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On the Discretization Error of the Numerical
Solution of the Third Boundary Value Problem*

MASTER

by

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1. INTRODUCTION

With the advent of electronic computers it has become common to approximate solutions of differential equations by use of finite difference methods. Unfortunately, in many cases the accuracy of such approximations is dependent on the behavior of the unknown solutions, or, worse yet, unknown.

Much attention has been paid to discretization errors arising from finite difference approximations to elliptic partial differential equations (see Forsythe and Wasow [1] for a relatively up-to-date bibliography). In most cases Dirichlet data were prescribed, any generality of results lying in the prescribed differential equation or the shape of the boundary. Batschelet [2] bounded the discretization error arising from prescribing Dirichlet and Neumann data alternatively on segments of the boundary; the bounds were dependent on the behavior of the solution of the differential equation. Wasow [3] bounded the discretization error arising from the Dirichlet problem for the rectangle by finding explicit solutions of the differential and difference equations. Giese [4] did the same for the Neumann problem for the rectangle. In both cases the

bounds were dependent only on the prescribed data. In this paper we use the methods of Wasow and Giese to bound the error arising from Laplace's equation for the rectangle, with Dirichlet data prescribed on two opposite sides and Neumann data on the other two sides. As with Wasow's and Giese's results, the bound is dependent only on the data.

2. THE PROBLEM

Let R be the rectangle with vertices at $(0,0)$, $(1,0)$, $(0,b)$ and $(1,b)$, $b > 0$. We consider the partial differential equation

$$\Delta w = 0$$

with boundary conditions

$$\frac{\partial}{\partial x} w(0,y) = -g_1(y)$$

$$\frac{\partial}{\partial x} w(1,y) = g_2(y)$$

$$w(x,0) = f_1(x)$$

$$w(x,b) = f_2(x).$$

Let h be a positive number such that $1/h = N$ and $b/h = bN$ are integers.

We consider mesh points (jh, kh) ($j=0, 1, \dots, N; k=0, 1, \dots, Nb$), and a solution $W(x, y)$ to the difference equations

$$\Delta_h W(x, y) = W(x+h, y) + W(x-h, y) + W(x, y+h) + W(x, y-h) - 4W(x, y) = 0$$

for $x = jh, y = kh; j = 0, 1, 2, \dots, N$ and $k = 1, 2, \dots, Nb-1$,

$$W(jh, 0) = f_1(jh)$$

$$W(jh, b) = f_2(jh)$$

for $j = 0, 1, 2, \dots, N$, and

$$W(h, kh) - W(-h, kh) = -2hg_1(kh)$$

$$W(1+h, kh) - W(1-h, kh) = 2hg_2(kh)$$

for $k = 0, 1, 2, \dots, Nb$. It can be easily shown that both w and W exist and are unique. We consider the function

$$\begin{aligned} v(x, y) = & g_1(0) \left[-x + \frac{1}{b} xy + \frac{1}{2} x^2 - \frac{1}{2} y^2 - \frac{1}{6b} (3x^2 y - y^3) \right] \\ & + g_2(0) \left[\frac{1}{2} x^2 - \frac{1}{2} y^2 - \frac{1}{6b} (3x^2 y - y^3) \right] + g_1(b) \left[-\frac{1}{2} xy + \frac{1}{6b} (3x^2 y - y^3) \right] \\ & + g_2(b) \frac{1}{6b} (3x^2 y - y^3). \end{aligned}$$

Since, by Taylor's theorem, $\Delta v = 1/h^2 \Delta_h v + (\text{fourth order derivatives of } v)$, and since v is a harmonic polynomial of degree < 4 , we have

$\Delta v = \Delta_h v = 0$. It will be convenient to assume that $\partial/\partial x w(x,y)$ vanishes at the four corners (and therefore that $g_1(y)$, $g_2(y)$, $f_1'(x)$ and $f_2'(x)$ do also). Since in the future we will only be concerned with the difference

$$(w(x,y) - v(x,y)) - (W(x,y) - v(x,y)) = w(x,y) - W(x,y),$$

we will make the above assumption.

We now simplify matters a little more by considering the four problems given by the vanishing of all but one of the functions, f_1 , f_2 , g_1 , and g_2 . By the linearity of the differential and difference equations it will be sufficient to treat each of the four problems separately and then add the results. By symmetry we can simplify things once again by treating only the problems $f_1(x) \neq 0$ and $g_1(y) \neq 0$; and we will do so, dropping subscripts.

The solution to $\Delta u = 0$, $u(x,0) = f(x)$, $\partial/\partial x u(0,y) = \partial/\partial x u(1,y)$, $u(x,b) = 0$, is given by

$$u(x,y) = \alpha_0 \frac{b-y}{b} + \sum_{n=1}^{\infty} \alpha_n \phi(y, n\pi) \cos n \pi x$$

where $f(x) = \sum_{n=0}^{\infty} \alpha_n \cos n \pi x$, and

$$\varphi(y,z) = \sinh(b-y)z / \sinh bz.$$

Next, the solution to $\Delta_h U = 0$, $U(jh,0) = f(jh)$, $U(jh,b) = 0$, $U(h,y) = U(-h,y)$ and $U(1+h,y) = U(1-h,y)$ is given by

$$U(x,y) = A_0 \frac{b-y}{b} + \sum_{n=1}^N A_n \varphi(y, a_n) \cos n \pi x$$

where

$$\sinh \frac{a_n h}{2} = \sin \frac{n \pi h}{2} \quad (2.1)$$

$$A_j = \frac{2}{N} \sum_{r=0}^N \cos jr \frac{\pi}{N} f(rh) \quad 0 < j < N$$

$$A_0 = \frac{1}{N} \sum_{r=0}^N f(rh)$$

$$A_N = \frac{1}{N} \sum_{r=0}^N \cos r \pi f(rh)$$

$$\text{where } \sum_{r=0}^N \mu_r = \frac{1}{2} \mu_0 + \frac{1}{2} \mu_N + \sum_{r=1}^{N-1} \mu_r.$$

The solution to $\Delta v = 0$, $v_x(0,y) = -g(y)$, $v_x(1,y) = v(x,0) = v(x,b) = 0$, is given by

$$v(x,y) = \sum_{n=1}^{\infty} \frac{b \beta_n}{n \pi} \psi(x, \frac{n \pi}{b}) \sin \frac{n \pi y}{b}$$

where

$$g(y) = \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi y}{b}$$

and

$$\psi(x, z) = \cosh z(1-x) / \sinh z.$$

Let

$$V(x, y) = \sum_{n=1}^{Nb} h B_n \operatorname{csch} b_n h \psi(x, b_n) \sin \frac{n\pi y}{b}$$

where

$$\sinh \frac{b_n h}{2} = \sin \frac{n\pi h}{2b} \quad (2.2)$$

$$B_n = \frac{2}{Nb} \sum_{r=1}^{Nb} g(rh) \sin \frac{n\pi r}{bN} \quad 0 < n \leq Nb$$

Then

$$\Delta_h V = 0$$

$$V(x, 0) = V(x, b) = 0$$

$$V(1+h, y) = V(1-h, y)$$

$$V(h, rh) - V(-h, rh) = -2hg(rh)$$

for $r = 0, 1, 2, \dots, bN$.

3. SOME INEQUALITIES

We now reproduce some inequalities due to Wasow and Giese. Because of the difference in notation we will, for the most part, indicate the proofs.

Integrating by parts the relation

$$\alpha_n = 2 \int_0^1 f(x) \cos n\pi x \, dx$$

we get

$$|\alpha_n| \leq K_1 n^{-3} \quad (3.1)$$

where

$$K_1 = \frac{4}{\pi^4} \max |f'''(x)|.$$

Similarly

$$|\beta_n| \leq K_2 n^{-3} \quad (3.2)$$

where

$$K_2 = 2 \frac{b^2}{\pi^3} (|g''(b)| + |g''(0)|) + 4 \frac{b^3}{\pi^4} \max |g'''(y)|.$$

Substituting in (2.1) $\sum \alpha_n \cos n\pi x$ for $f(x)$ we get

$$A_0 = \sum_{\nu=0}^{\infty} \alpha_{2\nu N}$$

$$A_N = \sum_{\nu=0}^{\infty} \alpha_{(2\nu+1)N}$$

$$A_n = \alpha_n + \sum_{\nu=1}^{\infty} (\alpha_{2\nu N-n} + \alpha_{2\nu N+n})$$

for $0 < n < N$, and thus

$$|A_0 - \alpha_0| \leq \sum_{\nu=1}^{\infty} |\alpha_{2\nu N}| \leq K_1 \sum_{\nu=1}^{\infty} (2\nu N)^{-3} < 1.5 K_1 h^3 \quad (3.3)$$

$$|A_N - \alpha_N| \leq K_1 h^3 \sum_{\nu=1}^{\infty} (2\nu+1)^{-3} \leq K_1 h^3 \int_0^{\infty} (2x+1)^{-3} dx < 1.7 K_1 h^3 \quad (3.4)$$

$$|A_n - \alpha_n| \leq K_1 \sum_{\nu=1}^{\infty} [(2\nu N-n)^{-3} + (2\nu N+n)^{-3}] < 1.7 K_1 h^3 \quad (3.5)$$

for $0 < n < N$.

Similarly

$$|B_n - \beta_n| \leq 1.7 K_2 \left(\frac{h}{b}\right)^3 \quad (3.6)$$

for $0 < n < N$.

By means of Taylor's formula Wasow showed that

$$0 < hn\pi - ha_n \leq \frac{1}{12} (n\pi h)^3 \quad (3.7)$$

and that

$$\frac{4}{3} nh \leq a_n h \leq n\pi h \quad (3.8)$$

for $0 < n \leq N$.

Similarly, one can show

$$0 < hn\pi/b - hb_n \leq \frac{1}{12} \left(\frac{n\pi h}{b} \right)^3 \quad (3.9)$$

and

$$\frac{4nh}{3b} \leq b_n h < \frac{n\pi h}{b} \quad (3.10)$$

for $0 < n < N$.

We will need an estimate of $|\varphi(y, a_n) - \varphi(y, n\pi)|$. We have

$$\frac{d}{dz} \varphi(y, z) = \frac{1}{2} \sinh^{-2} bz \left\{ (2b-y) \sinh yz - y \sinh(2b-y)z \right\}.$$

Since $\sinh t/t$ is increasing for $t > 0$ we have

$$\frac{d}{dz} \varphi(y, z) \leq y/2 \sinh^{-2} bz \sinh(2b-y)z \quad (3.11)$$

for $y, z \geq 0$.

The inequalities

$$\sinh t \leq \frac{1}{2}(1-e^{-2t})e^t \quad (3.12)$$

and

$$\sinh kt \leq e^{(k-1)t} \sinh t \quad (3.13)$$

$(0 < t \leq t, 0 \leq k \leq 1)$, follow (respectively) from the identities

$$\sinh t = \frac{1}{2}(e^t - e^{-t})$$

and

$$\sinh kt = \sinh t e^{(k-1)t} (1-e^{-2kt}) (1-e^{-2t})^{-1}.$$

Using (3.10-13) we obtain

$$\frac{d}{dz} \varphi(y, z) \leq (1-e^{-8b/3})^{-2} ye^{-yz} \quad (3.14)$$

for $0 < n \leq N$, $a_n < z < n\pi$, and $0 \leq y \leq b$.

Later we will need the identity

$$\frac{d}{dz} \psi(x, z) = \sinh^{-2} z \left(1 + (2-x) \sinh^2 \frac{xz}{2} + x \sinh^2 \left(1 - \frac{x}{2} \right) z \right). \quad (3.15)$$

4. TRUNCATION ERRORS

We have

$$\begin{aligned} |u(x, y) - U(x, y)| &\leq |\alpha_0 - A_0| \frac{(b-y)}{b} + \sum_{n=1}^N |\alpha_n - A_n| |\varphi(y, n\pi)| \\ &+ \sum_{n=1}^N |A_n| |\varphi(y, n\pi) - \varphi(y, a_n)| + \sum_{n=N+1}^{\infty} |\alpha_n| |\varphi(y, n\pi)| \\ &= R_1 + R_2 + R_3 + R_4. \end{aligned}$$

By (3.3)

$$R_1 \leq \frac{1}{5} K_1 h^3.$$

By (3.4), (3.5) and the inequality $\varphi(y, z) \leq 1$,

$$R_2 \leq 1.7 K_1 h^3 \sum_{n=1}^N |\varphi(y, n\pi)| \leq 1.7 K_1 h^2.$$

By (3.1), (3.4), (3.5), (3.9), (3.14), and the mean value theorem

$$R_3 = \sum_{n=1}^N \{1.7K_1 h^3 + K_1 n^{-3}\} \left(1 - e^{-\frac{8b}{3}}\right)^{-2} y e^{-yz} \frac{(n\pi)^3}{12} h^2$$

for some z with $a_n < z < n\pi$. Since $y e^{-\frac{4}{3}y} \leq \frac{3}{4}(1 - e^{-\frac{4}{3}y})$

$$R_3 \leq 2.7K_1 \frac{\pi^3 h^2}{12} \left(1 - e^{-\frac{8b}{3}}\right)^{-2} \sum_{n=1}^N y e^{-\frac{4}{3}ny}$$

$$\leq 2.7K_1 \frac{\pi^3 h^2}{12} \left(1 - e^{-\frac{8b}{3}}\right)^{-2} \frac{y e^{-\frac{4}{3}y}}{1 - e^{-\frac{4}{3}y}} < 2.1K_1 \frac{\pi^3 h^2}{12} \left(1 - e^{-\frac{8b}{3}}\right)^{-2}.$$

By (3.1) we have

$$R_4 \leq K_1 \sum_{n=N+1}^{\infty} n^{-3} \leq \frac{1}{2} K_1 h^2,$$

and thus

$$|u(x, y) - U(x, y)| \leq K_1(b) h^2$$

where $K_1(b)$ depends only on $f(x)$ and b .

Next,

$$\begin{aligned} |v(x, y) - V(x, y)| &\leq \sum_{n=1}^{Nb} |\beta_n| \left| \frac{b}{n\pi} \psi\left(x, \frac{n\pi}{b}\right) - h \operatorname{csch} b_n h \psi(x, b_n) \right| \\ &+ \sum_{n=1}^{Nb} |\beta_n - B_n| h \operatorname{csch} b_n h \psi(x, b_n) + \sum_{n=Nb+1}^{\infty} |\beta_n| \frac{b}{n\pi} \psi\left(x, \frac{n\pi}{b}\right) \\ &= S_1 + S_2 + S_3. \end{aligned}$$

First we bound S_1 . We have

$$S_1 \cong \sum |\beta_n| \frac{b}{n\pi} \left| \psi(x, \frac{n\pi}{b}) - \psi(x, b_n) \right| + \sum |\beta_n| \frac{\psi(x, b_n)}{b_n} \left| \frac{b_n b}{n\pi} - 1 \right| \\ + \sum |\beta_n| \left| \frac{1}{b_n} - h \operatorname{csch} b_n h \right| \psi(x, b_n) = J_1 + J_2 + J_3.$$

By (3.2) and the mean value theorem

$$J_1 \cong K_2 \frac{b}{\pi} \sum n^{-4} \left| \frac{n\pi}{b} - b_n \right| \left| \frac{d}{dz} \psi(x, z) \right|$$

for some z with $b_n < z < \frac{n\pi}{b}$. From (3.9) and (3.15) it follows that

$$J_1 \cong \frac{K_2}{12} \left(\frac{\pi h}{b} \right)^2 \sum_{n=1}^{Nb} n^{-1} \left| \frac{1 + (2-x) \sinh^2 \frac{xz}{2} + x \sinh^2 (1 - \frac{x}{2}) z}{\sinh^2 z} \right| \\ = \frac{K_2}{12} \left(\frac{\pi h}{b} \right)^2 [I_1 + I_2 + I_3].$$

First,

$$I_1 = \sum_{n=1}^{Nb} \frac{1}{n \sinh^2 z} \cong \sum_{n=1}^{Nb} \frac{1}{nz^2} \cong \left(\frac{3b}{4} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^3} = c_1(b).$$

Since $\sinh \frac{xz}{2} \cong (\sinh z) e^{-z/2}$ for $z \geq 0$, $0 \leq x \leq 1$,

$$I_2 \cong \sum_{n=1}^{Nb} (2-x) \frac{e^{-z}}{n} \cong \sum_{n=1}^{\infty} e^{-\frac{4n}{3b}} = c_2(b).$$

Since $\sinh z(1-x/2) \leq e^{-xz/2} \sinh z$,

$$I_3 \leq \sum_{n=1}^{Nb} \frac{x}{n} e^{-xz} \leq C_2(b),$$

and thus

$$J_1 \leq C_3(b)h^2.$$

Since $\psi(x, b_n) \leq \psi(x, 4n/3b)$, we have, by (3.2) and (3.9),

$$J_2 \leq \frac{K_2}{12} \left(\frac{\pi h}{b}\right)^2 \sum \frac{1}{nb_n} \psi(x, b_n) \leq \frac{K_2}{16} \left(\frac{\pi h}{b}\right)^2 (\coth \frac{4}{3b}) \sum_{n=1}^{\infty} \frac{1}{n^2} = C_4(b)h^2.$$

For $t \geq 0$, $1 - t/\sinh t \leq t^2/6$. Thus by (3.2) and (3.9)

$$\begin{aligned} J_3 &\leq K_2 \sum \frac{1}{n^3 b_n} \psi(x, b_n) \left| 1 - \frac{b_n h}{\sinh b_n h} \right| \\ &\leq \frac{K_2}{6} h^2 \sum \frac{b_n}{n^3} \psi(x, b_n) \leq \frac{\pi K_2}{6b} h^2 (\coth \frac{4}{3b}) \sum_{n=1}^{\infty} \frac{1}{n^2} = C_5(b)h^2. \end{aligned}$$

Thus $S_1 \leq C_6(b)h^2$.

Since $b_n h \leq \sinh b_n h$, and by (3.6), we obtain

$$\begin{aligned} S_2 &\leq 1.7K_2 h^4 b^{-3} \sum n^{-3} \frac{\psi(x, b_n)}{\sinh b_n h} \\ &\leq 1.7K_2 \left(\frac{h}{b}\right)^3 (\coth \frac{4}{3b}) \sum_{n=1}^{\infty} \frac{3b}{4} n^{-4} = C_7(b)h^3. \end{aligned}$$

Finally

$$\begin{aligned}
 S_3 &\cong \frac{K_2 b}{\pi} \sum_{n=Nb+1}^{\infty} n^{-4} \psi(x, \frac{n\pi}{b}) \cong \frac{K_2 b}{\pi} \coth \frac{\pi}{h} \sum_{n=Nb+1}^{\infty} n^{-4} \\
 &\cong \frac{K_2 h^3}{3\pi b^2} \coth \frac{\pi}{h} \cong C_9(b) h^3.
 \end{aligned}$$

Thus

$$|v(x,y) - V(x,y)| \leq C_9(b) h^2.$$

It will be noted that the constants $K_1(b)$ and $C_9(b)$ can be estimated, but we have not bothered to do this as the estimates take different forms according as to whether b is large or small in comparison with unity.

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