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REPRESENTATION OF CREEP OF METALS

E. Tutan Onat
F. Fardshisheh

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Reactor Division

REPRESENTATION OF CREEP OF METALS

E. Taran Onat F. Fardshisheh

Department of Engineering and Applied Science
Yale University

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PREFACE

This document is one of a series of interpretive, or state-of-the-art, reports prepared as a part of the Oak Ridge National Laboratory program entitled "High-Temperature Structural Design Methods for LMFBR Components." This program has as its goal the development of a verified design technology applicable to the high-temperature long-term operating conditions expected for LMFBR (liquid-metal fast breeder reactor) vessels, components, and core structures.

In addition to contributing to the establishment of the overall state of the art of elevated-temperature design technology, the interpretive reports are intended to assist in identifying and interpreting near-term needs in their respective areas. They will also contribute to the identification and recommendation of potential paths of approach to some of the longer-range needs associated with the generation of a verified technology. The author for each interpretive report was selected on the basis of his experience in that particular area and his overall recognized expertise.

This interpretive report on the representation of time-dependent behavior of metals was prepared under Contract No. AT-(40-1)-4075 between the United States Atomic Energy Commission and Yale University.

C. E. Pugh
Oak Ridge National Laboratory

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SUMMARY AND INTRODUCTION

The work reported herein constitutes a critical review of the subject of mathematical description of creep behavior of metals for the purposes of structural analysis. From the material presented it may be seen that the subject is a vast one. First, there is the physical background of the creep phenomenon; texts by Dorn,¹ Garofalo,² Gilman,³ and Sherby and Burke⁴ give accounts of the extensive progress made in this area in recent years. Second, there is the problem of representation of creep behavior that is nonlinear and hereditary. In this case a synthesis is needed of metallurgical knowledge, of results of phenomenological experiments, of mathematical methods available for the representation of nonlinear, hereditary behavior, and of the needs of the structural analyst. Texts by Rabotnov,⁵ Odqvist,⁶ and Kachanov⁷ cover large aspects of this broad problem. Our study of this and related work shows, however, that there is need for a more systematic presentation of the knowledge available. Moreover, we are convinced that further experimental and theoretical work is needed even in the most established parts of the subject. The present work is therefore not a review in the classical sense; it is rather a study of the subject, with due regard to background and history, from a (hopefully) systematic point of view.

Chapter 1 presents a review of experimental evidence, with particular emphasis on creep behavior under nonsteady loads and/or temperatures.

Chapter 2 is devoted to a detailed discussion of representation of creep by means of state variables and the associated system of differential equations. This representation is introduced in Section 2.3, where we discuss linear viscoelastic solids in the setting of state variable representation. We develop here geometrical notions (such as relaxation lines) associated with the flows in the state space of the material. Section 2.4 offers a fundamental representation [Eq. (2.27)] for creep of metals. We show next that classical laws of creep are of this form and discuss at length the strain-hardening law and point out its deficiencies. A major drawback of this law is that it is incapable of representing a softening of the material, which accompanies creep recovery. In Section 2.5 we abandon the main hypothesis of the strain-hardening law that the

total creep strain is the "inelastic" state variable while still retaining the number of state variables at 2. We show that by suitable choice of the functions $g_1(q_1, q_2)$ and $g_2(q_1, q_2)$ in Eq. (2.33), we can achieve a representation that is capable of representing the principal features of experiments. This discussion is carried out in geometrical terms (cf. Figs. 23 to 25). The influence of the ambient temperature on creep behavior is considered, and the intimate relationship that exists between creep and plasticity is brought to the foreground. Section 2.6 takes up the problem of the determination of the number of state variables and the growth laws from phenomenological experiments. Sections 2.5 and 2.6 offer new contributions to the subject, and they reflect the results of our studies on the representation of nonlinear systems.

Section 2.7 considers the question of the representation of mechanical behavior in the presence of time-dependent temperatures. The discussion proceeds along the classical lines, but the introduction of the fundamental form [Eq. (2.57)] enables one to discuss questions such as temperature-modified time scale with clarity.

Section 2.8 is concerned with the mechanical behavior of an initially isotropic material in the presence of small isothermal three-dimensional deformations. In this case we show that the state variables must be even-rank irreducible tensors (including zero-rank tensors, namely, scalars). The representation has the form given in Eq. (2.82), which is similar to (2.33), but now the inelastic state is defined by a number of tensors and the growth laws are subject to invariance requirements (2.83). We next show that appropriately planned phenomenological experiments could provide information on the rank of state variables. We discuss other types of tests that could be used in the determination of the growth laws. This section is rather brief but it is central to the subject and contains new results. We hope to offer an extended version of this section as a separate research paper.

Chapter 3 is devoted to the integral representation of creep behavior. The drawbacks of the Fréchet representation (3.6) and the Hammerstein-Uryson sums (3.10) are pointed out. The Pipkin-Rogers representation, which is a particular and promising form of (3.10), is studied in detail.

However, as is shown, the Pipkin-Rogers representation provides a poor description of creep of a large class of materials. The authors believe that the state variable method of representation is superior to integral representations both from conceptual and computational points of view.

Chapter 4 includes a brief analysis of structures in the presence of creep. We point out that in the case of nonsteady loads the intricate aspects of creep observed under multistep loading programs may contribute significantly to creep behavior of structures. It would be desirable to analyze some simple structures subjected to nonsteady loads with the help of more realistic creep laws to assess the influence of the phenomena such as creep recovery on the structural behavior. In conclusion, we point out the need for the development of methods for obtaining bounds on the deformation of creeping structures and also on the motions of the state points in the individual state spaces of the elements of the structure.

REPRESENTATION OF CREEP OF METALS*

E. Turan Onat[†] F. Fardshisheh[†]

1. A REVIEW OF EXPERIMENTAL EVIDENCE CONCERNING CREEP OF METALS

1.1 Introduction

The term creep generally refers to the time-dependent deformation of materials that occurs under constant stress and temperature. However, in the present work we assign a more general meaning to this term; by creep we mean deformations that materials undergo in the presence of elevated and possibly nonsteady temperatures and nonsteady stresses.

The purpose of this chapter is to review those experimental investigations that have contributed most to our knowledge of creep, especially in regard to the development of phenomenological descriptions of creep. The presentation will be concerned mostly with metals.

The experimental evidence to be considered in detail here consists of the results of phenomenological experiments, where each test consists in the application, say, of some given, generally time-dependent, stress and temperature and the observation of the resulting deformations.

Experiments of this type do not, however, exhaust our experimental knowledge of creep phenomena. Evidence of microscopic nature, such as microscopic investigations of polished surfaces of metals, x-ray photograms of the surfaces, study of edge pits, etc., must also be considered, especially in view of the fact that this type of evidence increased enormously our appreciation and evaluation of microscopic aspects of creep of metals within the last two decades. Undoubtedly this type of evidence will play an increasingly important role in the construction of relationships that describe the thermomechanical behavior of metals. But at present a direct use of microscopic evidence in the fulfillment

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[†]Department of Engineering and Applied Science, Yale University.

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of the above stated goal is difficult if not impossible. One reason for this is that microscopic studies have been concerned mostly with relatively pure metals or dilute alloys and therefore are not often directly applicable to complex heat-resistive alloys. Another reason is that the profusion of thermally activated processes that contribute to or cause creep discourages and bewilders the worker whose primary interest is to construct "stress-strain-temperature laws" that are simple enough to be tractable yet realistic enough to be useful. A knowledge of individual thermally activated processes is desirable, but, from the above point of view, one needs to know more about the interactions of these processes, which is not often studied in detail.

Nevertheless it is extremely important to have an appreciation of the physical basis of creep. We hope to demonstrate in the course of this work that considerations of the physics of creep are helpful even in the most abstract parts of the question of representation.

1.2 Typical Tests

A simple creep test involves the sudden application of stress which is then maintained at constant value. Figure 1 shows the results

Fig. 1. Creep of Al 1100 at 300°F at various stress levels (Wang and Onat⁸).

of a series of simple creep tests performed with 1100 Al at 300°F at various levels of stress.⁸ In Fig. 2 creep curves are shown for copper where the stress is maintained at a given value but the temperature is varied from test to test (Ref. 2, p. 14). The simple creep of metals is well documented and need not be repeated here. It is pointed out that simple creep behavior of austenitic steels has been studied.⁹⁻¹¹

In a systematic study of simple creep of a given material, the stress level σ and the ambient temperature θ (to which the specimens would be subjected for a sufficiently long time before the application of stress) would be the parameters to vary from test to test. For a class of such tests $\epsilon(t)$, the strain at time t , would be a function of σ , θ , and t :

$$\epsilon = f(\sigma, \theta, t) . \quad (1.1)$$

It is customary to write f in terms of more or less easily identifiable components:

$$f = \frac{\sigma}{E} + \epsilon_p + \epsilon_{tr} + \dot{\epsilon}_m t , \quad (1.2)$$

Fig. 2. Logarithmic creep curves for copper at various temperatures under a stress of 6 kg/mm² [Garofalo (2), p. 14].

where σ/E is the elastic strain; ϵ_p is the inelastic strain that is observed immediately after the application of stress and is commonly referred to as the plastic strain; ϵ_{tr} is the primary or transient creep that dominates, by its time derivative, $\dot{\epsilon}_{tr}$, the strain rate during primary creep ($\dot{\epsilon}_{tr} \rightarrow 0$ as $t \rightarrow \infty$); and $\dot{\epsilon}_m$ is the slope of the creep curve at large values of time. The representation [Eq. (1.2)] ignores the possibility of occurrence of tertiary creep. Indeed in the present work we leave the tertiary phase of creep out of consideration.

We note that the definition of ϵ_p depends somewhat on the measuring equipment; in fact, often ϵ_p is taken to be the first reliable strain reading that the measuring equipment could produce. Whether this component of strain really exists as a separate entity is discussed further in Chapter 2.

It is often accepted that $\dot{\epsilon}_{tr}$ behaves as $1/t^n$ for small t . The case $n = 1$, the logarithmic creep, has been observed to occur at low temperatures.¹² The case $0 < n < 1$, which occurs at intermediate or high temperatures, is known as parabolic creep.

It may be useful here to agree on the terminology concerning the temperature in regard to terms such as low, intermediate, or high temperatures. As is well known, room temperature must be regarded as elevated for lead, but it is low as far as steels are concerned. Following Conrad,¹² we compare the ambient temperature θ with the melting temperature θ_m of the material to arrive at the following classification. We mean by low, intermediate, and high temperatures, the temperatures that satisfy the following inequalities, respectively, $\theta < 0.25\theta_m$, $0.25\theta_m < \theta < 0.5\theta_m$, $0.5\theta_m < \theta$, although the limits that appear above are not as sharp as these inequalities imply.

The dependence of the rate of steady creep $\dot{\epsilon}_m$ on σ and θ has been well studied. The results of numerous experiments conducted with a wide selection of metals suggest that $\dot{\epsilon}_m$ has the following dependence on temperature and stress,

$$\dot{\epsilon}_m = f(\sigma) e^{-Q/R\theta}, \quad (1.3)$$

where Q is referred to as the activation energy for creep, R is the gas constant, and θ is the absolute temperature.

Often it is found that

$$f(\sigma) = A\sigma^n, \quad (1.4)$$

where A and n are constants. Weertman and his co-workers¹³ reported that at high temperatures and intermediate stresses, $n = 4$ to 5 for pure metals and 3 for dilute alloys.

In an early preliminary report on creep behavior, Blackburn¹¹ suggests that $n = 3.29$ for type 304 stainless steel over the stress range 10,000 to 40,000 psi, but above the 40,000-psi level n increases beyond this value. On the other hand, for type 316 stainless steel, Blackburn finds that the following expression,

$$f(\sigma) = A [\sinh(\alpha\sigma)]^n,$$

provides a good description of creep data with $n = 6.16$. For a more detailed discussion of the nature of $f(\sigma, \theta, t)$, we refer the reader to Garofalo² and Conrad.¹² For a convenient way of representing the data in Eq. (1.1), see Manjoine.¹⁴

Multistep tests

As we shall see later, it is essential from both physics and engineering standpoints that creep of metals is studied under time-dependent stresses. A particularly useful form of nonsteady stress history is found in multistep tests. The stress histories given in Figs. 3 and 4 are multistep tests that involve episodes of constant stressing separated by the abrupt changes in the stress level. Interrupted creep tests are a particular form of multistep tests. Figure 3 shows a test where a simple creep test with Cr-Mo-V steel at 800°F and $\sigma = 72,000$ psi was "interrupted" at $t = 23$ hr by suddenly decreasing the stress to zero and then maintaining it at that level. We note from this figure that upon sudden unloading, the elastic strain σ/E is recovered suddenly. The strain then decreases by a small amount and attains a constant asymptotic value. This time-dependent recovery of strain, which is sometimes called creep recovery,¹⁵ is a common aspect of mechanical behavior of viscoelastic solids. What is extremely interesting and significant is that in metals the strain recovery is unusually small (see also Fig. 4). Zener and his co-workers (Ref. 16, p. 153) proposed a theory to explain the strain recovery in terms of grain-boundary sliding.

Fig. 3. Creep and creep recovery in Cr-Mo-V steel at 800°F. 72,000 psi was applied for 23 hr and then removed (Lubahn¹⁵).

The question that may now be raised is: Does the smallness of the strain recovery in metals imply that no significant changes occur within these materials during the periods of total unloading? To answer this question one must study the future behavior of specimens. For this purpose we may increase the stress to its original value after a period of total unloading and then observe the resulting strains (Fig. 4). We note from the figure that in 1100 Al, as it is the case with many other metals, a period of total unloading generally affects the future behavior of the material. If this period is short, upon reloading, the material continues as before (Fig. 5). But if it is large the material develops primary creep upon reloading (Fig. 4). Thus we must conclude that at intermediate or high temperatures the internal state of a metal

Fig. 4. Interrupted creep tests with Al 1100 at 300°F. Duration of interruption: T = 16 hr (upper); T = 96 hr (lower) (Wang and Onat⁹).

Fig. 5. Interrupted creep tests where the time of interruption is short. (a) With Al 1100 at 300°F; T = 0.5 hr (Wang and Onat⁹). (b) On quenched and tempered Cr-Mo-V steel at 1000°F and 35,000 psi (Lubahn¹⁸).

that has experienced previous creep deformations may change in the absence of external stress.

Figure 6 shows the different types of behavior that may be observed in an interrupted creep test. In tests 1 and 2 the period of unloading has a softening effect on the material, whereas in test 3 it has the effect of increasing the creep resistance.

From a physical point of view, the change in the internal state of the material may be produced by the action of internal stresses (which develop upon unloading) on thermally activated processes such as dislocation climb (Ref. 17, p. 133). At high temperatures self-diffusion may occur at significant rates even without the presence of internal

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Fig. 6. Several types of behavior observed upon reloading (schematic).

1. Al 1100 at 300°F, $\sigma_0 = 3750$ psi, $T = 96$ hr (Wang and Onat⁸).
2. Cr-Mo-V steel at 1000°F, $\sigma_0 = 35,000$ psi, $T = 300$ hr (Lubahn¹⁵).
3. Cr-Mo-V steel at 800°F, $\sigma_0 = 72,000$ psi, $T = 45$ days (Lubahn¹⁵).
4. Prediction of the strain-hardening theory.

stresses. For a detailed discussion of the rearrangement and/or annihilation of dislocations that take place during the recovery process, see Ref. 1, p. 91.

In another important class of tests one applies incremental loading or unloading starting from a given stress level^{8,13,19} (Figs. 7 and 8). It is interesting to observe from these figures that incremental loading causes primary creep to occur, whereas, in the case of lead, incremental unloading has the effect of stopping creep for a period of time that is sometimes called the incubation period. It is recognized that tests of this type, which are also called differential tests (rapid small changes in stress and/or temperature), are useful for evaluating dislocation

Fig. 7. Effect of incremental loading on the creep behavior of lead. (Onat and Wang¹⁸).

Fig. 8. Incremental creep tests on Al 1100 at 300°F (Wang and Onat⁸).

mechanisms by creep. Section 2.4 shows that incremental tests constitute a powerful tool for the construction of the representation of creep behavior. It must also be pointed out that tests of this type are of fundamental importance in the study of the stability of structures composed of creeping materials (Ref. 5, p. 261; Ref. 18).

Another class of multistep tests involves periodic applications of stress. In Fig. 9 we show the results of such a test,²⁰ where the total strain is a function of the reduced time $\bar{t}$, which measures the total time spent under the stress. We observe that the softening caused by total unloading has the net effect of increasing the creep strains, especially in the case where the period of total unloading is much larger than the period of loading. The above observation and Fig. 9

Fig. 9. Creep of steel at 450°C under periodically varying stress (Taira²⁰).

indicate that it may be imprudent to analyze a structure subjected to periodically varying loads by using a creep law that is not capable of representing this phenomenon of softening.

In a related class of tests, small periodic stresses are superimposed on a given finite static stress. In some materials the addition of vibratory component to the stress increases the creep rate considerably (vibrocreep).²¹

Relaxation tests, which involve sudden application of a strain that is then maintained at constant value, are more difficult to perform than simple creep tests. Figure 10 shows a typical relaxation

Fig. 10. A typical relaxation test.

test where the output of the test is the time-dependent stress caused by the constant straining. The value of the relaxation tests lies in the fact that such a test is quite different from the tests where the stress is controlled and therefore constitutes a serious check on the validity of a creep law constructed with the help of multistep tests.

Hybrid tests

Sometimes it is useful to start a test as a simple creep experiment and then continue it as a relaxation test. An example of such a hybrid test is shown in Fig. 11. Recent work of Gibbs²² and MacEwen, Kupcis, and Ramaswami²³ has shown that hybrid tests are important in the study

Fig. 11. Hybrid test on Cr-Mo-V steel at 800°F, where total strain after unloading from 45,000 to 30,000 psi was kept constant (Lubahn¹⁸).

of internal stresses that develop in creeping materials. For further discussion of the value of hybrid tests, see Section 2.4.

It is hoped that the above examples have shown that creep of metals is an intricate nonlinear time-dependent phenomenon. The nonlinearity does not only occur in relations such as Eqs. (1.2) and (1.4), but it is evident in tests such as shown in Figs. 3 to 7. The response of linear time-dependent materials (i.e., viscoelastic solids) to the conditions depicted in these figures would be entirely different. For instance, if for such a linear material the softening effect shown in Fig. 4 is present, then this material must exhibit strong strain recovery. That this is not

the case in metals is an additional indication of nonlinearity of these materials.

We refrain here from discussing the results of tests that involve multi-axial stressing and proceed directly to the question of mathematical representation of mechanical behavior.

2. REPRESENTATION OF MECHANICAL BEHAVIOR BY MEANS OF STATE VARIABLES

2.1 What Is Meant by the Representation of Mechanical Behavior?

Suppose we wish to study the mechanical behavior of a given material at a given constant elevated temperature. We may start by preparing a number of "identical" test specimens, say, for simplicity, tensile specimens. These specimens may be cut at room temperature from a given homogeneous stock with due attention given to the problems of material symmetry. We may then heat these specimens in an identical fashion so that at a given time we have available a number of identical specimens that are at a given elevated temperature. We are now ready to perform tests with these specimens. Each test may involve the application of a chosen time-dependent stress $\sigma(\tau)$, where τ denotes the time, and the observation of the resulting strain $\epsilon(\tau)$ over the test interval $[0, T]$. We can say, using the terminology of the systems theory, that the outcome of each test is an input-output pair composed of the functions $\sigma(\tau)$; $\epsilon(\tau)$ defined on $[0, T]$. We have discussed in the previous chapter such input-output pairs in some detail. In some of these tests $\epsilon(\tau)$ was the input of the problem and $\sigma(\tau)$ was the output.

In the above-described setting the given material acts as a mathematical operator; that is, it assigns to each $\sigma(\tau)$ on $[0, T]$ a function $\epsilon(\tau)$ on $[0, T]$ through the process of specimen preparation and testing. However, for theoretical reasons, which may become clear when we discuss the three-dimensional deformations, we shall consider the strain histories $\epsilon(\tau)$ as the inputs of the problem and the stress histories $\sigma(\tau)$ as outputs, so that the mathematical operator of interest has the form

$$\sigma(\tau) = P[\epsilon(\tau)] \quad 0 \leq \tau \leq T, \quad (2.1)$$

which simply expresses symbolically the observation that the given material assigns to each $\epsilon(\tau)$ on $[0, T]$ a $\sigma(\tau)$ on the same interval.

The representation of mechanical behavior is then the specification of a number of mathematical rules and operations by which one can construct, with reasonable accuracy and for the given material, the experimentally observed output $\sigma(\tau)$ by mathematical operations and from the knowledge of the input $\epsilon(\tau)$.

For linear elastic solids the representation of mechanical behavior involves the rule that $\sigma(\tau)$ is simply proportional to $\epsilon(\tau)$:

$$\sigma(\tau) = E \epsilon(\tau) . \quad (2.2)$$

As shown in Chapter 1, for creeping materials the rule cannot be so simple as Eq. (2.2), since it must cope with the intricate nonlinearities of the creep phenomenon.

What are then the methods for representing the nonlinear relationships of the type in Eq. (2.1)?* We shall see that there are two major modes of representation. The first, to be discussed in this chapter, is based on a system of differential equations that governs the evolution of the state of the material of interest. The second, discussed in Chapter 3, is related to the first but involves integrations.

It will be clear that, in addition to the study of general methods for the representation of (2.1), we must be prepared to tackle a problem of interpolation. Indeed, by means of a program of testing we shall be able to accumulate only a finite amount of data in the form of a finite number of input-output pairs $[\epsilon_i(\tau), \sigma_i(\tau)]$, $i = 1, \dots, n$. Thus the action of the operator F will be known only for a finite number of argument functions; that is, $\epsilon_1(\tau), \epsilon_2(\tau), \dots, \epsilon_n(\tau)$. In the simpler case of a function $[f(x)]$ this is equivalent to knowing the value of function f at a number of points on the x axis. If the location and the number of these points are well chosen and if f is smooth enough, we can proceed to find an interpolating function $\hat{f}$ that is smooth enough and comes close at the appointed points to the known values of f . It should now be obvious that a similar procedure must be followed in the construction of a representation for F .

2.2 Mathematical Preliminaries

The operator F has certain properties that must be brought to the foreground. First, we must specify the input and output sets more precisely.

*It may be interesting to ask the same question for ordinary real valued functions, $y = f(x)$. The answer would be that Taylor's expansion, Weierstrass sums, Fourier series, differential equations that generate functions by their solutions, etc., are the tools for representing functions.

Here we confine ourselves to the case where the input set is composed of continuous and uniformly bounded functions* on $[0, T]$. We shall assume that the outputs also constitute a set of continuous and uniformly bounded functions on $[0, T]$. For metallic materials this assumption has strong experimental support.

We observe, because of causality that exists in the time domain, that F must be such that $\sigma(t)$ must depend only on the strains on the time interval $[0, t]$. Thus we may say that F is causal.

Next we assume that the materials considered here are nonaging; that is, if a virgin specimen is kept, in the presence of the test temperature θ , under zero stress, no changes will occur in its internal state. This means that a stress-free nonaging virgin specimen is in a state of equilibrium in the presence of θ . Often this condition can be realized by appropriate care in the initial heating of the specimen. The condition of nonaging brings the following restrictions on F . Let $\sigma(\tau)$ be the output created by F from the input $\epsilon(\tau)$. The property of lack of aging implies that the following input,

$$\epsilon(\tau) = \begin{cases} 0 & 0 \leq \tau \leq t \\ \epsilon(\tau - t) & t \leq \tau \end{cases},$$

must produce the output,

$$\sigma_1(\tau) = \begin{cases} 0 & 0 \leq \tau \leq t \\ \sigma(\tau - t) & t \leq \tau \end{cases}, \quad (2.3)$$

for any positive t . We shall further assume that the operator F is continuous, in the sense that for each $\epsilon > 0$ there exists a $\delta > 0$ such that any two strain histories satisfy the inequality

$$|\epsilon_1(\tau) - \epsilon_2(\tau)| < \delta \quad \text{on } [0, T],$$

we have

$$|\sigma_1(\tau) - \sigma_2(\tau)| = |F[\epsilon_1(\tau)] - F[\epsilon_2(\tau)]| < \epsilon.$$

*Note that $\epsilon(0) = 0$ and $\sigma(0) = 0$ for every input and output pair.

In physical terms the continuity of F implies that the stress response obtained in tests conducted with slightly different strain histories differ from each other only slightly.

After these preparatory remarks we are ready to consider the state variable representation of mechanical behavior. In the following section implicit use will be made of the properties that we have assigned here to F . For a deeper appreciation of the importance of these properties in the problems of representation, see Ref. 24.

2.3 Representation by Means of State Variables

We adopt here a microscopic and informal point of view to introduce the state variable representation. For simplicity in notation and mathematics we confine ourselves to the uniaxial stressing of initially identical tensile specimens.

In a test where the strain is prescribed, the stress acting on the specimen would depend upon what is happening currently within the specimen. On the other hand, the future behavior of the specimen would depend not only on the present internal state of the specimen but also on the future deformations applied to the specimen. Since the stress is a quantity that represents an average, it would not depend on all the details of the present arrangement of "particles" that make up the specimen, but rather on a number of average quantities that define the salient and relevant features of this arrangement. For a polycrystalline bar these quantities may be average interatomic distance in the direction of the load application, density of mobile dislocations, and other average quantities related to dislocation distribution, etc.

Let us now assume that we need n quantities, $q_1, \dots, q_n$, to specify the features of particle arrangement that are relevant to mechanical behavior. Thus, if these features are represented by the same set of quantities $q_1, q_2, \dots, q_n$ in two different specimens, these specimens would not only carry the same stress but would behave identically, from a mechanical point of view, under identical future deformations measured with respect to the present configurations of the specimens. [For a more formal definition of the notion of state and state variables the reader is referred to Refs. 25 and 26 and Sect. 2.6.]

Let $\underline{\tilde{q}}$ denote the vector with coordinates $q_1, \dots, q_n$ in the n -dimensional Euclidean space E_n . We call the q_i 's the state variables and the space E_n the state space.

It is obvious from the above comments that the stress in the specimen is a function of the state vector $\underline{\tilde{q}}$ or simply the state of the material:

$$\sigma = f(\underline{\tilde{q}}) . \quad (2.4)$$

Our next task is to relate the state $\underline{\tilde{q}}$ to the deformation applied to the bar. For this purpose we consider a specimen that is in state $\underline{\tilde{q}}(t)$ at time t . Suppose that we subject this specimen to further deformation

$$\begin{aligned} \tau &= t + \Delta t \\ \epsilon(\tau) \\ \tau &= t \end{aligned}$$

on the interval $[t, t + \Delta t]$. What can we say about the state of the specimen at time $t + \Delta t$? It should be clear that the state $\underline{\tilde{q}}(t + \Delta t)$ of the specimen at time $t + \Delta t$ will be a "function" of the state at time t and the applied strains $\epsilon(\tau)$ on $[t, t + \Delta t]$. Namely,

$$\begin{aligned} \tau &= t + \Delta t \\ \underline{\tilde{q}}(t + \Delta t) &= G[\underline{\tilde{q}}(t); \epsilon(\tau)] , \quad (2.5) \\ \tau &= t \end{aligned}$$

where G is a vector valued function of $\underline{\tilde{q}}$ and the input segment $\epsilon(\tau)$ on $[t, t + \Delta t]$.

Note that for sufficiently small Δt a smooth input segment on $[t, t + \Delta t]$ is defined with reasonable accuracy by $\epsilon(\tau) \simeq \epsilon(t) + (\tau - t)\dot{\epsilon}(t)$. On the other hand, the deformation of interest here is the one that is referred to the configuration at time t . Therefore, of the two quantities $\epsilon(t)$ and $\dot{\epsilon}(t)$, the rate of strain $\dot{\epsilon}(t)$ will "control" the "future" deformation for small Δt . It is shown in Ref. 24 that these considerations lead one, under certain assumptions, to the conclusion that Eq. (2.5) must reduce to the following relation as $\Delta t \rightarrow 0$:

$$\dot{\underline{\tilde{q}}}(t) = g[\underline{\tilde{q}}(t), \dot{\epsilon}(t)] . \quad (2.6)$$

It can also be shown that in the case where the operator F is differentiable in the sense of Fréchet,²⁷ Eq. (2.6) further simplifies to

$$\dot{\underline{q}}(t) = \underline{g}[\underline{q}(t)] + \underline{h}[\underline{q}(t)] \dot{\epsilon}(t) . \quad (2.7)$$

We shall refer to the above equations as the growth law for the state of the system and shall assert that the materials considered here obey growth laws of the type (2.7). We shall also have the occasion to show that in certain circumstances a growth law of type (2.6) may be used with advantage for some of the materials.

We now make the fundamental assertion that Eqs. (2.4) and (2.7) constitute a representation of the mechanical behavior of the solid. Indeed for a given strain history $\epsilon(\tau)$, we can integrate (2.7), starting with appropriate initial conditions such as

$$\underline{q}_i(0) = 0 \quad (2.8)$$

(which imply that the initial state of the material is represented by the origin of the state space), to obtain the state $\underline{q}_i(t)$ of the element at some time $t > 0$. The stress at time t is then obtained with the help of (2.4).

Observe that (2.4), (2.7), and (2.8) provide one with a bare outline of the representation. For a given material the dimension of the state space n and the functions $\underline{f}$, $\underline{g}$, and $\underline{h}$ must be determined by experiments and/or from the knowledge of the internal structure of the material. We defer the discussion of this important question until Sections 2.5 to 2.7.

We wish now to show that linear models employed in the study of linear viscoelastic solids give rise to the representation of the type (2.4) and (2.7). By studying these simpler representations in some detail we hope to prepare the groundwork for Sections 2.4 to 2.7.

Linear viscoelastic solids

We consider first the Maxwell model shown in Fig. 12. If E and μ denote the spring constant and the viscosity of the dashpot, respectively, and if q denotes the elongation of the spring, then the stress carried by the model is given by

$$\sigma = E q , \quad (2.9)$$

whereas $\dot{\epsilon}$, the rate of elongation of the model, is found by observing

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Fig. 12. Maxwell solid.

that the dashpot is subject to the same stress and hence

$$\dot{e} - \dot{q} = (Eq)/\mu$$

or

$$\dot{q} = -\frac{E}{\mu} q + \dot{e} . \quad (2.10)$$

We see that (2.9) and (2.10) are a very special case of (2.4) and (2.7). In the present case, $n = 1$ and the elongation of the spring plays the role of a state variable.

To introduce a model with two state variables, we next consider the well-known arrangement of springs and dashpots shown in Fig. 13. We let E and c_2 denote spring constants and μ_2 and μ_3 the viscosities of the dashpots. The force applied at point A will represent the stress, and the displacement of A will play the role of strain.

Fig. 13. Four-parameter model of linear viscoelastic solid.

We have, then, as a result of simple calculations,

$$\sigma = E q_1, \quad (2.11)$$

and

$$\dot{q}_1 = -\frac{1}{\mu_3} (E q_1 - c_2 q_2) - q_1 \frac{E}{\mu_3} + \dot{\epsilon}, \quad (2.12)$$

$$\dot{q}_2 = \frac{1}{\mu_2} (E q_1 - c_2 q_2).$$

where q_1 and q_2 are elongations of the springs.

In the above equations q_1 and q_2 should be interpreted as state variables, so that (2.11) now plays the role of (2.4), and (2.12) corresponds to the growth law (2.7) for the material under consideration.

Note that the growth law (2.12) has the vectorial form

$$\dot{\underline{q}} = \underline{g}(\underline{q}) \cdot \underline{h} \dot{\epsilon}, \quad (2.13)$$

where $(h_1, h_2) = (1, 0)$ and

$$g_i = \sum_{j=1}^2 A_{ij} q_j \quad i = 1, 2, \quad (2.14)$$

with

$$A_{ij} = \begin{bmatrix} -\frac{E}{\mu_0} + \frac{E}{\mu_0} & \frac{C_2}{\mu_0} \\ \frac{E}{\mu_0} & -\frac{C_2}{\mu_0} \end{bmatrix}.$$

We adjoin to the growth laws (2.12) the initial conditions

$$q_1 = q_2 = 0 \quad \text{at} \quad \tau = 0. \quad (2.15)$$

For a given strain history $\epsilon(\tau)$, Eq. (2.12) is integrated with the help of (2.15) to determine the state vector $\underline{q}(\tau)$ with components $q_1(\tau)$ and $q_2(\tau)$. As the strain $\epsilon(\tau)$ develops over the course of time, the end point of the vector $\underline{q}(\tau)$ describes a trajectory in the state space. The first coordinate of the state point (q_1, q_2) then determines the stress by (2.11).

Relaxation lines

For purposes of later discussions, it is desirable to convert the information in the growth law (2.12) into a geometrical form. This could be achieved by observing that $\underline{g}(\underline{q})$ and $\underline{h}$, which appear in (2.13), determine two vector fields in the state space. These vector fields create two families of curves in the state space by means of their envelopes. The family of curves created by the field $\underline{g}(\underline{q})$ will be called

the relaxation lines. If during a given phase of a test ϵ is kept constant, then the state point moves, during this phase, along a relaxation line; the motion of the point is defined with $\dot{\epsilon} = 0$ as

$$\dot{\underline{q}} = \underline{g}(\underline{q}) \quad (2.16)$$

or, equivalently,

$$\dot{q}_i = \sum_{j=1}^2 A_{ij} q_j.$$

As shown in Fig. 14 the relaxation lines converge to the origin. One

Fig. 14. Motions in the state space for the four-parameter model of Fig. 13.

relaxation line is straight; it corresponds to one of the eigen directions of the relaxation matrix A_{ij} . When the constants E , c_2 , μ_2 , and μ_3 are positive and finite, A_{ij} has two real negative eigenvalues. This indicates that A_{ij} can be diagonalized by a suitable transformation of coordinates.

Since $\underline{h}$ is the constant vector $(1, 0)$, the family of curves created by $\underline{h}$ are simply lines parallel to the q_1 axis. These lines may be called the lines of fast deformation. As can be seen from (2.12), when $\dot{\epsilon}$ is (infinitely) large the state point moves along a line of fast deformation.

We may now discuss the images of some simple tests in the state space. A relaxation test, which involves a suddenly applied strain that is maintained subsequently, has an image such as OACD in the state space for the material under consideration. The segment OA describes the sudden application of the strain; the relaxation segment ACD is the path that the state point traverses during subsequent times where ϵ is kept constant. Note that both the stress and the state of the material relax to zero during a relaxation test.

A creep test involves a suddenly applied stress that is maintained subsequently. In view of (2.11) and (2.12) a suddenly applied stress causes a sudden strain that brings the state point, say, to A (Fig. 14). During subsequent times the stress and hence, by (2.11), the state variable q_1 are kept constant. We thus have for $\tau > 0$,

$$q_1 = \frac{\sigma_0}{E},$$

$$q_2 = \left[1 - \exp \left(-\frac{c_2}{\mu_2} \tau \right) \right] \frac{\sigma_0}{c_2}, \quad (2.17)$$

$$\epsilon(\tau) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\mu_2} \tau + \frac{\sigma_0}{c_2} \left[1 - \exp \left(-\frac{c_2}{\mu_2} \tau \right) \right].$$

The second equation in (2.17) is obtained from the integration of the last equation in (2.12). The last equation in (2.17) then follows from a simple integration of the first equation in (2.12).

We observe from the above equations that the state point moves during a creep test along a line parallel to the q_2 axis and becomes practically stationary at point B for times greater than μ_2/c_2 . During this phase the strain increases "linearly" with time, and therefore steady creep takes place. We note that at point B, the vector $\underline{g}$ is parallel to the q_1 axis.

As indicated above, the material defined by the model of Fig. 13 exhibits full relaxation of stress under constant strain. Conversely, under constant stress, steady creep takes place for large values of time.

We now consider the modification of this model obtained by the removal of the dashpot with viscosity μ_2 . This can be achieved by letting $\mu_2 \rightarrow \infty$. The resulting model is the familiar three-element model of the linear viscoelasticity. Relaxation lines for this model are parallel and straight (Fig. 15). In contrast with the previous model the relaxation vector $\underline{g}$ is zero not only at the origin but also along the line

$$Eq_1 - c_2 q_2 = 0. \quad (2.18)$$

The images of typical relaxation and creep curves are shown in Fig. 15. The relaxation test now ends at point D of the relaxation line defined by (2.18). The stress does not relax completely at the end of a relaxation test.

The creep test has the image OAB in the state space. As the last equation of (2.17) shows (with $\mu_2 = \infty$), the development of strain stops at B, so that this material does not exhibit steady creep.

A generalization of the model shown in Fig. 13 can be obtained by increasing the number of those units that consist of a parallel arrangement of a spring and dashpot as shown in Fig. 16. Such a model is associated in the following representation:

$$\begin{aligned} \sigma &= \sum_{i=1}^n E q_i, \\ \dot{q}_1 &= \sum_{i=2}^n \frac{1}{\mu_i} (E q_i - c_i q_i) - \frac{E q_1}{\mu} + \dot{\epsilon}, \\ \dot{q}_i &= \frac{1}{\mu_i} (E q_i - c_i q_i) \quad i = 2, \dots, n. \end{aligned} \quad (2.19)$$

Fig. 15. Motions in the state space for the model shown above.

Fig. 16. n-parameter model of linear viscoelastic solid.

We write (2.19) in matrix notation,

$$\begin{aligned}\sigma &= E \underline{q} \\ \dot{\underline{q}} &= \underline{A} \underline{q} + \underline{h} \dot{\epsilon}\end{aligned}\quad (2.20)$$

and observe that the vector $\underline{h}$ has the components (1, 0, ..., 0). It can be shown that when the constants E , μ_i , c_i , and μ are positive, the eigenvalues of the matrix $\underline{A}$ are real and negative. This observation implies that $\underline{A}$ can be diagonalized by means of similarity transformation; that is, one can find an $n \times n$ nonsingular real matrix $\underline{B}$ such that

$$\underline{D} = \underline{B} \underline{A} \underline{B}^{-1} \quad (2.21)$$

is a diagonal matrix.

We wish now to convert (2.19), which is one of the basic modes of representation of linear viscoelastic solids, into another fundamental form by means of the coordinate transformation

$$\underline{r} = \underline{B} \underline{q}, \quad (2.22)$$

where $\underline{B}$ is a nonsingular matrix that diagonalizes $\underline{A}$ and $r_1, \dots, r_n$ are the new state variables.

With the help of (2.22) the representation (2.19) can be written as

$$\sigma = E \sum_{i=1}^n (\underline{B}^{-1})_{1i} r_i, \quad (2.23)$$

$$\dot{\underline{r}} = \underline{D} \underline{r} + \underline{B} \underline{h} \dot{\epsilon},$$

where $\underline{D} = \underline{B} \underline{A} \underline{B}^{-1}$ is a diagonal matrix. Now assume that there are n distinct eigenvalues of $\underline{A}$ and denote these by $-(1/T_i)$ with $T_i > 0$ ($i = 1, \dots, n$). Moreover, observe that we can take $B_{11} = 1$ without any loss of generality, so that (2.23) becomes

$$\sigma = \sum_{i=1}^n K_i r_i, \quad (2.24)$$

$$\dot{r}_i = -\frac{r_i}{T_i} + \dot{\epsilon} \quad i = 1, \dots, n,$$

where K_i and T_i are positive constants. The above fundamental representation has the advantages that its growth law has the decoupled form and it exhibits individual relaxation times T_i explicitly.

Another advantage of this form is the existence of the potential

$$\phi = \frac{1}{2} \sum_{i=1}^n \frac{r_i^2}{T_i}, \quad (2.25)$$

which enables one to write the growth laws in the attractive form

$$\dot{r}_i = -\frac{\partial \phi}{\partial r_i} + \dot{\epsilon}. \quad (2.26)$$

It goes without saying that the form (2.24) is the representation of the model shown in Fig. 17.

FIG. 17. n -parameter model of linear viscoelastic solid.

2.4 Representation of Creep of Metals by Means of State Variables

We have seen in Chapter 1 that creep of metals is a highly nonlinear and intricate phenomenon. In Section 2.3 we have shown that the phenomenon of creep could be represented by means of a system of differential equations that govern the changes in the state of the material. We now wish to make precise statements, in the sense explained below, on the nature of the functions f , g , and h , which define this mode of representation [cf. Sect. 2.3, Eq. (2.7)].

We ask first whether the above-mentioned functions must have certain general properties if they are to represent creep of metals at high temperatures. Next we raise the question as to whether the classical laws of creep fall within the fundamental representation to be adopted

here. This question, which is answered affirmatively, enables us to arrive at a rational comparison of these laws.

Finally, we shall consider the fundamental problem of improving the "best" of these laws so that a more faithful representation of mechanical behavior can be achieved. This section ends with a lengthy discussion of the experimental determination of the number of state variables and the associated growth laws.

We take the first fundamental representation of linear solids (2.19) as a starting point. As explained before, this representation singles out or emphasizes the elastic strain defined as $q_1 = \sigma/E$ and has the form

$$\begin{aligned} \sigma &= E q_1, \\ \dot{q}_1 &= g_1(q_j) + \dot{\epsilon}, \\ \dot{q}_j &= g_j(q_j) \quad \begin{matrix} i = 2, \dots, n \\ j = 1, \dots, n \end{matrix}, \end{aligned} \tag{2.27}$$

where g_j represents linear functions of the state variables q_j ($j = 1, \dots, n$).

We observe that $\dot{\epsilon}$ occurs only in the first growth law and simply as $\dot{\epsilon}$. This implies that when $|\dot{\epsilon}|$ is large enough, the only important change in the state of the material occurs in the first component of the state vector $\underline{q}$. In other words, the solid defined by (2.27) behaves, in the presence of high strain rates, in the manner of an elastic solid. Now we wish to argue that isothermal creep of metals can be represented by (2.27) again, but in this case g_j 's will be nonlinear functions of the state variables $q_1, \dots, q_n$.

We observe first that the present value of the stress, or more conveniently σ/E , can be chosen, without loss of generality, as the first state variable q_1 . This observation is based on our definition of state, which asserts, among other things, that the two elements which are in the same state must carry the same stress. It follows that we can retain the first equation in (2.27) without loss of generality.

We next consider the implications of certain experimental observations in metals on the form of the growth law. Sternglass and Stuart²⁸ [and others²⁹] studied propagation of small-amplitude waves in a plastically prestrained metal. These waves are found to propagate with the

speed based on the elastic modulus of the material, but not on the speed based on the tangent modulus as the classical theory of plasticity would predict. The reason for this seems to be that the average speed of dislocations and the rate of dislocation production in metals are such that the sudden change in the dimensions of the crystal lattice is not accompanied by a sudden change in the arrangement of dislocations, namely, in the plastic state of the material. The Sternglass and Stuart experiment could be repeated in a solid that is undergoing creep deformations. The present authors were not able to find any such tests in the literature. Nevertheless, there is ample evidence that the creeping metals behave elastically during sudden reduction of stress (Fig. 5). If, instead, a fast incremental loading is considered, the resulting creep rate is often so high that it is difficult to assert that during incremental phase the material behaves elastically. In fact, it is customary in these tests to separate the sudden increase in strain into its elastic and plastic components. We take the view that this is done for the sake of mathematical convenience and also because we are seldom able to measure creep strain within the short periods of time that follow the load application. We argue that during the short periods of fast loading, the physics of creep is the same as in plasticity, so that a sudden increase in strain should not cause any appreciable change in the inelastic state of the metal. It goes without saying that these arguments lead to the growth laws in (2.27), where $\dot{\epsilon}$ appears only in the first equation. It may be noted that similar arguments cannot be applied to materials such as graphite, where sudden changes in stress may cause sudden occurrence of new internal cracks. We shall later have the occasion to reintroduce the notion of sudden plastic deformations in the way of mathematical convenience [Sect. (2.5)].

Classical laws of creep

We are now in a position to review the classical laws of creep. We start with Söderberg's law, which has the form

$$\dot{\epsilon} = \frac{\sigma}{E} + f(\sigma), \quad (2.28)$$

which constitutes a direct generalization of the Maxwell model (cf.

Sect. 2.3). This law can be written in the form of Eq. (2.27) as

$$\begin{aligned}\sigma &= E\epsilon, \\ \dot{\epsilon} &= -f(E\epsilon) + \dot{\epsilon}_c.\end{aligned}\quad (2.29)$$

Although this law is useful in calculating asymptotic behavior of structures at large values of time, it is not realistic enough in that it cannot give rise to primary creep often observed in metals.

The time-hardening law has the form

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + f(\sigma) g(t). \quad (2.30)$$

It constitutes an unfortunate modification of (2.28). The purpose of this modification is to render the law capable of representing primary creep. The shortcomings of (2.30) are well known (see, e.g., Rabotnov,⁵ pp. 200-205). Its greatest shortcoming is the explicit appearance of t in (2.30), which contradicts the assumed but generally not stated assumption that the materials under consideration are nonaging [cf. Eq. (2.3)].

Odqvist's law is a more successful and interesting attempt to modify (2.28) so that primary creep could be represented approximately (cf. Fig. 18). We shall discuss this law a little later.

The strain-hardening law is a very attractive representation and has the classical form

$$\begin{aligned}\dot{\epsilon} &= \frac{\dot{\sigma}}{E} + \dot{\epsilon}_c, \\ \dot{\epsilon}_c &= f(\sigma, \epsilon_c),\end{aligned}\quad (2.31)$$

where ϵ_c is the inelastic component of strain. The second equation above contains the basic assertion of the law that $\dot{\epsilon}_c$ depends only on the present values of stress σ and inelastic strain ϵ_c .

This law can be written in the fundamental form (2.27) by identifying q_1 and q_2 with σ/E and ϵ_c respectively. Thus (2.31) is equivalent to

$$\begin{aligned}\sigma &= E q_1, \\ \dot{q}_1 &= -f(E q_1, q_2) + \dot{\epsilon}, \\ \dot{q}_2 &= f(E q_1, q_2).\end{aligned}\quad (2.32)$$

Several forms for f , which defines this representation [(2.31) or (2.32)],

Fig. 18. State space representation of Odqvist law.

have been suggested in the literature. A detailed and readable account of these can be found in Rabotnov [Ref. 5, pp. 210-15]. Instead of repeating this discussion we wish to look at (2.31) or (2.32) from a different and possibly fresh view.

We note by comparing (2.31) with (2.27) that in this law the number of state variables is 2 and moreover that $g_1 = -f$ and $g_2 = f$.

Thus the relaxation lines in the (q_1, q_2) space make an angle of 45° with the coordinate axes (Fig. 19). We note therefore a similarity between the present case and Fig. 15 which describes the behavior of the linear model discussed in Section 2.3.

Fig. 19. Motions in the state space for the strain-hardening law.

We next note that the function $f(\sigma, \epsilon_c)$ can be determined by performing creep tests at various stress levels. It is easily seen that a creep test at a given stress level enables one to determine $\dot{\epsilon}_c$ as a function of ϵ_c and hence $f(\sigma, \epsilon_c)$ for a given σ . A typical $\dot{\epsilon}_c$ vs ϵ_c curve has the form shown in Fig. 20. If this curve has a nonzero asymptote, then the material is said to reach the state of steady creep at large values of time. For such a material the variation of the magnitude of the $\underline{g}$ vector at a given level q_1 would be as shown in Fig. 19. Note also that $\underline{g} = 0$ on the $q_1 = 0$ line.*

*The dependence of $\underline{g}$ on $\underline{q}$ in Fig. 19 could be modified so that $\underline{g}$ may be zero above the q_2 axis.

Fig. 20. Creep rate vs creep strain in a creep test.

The attractiveness of the strain-hardening law lies in the fact that it is simple and its predictions are supported in a qualitative way by some experimental observations. However, a careful study shows that the predictions of the strain-hardening law show systematic and serious deviations from certain test results.

To study these systematic deviations we consider first a relaxation test. This test has, in view of (2.32), the image OAB in a state space (Fig. 19). Thus, according to this law the stress decreases with increasing time and eventually becomes zero during a relaxation test. In Fig. 21 we compare a typical experimental result with the prediction of the strain-hardening law that is constructed with the help of creep tests. As Fig. 21 and other similar results show, at large values of time, the strain-hardening law invariably predicts lower stresses than experimental results, although often not by appreciable amounts [cf. Rabotnov (5), p. 217; and Lubahn (15), p. 372].

Next we consider tests that involve incremental loading and/or unloading. Figure 22 depicts stress histories employed in two tests performed at 800°F with Cr-Mo-V steel (Lequear and Lubahn²⁰). The difference between these two tests lies in the application, for a rather

Fig. 21. Comparison of test results with the prediction of the strain-hardening theory in a relaxation test [Rabotnov (5), p. 217].

short time interval, of the incremental stress of 10,000 psi to specimen II at the stress level of 76,000 psi.

The observed $\dot{\epsilon}_c$ vs ϵ_c relations are shown in Fig. 22. As seen from these figures, at the same inelastic strain of 0.003, the two specimens carry the same stress (76,000 psi); yet

$$\dot{\epsilon}_c \text{ II} \approx 10 \dot{\epsilon}_c \text{ I} ,$$

indicating that $\dot{\epsilon}_c$ cannot depend only on σ and ϵ_c for this material at 800°F. Note also that the portion AB of test II is rather similar in shape to the portion CD of test I, but it is translated to the right. This implies that the considerable inelastic strain developed during the short-lived action of the stress at 86,000 psi did not affect the inelastic state of the material appreciably; thus upon unloading to 76,000 psi, the state of the second specimen is only slightly different from the state of the first specimen at time t_1 (Fig. 22). Therefore the conclusion follows that the inelastic strain is not a good measure of the inelastic state of this material.

Fig. 22. Creep of Cr-Mo-V steel at 800°F under multistep loading (Lequear and Lubahn³⁰).

We next consider three tests conducted with 1100 Al at 300°F (Wang and Onat⁸). Tests I and II (Fig. 8a) are ordinary creep tests performed at 4000 and 3750 psi respectively. Tests III involve an incremental unloading at some time t_1 from 4000 to 3750 psi. The resulting variation of ϵ with time is shown in Fig. 8b. As seen in this figure, when $t_1 = 1$ hr the prediction of the strain-hardening theory is good; but when $t_1 = 4$ hr the prediction of the strain-hardening theory (the dashed line) is markedly different from the experimental curve, the observed $\dot{\epsilon}_c$ being much smaller (~ 4 times) than the predicted value. Observe also that

the curvature of this particular curve is different in sign than the curvature of the creep curves, which decidedly shows that the strain-hardening law fails under detailed scrutiny for this material.

A more detailed comparison of experimental results with the predictions of the strain-hardening theory in tests performed with lead are given in Ref. 18. In this work the relevance of the above discussed discrepancies on the analysis of the stability of structures is studied. It is shown that the Rabotnov-Shesterikov theory of stability, which is based on the strain-hardening law, could produce erroneous results.

A final comparison is offered by tests that might be called interrupted creep tests (Fig. 6), where the stress is reduced to zero over a long period of time. The strain response of several kinds of material to this type of stress history are shown in Fig. 6 in a rather schematic way. The dashed line indicates the prediction of the strain-hardening theory in the case where the initial loading has brought the material near the state of steady creep. As seen from this figure the effect of a long period under the zero stress may result in either the strengthening of the material (test 3) or softening of the material, although the softening seems to occur more often. If we accept the idea that softening is a more usual occurrence, then we must conclude that the waiting under zero stress has the effect of moving the representative state point toward the origin of the state space without appreciably changing ϵ_c . This observation once more shows that ϵ_c may not be a good measure of the inelastic state of the material.

We have seen that the predictions of the strain-hardening theory deviate in significant ways from the results of some important experiments. These deviations must be considered serious, since, as noted above, they invalidate the stability analyses based on the strain-hardening law. They also warn us that the use of strain-hardening theory may result in poor prediction of deformations of a structure that is subjected to intermittent loading. Therefore it would be desirable to look for a better, more faithful representation of mechanical behavior. One's first reaction to this problem would be to add new state variables (cf. Rabotnov,⁵ pp. 235-40). However, before attempting this it may

be worthwhile to retain the number of state variable at 2 but to choose for the second state variable a quantity other than the inelastic strain ϵ_c . We shall show in the next section that the main features of the above discussed experimental results could be represented with two state variables, if the state-space picture shown in Fig. 19 is appropriately modified. We shall argue that for metals the relaxation lines must be rather like the ones shown in Fig. 23. In Section 2.6 we shall raise the question of experimental determination of the state-space shown in Fig. 23.

2.5 A More Faithful Representation of Mechanical Behavior with Two State Variables

We wish now to abandon the main hypothesis of the strain-hardening theory while still retaining the number of state variables at 2. We shall show that the difficulties encountered with the strain-hardening theory could be overcome by introducing as the second state variable a quantity that is not the inelastic strain and by suitably modifying the g field of the state space.

First, let us recall that in the case of two state variables the representation (2.27) becomes

$$\begin{aligned}\sigma &= E q_1, \\ \dot{q}_1 &= g_1(q_1, q_2) + \dot{\epsilon}, \\ \dot{q}_2 &= g_2(q_1, q_2).\end{aligned}\tag{2.33}$$

It will be remembered that in the state space the field defined by the vector g with components g_1 and g_2 gives rise to a family of curves called relaxation lines (cf. Sect. 2.3). It goes without saying that the shape of these lines and the values of $|g|$ on them completely determine the representation (2.33). Since it is easier to discuss the geometry of a family of lines, we choose to describe the general features of a representation appropriate to metals by means of relaxation lines. The explicit determination of the functions $g_1(q_1, q_2)$ and $g_2(q_1, q_2)$ from the test results will be considered in the last part of Section 2.6.

We consider first materials that exhibit steady creep. In metals this case seems to occur definitely when the test temperature is greater

Fig. 23. Motions in the state space in a more faithful description of mechanical behavior at elevated temperatures ($\theta > 0.50_m$).

than $0.5\theta_m$. From the arguments given below and on the strength of experiments discussed in Chapter 1 and in Section 2.4, we assert that for metals with steady creep the relaxation lines and the distribution of $|g|$ in the state space must have the general features shown in Fig. 23.

The image of simple creep test in the state space (Fig. 23) would be the path OAB, where OA corresponds to sudden loading and AB to creep deformations. It is easily seen from (2.33) that at a point on AB, the rate of change of state $\dot{q}$, the strain rate $\dot{\epsilon}$, and g are related to each other by the vector diagram shown in Fig. 23. Thus we observe that point B corresponds to a state of steady creep, where no changes take place in q but the strains are produced at the rate of $\dot{\epsilon} = -\dot{\epsilon}_1$. This shows that we have already assumed that the state of the material remains stationary during fully developed steady creep.* We decide from the shape of experimental creep curves and (2.33) that $|g|$ must be large in the vicinity of A but must decrease as the state point moves toward B, where $|g|$ becomes equal to the asymptotic rate of strain $\dot{\epsilon}_m$.

Next we consider the interrupted creep test shown in Fig. 4. This test would have an image such as OABKLMB in the state space; here the horizontal segment AB corresponds to creep at 4000 psi. Sudden unloading takes place on BK. Along KI the stress is zero, but the state of the material is changing because of recovery. On LM the stress is increased suddenly to its previous level. This causes further creep deformations to occur while the state point moves once more on the segment MB.

The experimental results shown in Fig. 6 enable us to make the following observations: The fact that no important changes take place in strain under zero stress implies, in view of the construction mentioned

*To check the validity of this assumption we may conduct simple creep tests with two different but initially identical specimens and under a common stress. Suppose that the first specimen is left under this stress for a long time, so that $\dot{\epsilon}$ has been nearly constant for a long period of time. Suppose further that for the second specimen the corresponding times are shorter. If we can establish by further experimentation that these specimens behave in nearly the same manner under identical future stresses, we could decide that the above assumption holds. It is surprising that no such tests have been reported in the literature.

above, that g_1 must be extremely small on KL. On the other hand, the softening effect observed in Fig. 4 becomes important only when the waiting time under zero load is large (see Fig. 5). This indicates that g_2 must also be small on KL. Moreover, we expect that the slope of relaxation lines passing through KL will be extremely small. This argument follows from (2.33) and the observation that the change in q_2 over KL is considerable, whereas hardly any change is observed in ϵ during the same episode.

Furthermore, the tests of the type shown in Fig. 4 indicate that the recovery observed under zero load is never complete. This means that even for very long periods of zero stress the point L does not reach the origin of the state space. In view of (2.33) this observation implies that g must be nearly zero on the q_2 axis in a finite neighborhood of the origin. That there are no appreciable creep deformations in the presence of small stresses provides support for the generalization of this observation; namely, that $|g|$ must be extremely small in a finite neighborhood of the origin.

We now study test III ($t_1 = 4$ hr) of Fig. 8a. This test may have the image OAB_1C_1D in Fig. 23. Sudden reduction in stress at $t_1 = 4$ hr moves the state point from B_1 to C_1 . As the stress is kept at this new level the state point moves toward the state of steady creep D. We see from Fig. 8b that it takes about 20 hr for the strain to reach its asymptotic shape. This implies that the motion of the state point along C_1D must be much slower than it is on AB_1 . We also note from Fig. 8b that $\dot{\epsilon}$ at C_1 is considerably smaller than what it was prior to time t_1 . These observations imply that the magnitude of g and the slope of the relaxation line at C_1 are "small." Observe in passing that in test III with $t_1 = 1$ hr, B_1 and C_1 would be much closer to the q_1 axis. This would have the consequence that upon incremental unloading the state of the material would be closer to D, and therefore it would not be surprising that the behavior observed in Fig. 8b could ensue.

We finally consider the tests shown in Fig. 22. The images of the latter and more important parts of these tests in the state space would be EHB for test I and EFGHB for test II. A comparison of Figs. 22 and 23

shows that g_1 grows extremely fast as the state point moves from E to F. Moreover, previous observations concerning Fig. 22 (cf. pp. 37-39) suggest that the slope of the relaxation curve must be much larger on FG than on EH, so that large changes in ϵ could be produced on FG with only a small increase in q_m .

It is the above type of considerations that lead us to Fig. 23. We believe that the relaxation lines and the $|g|$ distribution shown in Fig. 23 is fairly typical for metals that exhibit steady creep.*

In order to offer a check on the validity of the assertions contained in Fig. 23 we may consider the hybrid test shown in Fig. 11, which starts as a creep test and upon partial unloading is continued as a relaxation test. We see from Fig. 11 that for the first 50 hr after partial unloading the stress increases by about 5%, whereby it reaches a maximum, and then it decreases by about 9% over the next 1200 hr.

We now ask the following question: Is the above experimental observation consistent with the representation depicted in Fig. 23? To answer this question, consider the image of the test in the state space. The creep phase of the test will be described by a path such as OAB. Upon partial unloading the state point descends on a new relaxation line and moves on it for further times to describe the relaxation phase of the test. Figure 11 implies that this motion of the state point on a relaxation line must be slow, particularly at large values of time; moreover, it must at first increase and then decrease the stress by relatively small amounts. We see that a relaxation line such as TS in Fig. 23 meets all these requirements: It is flat, it has the desired curvature, and on it the value of $|g|$ is small. Note that by varying the magnitude of unloading in such a hybrid test one can investigate in detail the relaxation lines that pass through the segment BT.

We now study the effect of the ambient temperature on Fig. 23. If the ambient temperature is between 0.25 and 0.50_{m} , steady creep may not

*It will be seen in the next section that a more careful analysis of data may suggest that two state variables may not be enough to represent mechanical behavior of some metals. In such a case the above kind of thinking has to be applied to a state space of higher dimension.

be present, but changes in state under zero loads may be possible. In such a case there would be no need to have points in the state space where the g vector is parallel to the q_1 axis. All one needs is to have $|g|$ decrease as the state point moves along a relaxation line. Thus it would be reasonable to expect that in the case of $0.25\theta_m < \theta < 0.5\theta_m$ the relaxation lines would have the form shown in Fig. 24. Note that this case is similar to the linear model discussed in Fig. 15, but the dependence of g on q is now nonlinear, the nature of nonlinearity being indicated by comments on $|g|$ in Fig. 24.

When the test temperature is less than $0.25\theta_m$, most metals of interest do not exhibit any appreciable recovery upon complete unloading.

Fig. 24. Motions in the state space for creep at intermediate temperatures ($0.25\theta_m < \theta < 0.5\theta_m$).

However, in this temperature range metals still exhibit creep or relaxation. But now these processes may be so fast (especially at room temperature) that special care must be exercised in observing them.

These comments suggest that relaxation lines may have the form shown in Fig. 25 when $\theta < 0.25\theta_m$. Note that in this figure $g = 0$ on and below the line PQ, which indicates that we consider as negligible any recovery that might develop in the present temperature range. If we want to include in our representation the usually negligible hysteresis type of phenomena observed in metals (see Fig. 26), we would allow $|g|$ to be different from zero below PQ.

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Fig. 25. Motions in the state space for mechanical behavior of metals at low temperatures ($\theta < 0.25\theta_m$); hysteresis effects are neglected.

2

Fig. 26. Hysteresis and meta-recovery at room temperature (Conrad¹²).

It should be clear that Fig. 25 represents a visco-plastic solid of the type considered by Malvern³¹ and other workers.³² Suppose now that above the line PR in Fig. 25,

$$|g| > C, \quad (2.34)$$

where C is a constant. It can be shown by using (2.33) that the state point will not travel above the line PR if the applied strain rate $\dot{\epsilon}(\tau)$ satisfies the inequality

$$\dot{\epsilon}(\tau) < \dot{\epsilon}_{cr}, \quad (2.35)$$

where $\dot{\epsilon}_{cr}$ is a finite value of strain rate that depends on the constant C and on the distribution of g vectors. Moreover, one could show that in a test where

$$0 < \dot{\epsilon}(\tau) < \dot{\epsilon}_{cr} \quad 0 \leq \tau \leq T,$$

the state point will move first along the line OP and then will remain within the region QPR.

It may so happen that for a given g distribution and C , the region QPR may be quite narrow. In such a case the material will behave very much like a classical elastic-plastic solid for the strain rates satisfying (2.35). Indeed the narrowness of the region QPR implies that in tests where $\dot{\epsilon} < \dot{\epsilon}_{cr}$ the stress-strain curve will be nearly insensitive to the magnitude of the rate of straining during a loading process. Also, the unloading curve will follow the initial elastic slope in view of the fact that $g = 0$ below the PQ line.

If in the above situation we are willing to ignore the slight dependence of the stress-strain curve on the rate of straining, we can replace the narrow region QPR by the line PQ. Once this is done the growth law in (2.33) must also be modified. The modification must be such that the motion of the state point on the PQ line is assured during a loading process. This can be achieved by allowing $\underline{h}$ to be tangential to PQ. The resulting representation will have the following form:

$$\sigma = E q_1 ,$$

$$\dot{q}_1 = \begin{cases} \dot{\epsilon} \\ h_1 \dot{\epsilon} \end{cases} , \quad \dot{q}_2 = \begin{cases} 0 & \text{when } q_1 < \gamma(q_2) \text{ or } q_1 = \gamma(q_2) \text{ and } \dot{\epsilon} < 0 \\ h_2 \dot{\epsilon} & \text{when } q_1 = \gamma(q_2) \text{ and } \dot{\epsilon} = 0 \end{cases} , \quad (2.36)$$

where $q_1 = \gamma(q_2)$ is the equation that defines the curve PQ and the vector $\underline{h}$, with components (h_1, h_2) , is tangential to PQ at the point of interest.

We recognize that the above representation defines a classical elastic-plastic material. Here E_Y is the yield stress, which depends on the plastic state q_2 of the material; and $E h_1$ is the tangent modulus, which also depends, via h_1 , on the plastic state q_2 . Note that in the above description the $\underline{g}$ vector vanishes everywhere and the region above the PQ line becomes inaccessible (see Fig. 27).

A simpler representation

We have observed that in creeping metals $|\underline{g}|$ will assume large values above some line PR in the state space. If we now restrict our

Fig. 27. Mechanical behavior of a classical elastic-strain-hardening solid.

attention to strain rates $\dot{\epsilon}(\tau)$ that are bounded from above and therefore satisfy an inequality such as (2.35), we can expect that the state points will not go much above the line PR. In such a case we can simplify Fig. 23 by replacing the motions above the line PR by motions on PR as shown in Fig. 28. To do this, introduce the unit vector $\underline{n}$ which is normal to PR and denote by $\underline{h}_0$ the vector with components (1, 0).

Now suppose that at a given instant the state point of an element lies on PR. The strain rate $\dot{\epsilon}$ applied to the element will produce in view of (2.33) the rate of change of state,

$$\dot{\underline{q}} = \underline{g}(\underline{q}) + \underline{h}_0 \dot{\epsilon} . \quad (2.37)$$

If $\dot{\underline{q}}$ takes the state point below the PR line, that is, if

$$(\underline{g}(\underline{q}) + \underline{h}_0 \dot{\epsilon}) \cdot \underline{n} \geq 0 ,$$

no modification is needed in (2.33). But if

$$(\underline{g}(\underline{q}) + \underline{h}_0 \dot{\epsilon}) \cdot \underline{n} < 0 ,$$

Fig. 28. State space representation of mechanical behavior when a partitioning of total strain into elastic, plastic, and creep components is introduced.

then the state point will start its journey into the region of extremely high g . Here we want to approximate this journey by movements on PR. Thus we introduce the following modification to (2.33):

$$\dot{\underline{q}} = \underline{h}\dot{\epsilon} \quad \text{when } \underline{q} \text{ is on PR and } (\underline{g} + \underline{h}_0\dot{\epsilon}) \cdot \underline{n} < 0, \quad (2.38)$$

where $\underline{h}$ is tangential to the line PR. The magnitude of $\underline{h}$ is found by considering the case where

$$(\underline{g}(\underline{q}) + \underline{h}_0\dot{\epsilon}) \cdot \underline{n} = 0. \quad (2.39)$$

When (2.39) holds, $\dot{\underline{q}}$ produced by (2.37) and (2.38) must be equal if a continuous dependence of $\dot{\underline{q}}$ on $\dot{\underline{\epsilon}}$ is expected. Thus the magnitude of $\underline{h}$ must be such that

$$|\underline{h}\dot{\underline{\epsilon}}| = |\underline{g} + \underline{h}_0\dot{\underline{\epsilon}}| \quad \text{when } (\underline{g} + \underline{h}_0\dot{\underline{\epsilon}}) \cdot \underline{n} = 0. \quad (2.40)$$

We can find from (2.40) the following explicit expression for $\underline{h}$:

$$\underline{h} = \underline{e}_1 - \left(\frac{\underline{e}_1 \cdot \underline{n}}{\underline{g} \cdot \underline{n}} \right) \underline{g}, \quad (2.41)$$

where $\underline{e}_1$ is the unit vector in the q_1 direction.

It may be interesting to discuss the material defined by the above statements in some detail. We first note that when the element is subject to sufficiently high strain rates [so that the inequality in (2.38) would be satisfied subsequently] the state point will move, after traversing OP, along PR. This motion is exactly like the one observed in the plastic solid discussed above. Thus it would be reasonable to call the strain accumulated in such an episode as plastic strain. If at a point A the loading is stopped and thus $\dot{\underline{q}}_1$ is rendered zero, the motion of the state point takes place along the line AB according to the growth law (2.33). The strain developed during this phase may be called creep strain (see Fig. 28).

The above representation replaces the motions of the state point in the region of large $\underline{g}$ by motions on the line PR. This approximation is an acceptable one if the strain rates do not assume very large values encountered in wave propagation problems. In the analysis of creeping structures a law of the above type could be used with advantage.

It is worth pointing out that the above representation constitutes a generalization of the Odqvist law, which has the form

$$\dot{\underline{\epsilon}} = \frac{\dot{\sigma}}{E} + \dot{\underline{\epsilon}}_{po} + f(\sigma),$$

where $\underline{\epsilon}_{po}$ is the so called plastic strain which includes some or the strain developed during primary creep (cf. Fig. 18).

Creep under stress reversal. Until now we have considered the cases where the stress σ and hence the state variable q_1 was nonnegative. In structures subjected to nonsteady loads the stress will change sign; so it is highly desirable to extend the description contained, say, in Fig. 23, to negative values of q_1 and possibly of q_2 . In the case where n remains 2 even under stress reversal, we offer Fig. 29 as a possible

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Fig. 29. Motions in the state space under load reversal.

description of creep behavior under general loading programs that include stress reversals. Note that we take the relaxation lines to be roughly symmetrical with respect to the origin of the state space. The magnitude of g is assumed to be very large above the line P_1R_1 and below the line P_2R_2 . In order to see that Fig. 29 constitutes a reasonable description of the expected behavior, consider a two-step test in which the first step is a creep test under positive stress and the second involves the subsequent application of a negative stress. The image of such a test in Fig. 29 would be OABCD. Note that as the state point moves over CD, primary creep occurs in the reverse direction until the asymptotic state of steady creep D is reached. This kind of behavior is what would be expected to occur in an actual test. Note from the above example that by suddenly unloading the material at point C, we can bring it to its original state. This assertion is a net result of the assumption that $n = 2$. It can be seen that when $n = 3$ it may not be possible to return to the original state by a test of this type (cf. Fig. 37).

It has recently come to our attention that Pugh, Liu, Corum, and Greenstreet,³³ in their work on constitutive equations for austenitic steels, have observed rightly that the classical strain-hardening law (2.31) should be supplemented by additional statements in the case where stress reversal is present. These authors suggest a set of modifications in (2.31) (based on certain ad hoc assumptions) so as to make it applicable to cases where stress reversal occurs. Our modification of the strain-hardening law as represented in Fig. 29, and presumably constructed by experiments of the type discussed in Section 2.6, would have the advantage that it would produce realistic predictions for creep recovery and stress reversal. We make this assertion on the basis that Fig. 29 will be constructed by tests which include those that contain stress reversals (cf. test III, Fig. 33).

2.6 Determination of the Number of State Variables and the Growth Laws from Phenomenological Experiments.

We have seen in Sections 2.4 and 2.5 that a great deal on the state variable representation could be learned from a study of results of suitable phenomenological experiments. In this section we would like to

explore this matter in some depth. It may be appropriate to issue a warning here that a systematic study of the problem of determining a representation of mechanical behavior from phenomenological experiments is complex. On one hand, there are statistical aspects of the problem associated with the scatter and sensitivity of experimental data, and on the other hand, there is the problem of interpolation that was discussed at the end of Section 2.1. This problem is closely related to the purpose of representation. For example, if the representation is to be used in a situation where the stress is expected to be a monotonic and slowly varying function of time, there would be no need to include the intricate effects of unloading. In the following developments we ignore such aspects of the problem and assume that experimental data are perfect and are available in suitable forms and in large quantities. Thus the availability of an infinite number of test results is accepted without apologies. We do this in the hope of creating a problem that is more amenable to mathematical thinking. We obviously hope that the results of our efforts could be modified to meet the actual situations where questions of approximation occur at so many levels.

The main question raised in this section is whether one can establish for a given material the dimension and the form of the representation

$$\sigma = \text{Eq}_1$$

$$\dot{\underline{q}} = \underline{g}(\underline{q}) + \underline{h}(\underline{q}) \dot{\epsilon} \quad (2.42)$$

by studying the results of phenomenological tests (each result being in the form of a pair of functions $\epsilon(\tau)$ and $\sigma(\tau)$ on the test interval $[0, T]$).

We may start providing answers to this question by establishing the formal relationship that exists between the notions of state and dependence on history. For this purpose, we may consider an experiment where two distinct strain histories are applied on the time interval $[0, T]$ to two different but initially identical test specimens (Fig. 30). We obtain, say, the stress responses shown in this figure.

Let us assume that the two histories $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$ give rise to the same stress at time t . Under this assumption we might begin to wonder whether the two specimens might be in the same state at time t .

Fig. 30. Two distinct strain histories on $[0, t]$ that give rise to the same stress at time t .

In order to investigate the situation further, we would like to apply identical future deformations to the two specimens, that is, identical future strains with respect to the configurations at time t . If the strains are small this would mean that the future strains measured with respect to the initial configurations at time $\tau = 0$ must differ from each other by the constant $\epsilon_1(t) - \epsilon_2(t)$. Figure 31 shows a pair of strain histories for $\tau > t$ that are identical from the present point of view.

Suppose now that we observe the same stresses in the two test bars on $[t, T]$, Fig. 31. Suppose further that this remains to be true for any extensions of $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$ into the future satisfying the requirement of

$$\epsilon_1(\tau) - \epsilon_2(\tau) = \epsilon_1(t) - \epsilon_2(t) \quad \tau > t.$$

Fig. 31. Example of two strain histories on $[0, t]$ that give rise to the same state at time t .

If the above condition exists, then we say that the strain histories $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$ on $[0, t]$ give rise to the same state at time t .

We may next inquire as to how many distinct states can be produced at time t by the application of all strain histories of interest on $[0, t]$. In general, the number of distinct states produced will be much less than the number of distinct histories on $[0, t]$.

In order to discuss the variety of states produced at time t one must introduce a suitable notion of distance between the distinct states.³⁴ It was shown in Ref. 24 that the metric space thus created is separable, so that one could speak of its dimensions and could map it into a Euclidean space. Here we shall assume that the state space can be mapped onto an open set Σ_t of the n -dimensional Euclidean space E_n . If one introduces the additional hypothesis that the material is nonaging, one finds that the state sets belonging to times $\tau \leq t$ are in the state set of time t .

Thus the state of the material prior to time t could be represented, by the mapping mentioned above, by a vector $\underline{q}$ of E_n . One could then pursue a line of thinking exactly the same as given in Section 2.3 to arrive at the representation (2.4) and (2.7). We are now ready to discuss the problem of the determination of the dimension of the state space.

The number of state variables. What kind of tests should one perform to obtain information on the number of state variables that occur in the description of mechanical behavior of a nonlinear material?* We first show that tests with piecewise linear inputs shown in Fig. 32 can provide, in principle, a lower bound to the number of state variables. In the figure, t and the number of intervals m are fixed. Thus each strain input $\epsilon(\tau)$ on $[0, t]$ is defined with the help of m numbers, namely $\epsilon_1, \dots, \epsilon_m$, which are the values of $\epsilon(\tau)$ at times $t/m, 2t/m, \dots, t$ respectively. For definiteness we assume that

$$|\epsilon_i| < C, \quad (2.43)$$

where C is a constant.

*If the material is linear a creep or relaxation test is all that is needed, in principle, to establish the entire representation.³⁶ In practice one is forced to perform vibratory and wave propagation tests to study relaxation mechanisms with short relaxation times.

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Fig. 32. Piecewise linear strain history.

Thus we see that each test belonging to the above-defined class can be represented by a point with coordinates $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ in the Euclidean space E_m . In view of (2.43) these points would occupy an open subset J of E_m . Now suppose that the given material is such that each test belonging to the above class gives rise to a distinct state at time t . This fact, if true, can be established, in principle, by continuing each test beyond $\tau = t$ in the manner described in Fig. 31. If the above assertion is true, a one-to-one correspondence is established between J and a subset S of the state space E_n . If this correspondence is bicontinuous, as we assume here, then the dimension of the subset S would be m . Since $S \subset E_n$, the dimension n of the state space must be greater than or equal to m , the dimension of the particular input space. This observation proves the assertion that the piecewise linear inputs such as defined above can be used to obtain lower bounds on n .

We believe that these rather abstract observations are actually of some practical value. They suggest among other things that the number of steps in a multistep test must be as large as n , the dimension of the state space, if the representation is to be constructed using such tests.

We now describe a method that would enable one to obtain local information on the g field and the dimension of the state space in the neighborhood of a state of steady creep.*

Suppose that a material is in the state of steady creep, so that it carries the constant stress σ_0 and is deforming at the constant strain rate $\dot{\epsilon}_0$ while retaining its constant state $\underline{q}^0$. Thus $\underline{q}^0$ satisfies, in view of (2.42), the following equation

$$\sigma_0 = E q_1^0,$$

$$g_1(\underline{q}^0) + \dot{\epsilon} = 0, \quad (2.44)$$

$$g_i(\underline{q}^0) = 0 \quad i = 2, \dots, n.$$

*We assume, of course, that the material under consideration is capable of steady creep at the ambient temperature of interest.

Let us now imagine that at some time t we disturb this state of steady creep by applying superimposed vibrations to the specimen or by passing a wave of small amplitude through it. However, we will not discuss the realization of such experiments here (see Ref. 36).

The specimen or, more correctly, a typical element of the specimen will no longer be in the state of steady creep for $\tau > t$. Let $\underline{q}(\tau)$ denote the new state of the element. If the applied disturbances are small, the difference

$$|\underline{q} - \underline{q}^0|$$

will remain small, at least over a certain time interval. Assuming that $g(\underline{q})$ is a well-behaved function in the neighborhood of the state $\underline{q}^0$, we can use, in view of the above observation, the Taylor expansion to obtain

$$g(\underline{q}) \simeq g(\underline{q}^0) + \frac{\partial g}{\partial \underline{q}}(\underline{q}^0) \cdot (\underline{q} - \underline{q}^0), \quad (2.45)$$

where the last term is a shorthand notation for

$$\sum_{j=1}^n \frac{\partial g_i}{\partial q_j}(\underline{q}^0)(q_j - q_j^0). \quad (2.46)$$

If we denote the strains for $\tau > t$ by $\epsilon(\tau)$, we have, in view of (2.42) and (2.45),

$$\begin{aligned} \dot{q}_i &= g_i(\underline{q}^0) + g_{i,\underline{q}}(\underline{q}^0) \cdot (\underline{q} - \underline{q}^0) = \dot{\epsilon}, \\ q_i &= g_i(\underline{q}^0) + g_{i,\underline{q}}(\underline{q}^0) \cdot (\underline{q} - \underline{q}^0). \end{aligned} \quad (2.47)$$

But when compared with (2.44),

$$\begin{aligned} \dot{q}_i &= g_{i,\underline{q}} \cdot (\underline{q} - \underline{q}^0) + \dot{\epsilon} - \dot{\epsilon}_0, \\ \dot{\epsilon}_i &= g_{i,\underline{q}} \cdot (\underline{q} - \underline{q}^0), \end{aligned} \quad (2.48)$$

while the first equation in (2.42) yields

$$\sigma = E q_1^0 + E(q_1 - q_1^0) . \quad (2.49)$$

Let $\Delta\sigma$, $\Delta\epsilon$, and Δq_i denote the differences $\sigma - \sigma_0$, $\epsilon - \epsilon_0$, and $q_i - q_i^0$ respectively. Then (2.48) and (2.49) become

$$\Delta\sigma = E \Delta q_1 ,$$

$$\Delta\dot{q}_1 = \sum_{i=1}^n \frac{\partial g_1}{\partial q_i} \Delta q_i + \Delta\dot{\epsilon} , \quad (2.50)$$

$$\Delta\dot{q}_i = \sum_{j=1}^n \frac{\partial g_i}{\partial q_j} \Delta q_j .$$

Equation (2.50) shows that as far as incremental stress and strain, $\Delta\sigma$ and $\Delta\epsilon$, are concerned, the given material behaves as a linear viscoelastic solid [cf. (2.20)]. Thus by studying the incremental quantities and with the experimental methods of linear viscoelasticity, we can obtain, in principle, the number and nature of relaxation mechanisms that govern this incremental behavior. In the case when the $n \times n$ matrix $\left[\frac{\partial g_i}{\partial q_j} \right]$ [which corresponds to the matrix $\underline{A}$ of (2.20)] has distinct real and negative eigenvalues, say λ_i , the relaxation times obtained from the above study would coincide with $-(\lambda_i)^{-1}$. Thus such a study would provide the number of state variables and also the eigenvalues of $\left[\frac{\partial g_i}{\partial q_j} \right]$ at a state of steady creep.

It may be pointed out, however, that the matrix $\left[\frac{\partial g_i}{\partial q_j} \right]$ may be singular or ill-conditioned at the point of interest, so that in such a case only a modified and less strong version of the above result might hold.

Onat and Wang, in their work on creep of lead,¹⁶ have studied the incremental quantities $\Delta\sigma$ and $\Delta\epsilon$ systematically (cf. Fig. 7). This work, when interpreted from the present point of view, suggests that in the case of lead $\left[\frac{\partial g_i}{\partial q_j} \right]$ may exhibit discontinuities at a state of steady creep.

Nevertheless, it is interesting to note that in the case of $\Delta\sigma > 0$ these authors were able to obtain relaxation times associated with the incremental deformations.

Determination of the growth laws from phenomenological experiments ($n = 2$)

Let us assume that, on the strength of the tests of the type discussed above, we have decided that the dimension of the state space for a particular metallic material and at some temperature is 2, so that the representation of mechanical behavior has the form

$$\sigma = E q_1, \quad (2.51)$$

$$\dot{q}_1 = g_1(q_1, q_2) + \dot{\epsilon}, \quad (2.52)$$

$$\dot{q}_2 = g_2(q_1, q_2).$$

We inquire now whether the functions g_1 and g_2 , which define the growth of the state vector by means of (2.52), can be obtained in principle from phenomenological experiments. We shall show below that this is indeed possible, and it will be seen that the ideas developed in this section can be generalized to the case of $n > 2$.

We observe first that if we cannot identify the second state variable q_2 with a measurable physical quantity, then we have a certain freedom in the choice of q_2 . Indeed, any monotonically increasing function of a given q_2 could be taken as a new second state variable. In the strain-hardening theory such a freedom does not exist since q_2 is identified as the inelastic strain ϵ_c right at the beginning.

The starting point of our present investigation is this freedom of choice of q_2 . To exploit this point we consider a creep test under the given constant stress σ_0 . This test gives rise to the motion of the state point along the

$$q_1 = \frac{\sigma_0}{E} = q_1^I$$

line in the (q_1, q_2) space (Fig. 33). The development of q_2 in this particular motion can be taken arbitrarily, except that $q_2(\tau)$ must have the form shown* in Fig. 34 if, as it is assumed here, secondary creep takes place for large τ under a constant q_2 . It is now an easy matter to see that g_1 and g_2 at the $q_1 = q_1^I$ level, namely, $g_1(q_1^I, q_2)$ and $q_2(q_1^I, q_2)$, can now be determined. Indeed the first equation in (2.52) gives us, for

*For convenience we take $\lim_{T \rightarrow \infty} q_2 = 1$ in this test.

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Fig. 33. Tests that lead to the determination of the g field in the state space.

Fig. 34. Assumed growth of q_2 in a creep test with $q_1 = q_1^I = \frac{\sigma_0}{E}$.

this creep test,

$$\dot{q}_1 = g_1[q_1^I, q_2(\tau)] + \dot{\epsilon}(\tau) = 0, \quad (2.53)$$

where $\dot{\epsilon}(\tau)$ is available from measurements (see Fig. 35). In view of assumed form of $q_2(\tau)$, Fig. 34, the above equation determines $g_1(q_1^I, q_2)$. The second equation (2.52), which now has the form

$$\dot{q}_2(\tau) = g_2[q_1^I, q_2(\tau)], \quad (2.54)$$

determines $g_2(q_1^I, q_2)$ in view of the assumed knowledge of $q_2(\tau)$.

Next consider a test II, which starts as a creep experiment with $q_1 = q_1^{II} = \text{const.} < q_1^I$. At a given time t_1 before the occurrence of secondary creep in this test, let the stress be increased to σ_0 and hence q_1^{II} to q_1^I . In Fig. 33 the image of this test in the (q_1, q_2) space is denoted by II. We assert that we can find q_2 at this instant by measuring $\epsilon(\tau)$ for $\tau > t_1$. This assertion is based on the observation that $\epsilon(\tau)$ in test II will have the form shown in Fig. 36, where the portion BC of $\epsilon(\tau)$ vs τ curve must coincide with a certain portion of the creep curve under $\sigma = \sigma_0$ (Fig. 35). This observation is the net result of the assumption that $n = 2$. Indeed, as Fig. 33 shows, the growth laws for tests I and II are identical over the episode B'C' in the state space, and therefore the observation follows concerning the shape of $\epsilon^{II}(\tau)$ curve

Fig. 35. Strains produced in the creep test with $q_1 = q_1^I = \frac{\sigma_0}{E}$.

for $\tau > t_1$. [If there were three state variables we could not assert that the curve BC in Fig. 36 would be identical in shape with a portion of the creep curve under σ_0 , since the tests I and II would have the images shown in Fig. 37 and there would be no coincidence of paths in the q_1, q_2, q_3 space].

A comparison of Fig. 35 with Fig. 36 in connection with Fig. 33 shows that

$$q_2^{II}(t_1) = q_2^I(t),$$

where t is the time in Fig. 35 such that $\dot{\epsilon}(t)$ in this test is equal to $\dot{\epsilon}(t_1 + 0)$ in the second test. Thus we have determined the state at which the second specimen finds itself at time t_1 . Note as a corollary that we are not able to assign arbitrary values to $q_2(t)$ at the level of q_1^{II} once such an assignment has been carried out at the level of q_1^I . It should now be clear that we can find $q_2(\tau)$ that develops in a creep test at the $q_1 = q_1^{II}$ level by repeating the test II at other values of t_1 . Once $q_2(\tau)$ is obtained at the level of q_1^{II} , then $g_1(q_1^{II}, q_2)$ and $g_2(q_1^{II}, q_2)$ could be determined following the arguments given in connection with Eqs. (2.53) and (2.54).

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Fig. 36. Strain history associated with test II of Fig. 33.

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Fig. 37. The images of tests I and II of Fig. 33 in the case of $n = 3$.

Let now the line PQ be the locus of points of the g field where $g_2 = 0$. In other words, PQ is the locus of the states of secondary creep (Fig. 33). It is easily seen that the g field can be determined by the type of tests discussed above in the domain that lies between the PQ line and the $q_1 = q_1^I$ level (Fig. 33). For the determination of the g field above the q_1^I level, we must undertake similar tests with $q_1^{II} > q_1^I$ and must decrease the stress to the σ_0 level at a time t .

In order to determine the g field below PQ, we must make tests where $q_1 = q_1^I$ at the beginning. After a large value of time, q_1 is decreased to some value q_1^{II} and maintained at this value until it is brought up again at time t_2 to the q_1^I level (test III, Fig. 33). It can be shown that this test would enable one to determine the state of the specimen at time t_2 . By a repetition of the arguments associated with (2.53) and (2.54), we can conclude that the region below the PQ line can be constructed with the help of the tests of type III.

2.7 Representation of Mechanical Behavior in the Presence of Time-Dependent Temperatures

So far we have been concerned with the description of mechanical behavior of metals at constant (elevated) temperatures, although the effects of the magnitude of the ambient temperature on creep of metals were discussed in Section 2.5.

In this section we assume that the temperature is raised simultaneously with the loads or strains. As is well known the creep phenomenon is the end result of a variety of thermally activated processes. Some of these processes do not occur below a certain temperature level. For instance, there is strong evidence that grain-boundary sliding is not an important cause of creep deformations if the temperature is less than half of the melting temperature.³⁷ A review of the temperature dependence of thermally activated processes that cause creep is given in Garofalo [Ref. 2, pp. 66-101].

Here we shall have a brief look at the phenomenological basis of thermomechanics of creeping materials. From a phenomenological point of view we may consider as the input of the problem the temperature and

strain histories, $\theta(\tau)$ and $\epsilon(\tau)$ respectively, that are applied to a given specimen. [We assume that at the start of each test $\tau = 0$, $\theta(0) = \theta_0 = \text{const.}$, and $\epsilon(0) = 0$, and we envisage that the specimens are kept under constant temperature θ_0 and zero stress for a sufficiently "long" time prior to testing so that they may be considered, in the sense discussed before, as nonaging.] As the output we take the stress history $\sigma(\tau)$ and the amount of heat $Q(\tau)$ that is supplied to the specimen since the inception of the test at $\tau = 0$. As shown by Onat,³⁴ it is possible to replace $Q(\tau)$ by the internal energy $e(\tau)$ of the specimen by using the first law of thermodynamics. Note also that the roles of input and output pairs may be exchanged to suit the particular testing procedure employed.

The general study of the relationship that exists between the input and output pairs, $\theta(\tau)$; $\epsilon(\tau)$ and $e(\tau)$; $\sigma(\tau)$, is in the domain of thermodynamics of irreversible processes. It would be desirable to pursue the study of this relationship within this thermodynamic context with a view of obtaining the restriction that the second law of thermodynamics places on this relationship. However, we shall not pursue this path here and refer the reader to the work of Coleman,³⁸ Chu,³⁹ and Schapery.⁴⁰ Instead, we shall consider in some detail a particular aspect of the problem, that is, the dependence of stress upon the histories of temperature and strain. We propose to represent this dependence with the help of state variables, which we take as the current temperature θ of the specimen and the parameters $q_1, \dots, q_n$ which represent the salient features of the current internal structure of the specimen. As before, we take q_1 to be the "average" lattice strain in the direction of load application. The current stress then can be written as

$$\sigma = E[q_1 - \alpha(\theta - \theta_0)] , \quad (2.55)$$

where α is the coefficient of thermal expansion and E is the elasticity modulus. Both E and α may be given functions of θ . The growth laws (2.42) now take the form

$$\begin{aligned} \dot{\theta} &= \dot{\epsilon} , \\ \dot{q}_j &= g_j(\theta, q_j) + h_j(\theta, q_j)\dot{\epsilon} + k_j(\theta, q_j)\dot{\theta} \quad 1, j = 1, \dots, n . \end{aligned} \quad (2.56)$$

The first equation in (2.56), which follows from the fact that θ is both a component of the input and a state variable, could be omitted.

For metals it would be reasonable to assume that $h = 1, 0, \dots, 0$ and $k = 0, \dots, 0$, which is equivalent to saying that the rapid changes in the strain and temperature leave the basic inelastic state variables $q_1, \dots, q_n$ unchanged (cf. Sect. 2.4). This is quite close to the hypothesis used by Dorn¹² that the "internal structure" of the metal remains constant as the temperature is changed rapidly. Under these hypotheses the relationship between $\sigma(\tau)$ and $\theta(\tau)$ and $\epsilon(\tau)$ accepts the representation

$$\begin{aligned}\sigma &= E[q_1 - \alpha(\theta - \theta_c)] , \\ \dot{q}_j &= g_j(\theta, q_j) + \dot{\epsilon} \quad j = 1, \dots, n , \\ \dot{q}_i &= g_i(\theta, q_j) \quad i = 2, \dots, n .\end{aligned}\tag{2.57}$$

This form is quite similar to (2.42), but the temperature dependence of g is a new feature.

Note that if one wants to introduce the dependence of mechanical behavior on the ambient temperature that exists before the testing starts, then one could write g_j as

$$g_j = g_j(\theta_0; \theta - \theta_0; q_1) ,\tag{2.58}$$

with the condition that

$$g_j(\theta_0; 0; 0) = 0 ,\tag{2.59}$$

which expresses the requirement that a virgin specimen does not change its state, in view of previous remarks, under the constant ambient temperature and in the absence of any straining. When isothermal tests, discussed in the previous section, are repeated under various levels of temperature, they shed some light on the functions $g_i(\theta_0; 0; q_j)$. As discussed in Chapter 1 such tests suggest that $g_i(\theta_0; 0; q_j)$ may have the form

$$g_i(\theta_0; 0; q_j) = \hat{g}_i(q_j) \exp \left(-\frac{Q}{R\theta_0} \right), \quad (2.60)$$

where Q is the activation energy for the steady creep and R is the gas constant. In view of (2.57) this relationship implies that at higher temperatures the motions in the state space take place faster. But that the rate of change of each state variable is affected by the ambient temperature in the manner of (2.60) is a little difficult to defend, in view of the expectation that different thermal processes are associated with different activation energies. Incidentally, Eq. (2.60) predicts that the shape of relaxation lines in the state space remains fixed as the ambient temperature is varied. Equation (2.60) also implies that the magnitudes of g vectors will decrease by the same proportion throughout the state space as θ_0 is decreased. Whether this is true could be ascertained only when the g fields are constructed experimentally according to the ideas discussed in the previous chapter at various levels of ambient temperature. In any event our speculations concerning Fig. 37 suggest that we expect significant changes to occur in the shape of relaxation lines with decreasing ambient temperatures, so that we regard (2.60) only as a first approximation. Now when we consider variable temperatures, it is tempting to assume that in the general case of time-dependent temperatures $g(\theta; q)$ has the form

$$\underline{g} = \underline{\hat{g}}(q_j) \exp \left(-\frac{Q}{R\theta} \right) \quad i = 1, \dots, n, \quad (2.61)$$

with $\underline{g}(0) = 0$. This very strong assumption seems to imply that all thermally activated processes that cause creep have the same activation energy, which is very probably not true if the temperature is allowed to vary broadly. Nevertheless, for reasons that will become apparent we pursue the implications of (2.61).

If one now introduces a new time scale ξ , as is done in the literature (cf. Ref. 2, p. 69), by means of

$$d\xi = \exp \left[-\frac{Q}{R\theta(\tau)} \right] d\tau. \quad (2.62)$$

The growth laws corresponding to (2.61) become

$$\frac{dq_1}{d\xi} = \hat{g}_1(q) + \frac{d\epsilon}{d\xi}, \quad (2.63)$$

$$\frac{dq_i}{d\xi} = \hat{g}_i(q).$$

We observe that (2.63) has exactly the same form as (2.42). Thus the introduction of (2.62) reduced the problem to the isothermal case. We may point out that the idea of using a temperature-modified time scale has also been used in the study of polymers (Ferry⁴¹ and Schwarzl and Staverman⁴²).

In order to verify the validity of (2.63) we may perform creep tests under variable temperature $\theta(\tau)$. Let $\epsilon_0(\tau)$ be the strains obtained in an ordinary creep test under the stress σ_0 and the constant temperature θ_0 ; on the other hand let $\epsilon(\tau)$ denote the strains observed in a test where the stress is σ_0 but the temperature is varied; and $\theta = \theta(\tau)$ with the initial condition $\theta(0) = \theta_0$. If

$$\xi = f(\tau) \quad (2.64)$$

denotes the integrated form of (2.62) for the given temperature history $\theta(\tau)$, then we must have, if we neglect the usually small thermal term $\alpha(\theta - \theta_0)$ in (2.57),

$$\epsilon[f^{-1}(\xi)] = \epsilon_0 \left[\xi \exp \left(\frac{Q}{R\theta_0} \right) \right]. \quad (2.65)$$

An experimental comparison suggested by (2.65) was carried out by Dorn⁴³ with pure aluminum. The creep tests were carried out under the stress $\sigma_0 = 14.06 \text{ kg/mm}^2$, but temperature was varied rather arbitrarily from 420 to 480°K. The tests indicated that (2.65) holds to a very remarkable degree. The reason for the success of this experiment may be that within the rather narrow temperature range employed one of the thermally activated processes was dominant, so that the form (2.61) could be used without introducing large errors in the representation.

2.8 Representation of the Three-Dimensional Behavior

We are now ready to consider the representation of mechanical behavior of an initially isotropic solid in the presence of arbitrary but small isothermal deformations. The assumption of small deformations implies that an element of solid will undergo only small shape changes and small rotations. To introduce the problem of representation formally, we may envisage thought experiments* which involve the application of a tensor-valued strain history $\epsilon_{ij}(\tau)$ on $[0, \infty)$ to a virgin element of the material and the observation of the resulting stress tensor $\sigma_{ij}(\tau)$. Here ϵ_{ij} and σ_{ij} denote, respectively, the components of the tensors of infinitesimal strain and stress in a fixed rectangular frame with coordinates x_i ($i = 1, 2, 3$).

The material under consideration acts, as a result of such tests, as a tensor-valued operator F_{ij} and produces out of each input $\epsilon_{ij}(\tau)$ an output $\sigma_{ij}(\tau)$:

$$\sigma_{ij}(\tau) = F_{ij}[\epsilon_{kl}(\tau)] \quad 0 \leq \tau < \infty. \quad (2.66)$$

For economy in writing we shall employ the following shorthand notation for (2.66):

$$\underline{\sigma} = \underline{F}[\underline{\epsilon}(\tau)], \quad (2.67)$$

where $\underline{\sigma}$, $\underline{F}$, and $\underline{\epsilon}$ stand for σ_{ij} , F_{ij} , and ϵ_{ij} respectively.

As before, the operator F is causal (cf. Sect. 2.2), and it will again be assumed that F is time invariant (thus the material is nonaging) and continuous.[†]

If the material is isotropic in its initial state, then $\underline{F}$ must possess the following additional property: Suppose that $\underline{\epsilon}(\tau)$ gives rise to

*The difficulties encountered in the realization of such experiments will be discussed later.

[†]The formal definition of continuity in Sect. 2.2 must be modified in a suitable manner for the present case.

$\underline{\sigma}(\tau)$ by means of $\underline{F}$; then the isotropy of the initial state implies that the strain history $\underline{\epsilon}_1(\tau)$ with the component

$$(\epsilon_1)_{ij} = Q_{ip} Q_{jq} \epsilon_{pq}(\tau),$$

or, more concisely,

$$\underline{\epsilon}_1 = \underline{Q} \underline{\epsilon} \underline{Q}^T \quad \text{on } 0 \leq \tau < \infty \quad (2.68)$$

must give rise, by $\underline{F}$, to the stress history

$$\underline{\sigma}_1 = \underline{Q} \underline{\sigma} \underline{Q}^T \quad \text{on } 0 \leq \tau < \infty, \quad (2.69)$$

where $\underline{Q}$ is the matrix of an orthogonal transformation in the three-dimensional Euclidean space E_3 . In other words, initial isotropy of the material implies that $\underline{F}$ must possess the invariance property

$$\underline{Q} \underline{F} [\underline{\epsilon}(\tau)] \underline{Q}^T = \underline{F} [\underline{Q} \underline{\epsilon}(\tau) \underline{Q}^T] \quad (2.70)$$

for any orthogonal transformation $\underline{Q}$ and E_3 and for any strain history $\underline{\epsilon}(\tau)$.

The representation of $\underline{F}$ by means of state variables may be established by an appropriate study of input-output pairs $[\underline{\epsilon}^{(i)}(\tau), \underline{\sigma}^{(i)}(\tau)]$ and without any reference to the microscopic happenings within the solid. This was done in a series of papers by Onat^{25,26} and by Geary and Onat⁴⁴ in the general case of finite deformations. In Ref. 34 a formal definition of state was introduced in the thermodynamics of continuous media. In Refs. 26 and 44, which constitute a sequel to Ref. 34, the implications of the usual requirement that the superimposed rigid-body rotations leave the state of the material invariant are studied in the case of initially isotropic solids.

The present case of small deformations is simpler and can be treated independently. Here we give a brief summary of the work reported in Ref. 45.

Consider two sets of elements. The elements in the first set have been subjected to the strain history $\underline{\underline{e}}_1(\tau)$ on $[0, t]$ and the elements in the second set to $\underline{\underline{e}}_2(\tau)$ on $[0, t]$. These two sets of elements are said to be in the same state and orientation (or simply in the same state) if identical future strain histories measured with respect to the configurations at time t give rise to identical stress histories in these elements.

This definition, together with the introduction of a suitable notion of distance between distinct states, enables one to construct a metric state space Σ_t . It is then shown, using the nonaging property of $\underline{\underline{P}}$, that the states created at time $\tau \leq t$ belong to the state space Σ_t .

Now in order to explore the implications of the initial isotropy of the material, consider a strain history, say, $\underline{\underline{e}}(\tau)$ on $[0, t]$, that gives rise to the state $S \in \Sigma_t$. Next consider the strain history

$$\underline{\underline{e}}_1(\tau) = \underline{\underline{Q}} \underline{\underline{e}}(\tau) \underline{\underline{Q}}^T \quad \text{on } [0, t],$$

where as before $\underline{\underline{Q}}$ denotes an orthogonal transformation. Let S_1 be the state produced by $\underline{\underline{e}}_1(\tau)$. It can be shown by using (2.70) and the definition of state that S_1 will depend only on S and $\underline{\underline{Q}}$ but not on the particular choice of $\underline{\underline{e}}(\tau)$ provided that $\underline{\underline{e}}(\tau)$ belongs to S . Thus it would be permissible to write

$$S_1 = P_{\underline{\underline{Q}}} S, \quad (2.71)$$

where $P_{\underline{\underline{Q}}}$ can be interpreted as a mapping of Σ_t onto itself.

In order to understand the meaning of the transformation $P_{\underline{\underline{Q}}}$, we may observe that the strain histories $\underline{\underline{e}}(\tau)$ and $\underline{\underline{e}}_1(\tau) = \underline{\underline{Q}} \underline{\underline{e}}(\tau) \underline{\underline{Q}}^T$ are equivalent in some sense. Indeed if the strain history $\underline{\underline{e}}_1(\tau)$ is studied in the coordinate frame x'_i defined by

$$\underline{\underline{x}}' = \underline{\underline{Q}}^T \underline{\underline{x}}, \quad (2.72)$$

it will be found that the strain history $\underline{\underline{e}}_1(\tau)$ has the components $\underline{\underline{e}}(\tau)$ in the new coordinate frame x'_i . Therefore, since the material is isotropic, the observer who works in the new coordinate frame would decide

by further tests that from his point of view the material would be in the state S . But an observer who is working in the original coordinate frame x_1 will find that the element of interest is in the state and orientation S_1 and that, in general, $S_1 \neq S$. Clearly the states S and S_1 are closely related. One could say that they correspond to two elements that are in the same state but have different orientations.

One could easily establish that the set of transformations $\{P_Q\}$ (each transformation being created by an orthogonal transformation Q) constitutes a group that is homomorphic to the group O_3 of orthogonal transformations in E_3 . On the other hand, as a result of the particular notion of distance used in Ref. 34, each P_Q turns out to be a distance preserving transformation. Another property of P_Q is that it leaves the initial state S_0 invariant:

$$P_Q S_0 = S_0 \text{ for any } Q \in O_3.$$

It is shown in Ref. 44 that in the case where Σ_t is finite dimensional, the above properties of P_Q enable us to map Σ_t into a finite-dimensional Euclidean space E_m in such a way that in E_m the P_Q 's appear as linear transformations. This last observation enables us to assert that in the new state space E_m , $\{P_Q\}$ constitutes a representation of the full orthogonal group O_3 . This in turn brings the tools of the theory of group representations to within our reach.⁴⁶ In particular, we discover that we can use tensor-valued state variables to delineate the invariant subspaces of E_m under $\{P_Q\}$.

Following these arguments it is shown in Ref. 44 that the state S of a given element may be defined by a number of even-rank irreducible tensors, say, $q_1, \dots, q_n$. Irreducible tensors are tensors that possess certain properties of symmetry. For instance, second-rank irreducible tensors are traceless and are either antisymmetric or symmetric.

Thus the state of the material will be thought as the sum of n irreducible tensors,

$$S = q_1 + q_2 + \dots + q_n, \quad (2.73)$$

where each $\underline{q}_i$ defines the component of the state vector in a particular invariant subspace of E_m . (E_m is a direct sum of its invariant subspaces.) It is to be noted the notation S in (2.73) represents an m -dimensional vector in the E_m space and is not to be taken as an absolute scalar.

The action of the transformation P_Q on S is then given by

$$P_Q S = P_Q \underline{q}_1 + P_Q \underline{q}_2 + \dots + P_Q \underline{q}_n, \quad (2.74)$$

where $P_Q \underline{q}_i$ is a tensor of the same rank and symmetries as the $\underline{q}_i$. Indeed the action of P_Q on $\underline{q}_i$ constitutes an ordinary tensor transformation. Thus if $\underline{q}_1$ is a second-rank tensor, then $P_Q \underline{q}_1$ is defined explicitly in terms of its components as

$$(P_Q \underline{q}_1)_{ij} = Q_{ip} Q_{jq} (\underline{q}_1)_{pq}, \quad (2.75)$$

or, more concisely,

$$P_Q \underline{q}_1 = Q \underline{q}_1 Q^T.$$

If $\underline{q}_2$ is a fourth-rank tensor, then the components of $\underline{q}_2$ will transform according to the well-known rule, which is a generalization of (2.75).

The important consequence of these rather abstract considerations is that in the present case the state variables occur as even-rank tensors which possess certain symmetries. Thus the representations such as (2.27) and (2.33) discussed earlier can be adapted to the present general case by replacing q_i in an appropriate manner by the tensorial state variables $\underline{q}_i$. For instance, the representation (2.27), which is suitable for metals, can be generalized in the following way:

$$\begin{aligned} \underline{\sigma} &= 2\mu \underline{q}_1 + \lambda (\text{tr } \underline{q}_1) \underline{I}, \\ \underline{\dot{q}}_1 &= \underline{g}_1(\underline{q}_1, \dots, \underline{q}_n) + \underline{\dot{\epsilon}}, \\ \underline{\dot{q}}_i &= \underline{g}_i(\underline{q}_1, \dots, \underline{q}_n) \quad i = 2, \dots, n, \end{aligned} \quad (2.76)$$

where λ and μ are the Lamé constants, $\underline{q}_1$ is a second-rank symmetric tensor that measures the elastic component of the strain tensor, and $\underline{q}_2, \dots, \underline{q}_n$ are tensors of even rank that represent the inelastic state of the material element. Furthermore, in (2.76) $\underline{I}$ denotes the Kronecker delta, δ_{ij} , and $\text{tr } \underline{q}_1$ stands for the trace of tensor $\underline{q}_1$.

It will be found that initial isotropy of the material imposes further restrictions on the growth laws in (2.76). These restrictions are that for any $\underline{Q} \in O_3$ and for any state $\underline{q}_i$,

$$\underline{g}_1(\underline{P}_{\underline{Q}} \underline{q}_1, \underline{P}_{\underline{Q}} \underline{q}_2, \dots, \underline{P}_{\underline{Q}} \underline{q}_n) = \underline{P}_{\underline{Q}} \underline{g}_1(\underline{q}_1, \dots, \underline{q}_n), \quad (2.77)$$

which imply that the growth laws must be form invariant under the transformations $\{\underline{P}_{\underline{Q}}\}$.

The representation can be simplified further if one introduces the generally accepted hypothesis that in metals volume changes do not affect, and are not affected by, the inelastic state of the material. Under this hypothesis, we retain the elastic relation for the average normal stress and volume change,

$$\text{tr } \underline{\sigma} = 3K \text{tr } \underline{\epsilon}, \quad (2.78)$$

where K is the bulk modulus of the material.

If we now introduce the deviatoric components of stress and strain, namely,

$$\underline{s} = \underline{\sigma} - \frac{1}{3} \underline{I} \text{tr } \underline{\sigma},$$

and

$$\underline{e} = \underline{\epsilon} - \frac{1}{3} \underline{I} \text{tr } \underline{\epsilon}, \quad (2.79)$$

we can write the rest of the representation as

$$\begin{aligned} \underline{s} &= 2\mu \underline{q}_1, \\ \dot{\underline{q}}_1 &= \underline{g}_1(\underline{q}_1, \dots, \underline{q}_n) + \dot{\underline{e}}, \\ \dot{\underline{q}}_i &= \underline{g}_i(\underline{q}_1, \dots, \underline{q}_n) \quad i = 2, \dots, n, \end{aligned} \quad (2.80)$$

where $\underline{q}_1$ is a traceless second-rank symmetric tensor. Equations (2.80) imply that inelastic behavior lets itself be felt through the deviatoric components of stress and strain.

For conciseness it will be convenient to refer to the sum

$$\underline{q}_2 + \dots + \underline{q}_n$$

as the inelastic state of the material. We shall denote the inelastic state of the material by $\underline{S}$, so that the state of the material is given by

$$\underline{S} = \underline{q}_1 + \underline{S} . \quad (2.81)$$

It would be desirable to write the fundamental representation proposed for materials [i.e., (2.78) and (2.80)] in its entirety with the help of the notation just introduced:

$$\text{tr } \underline{\sigma} = 3K \text{ tr } \underline{\epsilon} , \quad (2.82a)$$

$$\begin{aligned} \underline{s} &= 2\mu \underline{g} , \\ \dot{\underline{g}} &= \underline{g}_1(\underline{g}, \underline{S}) + \dot{\underline{\epsilon}} , \\ \dot{\underline{S}} &= \underline{g}_S(\underline{g}, \underline{S}) , \end{aligned} \quad (2.82b)$$

where $\underline{g}_1$ and $\underline{g}_S$ are subject to the invariance requirements

$$\begin{aligned} P_Q \underline{g}_1(\underline{g}, \underline{S}) &= \underline{g}_1(P_Q \underline{g}, P_Q \underline{S}) , \\ P_Q \underline{g}_S(\underline{g}, \underline{S}) &= \underline{g}_S(P_Q \underline{g}, P_Q \underline{S}) , \end{aligned} \quad (2.83)$$

where $\underline{S} = \underline{q}_2 + \dots + \underline{q}_n$ and $P_Q \underline{S} = P_Q \underline{q}_2 + \dots + P_Q \underline{q}_n$.

Equation (2.82a) states that the relationship between the volume change and the average normal stress is elastic; (2.82b) constitutes the state variable representation of the relationship that exists between deviatoric stress and strain histories $\underline{s}(\tau)$ and $\underline{\epsilon}(\tau)$. Here $\underline{g}_S$ is a shorthand notation for $\underline{g}_2, \dots, \underline{g}_n$. Note that (2.82b) is quite similar formally to the much discussed representation (2.33), but it must be remembered that $\underline{S}$ stands for the entire inelastic state composed of tensors $\underline{q}_2, \dots, \underline{q}_n$.

The task that lies ahead is the determination of the number and the tensorial nature of the inelastic state variables $q_2, \dots, q_n$ and the associated growth laws for a given metal or for a given class of metals. This is a difficult task. An obvious source of difficulty lies in the realization of the thought experiments mentioned at the beginning of this section. Reliable experiments could be performed only in simpler situations. It would be difficult to think of a more sophisticated experiment than the combined tension, torsion, and internal pressure of a thin-walled cylindrical tube.* Yet the input-output pairs produced in such experiments contain two or three components (instead of six) that can be "controlled" at will. Nevertheless, these tests provide a wealth of information; most of our knowledge of creep and plasticity of metals in the presence of three-dimensional deformation comes from these tests.

Rank of state variables. The problem of representation of mechanical behavior of a solid in the presence of three-dimensional deformations not only involves the question of number of state variables but also the tensorial nature of these variables (their rank and symmetries). Here we shall inquire as to whether phenomenological experiments of the type just discussed can shed any light on the tensorial nature of the inelastic state variables.

We may initiate this discussion by considering some simple materials. It may be that for a given material, $\underline{s} = 0$ throughout its deformation. This is equivalent to saying that no inelastic state develops during the deformation of this particular material. It is easily seen that in this case the representation (2.82b) becomes

$$\underline{s} = 2\mu \underline{q}_1, \quad (2.84)$$

$$\dot{\underline{q}}_1 = \dot{\epsilon} + B \underline{q}_1 - \frac{3A}{\text{tr } \underline{q}_1^2} \underline{q}_1^2 + \dot{\underline{\epsilon}},$$

where the particular form of the growth law follows from the invariance requirement (2.83) and from the fact that $\text{tr } \dot{\underline{q}}_1 = \text{tr } \dot{\underline{\epsilon}} = 0$. In (2.84) the terms A and B are functions of $\text{tr } \underline{q}_1^2$ and $\text{tr } \underline{q}_1^3$. By eliminating $\underline{q}_1$

*For a description of such a testing procedure see, for instance, Ref. 47.

from (2.84) we obtain

$$\dot{\underline{s}} = \frac{\dot{\underline{s}}}{2\mu} + a\underline{I} + b\underline{s} - \frac{3a}{\text{tr } \underline{s}^2} \underline{s}^2, \quad (2.85)$$

where a and b are functions of $\text{tr } \underline{s}^2$ and $\text{tr } \underline{s}^3$. We recognize immediately that (2.85), together with (2.82a), represents a generalized Maxwell solid.

The following simple form of (2.85) is often used in the analysis of creeping bodies (cf. Ref. 5, p. 305):

$$\dot{\underline{s}} = \frac{\dot{\underline{s}}}{2\mu} + F(J_2) \underline{s}, \quad (2.86)$$

where $J_2 = 1/2 \text{tr } \underline{s}^2$.

Next we consider linear materials. A material is said to be linear if the operator F in (2.66) is additive and homogeneous. It can be shown that for a linear material the representation (2.82b) is also linear; namely, the tensor-valued functions $\underline{g}_1$ and $\underline{g}_2$ are linear functions of their arguments. What is interesting is that the linearity of (2.82b) together with (2.83) implies that each state variable q_i must be either a scalar or a traceless symmetric second-rank tensor.⁴⁴ This important observation may explain the tendency by certain authors to take the state variables as scalars or second-rank symmetric tensors even in the case of nonlinear materials. However, a consideration of the distribution of dislocations within a deformed element shows [Kröner (48)] that some of the state variables may be antisymmetric second-rank tensors or tensors of higher rank.

Now we may return to nonlinear materials. We shall claim that information on the rank of tensors $\underline{q}_1, \dots, \underline{q}_n$ which define the inelastic state of the material, may be obtained by studying symmetries of a deforming element. For this purpose consider first a test that involves the application of the stress history

$$\underline{s}_1(\tau) = \underline{a}(\tau) \quad 0 \leq \tau < \infty \quad (2.87)$$

and a second test with the stress history

$$\underline{s}_2(\tau) = \begin{cases} \underline{a}(\tau) & 0 \leq \tau \leq t \\ \underline{Q} \underline{a}(\tau) \underline{Q}^T & t < \tau < \infty \end{cases} \quad \text{when} \quad , \quad (2.88)$$

where $\underline{Q}$ is a fixed orthogonal transformation. We denote the deviatoric strain rates observed in the two tests by $\dot{\underline{e}}_1(\tau)$ and $\dot{\underline{e}}_2(\tau)$ respectively.

Note* that in both tests the material would be at the same inelastic state $\underline{s}(t)$ at time t .

Now we compare $\dot{\underline{e}}_1(\tau)$ and $\dot{\underline{e}}_2(\tau)$ for $\tau > t$. Can we expect that

$$\dot{\underline{e}}_2 = \underline{Q} \dot{\underline{e}}_1 \underline{Q}^T \quad \text{for } \tau > t ? \quad (2.89)$$

The answer to this question would be affirmative if

$$\underline{P}_{\underline{Q}} \underline{s}(t) = \underline{s}(t) . \quad (2.90)$$

To prove this assertion, consider a third test with the stress history

$$\underline{s}_3(\tau) = \underline{Q} \underline{a}(\tau) \underline{Q}^T \quad 0 \leq \tau < \infty . \quad (2.91)$$

In this last test the material element would be, by definition, at the state $\underline{P}_{\underline{Q}} \underline{s}(t)$ at time t , and, moreover, the strain rates observed in this test would be related to $\dot{\underline{e}}_1(\tau)$ by

$$\dot{\underline{e}}_3 = \underline{Q} \dot{\underline{e}}_1 \underline{Q}^T \quad 0 < \tau < \infty \quad (2.92)$$

in view of initial isotropy of the material. Now comparing (2.92) with (2.89) we decide that (2.89) would follow from (2.90). On the other hand whether (2.90) holds for a given $\underline{s}(t)$ and for some $\underline{Q}$ would depend, in general, upon the rank of tensors that make up $\underline{s}(t)$.

For instance, in the case where $\underline{s}(t)$ is composed only of scalars, we have, by definition of a scalar,

$$\underline{P}_{\underline{Q}} \underline{a}_i = \underline{a}_i \quad i = 2, \dots, n$$

*It goes without saying that $\dot{\underline{e}}_1(\tau) = \dot{\underline{e}}_2(\tau)$ when $\tau \leq t$.

for any $\underline{Q} \in O_3$, and hence (2.90) would be valid for any $\underline{Q}$ and $\underline{S}$. Incidentally, this observation implies that when the inelastic state variables are all scalars, a previously deformed but now stress-free element would be isotropic.

On the other hand, if $\underline{S}$ is composed of scalars and of a single second-rank symmetric tensor, then (2.90) would not be true for any $\underline{Q} \in O_3$. In fact, in this case Eq. (2.90) would hold only for those $\underline{Q}$ that represent 180° rotations about the principal axes of the tensorial state variable.

If, however, there are two second-rank state variables, then in general (2.90) would hold only for $\underline{Q} = \underline{I}$ unless principal directions of these state variables coincide or these variables are zero.

Note that to ask whether for an element in the state $\underline{S}(t)$ there exists values of $\underline{Q}$ which satisfy (2.90) is equivalent to the question: Are there symmetries in the inelastic state of a given element? Note also that the answer to these questions could be obtained by performing tests of the type (2.87) and (2.88) and by making comparisons suggested by (2.89). To explore the importance of this comment let us consider the case where

$$\underline{a}(\tau) = 0 \quad \text{for } \tau > t. \quad (2.93)$$

This means that the tests (2.87) and (2.88) would now be identical. Moreover, since the stress is zero for $\tau > t$, we are dealing with a recovery test. We wish now to see whether there exists $\underline{Q}$ which satisfy (2.90) at time $t + 0$. To answer this question we investigate whether (2.89) holds. But in the present case $\underline{e}_1(\tau) = \underline{e}_2(\tau)$, so that (2.89) becomes

$$\underline{\dot{e}}(\tau) = \underline{Q} \underline{\dot{e}}(\tau) \underline{Q}^T \quad \text{for } \tau > t. \quad (2.94)$$

Thus we simply measure the strain $\underline{e}(\tau)$ in the recovery part of the test and try to find a $\underline{Q}$ that satisfies (2.94).

Now consider the following possibilities: If $\underline{S}$ is composed of scalars, then (2.90) and hence (2.94) must hold for any $\underline{Q} \in O_3$. But the only deviatoric tensor that would satisfy (2.94) for any $\underline{Q}$ is the

one that vanishes:

$$\dot{\underline{\epsilon}}(\tau) = 0 \quad \tau > t .$$

Thus we arrive at the following important observation. If in a recovery test $\dot{\underline{\epsilon}}(\tau) \neq 0$ under $\underline{s} = 0$, then the state variables cannot be all scalars. Recall now the recovery test discussed in Section 2.5, where it was found that for most metals $\dot{\underline{\epsilon}}(\tau)$ is nonzero although it is small in magnitude. This implies that some of the state variables for these materials must be tensors of higher rank than zero.

Next, observe that (2.94) would have solutions if the principal directions of $\dot{\underline{\epsilon}}(\tau)$ remain fixed on the time interval (t, ∞) . Indeed the Q 's that represent rotations by 180° about these fixed directions would satisfy (2.94). This case would definitely occur if $\underline{s}$ is composed of scalars and of a second-rank symmetric traceless tensor (see our previous comment about this case). If, however, principal directions of $\dot{\underline{\epsilon}}(\tau)$ rotate over the course of time, we can decide that there must be more than one state variable with rank higher than zero.

We have thus shown that by studying the strains observed in a recovery test, we can obtain significant information on the rank of state variables. Similar ideas can be applied to other types of tests to obtain further information on the rank of tensorial state variables.

The present authors are not aware of the existence of any other systematic work in the literature that deals with the determination of the rank and number of state variables. The generalization of strain-hardening theory proposed by Rabotnov (Ref. 5, p. 313) seems to contain only scalar state variables. It may be of interest to record here the simplest case discussed by Rabotnov (ibid., p. 312) in our notation. He suggests that the inelastic state variable is a scalar q_0 and offers the representation

$$\begin{aligned} \underline{s} &= 2\mu \underline{q}_1 , \\ \dot{\underline{q}}_1 &= -\frac{3}{2} \frac{\partial \Phi}{\partial \sigma_0} \frac{q_1}{\sigma_0} + \dot{\underline{\epsilon}} , \\ \dot{q}_0 &= \frac{\partial \Phi}{\partial \sigma_0} , \end{aligned} \tag{2.95}$$

where

$$\sigma_0^2 = \frac{3}{2} \text{tr } \underline{s}^2 = \frac{3}{2} (2\mu)^2 \text{tr } \underline{g}^2$$

and

$$\Phi = \Phi(\sigma_0, q_2) .$$

Note from (2.95) that

$$\dot{\underline{g}} = \sqrt{\frac{2}{3} \text{tr } \dot{\underline{e}}_c^2} , \quad (2.96)$$

where $\dot{\underline{e}}_c$ is the creep strain defined by

$$\dot{\underline{e}}_c = \dot{\underline{e}} - \dot{\underline{g}} .$$

In view of (2.96) the inelastic state variable q_2 is related to the creep strain in the manner of the strain-hardening theory. It can be shown by arguments similar to those on pp. 33-34 that the law (2.95) is not capable of representing the softening of the material that may occur in a recovery test.

It is interesting to note the occurrence of the "potential" Φ in (2.95). This is rather reminiscent of the canonical form (2.26) discussed in Section 2.3. It would be interesting to inquire whether such a potential could be introduced in the growth law of (2.82b) in the present nonlinear case.

Simple creep test. At this stage it would be useful to review the generalization of uniaxial tests such as a simple creep test, an interrupted creep test, etc., to the present case. It was seen in Sections 2.5 and 2.6 that these tests provided vital information on the number of state variables and also on the growth laws in the representation of uniaxial behavior. We would expect that their three-dimensional counterparts may also serve a similar purpose.

We start with a simple creep test that involves the application of a constant deviatoric stress at $t = 0$; in other words, we consider an experiment where

$$\underline{g} = \frac{\underline{s}(t)}{2\mu} = \underline{a} = \text{const. for } t > 0.$$

We wish now to determine the resulting deviatoric strain $\underline{e}(t)$ from (2.82). The last equation in (2.82b) could be integrated with the initial condition $\underline{S}(0) = 0$ to yield

$$\underline{S}(t) = \underline{f}(\underline{a}, t). \quad (2.97)$$

In view of the invariance requirement (2.70) or, equivalently (2.83), $\underline{f}$ would possess the property

$$\underline{f}(P_Q \underline{a}, t) = P_Q \underline{f}(\underline{a}, t). \quad (2.98)$$

Now substituting (2.97) into the first equation of (2.82b) we obtain, for the strains developed in a creep test,

$$\dot{\underline{e}}(t) = -\underline{g}[\underline{a}, \underline{f}(\underline{a}, t)]. \quad (2.99)$$

In view of (2.98) and (2.83) we can decide that

$$\underline{g}[\underline{a}, \underline{f}(\underline{a}, t)]$$

is a form-invariant function of $\underline{a}$, so that (2.99) can be written as

$$\dot{\underline{e}}(t) = A \underline{I} + B \underline{a} - \frac{3A}{\text{tr } \underline{a}^2} \underline{a}^2, \quad (2.100)$$

where A and B are functions of $\text{tr } \underline{a}^2$, $\text{tr } \underline{a}^3$, and time t .

Equation (2.100) shows that principal direction of $\dot{\underline{e}}(t)$ will coincide with those of $\underline{a}$ or $\underline{g}$. If experiments do not confirm this finding, then one must decide that the material is not isotropic in its initial state. We see also that if $A = 0$, then $\dot{\underline{e}}$ is proportional to $\underline{a}$. If in an experiment it is found that $\dot{\underline{e}}$ is not proportional to $\underline{a}$, then the implication should be that $A \neq 0$. Johnson⁴⁹ in his much-quoted, careful experiments found that for a particular material $\dot{\underline{e}}$ is not proportional to $\underline{a}$, and he attributed this to anisotropy that may develop during deformation (cf. Fig. 104 of Rabotnov⁵ and the accompanying discussion). The presence of A in (2.100) may very well give rise to the results described in the

above-mentioned figure, so that a reinterpretation of Johnson's results becomes desirable.

If the material is capable of steady creep, then A and B must tend to constant values as t becomes large. If we denote the steady creep rate by $\dot{\underline{\underline{e}}}_c$, then we obtain, from (2.100) and with $\underline{\underline{s}} = 2\mu \underline{\underline{a}}$,

$$\dot{\underline{\underline{e}}}_c = a\underline{\underline{I}} + b\underline{\underline{s}} - \frac{3a}{\text{tr } \underline{\underline{s}}^2} \underline{\underline{s}}^2, \quad (2.101)$$

where a and b are functions of $\text{tr } \underline{\underline{s}}^2$ and $\text{tr } \underline{\underline{s}}^3$. There is some experimental evidence that for a class of metals the function a may vanish and b may depend only on $\text{tr } \underline{\underline{s}}^2$ (cf. Ref. 5, p. 329).

Now let $\underline{\underline{Q}}$ denote one of the coordinate transformations corresponding to 180° rotations about the principal axes of $\underline{\underline{a}}$. We have then

$$\underline{\underline{P}}_{\underline{\underline{Q}}} \underline{\underline{a}} = \underline{\underline{Q}} \underline{\underline{a}} \underline{\underline{Q}}^T = \underline{\underline{a}}. \quad (2.102)$$

This observation, together with (2.97) and (2.98), implies that the inelastic state $\underline{\underline{S}}(t)$ produced in a creep test would have the property

$$\underline{\underline{P}}_{\underline{\underline{Q}}} \underline{\underline{S}}(t) = \underline{\underline{S}}(t) \quad (2.103)$$

for the above-mentioned $\underline{\underline{Q}}, \underline{\underline{a}}$. Equation (2.103) has some important implications. It shows that antisymmetric second-rank state variables (if they are present) cannot grow in a creep test (the proof follows from the observation that an antisymmetric second-rank tensor is equivalent to an axial vector). Equation (2.103) also shows that the principal directions of the second-rank symmetric state variables that develop during a simple creep test must coincide with those of $\underline{\underline{a}}$.

A two-step test. We next consider a test where

$$\frac{\underline{\underline{s}}(t)}{2\mu} = \underline{\underline{g}}(t) = \begin{cases} \underline{\underline{a}} & \text{for } 0 < t < t_1 \\ \underline{\underline{b}} & \text{for } t_1 < t \end{cases} \quad (2.104)$$

and $\underline{\underline{a}}$ and $\underline{\underline{b}}$ are constants.

We can show, using arguments similar to the ones employed in connection with (2.97), (2.98), and (2.99), that in such a test

$$\underline{S}(t) = \underline{f}(\underline{a}, \underline{b}, t_1, t) \quad \text{when } t > t_1, \quad (2.105)$$

where $\underline{f}$ is a form-invariant function of $\underline{a}$ and $\underline{b}$, and

$$\dot{\underline{e}}(t) = \underline{g}[\underline{b}, \underline{f}(\underline{a}, \underline{b}, t_1, t)]. \quad (2.106)$$

The invariance requirement (2.83), together with the form invariance of $\underline{f}$, implies that $\dot{\underline{e}}(t)$ is a form-invariant function of $\underline{a}$ and $\underline{b}$. This, in turn, implies⁵⁰ that when $t > t_1$,

$$\begin{aligned} \dot{\underline{e}}(t) = & c_0 \underline{I} + c_1 \underline{b} + c_2 \underline{b}^2 + c_3 \underline{a} + c_4 \underline{a}^2 \\ & + c_5 (\underline{a} \underline{b} + \underline{b} \underline{a}) + c_6 (\underline{a}^2 \underline{b} + \underline{b} \underline{a}^2) \\ & + c_7 (\underline{b}^2 \underline{a} + \underline{a} \underline{b}^2) + c_8 (\underline{a}^2 \underline{b}^2 + \underline{b}^2 \underline{a}^2), \end{aligned} \quad (2.107)$$

where c_i 's are functions of $\text{tr } \underline{b}^2$, $\text{tr } \underline{b}^3$, $\text{tr } \underline{a}^2$, $\text{tr } \underline{a}^3$, $\text{tr } \underline{a} \underline{b}$, $\text{tr } \underline{a} \underline{b}^2$, $\text{tr } \underline{a}^2 \underline{b}$, $\text{tr } \underline{a}^2 \underline{b}^2$ and times t_1 and t . (We refrain from discussing explicitly the minor restrictions imposed upon the coefficients c_i , by $\dot{\underline{e}}(t)$ being traceless.) One reason for writing (2.107) in this explicit form is to point out that the dependence of $\dot{\underline{e}}(t)$ on the past history of stress may, in general, be quite complicated, even for an initially isotropic solid.

Now observe that if the state variables are all scalars, then $\underline{S}(t)$ would be a function of the invariants of $\underline{a}$ and $\underline{b}$ and therefore, in view of (2.106), (2.107) would be much simpler. It can easily be shown that in the case of scalar state variables the coefficients c_3 to c_8 would be zero in (2.107). This observation would enable one to assess experimentally the contributions of the higher-rank state variables to the behavior of a given solid. For instance, if the experiments suggest that the coefficients c_3 to c_8 are "small," then it may be appropriate to ignore the presence of higher-rank state variables.

Tests under constant inelastic state. Consider a thin-walled cylindrical tube that has been subjected to a given history of combined

tension and twist. For simplicity assume that the relevant stress components σ and τ are zero at a given time t . Let the inelastic state of the material at this instant be $\underline{S}(t)$. We want to explore the dependence of $\dot{g}_2$ on g_1 at the constant value of $\underline{S}(t)$ [cf. (2.82b)]. We can achieve this by suddenly changing g_1 and hence $\underline{S}$ and therefore* σ and τ . In the relevant stress space (σ, τ) the representative point would move from the origin to some other point. Let the stress point be kept at this new location for a short interval of time. Over this interval $\dot{g}_1 = 0$ and hence the measured strain rate $\dot{\underline{e}} = \underline{\gamma}_1$. We can then move the stress point to another location to measure $\dot{\underline{e}}$ and hence $\dot{g}_2$ at another value of g_1 . If we do these explorations within a short period of time, $\underline{S}$ will not change appreciably and hence the dependence of $\dot{g}_2$ on g_1 at a constant value of $\underline{S}$ will be studied.

By such an exploration we may discover that if the stress point (σ, τ) lies within some closed curve, say curve 1 in Fig. 38, then no measurable strain rate develops and hence $\dot{g}_2 \approx 0$ within this curve. We may also discover that if the stress point falls outside of another closed curve, say curve 2, very large values of $\dot{\underline{e}}$ will be observed.

The tests of Phillips and co-workers⁵¹ on pure aluminum are somewhat similar to the above proposed tests. However, in Phillips' tests the explorations in the stress space are carried out slowly and incrementally, so that the inelastic state $\underline{S}$ of the material may change appreciably while the points of the curve 1 are being determined. In fact, the yield curve found by Phillips, which corresponds to our curve 1, does not pass through the point of prestressing. This indicates that the inelastic state of the material has changed or, equivalently, recovery has taken place during testing. Nevertheless, Phillips' tests give us a good idea of how the region $\dot{g}_2 \approx 0$ would look like for the particular material considered.

By studying the shape and orientation of the regions defined by curves 1 and 2 one can gain information on the rank of the state variables. For instance, Phillips' tests show that the $\dot{g}_2 \approx 0$ region is not, in

*Note that, by the second equation of (2.82b), a sudden change in g_1 is not accompanied by a sudden change in $\underline{S}$.

Fig. 38. Tests under constant inelastic state.

general, symmetric with respect to the σ axis. This implies that the state variables for this material may not be all scalars. In fact, there are indications that a single second-rank symmetric traceless tensor may be sufficient to describe the inelastic state of the material used in these tests. (For further discussion of this part see Ref. 45.)

The above proposed tests can be used to study the phenomenon of recovery in the following way. After having determined the curves 1 and 2 at a particular time, we may unload the material. After a sufficiently "long" waiting period under zero stress, we may increase the stress again and repeat what was done before to establish the new curves 1 and 2 and the values of g_1 between these curves. By studying these two sets of data we can obtain quantitative information on recovery.

Tests under constant stress. In order to obtain further information on $g_1(q, S)$ and $g_3(q, S)$ one can conduct tests that contain episodes where q and hence S is constant. A simple creep test is entirely composed of such an episode. A two-step test [cf. (2.104)] with $\text{tr } b^2 < \text{tr } a^2$ and "large" t_1 would enable one to study the portion of the state space that corresponds to the phenomenon of recovery. (It may be recalled that by similar one-dimensional tests, we were able to study the part of the state space that is below the line PQ in Fig. 33). One would like to learn whether in such a test the state of the material will tend to the state of steady creep under the stress $2\mu b$. It will be seen from Fig 33 that in the one-dimensional case this depended on the magnitude of $2\mu b$. If this magnitude is sufficiently near to $2\mu a$, then the state tended toward the state of steady creep under the stress $2\mu b$ (cf. also Fig. 8). But if $2\mu b$ is sufficiently small, then the motion in the state space terminates on the boundary of the $g_3 \approx 0$ region.

Conclusions

In this section we have introduced (2.82) and (2.83) as a fundamental representation for metals. We have stressed the fact that the inelastic state is composed in general of a number of scalars and even-rank irreducible tensors. We have suggested tests that would produce information on the rank of these tensors. Finally, we discussed briefly tests aimed at partial determination of g_1 and g_3 .

Admittedly these discussions have been brief, and concrete considerations of Section 2.6 on the determination of the number of state variables and the growth laws have not been generalized to the present case. However, we believe that an interested reader can establish a deeper connection between Sections 2.5 to 2.7 and this section.

Certainly we have not been able to make any specific statements concerning the representation of mechanical behavior of austenitic steels. This is largely due to lack of experimental evidence of the types discussed in this section. We may, however, offer a preliminary guess as to what a representation for these materials may look like. We have a representation in mind which is suitable for the analysis of structures that are subject to nonsteady loading (see Chap. 4). In such a situation

we shall have to tackle not only the question of representation of mechanical behavior but also the question of failure.

We may choose to handle the failure question in the manner of Kachanov.⁷ This would involve the introduction of the state $\underline{d}$ of internal damage. In the work of Kachanov⁷ and Odqvist⁸ $\underline{d}$ is taken to be a simple scalar. The total representation of isothermal behavior will then be of the form

$$\begin{aligned}\text{tr } \underline{\sigma} &= 3K \text{ tr } \underline{\epsilon}, \\ \underline{s} &= 2\mu \underline{q}; \\ \dot{\underline{q}} &= \underline{g}_1(\underline{q}, \underline{s}) + \dot{\underline{\epsilon}}, \\ \dot{\underline{s}} &= \underline{g}_2(\underline{q}, \underline{s}); \\ \dot{\underline{d}} &= \underline{g}_3(\underline{q}, \underline{s}; \underline{d}).\end{aligned}\tag{2.108}$$

Note that in (2.108), $\underline{d}$ does not influence the mechanical behavior, but it develops by means of the last equation. Note also that invariance requirements similar to (2.83) must be added to (2.108).

We believe that it would be possible to construct a simple but accurate enough representation by taking $\underline{s}$ as a second-rank traceless symmetric tensor. In such a case the dependence of $\underline{g}_1$ and $\underline{g}_2$ on $\underline{q}$ and $\underline{s}$ will have the general form shown in (2.107). Whether a form similar to (2.107) could be further simplified, say by the introduction of a potential [cf. (2.95)], remains to be established.

3. INTEGRAL REPRESENTATIONS OF MECHANICAL BEHAVIOR

3.1 Introduction

As we have seen in Section 2.1, the representation of mechanical behavior involves (in the setting of the uniaxial isothermal stressing) the mathematical description of an operator F that assigns, on the test interval $[0, T]$, to, say, every continuous strain history $\epsilon(\tau)$, a continuous stress response $\sigma(\tau)$:

$$\sigma(\tau) = F[\epsilon(\tau)] . \quad (3.1)$$

The operator F has the property of a causality (cf. Sect. 2.2), and it is assumed that it is time independent in the sense of (2.3).

In Chapter 2 we discussed at length the differential equation representation of F by means of state variables.

In this chapter we offer a critical review of integral representations of F . To introduce this mode of representation and to establish its relationship to the results of Chapter 2, we reconsider briefly the Maxwell solid defined in Section 2.3. This solid accepts the state variable representation

$$\sigma = E q , \quad (3.2)$$

$$\dot{q} = -\frac{q}{T} + \dot{\epsilon} ,$$

with the initial conditions

$$q(0) = \epsilon(0) = 0 . \quad (3.3)$$

Now for a given strain history $\epsilon(\tau)$, we can integrate the growth law in (3.2) and use (3.3) to obtain

$$\sigma(t) = \int_0^t E \exp\left(-\frac{t-\tau}{T}\right) \dot{\epsilon}(\tau) d\tau . \quad (3.4)$$

Equation (3.4) is said to be the integral representation of the Maxwell solid. This representation has the essential simplicity and attractiveness that it provides a direct relationship between the inputs and outputs of the problem with no recourse to an auxiliary variable such as q .

Moreover, this attractive feature of the representation persists even for more complicated solids. Indeed if the solid is such that the operator F is additive [in addition to being causal, nonaging, and continuous; namely, if two arbitrary strain histories $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$ give rise via F to $\sigma_1(\tau)$ and $\sigma_2(\tau)$, respectively, then $\epsilon_1(\tau) + \epsilon_2(\tau)$ must produce $\sigma_1(\tau) + \sigma_2(\tau)$], then F accepts the representation (cf. Christensen³⁵ and Gurtin and Sternberg³²)

$$\sigma(t) = \int_0^t E(t - \tau) \dot{\epsilon}(\tau) d\tau, \quad (3.5)$$

where $E(t)$, which is called the relaxation modulus, is once and for all fixed for a given material. A solid that accepts the representation (3.5) is said to be linear viscoelastic. We observe that for linear solids considered in Section 2.3, $E(t)$ has the form

$$E(t) = \sum_{i=1}^n K_i \exp\left(-\frac{t}{T_i}\right).$$

Experimental determination of $E(t)$ in a given linear solid is a well-studied problem (cf. Ferry⁴¹; Kolsky⁵³) and will not be discussed here.

3.2 Integral Representation of Nonlinear Solids

When the given solid is nonlinear, that is, when the operator F defining the solid does not have the property of additivity, the representation (3.5) no longer holds.

We now look for an integral representation of a nonlinear solid (or equivalently of the nonlinear operator F). We may start our search by

considering a much simpler but analogous situation, namely, the representation of the function

$$y = f(x) , \quad (3.6)$$

where y and x are real numbers and $x \in [0, 1]$. When f is linear, (3.6) has the simple representation

$$y = ax , \quad (3.7)$$

where a is a constant. Note the formal similarity between the linear function (3.7) and the linear operator (3.5).

In the case where the function is nonlinear we have a variety of means for representing it. Among these we may mention Taylor's expansion, Weierstrass polynomials, Fourier series, splines (piecewise polynomial functions), etc. It would be reasonable to expect that each of these could be generalized to the present nonlinear operator F . It turns out, however, that only the first two representations could be generalized with relative ease to the present case.

We shall now discuss the generalization of the Weierstrass expansion. The theorem of Weierstrass asserts that a continuous function $f(x)$ defined on a closed interval can be represented to any desired degree of accuracy by a polynomial:

$$f(x) \approx f_0 + a_1 x + \dots + a_n x^n \cdot x \in [0, 1] . \quad (3.8)$$

This remarkable result, which is different from Taylor's expansion [although when $f(x)$ is smooth enough to possess derivatives of any desired order the two representations become related], is of limited value in the representation of functions that are continuous but are not smooth enough. For instance, if $f(x)$ exhibits discontinuities in its first derivative, the degree of the approximating polynomial may become prohibitively large if higher accuracy is required in the representation (3.8) (see Ref. 25). Another drawback of (3.8) is the lack of useful algorithms for determining n and the constants a_i when the values of $f(x)$ are available at n locations. To be sure in such a case, one can use Lagrange's interpolation polynomial (Ref. 54, p. 86); but, if no estimates of higher derivatives of $f(x)$ are available or if these

derivatives do not exist, one cannot judge the resulting degree of approximation a priori, and in fact it is known that by increasing the number of data points one may deteriorate the approximation. We may note in passing that Fourier series and spline approximations do not suffer from many of these drawbacks.

The generalization of (3.8) to the type of operators considered here is due to Fréchet.⁵⁵ A more abstract formulation of the representation question is known as the Stone-Weierstrass theorem (Ref. 56, p. 157).

Fréchet's result takes the following particular form for the operator considered here. If the inputs, namely, the continuous functions $e(\tau)$ defined on $[0, T]$ which are of interest here, constitute a compact set (Ref. 56, p. 110), a minor restriction as far as physical situations are considered, then the continuous nonaging and causal operator F can be represented to any desired degree of accuracy by a sum of the type

$$\begin{aligned}
 F_0 + \int_0^t K_1(t - \tau) e(\tau) d\tau + \int_0^t \int_0^t \\
 \times K_2(t - \tau_1, t - \tau_2) e(\tau_1) e(\tau_2) d\tau_1 d\tau_2 + \dots \\
 + \int_0^t \dots \int_0^t K_n(t - \tau_1, t - \tau_2, \dots, t - \tau_n) \\
 \times e(\tau_1) e(\tau_2) \dots e(\tau_n) d\tau_1 d\tau_2 \dots d\tau_n, \quad (3.9)
 \end{aligned}$$

where K_0 is a constant and $K_1, K_2, \dots, K_n$ are continuous functions of their arguments.

It must be pointed out that in the derivation of (3.9) the Weierstrass sum (3.8) plays a central role (Onat, Ref. 25). Thus it is not surprising that (3.9) suffers from drawbacks similar to the one discussed in connection with (3.8). For instance it is shown⁵⁷ that if one wishes to use a sum of the type (3.9) to describe the classical elastic-plastic behavior discussed in Section 2.5, a prohibitively large number of terms is needed. Since, as shown before, creep of metals is so intimately

connected with plasticity of metals, we expect that (3.9) would remain a cumbersome tool for the description of creep behavior. However, Eq. (3.9) proves to be a useful representation for polymers in certain circumstances.⁵⁸

Thus we may try to modify (3.9) to make it more suitable for the description of creeplike behavior. A broad class of modified integral representations that may be of help here is the Hammerstein-Uryson sums. These sums have the typical forms

$$\int_0^t K_1(t-\tau) f[\epsilon(\tau)] d\tau + \int_0^t \int_0^t K_2(t-\tau_1, t-\tau_2) \times g[\epsilon(\tau_1)] g[\epsilon(\tau_2)] d\tau_1 d\tau_2 + \dots \quad (3.10a)$$

or

$$\int_0^t K_1\{t-\tau, f[\epsilon(\tau)]\} d\tau + \int_0^t \int_0^t K_2\{t-\tau_1, t-\tau_2, g[\epsilon(\tau_1)], g[\epsilon(\tau_2)]\} \times d\tau_1 d\tau_2 + \dots, \quad (3.10b)$$

where the functions K_1 , f , g , etc., are assumed to be known. A major advantage of these forms is the appearance of nonlinearity in the first hereditary integrals, which leads us to expect that a given operator F could be represented by a few terms in (3.10a) or (3.10b). Obviously each sum of the above type defines a nonlinear operator $F[\epsilon(\tau)]$. But whether a given operator F could be represented conveniently by one of the expressions in (3.10) is not an easy question to answer. A difficulty that persists is that of the lack of algorithms for determining the functions K_1 , K_2 , ... and f , g from the given experimentally observed input-output pairs $[\epsilon_1(\tau); \sigma_1(\tau)]$.

The sums of the type (3.10) have also attracted attention of workers in mechanics. Bernstein, Kearsley, and Zapas⁵⁹ and Findley and co-workers,^{60,61} considered representations based on the first term in (3.10a)

or in (3.10b). Relative success of these representations led Pipkin and Rogers^{8,2} to introduce a representation essentially of the type (3.10b) that could be constructed by the direct use of certain experimental results. This means that the Pipkin and Rogers representation has its own algorithm, which is very commendable. This representation is discussed further below.

3.3 The Pipkin-Rogers Representation

It will be convenient for later purposes of comparison to exchange the roles of inputs and outputs and to consider the stress histories as inputs and the strain histories as outputs.

To introduce the first term in the Pipkin-Rogers representation, we consider a creep experiment with the stress history

$$\sigma(t) = 0 \quad \text{when } t < \tau$$

$$\sigma(t) = c = \text{const.} \quad \text{when } t > \tau$$

and with the corresponding strain response $\epsilon(t)$. For a given material the strain at time t ($t > \tau$) would be a function of the constant stress intensity σ and the time $t - \tau$ that elapsed since the application of the stress σ :

$$\epsilon(t) = R_1(\sigma; t - \tau) . \quad (3.11)$$

For reasons given below it is tempting to offer the following relationship* as a first approximation to the operator $F[\sigma(\tau)]$, which characterizes the mechanical behavior of the given solid:

$$\epsilon_1(t) = \int_0^t \frac{\partial R_1}{\partial \sigma} [\sigma(\tau); t - \tau] \dot{\sigma}(\tau) d\tau , \quad (3.12)$$

*For discontinuous stress histories, (3.12) must be interpreted as a Stieltjes integral.

where R_1 has the meaning of (3.11). The reasons for offering (3.12) as an approximation to $F[\sigma(\tau)]$ are as follows. First, we see that if F is linear, then (3.12) is the correct representation. This assertion follows from the observation that for linear solids

$$R_1 = \sigma \cdot J(t - \tau), \quad (3.13)$$

where $J(t)$ is the creep compliance. Now introducing (3.13) into (3.12) we find

$$\epsilon(t) = \int_0^t J(t - \tau) \dot{\sigma}(\tau) d\tau, \quad (3.14)$$

which proves the assertion [cf. (3.5)].

On the other hand, a nonlinear material of the Söderberg type [cf. (2.28) and (3.12)] is also an exact representation, since for this material we have from (2.28)

$$R_1[\sigma; t - \tau] = \frac{\sigma}{E} + f(\sigma)(t - \tau), \quad (3.15)$$

which when combined with (3.12) gives, upon integration by parts,

$$\epsilon(t) = \frac{\sigma(t)}{E} + \int_0^t f[\sigma(\tau)] d\tau,$$

which is nothing but the integrated form of (2.28).

It is found that even for nonlinear polymers, (3.12) does not constitute an entirely satisfactory representation⁶² in spite of its complete success for linear materials and for the Söderberg material. We shall see that (3.12) is not satisfactory for creeping metals in general. Of course for simple creep tests, (3.12) is exact for any material since for such tests (3.12) becomes (3.11) upon integration.

Pipkin and Rogers⁶² propose an improvement to (3.12) that has its genesis in the last remark above. They propose that we consider the

stress histories of the type

$$\sigma = \begin{cases} 0 & 0 < t < \tau_1 \\ \sigma_1 & \text{when } \tau_1 < t < \tau_2, \\ \sigma_2 & \tau_2 < t \end{cases} \quad (3.16)$$

where σ_1 and σ_2 are constants.

The representation (3.12) would produce for the above history, the strain $e_1(t)$, $t > \tau_2$,

$$e_1(t) = R_1(\sigma_1; t - \tau_1) + R_1(\sigma_2; t - \tau_2) - R_1(\sigma_1; t - \tau_2). \quad (3.17)$$

Now if $e(t)$ denotes the actual strain response to the stress history (3.16), we would expect that for an arbitrary solid

$$e(t) - e_1(t) \neq 0;$$

otherwise (3.12) would be valid for any solid.

Moreover, for a given solid the above difference would depend upon $\sigma_1, \sigma_2; t - \tau_1, t - \tau_2$, which leads us to introduce

$$e(t) - e_1(t) = R_2(\sigma_1, t - \tau_1; \sigma_2, t - \tau_2). \quad (3.18)$$

Note that by definitions of R_1 and R_2 we have

$$R_2 = 0 \text{ when } \sigma_1 = \sigma_2 \text{ or when } \sigma_1 = 0. \quad (3.19)$$

We would expect in addition that

$$R_2 \rightarrow 0 \text{ as } \tau_1 \rightarrow \tau_2 \quad (3.20)$$

for materials that behave elastically under fast straining or stressing (cf. Sect. 2.3). However, (3.20) would not hold for a classical elastic-plastic solid.

For purposes of symmetry we can extend the range of the function R_2 by insisting that

$$R(\sigma_1, t - \tau_1; \sigma_2, t - \tau_2) = R(\sigma_2, t - \tau_2; \sigma_1, t - \tau_1). \quad (3.21)$$

Now, Pipkin and Rogers introduce the following expression as a second approximation to the operator $F[\sigma(\tau)]$:

$$\begin{aligned} \epsilon_2(t) = & \int_0^t \frac{\partial R_1}{\partial \sigma} [\sigma(\tau), t - \tau] \delta(\tau) d\tau + \frac{1}{2!} \int_0^t \int_0^t \frac{\partial^2 R_2}{\partial \sigma_1 \partial \sigma_2} \\ & \times [\sigma(\tau_1), t - \tau_1; \sigma(\tau_2), t - \tau_2] \delta(\tau_1) \delta(\tau_2) d\tau_1 d\tau_2. \end{aligned} \quad (3.22)$$

It can be shown by using (3.19) and (3.21) that the above representation is exact whenever the stress history is composed of, at most, two steps.

It should now be clear as to how the representation (3.22) could be further improved by considering stress histories composed of three steps,

$$\sigma(t) = \begin{cases} 0 & 0 \leq t < \tau_1 \\ \sigma_1 & \tau_1 < t < \tau_2 \\ \sigma_2 & \tau_2 < t < \tau_3 \\ \sigma_c & \tau_3 < t \end{cases} \quad \text{when} \quad (3.23)$$

and by introducing the function $R_n = \epsilon(t) - \epsilon_n(t)$. This procedure could be continued to arrive at the general form of the Pipkin-Rogers sums, namely,

$$\begin{aligned} \epsilon_n(t) = & \int_0^t \frac{\partial R_1}{\partial \sigma} [\sigma(\tau), t - \tau] \delta(\tau) d\tau + \dots \\ & + \int_0^t \dots \int_0^t \frac{\partial^n R_n}{\partial \sigma_1 \dots \partial \sigma_n} [\sigma(\tau_1), t - \tau_1; \sigma(\tau_2), t - \tau_2, \dots] \\ & \times \delta(\tau_1) \delta(\tau_2) \dots \delta(\tau_n) d\tau_1 \dots d\tau_n. \end{aligned} \quad (3.24)$$

For a given material, (3.24) may be considered to be a successful and tractable representation if for all tests of interest the difference

$$\epsilon(t) - \epsilon_n(t)$$

is sufficiently small for some n , 2 or 3.

3.4 Application of (3.24) to Creep of Metals

We wish now to show that the representation (3.24) with $n = 1$ or 2 provides a poor description of creep of a large class of metals. We shall further assert that the use of higher n does not provide a remedy.

For this purpose consider a three-step test (3.23) with

$$\sigma_1 = \sigma; \sigma_2 = 0; \sigma_3 = \sigma \quad (3.25)$$

and the resulting strain response $\epsilon(t)$. In Fig. 4 we have shown the results of such a test for Al 1100. We shall now compare these experimental results with the predictions of (3.24).

We find that in the presence of (3.25), (3.24) with $n = 1$ yields

$$\epsilon_1(t) = \begin{cases} R_1(\sigma, t - \tau_1) & \tau_1 < t < \tau_2 \\ R_1(\sigma, t - \tau_1) - R_1(\sigma, t - \tau_2) & \text{when } \tau_2 < t < \tau_3 \\ R_1(\sigma, t - \tau_1) + R_1(\sigma, t - \tau_3) - R_1(\sigma, t - \tau_2) & \tau_3 < t, \end{cases} \quad (3.26)$$

where $R_1(\sigma, t - \tau_1)$ is by definition the creep strain developed at time t under the stress σ applied at time τ_1 . With this observation and by (3.26), it becomes clear that $\epsilon_1(t)$ has the form shown in Fig. 39 for the particular test considered above. We see that $\epsilon_1(t)$ coincides with the actual response when $t < \tau_2$ but differs from it in an essential way for $t > \tau_2$.

Now we consider the case of $n = 2$. We have by definition $\epsilon(t) = \epsilon_2(t)$ for $t < \tau_2$. For $t > \tau_2$, however, $\epsilon_2(t)$, namely (3.24) with $n = 2$, becomes, in view of the fact that $\sigma_1 = \sigma_3 = \sigma$ and $\sigma_2 = 0$,

$$\begin{aligned} \epsilon_2(t) = & R_2(\sigma, t - \tau_1) - R_2(\sigma, t - \tau_2) + R_2(\sigma, t - \tau_3) \\ & + R_2(\sigma, t - \tau_1; 0, t - \tau_2) + R_2(\sigma, t - \tau_3; 0, t - \tau_2) \\ & - R_2(\sigma, t - \tau_1; 0, t - \tau_3) \end{aligned} \quad (3.27)$$

Fig. 39. Comparison of predictions of Pipkin and Rogers representation with experimental results [Al 1100 at 300°F (schematic), Wang and Onat⁶].

for $\tau_2 < t$. On the other hand, by definition,

$$R_2(\sigma, t - \tau_1; 0, t - \tau_2) = \epsilon(t) - R_1(\sigma, t - \tau_1) + R_1(\sigma, t - \tau_2),$$

where $\epsilon(t)$ is the measured strain response for $t > \tau_2$ to the two-step stress history (3.16) with $\sigma_1 = \sigma$ and $\sigma_2 = 0$.

Since for the material of interest (cf. Fig. 39)

$$\epsilon(t) \approx R_2(\sigma, \tau_2 - \tau_1) - \frac{\sigma}{E} \quad \text{for } t > \tau_2,$$

we have

$$\begin{aligned} R_2(\sigma, t - \tau_1; 0, t - \tau_2) &\approx R_1(\sigma, \tau_2 - \tau_1) \\ &\quad - R_1(\sigma, t - \tau_1) + R_1(\sigma, t - \tau_2) - \frac{\sigma}{E} . \end{aligned}$$

Using the last result repeatedly in (3.27) we obtain for the stress history (3.25) and for the material of interest

$$\begin{aligned} \epsilon_2(t) &\approx R_1(\sigma, t - \tau_1) - R_1(\sigma, t - \tau_2) + R_1(\sigma, t - \tau_3) \\ &\quad - \frac{\sigma}{E} + R_1(\sigma, \tau_2 - \tau_1) - R_1(\sigma, t - \tau_1) + R_1(\sigma, t - \tau_2) \\ &\quad + R_1(\sigma, \tau_3 - \tau_2) - R_1(\sigma, t - \tau_2) + R_1(\sigma, t - \tau_3) \\ &\quad - [R_1(\sigma, \tau_2 - \tau_1) - R_1(\sigma, \tau - \tau_1) + R_1(\sigma, t - \tau_3)] \end{aligned}$$

for $\tau_3 < t$ and, finally,

$$\begin{aligned} \epsilon_2(t) &\approx \epsilon_1(t) - \frac{\sigma}{E} + R_1(\sigma, \tau_2 - \tau_1) \\ &\quad + R_1(\sigma, \tau_3 - \tau_2) - R_1(\sigma, \tau_3 - \tau_1) . \quad (3.28) \end{aligned}$$

In the case where $\tau_2 - \tau_1$ is large compared with the time over which primary creep occurs in a creep test, we have

$$R_1(\sigma, \tau_2 - \tau_1) \approx a + b(\tau_2 - \tau_1) , \quad (3.29)$$

where a is the intercept of the asymptote of the creep curve and b is its slope. In the present experiment $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_3 - \tau_1$ are large, so that (3.29) and similar relations for $R(\sigma, \tau_3 - \tau_2)$ and $R(\sigma, \tau_3 - \tau_1)$ are valid (Fig. 39). Using these we obtain from (3.29):

$$\epsilon_2(t) = \epsilon_1(t) - \frac{\sigma}{E} + a \quad \text{when } t > \tau_3 .$$

In Fig. 39, $\epsilon_1(t)$ and $\epsilon_2(t)$ are shown together with the actual strain response $\epsilon(t)$. We see that both $\epsilon_1(t)$ and $\epsilon_2(t)$ ($t > \tau_3$) differ from the actual strain response $\epsilon(t)$ in a fundamental way. Whereas $\epsilon(t)$

has a mild curvature for $t > \tau_3$, $\epsilon_1(t)$ and $\epsilon_2(t)$ have the curvature of the initial creep curve.

Now the question must be raised as to whether the situation would improve by using a representation with $n = 3$. In this case comparisons must be made with the experimental results involving stress histories with four steps. Such results were not available for Al 1100. However, a study of the representation of elastic-plastic solids by the Pipkin and Rogers method shows that the difficulties of the type shown in Fig. 39 must persist even when $n \geq 3$. Thus we conclude that the representation (3.24) must be used with extreme caution (if at all) for metals.

4. ANALYSIS OF STRUCTURES IN THE PRESENCE OF CREEP

4.1 Introduction

We consider a structure such as a truss, frame, or shell or a composite of these that is subjected to time-dependent loads in the presence of elevated temperatures. For economy in material consumption or for other reasons, we may be forced to allow "high" stresses to develop in this structure so that creep deformation may be present.

We can ask two related questions about the integrity of such a structure:

1. Will the deformation of the structure remain below prescribed limits during its lifetime?
2. Will an element of the structure fail due to causes such as fatigue, fracture, embrittlement, contamination, etc.

To answer these questions, a model of the proposed structure may be tested under realistic conditions of loading and support. In certain structures, such as nuclear reactors, the model has to be almost as complicated as the actual structure if the testing is to be realistic enough to include radiation effects, etc. For this type of structure, the mathematical analysis of the proposed structure may become a necessity. Even in more agreeable situations where realistic tests could be carried out with simpler models the analysis would be needed to achieve a better design.

The question is therefore not "Why analyze structures?" but rather "How?" For structures that are subject to creep the main problem is in the choice of relationships that describe the mechanical behavior of the structure. In the previous chapters we have discussed ways of constructing such relationships from experiments. It became clear in these discussions that for a given material one could produce a hierarchy of such relationships. Of these the simpler ones would not be capable of representing every aspect of creep behavior. For instance, Söderberg's law is quite satisfactory in the presence of steady stresses and at large times, but it could not predict any primary creep.

Thus the problem of the analyst is to choose among this hierarchy of relationships the one that represents only those aspects of creep of importance with regard to the two questions raised above. It is hoped that the following remarks may be of help to the analyst in the fulfillment of this difficult task.

4.2 Structures Subject to Steady Loads

When a constant load is applied to a structure in the presence of elevated temperature, the structure responds to this load first in a purely elastic manner, but later the stress distribution changes gradually to the one that would be obtained by analyzing the structure using a steady creep law, say $\dot{\epsilon} = c\sigma^n$ or a suitable generalization of it. This redistribution of stress is caused to a large degree by the geometry of structure, and, in statically indeterminate structures, it takes place even in the absence of primary creep. In Fig. 40 we show the displacement of a typical point of the structure as a function of time. Note that point A of this diagram can be obtained by analyzing the structure elastically. On the other hand, the slope of this curve could be obtained by analyzing the structure as if it were composed of a nonlinear viscous solid. It is, of course, desirable to calculate the quantity δ_s shown in

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Fig. 40. Deflection-time relation for a typical point of a creeping structure.

Fig. 40, which measures the displacement caused by the stress redistribution. It is fortunate that in many cases δ_0 could be estimated by simple calculations. For a detailed discussion of this and related questions the reader is referred to the work of Leckie and Martin,⁶³ Hoff,⁶⁴ and Calladine.⁶⁵

It is to be noted that in the problem discussed above the stress in a given element of the structure will vary over the course of time. But we expect, with justification, that this variation will be monotonic and slow.* Under these circumstances creep behavior of the material may be represented satisfactorily by one of the simpler laws; even the time-hardening law may be appropriate for the description of this situation (Rabotnov,⁵ p. 200).

4.3 Nonsteady Loads

When a structure capable of creep is subjected to nonsteady loads, the stress at a generic point of structure will no longer be a monotonic function of time. Therefore the question arises as to whether the intricate aspects of creep behavior that were observed to occur under multistep loading programs (cf. Fig. 4) should be considered in the analysis of structures subject to nonsteady loads. That, in general, the answer to this question would be affirmative can be seen from the fact that any tensile specimen under the action of a multistep loading program could be considered as a structure. To elaborate on this point, consider a structure subject to the periodic loading shown in Fig. 41. The analyst would ask whether a simple creep law, say that of Söderberg, would be appropriate to use in the analysis of this structure. In order to obtain a first answer to this question he may consider, if the structure is composed of steel, the tests conducted by Taira (Fig. 9) with tensile specimens of steel at 450°C. It was pointed out that in the case where the ratio τ/p is small the "deformation vs time under load" curve in such a test was considerably above the simple creep curve. This observation

*In this section we assume that inertia forces are negligible, and therefore dynamic effects will not be considered.

Fig. 41. A periodically varying load (moment) that may be applied to a typical point of a creeping structure.

implies that a simple creep law that is not capable of representing the softening effect discussed in Section 2.4 would seriously underestimate total deformations of the test specimen when the ratio τ/p is much less than 1. If this is the state of affairs for the tensile specimen, which is a simple structure, then it would be prudent to expect that similar effects would be observed in more complicated structures such as trusses.

We know, of course, that in statically indeterminate structures internal stresses that develop upon unloading may give rise to an overall behavior for the structure that is more complicated than the individual behavior of its elements. The shakedown phenomenon, which was first discovered in the context of elastic-ideally plastic structures, could be used to illustrate the above point. It was shown by Melan⁶⁶ that for an elastic-plastic structure subject to a loading history of the type shown in Fig. 41, the critical load intensity is not the limit load P_L . It may be recalled that P_L is the load at which the given structure would lose its load-carrying capacity under monotonically increasing loads. When the loading history is of the type shown in Fig. 41, the shakedown load P_S , which in general is less than P_L , becomes a measure of the load-carrying capacity of the structure. It is known that when the maximum load $P_{\max}$ is less than P_S the structure may suffer plastic deformation in the early stages of loading but later would settle to purely elastic behavior. But when $P_S < P_{\max} < P_L$, plastic deformation will

occur within the structure during each cycle of loading. Often these plastic deformations give rise to an increase of the total maximum deformation after each cycle of loading, and that leads to "incremental collapse." Even when incremental collapse does not take place, the structure would fail after a number of cycles due to "gross fatigue" caused by plastic deformation of some of its elements, these deformations taking place periodically in alternating directions. As an example of the shakedown phenomenon, consider the bending of a beam of T section (Armstrong and Townley⁶⁷). The bending moment has been varied over the course of time in the manner of Fig. 41. The corresponding bending moment vs curvature behavior is shown in Figs. 42 and 43 for various

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Fig. 42. Moment curvature behavior for T - Section beam, $M_{max} < M_S$ (Armstrong and Townley⁶⁷).

Fig. 43. Moment-curvature behavior for T-Section beam, $M_{\max} = M_s$ (Armstrong and Townley⁶⁷).

levels of maximum bending moment. We see that when $M_{\max} < M_s$ the behavior is as shown in Fig. 42; after a few cycles elastic behavior ensues. But when $M = M_s$, each cycle adds to the growth of permanent curvature (Fig. 43).

It has been recognized recently that it would be highly desirable to perform shakedown analysis in the design of pressure vessels (Townley,⁶⁸ Proctor and Flinders,⁶⁹ Lockie,⁷⁰ and Bree⁷¹)

It would be interesting and desirable to consider the shakedown question for creeping structures. Bree⁷¹ was among the first who studied the effect of creep deformations on the shakedown of structures. Recently,

Leckie and Ponter^{7,2} have considered the same problem in a series of papers. It is important to note that in all this work simple creep laws are used to describe the mechanical behavior of individual elements of the structure. It is not known at present how a phenomenon such as creep recovery (which is not represented by simple creep laws) would affect the shakedown behavior of structures.

It may be useful at this stage to review briefly the mathematical aspects of the analysis of creeping structures. From a mathematical point of view, a creeping structure can be analyzed as follows: When a structure is subjected to loading the state point of each element will move in the state space of the element. These motions will be governed by an equation of the type (2.42) and will be initiated and controlled by the strain of the element. The stress in an element will be the end result of this motion [cf. (2.42)]. On the other hand, throughout this structure the strain distribution must be compatible and the stress distribution must satisfy equilibrium. Thus we see that the analysis of the structure involves the problem of meeting the requirements of compatibility, equilibrium, and the growth laws (2.42).

We now show that this problem could be solved stepwise. At a given instant the rate of stress at a given element may be found by a combination of the first and second equations in (2.42):

$$\dot{\sigma} = E[g_1(q_j) + h_1(q_j)\dot{\epsilon}] . \quad (4.1)$$

Now suppose that at the instant of interest one knows q throughout the structure. Equation (4.1) then becomes, with $h_1 = 1$,

$$\dot{\sigma} = E\dot{\epsilon} + B , \quad (4.2)$$

where the distribution of E and B throughout the structure is known. Now $\dot{\sigma}$ must satisfy the equation of equilibrium, and $\dot{\epsilon}$ must be derivable from the velocity distribution $v(x)$ by some kinematical relationship. Thus the determination of the distributions of $\dot{\sigma}(x)$ and $v(x)$ would require, in view of (4.2), the solution of a problem of elasticity where certain spatial derivatives of B would play the role of a body force. Once this "elastic" problem is solved, we would know $\dot{\epsilon}$ at each element of the

structure and therefore could determine $\dot{q}_1$ by the growth law. With the help of these calculated values of $\dot{q}_1$, we can compute q_1 an instant later and repeat the above step once more, etc. Note that this step-by-step procedure could be started conveniently at $t = 0$, since each element of the structure would be at its known virgin state at this time.

The above remarks indicate that the analysis of creeping structures reduces to the solution of a series of "elastic" problems. As we know, this procedure could be carried out numerically with the help of a computer. We must stress here the computational advantage of the state variable representation. The procedure described above requires only the knowledge of the updated values of the state variables at each stage of computation. Whereas, if an integral representation is used, the computation of B in (4.2) requires the knowledge of the entire strain history for each element of the structure. This is, of course, a heavy burden on the memory of the computer.

The results of such an analysis would enable us to answer the two questions raised at the beginning of this chapter. In particular, we would learn from these results the motion of the state point of each element in its state space. If for the loading history considered the state point of each element remains within a prescribed bounded region containing the origin, we could say that the structure would not fail. This is especially true if some of the state variables measure the amount of the internal damage that has taken place within the element. If for an element these particular variables keep on growing during a loading process, the indication would be that the structure could fail by the "fracture" of this element.

At this point one should raise the question as to whether this vital information on the happenings in the state space of elements could be obtained without the benefit of a laborious, detailed numerical solution.

This is a legitimate question since we know that for elastic-plastic structures, bounds for the critical loads P_L and P_S could be found with relative ease without detailed analysis.

In fact Leckie⁷² has suggested that for creeping structures P_L and P_S may be replaced by $\frac{E}{n+1} P_L$ and $\frac{n}{n+1} P_S$, where n is the creep exponential [cf. (1.4)]. This is probably an oversimplification, and the arguments leading to the suggestion of Leckie is based on a simple creep law (for a further discussion of Leckie's work see Ref. 73). Nevertheless the need is obvious for the development of methods for obtaining bounds on the deformations of a creeping structure and also on the motions of the state points in the individual state spaces of the element. The work of Williams and Leckie,⁷⁴ Ponter,⁷⁵ and Martin and Williams⁷⁶ constitute a first step in this direction. It is hoped that the future work in this area will be based on a more realistic description of mechanical behavior (cf. Sect. 2.5). Moreover, it is hoped that in future work use could be made of the modern results concerning the vibrations and stability of nonlinear systems (Hahn⁷⁷).

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