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A TRANSFER FUNCTION FOR D. C. OPERATIONAL AMPLIFIERS WITH
GENERALIZED EXTERNAL NETWORKS

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GENERALIZED EXTERNAL NETWORKS

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SUMMARY

A transfer function for D. C. operational amplifiers with generalized input and feed-back networks is derived by application of network theory. This transfer function is more general than others in common use in that it applies to three and four-terminal networks. It contains, as a special case, the well-known transfer function for D. C. operational amplifiers with two-terminal networks.

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A Transfer Function for D. C. Operational
Amplifiers With Generalized External Networks

By S. R. Gossmann

I. INTRODUCTION

D. C. operational amplifiers are presently being used in a variety of applications. The main application is probably as summing and integrating amplifiers in general purpose analog computers. They are also used as components in automatic control systems and as the basic elements in electronic simulators for numerous types of systems. The properties of the drift-stabilized, D. C. operational amplifier that make it so successful in the above applications are:

1. Extremely high D. C. gain;
2. Very low levels of noise and drift at the output;
3. Negligible grid current;
4. Virtually infinite input impedance when fed back;*
5. Virtually zero output impedance when fed back.**

Because of properties 1, 3, 4 and 5 the external networks used with the amplifier essentially determine the amplifier transfer function at low frequencies. At high frequencies (above 1KC) interaction between the amplifier and the external networks starts to become significant. The result of such interaction is most certainly a deviation of the amplifier response from that desired and in some instances high frequency oscillation of the amplifier has been observed.

The purpose of this paper is to present a transfer function, more general than is available in the literature, for a D. C. operational amplifier when used with a wide class of linear and passive external networks. The use of this transfer function for investigation of the amplifier stability and high frequency response will give more accurate estimates of these quantities than the transfer functions now commonly in use. This transfer function applies primarily when complex external networks are used with an amplifier to obtain high frequency compensation in control systems or in the design of high speed special purpose computers or simulators requiring the use of complex networks.

* See Appendix C for proof.

** See Appendix D for proof.

II. DERIVATION OF THE TRANSFER FUNCTION

The basic circuit configuration to be analyzed is shown in Fig. 1. This circuit contains two generalized networks Z and Z' and the amplifier. The signal X_1 is the input to the circuit. The signal X_2 is the circuit output. The amplifier is characterized by a gain A, input and output impedance Z_I and Z_O , respectively, and by the load impedance Z_L . The analysis will proceed by separating the components of Fig. 1 as shown in Fig. 2 and by application of network theory to the separate circuits of Fig. 2. In Fig. 2 the amplifier input impedance, which acts as a load on Z or as a common element of Z and Z', has been equally divided between Z and Z'. The amplifier can then be considered as a device with infinite input impedance whose input cannot interact with either the output, X_2 , of Z or the input, X_1 , of Z'. That this is a valid assumption follows from property 3.

It is interesting to note that the impedance Z_I identified as the amplifier input impedance can actually be the parallel combination of amplifier input impedance, impedance due to connecting cable capacitance, and the impedance of any two-terminal shaping network that might be placed at the amplifier input (X_2).

The following network equations assume that Z and Z' are flat, linear, passive, 4-terminal networks. Network Z is described by the equations:

$$I_n - y_I \delta_{2n} X_n = \sum_{m=1}^2 y_{nm} X_m; \quad n=1, 2. \quad (1)$$

Similarly, the equations for network Z' are:

$$I'_n - y'_I \delta_{1n} X'_n = \sum_{m=1}^2 y'_{nm} X'_m; \quad n=1, 2. \quad (2)$$

Referring to Fig. 2, it is seen that the amplifier can be described by the equation

$$I'_2 = -y_O(X'_2 + AX'_1) - y_L X'_2. \quad (3)$$

In Eq. (1) and (2) the admittances y_{nm} and y'_{nm} are the short-circuit admittances for Z and Z', respectively. The reciprocity theorem holds for Z and Z' and therefore

$$\left. \begin{aligned} y_{nm} &= y_{mn} & ; & \quad n \neq m, \\ y'_{nm} &= y'_{mn} & ; & \quad n \neq m. \end{aligned} \right\} \quad (4)$$

The admittances y_I , y_O and y_L are simply

$$\left. \begin{aligned} y_I &= 1/2Z_I, \\ y_O &= 1/Z_O, \\ y_L &= 1/Z_L. \end{aligned} \right\} \quad (5)$$

The symbol δ_{nm} is the Kronecker delta satisfying the conditions:

$$\delta_{nm} = \begin{cases} 0 & ; m \neq n, \\ 1 & ; m = n. \end{cases} \quad (6)$$

Comparing Fig. 2 with Fig. 1 it is seen that Z and Z' are connected in cascade and that Z' and the amplifier are connected in parallel. This requires the relations

$$\text{and} \quad \left. \begin{aligned} I_2 &= -I'_1, \\ X_2 &= X'_1. \end{aligned} \right\} \quad (7)$$

Solving Eq. (1), (2), and (3) subject to the conditions (4), (6) and (7) results in the amplifier transfer function,

$$\frac{X'_2}{X_1} = \frac{-y_{12}/y'_{12}}{1 + \left[\frac{1 + y_L/y_0 + y'_{22}/y_0}{A + y'_{12}/y_0} \right] \left[\frac{y'_{11} + y_{22} + 2y_I}{-y'_{12}} \right]} \quad (8)$$

Alternately, Eq. (8) can be written in terms of the short-circuit impedances,

$$\frac{X'_2}{X_1} = \frac{-Z'_{12}/Z_{12}}{1 + \left[\frac{1 + Z_0/Z_L + Z_0/Z'_{22}}{A + Z_0/Z'_{12}} \right] \left[-Z'_{12} \left(\frac{1}{Z'_{11}} + \frac{1}{Z_{22}} + \frac{1}{Z_I} \right) \right]} \quad (9)$$

The short-circuit admittances or impedances are calculated by means of the relationships;

$$\left. \begin{aligned} Z_{11} &= 1/y_{11} = X_1/I_1 ; X_2 = 0, \\ Z_{12} &= 1/y_{12} = X_2/I_1 ; X_1 = 0, \\ Z_{22} &= 1/y_{22} = X_2/I_2 ; X_1 = 0. \end{aligned} \right\} \quad (10)$$

Calculation of Z_{12} by Eq. (10) results in a negative value for Z_{12} . This is due to the direction assigned to I_2 in Fig. 2 and Eq. (1). This difficulty is readily overcome by calculating $-y_{12}$ or $-Z_{12}$ for use in Eq. (8) and (9). The admittances and impedances for the Z' network can, of course, be calculated by Eq. (10).

All the impedances and the amplifier gain in Eq. (9) may be functions of frequency; thus, this equation can be used for investigating the high frequency amplifier response or for stability calculations.

Eq. (8) or (9) reveal that the effective load on the amplifier is the parallel combination of the load impedance Z_L and the short-circuit output impedance, Z_{22}' , of the Z' network. Also the amplifier input impedance, Z_I , and the short-circuit impedances Z_{11}' and Z_{22}' appear as a parallel combination at the amplifier input.

To facilitate the use of this transfer function, Fig. 3 has been prepared giving four useful networks and their short-circuit impedances. Each impedance Z_i in each of the networks can be made up of any combination of resistors and capacitors* such that the combination results in a two-terminal network with impedance Z_i .

* Inductances may also be used if desired.

III. ILLUSTRATIVE EXAMPLES

A. Computing Amplifiers

In analog computer applications the D. C. operational amplifiers are used as summing and integrating amplifiers. The circuit in Fig. 4 shows the type of external impedances used for analog computing. Fig. 4a is the summing amplifier, shown here with only one input signal. Fig. 4b is the integrating amplifier. Obviously in this application the networks Z and Z' are simple two-terminal networks consisting of a single element each.

For the summing amplifier;

$$\begin{aligned} Z_{11} &= Z_{22} = R, \\ Z_{12} &= -R, \\ Z'_{11} &= Z'_{22} = R', \\ Z'_{12} &= -R'. \end{aligned}$$

Using these values in Eq. (9), the summing amplifier has the transfer function,

$$\frac{X'_2}{X_1} = \frac{-R'/R}{1 + \frac{1 + Z_O/Z_L + Z_O/R'}{A - Z_O/R'} \left[1 + R'/R + R'/Z_I \right]}. \quad (11)$$

If the amplifier output impedance is small in relation to the amplifier load and to the feedback resistor R' and if the amplifier input impedance is considered infinite, Eq. (11) becomes

$$\frac{X'_2}{X_1} = \frac{-R'/R}{1 + \frac{1}{A} (1 + R'/R)}. \quad (12)$$

If the amplifier gain is such that

$$A \gg 1 + R'/R,$$

$$\frac{X'_2}{X_1} = -R'/R. \quad (13)$$

Eq. (11), (12) and (13) are well known expressions for computer amplifier transfer functions. They are derived here to show that the more general expression, Eq. (9), reduces to the proper form in the case of two-terminal networks and therefore Eq. (11) is a special case of Eq. (9).

B. Complex External Networks

This section will demonstrate the design of a lead-lag filter using the external networks shown in Fig. 5. The networks Z and Z' are T-networks for which the short-circuit impedances are given in Fig. 3.

Referring to Fig. 3

$$\begin{aligned} -Z_{12} &= (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) / Z_3, \\ Z_{11} &= (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) / (Z_2 + Z_3), \\ Z_{22} &= (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) / (Z_1 + Z_3). \end{aligned}$$

In this example, $Z_1 = Z_2 = R/2$ and $Z_3 = 1/4CS$. It then follows that

$$Z_{11} = Z_{22} = R(RCS + 1) / (2RCS + 1),$$

and

$$-Z_{12} = R(RCS + 1).$$

The short-circuit impedances for the Z' network are identical in form to those of the Z network.

Using these quantities in Eq. (9) the amplifier transfer function is

$$\frac{X_2'}{X_1} = \frac{-R'(R'C'S + 1) / R(RCS + 1)}{1 + 1/\mu\beta},$$

where;

$$1/\mu = \frac{\frac{Z_0}{Z_L} + \frac{Z_0/R'}{(R'C'S + 1)(2R'C'S + 1)}}{A - \frac{Z_0/R'}{R'C'S + 1}},$$

$$1/\beta = \frac{1}{2R'C'S + 1} + \frac{R'(R'C'S + 1)}{R(RCS + 1)(2RCS + 1)} + \frac{R'(R'C'S + 1)}{Z_I}.$$

Obviously the low frequency response is basically that of a lead-lag filter if the amplifier is stable and A is sufficiently large. Since the gain, A, and the impedances Z_L and Z_I are generally functions of frequency, the stability and high frequency response will depend on these parameters.

The symbol μ represents the "effective forward gain" of the amplifier with feed-back networks and load connected and β is the "feed-back ratio". The quantity $\mu\beta$ is the "effective open-loop transfer function of the circuit", a useful function for stability studies.

IV. CONCLUSION

A transfer function for D. C. operational amplifiers with generalized input and feed-back networks has been derived. The transfer functions appearing in the literature and in common use apply only in cases where the networks are two-terminal networks. The work reported here extends the theory to three-terminal networks for amplifiers taking a single ended input and to four-terminal networks for amplifiers accepting a floating input.

It has been shown (Appendix C and D) that, at low frequencies, the effective input impedance of the amplifier is the short-circuit input impedance of the input network and that the effective output or source impedance of the amplifier is essentially zero.

Fig. 1
Amplifier With Generalized
External Networks

Fig. 2
Equivalent Circuit Decomposed

NETWORK	INPUT (Z_{11})	OUTPUT (Z_{22})	TRANSFER ($-Z_{12} = -Z_{21}$)
	$\frac{Z_4 (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{Z_4 (Z_2 + Z_3) + (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}$	$\frac{Z_4 (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{Z_4 (Z_1 + Z_3) + (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}$	$\frac{Z_4 (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{Z_3 Z_4 + (Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}$
	$\frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_2 + Z_3}$	$\frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_1 + Z_3}$	$\frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_3}$
	$Z_1 + \frac{Z_3 (Z_2 + \frac{Z_4 Z_5}{Z_4 + Z_5})}{Z_2 + Z_3 + \frac{Z_4 Z_5}{Z_4 + Z_5}}$	$Z_4 + \frac{Z_5 (Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3})}{Z_2 + Z_5 + \frac{Z_1 Z_3}{Z_1 + Z_3}}$	$\frac{Z_1 [(Z_2 + Z_3)(Z_4 + Z_5) + Z_4 Z_5] + Z_3 Z_5 (Z_2 + Z_4) + Z_2 Z_3 Z_4}{Z_3 Z_5}$
	$\frac{Z_2 Z_3}{Z_2 + Z_3}$	$\frac{Z_2 Z_5}{Z_2 + Z_5}$	Z_2

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Fig. 3
Generalized Network Short-Circuit Impedances

Fig. 4
Analog Computer Amplifier

Fig. 5
A Lead-Lag Filter

V. APPENDIX

A. Multiple Input Networks

In many applications, D. C. operational amplifiers are used for summing various independent signals. The transfer function derived above (Eq. (9)) considers a single input signal. It is not difficult to extend the analysis to an arbitrary number of input signals and hence an arbitrary number of input networks. Fig. 6 shows the circuit for the case of the multiple input signals, X_1^k , the corresponding networks, Z^k , and the feedback network, Z' . The amplifier is represented exactly as before.

The short-circuit network equations for the k^{th} input network are

$$I_n^k - y_I \delta_{2n} X_n^k = \sum_{m=1}^2 y_{nm}^k X_m^k; \begin{cases} n = 1, 2, \\ k = 1, 2, \dots, N. \end{cases} \quad (14)$$

The feed-back network and amplifier are described, exactly as before, by the equations

$$I_n' - y_I \delta_{1n} X_n' = \sum_{m=1}^2 y_{nm}' X_m'; \quad n = 1, 2, \quad (15)$$

$$I_2' = -y_O A X_1' - (y_O + y_L) X_2' \quad (16)$$

As before, the amplifier input impedance, Z_I , has been equally distributed among the N input networks and the feed-back network so that

$$y_I = 1/(N + 1) Z_I \quad (17)$$

Since the amplifier draws no grid current, the following conditions hold at the amplifier input;

$$\sum_{k=1}^N I_2^k = -I_1', \quad (18)$$

$$X_2^k = X_1' \quad (19)$$

Because of reciprocity it is possible to write

$$\left. \begin{aligned} y_{nm}^k &= y_{mn}^k; \quad m \neq n, \\ y_{nm}' &= y_{mn}'; \quad m \neq n. \end{aligned} \right\} \quad (20)$$

Solving Eq. (14), (15) and (16) subject to the conditions (18), (19) and (20) results in the transfer functions for the case of N independent inputs:

$$X_2' = \frac{- \sum_{k=1}^N Z_{12}' X_1^k / Z_{12}'}{1 + 1/\mu\beta}, \quad (21)$$

where, in general;

$$\left. \begin{aligned} 1/\mu &= \frac{1 + Z_0/Z_L + Z_0/Z_{22}'}{A + Z_0/Z_{12}'} \\ 1/\beta &= -Z_{12}' \left[\frac{1}{Z_{11}'} + \frac{1}{Z_I} + \sum_{k=1}^N 1/Z_{22}^k \right] \end{aligned} \right\} \quad (22)$$

B. Alternate Amplifier Equivalent Circuit

The four-terminal network used for representing the amplifier gain, input, output and load impedances in Fig. 1 or 6 may be replaced by the more familiar representation of Fig. 7. In this representation the amplifier is considered as a voltage generator with internal (output) impedance Z_0 .

C. Effective Input Impedance With Feed-back

It has been stated that the amplifier input impedance, Z_I , appears to be infinite when feed-back is applied through the network Z' . The result of this property is that the k^{th} voltage source, X_k^k , sees a load impedance approximately equal to the short-circuit input impedance, Z_{11}^k , of the k^{th} input network. That this is so will now be shown.

Let the voltage X_1^k be applied to the input terminals of the corresponding input network, Z^k . X_1^k will supply a current I_1^k . The "effective input impedance" of the fed-back amplifier is then given by

$$Z_{IN}^k = X_1^k / I_1^k ; X_1^m = 0, m \neq k. \quad (23)$$

The current I_1^k is obtained from Eq. (14) through (20) and found to be

$$I_1^k = y_{11}^k \left[1 + \beta \frac{y_{12}^k y_{12}^k}{y_{11}^k y_{12}'} \right] / (1 + \mu\beta) X_1^k. \quad (24)$$

Therefore, the "effective input impedance," Z_{IN}^k , is simply,

$$Z_{IN}^k = \frac{1/y_{11}^k}{1 + \beta \frac{y_{12}^k y_{12}^k}{y_{11}^k y_{12}'}}. \quad (25)$$

Z_{IN}^k can be written in terms of the short-circuit impedances as follows,

$$Z_{IN}^k = \frac{Z_{11}^k}{1 + \beta \frac{Z_{11}^k Z_{12}'}{Z_{12}^k Z_{12}}} / (1 + \mu\beta) \quad (26)$$

If the amplifier gain, A , is very large, the "effective forward gain", μ , will be very large and Z_{IN}^k is approximately

$$Z_{IN}^k \approx Z_{11}^k \quad (27)$$

Eq. (27) shows that the actual input impedance, Z_I , of the amplifier does not load the voltage source X_1^k if the amplifier gain is very large. For this reason Z_I is usually considered to be infinite. It is not proper to consider Eq. (27) as valid for all frequencies. The amplifier gain, A , decreases with frequency so that at high frequencies Eq. (26) should be used for calculating Z_{IN}^k .

D. Effective Output Impedance With Feed-back

In this section it is shown that, with feed-back, the amplifier behaves as a voltage source with "effective output or source impedance", Z_s , and that Z_s is extremely small, being inversely proportional to the amplifier gain.

Let the voltage X_{2NL}' represent the amplifier output in the absence of a load ($Z_L = \infty$). X_{2NL}' is then given by

$$X_{2NL}' = \lim_{Z_L \rightarrow \infty} X_2' = \frac{-Z_{12}' \sum_{k=1}^N X_1^k / Z_{12}^k}{1 + 1/\mu_{NL}\beta}, \quad (28)$$

where

$$\mu_{NL} = \lim_{Z_L \rightarrow \infty} \mu.$$

The load impedance, Z_L , attenuates the voltage X_{2NL}' * yielding the output voltage

$$X_2' = \frac{Z_L}{Z_L + Z_s} X_{2NL}' \quad (29)$$

As previously shown, X_2' is given by

$$X_2' = \frac{-Z_{12}' \sum_{k=1}^N X_1^k / Z_{12}^k}{1 + 1/\mu\beta} \quad (30)$$

* This statement is equivalent to Thévenin's theorem.

Combining Eq. (28-30) results in

$$(Z_L + Z_S) (1 + 1/\mu_{NL}\beta) = Z_L(1 + 1/\mu\beta) . \quad (31)$$

Solving Eq. (31) for Z_S yields

$$Z_S = \frac{Z_L(\mu_{NL}-\mu)}{\mu(1 + \mu_{NL}\beta)} \quad (32)$$

It is not difficult to show that

$$\mu_{NL}-\mu = \frac{\mu Z_O/Z_L}{1 + Z_O/Z'_{22}} . \quad (33)$$

The fed-back amplifier source impedance is thus

$$Z_S = \frac{1}{1 + \mu_{NL}\beta} \cdot \frac{Z_O}{1 + Z_O/Z'_{22}} \quad (34)$$

If the amplifier gain, A , and hence μ_{NL} is very large, Z_S is given approximately by

$$Z_S \approx \frac{Z_O/\mu_{NL}\beta}{1 + Z_O/Z'_{22}} , \quad (35)$$

or in the limit as A or μ_{NL} is considered infinite,

$$Z_S \approx 0 . \quad (36)$$

At high frequencies Z_S should be calculated according to Eq. (34) rather than by Eq. (35) or (36) since, as previously mentioned, A decreases with frequency.

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