Simple Calculation of Critical
Parameters of Neutron Stars

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Simple Calculation of Critical Parameters of Neutron Stars

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Abstract

The instability of neutron stars comes about because of the softening of the equation of state for densities $\rho \approx 1$ to 4×10^{15} gm/cc. For soft equations of state, such as the Independent Particle Model, this happens when $\gamma(\rho) = \rho dP/d\rho$ falls below $\gamma_{CR} \approx 4/3$ the critical value on Newtonian gravitational theory: the equation of state softens where the neutrons become special relativistic. For stiffer equations of state, generally relativity leads to instability at γ_{CR} values somewhat greater than $4/3$; nevertheless, the softening of the equation of state, for causality reasons or otherwise, is what brings $\gamma(\rho)$ down below γ_{CR} . Particularly for relatively stiff equations of state, once $\gamma(\rho)$ is approximately

known in the region of densities about $\rho \approx 10^{15}$ gm/cc, the relativistic polytrope model considered in the preceding paper permits a quick estimate of the critical density, mass and radius to about 20% accuracy.

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I. INTRODUCTION: GENERAL RELATIVITY AND EQUATION OF STATE.

In this paper we discuss the critical parameters of neutron stars, the maximum mass M_{CR} and central density ρ_{CR} above which the star is unstable against gravitational collapse. The stellar structure has heretofore been calculated by inserting an equation of state $P = P(\rho)$ into the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium

$$-\frac{dP}{dr} = G \frac{(m + 4\pi r^3 \frac{P}{c^2})}{r^2 (1 - 2\frac{Gm}{c^2 r})} (\rho + P/c^2)$$
$$m(r) = \int_0^r \rho \cdot 4\pi r^2 dr \quad (1.1)$$

The TOV equation relates the pressure gradient $-dP/dr$ to the gravitational force, which in general relativity is produced by mass $m(r)$ and by pressure P , and acts on mass density $\rho(r)$ and on pressure. Our purpose is to disentangle the features of equation of state and of general relativity which fix the dimensions of neutron stars. In particular, we want to estimate the stellar mass M and radius R , which determine the moment of inertia of rotating neutron stars (pulsars), and M_{CR} the maximum mass above which a neutron star must shed mass or end as a black hole. At the present stage, we are not concerned with the detailed structure of neutron stars nor with obtaining great accuracy.

The gravitational force in the TOV equation (1.1) differs

from the Newtonian force $G m_p/r^2$ by the factors

$$\left(1 + \frac{4\pi\rho r^3}{m} \sigma\right) (1 + \sigma), \quad \sigma \equiv P/\rho c^2 \quad (1.2)$$

in the numerator and

$$1 - 2 \frac{G m}{c^2 r} \quad (1.3)$$

in the denominator. We call the denominator factor kinematic, because it originates in space curvature, and the numerator factors dynamic because they express the fact that in general relativity pressure as well as mass or energy density acts as a source of gravity and is acted on by gravity. In a typical massive neutron star $\sigma \sim 1/3$ and $GM/c^2 R \sim 1/3$, so that for massive neutron stars, general relativity enhances the gravitational force by about $(1 + 3 \cdot 1/3)(1 + 1/3)/(1 - 2/3) \sim 8$. The effect of general relativity on the hydrostatic equilibrium is thus to require a steeper pressure gradient. This requires in the equation of state $P \sim \rho^\gamma$, an exponent γ greater than the value $\gamma_{CR} = 1/3$ required for stability in Newtonian theory, and reduces the maximum stellar mass by about 8 in the most massive stars. In lower mass neutron stars, the effects of general relativity are weaker.

The fundamental problem is to obtain a reliable equation

of state. At densities $\rho \lesssim 5 \times 10^{14}$ gm/cc the nuclear forces are presumably well enough known and Brueckner many-body theory applies. Important questions of nuclear and phase composition have been considered, but the nuclear many-body theory is, in any case, adequate only up to neutron star masses of about $0.17 M_0$ (Baym et al. 1971; Cameron 1970).

At higher densities hyperons are present, many body forces are important, Brueckner theory does not converge and the very notion of nuclear potential is inapplicable. Nor are useful restrictions imposed on the equation of state by general principles of causality, relativity, statistical mechanics or bootstrap dynamics at least until densities of $10^{17} - 10^{19}$ gm/cc are reached. We are interested, however only in stable neutron stars for which the maximum central density is about $\rho_{CR} \sim 4 \times 10^{15}$ gm/cc, or $10 \times$ ordinary nucleon density.

The approximately 20 neutron star calculations published exhibit values of M_{CR} from $0.3 M_0$ to $2.4 M_0$ and of ρ_{CR} from 1×10^{15} gm/cc to 4×10^{15} gm/cc. (Cox and Giuli 1968; Tsuruta 1970). In fact, published calculations differ even in the low density ($\sim 5 \times 10^{14}$ gm/cc), low mass ($\sim 0.2 M_0$) region where the nuclear forces are presumably agreed upon and many body theory applies!

Our purpose is to find simply and approximately what determines the

instability of neutron stars at densities ρ_{CR} and their maximum mass M_{CR} . We take the attitude that the nonrotating, nonmagnetic neutron stars considered are still gross oversimplifications of the pulsars whose properties are to be explained, and that no equation of state is reliable in detail at densities about $1 - 4 \times 10^{15}$ gm/cc. We therefore will be satisfied at present with simple calculations that avoid the complexities of nuclear physics, many body theory and computer integration of the TOV equation and are probably 50% accurate.

II. SOFTENING OF EQUATION OF STATE AND GENERAL RELATIVISTIC INSTABILITY.

For convective stability $dM/d\rho_c > 0$ is necessary. (The subscript c always refers to values at the center of the star.)

In Newtonian gravitational theory this requires that

$\gamma \equiv \rho dP/d\rho$ averaged over the stellar volume exceed the critical value $\gamma_{CR} = 4/3$. In general relativity, the conditions are more complicated but require that some effective γ exceed a γ_{CR} that somewhat exceeds the Newtonian value $4/3$. If the speed of low frequency sound waves

$$c_s^2 = dP/d\rho = c^2 \gamma (P/\rho c^2) \quad (2.1)$$

is not to exceed the speed of light, then

$$\gamma < (P/\rho c^2)^{-1} < 1/2 \quad (2.2)$$

where $\sigma = (P/\rho c^2)_c$. Whether such causality arguments are explicitly invoked or not, for any equations of state recently published the bound (2.2) is approached at $\rho_c \approx 2 \times 10^{15} \text{ gm/cc}$. Thus stars destabilize when the equation of state softens sufficiently. General relativity raises γ_{CR} somewhat above the Newtonian value $4/3$ so that the instability occurs at somewhat lower densities than it would in Newtonian theory. How important these general relativity corrections are depends on the magnitude of σ and the softness of γ .

A. Ideal Gas of Fermions: Independent Particle Model

In an ideal gas of noninteracting fermions of mass m_N , so long as the fermions move non-relativistically ($p_F \ll mc$)

$$P = (3\pi^2)^{1/3} \hbar c \frac{1}{4} \tilde{n}^{4/3} \quad (2.3)$$

where

$$\tilde{n} = \rho/m_N$$

is the number density. When the particles move relativistically

$$P = \frac{1}{3} \rho c^2 \quad (2.4)$$

as for a gas of photons. Thus $P \sim \rho^\gamma$ with γ decreasing from $5/3$ in the nonrelativistic region to 1 in the relativistic region, the two regions being separated approximately at $p_F = mc$ where

$$\tilde{n} = \frac{1}{3\pi^2} \left(\frac{p_F}{\hbar} \right)^3 \approx \frac{1}{3\pi^2} \left(\frac{m_N c}{\hbar} \right)^3 \quad (2.5)$$

and $\rho \approx \tilde{n} m_N = \frac{1}{3\pi^2} \left(\frac{m_N c}{\hbar} \right)^3 m_N = 6 \times 10^{15} \text{ gm/cc}$,
 if we use for m_N the free neutron mass. At this density γ passes
 through $4/3$, the minimum value necessary for stability on Newtonian
 theory.

Oppenheimer and Volkoff (Oppenheimer and Volkoff 1939),
 numerically integrating the TOV equation with the exact (partially
 relativistic) ideal gas equation of state, found

$$\rho_{CR} = 6 \times 10^{15} \text{ gm/cc}, \quad M_{CR} = 0.72 M_{\odot}. \quad (2.7)$$

The Oppenheimer-Volkoff star becomes unstable at that density
 where the non-interacting neutrons become special relativistic.
 While our estimate (2.5) shows that this critical density, at
 which $M(\rho_c)$ is a maximum, is essentially independent of general
 relativity, $M_{CR} = M(\rho_c = \rho_{CR})$ is severely reduced by general
 relativity. In the absence of general relativity, we would find,
 as for a white dwarf of mean molecular weight $\mu = 1$, a limiting
 mass $M = 6M_{\odot}$. As explained in Sec. ., the factor 8 reduction
 from $6M_{\odot}$ to $0.72 M_{\odot}$ is due to the enhancement of gravitational
 forces in general relativity. The small values $\sigma \leq 0.095$ and
 $GM/c^2 R \leq 0.11$ are additional indications that, despite the
 dependence of the numerical value of M_{CR} on general relativity,
 the ideal gas neutron star is not very relativistic.

The noninteracting gas model remains a reasonable one in independent particle approximation where the effect of nuclear interactions is to replace the free neutron mass by some effective mass m_{eff} . Because of how quantities scale, we have from Oppenheimer and Volkoff's result (2.7),

$$\begin{aligned} \rho_{CR} &= 6 \times 10^{15} (m_{\text{eff}}/m_N)^4 \text{ gm/cc} \\ M_{CR} &= 0.72 (m_N/m_{\text{eff}})^2 M_\odot \end{aligned} \quad (2.8)$$

Wang et al. (Wang 1970) find that in the region of attractive nuclear forces $m_{\text{eff}} = 1.6 m_N$ and in the region of repulsive forces $m_{\text{eff}} = 0.7 m_N$. Applying the latter quantity at the densities involved, we obtain from (2.7)

$$\rho_{CR} = 1.5 \times 10^{15} \text{ gm/cc}, \quad M_{CR} = 1.5 M_\odot \quad (2.9)$$

These results are tabulated in column (1) of Table 1. The results of a more relativistic ($\sigma = 0.5$), stiffer ($\gamma = \infty$) model are given in column (2) for comparison.

B. Polytropic Equation of State.

Over each density range, any equation of state is approximately polytropic, $P = K\rho^\gamma$, with $\gamma \equiv 1 + 1/n \approx \text{constant}$. In massive neutron stars, particularly if they are fairly stiff, the polytropic approximation will apply over that volume of the star where most of the mass is contained. The preceding paper (Bludman 1972)

tabulates and plots γ_{CR} , the minimum γ as a function of σ , or σ_{CR} , the maximum σ as a function of γ or n . In polytropes, $\sigma = (P/\rho c^2)_c$ characterizes all the effects of general relativity. To a few percent

$$\gamma_{CR} - 4/3 = \frac{7}{5} \sigma \quad ; \quad (2.10)$$

the effect of general relativity is to raise above 4/3 the minimum stiffness exponent necessary for stability. Polytropes will be stable and "causal" only for the small σ , large γ region.

$$1/\sigma > \gamma \gtrsim \frac{4}{3} + \frac{7}{5} \sigma \quad (2.11)$$

indicated in Fig. 3 of the preceding paper (Bludman 1972). In particular, for $\gamma < 2.084$, σ is limited to plotted values of $\sigma < 0.48$, which already limits the central density $\rho_c = (\sigma c^2/K)^n$.

We will now illustrate how in more "realistic" equations of state, the stiffness exponent γ decreases in the region

$\rho_c \approx 2 \times 10^{15}$ gm/cc, and how the $\gamma_{CR}^{vs} - \sigma$ curve and the relations

$$M = B \tilde{M} \quad (2.12)$$

with

$$B(n, k) \equiv \left[\frac{1}{4\pi} \left(\frac{(n+1)c^2}{G} \right)^3 \left(\frac{K}{c^2} \right)^n \right]^{1/2} \quad (2.13)$$

$$(K/c^2)^n = \sigma^n / \rho_c \quad (2.14)$$

$$\tilde{M}(n, \sigma) = \sigma^{\frac{3-n}{2}} v(\xi,) \quad (2.15)$$

$$GM/c^2 R = \sigma(n+1) v(\xi_1)/\xi_1, \quad (2.16)$$

calculated for polytropes, determine the critical parameters of neutron stars simply and approximately.

Cohen et al. (Cohen 1970), using a Levinger-Simmons potential and allowing for muon production and for nucleon clustering, calculate an equation of state that is approximately

$$p = m\tilde{n} + A\tilde{n}^\Gamma, \quad 10^{14} < p < 4 \times 10^{15}, \quad (2.17)$$

with $\Gamma \equiv \tilde{n} dp/p d\tilde{n} = \gamma(1+\sigma) \approx 2.9$.

This equation of state is not of the polytropic form $p = K\rho^\gamma$.

In fact, γ decreases through the values 2.04, 1.86, 1.64, 1.52, 1.44, 1.42 as ρ_c (in units 10^{15} gm/cc) and σ go through 1.50, 2.00, 3.01, 4.01, 5.00, 5.35 and 0.41, 0.55, 0.74, 0.87, 0.96, and 0.99 respectively. $((c_s/c)^2$ increases through 0.83, 1.02, 1.21, 1.32, 1.38, 1.41, passing through unity precisely at that density $\rho_c = 2 \times 10^{15}$ gm/cc for which Cohen et al. find the star to destabilize!)

The Cohen et al. decreasing γ vs σ curve intersects the polytropic $\gamma_{CR} - \sigma$ curve at $\gamma = 2.01$, $\sigma = 0.43$ where $\xi_1 = 1.94$, $v(\xi_1) = 0.615$, $\tilde{M} = 0.25$. The polytropic equations (2.12) - (2.16) therefore give $B(n,K) = 7.6 M_\odot$, $M_{CR} = 1.9 M_\odot$,

$GM/c^2 R = 0.27$, $R = 10.3$ km in 20% agreement with the values Cohen et al. obtained from the "realistic" equation of state and numerical integration of the TOV equation. These results are summarized in columns 3 and 4 of Table 1, which also gives in column 5 the results of a recent calculation (Baym 1971) using an equation of state (Pandharipande 1971) softer than that of Cohen et al.

III. SUMMARY AND CONCLUSIONS

In Table 1 we summarize the critical parameters of the neutron star models calculated for our three simple equations of state and for two more realistic equations of state. The Independent Particle Model (1) and Constant Density Model (2) were calculated for an effective mass $m_{\text{eff}} = 0.7 m_n$ corresponding to repulsive nuclear forces; this fixes the central pressure $\rho_c = 1.5 \times 10^{15}$ gm/cc. For the Constant Density Model (2), $\sigma \equiv (P/\rho c^2)_c = 0.5$ was chosen for comparison with models (3), (4), (5). Among the simple models, (1) and (2) represent extremes of soft and stiff equations, just as (5) and (4) represent such equations of state between the detailed "realistic" models considered (Cohen 1970, Baym 1971). The Polytrope Model (3) with $\gamma = 2$ was chosen as a simple representation of model (4) that nevertheless is more realistic than models (1) and (2).

Neutron stars become unstable when the exponent γ is sufficiently reduced, as it must ultimately be at high densities if the

sound speed $c_s^2 = \gamma P / \rho$ is not to exceed light speed. In this sense, the onset of instability in neutron stars at $\rho_{CR} \sim 2 \times 10^{15}$ gm/cc is not a qualitative effect of general relativity. (On the other hand, the inevitability and irreversibility of gravitational collapse is a qualitatively novel effect of general relativity, caused by the gravitational effects of pressure and the non-linear effects of space curvature. These effects are qualitatively decisive in black holes and at densities $\rho \sim 10^{19}$ gm/cc). The maximum mass M_{CR} of stable neutron stars is particularly reduced quantitatively by the dynamic and kinematic effects of general relativity.

We conclude that the instability of neutron stars always comes about because of the softening of equation of state. For soft equations of state, this happens when $\gamma \approx 4/3$, the critical value on Newtonian gravitational theory: the equation of state softens where the fermions become special relativistic, $P_F \approx mc$ at $\rho_c \approx 1.5$ to 6.0×10^{15} gm/cc. General relativity plays an important part only in reducing M_{CR} . The Independent Particle Model is a simple way to include, by replacing M_N by an effective mass M_{eff} , some of the effects of nuclear forces.

For stiff equations of state, general relativity leads to instability at γ_{CR} values somewhat larger than $4/3$. Nevertheless, the softening of the equation of state at densities about 10^{15} gm/cc, for causality reasons or otherwise, is what brings γ down below γ_{CR} . Particularly for stiff equations of state, once $\gamma(\rho)$ is approximately known in the region of densities about 10^{15} gm/cc,

the relativistic polytrope model considered in the preceding paper permits a quick estimate of critical parameters to about 20% accuracy.

These considerations explain why on all models considered instability ensues at $\rho_C = 1$ to 4×10^{15} gm/cc, and permit a quick estimate of the maximum neutron star mass M_{CR} . At lower densities, detailed calculations of nuclear and phase composition are relevant and have been done. (Cohen 1970; Tsuruta 1970; Baym 1971)

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Table 1. Critical Parameters of Neutron Star Models

	Simple Equations of State			"Realistic" Equations of State	
	(1) Independent Particle	(2) Constant Density	(3) Polytrope	(4) Stiff	(5) Soft
σ	0.095	0.5	0.43	0.55	0.6
$\frac{GM}{c^2 R}$	0.11	0.32	0.27	0.30	0.3
γ	$\frac{5}{3}$ to 1	∞	2	2.9 to 1.85	NA
<hr/>					
ρ_c (gm/cc)	$\leftarrow 1.5 \times 10^{15}$	$\rightarrow$	2×10^{15}	2×10^{15}	6.2×10^{15}
R(km)	19	8.3	10.3	11.9	7.0
M(km)	2.1	2.65	2.8	3.6	2.07
M/M ₀	1.4	1.79	1.9	2.4	1.41

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