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Argonne National Laboratory

DYNAMIC BEHAVIOR OF
CIRCULAR PLATES AND BEAMS
WITH INTERNAL AND EXTERNAL DAMPING

by

G. Cinelli

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

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ANL-7239
Physics (TID-4500)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

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Reactor Engineering Division

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NOMENCLATURE

NOMENCLATURE

<u>Symbol</u>	<u>Description</u>
ω	Frequency of the forcing function
ω_n	Natural frequencies of the elastic system
ζ	Viscous damping ratio of elastic system
D, D_{CR}	Viscous damping coefficient and critical damping coefficient, respectively
ϕ	Phase angle between forcing function and system response
Q	'Quality' factor under resonance
C	Load damping coefficient
E	Young's Modulus of the material
∇^2	Laplacian in orthogonal coordinates
∇^4	Square of the Laplacian
h	Plate thickness
μ	Poisson's ratio
N	Flexural rigidity of the plate
w	Displacement of the plate
m	Mass per unit area of the plate (unit length of the beam)
P	Loading on plate or beam
M	Bending moment on plate edge
λ_m	Eigenvalues of plate or beam frequency equation
J_ν	Bessel function of the first kind and of order ν
I_ν	Modified Bessel function of the first kind and of order ν
Y_ν	Bessel function of the second kind and of order ν
a	Radius of the plate
z	Inertial displacement of cantilevered beam
Y	Base displacement
y	Displacement of beam from central axis
I	Moment of inertia of cross section of beam (normal to central axis) about the central axis
ℓ	Length of the beam
x	Spatial distance along central axis of the beam
w_n	Eigenfunctions of a cantilevered beam
Z	Mechanical impedance
m_e	Effective mass

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DYNAMIC BEHAVIOR OF CIRCULAR PLATES AND BEAMS WITH INTERNAL AND EXTERNAL DAMPING

by

G. Cinelli

I. INTRODUCTION

Nuclear reactors give rise to a myriad of problems involving the dynamic behavior of elastic systems. Some of the more prominent areas are:

- (1) rapid startup or shutdown of reactors;
- (2) control-rod scram;
- (3) containment studies involving postulated accidents;
- (4) rapid opening and closing of pressure valves;
- (5) fluid-structure interaction in the core and heat transfer system;
- (6) SNAP and nuclear rocket vehicles.

The types of loads, that is, their magnitude and physical origin, vary widely for each case but must be accounted for if an effective design is to be attained.

The question may be raised as to whether dynamic behavior should even be considered especially in the case of large land-based power reactors. Although such an approach is feasible, in this case the extra cost involved, especially with respect to containment studies, is not insignificant. In the case of mobile reactor and space vehicles, where weight is at a premium, complete system failure can result if inadequate consideration is given to vibration problems.

The purpose of this memorandum is twofold, namely,

- (1) to indicate the areas in which there is a need for making structural dynamic problems an integral part of nuclear reactor design;
- (2) to develop some new versions of a mathematical technique, known as finite integral transforms, used in solving dynamic problems.

In this study no attempt is made to define the type or magnitude of the loads. Damping is included in order to correlate the theory with design, especially under the condition of 'resonance.'

The paper is divided into four parts. The first considers the dynamic response of a circular plate under various boundary conditions. The second section treats the problem of a cantilevered beam. In the third portion the beam and plate are treated as a complete system. The final part presents the results and conclusions, recommended as developed by this study. The theory of a single-degree-of-freedom system is reviewed at the start of this paper in order to make the memorandum self-contained.

II. PRELIMINARY DISCUSSION

As a prelude to the dynamics of continuous elastic systems, it seems pertinent to recount the solution for a single-degree-of-freedom system. Many of the results for this case play a prominent role later on in discussing the solutions for continuous systems. This fact will become more apparent during the study of the plate and beam problems.

The equation of motion for a single-degree-of-freedom system is

$$m\ddot{x} + D\dot{x} + kx = F(t), \quad (2.1)$$

where

m = the mass of body (lb_m);

x = the displacement of the body (ft);

D = the viscous damping coefficient ($\text{lb}_F\text{-sec/ft}$);

k = the spring constant (lb_F/ft);

$F(t)$ = the arbitrary forcing function (lb_F);

Equation (2.1) can be put into the following form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = F(t)/m; \quad (2.2)$$

$$\zeta = D/D_{CR}; \quad D_{CR} = 2m\omega_n; \quad (2.3)$$

$$\omega_n^2 = k/m. \quad (2.4)$$

For sinusoidal vibrations the forcing function takes the form

$$F(t) = Fe^{i\omega t}. \quad (2.5)$$

Because of the damping, the free vibrations die out rapidly and only the particular solution (forced vibration) remains. Thus, the solution for $x(t)$ is taken to be

$$x(t) = Ae^{i\omega t} \quad (2.6)$$

Substituting equation (2.5) and (2.6) into equation (2.2) and solving for A gives

$$A = \frac{F}{m} \frac{1}{(\omega_n^2 - \omega^2) + i 2\zeta\omega\omega_n} \quad (2.7)$$

Placing equation (2.7) into (2.6) and rationalizing the denominator gives the solution for $x(t)$ as

$$x(t) = \frac{F}{k} \frac{e^{i(\omega t - \phi)}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}}; \quad (2.8)$$

$$\phi = \tan^{-1} \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \quad (2.9)$$

In equation (2.8) we note the following relationship for the static displacement of the body

$$x_{ST} = F/k \quad (2.10)$$

Putting (2.10) into (2.8) and dividing gives the ratio of dynamic to static displacement as

$$\frac{x}{x_{ST}} = \frac{e^{i(\omega t - \phi)}}{\left\{ \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right\}^{1/2}} \quad (2.11)$$

The magnitude and phase plots of equation (2.11) are shown in Figs. 1 and 2, respectively.

Fig. 1. Response Factors for a Viscous-damped Single-degree-of-freedom System

Fig. 2. Phase Angle Shift for a Single-degree-of-freedom System

When $\omega = \omega_n$ the phenomenon known as 'resonance' occurs. In this case equation (2.11) reduces to

$$\left(\frac{x}{x_{ST}}\right)_{RES} = \frac{1}{2\zeta} = Q, \quad (2.12)$$

where Q is the quality factor of the resonance curve. Thus, we see that when there is no damping the ratio becomes infinite. Physically this cannot happen, but it shows that the only factor limiting the displacement (strain) is the amount of damping. To obtain an idea of the damping in usual engineering materials the following relationship is used:

$$\Delta = 2\pi\zeta / (1 - \zeta^2)^{1/2}, \quad (2.13)$$

where Δ is the measured logarithmic decrement. From experiments equation (2.13) shows that $\zeta < 0.01$ for most engineering materials. Thus, from (2.12)

$$\left(\frac{x}{x_{ST}}\right)_{RES} > 50. \quad (2.14)$$

This shows that under conditions of resonance most engineering materials will suffer large displacements (strains) that can lead to the failure of parts unless some means of limiting the displacement is provided.

The solution to equation (2.2) for a general forcing function $F(t)$ is

$$x(t) = \frac{e^{-\zeta\omega_n t}}{m\omega} \int_0^t e^{\zeta\omega_n \tau} F(\tau) \sin \omega(t-\tau) d\tau; \quad (2.15)$$

$$\omega = \omega_n \sqrt{1-\zeta^2}. \quad (2.16)$$

In the field of vibration one of the most difficult problems is the determination of damping. The reason for this is that there are a multitude of physical mechanisms which produce small amounts of damping and knowledge about many of them is far from adequate. When elements vibrate, damping may arise externally due to windage loss or accoustical radiation, and it may arise as a result of slippage at or of energy transmission across an interface where the element is connected to a foundation. In addition, damping arises internally because of the fluctuating stress-strain history. The relative importance of internal vs. external damping varies widely with the application. In the cases considered here it is assumed that the internal damping mechanisms can be expressed by an equivalent load-type damping and the external damping can be lumped into an equivalent viscous-type of damping. The reason for doing this is that under many conditions these damping mechanisms give good results and that the equations are mathematically tractable. However, it is also true that for many engineering materials the damping is nonlinear. In such a case closed solutions cannot be attained. Thus, in the final analysis whether the damping postulated here gives good approximation must depend upon experiment for each individual case. Meanwhile, the results of this study will provide a starting point for further theory and experiment.

III. DYNAMIC RESPONSE OF CIRCULAR PLATES

In this section the transverse deflection of a circular plate under various boundary conditions will be found. Asymmetrical vibration modes are considered in order to permit general loading functions to be applied to the plate. The method used in developing the finite transforms for these cases is given in references [1-4],¹ so that only the results will be given here.

The differential equation and initial conditions of the transverse vibrations of a circular plate with load and viscous damping are as follows:

$$N\left(1 + \frac{C}{E} \frac{d}{dt}\right) \nabla^4 w + D \frac{\partial w}{\partial t} + m \frac{\partial^2 w}{\partial t^2} = P(r, \theta, t); \quad (3.1)$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2; \quad (3.2)$$

¹Number in square brackets refer to references.

$$0 \leq r \leq a; \quad -\pi \leq \theta \leq \pi; \quad t \geq 0; \quad (3.3)$$

$$N = Eh^3/12(1 - \mu^2); \quad m = \rho h. \quad (3.4)$$

The initial conditions are

$$w(r, \theta, 0) = 0; \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = 0. \quad (3.5)$$

As a preliminary to the solution, it is necessary to recall that any function $f(x)$ in the interval $(-\pi, \pi)$ can be written as the sum of an even and odd function by means of the identity

$$f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)]. \quad (3.6)$$

Let $F(x)$ be any function in $(0, \pi)$; then an even (odd) function $f(x)$ exists in $(-\pi, \pi)$ which is identical with $F(x)$ in $(0, \pi)$. If the even (odd) function $f(x)$ is represented by its Fourier series in $(-\pi, \pi)$, so is $F(x)$ in $(0, \pi)$. Thus, use of the finite sine and cosine transforms will resolve the angular variable. Because of this, the solution of the problem is accomplished in two steps, one involving the sine transform and the other the cosine transform. The complete solution is then obtained by adding the two separate solutions, according to equation (3.6).

Case I: Rigidly Clamped Plate

The boundary conditions for this case are

$$w(a, \theta, t) = \left. \frac{\partial w}{\partial r} \right|_{r=a} = 0. \quad (3.7)$$

The appropriate finite sine-Hankel transform, inverse infinite series, and operational properties are:

$$\begin{aligned} \bar{w}_s = \bar{w}_s(\lambda_n, \nu, t) &= \int_0^\pi \int_0^a r w(r, \theta, t) \sin \nu \theta \\ &\cdot [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)] dr d\theta; \end{aligned} \quad (3.8)$$

$$w(r, \theta, t) = \frac{2}{\pi} \sum_{\nu=1}^{\infty} \sum_{\lambda_n} \frac{\bar{w}_s}{A(\lambda_n)} \sin \nu \theta [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]; \quad (3.9)$$

$$A(\lambda_n) = \int_0^a r [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]^2 dr, \quad (3.10)$$

where λ_n is a positive root of

$$J_\nu(\lambda a) I'_\nu(\lambda a) - J'_\nu(\lambda a) I_\nu(\lambda a) = 0 \quad (3.11)$$

and

$$\lambda_n^4 = \frac{\omega_n^2 m}{N}; \quad \nu = 1, 2, 3. \quad (3.12)$$

Now

$$\int_0^\pi \int_0^a r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 w(r, \theta, t) \sin \nu \theta \cdot [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)] dr d\theta = \lambda_n^4 \bar{w}_s. \quad (3.13)$$

Applying equation (3.8) to equation (3.1) and using the results of (3.13) we obtain

$$m \frac{d^2 \bar{w}_s}{dt^2} + \left(\frac{N}{E} C \lambda_n^4 + D \right) \frac{d \bar{w}_s}{dt} + N \lambda_n^4 \bar{w}_s = \bar{P}_s(\lambda_n, \nu, \tau); \quad (3.14)$$

$$\bar{P}_s(\lambda_n, \nu, t) = \int_0^\pi \int_0^a r P(r, \theta, t) \sin \nu \theta [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)] dr d\theta. \quad (3.15)$$

Upon division of (3.14) by m and letting

$$\zeta = \frac{\left(\frac{N}{E} C \lambda_n^4 + D \right)}{\left(\frac{N}{E} C \lambda_n^4 + D \right)_{CR}}; \quad \left(\frac{N}{E} C \lambda_n^4 + D \right)_{CR} = 2m\omega_n, \quad (3.16)$$

equation (3.14) becomes

$$\frac{d^2 \bar{w}_s}{dt^2} + 2\zeta\omega_n \frac{d \bar{w}_s}{dt} + \omega_n^2 \bar{w}_s = \frac{1}{m} \bar{P}_s(\lambda_n, \nu, t). \quad (3.17)$$

Comparison of equation (3.17) with equation (2.2) shows that they are the same except that the damping ratio and natural frequency are different.

Thus, the solution for $\bar{w}_s$ in the transform domain is obtained in exactly the same manner as for the single-degree-of-freedom system. This serves to re-emphasize the point made in section II about the importance of understanding the results for a single-degree-of-freedom system.

Use of equation (2.15) leads to the solution for (3.17):

$$\bar{w}_s = \frac{e^{-\xi\omega_n t}}{m\omega} \int_0^t e^{\xi\omega_n \tau} \bar{P}_s(\lambda_n, \nu, \tau) \sin \omega(t - \tau) d\tau. \quad (3.18)$$

Putting (3.18) into (3.9) gives the solution for the odd function part of $w(r, \theta, t)$.

The appropriate finite cos-Hankel transform and infinite series are

$$\bar{w}_c = \bar{w}_c(\lambda_n, \theta, t) = \int_0^\pi \int_0^a r w(r, \theta, t) \cos \nu \theta [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)] dr d\theta; \quad (3.19)$$

$$\begin{aligned} w(r, \theta, t) = & \frac{1}{\pi} \sum_{\lambda_n} \frac{\bar{w}_c(\lambda_n, 0, t)}{A(\lambda_n)} [J_0(\lambda_n r) I_0(\lambda_n a) - J_0(\lambda_n a) I_0(\lambda_n r)] \\ & + \frac{2}{\pi} \sum_{\nu=1}^{\infty} \sum_{\lambda_n} \frac{\bar{w}_c}{A(\lambda_n)} \cos \nu \theta [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]. \end{aligned} \quad (3.20)$$

The result of using (3.14) on (3.2) is exactly the same as (3.13). Thus, the solution for $\bar{w}_c$ follows in exactly the same manner for $\bar{w}_s$ and is

$$\bar{w}_c = \frac{e^{-\xi\omega_n t}}{m\omega} \int_0^t e^{\xi\omega_n \tau} \bar{P}_c(\lambda_n, \nu, \tau) \sin \omega(t - \tau) d\tau. \quad (3.21)$$

The complete solution to equation (3.1) is obtained by using (3.6), (3.9), (3.18), (3.20), and (3.21), and is

$$\begin{aligned} w(r, \theta, t) = & \frac{1}{2\pi} \sum_{\lambda_n} \frac{\bar{w}_c(\lambda_n, 0, t)}{A(\lambda_n)} [J_0(\lambda_n r) I_0(\lambda_n a) - J_0(\lambda_n a) I_0(\lambda_n r)] \\ & + \frac{1}{\pi} \sum_{\nu=1}^{\infty} \sum_{\lambda_n} \frac{1}{A(\lambda_n)} (\bar{w}_s \sin \nu \theta + \bar{w}_c \cos \nu \theta) [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]. \end{aligned} \quad (3.22)$$

The case of symmetric vibrations is obtained from (3.22) by letting $\nu \rightarrow 0$, thereby leaving only the first term. Hence, another way of interpreting (3.22) is that with respect to the angular variable the case of symmetric vibrations is the constant term in a Fourier series expansion. For specific loads the integrals in (3.18) and (3.21) must be evaluated. Knowing the displacement the corresponding moments and stresses can readily be found using the appropriate formula.

The plate response to forced harmonic vibration is best studied by returning to equation (3.17). The loading now becomes

$$\bar{P}_s(\lambda_n, \xi, t) = \bar{P}_s^*(\lambda_n, \nu) e^{i\omega t}. \quad (3.23)$$

The solution for $\bar{w}_s$ is obtained in exactly the same manner as for $X(t)$ in section II and is

$$\bar{w}_s = \frac{\bar{P}_s^*(\lambda_n, \nu) e^{i(\omega t - \phi)}}{\omega_n^2 \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}} \quad (3.24)$$

Putting this into (3.9) gives

$$w_s(r, \theta, t) = \frac{2}{\pi} e^{i\omega t} \sum_{\lambda_n} \sum_{\nu=1}^{\infty} \frac{\bar{P}_s^*(\lambda_n, \nu) e^{-i\phi_n}}{A(\lambda_n) \omega_n^2 \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}} \sin \nu \theta [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]. \quad (3.25)$$

The solution for the cosine is obtained in exactly the same manner and is

$$w_c(r, \theta, t) = \frac{e^{i\omega t}}{\pi} \sum_{\lambda_n} \frac{\bar{P}_c^*(\lambda_n, 0) e^{-i\phi_n} [J_0(\lambda_n r) I_0(\lambda_n a) - J_0(\lambda_n a) I_0(\lambda_n r)]}{A(\lambda_n) \omega_n^2 \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}} + \frac{2}{\pi} e^{i\omega t} \sum_{\lambda_n} \sum_{\nu=1}^{\infty} \frac{\bar{P}_c^*(\lambda_n, \nu) e^{-i\phi_n} \cos \nu \theta}{A(\lambda_n) \omega_n^2 \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}} [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)]. \quad (3.26)$$

The complete solution for forced vibration is obtained by use of equation (3.6). The interpretation of the solutions in (3.25) and (3.26) leads to the following conclusions:

(1) There are an infinite number of vibration modes in the plate. However, for each individual mode the amplitude of vibration as a function of frequency is exactly the same as for a single-degree-of-freedom system. This means that Figs. 1 and 2 can be used in finding the forced vibrations of a circular plate.

(2) Letting the damping ratio ζ approach zero gives the solution to the undamped case and if $\omega \rightarrow \omega_n$ the phenomenon of resonance occurs. On the other hand, letting $\omega \rightarrow 0$ gives the solution to the static case. Hence, for each mode an expression similar to (2.11) can be written which is the transfer function for that mode.

(3) In this solution the phase shift (damping) is different for each mode and increases with the mode number. Hence, if the damping in actual experiments turns out to be independent of the mode, then some other type of damping besides that shown here must be postulated. One case of (3.25) and (3.26) which is of great interest is when $\omega = \omega_n$. For this condition,

$$\begin{aligned}
 w(r, \theta, t) = & \frac{e^{i\omega t}}{\pi} \sum_{\lambda_n} \frac{\bar{P}_c^*(\lambda_n, 0)}{A(\lambda_n)\omega_n^2} [J_0(\lambda_n r) I_0(\lambda_n a) - J_0(\lambda_n a) I_0(\lambda_n r)] Q_n^{-i\phi_n} \\
 & + \frac{2}{\pi} e^{i\omega t} \sum_{\lambda_n} \sum_{\nu=1}^{\infty} \frac{1}{A(\lambda_n)\omega_n^2} [\bar{P}_s^*(\lambda_n, \nu) \sin \nu \theta + \bar{P}_c^*(\lambda_n, \nu) \cos \nu \theta] \\
 & \cdot [J_\nu(\lambda_n r) I_\nu(\lambda_n a) - J_\nu(\lambda_n a) I_\nu(\lambda_n r)] Q_n e^{-i\phi_n}, \quad (3.27)
 \end{aligned}$$

where Q_n is the 'Q' of each mode of vibration. Equation (3.27) represents the static case except for the factor $Q_n e^{-i\phi_n}$. Thus, we see that for each mode the static displacement is multiplied by the Q in order to obtain the dynamic deflection for each mode as for a single-degree-of-freedom system, thereby again pointing out the need for considering the dynamics of a problem.

Case II: Simply Supported Plate

The boundary conditions for this case are

$$w(a, \theta, t) = 0; \quad (3.28)$$

$$M_r = -N \left[\frac{\partial^2 w}{\partial r^2} + \mu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \bigg|_{r=a} = 0. \quad (3.29)$$

The second boundary condition, (3.29), can be modified* by writing it in the form

$$M_r(r, \theta, t) = -N \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \bigg|_{r=a} = 0. \quad (3.30)$$

Such a modification has negligible influence on the final results, but simplifies the solution considerably. The appropriate finite sine-Hankel transform and properties are:

$$\bar{w}_s = \bar{w}_s(\lambda_n, \nu, t) = \int_0^\pi \int_0^a r w(r, \theta, t) \sin \nu \theta J_\nu(\lambda_n r) dr d\theta; \quad (3.31)$$

$$w_s(r, \theta, t) = \frac{4}{\pi a^2} \sum_{\nu=1}^{\infty} \sum_{\lambda_n} \frac{\bar{w}}{[J'_\nu(\lambda_n a)]^2} \sin \nu \theta J_\nu(\lambda_n r), \quad (3.32)$$

where λ_n is a positive root of

$$J_\nu(\lambda a) = 0 \quad (3.33)$$

and

$$\int_0^\pi \int_0^a r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) w(r, \theta, t) \left(\frac{\sin \nu \theta}{\cos \nu \theta} \right) J_\nu(\lambda_n r) dr d\theta = \lambda_n^4 \bar{w}. \quad (3.34)$$

By use of (3.31) and (3.34), equation (3.1) transforms again into (3.14), except that

$$\bar{P}_s(\lambda_n, \nu, t) = \int_0^\pi \int_0^a r P(r, \theta, t) \sin \nu \theta J_\nu(\lambda_n r) dr d\theta. \quad (3.35)$$

Thus, for this case the solution for $\bar{w}_s$ is the same as (3.18) which, when put into (3.32), gives the complete solution for $w_s(r, \theta, t)$.

*See Reference [1], p. 151.

The appropriate finite cosine-Hankel transform and inverse series are:

$$\bar{w}_c = \bar{w}_c(\lambda_n, \nu, t) = \int_0^\pi \int_0^a r w(r, \theta, t) \cos \nu \theta J_\nu(\lambda_n r) dr d\theta; \quad (3.36)$$

$$\begin{aligned} w_c(r, \theta, t) = & \frac{2}{\pi a^2} \sum_{\lambda_n} \frac{\bar{w}_c(\lambda_n, 0, t)}{[J_1(\lambda_n a)]^2} J_0(\lambda_n \nu) \\ & + \frac{4}{\pi a^2} \sum_{\lambda_n} \sum_{\nu=1}^{\infty} \frac{\bar{w}_c(\lambda_n, \nu, t)}{[J'_\nu(\lambda_n a)]^2} \cos \nu \theta J_\nu(\lambda_n r). \end{aligned} \quad (3.37)$$

The solution for the cosine transform follows exactly the same path as in Case I. Therefore the solution for $\bar{w}_c$ is equation (3.21) except that $\bar{P}_c$ is now

$$\bar{P}_c(\lambda_n, \nu, t) = \int_0^\pi \int_0^a r P(r, \theta, t) \cos \nu \theta J_\nu(\lambda_n r) dr d\theta. \quad (3.38)$$

Putting (3.21) into (3.37) gives the complete solution for the cosine transform. The complete solution to equation (1) for Case II is thus obtained by adding (3.32) and (3.37) according to (3.6).

From the foregoing work it is seen that the method of solution for Case II is the same as for Case I. The only difference between the two occurs in the finite Bessel kernel. Hence, all the results and conclusions for general loads and forced harmonic vibrations in Case II are exactly the same as in Case I and need not be repeated here.

Case III: Clamped Plate with an Edge Moment

In the first case a rigidly clamped plate, which allows a zero moment at the edge, was considered. This is a good approximation for static loads but not necessarily so for dynamic loads, especially under resonant conditions. Thus, a more realistic model for dynamic loads is to allow a nonzero moment at the edge. The boundary conditions for this case are

$$w(a, \theta, t) = 0; \quad (3.39)$$

$$M_r(a, \theta, t) = -N \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \Big|_{r=a} = A(\theta, t). \quad (3.40)$$

The finite transforms for this case are the same as for Case II. However, the operational result on the derivative is changed:

$$\begin{aligned} \int_0^\pi \int_0^a r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 w(r, \theta, t) \sin \nu \theta J_\nu(\lambda_n r) dr d\theta = \\ \frac{-a\lambda_n}{N} \overline{M}_r(a, \nu, t) + a\lambda_n^3 J'_\nu(\lambda_n a) w(a, \theta, t) + \lambda_n^4 \overline{w}_s. \end{aligned} \quad (3.41)$$

The use of (3.39), (3.40), and (3.41) transforms equation (3.1) to

$$m \frac{d^2 \overline{w}_s}{dt^2} + \left(\frac{N}{E} C \lambda_n^4 + D \right) \frac{d \overline{w}_s}{dt} + N \lambda_n^4 \overline{w}_s = \overline{P}_s(\lambda_n, \nu, t) - a \lambda_n \left(1 + \frac{C}{E} \frac{d}{dt} \right) \overline{A}_s(\nu, t), \quad (3.42)$$

where

$$\overline{A}_s(\nu, t) = \int_0^\pi A(\theta, t) \sin \nu \theta d\theta. \quad (3.43)$$

The solution to (3.42) is found in the same manner as in Case I. Hence,

$$\overline{w}_s = \frac{e^{-\xi \omega n t}}{m \omega} \int_0^t e^{\xi \omega n \tau} \left[\overline{P}_s(\lambda_n, \nu, \tau) - a \lambda_n \left(1 + \frac{C}{E} \frac{d}{d\tau} \right) \overline{A}_s(\nu, \tau) \right] \cdot \sin \omega(t - \tau) d\tau. \quad (3.44)$$

This solution differs from (3.18) in two respects. First, the edge moment either adds to or subtracts from the load, depending upon the sign of the moments. Secondly, there is a time rate of change of the moment which also affects the load. Hence, the displacement (strain) has included in it a viscoelastic property due to the internal damping. This means that the stresses in the plate are dependent not only on the strain, but also upon the strain rate. The solution for the cosine transform is

$$\overline{w}_c = \frac{e^{-\xi \omega n t}}{m \omega} \int_0^t e^{\xi \omega n \tau} \left[\overline{P}_c(\lambda_n, \nu, \tau) - a \lambda_n \left(1 + C \frac{d}{d\tau} \right) \overline{A}_c(\nu, \tau) \right] \cdot \sin \omega(t - \tau) d\tau; \quad (3.45)$$

$$\bar{A}_c(\nu, \tau) = \int_0^\pi A(\theta, \tau) \cos \nu \theta d\theta. \quad (3.46)$$

Putting (3.44) and (3.45) into the appropriate inverse series gives the complete solution for (3.11).

The solution for the forced vibration case is obtained from (3.42) in the same manner as in Case I, using the relationship

$$\bar{A}_s(\nu, t) = \bar{A}_s^*(\nu) e^{i\omega t}. \quad (3.47)$$

The solution for the sine transform is

$$\bar{w}_s = \frac{\left[\bar{P}_s^*(\lambda_n, \nu) - a\lambda_n \left(1 + i\omega \frac{C}{E} \right) \bar{A}_s^*(\nu) \right] e^{i(\omega t - \phi_n)}}{\omega_n^2 \left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}}. \quad (3.48)$$

It is seen from (3.48) that the effect of the edge moment is to have a component of it in phase and out of phase with the load. The out-of-phase component of $\bar{A}_s^*(\nu)$ is proportional to frequency. Since the load damping coefficient C is small, at low frequencies the in-phase component overrides the out-of-phase component. However, at higher frequencies this is not true. Thus, for higher modes or rapidly applied loads the out-of-phase component becomes important, and therefore the stress will depend upon the strain rate under these conditions.

IV. DYNAMIC RESPONSE OF A CANTILEVERED BEAM

The problem under consideration is shown in Fig. 3. Two coordinate systems, one inertial (z, x) and one moving (y, x), are used. The relationship between the coordinate systems is

$$y(x, t) = z(x, t) - Y(t). \quad (4.1)$$

The differential equation for the beam in the inertial system is

$$EI \frac{\partial^4 z}{\partial x^4} + CI \frac{\partial^5 z}{\partial x^4 \partial t} + D \frac{\partial z}{\partial t} + m \frac{\partial^2 z}{\partial t^2} = P(x, t). \quad (4.2)$$

The initial conditions are

$$z(x, 0) = \left. \frac{\partial z}{\partial t} \right|_{t=0} = 0, \quad (4.3)$$

and the boundary conditions

$$z(0, t) = Y(t); \quad z_x(0, t) = 0; \quad (4.4)$$

$$z_{xx}(l, t) = 0; \quad z_{xxx}(l, t) = 0. \quad (4.5)$$

Fig. 3. Cantilevered Beam of Length L Excited by Motion $Y(t)$ of the Base

For lightly damped systems ($\xi < 0.01$), equation (4.1) can be approximated by

$$y(x, t) \cong z(x, t). \quad (4.6)$$

Thus, the differential equation of the beam in the moving coordinate system is approximated by

$$EI \frac{\partial^4 y}{\partial x^4} + CI \frac{\partial^5 y}{\partial x^4 \partial t} + D \frac{\partial y}{\partial t} + m \frac{\partial^2 y}{\partial t^2} = P(x, t), \quad (4.7)$$

with initial conditions

$$y(x, 0) = \left. \frac{\partial y}{\partial t} \right|_{t=0} = 0 \quad (4.8)$$

and boundary conditions

$$y(0, t) = Y(t); \quad y_x(0, t) = 0; \quad (4.9)$$

$$y_{xx}(\ell, t) = 0; \quad y_{xxx}(\ell, t) = 0. \quad (4.10)$$

The appropriate finite transform, inverse infinite series, and operational properties are:

$$\bar{y} = F[y(x)] = \int_0^\ell y(x) w_n(x) dx; \quad (4.11)$$

$$y(x) = \frac{1}{\ell} \sum_{\lambda_n} \bar{y} w_n(x); \quad (4.12)$$

$$w_n(x) = (\cosh \lambda_n x - \cos \lambda_n x) - \sigma_n (\sinh \lambda_n x - \sin \lambda_n x); \quad (4.13)$$

$$\sigma_n = \frac{\sinh \lambda_n \ell - \sin \lambda_n \ell}{\cosh \lambda_n \ell + \cos \lambda_n \ell}, \quad (4.14)$$

where λ_n is a positive root of

$$\cosh \lambda \ell \cos \lambda \ell + 1 = 0; \quad (4.15)$$

$$F \left[\frac{d^4 y}{dx^4} \right] = 2y_{xxx}(\ell) - \lambda_n [T(\lambda_n \ell) - \sigma_n U(\lambda_n \ell)] y_{xx}(\ell) + y_x(0) - 2\sigma_n \lambda_n^3 y(0) + \lambda_n^4 \bar{y}; \quad (4.16)$$

$$T(\lambda_n \ell) = \sinh \lambda_n \ell + \sin \lambda_n \ell; \quad (4.17)$$

$$U(\lambda_n \ell) = \cosh \lambda_n \ell - \cos \lambda_n \ell. \quad (4.18)$$

By use of (4.16) with (4.9) and (4.10), equation (4.7) is transformed into

$$\begin{aligned} m \frac{d^2 \bar{y}}{dt^2} + (CI\lambda_n^4 + D) \frac{d\bar{y}}{dt} + EI\lambda_n^4 \bar{y} &= \bar{P}(\lambda_n, t) \\ &+ \sigma_n \lambda_n^3 I \left(E + C \frac{d}{dt} \right) Y(t); \end{aligned} \quad (4.19)$$

$$\bar{P}(\lambda_n, t) = \int_0^\ell P(x, t) w_n(x) dx. \quad (4.20)$$

Upon dividing (4.19) by the mass and using the relationship

$$\lambda_n^4 = \omega_n^2 m / EI, \quad (4.21)$$

equation (4.19) becomes

$$\frac{d^2 \bar{y}}{dt^2} + 2\xi \omega_n \frac{d\bar{y}}{dt} + \omega_n^2 \bar{y} = \frac{1}{m} \bar{P}(\lambda_n, t) + \frac{\sigma_n}{\lambda_n} \omega_n^2 \left(1 + \frac{C}{E} \frac{d}{dt} \right) Y(t); \quad (4.22)$$

$$\xi = \frac{(CI\lambda_n^4 + D)}{(CI\lambda_n^4 + D)_{CR}}; \quad (CI\lambda_n^4 + D)_{CR} = 2m\omega_n. \quad (4.23)$$

Equation (4.22) is similar to (2.2). Thus, the solution for $\bar{y}$, using (2.15), is

$$\bar{y} = \frac{e^{-\xi \omega_n t}}{\omega} \int_0^t e^{\xi \omega_n \tau} \left[\frac{1}{m} \bar{P}(\lambda_n, \tau) + \frac{2\sigma_n}{\lambda_n} \omega_n^2 \left(1 + \frac{C}{E} \frac{d}{d\tau} \right) Y(\tau) \right] \cdot \sin \omega(t - \tau) d\tau. \quad (4.24)$$

Putting (4.24) into (4.12) gives the solution as

$$\begin{aligned} y(x, t) = \sum_{\lambda_n} \frac{e^{-\xi \omega_n t}}{\omega \ell} &\left\{ \int_0^t e^{\xi \omega_n \tau} \left[\frac{1}{m} \bar{P}(\lambda_n, \tau) + \frac{2\sigma_n}{\lambda_n} \omega_n^2 \left(1 + \frac{C}{E} \frac{d}{d\tau} \right) Y(\tau) \right] \right. \\ &\cdot \sin \omega(t - \tau) d\tau \left. \right\} w_n(x). \end{aligned} \quad (4.25)$$

It is seen in (4.25) that the displacement is dependent not only upon the load but also upon the fixed end displacement and its rate of change. Thus, the

displacement (strain) has a time-rate-of-change which causes the stress in the beam to become strain-rate dependent. This is similar to the case of the circular plate, and all the statements made there apply equally as well to the beam.

The case of forced vibration is solved by using the following relationships:

$$\bar{P}(\lambda_n, t) = \bar{P}(\lambda_n) e^{i\omega t}; \quad Y(t) = Y_0 e^{i\omega t} \quad (4.26)$$

in equation (4.22). Solution of this case by the same method as in section II yields

$$\bar{y} = \frac{1}{m\omega_n^2} \frac{P(\lambda_n) e^{i(\omega t - \phi_n)}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}} + \frac{2\sigma_n}{\lambda_n} Y_0 \frac{\left[1 + \left(\omega \frac{C}{E}\right)^2\right]^{1/2} e^{i(\omega t + \theta - \phi_n)}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2 \right]^{1/2}}; \quad (4.27)$$

$$\theta = \tan^{-1} \left(\omega \frac{C}{E} \right). \quad (4.28)$$

Putting (4.28) into (4.12) gives the solution for forced vibrations.

Of particular interest are the cases of steady state and resonance which are, respectively,

$$\bar{y}_{ss} = \frac{\bar{P}(\lambda_n)}{m\omega_n^2} + \frac{\sigma_n Y_0}{\lambda_n}; \quad (4.29)$$

$$\bar{y}_{RES} = \left\{ \frac{P(\lambda_n)}{m\omega_n^2} + \frac{2\sigma_n Y_0}{\lambda_n} \left[1 + \left(\omega \frac{C}{E} \right)^2 \right] e^{i\theta} \right\} Q_n e^{i\frac{\pi}{2}}. \quad (4.30)$$

Thus, for low frequencies $\left(\omega \frac{C}{E} \ll 1 \right)$,

$$y_{RES}/y_{ss} = Q_n, \quad (4.31)$$

which shows that for each mode the ratio of the dynamic to the static displacement is the Q of each mode. For lightly damped systems ($\zeta < 0.01$), this means a magnification ratio of 50 or more in the first mode.

Under conditions of steady acceleration

$$P(X, t) = ma; \quad Y(t) = 0, \quad (4.32)$$

for which the solution is

$$y_s(x) = \sum_{\lambda_n} \frac{a}{\omega_n^2 \ell} \left[\int_0^\ell w_n(\lambda_n y) dx \right] w_n(\lambda_n x). \quad (4.33)$$

For the case where the excitation is sinusoidal at the fixed end,

$$P(x, t) = 0; \quad y = Y_0 e^{i\omega t}, \quad (4.34)$$

the solution is

$$\frac{y(x, t)}{Y_0 e^{i\omega t}} = \sum_{\lambda_n} \frac{2\sigma_n}{\lambda_n \ell} \frac{\left[1 + \left(\omega \frac{C}{E} \right)^2 \right]^{1/2} e^{i(\theta - \phi_n)}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}} w_n(\lambda_n x). \quad (4.35)$$

For lightly damped systems the response due to all other modes besides the first are negligible, since they contribute only between 1 and 2% of the deflection. Thus, equations (4.33) and (4.35) reduce to, respectively,

$$y_s(x) = \frac{a}{\omega_1^2 \ell} \left[\int_0^\ell w_1(\lambda_1 y) dx \right] w_1(\lambda_1 x); \quad (4.36)$$

$$\frac{y(x, t)}{Y_0 e^{i\omega t}} = \frac{2\sigma_1}{\lambda_1 \ell} \frac{e^{-i\phi_1}}{\left[\left(1 - \frac{\omega^2}{\omega_1^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_1} \right)^2 \right]^{1/2}} w_1(\lambda_1 x). \quad (4.37)$$

Now

$$\frac{1}{\ell} \int_0^\ell w_1(\lambda_1 x) dx = \frac{2\sigma_1}{\lambda_1 \ell}, \quad (4.38)$$

so that (4.36) and (4.37) become

$$y_s(x) = 2 \frac{a}{\omega_1^2} \frac{\sigma_1}{\lambda_1 \ell} w_1(\lambda x); \quad (4.39)$$

$$\frac{y(x, t)}{y_0 e^{i\omega t}} = \frac{\omega_1^2}{a} \frac{e^{-i\phi_1}}{\left[\left(1 - \frac{\omega^2}{\omega_1^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}} y_s(x). \quad (4.40)$$

At resonance (4.40) becomes

$$\frac{y(x, t)}{Y_0 e^{i\omega t}} = \frac{\omega_1^2}{a} Q_1 e^{i\frac{\pi}{2}} y_s(x). \quad (4.41)$$

Equation (4.41) establishes the relationship between fixed end displacement and dynamic displacement in terms of the steady acceleration displacement. Again it is seen that Q plays a major role in determining the displacement of the beam.

Using the following relationships for a cantilever beam in the first mode:

$$2\sigma_1/\lambda_1 \ell = 0.783; \quad w_1(\lambda_1 \ell) = 2, \quad (4.42)$$

the steady acceleration at the end of the beam is, from (4.39),

$$y_s(\ell) = 1.57 a/\omega_1^2 \quad (4.43)$$

and the free end displacement at resonance is, from (4.37),

$$\frac{y(\ell, t)}{Y_0 e^{i\omega t}} = 1.57 Q_1 e^{-i\frac{\pi}{2}}. \quad (4.44)$$

For lightly damped systems ($\zeta < 0.01$), (4.44) gives a ratio of 157 or more. Hence, unless care is taken to limit the displacement, large stresses can be engendered in the beam that can cause failure due to fatigue.

It can be shown that the ratio of velocity (acceleration) along the beam to fixed end velocity (acceleration) is the same as for the displacement in equation (4.35). Hence, for lightly damped systems at resonance, the displacement, velocity, and acceleration under sinusoidal excitation have the same magnification factor.

V. DYNAMIC INTERACTION OF A CIRCULAR PLATE AND CANTILEVERED BEAM

In practice, from the structural point of view, a reactor design consists of a number of elastic systems joined together to form a workable concept. Thus, when subjected to various dynamic, fluid, and thermal loads the entire structure should be treated as a unit. Such an approach is entirely feasible but requires considerable work and time. What is needed from a design point of view is a means for determining when it is possible to use an isolated elastic system and some estimation of the error involved. If such an approximation fails, recourse must be made to more advanced mathematical techniques involving computers.

The concept that provides the desired means is that of mechanical impedance. For a single-degree-of-freedom system impedance is defined as the ratio of applied force to velocity for sinusoidal excitation in the following manner:

$$Z(i\omega) = P(i\omega)/v(i\omega). \quad (5.1)$$

For a continuous elastic system the impedance is

$$Z(\xi_1, \xi_2, \xi_3, i\omega) = F(\xi_1, \xi_2, \xi_3)/v(\xi_1, \xi_2, \xi_3, i\omega), \quad (5.2)$$

where ξ_1, ξ_2, ξ_3 are the spatial coordinates of an orthogonal coordinate system. Another quantity found to be useful is the so-called effective mass

$$m_e = (i/\omega) Z. \quad (5.3)$$

The main result of the impedance concept that provides a design answer is: If two impedances are complex conjugates of each other, then the energy (force) transfer between them is a maximum. This result leads to the following design procedure:

- (1) Find the mechanical impedance for each elastic system at all points of attachment to other elastic systems.
- (2) Do the same for the effective mass.
- (3) Compare the impedances and effective mass for all elastic systems at each point of attachment.
- (4) If the amplitudes of any two impedances (effective masses) being considered differ at least by a factor of 10 over the appropriate bandwidth, then the elastic system with the higher impedance (lower effective mass) can be considered as having no effect on the dynamic behavior of the elastic system with the low impedance.

(5) Under the conditions in (4) the two coupled elastic systems are separated and the dynamic response of the system with the lower impedance is found independently of the other system. For the elastic system with the higher impedance it is necessary to consider the lower-impedance system since its behavior influences the dynamic response of the higher-impedance system. Here we have a coupled problem but, since the response of the lower-impedance system is known, its influence appears in the dynamic behavior of higher-impedance systems as a known forcing function. Thus, in this manner we are led to a complete solution of the problem.

(6) If the conditions in (4) are not satisfied, recourse must be made to the actual physical problem.

To clarify the issue, the dynamic behavior of a beam cantilevered from the center of a circular plate which in turn is mounted to a support frame is considered. Dynamic loads are applied axially and transverse to the plate. For a cantilevered beam with high Q the impedance is

$$Z(x, \omega) = \frac{Y_0 e^{i\omega t}}{y(x_1 t)} = \frac{\lambda_1 \ell}{\sigma_1 \omega_1} \frac{\left(2\zeta \frac{\omega}{\omega_n}\right) e^{i(\phi_1 - \pi/2)}}{\left\{1 + \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_1^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_1}\right)^2\right]^{1/2}} w_1(\lambda_1 x)}\right\}} \quad (5.3)$$

At resonance this becomes

$$Z(x, \omega) = \frac{\lambda_1 \ell}{\sigma_1 \omega_1} \frac{1}{Q[1 + QW_1(\lambda, x)]} \quad (5.4)$$

Since the impedance at the center of the circular plate is desired, all the terms in the circular-plate solutions involving angular variation vanish, leaving only the symmetrical case. In addition, because of the low damping, only the first mode of symmetrical vibration need be considered. Therefore, the impedance of the circular plate at resonance is:

Rigidly Clamped Plate:

$$Z(r, \omega) = \frac{\pi A(\lambda_n) \omega_1}{[J_0(\lambda_1 r) I_0(\lambda_1 a) - J_0(\lambda_1 a) I_0(\lambda_1 r)] Q} \quad (5.5)$$

Simply Supported Plate

$$Z(r, \omega) = \frac{\pi \omega_1 J_1^2(\lambda_1 a)}{J_0(\lambda_1 r) Q} \quad (5.6)$$

The clamped plate with elastic edge restraint has the same impedance as in (5.7).

The point of attachment of the beam to the circular plate has the coordinates $r = 0$, $x = 0$. Putting these values into (5.4), (5.5), and (5.6) and taking the ratios of (5.5) and (5.6) to (5.4), we obtain, respectively (assuming equal Q 's),

Rigidly Clamped Plate

$$\frac{Z_P(0, \omega)}{Z_B(0, \omega)} = \frac{\pi A(\lambda_B) \omega_P}{[I(\lambda_B a) - J_0(\lambda_B a)] \omega_B} \frac{\sigma_P \pi}{\lambda_P \ell}; \quad (5.7)$$

Simply Supported Plate

$$\frac{Z_P(0, \omega)}{Z_B(0, \omega)} = \frac{\sigma_P \pi}{\lambda_P \ell} J_1^2(\lambda_1 a) \frac{\omega_P}{\omega_B}, \quad (5.8)$$

where

ω_P, ω_B = the first natural frequency of the plate and beam, respectively;

Z_P, Z_B = the impedance of the plate and beam, respectively;

a = the radius of the circular plate;

λ_P, λ_B = the first eigenvalue of vibration of the plate and beam, respectively.

It turns out that for normal vibration testing the ratios in (5.8) and (5.9) are greater than ten. This means that the dynamic response of the beam can be determined independently of the plate.

However, the dynamic behavior of the plate does depend upon the beam response under resonant conditions. This means in designing fixtures for vibration testing care must be taken to strengthen points of attachment to the test fixture of the objects to be vibrated.

To account for the effect of the beam on the plate, we must note that, knowing the beam displacement, one can calculate the moment at the point of attachment. Hence, the effect of the beam is accounted for by finding the moment at the point of attachment and regarding this moment as a load applied to the plate. In this manner the response of the plate can then be determined.

VI. RESULTS AND CONCLUSIONS

The main results and conclusions of the study are:

(1) Use of the finite transform technique enables the solution of dynamic problems to be expressed in terms of normal modes. The solution thus appears as the sum of an infinite number of single-degree-of-freedom systems which leads to easy physical interpretation and calculation. For lightly damped systems the solution can be approximated by the first-mode term to within a couple of per cent accuracy.

(2) The finite transform approach handles static, forced vibration, aperiodic loads, and time-dependent boundary conditions in a direct and simple manner, thereby avoiding the complex integration usually associated with the Laplace transform technique. However, the breadth of problems solved is not as wide as those of the Laplace transform. The main advantage, therefore, lies in solving problems with several spatial variables in a straightforward manner.

(3) Use of impedance concepts can play an important role in the analysis and synthesis of dynamic design problems. It has been shown how this concept can be used to simplify the solution of coupled elastic systems.

(4) In lightly damped systems it has been shown that the displacement (stress) is limited solely by the amount of damping. In addition, the inclusion of internal damping leads to the result that the stress is dependent upon strain rate as well as the strain, even for systems remaining within the elastic limit.

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