_l}$ in Eq. (21), P_{i-1} carries indices $\alpha_{n_k + n_{k-1} + \dots + n_{i+1} + 1}$ to $\alpha_{n_k + n_{k-1} + \dots + n_{i+1} + n_i}$.

A term in $R^{\alpha_1 \alpha_2 \dots \alpha_q}$ when dotted into $L^{\alpha_1 \alpha_2 \dots \alpha_q}$ will give $g^{T_P} \prod_{i,j} \sigma_{ij}^{n_{ij}}$ if the permutation corresponds to n_{ij} P_j' vectors carrying indices that will match the P_{i-1} vectors for every i and j . T_P is the number of transpositions associated with the permutation. The problem then is to compute g^{T_P} and to sum over the coefficients g^{T_P} of all terms in $R^{\alpha_1 \alpha_2 \dots \alpha_q}$ that give rise to the product of σ_{ij}' 's raised to the same powers n_{ij} .

We will now compute the coefficient $g^{T_{\bar{P}}}$ of a particular permutation $\bar{P}$ which is depicted in Fig. 2. In the picture, the integers $k+1, k+1, \dots, N-1$ in the middle row correspond to the momenta $P_{k+1}'^{\alpha_1}, P_{k+2}'^{\alpha_2}, \dots, P_{N-1}'^{\alpha_q}$. This particular permutation has the feature that all P' momenta in the subgroups of size n_i appear in increasing order.

The integers n_{ij} in the first row represent the number of times the vector P_j' appear in the n_i subgroup. To compute $g^{T_{\bar{P}}}$ we simply have to count the number of transpositions required to bring the configuration of Fig. 2 back to its unpermuted configuration represented by Fig. 3. Thus, we start by bringing the vectors P_{k+1}' in the " n_{k-1} " compartment to the front. The number of transpositions is equal to $n_{k-1} (n_{k+1} + \sum_{s=k+2}^{N-1} n_{ks})$. If we keep repeating this process with all the primed vectors, we can quickly convince ourselves that the total number of transpositions

is given by

$$T_P = \sum_{i=2}^k \sum_{j=k+1}^{N-1} \sum_{r=i+1}^k \sum_{s=j+1}^{N-1} n_{ij} n_{rs}$$

A term in $R^{\alpha_1 \alpha_2 \dots \alpha_\ell}$ will contribute to $\prod_{i,j} \sigma_{ij}^{n_{ij}}$

if it can be represented by a configuration like Fig. 2, except that within the " n_i " compartment, the primed vectors are permuted.

Let T_{P_i} be the number of transpositions associated with the permutation within the " n_i " compartment. It is easy to see that the total number of transpositions associated with a term in $R^{\alpha_1 \alpha_2 \dots \alpha_\ell}$ may then be considered as a sum of several contributions, first, a contribution T_{P_i} from each compartment, and secondly, a contribution T_P from bringing the configuration of Fig. 2 to Fig. 3.

Therefore, the q factor may be decomposed as

$$q^{T_P} = q^{T_P} \prod_i q^{T_{P_i}}$$

Now, summing over all contributing configurations, we obtain

$$\sum q^{T_P} = q^{T_P} \prod_i \left(\sum_{dP_i} q^{T_{P_i}} \right)$$

Now, we apply Lemma 1 to the permutations within each compartment, and we have

$$\sum q^{T_P} = q^{T_P} \prod_i \frac{f(n_i)}{\prod_j f(n_{ij})}$$

Recalling that the "left" tensor $L_{\alpha_1 \alpha_2 \dots \alpha_\ell}$ has associated with it factors of $f(n_i)$, we are finally led to the coefficient of $\prod_{i,j} \sigma_{ij}^{n_{ij}}$ given by

$$\prod_{i=2}^k \frac{1}{f(n_i)} \sum q^{T_P} = \frac{q^{T_P}}{\prod_{i,j} f(n_{ij})} = \prod_{i=2}^k \prod_{j=k+1}^{N-1} \frac{1}{f(n_{ij})} \prod_{r=i+1}^k \prod_{s=j+1}^{N-1} q^{n_{ij} n_{rs}}$$

This is precisely the coefficient of the term $\prod_{i,j} \sigma_{ij}^{n_{ij}}$ in Eq. (14). Thus, we have proved Eq. (24).

To prove the equivalence of Eqs. (24) and (20), we need only to introduce the identity

$$R^{\alpha_1 \alpha_2 \dots \alpha_\ell} = N_{\beta_1 \beta_2 \dots \beta_\ell}^{\alpha_1 \alpha_2 \dots \alpha_\ell} L_{\beta_1 \beta_2 \dots \beta_\ell} (P'_{k+1} m_{k+1}, \dots; P'_{N-1} m_{N-1}) \quad (29)$$

At this point, the proof of Eq. (29) is trivial. $N_{\beta_1 \beta_2 \dots \beta_\ell}^{\alpha_1 \alpha_2 \dots \alpha_\ell}$, when acting on the primed vectors in $L_{\beta_1 \beta_2 \dots \beta_\ell}$, produces all possible permutations of the indices. It is easy to see that the permutation of indices is equivalent to the permutation of vectors. $R^{\alpha_1 \alpha_2 \dots \alpha_\ell}$, as defined in Eq. (25), includes only the distinct permutations of vectors. As we have learned in the proof of Lemma 1, the difference can be accounted for by multiplying by the expression $\frac{[f(1)]^\ell}{\prod_j f(m_j)}$. The factor $[f(1)]^\ell$ has been incorporated into the definition of $N_{\beta_1 \beta_2 \dots \beta_\ell}^{\alpha_1 \alpha_2 \dots \alpha_\ell}$, and the remaining factors are associated with $L_{\beta_1 \beta_2 \dots \beta_\ell}$.

SECOND FACTORIZATION

From factorization, we obtained the scattering amplitude

$B_N \alpha_1 \alpha_2 \dots \alpha_\ell$ in which one of the particles is in a high spin

state. From the definition Eq. (23) of $B_N \alpha_1 \alpha_2 \dots \alpha_\ell$, we see that this amplitude is manifestly dual, i.e., $B_N \alpha_1 \alpha_2 \dots \alpha_\ell$ may be written as a sum of poles corresponding to any tree graph

with one external excited particle. However, the multiperipheral graph of Figure 4 plays a special role since the 6-vectors which appear in $B_{N\alpha_1\cdots\alpha_l}$ correspond to the internal lines of this graph. In this section, we will consider factorization at a pole σ_{1k} of this multiperipheral graph. As a result, we will arrive at the N-point amplitude with two adjacent nonzero external particles. This procedure will be called second factorization.

The proof of second factorization is similar to that of first factorization. First, we write down the N-point amplitude involving one excited particle of Fig. 4, given by⁹

$$B_N^{\alpha_1\alpha_2\cdots\alpha_l} = \sum_{n_{i|l}} B_N(\sigma_{ij} g_{r=i}^{\sum_{r=i}^{j-1} n_r}) \frac{(P_1^\alpha)^{n_1}}{f(n_1)} \frac{(P_2^\alpha)^{n_2}}{f(n_2)} \cdots \frac{(P_{N-2}^\alpha)^{n_{N-2}}}{f(n_{N-2})}, \quad (30)$$

where we have introduced the notation

$$(P_1^\alpha)^{n_1} (P_2^\alpha)^{n_2} \cdots (P_{N-2}^\alpha)^{n_{N-2}} = (P_1^{\alpha_1} \cdots P_1^{\alpha_{n_1}}) (P_2^{\alpha_{n_1+1}} \cdots P_2^{\alpha_{n_1+n_2}}) \cdots (P_{N-2}^{\alpha_{l-n_{N-2}+1}} \cdots P_{N-2}^{\alpha_l}) \quad (31)$$

We exhibit the pole in σ_{1k} by summing over n_{1k} . This gives

$$B_N^{\alpha_1\alpha_2\cdots\alpha_l} = \sum_{l'=0}^{\infty} \frac{K_{l'}^{\alpha_1\alpha_2\cdots\alpha_l}}{G(\sigma_{1k} g^{l'})}, \quad (32)$$

which is analogous to Eq.(6). Here, $K_{l'}^{\alpha_1\alpha_2\cdots\alpha_l}$ is a function of σ_{ij} variables and 6-vectors. In the special case of $l=0$ the tensor $K_{l'}^{\alpha_1\alpha_2\cdots\alpha_l}$ reduces to the scalar $K_{l'}$ of Eq.(6).

We will now write down the final factorized expression for $K_{l'}^{\alpha_1\alpha_2\cdots\alpha_l}$:

$$K_{l'}^{\alpha_1\alpha_2\cdots\alpha_l} = B_{k+1}^{\beta_1\beta_2\cdots\beta_l} N_{\delta_1\delta_2\cdots\delta_{l'}}^{\beta_1\beta_2\cdots\beta_{l'}} \delta_{\delta_1\delta_2\cdots\delta_{l'}}^{\alpha_1\alpha_2\cdots\alpha_l} B_{N-k+1} \quad (33)$$

where $B_{k+1, \beta_1 \beta_2 \dots \beta_{l'}}$ is again the amplitude with one excited

particle associated with the left hand side of Fig. 5, and

$B_{N-k+1, \gamma_1 \gamma_2 \dots \gamma_{l'}, \alpha_1 \alpha_2 \dots \alpha_l}$ is the amplitude with two spinning

particles, associated with the right hand side of Fig. 5, given by

$$\begin{aligned}
 B_{N-k+1, \gamma_1 \gamma_2 \dots \gamma_{l'}, \alpha_1 \alpha_2 \dots \alpha_l} &= \sum_{d=0}^{\min(l, l')} \sum_{n_i | l-d} \sum_{d_j | l'-d} B_{N-k+1}(\sigma_{ij} g^{k_{ij}}) \\
 &\times N^{-1} \gamma_1 \gamma_2 \dots \gamma_d, \alpha_1 \alpha_2 \dots \alpha_d L^{\alpha_{d+1} \alpha_{d+2} \dots \alpha_l} (P_{k, n_k} \dots P_{N-2, n_{N-2}}) \\
 &\times L^{\gamma_{d+1} \gamma_{d+2} \dots \gamma_{l'}} (P'_{N-1, d_{N-1}} \dots P'_{k+1, d_{k+1}}) \prod_{i=k}^{N-2} g^{n_i (\sum_{s=i+1}^{N-1} d_s)},
 \end{aligned} \tag{34}$$

where $L^{\alpha_1 \alpha_2 \dots \alpha_l}$ is defined in Eq. (21). The matrix $N^{-1} \gamma_1 \gamma_2 \dots \gamma_d, \alpha_1 \alpha_2 \dots \alpha_d$, the inverse of the matrix N , introduced in Eq. (19) is defined by

$$N^{\alpha_1 \alpha_2 \dots \alpha_d} N^{-1}_{\gamma_1 \gamma_2 \dots \gamma_d, \beta_1 \beta_2 \dots \beta_d} = \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2} \dots \delta^{\alpha_d}_{\beta_d} \tag{35}$$

and K_{ij} , the indices in the satellite Born term, are given by

$$\begin{aligned}
 K_{ij} &= \sum_{s=i}^{j-1} (n_s + d_s) \quad \text{for } k < i < j < N \\
 K_{ij} &= K_{j+1, N} = \sum_{r=k}^{j-1} n_r + \sum_{s=j+1}^{N-1} d_s + d
 \end{aligned} \tag{36}$$

For $l' = 0$, $B^{\delta_1 \delta_2 \dots \delta_{l'}; \alpha_1 \alpha_2 \dots \alpha_l} \rightarrow B^{\alpha_1 \alpha_2 \dots \alpha_l}$,
the amplitude with one spinning particle.

We now proceed with the proof of Eq. (33). As in the previous section, we separate the σ variables in $K_{l'}^{\alpha_1 \alpha_2 \dots \alpha_l}$ into three sets: the left set, the right set, and the crossed set. We wish to make a similar grouping of 6 vectors. The set of 6-vectors $\{P_{i\alpha} | i = k, k+1, \dots, N-2\}$ appearing in Eq. (30) involve momenta of the internal lines of the right hand graph of Fig. 5, and carry the spin indices of particle N, which is an external line of the right hand graph. Therefore, the set of vectors can be categorized as right variables. The vectors $\{P_{i\alpha} | i = 1, 2, \dots, k-1\}$ are slightly more tricky. They involve momenta of the internal lines of the left hand graph in Fig. 5, but they carry spin indices of the right hand line N. Since these vectors do not belong exclusively to the right hand graph or to the left hand graph, they will, for now, be considered as "crossed" variables.

As in the previous section, we can sum over the indices associated with the left set of variables and the indices associated with the right set of variables independently. Summing over the left group, we obtain

$$B_{k+1} (\sigma_{ij} g^{\sum_{r=k+1}^j C_r}) \quad (37)$$

where

$$C_r = \sum_{s=k+1}^{N-1} n_{rs} + n_r \quad (38)$$

Equations (37) and (38) are analogous to Eqs. (8) and (9) of Section II. n_r enters Eq. (38) because the satellite $B_N(\sigma_{ij} g^{\sum n_r})$ in Eq. (30) has q coefficients of the form $q^{n_r n_{ij}}$. The right group, in addition to the satellite Born term, will consist of 6-vectors P_i^α , $i = k, k+1, \dots, N-2$

Similarly, we can sum over the right group to give

$$\begin{aligned} \bar{B}_{N-k+1}^{\alpha_{d+1} \alpha_{d+2} \dots \alpha_d} &= \sum_{n_i | l-d} B_{N-k+1}(\sigma_{ij} g^{K_{ij}}) \frac{(P_k^\alpha)^{n_k}}{f(n_k)} \frac{(P_{k+1}^\alpha)^{n_{k+1}}}{f(n_{k+1})} \dots \frac{(P_{N-2}^\alpha)^{n_{N-2}}}{f(n_{N-2})} \\ &\times \prod_{i=k}^{N-2} g^{n_i} \left(\sum_{s=i+1}^{N-1} d_s \right) \end{aligned} \quad (39)$$

where

$$d_s = \sum_{r=2}^k n_{rs} \quad k+1 \leq s \leq N-1,$$

and

$$\begin{aligned} K_{ij} &= \sum_{s=i}^{j-1} (n_s + d_s) \quad i, j > k \\ K_{1j} &= \sum_{s=k}^{j-1} n_s + \sum_{s=j+1}^{N-1} d_s + d \end{aligned} \quad (40)$$

With this regrouping we can now rewrite $K_{l'}^{\alpha_1 \alpha_2 \dots \alpha_d}$ in a form which is analogous to Eq. (12)

$$K_{l'}^{\alpha_1 \alpha_2 \dots \alpha_d} = \sum_{c_i | l'} \sum_{d_i | l'-d}^{\min(l, l')} \sum_{d=0}^{\sum_{r=i+1}^j c_r} B_{k+1}(\sigma_{ij} g^{\sum_{r=i+1}^j c_r}) C^{\alpha_1 \alpha_2 \dots \alpha_d} \bar{B}_{N-k+1}^{\alpha_{d+1} \dots \alpha_d} \quad (41)$$

where $C^{\alpha_1 \alpha_2 \dots \alpha_d}$ is the object which contains all the crossed variables.

$$C^{\alpha_1, \alpha_2, \dots, \alpha_d} = \sum_{n_{ij}, n_i | c_i, d_j} \frac{(P_1^\alpha)^{n_1}}{f(n_1)} \frac{(P_2^\alpha)^{n_2}}{f(n_2)} \dots \frac{(P_{k-1}^\alpha)^{n_{k-1}}}{f(n_{k-1})} \\ \times \prod_{\lambda=2}^k \prod_{j=k+1}^{N-1} \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{r=\lambda+1}^k \prod_{s=j+1}^{N-1} g^{n_{ij}(n_{rs} + n_{r-1})}, \quad (42)$$

where $\sum_{n_{ij}, n_i | c_i, d_j}$ implies summation over all crossed indices n_{ij} and indices n_i $i=1, 2, \dots, k$ subject to the conditions (38) and (40).

In the case $d = 0$ Eq. (42) reduces to the C of Eq. (14).

The above regrouping of variables, though notationally complex, is nevertheless straightforward. As in first factorization, we now wish to factorize $C^{\alpha_1, \alpha_2, \dots, \alpha_d}$. The σ_{ij} variables in $C^{\alpha_1, \dots, \alpha_d}$ may again be decomposed into a scalar product of 6-vectors as in Eq. (18). In order to separate the spin indices of the N th particle from the left 6-momenta P_i , we simply write

$$P_{i\alpha} = P_{i\beta} g^{\beta\alpha} \quad (43)$$

where $g_{\alpha\beta}$ is just the 6-dimensional metric defined in the previous section.

Having made this trivial separation of variables, we now note that as far as the q coefficients are concerned, Eq. (42) is structurally analogous to Eq. (14). Consider the case of factorization at σ_{ik} of the $N+1$ point Born term with all spin zero external particles. Equation (14) gives

$$C = \sum_{n_{ij} | c_i, d_j} \prod_{\lambda=2}^k \prod_{j=k+1}^N \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{r=\lambda+1}^k \prod_{s=j+1}^{(N-1)+1} g^{n_{ij} n_{rs}} \quad (44)$$

It is quite easy to see that if we take Eq. (42), and make the substitutions

$$\begin{aligned} P_{i-1}^\alpha &\Rightarrow \sigma_{iN} \\ n_{i-1} &\Rightarrow n_{iN} \end{aligned} \quad (45)$$

that Eq. (42) goes into Eq. (44).

But we have already learned that Eq. (44) may be factorized in the form⁹

$$C = \tilde{L}_{\beta_1 \beta_2 \dots \beta_{l'}} \tilde{R}^{\beta_1 \beta_2 \dots \beta_{l'}} \quad (46)$$

where

$$\tilde{L}_{\beta_1 \beta_2 \dots \beta_{l'}} = \frac{(P_{1\beta})^{c_2}}{f(c_2)} \frac{(P_{2\beta})^{c_3}}{f(c_3)} \dots \frac{(P_{l-1\beta})^{c_k}}{f(c_k)} \quad (47)$$

and

$$\tilde{R}^{\beta_1 \beta_2 \dots \beta_{l'}} = \sum_{dP} g^{TP} (P'_{P(N)}^\beta)^{d_N} (P'_{P(N-1)}^\beta)^{d_{N-1}} \dots (P'_{P(k+1)}^\beta)^{d_{k+1}} \quad (48)$$

From (45), (43), and (18), it is then obvious that we can get the factorized form of $C^{\alpha_1 \dots \alpha_d}$ by replacing $(P'_{P(N)}^\beta)^{d_N}$ in Eq. (48) by

$$(g^{\alpha\beta})^d. \quad \text{Therefore, we have}$$

$$C^{\alpha_1 \dots \alpha_d} = \tilde{L}_{\beta_1 \beta_2 \dots \beta_{l'}} \tilde{R}^{\beta_1 \beta_2 \dots \beta_{l'}; \alpha_1 \alpha_2 \dots \alpha_d} \quad (49)$$

where

$$\begin{aligned} \tilde{R}^{\beta_1 \dots \beta_{l'}; \alpha_1 \dots \alpha_d} &= \sum_{dP} g^{TP} (g^{\beta_1 \alpha_{P(1)}} g^{\beta_2 \alpha_{P(2)}} \dots g^{\beta_d \alpha_{P(d)}}) \\ &\times (P'_{P(N-1)}^{\beta_{d+1}} \dots P'_{P(N-1)}^{\beta_{d+d_{N-1}}}) \dots (P'_{P(k+1)}^{\beta_{(d-d_{k+1}+1)}} \dots P'_{P(k+1)}^{\beta_{l'}}) \end{aligned} \quad (50)$$

Substituting this factorized form of $C^{\alpha_1 \dots \alpha_d}$ in Eq. (49) back in Eq. (41), we obtain $K^{\alpha_1 \alpha_2 \dots \alpha_d}$ in factorized form.

To get the symmetrized factorized form of $C^{\alpha_1 \alpha_2 \dots \alpha_d}$, we will write $\bar{R}$ in a form analogous to Eq. (29). We introduce a tensor $N^{-1} \alpha_1 \alpha_2 \dots \alpha_d$ defined by

$$N^{\alpha_1 \alpha_2 \dots \alpha_d}_{\delta_1 \delta_2 \dots \delta_d} N^{-1}_{\beta_1 \beta_2 \dots \beta_d} = g^{\alpha_1}_{\beta_1} g^{\alpha_2}_{\beta_2} \dots g^{\alpha_d}_{\beta_d} \quad (51)$$

$N^{\alpha_1 \alpha_2 \dots \alpha_d}_{\delta_1 \delta_2 \dots \delta_d}$, when acting on vectors, generates all permutations of the vectors. $N^{-1}_{\beta_1 \beta_2 \dots \beta_d}$, the inverse operator, puts the permuted vectors back into their original form. Thus N^{-1} is the generalized operator which plays the role of the $\frac{(1-q)^d}{f(d)}$ factors in the q -generalized multinomial coefficients in Lemma 1. Now, we can write in analogy to Eq. (29)

$$\begin{aligned} \bar{R}_{\beta_1 \beta_2 \dots \beta_d}^{\alpha_1 \alpha_2 \dots \alpha_d} &= N^{\beta_1 \beta_2 \dots \beta_d}_{\delta_1 \delta_2 \dots \delta_d} \left\{ (N^{-1} \delta_1 \dots \delta_d, \alpha_1 \dots \alpha_d) \right. \\ &\quad \times \frac{(P'_{N-1}^{\delta_{d+1}} \dots P'_{N-1}^{\delta_{d+d_{N-1}}})}{f(d_{N-1})} \dots \frac{(P'_{k+1}^{\delta_{d+1}} \dots P'_{k+1}^{\delta_{d+d_{k+1}}})}{f(d_{k+1})} \left. \right\} \quad (52) \end{aligned}$$

Summarizing these results, we write

$$\begin{aligned} K_{\alpha_1 \alpha_2 \dots \alpha_d}^{\beta_1 \beta_2 \dots \beta_d} &= \left\{ B_{k+1} \beta_1 \beta_2 \dots \beta_{k+1} \right\} N^{\beta_1 \beta_2 \dots \beta_{k+1}}_{\delta_1 \delta_2 \dots \delta_{k+1}} \\ &\quad \times \left\{ B_{N-k+1}^{\delta_1 \delta_2 \dots \delta_{N-k+1}} \alpha_1 \alpha_2 \dots \alpha_{N-k+1} \right\} \quad (53) \end{aligned}$$

where $B_{N-k+1}^{\delta_1 \delta_2 \dots \delta_{N-k+1} \alpha_1 \alpha_2 \dots \alpha_{N-k+1}}$, the $(N-k+1)$ -point amplitude with two spinning particles is given by

$$B_{N-k+1}^{\delta_1 \dots \delta_{\ell'}, \alpha_1 \dots \alpha_{\ell}} = \sum_{d_i, d | \ell'} \left(\bar{B}_{N-k+1}^{\alpha_{d+1} \alpha_{d+2} \dots \alpha_{\ell}} \right) (N^{-1} \delta_1 \dots \delta_d \alpha_1 \dots \alpha_d) \\ \times \frac{(P'_{N-1}^{\delta_{d+1}} \dots P'_{N-1}^{\delta_{d+d_{N-1}}})}{f(d_{N-1})} \dots \frac{P'_{k+1}^{\delta_{\ell'} - d_{k+1} + 1} \dots P'_{k+1}^{\delta_{\ell'}}}{f(d_{k+1})} \quad (54)$$

where $\bar{B}^{\alpha_{d+1} \dots \alpha_{\ell}}$ is given in Eq. (39).

The special case of the three-point vertex with two nonzero spins corresponding to Fig. 6 is given by

$$\sum_{d=0}^{\infty} N^{-1} \beta_1 \beta_2 \dots \beta_d \alpha_1 \alpha_2 \dots \alpha_d \frac{(P'_2{}^{\beta_{d+1}} \dots P'_2{}^{\beta_{\ell'}})}{f(\ell' - d)} \\ \times \frac{(P_1^{\alpha_{d+1}} \dots P_1^{\alpha_{\ell}})}{f(\ell - d)} g(\ell - d)(\ell' - d) \quad (55)$$

LINEAR DEPENDENCE RELATIONS

We will now show that there are two sets of linear dependence relations among the factorized states. These are most conveniently expressed in terms of "improper amplitudes" defined as

$$(NB)_N^{\alpha_1 \alpha_2 \dots \alpha_{\ell}} = N_{\beta_1 \beta_2 \dots \beta_{\ell}}^{\alpha_1 \alpha_2 \dots \alpha_{\ell}} B_N^{\beta_1 \beta_2 \dots \beta_{\ell}} \quad (56)$$

where $B_N^{\beta_1 \beta_2 \dots \beta_{\ell}}$ is given in Eq. (23).

Let $P_{N\alpha}$ be the 6-vector associated with the particle in the "1th state." (See Fig. 4.) The linear dependence relations are given by the following:

Relation 1

$$(NB)^{\alpha_1 \alpha_2 \dots \alpha_{\ell}} P_{N\alpha_{\ell}} = (NB)^{\alpha_1 \alpha_2 \dots \alpha_{\ell-1}} \quad (57)$$

Relation 2

$$(NB)^{\alpha_1 \alpha_2 \dots \alpha_l} = (NB)^{\alpha_2 \alpha_3 \dots \alpha_l} \quad (58)$$

These relations are true only if p_1 is confined to lie on the internal trajectory, i.e.

$$\sigma_{11} \equiv a p_1^2 + b = 1 \quad (59)$$

Relation 1 has also been proved in operator form.³ However, as yet, we have not been able to put Relation 2 in operator form.

From Eqs. (56), (23), (29), and (25) we write down the explicit form for $(NB)_N^{\alpha_1 \alpha_2 \dots \alpha_l}$.

$$\begin{aligned} (NB)_N^{\alpha_1 \alpha_2 \dots \alpha_l} &= \sum_{n_i | l} B_N \left(\sigma_{ij} g^{\sum_{r=1}^{j-1} n_r} \right) \sum_{dP} g^{TP} \\ &\times \left(P_{P(1)}^{\alpha_1} \dots P_{P(1)}^{\alpha_{n_1}} \right) \left(P_{P(2)}^{\alpha_{n_1+1}} \dots P_{P(2)}^{\alpha_{n_1+n_2}} \right) \dots \left(P_{P(N-2)}^{\alpha_{l-n_{N-2}+1}} \dots P_{P(N-2)}^{\alpha_l} \right) \end{aligned} \quad (56')$$

We will first prove Relations 1 and 2 in the special case $l = 1$. From Eq. (56'), we see that $(NB)^{\alpha_1}$ is a sum of $N - 2$ terms.

The i^{th} term of the sum is characterized by $n_i = 1$ and $n_j = 0$, for

$j \neq i$. In the satellite $B_N \left(\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r} \right)$, the index

$\sum_{r=j}^{k-1} n_r$ is either 1 or 0, depending on whether the value of i lies in between j and k . Thus, if we define

$$\begin{aligned} M(i, j, k) &= 1 \quad \text{if} \quad j \leq i \leq k - 1 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (60)$$

then

$$(NB)_N^{\alpha_i} = \sum_{i=1}^{N-2} P_i^{\alpha_i} B_N(\sigma_{jk} q^{M(i,j,k)}) \quad (61)$$

Now

$$P_i^{\alpha_i} = \sigma_{1i} \quad , \quad (62)$$

and

$$P_i^{\alpha_i} P_{N-i}^{\alpha_{N-i}} = \sigma_{i+1, N-i} \quad . \quad (63)$$

Therefore, Relations 1 and 2 read in the special case of $l = 1$ as

$$\sum_{i=1}^{N-2} \sigma_{i+1, N-i} B_N(\sigma_{jk} q^{M(i,j,k)}) = B_N(\sigma_{jk}) \quad (64)$$

and

$$\sum_{i=1}^{N-2} \sigma_{1i} B_N(\sigma_{jk} q^{M(i,j,k)}) = B_N(\sigma_{jk}) \quad (65)$$

respectively.

We will first prove Eq. (65). The satellite $B_N(\sigma_{jk} q^{M(i,j,k)})$ when written in the form of a multiple sum, differs from $B_N(\sigma_{jk})$ only in that the satellite term has some extra q factors in the summand. Thus,

$$\sigma_{1i} B_N(\sigma_{jk} q^{M(i,j,k)}) = \sigma_{1i} \sum_{n_{ij}=0}^{\infty} \prod_{i=1}^{N-2} \prod_{j=i+1}^{N-1} \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{k>i} \prod_{l>j} q^{n_{ij} n_{kl}} F_i \quad (66)$$

where

$$F_i = q^{\sum_{j=1}^i \sum_{k=i+1}^{N-1} n_{jk}} \quad (67)$$

Let us study the n_{1i} sum in Eq. (66). The n_{1i} dependent part of the summand may be written as

$$\frac{\sigma_{1i}^{n_{1i}}}{f(n_{1i})} q^{n_{1i} \left(\sum_{j=2}^i \sum_{k=i+1}^{N-1} n_{jk} \right)} \quad (68)$$

If we include the extra factors σ_{1i} and F_i of Eq. (66) in Eq. (68), we have

$$\frac{\sigma_{1i}^{(n_{1i}+1)}}{f(n_{1i})} q^{(n_{1i}+1) \left(\sum_{j=2}^i \sum_{k=i+1}^{N-1} n_{jk} \right)} q^{\sum_{k=i+1}^{N-1} n_{1k}}$$

Then replacing $\frac{1}{f(n_{1i})}$ by $\frac{1 - q^{n_{1i}+1}}{f(n_{1i}+1)}$, and using $(n_{1i} + 1)$ as a new summation index for σ_{1i} , we obtain

$$\sigma_{1i} B_N(\sigma_{jk} q^{M(ijk)}) = \sum_{n_{ij}} \prod_{i,j} \frac{\sigma_{ij}^{n_{ij}}}{f(n_{ij})} \prod_{k,l} q^{n_{ij} n_{kl}} \times (1 - q^{n_{1i}}) q^{\sum_{k=i+1}^{N-1} n_{1k}} \quad (69)$$

Furthermore,

$$\sum_{i=2}^{N-1} (1 - q^{n_{1i}}) q^{\sum_{k=i+1}^{N-1} n_{1k}} = 1 - q^{\sum_{k=2}^{N-2} n_{1k}} \quad (70)$$

Eq. (70) follows immediately from the fact that all terms on the

left hand side cancel in pairs except the first and last terms.

Equations (69) and (70) then give

$$\sum_{i=2}^{N-2} \sigma_{1i} B_N(\sigma_{jk} g^{M(ijk)}) = B_N(\sigma_{jk}) - B_N(\sigma; \sigma_{1k} g \text{ for } 2 \leq k \leq N-2) \quad (71)$$

Now, the second term on the RHS of Eq. (71) is $B_N(\sigma_{jk} g^{M(1jk)})$.

Therefore, if $\sigma_{11} = 1$, we have

$$\sum_{i=1}^{N-2} \sigma_{1i} B_N(\sigma_{jk} g^{M(ijk)}) = B_N(\sigma_{jk})$$

This proves Eq. (65). The proof of Eq. (64) is analogous. In

fact, Eq. (64) follows from Eq. (65) by making the reflection

$$p_i \rightarrow p_{N-i}, \text{ and noting that } B_N(\sigma_{jk}) \text{ is invariant}$$

under this reflection transformation.

We will now go on to prove Relation 2 in general.

$(NB)_{N}^{\alpha_1 \alpha_2 \dots \alpha_l}$ in Eq. (56') contains a part which is a sum of distinct permutation of six vectors. Each term in this sum is a permutation P of l 6-vectors with a coefficient q^{T_P} .

For a term in which the vector carrying the index α_1 is p_i , we can write

$$q^{T_P} = q^{T_{P(l-1)}} q^{\sum_{r=1}^{i-1} n_r} \quad (72)$$

where $T_{P(l-1)}$ is the number of transpositions associated only with the permutation of the last $l-1$ 6-vectors, carrying indices

$$\alpha_2, \alpha_3, \dots, \alpha_l$$

In Eq. (56') the 6-tensor is multiplied by a satellite term characterized by indices n_r . Now let $n_i - 1 \rightarrow n_i$, and all other n_r 's remain the same. Then

$$B_N(\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r}) \rightarrow B_N(\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r + M(ijk)}) \quad (73)$$

where $M(i, j, k)$ is defined in Eq. (60). With some rearrangements of the sums, we can now rewrite Eq. (56') to give

$$\begin{aligned} (NB)_N^{\alpha_1 \alpha_2 \dots \alpha_l} &= \sum_{n_r | l-1} \left[\sum_{i=1}^{N-2} P_i^{\alpha_1} g^{\sum_{r=1}^{i-1} n_r} B_N(\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r + M(ijk)}) \right] \\ &\times \sum_{dP} g^{T_{P(l-1)}} P_{P(1)}^{\alpha_2} P_{P(1)}^{\alpha_{n+1}} \dots P_{P(N-2)}^{\alpha_{l-n-2+1}} P_{P(N-2)}^{\alpha_l} \end{aligned} \quad (74)$$

Now for $\alpha_1 = 4$, we have $P_i^4 = \sigma_{ii}$. Substituting this value of P_i^4 in Eq. (74), we see that the expression in the square brackets is identical to the left-hand side of Eq. (65), if we make the replacement $\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r} \rightarrow \sigma_{jk}$. Thus, making use of

Eq. (65), we can write

$$\begin{aligned} (NB)_N^{4 \alpha_2 \dots \alpha_l} &= \sum_{n_r | l-1} B_N(\sigma_{jk} g^{\sum_{r=j}^{k-1} n_r}) \sum_{dP} g^{T_P} P_{P(1)}^{\alpha_2} \dots P_{P(N-2)}^{\alpha_l} \\ &= (NB)_N^{\alpha_2 \alpha_3 \dots \alpha_l} \end{aligned}$$

This proves Relation 2. The proof of Relation 1 is analogous.

DISCUSSION

Perhaps the most interesting aspect of the results obtained in this paper is that many of the remarkable features of the Veneziano model are also present in a wider class of dual models with trajectories of the more general form

$$\alpha(s) = - \frac{\ln(as+b)}{\ln g} = - \frac{\ln [1+(1-g)(a's+b')]}{\ln g}.$$

Factorization in the nonlinear theory has been performed with no on-mass-shell restrictions⁷. By analogy with Feynman diagrams, factorization subject to mass shell constraints seems quite unnatural. Therefore, the method of factorization reported here appears to be very natural from a physical point of view.

The greatest phenomenological difficulty of the nonlinear theory has to do with its bad fixed-angle behavior.² An argument has been presented as to why the $g < 1$ case should not be discarded on the basis of this feature alone². Also, there are ghosts on the first daughter trajectory. This means that when higher order unitarity corrections are included, these poles should produce no prominent structure on the real axis in the physical region. This might be correlated with the experimentally apparent absence of resonances on the first daughter trajectories. However, the question of how to calculate these higher order corrections in detail is still open.

ACKNOWLEDGMENTS

One of us (S.Y.) gratefully acknowledges partial support of the Battelle Memorial Institute. This work was carried out while one of the authors (D.D.C.) was at University of Minnesota and University of Washington. He would like to thank colleagues at both institutions for helpful conversations.

References

1. M. Baker, S. Yu, and D. D. Coon, "Twist Relation, Third Factorization, and the General Vertex, in a Dual Multiparticle Theory with Nonlinear Trajectories," University of Washington preprint.
2. D. D. Coon, Phys. Lett. 29B, 669; M. Baker and D. D. Coon, Phys. Rev. D2, 2349 (1970); *ibid.*, D4, 1234 (1971).
3. D. D. Coon, S. Yu, and M. Baker, "Operator Formulation of a Dual Multiparticle Theory with Nonlinear Trajectories," to be published, Phys. Rev. D (March) 1971).
4. The corresponding developments in the Veneziano theory are as follows:

The N-point amplitudes: G. Veneziano, Nuovo Cimento 57A, 190 (1960); K. Bardakci and H. Ruegg, Phys. Lett. 28B, 342 (1968); M. A. Virasoro, Phys. Rev. Lett. 22, 37 (1969); K. Bardakci and H. Ruegg, Phys. Rev. 181, 1884 (1969); H. M. Chan and S. T. Tsou, Phys. Lett. 28B, 485 (1969); C. G. Goebel and B. Sakita, Phys. Rev. Letters 22, 257 (1969); Z. Koba and H. B. Nielsen, Nucl. Phys. B10, 633 (1969).

Factorization and generalized Ward identities: S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Bardakci and S. Mandelstam, Phys. Rev. 184, 1640 (1969).

Operator formalism: S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969).

Unsymmetric vertex: S. Sciuto, Lett. Nuovo Cimento 2, 411 (1969).

Twist operator and symmetric three-point vertex: L.

Caneschi, A. Schwimmer, and G. Veneziano, Phys. Lett. 30B, 356 (1969); L. Caneschi and A. Schwimmer, Lett. Nuovo Cimento 3, 213 (1970).

5. The four-point function of the nonlinear theory is given by

$$B_4(\sigma, \tau) = \frac{(1-\sigma\tau)(1-\sigma\tau g)(1-\sigma\tau g^2)\dots}{(1-\sigma)(1-\sigma g)(1-\sigma g^2)\dots(1-\tau)(1-\tau g)(1-\tau g^2)\dots}$$

If we let $\sigma = 1 - g(s-s_0)$ and $\tau = 1 - g(t-t_0)$

then,

$$\lim_{g \rightarrow 0} g B_4(\sigma, \tau) = \frac{1}{s-s_0} + \frac{1}{t-t_0}$$

6. L. J. Slater, "Generalized Hypergeometric Functions" (Cambridge University Press, London, 1966); J. E. Littlewood, Proc. London Math. Soc. 5, 361 (1907); G. N. Watson, Trans. Cambridge Phil. Soc. 11, 281 (1910); E. Heine, Handbuch der Kugelfunktionen (G. Reimer, Berlin, 1878); F. H. Jackson, Am. J. Math. 32, 305 (1910); Quart. J. Pure Appl. Math. 41, 193 (1910); W. N. Bailey, Generalized Hypergeometric Series (Cambridge University Press, London, 1935); D. B. Sears, Proc. London Math. Soc. (2) 53, 158 (1951); 53, 181 (1951).

7. Y. Nambu, Phys. Rev. D4, 1193 (1971); I. T. Drummond, Nucl. Phys. B35, 269 (1971); F. Csikor, ITP-Budapest Report No. 292 (1971).
8. As an example, we consider the case $l=3$, $m_{k+1}=2$, $m_{k+2}=1$, $m_j=0$ otherwise. Then, we have

$$R^{\alpha_1 \alpha_2 \alpha_3} = P_{k+1}^{\alpha_1} P_{k+1}^{\alpha_2} P_{k+2}^{\alpha_3} + g P_{k+1}^{\alpha_1} P_{k+2}^{\alpha_2} P_{k+3}^{\alpha_3} + g^2 P_{k+2}^{\alpha_1} P_{k+1}^{\alpha_2} P_{k+1}^{\alpha_3}$$

9. In Eq. (24), C , the polynomial in the crossed variables, was written in a factorized form. If we invert the ordering of vectors in both $L_{\alpha_1 \alpha_2 \dots \alpha_l}$ and $R^{\alpha_1 \alpha_2 \dots \alpha_l}$, the result will remain valid. i.e. we have

$$C = \tilde{L}_{\alpha_1 \alpha_2 \dots \alpha_l} \hat{R}^{\alpha_1 \alpha_2 \dots \alpha_l} \quad (24')$$

where

$$\tilde{L}_{\alpha_1 \alpha_2 \dots \alpha_l} = \frac{(P_{1\alpha_1} P_{1\alpha_2} \dots P_{1\alpha_{n_2}})}{f(n_2)} \frac{(P_{2\alpha_{n_2+1}} P_{2\alpha_{n_2+n_3}}) \dots (P_{k-1\alpha_{l-n_{k+1}}} P_{k-1\alpha_l})}{f(n_k)} \quad (20')$$

and

$$\hat{R}^{\alpha_1 \alpha_2 \dots \alpha_l} = \sum_{dP} g^{TP} (P_{P(N-1)}^{\alpha_1} \dots P_{P(N-1)}^{\alpha_{m_{N-1}}}) \dots (P_{k+1}^{\alpha_{l-m_{k+1}+1}} \dots P_{k+1}^{\alpha_l}) \quad (25')$$

The amplitude of Eq. (30) follows from the factorized form of Eq. (24'). For notational convenience, we have also made the change $n_i \rightarrow n_{i-1}$.

FIGURE CAPTIONS

Figure 1. Factorization at σ_{1k} .

Figure 2. A term in $R^{\alpha_1 \alpha_2 \dots \alpha_\ell}$ corresponding to the permutation $\bar{P}$ considered in the proof of first factorization.

The integers in the middle row represent P' vectors. Thus, the first integer $k+1$ represents $P'_{k+1}{}^{\alpha_1}$, the second integer $k+1$ represents $P'_{k+1}{}^{\alpha_2}$, etc. until we come to the integer $N-1$ at the end of the row which represents $P'_{N-1}{}^{\alpha_\ell}$. The integers n_{ij} in the top column represent the number of times a primed vector appears. The primed vectors are divided into subgroups of size n_i . Within each subgroup, the primed vectors are arranged in ascending order.

Figure 3. The term in $R^{\alpha_1 \alpha_2 \dots \alpha_\ell}$ corresponding to the identity permutation. As in Figure 2, there are ℓ primed vectors. They are all arranged in ascending order.

Figure 4. The N-point tree graph with one nonzero spin.

Figure 5. Second factorization at σ_{1k} .

Figure 6. The vertex with two nonzero spins.

Figure 1

$$\begin{array}{cccccccc}
 \overleftarrow{n_{kk+1}} & \overleftarrow{n_{kk+2}} & \overleftarrow{n_{kN-1}} & \overleftarrow{n_{k-1k+1}} & \overleftarrow{n_{k-1k+2}} & \overleftarrow{n_{k-1N-1}} & \overleftarrow{n_{2k+1}} & \overleftarrow{n_{2k+2}} & \overleftarrow{n_{2N-1}} \\
 k+1 & k+1 \cdots k+1 & k+2 & k+2 \cdots k+2 \cdots N-1 \cdots N-1 & k+1 & k+1 \cdots k+1 & k+2 \cdots k+2 \cdots N-1 \cdots N-1 & \cdots k+1 \cdots k+1 & k+2 \cdots k+2 \cdots N-1 \cdots N-1 \\
 \overleftarrow{\hspace{1.5cm}} & n_k & \overrightarrow{\hspace{1.5cm}} & \overleftarrow{\hspace{1.5cm}} & n_{k-1} & \overrightarrow{\hspace{1.5cm}} & \overleftarrow{\hspace{1.5cm}} & n_2 & \overrightarrow{\hspace{1.5cm}}
 \end{array}$$

Figure 2

$$\begin{array}{ccc}
 \overleftarrow{m_{k+1}} & \overleftarrow{m_{k+2}} & \overleftarrow{m_{N-1}} \\
 k+1 & k+1 \cdots k+1 & k+2 & k+2 \cdots k+2 \cdots N-1 & N-1 \cdots N-1
 \end{array}$$

Figure 3

Figure 4

Figure 5

Figure 6