REVIEW OF PHYSICS METHODS USED FOR
COMPUTING THE KINETIC BEHAVIOR
OF LARGE POWER REACTORS

N. E. Carter

JANUARY 1968

AEC RESEARCH &
DEVELOPMENT REPORT

POINTS TO		BY	

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

PACIFIC NORTHWEST LABORATORY

RICHLAND, WASHINGTON

operated by

BATTELLE MEMORIAL INSTITUTE

for the

UNITED STATES ATOMIC ENERGY COMMISSION UNDER CONTRACT AT(45-1)-1830

PRINTED BY/ FOR THE U. S. ATOMIC ENERGY COMMISSION

3 3679 00061 2996

BNWL-738

UC-80, Reactor Technology
~~Special Distribution~~

REVIEW OF PHYSICS METHODS USED
FOR COMPUTING THE KINETIC BEHAVIOR OF LARGE POWER REACTORS

By

N. E. Carter

Nuclear Experiments and Analyses Section
Reactor Physics Department

January 1968

NOT INVESTIGATED
DISTRIBUTION

PACIFIC NORTHWEST LABORATORY
RICHLAND, WASHINGTON

Printed in the United States of America
Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U.S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.65

REVIEW OF PHYSICS METHODS USED
FOR COMPUTING THE KINETIC BEHAVIOR OF LARGE POWER REACTORS

ABSTRACT

The status of the technology for theoretically predicting the neutron kinetic behavior of large power reactors is presented. The need for developing analytical techniques capable of predicting both the spatial and temporal behavior of a large reactor is pointed out. The various techniques which have been developed are described in detail and their relative effectiveness evaluated. The problem of xenon instability is also discussed along with the methods of solution. Computer codes which have been developed to date are listed.

PREFACE

This review was funded through the Heavy Water Reactor Program Office of the USAEC. Summary and conclusions which have been made in the review appear as part of the Reactor Physics section of the Program Plan (BNWL-656) issued by the Program Office. A technical approach to problems in the kinetics area was established in the Program Plan, based on the findings of this review.

TABLE OF CONTENTS

	<u>Page No.</u>
INTRODUCTION	1
STATUS OF POWER REACTOR DYNAMICS	2
XENON INSTABILITIES	18
CONCLUSIONS	28
APPENDIX A - THE EXACT METHOD	A-1
APPENDIX B - THE ADIABATIC METHOD	B-1
APPENDIX C - NODAL ANALYSIS	C-1
APPENDIX D - MODAL ANALYSIS	D-1
APPENDIX E - FLUX SYNTHESIS METHOD	E-1
BIBLIOGRAPHY	V-1

REVIEW OF PHYSICS METHODS USED
FOR COMPUTING THE KINETIC BEHAVIOR OF LARGE POWER REACTORS

N. E. Carter

I. INTRODUCTION

Economic pressures are currently motivating the power reactor industry to design and build reactors of the largest sizes permitted by present-day technology. The advent of these large reactor cores has necessitated a critical re-evaluation of the traditional methods used in reactor dynamics analysis. Cores with dimensions that exceed considerably the neutron migration length have neutron flux distributions that are very sensitive to instantaneous local conditions; that is, a small local perturbation can cause a significant flux redistribution. Thus, the crucial assumption of constant flux shape, which led to the derivation of the conventional "point" reactor kinetics equations and the consequent ability to analyze the dynamics of small reactors, cannot be applied to large reactors. In order to correctly describe the dynamic behavior of large core reactors it is necessary to treat both the space and the time variables.

Considerable effort is underway today to develop analytical techniques capable of predicting both the spatial and temporal behavior of a large reactor. Of the many approaches which have been attempted, only one has been reduced to widespread practice - that of solving by direct finite difference methods the space- and time-dependent, few-group, neutron diffusion equations. However, even this approach has been wholly successful for only one space dimension and very few energy groups. The need for techniques capable of handling the space-time reactor dynamics problem in two- and three-space dimensions is present

today, and numerical codes cannot now satisfy this need in reasonable amounts of digital computer time. Until a satisfactory numerical code is available, other techniques must be employed. The nodal, modal, adiabatic, and synthesis techniques are analytical approaches that have been suggested as a means of solving the space-time kinetics equations. These methods and their effectiveness relative to the numerical method are discussed in Sections II and III of this report. Specific details of each method are contained in the Appendix.

The goal of this report is to describe the current status of the technology used to analyze the dynamic behavior of a D_2O power reactor. It is to be noted that only a small part of the effort to develop methods to treat reactor dynamics problems has been accomplished relative to D_2O reactors; instead, the effort has been centered mainly around H_2O moderated reactors. However, from a dynamics point of view, most of the technology used to treat the dynamics of H_2O power reactors is directly applicable to D_2O power reactors. Hence, the theme of this report is to describe, in general, the current state-of-the-technology for reactor dynamics analysis.

II. STATUS OF POWER REACTOR DYNAMICS

Much effort has been made to simplify the neutron kinetic equations. These efforts have naturally increased in complexity and generality with time. Early analyses were based upon the continuous slowing-down diffusion model, where the neutron flux distribution was assumed to be separable in space and time. The classic space-independent zero power kinetic equations are then derived by assuming that the space dependency can be represented by some average distribution whose shape does not change with time.^(1,2,3)

To place these analyses on a more firm theoretical footing, Henry⁽⁴⁾ rederived the kinetic equations from the more general transport theory. The purpose of his paper was to obtain a precise statement of the approximations required to derive the reduced equation and to clarify the physical meaning of the parameters appearing in such a reduction. The neutron density is expressed as an amplitude $T(t)$ times a shape function, $\eta(r,t)$, which is itself time dependent. By the introduction of a static adjoint flux, the spatial dependency of the separable neutron transport equation could be eliminated. The resulting time-dependent equation was indeed of the same form as the classic kinetic equations that had previously been obtained, but the parameters - such as reactivity, neutron lifetime, and the effective delayed neutron fraction - were precisely defined. This reformulation is only a formality since the unknown space shape, $\eta(r,t)$, is involved in the evaluation of the reactor parameters. Except in the few cases where the shape is constant and known during the transient, it is necessary to approximate $\eta(r,t)$. A very common approximation is to use the shape $\eta_0(r)$, corresponding to the steady state. In this case, use of the static adjoint flux, $\phi_0^*(r)$, as a weighting function causes the first order part of the error in the computation of reactivity to vanish.^(4,5) Other choices of the weighting function that have been made are discussed and compared by Gozani⁽⁶⁾ and Henry;⁽⁷⁾ however, the adjoint flux has been used the most. The physical interpretation of the adjoint function has been discussed by Hurwitz⁽⁸⁾ and Ussachoff⁽⁹⁾ in the critical case, Selengut⁽¹⁰⁾ and Kadomtsev⁽¹¹⁾ in the subcritical case, and Lewins⁽¹²⁾ in the time-dependent case.

The numerical solution of the space-independent reactor kinetic equations has been treated by various authors.^(13,14,15,16,17) The various techniques have had varying degrees of success in handling the problems of numerical stability, truncation error, and machine running time.

"More recently, it has become both necessary and possible to describe in some detail the space-time kinetic behavior of a nuclear reactor. The necessity has arisen for two reasons. First, the current trend in power reactors is towards very large sizes for which space-time effects can become limiting design considerations. Second, certain useful experimental techniques for measuring basic reactor parameters and for verifying the adequacy of theoretical models require that space-time phenomena be accounted for, either as an essential part of the analysis or to insure that they are not disturbing the experimental conditions. The possibility of providing a detailed description of space-time effects has arisen from an improved understanding of the basic theory and from an increased capacity to determine numerical solutions to the governing equations through the use of digital computers."⁽¹⁸⁾

It has not been clear exactly when the point kinetics equations become seriously in error. Most time-dependent reactor analysis in the past has used the conventional point kinetics, which assumes that the distribution of the flux is independent of its time behavior. In the case of a reactor core whose size is large compared with the neutron migration area, a local change of reactivity produces a marked deviation in the flux from its asymptotic distribution for many neutron lifetimes.

In a rigorous formulation, reactivity is determined by this flux shape. Hence, for a given perturbation the solution to the kinetics equations can change (sometimes substantially) if flux shape changes are taken into account. But quantitative information concerning the magnitude of errors and the conditions which lead to them has not been generally available. With the development of computer programs capable of solving the space-time diffusion equations in detail,^(19,20) a more precise evaluation of the accuracy of the point kinetic equations is becoming possible.

Several studies have recently been made to determine the range of validity of the point kinetics. Yasinsky and Henry⁽²¹⁾ investigated the kinetic behavior of two simple slab cores having material compositions typical of light-water moderated power reactors. The first core (240 cm thick) was thought to be representative of a large loosely coupled reactor. The second (60 cm thick) was considered to be a tightly coupled core. In the numerical experiments presented, both of the reactors, initially critical, were perturbed by certain local changes in fission cross sections in such a manner as to accentuate nonseparable space-time effects. The resulting transients were described exactly by WIGLE, a two-energy-group, space-time, diffusion-theory code. The same transients were analyzed by conventional point kinetics, by the adiabatic approximation, by a flux synthesis technique, and by two nodal methods. It was felt that the lack of feedback in the analysis permitted distinguishing between situations where the space-time description is intrinsically accurate and those where it is poor but unimportant. Results were presented for both prompt critical and

delayed supercritical transients. Perhaps the most fundamental result of the entire investigation was the total inadequacy of the point approximation when it is applied to large cores. Predictions of the point kinetics for the large, 240 cm core were off by a factor of more than 10^4 . Even when applied to the 60 cm core, the point kinetics predicted the peak power too small by a factor of ~ 4 , the conclusion being that small changes in flux shape can result in complete deterioration of the point model.

Johnson and Gyftopoulos⁽²²⁾ performed an analysis which was very comparable to that of Yasinsky and Henry.⁽²¹⁾ They analyzed the behavior of a 60 cm slab core for various perturbations using WIGLE and the conventional point kinetics. The unique point about this numerical experiment was that inherent feedback was included by changing the thermal-absorption cross section in each region in proportion to the fission-energy density. Again the complete inadequacy of point kinetics was demonstrated.

The importance of spatial effects for reactivity changes $\ll \beta/2$ per sec was demonstrated by Hamric⁽²³⁾ using the one dimensional kinetics code CLUMSY I.⁽²⁴⁾ The reactor chosen for investigation was the zero-power Hanford Test Reactor modified theoretically to include cooling and to produce about 500 MW(th). The 18-ft cube of graphite was doubled in all dimensions to represent a large reactor, and the power feature was included so that Doppler feedback could be utilized. A series of numerical experiments again demonstrated the importance of spatial effects in large reactors even for very small reactivity changes.

The transfer function of the zero power reactor NORA has been measured as a function of position.⁽²⁵⁾ In spite of the small core dimensions, considerable space dependence was observed for frequencies above the break frequency β/ℓ . A theoretical interpretation using two-group theory was obtained by solving the equations by the finite differences method, and the qualitative agreement between theory and experiment was good. The conclusion of the investigation was that spatial neutron kinetics effects may be very important in certain types of reactors, especially in large D₂O reactors.

A survey paper was presented at the 1964 Geneva Conference which correlated the work in space-time dynamics as of 1964.⁽¹⁸⁾ The paper is divided into three sections. Work relating to the basic mathematical theory of the reactor space-time equations is reviewed in the first section; approximate methods for obtaining numerical solutions to these equations are described in the section following; the applications of space-time analysis that have been made to various experiments and to the problem of xenon spatial stability are discussed in the last section. An excellent bibliography is also included.

The status of space-time reactor analysis as of 1965 was reviewed by Kerlin.⁽²⁶⁾ This discussion was restricted to methods used primarily for "fast transients." The methods for analyzing "slow transients" have been thoroughly reviewed by Wilson⁽²⁷⁾ for the xenon stability problem. The discussion that follows is an extension of these review articles which hopefully will indicate the present "state-of-the-technology" in the various aspects of neutron space-time dynamics and will indicate what problem areas remain to be solved.

To describe in detail the space-time behavior of the neutron flux in a reactor is difficult. The physical source of this difficulty is the fact that the prompt component of the neutrons emitted in fission reacts to changes in local environment almost instantaneously while the component due to neutrons from delayed emitters changes relatively slowly. Thus, numerical methods for solving the equations which specify the space-time behavior must use very small time steps to account for the prompt neutron behavior, yet must describe a time span sufficiently long that delayed neutron effects are included. Since the major part of the numerical difficulty is associated with the spatial part of the problem, the traditional approach, as discussed previously, has been to neglect changes in flux shape and to concentrate on describing accurately the behavior in time of the overall neutron population or power level. However, as has already been pointed out, the flux shape is practically never constant during a transient, and assuming that it is can lead to very serious errors. Hence, the necessity of solving the complete space-time problem is evident.

The so-called exact method is a brute force technique in which the time-dependent group diffusion theory equations, together with those equations required to represent the inherent feedback effects, are simply reduced to difference equations and solved by numerical techniques. The spatial differencing intervals, or node spacings, are typically required to be quite small in order to obtain accurate results, and are equivalent in size to the node spacings required in static calculations. This method is the only method of space-time reactor dynamics analysis which has been

reduced to widespread practice, and it is often used as the basis for evaluating the accuracy of other, more approximate methods. The most typical examples of the method are the various versions of the one-dimensional WIGLE program.^(19,20,28) The limitations of the WIGLE programs are that they treat only one spatial dimension and only two neutron energy groups; however, xenon and temperature feedback effects are included. In principle, extensions of the method to include nonlinear feedback, several energy groups, and three-space dimensions are straightforward. In practice, however, such extensions would lead to long running times on very large computers. Although two-dimensional codes have been written^(29,30) and schemes have been developed to increase the speed of numerical integration, a practical, accurate code does not exist. K. F. Hansen⁽³¹⁾ has made a detailed analysis of possible approaches to a two-dimensional kinetics code and has evaluated their relative merits. The details of his analysis are given in the Appendix. Using the UNIVAC-1108 as his standard present-day computer, he concluded that a practical two-dimensional code could be developed. The most favorable method of solution is probably the alternating direction implicit approach (ADI), with the GRCORK⁽³²⁾ and GAKIN⁽³³⁾ approaches less attractive, but still potentially useful. The results of Hansen's study, the success of WIGLE codes in one-dimension, and the promise of larger and faster computers in the future probably lead one to conclude that a finite difference code will eventually provide the exact answer for multi-dimensional, space-time kinetics studies. However, these kinetics problems exist today and other, more approximate techniques must provide this information until the time when a suitable finite difference code is available. Even when an exact code is available,

these other more approximate techniques should still remain useful since they will inevitably be cheaper to use than the exact code.

The usual approach that is taken to treat the space-time dynamics problem approximately is to attempt to break the space-time equations into separate space equations and time equations. The space equations are usually solved on digital computers, and the time equations are solved either digitally, by discretizing the time variable, or by an analog computer with time varying continuously. Several methods which have been used or proposed for accomplishing this breakdown of the space-time equations are the adiabatic, nodal, modal, and synthesis techniques.

In the adiabatic method,^(34,35,36,37,38,39) the approximation is made that the rate-dependent terms in the diffusion theory equations have a negligible effect on the flux shape, i.e., that the flux shape adjusts instantaneously to all perturbations. At each time step during a computer solution, the shape of the flux is determined by means of a conventional reactor statics code. The resultant flux shape is then employed in the computation of the instantaneous reactivity of the core. The reactivity is used in conjunction with the conventional-point-kinetic equations to obtain a new value of the flux amplitude.

The principal advantage of the adiabatic method is that it requires less computer time than the exact method while still maintaining a significant improvement in accuracy over the point kinetics. However, the approximation that the flux shape adjusts instantaneously does restrict its range of validity, particularly when the flux is undergoing significant redistribution. The possible shortcomings of the adiabatic method were clearly

pointed out by Yasinsky and Henry.⁽²¹⁾ Attempts to improve upon the basic adiabatic technique by a more realistic treatment of the delayed neutron source have been quite successful in limited applications for both prompt and delayed supercritical reactors. Very little work has been done on applying the adiabatic technique to a two-dimensional situation.

The nodal,^(40,41,42,43,44,45) or coupled region, method can be viewed as an attempt to overcome the principal disadvantage of the exact method. The reactor is divided into a few large regions rather than many small ones. A flux shape is chosen for use within each region, and the amplitude of each shape function is then determined by space-independent equations. The equations are of the form of the point-kinetic equations, but have incorporated in them additional terms to account for the effects of the transfer of neutrons from one region to another.

The principal difficulty encountered in the use of the nodal method has been the determination of the coupling parameters. Various approaches to this problem have been suggested and are discussed in more detail in the Appendix. No really practical method for obtaining suitable coupling coefficients has yet been found which would allow the nodal approach to be useful in three-dimensional kinetics applications. However, the technique is desirable because of its resemblance to exact methods and because feedback effects are very easily included in the analysis.

Wachspress^(44,45) has been working on a nodal approach in which the coupling coefficients are determined from a variational principle. The

method requires selecting shape functions for the flux and current, and no best method exists for choosing these shapes. Wachspress has applied the technique only to some simple models, but it is probably the type of nodal analysis that has the best possibility at present for solving three-dimensional kinetics problems.

Modal analysis⁽⁴⁶⁾ is an approach to space-time kinetics that allows for the most improvisation by the analyst. Modal methods are characterized by some form of expansion of the space- and time-dependent reactor variables into a sum of products of time functions and space functions. The general system equation describing the dynamic behavior of a reactor is

$$\frac{\partial \phi(\vec{r}, t)}{\partial t} = L(\vec{r}, t) \cdot \phi(\vec{r}, t)$$

where $\phi(\vec{r}, t)$ is the solution vector and $L(\vec{r}, t)$ is an operator which describes the system. To obtain a solution the flux vector is expressed in the form

$$\phi(\vec{r}, t) = \sum_{j=1}^N \psi_j(\vec{r}) \cdot T_j(t)$$

The improvisation in the approach involves the choice of the space functions $\psi_j(\vec{r})$. Once the form of the space function is chosen, the difficulties of the method consist of actually calculating the values of the ψ_j 's and of calculating the time coefficients $T_j(t)$ that apply for the given choice of space function. Another consideration is the determination of the number, N , of modes required for an adequate analysis. Several different forms of modal analysis have been suggested and used. The differences among the various modal methods derive from the many possible choices of the shape functions $\psi_j(\vec{r})$.

The eigenfunctions of the Helmholtz or wave equation can be used for the space functions. The Helmholtz equation is

$$\nabla^2 \psi + B^2 \psi = 0.$$

The solutions of the Helmholtz equation are not difficult to find, and the eigenfunctions have orthogonality properties that may be used in calculating the time coefficients. The Helmholtz modes have been used for nonuniform systems with delayed neutrons and two energy groups.^(47,48,49) However, complex geometries for which the flux shape deviates markedly from the shape of a uniform system require a large number of Helmholtz modes for adequate representation. This makes the computation excessively long and probably eliminates the Helmholtz modes from consideration for multi-dimensional problems.

The obvious method to overcome the disadvantages of the Helmholtz modal approach is to attempt to reduce the required number of modes by finding a set of shape functions which are more natural to the system. Omega modes⁽⁵⁰⁾ and lambda modes⁽⁵⁰⁾ are of this type. They are defined as the eigenfunctions of the set of equations describing the entire system, including delayed neutrons and feedback effects. The fundamental mode is then the steady-state flux shape. The main advantage of this approach is that relatively few shape functions are required in the modal expansion. The natural modes also possess orthogonality properties that are useful in determining time coefficients. The main disadvantage of the method is that it is very difficult to determine the natural modes for multi-dimensional problems.

Dougherty and Shen^(51,52) have developed a modal expansion method that avoids the use of orthogonality properties for finding the time coefficients. The method permits analysis of complex geometries using existing computer codes and is theoretically applicable to multi-dimensional problems. The procedure is to first run a standard static calculation of the flux and the adjoint flux for the reactor. To obtain the modes, one then divides the reactor into regions and sets the fission cross section everywhere equal to zero. One then inserts a distributed fission source, obtained from the static source distribution, into one of these regions and solves for the resulting flux shape. This process is repeated for each region, thus generating a set of spatial shapes which Dougherty and Shen refer to as "Green's function modes." A variational procedure is then used to find suitable time coefficients. The Green's modes are easier to find than the natural modes, and they should still possess better convergence properties than the Helmholtz modes. Results of one-dimensional studies with the Green's modes indicate that they are capable of rapid convergence to a solution that agrees with an exact solution.^(53,54) Extension to multi-dimensions should be straightforward, but no studies have yet been made to determine the practicality of such an extension.

Another approach that avoids the restriction to orthogonal eigenfunctions is the synthesis method.^(55,56,57,58,59,60) This technique employs the idea of blending guessed solutions which in some sense bracket the true solution. It may be viewed as a procedure for breaking down problems with many independent variables into problems with fewer variables.

Most reactors have great complexity of composition in the horizontal plane. Vertically, however, the material properties are usually uniform, or at least slowly varying, except for a small number of discontinuities occurring at the interfaces of reflectors and at the tips of partially inserted control rods. It is this simplicity in the vertical direction that encouraged the consideration of synthesis-type solutions. In space-synthesis approximations, the three-dimensional flux solution for energy group g is expressed in the form

$$\phi^g(x,y,z) = \sum_{k=1}^K H_k^g(x,y) \cdot Z_k^g(z),$$

where the H_k^g are two-dimensional static solutions in the horizontal plane. These trial functions, as they are called, are selected by the intuition and experience of the analyst. Synthesis approximations of this type are described in more detail in the Appendix. Analogous expressions may be employed in multi-dimensional depletion studies, using what is termed time-synthesis. Space- and time-synthesis results can be compared with results from existing finite difference codes. Sufficient comparisons of this type have been made to conclude that the synthesis technique is an accurate and inexpensive design tool to use in multi-dimensional static and depletion problems.

Synthesis methods may also be used to determine the space-time behavior of a reactor during a transient. In this case the g 'th group flux is expressed as

$$\phi^g(\vec{r},t) = \sum_{k=1}^K \psi_k^g(\vec{r}) T_k(t),$$

where the ψ_k^g are known spatial trial functions which are found by solving static group-diffusion problems using values of reactor parameters representative of reactor conditions at several times during the transient. The T_k are unknown mixing coefficients which are obtained by a variational technique or some other weighted residual technique.

The basic idea behind choosing time-synthesis trial functions is that of bracketing the spatial flux shape in time. For example, consider an initially critical reactor for which one desires to predict the flux space-time behavior when a control rod is being withdrawn. Initially, the critical flux shape is denoted by $\psi_1(\vec{r})$. After the rod has been totally withdrawn, the flux will achieve an asymptotic shape, call it $\psi_K(\vec{r})$. The idea then is to express the dynamic flux behavior in the form

$$\phi(\vec{r}, t) = \psi_1(\vec{r})T_1(t) + \psi_K(\vec{r})T_K(t),$$

where the unknown mixing coefficients T_1 and T_K blend the bracketing flux shapes ψ_1 and ψ_K . Additional trial functions may be determined by static solutions using values of reactor parameters that would be achieved at several instants during the transient.

An exact finite difference solution to the neutron space-time equations exists only for one-dimensional situations, i.e., WIGLE. In one-dimensional comparisons with WIGLE, synthesis methods have provided better results than any of the other approximate methods - nodal, modal, or adiabatic. Two advantages of the synthesis method are that trial functions are easy to obtain and the trial functions can be tailored to the problem at hand.

Within the continuous synthesis framework, all K trial functions are used in the calculations at any specific time, even though many of these trial functions may be associated with reactor conditions vastly different from those present at this specific time. Hence, for problems where a large number of trial functions are required, the synthesis technique can become expensive. This difficulty can be circumvented by using a synthesis technique which provides the possibility of adding or deleting spatial trial functions at specified times during a transient analysis.^(61,62) Thus, one can require that the set of spatial trial functions being used at any particular time be composed only of those spatial shapes most representative of reactor conditions near that time. Such an approximation necessitates the use of spatial trial functions which display a discontinuous time behavior; hence, this method is referred to as time-discontinuous synthesis. One-dimensional applications of the discontinuous method have indicated that it is an inexpensive and fairly accurate method for solving the space-time kinetics equations.

Synthesis techniques seem to offer the best possibilities for obtaining approximate solutions to multi-dimensional kinetics problems. Many tests of the accuracy of synthesis methods have been made, and they have proven synthesis methods to be superior to all other approximations. The major fault with a synthesis technique is the lack of error bounds. There really is no way of determining whether a synthesis result is accurate except by performing an exact, finite difference analysis. Thus, again one can see that finite difference codes will probably be the final answer to space-time analyses, but synthesis techniques will still be useful because they

are cheaper. Using synthesis methods to perform parameter studies, with an occasional check on accuracy by an exact code, should be a useful design technique. Until an exact, three-dimensional code is available, one can use synthesis methods to perform design analyses, using intuition and experience to evaluate the results.

III. XENON INSTABILITIES

A very important consideration in the design and operation of nuclear reactors is the problem of xenon oscillations. A large thermal reactor operating at sufficiently high flux levels is subject to oscillations in the power distribution that are associated with a periodic redistribution of xenon poison. To see how this variation in xenon occurs, consider a reactor that is operating at constant power and at xenon equilibrium. The introduction of a small tilt in the flux shape,⁶ for example from top to bottom, causes the xenon to burn out more rapidly at the top and less rapidly at the bottom. However, the increase in flux also increases the rate of formation of fission product ^{135}I which, after a delay due to its 6.7-hr half life, decays and causes a later increase in xenon. The key factor here is the point that the decrease in xenon due to burnout is prompt, whereas the increase in xenon due to ^{135}I decay is delayed. Hence, initially the xenon concentration is decreased at the top, the reactivity then increases at the top, and the flux tilt is amplified. Ultimately, the growth of xenon from the iodine that is now forming more rapidly at the top and the radioactive decay of xenon at the bottom reverse the reactivity distribution, and the power then peaks in the bottom of the reactor. These oscillations can persist unless some action is taken to counteract them.

Xenon poisoning in reactors was discovered when the first Hanford production reactor operated, and xenon instability was first observed when local hot spots were detected in the Hanford reactors.⁽⁶³⁾ The first clear demonstration of flux-tilt oscillations was obtained at Savannah River,⁽⁶⁴⁾ and oscillations have since been reported for many reactors, such as Shippingport.⁽⁶⁵⁾ Characteristically, the spatial oscillations have long periods and, therefore, there is ample time for a reactor operator to take corrective action. The scheme has been to monitor the power level and trend of the power level in various regions of the reactor and then to control the multiplication factor in each region. To accomplish control of xenon oscillations by this technique it is necessary to provide instrumentation all around the core to detect oscillations, and then the reactor control system must necessarily be designed to allow rods in various core regions to move independently. This method of operator control has worked in many situations, especially for large graphite reactors,⁽⁶⁶⁾ but a number of investigations have shown that this mode of control does not suppress xenon oscillation.^(67,68,69) In fact, Margolis⁽⁷⁰⁾ has demonstrated a situation where this conventional control may actually excite axial oscillations in a core which would be stable if reactivity were compensated for by temperature drift instead of by control rod motion. Thus, a complete understanding of the basic theory of xenon spatial oscillations is required if one is to develop control systems that can stabilize the oscillations.

The first efforts to analytically describe the xenon phenomena were designed primarily to obtain threshold conditions for xenon oscillations, i.e.,

to see if reactors could be built such that xenon oscillations could be eliminated. The basic theory was first analyzed using a modal method by Ward⁽⁷¹⁾ and a nodal method by Henry and Germann.⁽⁷²⁾ The modal method has been pursued by Randall and St. John,⁽⁷³⁾ Wick,⁽⁷⁴⁾ Harris and Lacy⁽⁷⁵⁾ and others.

The equations governing xenon oscillations are nonlinear; however, within a small neighborhood of the equilibrium point a linear approximation may be used. This linearization technique has been used in almost all analytical analyses of xenon feedback with the justification that reactor control systems must keep the xenon perturbations small. The procedure is to write the differential equations for the neutron flux, iodine concentration, and xenon concentration and then to introduce the linearization, i.e., all variables are expressed as the sum of a steady state component and a small, time-dependent component in the form

$$\phi(\vec{r},t) = \phi_o(\vec{r}) + \delta\phi(\vec{r},t)$$

$$X(\vec{r},t) = X_o(\vec{r}) + \delta X(\vec{r},t).$$

Second order terms like $\delta\phi \cdot \delta X$ are then neglected and the resulting equations are linear.

An excellent summary of all investigations of xenon instabilities that have been performed as of 1964 has been published by J. W. Wilson.⁽²⁷⁾ The conclusions that follow may also be found in that report; in addition, information pertaining to xenon that has been obtained since 1964 is also now reported.

The various analyses of xenon feedback have shown that instability is enhanced as

- the flux level is raised
- the distribution of the flux in the reactor is flattened
- the reactor size is increased

and as

- the temperature coefficient of reactivity increases algebraically.

The conclusion relative to size has been that spatial instability may occur when the cross sectional area of the reactor expressed in square feet is greater than the neutron migration area expressed in square centimeters, unless the flux level is very low or there is a sizable negative temperature coefficient.⁽⁷³⁾ Physically, this means that xenon oscillations can occur in reactors that are so large that different regions begin to act independently, with almost none of the fissions in one region being caused by neutrons born in another region.

The usual procedure used to analyze the xenon problem is to express the neutron-flux distribution and xenon distribution as a sum of modes or harmonics in the form

$$\delta\phi(\vec{r},t) = \sum_{i=1}^N \psi_i(\vec{r})T_i(t).$$

Kaplan⁽⁷⁶⁾ has shown that, for the linearized perturbation analysis of the two-group reactor with xenon present, it is possible to find modes that have the property of finality. For such modes, the time behavior of each mode is independent of the other modes. However, these modes are described by solving for complex eigenfunctions and eigenvalues and are usually difficult to determine. Consequently, almost all analytical treatments have used either the μ modes and λ modes or the Helmholtz modes. The μ modes were developed by Harris and Lacy⁽⁷⁷⁾ as an approximation to the Kaplan modes and the Helmholtz modes have been discussed before. When the flux is described by modes lacking finality, then a very important consideration is the interaction between such modes. It is usually convenient, and often necessary, to assume that the modes do not interact in order to facilitate the mathematical analysis. But Gyorey⁽⁷⁸⁾ and Pearce⁽⁷⁹⁾ have shown that this interaction between modes can be very large and may seriously affect the conclusions reached.

In the modal analysis the time variation of each mode is usually found to be of the form $\exp(\alpha t + i\beta t)$, where α and β are constants and t is time. The stability criterion is $\alpha < 0$, and the instability may be of two types:

- A fundamental instability in which, if uncontrolled, the power throughout the reactor would rise or fall in equal proportions.
- A spatial instability in which, if uncontrolled, the power variations would have magnitude and direction that would depend on the position in the reactor.

It has been shown^(67,68,69,70) that action of the reactor control system can either suppress or excite xenon spatial instabilities. Thus, a suitable program is necessary for operating the control rods.

The nonlinear stability characteristics of fundamental-mode oscillations of a point thermal reactor with xenon feedback but no temperature feedback have been studied by Smets⁽⁸⁰⁾ and by Gyftopoulos⁽⁸¹⁾ using Liapunov's second method. The general conclusions of these studies have been that the nonlinearities become increasingly important at high fluxes and that the general nonlinear equations are very difficult, if not impossible, to solve analytically. Thus, in treating problems of complex geometry and in handling nonlinear problems, numerical solutions using high-speed computers are necessary.

Digital simulation of the space-time behavior of a reactor undergoing xenon feedback is usually accomplished by a stepwise procedure; for some initial time the static, space-dependent reactor problem is solved; then the xenon and iodine behavior for this reactor during a prescribed time increment is determined and represented as a change in reactor parameters; then a new static problem is solved for the reactor with the new parameters; etc. The time behavior is determined by comparing the successive solutions. WIGL2⁽²⁰⁾ treats the xenon problem in one-dimension and two-energy groups. Barss⁽⁸²⁾ has described a program using a one-group, two-dimensional model, and Pearce and Roth⁽⁸³⁾ have developed a program, called XENOSC, using a two-group two-dimensional model. STAB^(84,85) is a program that solves the one-group time-dependent three-dimensional diffusion equation, together with equations representing feedback effects from heat transfer, xenon production, temperature, and control-rod motion. The digital computer approach is more general than the analytical approach in that it can handle complex geometries and can include nonlinear effects. In principle, it can give the complete

space-time response of the reactor to any operating change. However, small time steps are often required, and the number of space points that can be handled is limited. Hence, as in the analysis of fast transients, a digital code is probably the best method to use for xenon analyses, but the speed and memory capacity of present day computers are not sufficient to allow for detailed multi-dimensional codes.

The analytical approaches that have been previously described in this report have served to give a great deal of insight into the causes of xenon instability and have permitted calculation of some threshold conditions of xenon oscillations. The dependence of xenon instability on reactor size, flux level, temperature coefficient, etc. is now fairly well understood. It is also fairly well agreed that future design objectives of high flux and high power will make necessary the construction of reactors that are inherently unstable against xenon oscillations. The control problem in these large high-flux reactors is not to calculate stability thresholds but to control the flux despite the inherent xenon instability. The realization of the rather involved nature of xenon oscillations and the discovery that the action of a reactor control system can actually excite xenon spatial instabilities rather than suppress them have made apparent the need for detailed multi-dimensional analyses of reactors and their control systems. Since computers are not yet capable of handling a three-dimensional xenon code of sufficient detail, other approximate methods are being used.

The modal analysis technique has been used extensively in the analysis of xenon oscillations of practical reactor models. Westinghouse Bettis has done a

great deal of work using the natural modes (i.e., those having finality), as originally described by Kaplan,⁽⁷⁶⁾ and the μ -modes, as originally described by Harris and Lacy.⁽⁷⁷⁾

Westinghouse Bettis has developed⁽⁸⁶⁾ a practical quantitative model that has successfully predicted the observed stability variation of both Shippingport Core 1 and Core 2, and has given consistent comparisons with extensive analog simulations of a large seed-blanket reactor. The analog model used a nodal technique in which the core was represented as a three-dimensional array of coupled nodes, each described explicitly as to power, xenon, and fuel and moderator feedback. A very analogous nodal analysis has been performed using a digital code called X-WAVE. The model used to analyze the Shippingport cores evolved from μ -mode-type approximations to the basic p-mode, or natural mode, equations. The model includes all of the major factors affecting xenon stability in reactors; size, thermal flux spectrum and level, heterogenities, X^{135} yield, and fuel and moderator temperature feedback. This work is probably the most recent and complete analysis of xenon instability using μ -mode or nodal techniques.

Chock, Kaplan, and Margolis⁽⁸⁸⁾ have done some analog studies of the effect of coolant recirculation on spatial xenon stability. In reactors with negative temperature coefficient of reactivity the recirculation of the coolant from heat exchanger to core constitutes a feedback loop which carries an error signal that grows at a rate proportional to the mismatch between the reactor power and the power demand. The signal grows to whatever value is necessary to reduce the mismatch to zero. In this way

the recirculation enhances the temperature coefficient and stabilizes the reactor against xenon oscillations in the fundamental mode. Chock, et al., have pointed out that this same effect may be exploited to stabilize the reactor against xenon oscillations in higher spatial modes. This may be done by separating, rather than mixing, the flows in the plena and by feeding the coolant from each section of the core back into that same section. The model used to illustrate this effect was an analog simulation of a large reactor using a coarse three-dimensional, r, θ, z mesh. Kaplan and Yasinsky⁽⁸⁹⁾ have investigated this question further using a modal analysis based upon the natural modes for a one-group, uniform-ring reactor.

Stacey⁽⁹⁰⁾ at KAPL has been investigating xenon spatial oscillations using practical reactor models. The basic mathematical model consists of the three-group diffusion equations and the xenon and iodine equations. These equations are linearized and Laplace transformed. Then the spatial dependence of the deviation (from equilibrium value) of the flux and xenon concentration is expanded in the first harmonic eigenfunction of the equilibrium neutron balance operator (i.e., a λ -mode expansion). Use of orthogonality relations result in a transfer function relation between the flux or xenon deviation and some driving function. The condition that the deviations do not grow in time is that the poles of the transfer function lie in the left half complex plane. In order to test the stability criterion, a zoned two-dimensional model was analyzed. Results were compared with the SPRITE code, a code which solves the 4-group diffusion and xenon equations numerically, and agreement was excellent.

J. V. Wilson⁽⁹¹⁾ has been attempting to analyze the xenon stability problem using Helmholtz modal expansions of the variables in a linearized, two-energy group model. He has been very successful in one-dimensional situations, but the convergence rate of the Helmholtz modes seems to be very slow for multi-dimensional situations. In this type of analysis, one can express the time dependence of each xenon harmonic in the form $\exp(\alpha t + i\beta t)$, where the values of α, β are found by solving a complex eigenvalue problem. It is not clear at this time whether this type of analysis will be successful in determining optimal control systems for practical reactors.

Stacey⁽⁹²⁾ at KAPL has reported some success in determining an optimal control system for xenon transients. Stacey notes that due to the delayed reactivity effect associated with xenon production and decay, any control action taken to suppress an instantaneous power tilt may actually aggravate the spatial power transient. In developing a control program, the designer must account for the eventual, as well as the immediate, consequences of the control action. This general problem is, in principle, amenable to solution by the techniques of optimal control theory.^(93,94) Some exploratory work in this area has been done by Wiberg.^(95,96) In Stacey's work, dynamic programming is applied to obtain a computing algorithm for the optimal control of xenon-induced spatial power transients. XENOP, a digital code, has been written to solve the flux-xenon-iodine-temperature problem in three spatial dimensions (777 nodes), using modified (fast-neutron coupling included) one-group diffusion theory. Different groups of control rods are allowed to move, and an optimal control program is determined.

IV. CONCLUSIONS

Economic considerations have motivated the power reactor industry to design and build reactors where power output is as large as present-day technology permits. These reactors necessarily have physical dimensions that are large compared to their representative neutron migration lengths. Many numerical and experimental studies have indicated that conventional point kinetics techniques are completely inadequate for describing the dynamic behavior of large reactors. Individual regions of such large reactors can react independently to perturbations; as a result, the flux distributions can undergo significant spatial redistributions. The fundamental assumption of point kinetics is that the flux shape does not change during a reactor transient. Hence, in order to correctly describe the dynamic behavior of a large power reactor it is necessary to determine both the space and the time-dependence of the neutron flux.

The gross dynamic behavior of a nuclear power plant may be represented by a set of nonlinear differential equations, i.e., the space-time kinetics equations. Since an exact solution of these equations is not attainable, it is necessary to construct some realistic approximate models. The approximate models that have been discussed in this report are the

- exact method,
- adiabatic method,
- nodal method,
- modal method, and
- flux synthesis method.

The exact method is a technique whereby the space-time kinetic equations are reduced to difference equations and solved by numerical techniques. Many one-dimensional digital computer codes have been written based upon the exact technique:

- WIGLE (28)
- WIGL2 (20)
- CLUMSY-I (24)
- GAKIN (33)
- GRCORK (32)

These codes are generally considered to provide the best solutions to the one-dimensional space-time equations that are currently attainable. However, present-day computers are not capable of solving the kinetics difference equations for complex reactor geometries in two- and three-dimensions; therefore, more approximate methods must be employed.

Each of the approximate techniques discussed in this report was found to have its relative advantages and disadvantages, but, at the present time, the flux synthesis method appears to be the most promising approach for multi-dimensional situations. Westinghouse Bettis has done most of the development work on the synthesis technique, and they have written synthesis codes for three-dimensional static and burn-up analyses that have proven to be very accurate and relatively inexpensive to use. One-dimensional synthesis kinetic codes have yielded results that compare very favorably with exact (WIGLE) results at greatly reduced machine running times. The major disadvantage of the synthesis approach is that there is

no basis from which to determine its accuracy and range of validity unless the correct solution is already known. However, the success of synthesis to date seems to justify faith in the method.

From a design point of view, all the approximate techniques suffer from two shortcomings. First, none of the approximate techniques have been used to generate a kinetics computer code wherein all the important reactor feedback effects are included. Second, none of the techniques used for determining approximate solutions to the space-time equations can be used with confidence until their accuracy and range of validity have been established. To determine this range of confidence, it would be very helpful if comparisons between analytical results and experimental data could be made; however, very little space-time data exist which could be used for these comparisons. A series of control-rod-induced power excursions in a reactor with large spatial dimensions is one type of experiment that would be useful.

In order to systematically evaluate the relative merits and range of validity of all possible approaches to space-time problems, it would seem appropriate to analyze a "typical" reactor via all methods. That is, to date each approximate method has been applied to a different model; therefore, it is very difficult to make comparisons. The "typical" reactor could be a numerical model of a 1000 Mwe D_2O reactor. Since no such reactor exists, one would have to scale-up a smaller D_2O reactor. This large D_2O reactor could then be analyzed for various control rod movements, reactivity accidents, xenon stabilities, etc., using all available methods to solve the relevant space-time kinetics equations. The same set of basic cross sections should be used in all calculations.

Finally, the xenon stability problem for large power reactors has been discussed in this report. The major conclusion reached is that control of xenon oscillations is a very difficult problem that requires two- and three-dimensional analysis. The major goal of such an analysis should be to predict where control rods should be located and in what sequence they should be operated in order to insure controllability of the reactor. Previous efforts on the xenon problem have generated oscillation threshold conditions for reactor size, power level, etc. But, it is now conceded that economic considerations will demand that reactors be built that are inherently unstable relative to xenon oscillations. Hence, since the problem cannot be eliminated, it must be controlled. Many analytical techniques have been applied to this control problem, but none has been successful in generating control rod design criterion. Stacy⁽⁹²⁾ has quite recently published some information on this problem that might prove helpful if extended.

Many computer codes have been mentioned in this report. In an effort to more simply demonstrate what kinetics codes are available, these codes and their pertinent characteristics are summarized in Table I.

TABLE I
DIGITAL COMPUTER KINETICS CODES

<u>Type</u>	<u>Name</u>	<u>Dimensions (Number)</u>	<u>Feedback Included</u>	<u>Energy Group (Number)</u>	<u>Reference (Number)</u>
Exact	WIGL2	1	Yes	2	20
Exact	WIGLE	1	No	2	28
Exact	CLUMSY-I	1	Yes	1	24
Exact	GAKIN	1	No	≤ 10	33
Exact	GRCORK	2	No	2	32
Exact	TWIGLE	2	Yes	2	117
Adiabatic	QX-1	1	Yes	30	121, 122
Synthesis	PART2	3	Yes	> 2	119
Synthesis	MUMKIN	1	No	4	120
Exact (Xenon)	XENOSC	2	Yes	2	83
Exact	STAB	3	Yes	1	84, 85
Nodal (Xenon)	X-WAVE	2	Yes	2	87
Exact	RTS	0	Yes	1	16
Exact (Xenon)	SPRITE	3	Yes	4	90
Nodal (Xenon)	XENOP	3	Yes	1	92
Synthesis (Xenon)	XETRAN-1	1	Yes	2	91
Modal (Xenon)	XEP	1	Yes	2	105
Modal	MULE	1	No	2	50
Synthesis	VARI-QUIR	2	Yes	2	29
Exact	STREAK	1	Yes	2	98

BNWL-738

APPENDIX A
THE EXACT METHOD

A. The Exact Method

The "exact" method is a brute force technique whereby the time-dependent, multigroup diffusion theory equations, together with the equations required to represent any inherent feedback effects, are simply reduced to difference equations and solved by some numerical technique. The success of the method is largely dependent upon the efficiency of the numerical technique employed. A digital computer is typically used to perform the numerical operations, and the efficiency of the numerical technique is measured in terms of speed of operation, truncation error, and core memory requirements. The "exact" method is given this name because it is generally considered to involve the least amount of approximation compared to other techniques that could be utilized.

In principle, the "exact" method could be extended to include nonlinear feedback, several energy groups, and three space dimensions. The range of applicability of the technique is limited only by computer memory capacities and the cost of machine running time. One-dimensional applications of the technique are the least demanding and have correspondingly met with the greatest degree of success. The various numerical methods that have been used in one-dimensional situations are discussed below and their relative merits evaluated. The possibilities for extending the numerical methods to two- or three-dimensions are then discussed, and it will be shown that some techniques that are efficient enough for one-dimensional applications are not suitable for multi-dimensional situations.

The simplest numerical scheme that can be employed is the explicit forward difference technique. If the system of equations to be solved

can be written as

$$\frac{d\underline{\psi}}{dt} = A \cdot \underline{\psi}$$

and if $\underline{\psi}^j$ denotes the solution vector at time t_j , then the explicit forward difference solution of lowest order approximation is

$$\underline{\psi}^{j+1} = \underline{\psi}^j + hA \cdot \underline{\psi}^j.$$

However, there are severe numerical stability problems connected with this method.⁽¹⁷⁾ The source of this numerical difficulty is the widely separated characteristic times at which nuclear events occur. The result of the stability problem is that the time step h must be very small, on the order of a prompt neutron lifetime. Variations of this simple method include higher order explicit methods such as Runge-Kutta or Adams-Bashforth methods. Each of these methods will multiply the machine time used per time step by a factor equal to the order of the approximation. At the same time the improvement in step size is quite small, certainly less than the order of the approximation. The conclusion is that there are probably better techniques to use than Taylor series expansions.

Perhaps the most commonly used computer program for one-dimensional situations are the WIGLE⁽¹⁹⁾ and WIGL2⁽²⁰⁾ routines. WIGLE is limited to slab geometry, two neutron energy groups, and includes no feedback effects. WIGL2 is the more general program in that it will treat one-dimensional slab, cylindrical, and spherical geometries and includes non-boiling heat transfer. Xenon feedback and feedback effects due to fuel and coolant temperature are also accounted for. Control rod motion and control system

feedback based on total core power or outlet coolant temperature can be simulated. Transients may be excited by prescribed changes in inlet coolant temperature, coolant flow rate, or rod position. The program was written basically to describe the behavior of pressurized water reactors so some changes are necessary before it can be applied to D_2O reactors.

The WIGL2 approach is based upon an ingenious implicit difference scheme, called θ weighting, where the weight factors are chosen so as to minimize truncation error and improve stability of the method. Experience with WIGL2 in two-group, one-dimensional problems indicates that the equations are numerically stable for time steps at least up to 100 prompt lifetimes. The truncation error is usually tolerable for such step sizes. For instance, systems with lifetimes of about 10^{-5} sec. can be treated with step sizes of about 10^{-3} sec. with truncation errors of about 10%. Compared to the explicit method WIGL2 is much preferred. The computing time/step for the WIGL2 approach (2 group problems) is about 10 times that of the explicit method. However, the step sizes are 100 times that of the explicit step sizes. Overall, the WIGL2 method is nearly an order of magnitude faster than the explicit method for most problems.

Another technique for solving the one-dimensional, multi-group diffusion equations has been suggested by Hansen and Andrews.^(33,97) The technique has been incorporated into a production code at General Atomics called GAKIN, and it is now available from Argonne Code Center. The reactor equations are written as

$$\frac{d\phi}{dt} = A\phi + R$$

where $\underline{\phi}$ is the $G + I$ dimensional vector of the group fluxes, G in number, and the precursor concentrations, I in number. The spatial derivative is approximated by a three-point difference relation at points x_k , $1 \leq k \leq K$. The $(G + I)K$ dimensional vector $\underline{\psi}$ is then defined as

$$\underline{\psi}^T = [\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,K}, \phi_{2,1}, \dots, \phi_{G,K}, C_1^1, \dots, C_K^I]$$

and the entire set of equations becomes

$$\frac{d\underline{\psi}}{dt} = B\underline{\psi} + S.$$

The matrix B is factored into a strictly lower triangular matrix L , a strictly upper triangular matrix U , a diagonal matrix Γ , and a tridiagonal matrix H representing the three-point difference approximation; i.e.,

$$B = (L + U + \Gamma + H)$$

The set of equations is written as

$$\frac{d\underline{\psi}}{dt} - \Gamma\underline{\psi} = (L + U + H) \underline{\psi} + S.$$

and integrated over one time step,

$$\begin{aligned} \underline{\psi}^{j+1} &= \exp(\Gamma h) \underline{\psi}^j + \int_0^h d\xi \exp[\Gamma(h-\xi)] (L + U + H) \underline{\psi}^j(t_j + \xi) \\ &\quad + \Gamma^{-1} [\exp(\Gamma h) - I] S_j. \end{aligned}$$

It is assumed that $\underline{\psi}(t_j + \xi) = \exp(\omega \xi) \underline{\psi}_j$, with ω a free parameter.

Further manipulations lead to writing the equations in the form

$$F_1 \cdot \underline{\psi}_{j+1} = F_2 \cdot \underline{\psi}_j + F_3 \cdot S_j$$

where the operator F_1 is $F_1 = [I - (\omega I - \Gamma)^{-1} (I - e^{(\Gamma - \omega I)h}) \bar{H}]$.

In order to solve for $\underline{\psi}_{j+1}$ it is necessary to invert the operator F_1 ; the inversion is readily performed in one-dimensional cases by the method of matrix factorization. The final form of the algorithm is

$$\underline{\psi}_{j+1} = G \cdot \underline{\psi}_j + P \cdot S_j$$

Using the theory of matrices and their properties, it can be shown that for all ω and h the method is numerically stable. In addition, in the case of constant reactivity, fixed bounds on the truncation error are readily established. The parameter ω is used to adjust the truncation error.

Test problems of the method run on the IBM-7094 have indicated that computation times are of the order of 0.6 sec/time step/1000 unknowns. Problems involving 4 energy groups, 6 delayed groups, and 100 space points require about 4 min. of computer time.

The principal virtue of the GAKIN approach is the slow increase in computing time with the number of energy groups. This seems to be the program's main advantage over WIGLE.

STREAK⁽⁹⁸⁾ is a computer program designed for analysis of prompt neutron flux transients such as those of pulsed research reactors, for hazard analysis of large power reactors, and for analyzing pulsed neutron experiments in multiplying or non-multiplying media. STREAK solves the nonlinear, one-dimensional, time-dependent neutron diffusion equations in two energy groups. Separability of space and time is not assumed. Up to six groups of delayed neutrons are allowed, and almost any feedback mechanism can be incorporated.

A new method of direct numerical integration of the kinetics equations is employed in STREAK which allows interaction time steps considerably larger than those normally allowed in finite differencing integration schemes. Time savings by factors of 3 to 10 or more over the usual linear finite differencing can be obtained. The proposed method begins with a neutron balance on a region I (see figure below), in contrast to methods in which the differential equations are finite differenced on a space-time grid of points. For interaction rates, absorptions, fissions, and similar calculations, the flux is assumed to be constant over the volume of each region. For calculating leakage between regions, the average value of the flux is assumed to be located at mid-points of the regions.

The "D/L" method referred to in the literature⁽²¹⁾ also uses a grid of region volumes as opposed to a set of node points, but it was proposed for a very coarse grid with only a few regions covering the entire reactor. In the method here described, the volume-node grid has the same effect as the usual grid-point scheme.

STREAK has been compared to several more standard methods - such as TWENTY-GRAND in steady state case, and modal⁽⁴⁸⁾ solutions for a transient situation - and good agreement was obtained. Theoretically, the method could be extended to two or three spatial dimensions and to more energy groups, but these possibilities have not been explored.

The methods considered previously were based upon fine-mesh finite difference approximations to the kinetics equations. The different means of integrating the kinetics equations distinguish the various approaches. The basis of the VARI-QUIR method, a two-dimensional code,⁽²⁹⁾ is to use a higher order difference approximation in order to reduce the number of unknowns. In particular, it is assumed that the spatial shape of the flux can be represented as a biquadratic function over homogeneous regions. As a consequence of the assumed shape, the number of mesh points can be greatly reduced at the expense of dense coefficient matrices.

The VARI-QUIR difference equations are derived starting from the variational integral

$$I = \sum_{g=1}^{G+I} \int dvol \int dt \left\{ -\phi_g^* \frac{1}{v_g} \frac{\partial \phi_g}{\partial t} - D_g (\nabla \phi_g^* \cdot \nabla \phi_g) + \sum_{g'=1}^{G+I} \phi_g^* \sigma_{gg'} \phi_{g'} \right\}$$

The flux and adjoint (ϕ_g^*) are expanded in terms of biquadratic functions of the form

$$\phi_g^{(r)}(x,y) = \sum_{i,j} W_{ij}^{(r)}(x,y) \phi_g(x_i, y_j)$$

where $\phi_g^{(r)}$ is the flux in group g and region r . The weighting function, $w_{ij}^{(r)}(x,y)$ is biquadratic. The coefficients of the polynomial expansion are the values of the flux at the interpolation points themselves. A similar expansion is used for the adjoint function. Continuity of flux is assured by matching the expansion functions at boundary points between regions.

The expansions for the flux and adjoint are inserted into the variational integral and volume integration performed. The variation with respect to the adjoint is taken at each mesh point. A set of simultaneous equations is generated, one for each mesh point. The coefficient matrices are relatively dense, the precise number of nonzero entries depending upon the mesh point location. The difference equations can be reduced to the form

$$\frac{d\psi_g}{dt} = v_g (A_g^{-1} B_g) \psi_g + v_g \left\{ \sum_{g'} \sigma_{gg'} \psi_{g'} + \sum_i F_{gi} \psi_{G+i} \right\}.$$

Experience has indicated that the problem of separated roots appears in this form of the difference equations; hence, numerical stability problems do exist. Unfortunately, implicit methods are very difficult as $(A_g^{-1} B_g)$ is dense. The only practical solution is probably to use constant time steps.

Nevertheless, VARI-QUIR is a working, two-dimensional, multigroup kinetics code. The most important factor in determining computing time is the number of regions adequate to describe the physical problem. By use of an implicit method, and constant time steps, it appears that the method is stable or nearly so. Feedback can be incorporated if core memory is still available after the main code has been input.

A very good analysis of possible methods for solving two-dimensional, multigroup reactor kinetics problems has been made by Hansen.⁽³¹⁾ Based upon the analysis of the computing requirements of the various techniques, and the properties of the UNIVAC-1108 computer equipment at General Atomic, certain recommendations are made with respect to what methods should be pursued to develop a production code. The methods reviewed are:

1. Explicit methods
2. WIGL2 method
3. GAKIN method
4. ADI methods (alternating direction implicit)
5. VARI-QUIR method
6. GRCORK method

The methods are analyzed as to how much time would be required on the 1108 to perform the numerical operations required for a two-dimensional situation and as to core memory requirements. The study is very complete and the conclusions reached therein are now repeated.

The explicit methods were discussed previously in this report, and the conclusions drawn then are comparable to those reached by Hansen. On the pro side, explicit methods offer small computing time per step, simple programming, and easy adjustment of time step size during the problem. However, the disadvantages of the methods are sufficient to remove explicit techniques from consideration for an economic code. These disadvantages are very poor numerical stability - hence very small time steps required - and very little improvement in stability by going to higher order methods. Hence, implicit numerical methods become necessary, and several techniques of this type are discussed next.

The one-dimensional code WIGL2 was discussed previously. Hansen extended the WIGL2 technique to a two-dimensional situation. In so doing, he reached the following conclusions. The θ -weighting idea results in very good numerical stability and very good truncation error, hence reasonably large time steps are allowed. The technique is difficult to program and requires large computing times per step. Hansen concluded that a two-group, 6 delayed group, two-dimensional problem would require about 5 to 25 hours of computing time on the UNIVAC-1108, and he thought that this price was probably too high for production use. It should be noted that a two-dimensional version of WIGL2, called TWIGLE, has been developed by Westinghouse Bettis.⁽¹¹⁷⁾ The computing time required by the program appears to be somewhat less than that anticipated by Hansen. The WIGL2 approach becomes completely impractical for problems of more than two energy groups.

The extension of the GAKIN approach to two-dimensional problems is straightforward. This method was previously described for the one-dimensional case. Recall that the method demanded inversion of an operator F_1 of the form

$$F_1 = \left[I - (\omega I - \Gamma)^{-1} (I - e^{(\Gamma - \omega I)h}) \right] H .$$

In the two-dimensional case inversion of this matrix is impractical. Various alternative procedures around this problem are discussed by Hansen, and it appears that the problem can be resolved. Conclusions would be that the unconditional stability of the GAKIN method permits very large time steps,

but the truncation error in the method can be quite large, which limits the step size. In comparison with WIGL2 for 2 energy group, 6 delayed group problems, it is expected that the two approaches would require about the same computing time. However, a very important virtue of the GAKIN approach is the slow increase in computing time as the number of energy groups is increased. Hansen concludes that if an improvement in truncation error/step can be achieved, or if larger step sizes can be allowed, then the method is on the threshold of a practical production code. The approach probably merits further study.

Hansen mentions that alternating direction implicit⁽¹¹⁸⁾ methods have been widely used for two-dimensional diffusion and heat transfer problems, but that there was apparently no use of ADI methods in connection with the reactor kinetics equations. In a simple ADI method, the kinetics equations for g^{th} neutron group are written as

$$\frac{d\psi_g}{dt} = B_{gg} \cdot \psi_g + \sum_{\substack{g'=1 \\ g' \neq 1}}^{G+I} T_{gg'} \psi_{g'} \quad \text{where } B_{gg} \equiv H_g + V_g.$$

For the time step from t_j to t_{j+1} we use the algorithm

$$\psi_g^{j+1} - \psi_g^j = hH_g \psi_g^{j+1} + hV_g \psi_g^j + h \sum_{g'} T_{gg'} \psi_{g'}^j$$

or

$$\underline{\psi}_g^{j+1} = (I - hH_g)^{-1} (I + hV_g) \underline{\psi}_g^j + h(I - hH_g)^{-1} \underline{S}_g^j$$

with

$$\underline{S}_g^j = \left[\sum_{\substack{g'=1 \\ g' \neq 1}}^G B_{gg'} \underline{\psi}_{g'}^j + \sum_{i=1}^I F_{gi} \underline{\psi}_{G+1}^j \right] .$$

For the next time step we alternate the direction, keeping h constant, to obtain

$$\underline{\psi}_g^{j+2} = (I - hV_g)^{-1} (I + hH_g) \underline{\psi}_g^{j+1} + h(I - hV_g)^{-1} \underline{S}_g^{j+1} .$$

With suitable definitions and recombinations, these equations can be reduced to the form

$$\underline{\psi}_g^{j+2} = P_g^T R_g Q_g \underline{\psi}_g^j + h P_g^T R_g \underline{S}_g^j + h P_g \underline{S}_g^{j+1} .$$

In general, it cannot be shown that this algorithm is stable; however, much experience with diffusion calculations indicates that the method may well be stable. Hansen makes a strong case for this method in his report, indicating that a two-energy group 6 delayed group, two-dimension problem could be solved in about 1 to 5 hours on the UNIVAC-1108 machine. The advantages of the technique are very small computing times per step, probable numerical stability, and programming relatively straightforward. The

disadvantages are uncertain truncation error and possible difficulties with non-Cartesian geometries. These disadvantages are really a result of the need for further development of the method.

Hansen also discusses the VARI-QUIR approach and then describes a variant of this approach which seems to offer a considerable improvement in computing time. The variant of the V-Q method, called GRCORK,⁽³²⁾ simplifies the problem associated with large dense matrices at the expense of increasing the number of unknowns. A biquadratic expansion is used, but flux discontinuities across region boundaries are permitted. The weighting functions are assumed discontinuous, going to zero outside region boundaries. Thus an expansion is assumed of the form

$$\phi_g^{(r)}(x,y) = \sum_{i,j} W_{ij}^{(r)}(x,y) \phi_g(x_i,y_j) ,$$

where the weighting function is biquadratic. However, for x,y outside of region r , $W_{ij}^{(r)}(x,y) \equiv 0$. This expansion is substituted into the multigroup equations; then the equations are multiplied by $W_{ij}^{(r)}(x,y)$ and integrated over all space. Since $W_{ij}^{(r)}(x,y)$ is discontinuous, only the integral over region r contributes. Thus the set of equations is "almost" uncoupled from all other regions. The integral over diffusion terms introduces boundary terms which must be evaluated. If continuity of partial currents is assumed, then the boundary terms can be evaluated. The equations thus produced involve all 9 interpolation points of region r plus the contributions over the surface bounding the region. Hansen concludes that the method offers very good possibilities as far as computing time is concerned. He estimates about one hour for a 2 energy group problem and 4 to 7 hours for a 10 energy group problem. The

principal disadvantages are large storage requirements and the question of the number of regions required to adequately describe the assembly. If the number of regions is doubled in each direction, the computing time is quadrupled. Thus, about 100 regions is probably a maximum for the UNIVAC-1108.

APPENDIX B
THE ADIABATIC METHOD

B. The Adiabatic Method

In the adiabatic method,⁽³⁴⁾ or the instantaneous tilt method as it is sometimes called, the assumption is made that the flux shape adjusts instantaneously to all perturbations. Hence, the flux shape at any instant can be related to the nuclear properties of the reactor at that instant. A series of static spatial calculations is performed to specify this relation. The time equations in this method are the conventional kinetic equations.

The method is an application of the formal derivation of the kinetics equations from the general transport theory.⁽⁴⁾ In this formalism, the time-dependent directional flux density $\phi(\vec{r}, u, \vec{\Omega}, t)$ is written as a product $\psi(\vec{r}, u, \vec{\Omega}, t) \cdot T(t)$ of a "shape function" ψ , the time dependence of which is not separable, and an amplitude function T of time alone.

Although ψ is time dependent, it is normalized so that

$$\int_V \int_u \frac{\int_V \psi_o^*(\vec{r}, u, \vec{\Omega}) dV \int_V \psi(\vec{r}, u, \vec{\Omega}, t) dV}{v(u)} dV du$$

is constant in time. $\psi_o^*(\vec{r}, u, \vec{\Omega})$ is the adjoint flux for some critical state of the reactor. With this normalization for ψ , $T(t)$ obeys the conventional reactor kinetics equations with the reactivity, effective delayed neutron fraction, and prompt neutron lifetime being defined by integrals involving ψ_o^* , ψ , and the reactor parameters.

The formalism thus casts the time-dependent transport equation without approximation into the form of the point kinetic equations. This reformulation could be considered only a formality since the unknown space

shape ψ is involved in the evaluation of the reactor parameters. What distinguishes this circumstance from an interesting curiosity is the fact that it is frequently possible to compute an adequate approximation to the shape function by a series of static calculations. This possibility exists since the shape of the neutron flux (as distinct from its amplitude) arises from the shape of the source term, which is comprised mostly of prompt neutrons and hence adjusts itself to changing reactor conditions in a few prompt neutron lifetimes. Thus, the true shape function is replaced by the static flux shape corresponding to the instantaneous values of the nuclear properties of the reactor. (The instantaneous reactor must be made critical for these static calculations by adjustment of some designated parameter.) This replacement is referred to as the instantaneous tilt approximation since it assumes, in effect, that the flux shape responds instantaneously to changes in nuclear properties. In actual fact, the presence of delayed neutrons causes the shape change to lag somewhat behind the change in properties.⁽⁹⁹⁾

The procedure for solving a nonseparable space-time problem via the adiabatic approximation is then the following:

1. Determine by solving static problems and using the definitions of ρ , β , and Λ the values of these quantities appropriate to various reactor configurations that will be attained during the transient.
2. Express these quantities and the flux shape directly as functions of the reactor conditions. (Generally, a division of the core into regions, followed by a power series fit of the desired quantities to the average properties of these regions will be necessary.)

3. Solve the conventional kinetics equations for the amplitude function $T(t)$ using a group of coupled time-dependent equations in which the reactor conditions at any instant yield the adiabatic shape function $\psi(t)$, ρ , $\bar{\beta}$, and Λ ; ρ , β , Λ in turn yield $T(t)$; and $T(t)$ allows computation of reactor conditions (generally through some intermediate heat transfer or control equations).

The adiabatic approximation was originally suggested by Henry and Curlee⁽³⁴⁾ and applied to a very much simplified two-point model of a reactor. This model, consisting of two point reactors separated by a pure scatterer, was chosen for study because the nonseparable space-time behavior of the system could be described exactly by analytical means as well as by the adiabatic approximation. The results of the exact and approximate methods agreed very well and encouraged application of the technique to more complex and realistic reactors in which an exact solution was not readily available. An example of such an application was a study by Curlee⁽³⁷⁾ in which the adiabatic approximation was applied to a two-dimensional model of a large power reactor (the Shippingport reactor). The problem was to compare the transient behavior of the reactor when the inlet temperature to all parts of the seed is lowered with the behavior when the inlet temperature in only one region is lowered. Results obtained indicated spatial effects were present but were not compared with any "exact answers."

Kaplan and Margolis⁽⁹⁹⁾ demonstrated that the adiabatic approximation was not valid for core geometries whose characteristic dimension exceeded about 20 neutron migration lengths. They showed that after a change in

the material properties of the core, the flux shape changes part way towards its asymptotic form in a few prompt neutron lifetimes, but the remaining change takes place in times characteristic of the delay precursors. The "prompt" jump component of the asymptotic flux shape decreases as the core size increases. Since the adiabatic approximation assumes that the flux shape responds instantaneously to changes in reactor parameters, it may be a poor method to apply to large reactors.

Previously, an investigation by Henry and Yasinsky⁽²¹⁾ was described wherein the various analytical techniques for solving the space-time equations were compared. One conclusion of this study was that the adiabatic model may lead to considerable deviations from the correct result in situations where the spatial dependence of the transient is large. Following this conclusion, there have been many efforts to improve upon the basic adiabatic model so as to remove its deficiencies.

Many transients of interest are in the delayed supercritical and subcritical range; in this range delayed neutrons are important. It has been noted that the adiabatic model assumes that the delayed-neutron precursor distributions are in equilibrium with the transient flux distribution. Stacey and Adams⁽³⁵⁾ have presented an improved method of treating the delayed neutron source in the adiabatic model. Using their improved technique, they performed several numerical experiments involving typical control-rod induced transients, and they compared these results with exact (WIGLE) and conventional adiabatic model results. In general, delayed supercritical and subcritical transient results obtained with the improved adiabatic model and WIGLE agreed in all particulars to within about 2%. Conventional adiabatic model errors were an order of magnitude larger. Work is now

progressing on a two-dimensional application of the method.

Madell⁽³⁶⁾ also noted Yasinsky and Henry's⁽²¹⁾ conclusion that the adiabatic and exact (WIGLE) solutions of a space-dependent kinetics problem yield significantly different flux distributions for a loosely coupled system undergoing a delayed critical excursion. In the adiabatic solution of the space-time kinetics equations, the distribution of the precursors is identical to the flux distribution throughout the excursion. In the exact solution, the precursor concentration at each point depends upon a previous flux distribution. This situation suggests that the major contribution to the difference in flux distribution obtained by the two methods may be in the treatment of the delay-neutron precursors. To investigate this question, Madell altered the exact (WIGLE) solution to Henry and Yasinsky's model problem so that the treatment of the delayed-neutron precursors was the same as that in the adiabatic solution. The agreement between the adiabatic solution and the altered version of WIGLE demonstrates that the two solutions to this particular problem differ only in the treatment of the delayed-neutron precursors. Thus, it is concluded that the approximate solution of the space-time kinetics equations for delayed critical excursions by the adiabatic method can be improved by a more accurate treatment of the precursor distribution.

A quasistatic (adiabatic) excursion code, called QX1, is being programmed at ANL.⁽³⁶⁾ The code employs the improved adiabatic technique wherein the delayed neutron source is used in its actual shape and is not approximated by the shape of the prompt source as in the conventional adiabatic model. The feedback block of QX1 calculates the changes in

material densities, geometry, and temperatures from the heat source distributions. These changes are fed into the neutronics module where for every spatial region new Doppler broadened group constants are calculated. The code is designed to analyze an excursion transient of a one-dimensional fast reactor. Maximum problem dimensions are 20 material regions, 150 mesh points, and 30 energy groups. The code is not yet operational, but it does serve to indicate the extent of current interest in the adiabatic technique.

GE-San Jose⁽³⁸⁾ has reported an adiabatic approximation that yields relatively accurate results for prompt-critical, large-core reactor excursion analyses. The algorithm includes spatial Doppler temperature feedback. To check the validity of the model, the basic WIGLE space-time code was modified to compute temperature feedback. Comparisons between the adiabatic model and WIGLE as well as comparisons between the adiabatic model and an experimental SPERT 3.2 msec-period both served to illustrate the validity of the model.

The adiabatic method has been applied to axially space-dependent boiling water reactor kinetics.⁽³⁹⁾ The feedback equations were developed for a multiregion reactor core from the balance relations of mass, energy, and momentum. These equations were coupled to the neutron kinetics equations by the adiabatic approximation. For the one-dimensional model studied, the reactor core is divided into five regions in addition to the top and bottom reflector regions. The number of regions must be sufficiently large to provide for the space-dependent information, but is limited by the correlation procedure for all the variables. A division of five regions

gave almost identical results to divisions of twenty or more. An arbitrary set of void distributions was chosen within a reasonably probable range of conditions for the reactor during the transient. When additional reactivity was introduced in steps of 90 cents ($0.9 \bar{\beta} \delta k/k$) to the model boiling water reactor, a fairly large space-dependent excursion in the reactor variables was observed. The transient died out within a few seconds and a new steady state was attained. When sinusoidal reactivity was introduced with an amplitude of 10 cents, negligible space-time dependence was observed. This was thought to be due to the fact that the shape of the void distribution was almost unchanged in the observed frequency range (0.1 to 2 cps). The observed space-independence agrees with experimental results on the Dresden BWR.⁽¹⁰⁰⁾ No comparisons of the adiabatic results with other analytical techniques were made.

APPENDIX C

NODAL ANALYSIS

C. Nodal Analysis

Another approach to space-time analysis is accomplished by the division of the reactor into regions or "nodes". Each region is then treated as a point, and space-independent equations for the average flux or power at each node and for the feedback effects are written. This method is generally referred to as the nodal method.

Finite difference techniques (the "exact" method) may be considered as a limiting form of nodal analysis in which the nodes are small and closely spaced. However, as has been discussed previously, this approach leads to programs with long machine running times and has therefore been restricted to models having only one spatial variable, i.e., WIGLE. For somewhat larger nodes a "higher order" difference approximation such as that used in VARI-QUIR⁽²⁹⁾ is appropriate. Recall that in this approach the flux within each region is approximated by a simple function such as a polynomial.

However, the term nodal analysis usually refers to a situation where the nodes are few in number and represent gross regions of the core. The nodes are interconnected by allowance for neutron diffusion, heat transfer, and material transport from node to node. The space part of the problem consists of the determination of parameters governing neutron leakage from node to node. Finding a suitable technique for evaluating such parameters has been the main point of difficulty with the nodal approach. If leakage parameters can be determined which do actually describe this interaction between nodes, the nodal approach offers good potential. The time part of the problem consists merely of

solving the "point" kinetics equations for each node, and feedback is easily included by writing suitable equations for each node.

In Avery's theory of coupled reactors,⁽⁴⁰⁾ a scheme is described for evaluating the leakage parameters from fine mesh, time independent spatial calculations. In this formulation the individual reactors of the coupled system can be arbitrarily defined by any prescription which specifies the reactor in which each fission neutron is emitted. The term "coupled" is taken to mean that in each of the reactors some of the fission neutrons are emitted in fissions induced by neutrons born in other reactors. The formalism developed by Avery treats the system in terms of integral parameters which explicitly characterize the individual reactors and the coupling between them. A k_{ij} is defined as the expectation value that a fission neutron born in reactor j gives rise to a next generation fission neutron in reactor i , and ℓ_{ij} is defined as the average prompt neutron lifetime for the process. The k_{ij} for $i \neq j$ is a measure of the cross coupling from reactor j to reactor i , and in general $k_{ij} \neq k_{ji}$. The total fission neutron source in reactor i which results from fissions caused by neutrons which originate in reactor j is defined as S_{ij} . We then have

$$S_i = \sum_{j=1}^N S_{ij}$$

where N is the number of reactors in the system. The definitions of the various integral parameters, k_{ij} , ℓ_{jk} , S_i and S_{ij} , are applicable for both the steady state and time dependent situation. Their quantitative definition will require some sort of averaging or weighting procedure.

In terms of these parameters and the further definition that $N_{jk} \equiv S_{jk} \cdot \ell_{jk}$,

the following equations describe the kinetic behavior of the coupled system:

$$\frac{dN_{jk}}{dt} = k_{jk}(1-\beta) \sum_{m=1}^N \frac{N_{km}}{\ell_{km}} - \frac{N_{jk}}{\ell_{jk}} + k_{jk} \sum_{i=1}^D \lambda_i C_{ki}$$

$$\frac{dC_{ki}}{dt} = \beta_i \sum_{m=1}^N \frac{N_{km}}{\ell_{km}} - \lambda_i C_{ki}.$$

The usual procedure for solving the coupled kinetics equations is to assume that the k_{jk} , but not the ℓ_{jk} may be time dependent, and then to solve for the time dependence of the N_{jk} and C_{ki} . Using the theory of adjoint formulation, and the corresponding idea of neutron importance, Avery determines exact, quantitative definitions of the various coupling parameters and shows how to explicitly evaluate them. For example,

$$k_{ij} = \frac{\int \left[v\sigma_f(r,v) \right]_j \phi(r,v) dr dv}{\int \left[v\sigma_f(r,v) \right]_k \phi(r,v) dr dv} \times \frac{\int \chi(v') \phi^*(r,v') \left[v\sigma_f(r,v) \right]_j \phi_k(r,v) dr dv dv'}{\int \chi(v') \phi^*(r,v') \left[v\sigma_f(r,v) \right]_j \phi(r,v) dr dv dv'}$$

where $\phi(r,v)$ = neutron flux as function of position and velocity

$\phi^*(r,v)$ = adjoint flux

$[v\sigma_f(r,v)]_j$ = part of $v\sigma_f(r,v)$ which prescribes reactor j

$\chi(v)$ = fission spectrum

$\phi_j(r,v)$ = that part of flux distribution that results from

$$\text{neutrons born in reactor } j. \left(\sum_j \phi_j(r,v) = \phi(r,v) \right).$$

A similar definition exists for k_{ij} .

Now we can see that we have a situation comparable to the one incurred in Henry's⁽⁴⁾ formulation from the transport equation. That is, the value of k_{ij} at any instant during a transient is dependent upon the flux, $\phi(r,v)$, at that instant; but $\phi(r,v)$ is precisely what is to be evaluated. Hence either an adiabatic type of calculation of the coupling parameters throughout a transient or some other even more approximate technique must be employed. Some applications of Avery's formulism using certain approximate methods to evaluate the k_{ij} are now described.

Pluta^(41,101) employed the basic Avery formulation to describe the dynamic behavior of a coupled thermal-fast spectrum superheat nuclear reactor. The reactor was assumed to be described by two nodes, one describing the thermal region and one describing the fast region. A simple approximate technique was used to evaluate the k_{ij} , and the values were updated during a transient only when they had changed appreciably from their base values. An extensive digital code was written which combines this neutronic formulation with plant feedback to calculate the transient performance of a superheat reactor plant concept. Many evaluations of the code were reported, and the results seem to indicate that the technique works quite well for this situation. It is not clear from this report whether the technique would work as well when large spatial effects are present.

Recently Rubin and Wade⁽⁴²⁾ have reported a nodal calculation of a space-time transient using coupling coefficients which account for changing internodal leakages during the transient. In the nodal treatment of space-time kinetics, the nodal average sources S_i are assumed to influence one another by means of coupling coefficients k_{ij} , which are a measure of the contribution to the integral fission rate in node i of the fission source integrated over node j . In loosely coupled (large) power reactors, the overall power density distribution is quite "tiltable" so that the intranodal source distributions are strongly dependent on the reactor material configuration. Since the coupling coefficient k_{ij} depends on the node j source distribution, it also changes with material configuration. Rubin and Wade suggest that defining k_{ij} as

$$k_{ij} = \left(\begin{array}{l} \text{absorption rate in node } i \\ \text{of neutrons which were} \\ \text{born in node } j \end{array} \right) / \left(\begin{array}{l} \text{Total source} \\ \text{in} \\ \text{node } j \end{array} \right)$$

would be characteristic of most techniques that have been proposed for generating coupling coefficients in that it is based on a single choice of the intranodal source distribution. However, coupling coefficients based on a single choice of intranodal source distributions cannot accurately represent nodal coupling which changes during a transient in which spatial effects are important. The technique suggested is to attempt to construct coefficients that are applicable over the range of material configurations to be encountered during the transient. For several known static configurations, denoted by index n , the static form of the nodal balance equations can be viewed as a simultaneous set of linear equations in the unknowns k_{ij} , and the nodal

nonleakage probabilities ℓ_i :

$$S_i(n) - \frac{k(n)}{k_{\infty i}(n)} S_i(n) \frac{1}{\ell_i} + \sum_{j \neq i}^J S_j(n) k_{ij} = R_i(n)$$

$n = 1, 2, \dots, N$

$i = 1, 2, \dots, J.$

The $k_{\infty i}$, k , and S_i are known from previously made static diffusion calculations, and $R_i(n)$ is a residual in node i which is nonzero when the unknowns k_{ij} fail to satisfy the nodal balance equations exactly. For a particular reactor model, Rubin and Wade used this technique to generate a set of coupling parameters using three static configurations which bracketed the configurations to be encountered in a particular space-time transient. The reactor analyzed was a one-dimensional axial model of a pressurized water reactor. Results of this analysis compared with the exact (WIGLE) results indicate that the technique offers a significant improvement over methods based on a single choice of intranodal source distribution.

As described above, a nodal approach to reactor kinetics is implied when a reactor is treated as an interacting array of coupled subsystems. In his original work on coupled-reactor kinetics Avery⁽⁴⁰⁾ pointed out that there is no unique way of formulating coupled-reactors kinetics equations. Kohler and Plaza⁽⁴³⁾ have presented a new approach to formulating these kinetics equations. The method consists of specifying the neutron population within a reactor by a single equation and classifying neutrons on the surface of the reactor by groups according to the direction

in which they travel with respect to a unit normal at the surface. This permits the use of the concept of surface coupling to account for the neutronic interaction between the reactors. The derivation of the coupled-reactors kinetics equations is accomplished within the frame of the general neutron transport equation and is very analogous to the derivation of the generalized point kinetics equations by Henry;⁽⁴⁾ the adjoint flux of an arbitrary stationary state of the reactor is used as a weighting function. An exact formulation of the coupled kinetics equations is obtained and a corresponding exact definition of the coupling parameters is given. However, the expression for the coupling coefficient k_{ij} involves the unknown flux distribution. Kohler and Plaza suggest using the adiabatic approximation to eliminate this problem and also suggest various techniques for evaluating the coupling parameters. No results have as yet been presented for this formulation. Its interest now is probably that it is exact and completely general; thus it applies to multi-dimensional situations. Whether it will be practical in actual applications remains to be determined.

Wachspress⁽⁴⁴⁾ has been working on a technique which he calls "multichannel flux synthesis". The method involves dividing the reactor into planes and each plane into channels. Each channel is then a node in a three-dimensional mock-up of the reactor. An important aspect of this model is the coupling between channels. Originally this coupling was taken to be associated with the net current at channel interfaces, and corresponding expressions for the coupling coefficients were obtained. More recently, Wachspress⁽⁴⁵⁾ refined his multi-channel formulation by developing variational expressions for the coupling

coefficients. Since the multi-channel method is a nodal method that permits flux discontinuities, conventional variational procedures are not applicable. However, Selengut⁽¹⁰²⁾ has described a variation principle wherein discontinuous trial functions are allowed. The technique involves replacing the diffusion equation, a second order differential equation, with two coupled first order differential equations. Wachspress has applied this variational principle to his multi-channel technique to obtain expressions for the coupling coefficients. The disadvantage of the method is that trial functions must be chosen for both the flux and the current, and there is no unique formula for this selection. However, the technique permits three-dimensional kinetics studies to be made via one- and two-dimensional calculations. Wachspress has applied the technique to various models and seems very enthusiastic about the results, but he has not developed the method to the extent that a working code is available for application to realistic reactor models.

Combustion Engineering, Inc. has recently suggested a unique method for determining the nodal coupling parameters. The power density at each node is calculated using one-group diffusion theory. The parameters governing neutron leakage from node to node are input quantities. Selection of these leakage parameters is based upon an empirical fit of a source condition that best represents a desired initial condition. This distribution may come from static fine mesh diffusion theory or from experiments. These statically fitted leakage parameters are then retained for future transient calculations. It is possible in the analysis to control each node separately. The technique is to be applied specifically to the xenon control problem, but no results are yet available.

Another method which can be used to determine the leakage of neutrons from node to node is to apply the gross coupling concept^(21,123) used in the multichannel synthesis technique.⁽⁴⁴⁾ Again the reactor volume, R , is divided into subregions or nodes, R_i , with volumes V_i . The task is to determine the time behavior of some average flux in each of the regions, R_i . The nodal equations are obtained by integrating the diffusion equations over each node and defining average nodal fluxes, i.e.,

$$\phi_i = \frac{1}{V_i} \int_{R_i} \phi \, dR,$$

and average nodal parameters, i.e.,

$$\Sigma_{ai} = \frac{1}{\phi_i V_i} \int_{R_i} \Sigma_a \phi \, dR.$$

The essential assumption of the gross coupling scheme is that the gross current from node i to node j is proportional to the average flux in node i ; that is,

$$J_{ij}^+ = (p_{ij}) \cdot V_i \phi_i.$$

For diffusion theory,

$$p_{ij} = \frac{1}{V_i \psi_i} \left[\frac{1}{4} \oint_{S_{ij}} \psi \, dS - \frac{1}{2} \oint_{S_{ij}} D \nabla \psi \cdot \mathbf{N}_i \, dS \right]$$

where

S_{ij} denotes interface between i, j nodes

ψ_i are average fluxes resulting from a static calculation of the unperturbed problem.

The resulting nodal equations, for one group diffusion theory, would be of the form

$$- \left[\sum_{j=0}^J \frac{p_{ij}}{V_i} + \Sigma_{ai} - v \Sigma_{fi} \right] \phi_i V_i + \sum_{j=1}^J p_{ji} \phi_j = \frac{1}{v_i} \frac{d\phi_i}{dt} .$$

A very similar approach for determining internodal leakage is called the "effective D/L" method.^(21,123) In this method the net current between the i^{th} and j^{th} node is approximated by

$$J_{ij} = - \frac{\bar{D}_{ij}}{L_{ij}} (\phi_j - \phi_i) S_{ij}$$

where

S_{ij} is area of the i, j interface

L_{ij} is distance from center of i^{th} node to center of j^{th} node

$\bar{D}_{ij}$ is an effective diffusion coefficient.

The effective diffusion coefficients, $\bar{D}_{ij}$, are determined by multiplying the volume-averaged diffusion constants for each node by a factor which is adjusted so that the few-node fundamental eigenvalue is the same as that of an unperterbed fine-mesh solution . In the study by Henry and Yasinsky,⁽²¹⁾ the D/L method gave somewhat better results than the gross coupling method.

BNWL-738

APPENDIX D

MODAL ANALYSIS

D. Modal Analysis

The basic idea of modal analysis is to approximate unknown functions of space and time by a linear combination of known space functions (also called modes) with suitable time-dependent coefficients, i.e.,

$$\phi(\vec{r}, t) = \sum_{i=1}^N \psi_i(\vec{r}) \cdot T_i(t).$$

There are two major steps in such an analysis: the selection of the space functions and, subsequently, the determination of the coefficients. There are a great many versions of modal analysis corresponding to different ways of handling these two steps.

Perhaps the most basic and common version of modal analysis is to expand each flux and precursor group in a series of orthogonal space modes. A very common set of functions that has been used is the set of eigenfunctions generated by solving the Helmholtz equation over the reactor geometry. For the case of a one-dimensional slab reactor of width R , the eigenfunctions are generated by solving the boundary value problem

$$\frac{d^2}{dx^2} \psi_n(x) + B_n^2 \psi_n(x) = 0; \quad \psi_n(0) = \psi_n(R) = 0.$$

Thus, $\psi_n(x) = \sin \frac{n\pi x}{R}$. The advantages of this set of eigenfunctions are two-fold: the orthogonality of the modes permits a straightforward derivation of the coefficients $T_n(t)$, and the completeness⁽¹⁰³⁾ of the set guarantees that if the summation

$$\sum_{i=1}^N \psi_i(\vec{r}) T_i(t) \text{ is taken for } N = \infty,$$

the resulting expression for $\phi(\vec{r}, t)$ is mathematically valid. The Helmholtz

modes have been used to solve the two-group diffusion equation for a one-dimensional reflected slab reactor by Foderaro and Garabedian.⁽⁴⁸⁾ The reactor model is shown below.

The two-group fluxes are expressed as

$$\phi_S(x,t) = \sum_{v=1}^m \cos \left[(2v-1) \frac{\pi x}{2b} \right] \sum_{\mu=1}^{3m} A_{v\mu}^S e^{\omega_{\mu} t},$$

and

$$\phi_F(x,t) = \sum_{v=1}^m \cos \left[(2v-1) \frac{\pi x}{2b} \right] \sum_{\mu=1}^{3m} A_{v\mu}^F e^{\omega_{\mu} t},$$

where the $A_{\mu v}$'s and ω_{μ} 's are derived as eigenvectors and eigenvalues of a $3m \times 3m$ matrix. The approximation in the technique is due to the termination of the series expansions at m terms. Recall that if $m = \infty$, the expressions above are rigorous. However, it is impossible to evaluate these series for $m = \infty$; one must hope that the series expansions will converge for m equal some small number. The convergence criterion that is commonly used for modal expansions is to compare a $(m+1)$ -term solution with a m -term solution and note if there is any significant difference. Foderaro and Garabedian assumed a step removal of poison in the region $-p \leq x \leq p$ and determined the resulting time-dependent flux distributions.

One problem with the Helmholtz method is that if the reactor geometry

is complex or if nonsymmetric flux distributions are present, the series expansions converge very slowly. Thus a large number of modes is required to describe the flux, and the $3m \times 3m$ matrix eigenvalue problem likewise becomes very difficult to solve. The final result of this situation is excessive computation time. However, for the simple reactor geometry investigated by Foderaro and Garabedian, the results were very good.

If the reactor parameters are time-dependent functions of some form other than a step-function, then the equations for the time coefficients of the Helmholtz expansion are nonlinear differential equations. This situation is even more difficult to resolve than the eigenvalue problem incurred above for step removals of poison. Garabedian and Lynch⁽⁴⁹⁾ have extended the analysis of Foderaro and Garabedian to include time-dependent coefficients and nonlinear terms. The method described can be used in problems in which the cross sections are given as linear combinations of powers of the fluxes or as powers of the integrated fluxes, i.e., a feedback term.

Helmholtz expansions have been used by Garabedian and Thomas⁽¹⁰⁴⁾ to solve the steady state two-group diffusion equations for a two-dimensional fully reflected reactor. This analysis involved expansions of each of the fluxes in a double series of eigenfunctions valid over the entire reactor. The results of the analysis compared favorably with solutions obtained by standard finite difference codes.

In the analysis of a reactor space-time problem by the method of modal expansion, the labor is reduced and accuracy improved if the modes are chosen such that the expansion has the property of finality. In the usual space-time problem, if the expansion displays finality, each expansion

coefficient is found separately as the solution of a single ordinary differential equation. If there is no finality, it is usually necessary to solve N equations for the N coefficients simultaneously. In this case the value obtained for any given expansion coefficient changes with the number N , the number of terms in the truncated series. Thus, it is very desirable, if possible, to choose an expansion having the finality property.

Kaplan⁽⁷⁶⁾ has identified modes having this property for a zero power (no feedback) reactor and for a reactor with linearized xenon feedback. Initially Kaplan examines several expansion schemes which have been proposed for reactor space-time problems to see when and if they display finality, and then he devises expansion schemes which will display finality in a given problem. The modes displaying finality are the set of eigenfunctions which diagonalize the differential operator of the basic differential equations. That is, if the kinetics equations are written in the matrix form

$$L\phi = \dot{\phi}$$

and L is broken into an initial and perturbed part

$$L_0\phi = \dot{\phi} - L_1\phi ,$$

then the finality modes are the eigenvectors of the L_0 operator, i.e.,

$$L_0\psi_n = \omega_n\psi_n .$$

Thus, the flux is expanded in the form

$$\phi(\vec{r},t) = \sum_{i=1}^N a_n(t) \psi_n(\vec{r}) ,$$

where the $\psi_n(\vec{r})$ are eigenvectors of L_0 . The coefficients $a_n(t)$ are then

determined by solving N independent differential equations in which the eigenvalues ω_n are parameters. The eigenfunctions $\psi_n(\vec{r})$ are often called the "natural" modes of the reactor system. One would expect expansions in these modes to converge very rapidly.

The advantages of modes displaying finality are obvious, but the major difficulty with using them in practice is that they are very difficult to determine. The equation

$$L_o \psi_n = \omega_n \psi_n$$

has been programed for one-dimensional geometry, two energy groups, and xenon feedback by Ewen.⁽¹⁰⁵⁾ The program is called XEP and is capable of finding overtone ($n > 1$) modes and associated eigenvalues. The equation cannot be solved by existing few-group diffusion programs because its solutions are complex for cases of interest and because the components of $\psi(\vec{r})$ do not all satisfy the same interface boundary conditions as the flux components. Thus some approximation is necessary if problems in more than one dimension and two energy groups are to be solved using existing programs; no code to solve this equation in more than one dimension now exists.

Several other attempts have been made to reduce the required number of modes in a modal expansion by finding a set of space functions which are natural to the reactor system. In the limiting case of this approach the modes are defined as the eigenfunctions of the set of equations describing the entire system, including delayed neutron and feedback effects. The fundamental mode is then the steady-state flux shape. The main advantage of this natural mode approach is that relatively few modal shape functions are required. Kaplan⁽⁷⁶⁾ has shown

these modes display the finality property.

If one writes the reactor equations in the operator form

$$T\dot{\phi} + L\phi = M\phi ,$$

then the omega modes are defined to be the eigenfunctions of the problem

$$[L - M] = \omega_n T \cdot \psi_n ,$$

and the lambda modes are the eigenfunctions of the problem

$$L\psi_n = \frac{1}{\lambda_n} M\psi_n .$$

A computer code (MULE) has been written to find the lambda and omega modes for one dimensional slab and cylindrical geometry and two energy groups.⁽⁵⁰⁾

Generally speaking the more effort put into finding the modes of a modal expansion the less is required in finding the time dependent coefficients. Thus if the natural modes of the system are used, ordinarily only a few modes are needed, and the time coefficients are determined via the useful properties of biorthogonality and finality. However, finding these modes in two- and three-dimensional situations is a very difficult problem that has not yet been solved. Applications of the omega and lambda modes have been somewhat limited, but some results have been obtained and published by Kaplan and Bewick⁽¹⁰⁶⁾ and by Henry.⁽²¹⁾

The question of the completeness of the lambda and omega modes is not fully settled.⁽¹⁰⁷⁾ However, to the designer this question is academic since for practical computations he is able to use only a small number of modes anyway. Therefore the idea is generated that one need not necessarily use eigenfunctions as expansion functions. Indeed, a set of functions that might yield an adequate approximation could be motivated

only by intuition or experience. Such functions would probably possess no special mathematical properties, but hopefully they would converge rapidly to the solution merely because of physical arguments. Thus Dougherty and Shen⁽⁵¹⁾ have proposed the use of "Green's function modes" obtained by solving the diffusion equations with sources fixed in various subregions of the reactor. These modes, although not orthogonal, can be adapted to perturbations in the diffusion parameters about which one has some a priori information.

In the Green's mode approach the kinetic equations for a given reactor system are written in the matrix form

$$L\phi = v^{-1} \partial\phi/\partial t.$$

In order to write the kinetic equations in integral equation form the multigroup operator L is redefined in terms of a removal operator L_r and a production operator vM such that $L \equiv L_r - vM$. Thus, the kinetics equations can be expressed as⁽¹⁰³⁾

$$\phi(x,t) - \phi_0(x) = \int_0^t \int_{x_0}^{x_n} dx' dt' \left\{ G(x,t;x',t') vM(x',t') \phi(x',t') \right\}$$

where the kernel $G(x,t;x',t')$ is defined by

$$\left(L_r - v^{-1} \frac{\partial}{\partial t} \right) G(x,t;x',t') = \delta(x-x') \delta(t-t')$$

plus the same homogeneous boundary conditions as required for $\phi(x,t)$.

The integrand of the integral equation is now approximated by a finite sum in the manner

$$G(x, t; x', t') \nu M(x', t') \phi(x', t') \cong \sum_{i=1}^N C_i(t') G_0(x, x') \nu M_0(x') \phi_0(x') \Delta_i(x)$$

where

zero subscripts denote initial values

$G_0(x, x')$ is the initial, steady state Green's function,
defined by

$$Lr_0 G_0(x, x') = \delta(x, x')$$

and

$\Delta_i(x)$ = unity in i^{th} reactor region and zero elsewhere.

By substituting this approximation back into the integral kinetics equation, the following is obtained:

$$\phi(x, t) - \phi_0(x) \cong \sum_{i=1}^N \left[\int_0^t C_i(t') \cdot \left\{ \int_{x_0}^{x_N} dx' G_0(x, x') \nu M_0(x') \phi_0(x') \Delta_i(x') \right\} dt' \right]$$

The space modes are now defined to be

$$\psi_i(x) = \int_{x_0}^{x_N} dx' G_0(x, x') \nu M_0(x') \phi_0(x') \Delta_i(x').$$

It is apparent from the definition of $G_0(x, x')$ that the space modes

$\psi_i(x)$ also satisfy the differential equations

$$Lr_0 \psi_i(x) = \nu M_0(x) \phi_0(x) \cdot \Delta_i(x).$$

It is also noted that

$$\text{Lr}_0 \sum_{i=1}^N \psi_i(x) = v M_0 \phi_0;$$

so the initial steady state flux is equal to the sum of the space modes.

The time dependent flux may now be expressed in terms of these space modes as

$$\phi(x,t) = \sum_{i=1}^N A_i(t) \psi_i(x).$$

A similar analysis will result in the adjoint flux being expressed as

$$\phi^*(x,t) = \sum_{i=1}^N A_i^*(t) \psi_i^*(x)$$

where

$$\text{Lr}_0^* \psi_i^*(x) = v M_0^*(x) \phi_0^*(x) \Delta_i(x).$$

The choice of dividing the reactor into N regions will depend upon the spatial and temporal variations of the core parameters. Note that the calculation of the space modes does not require the usual source iteration. Also, for a one-group analysis, it is not necessary to determine adjoint modes. A theoretical extension of the technique to two- or three-space dimensions is straightforward.

Since the Green's function modes do not possess the property of orthogonality, some other means must be found to determine the time dependent coefficients of the modes. Dougherty and Shen suggested using a variational principle to derive a consistent method of finding these coefficients. Briefly, a Lagrangian is chosen whose Euler equations are given by the kinetic equations and the adjoint kinetic

equations. A separable trial function is substituted into the functional and integrated where possible before variation. The subsequently derived first order differential equations have as their solution a set of time dependent coefficients suitable for the space modes.

It is to be noted that the functional used in the variational principle mentioned above is not selfadjoint; hence, one can only expect to get a saddle point as the stationary point of the functional. This means that the methods that exist for the minimum-maximum case for assessing the errors made in the calculations with the variational principle do not apply here. However, the useful property still exists that first order errors in the flux and adjoint flux do not appear in the numerical values of the functional.

Carter⁽⁵³⁾ has extended the basic modal technique of Dougherty and Shen in order to analyze the dynamic behavior of a coupled reactor model very similar to that used by Foderaro and Garabedian.⁽⁴⁸⁾ The analysis was one-dimensional, two energy groups with no feedback, and time dependent flux distributions were obtained for step and ramp reactivity insertions. Carter also obtained the space dependent frequency response of the reactor to an oscillating absorber in certain regions.

Robinson⁽⁵⁴⁾ has applied the Green's mode approach to a one-dimensional model of the TREAT reactor.⁽¹⁰⁸⁾ The analysis was multi-group, and feedback effects were included. Comparisons with experimental data indicate that the technique gave excellent results.

BNWL-738

APPENDIX E

FLUX SYNTHESIS METHOD

E. Flux Synthesis Method

The general approach for the flux synthesis method is similar to that for the modal method; however, the shape functions employed in the flux synthesis method are not eigenfunctions. The idea behind this approach is that the flux shapes which occur during a reactor transient can be bracketed by some set of shape functions. A rigorous method of determining the shape functions is not used. Rather, the intuition and experience of the user is employed to develop the functions through static calculations or other means. It is assumed that the flux behavior can be expressed as a linear combination of products of shape and amplitude functions, i.e.,

$$\phi(\vec{r}, t) = \psi_1(\vec{r})T_1(t) + \psi_2(\vec{r})T_2(t) + \dots$$

Some form of the weighted residual method, such as a variational principle, is used to determine the amplitude functions. Most of the evaluation of the flux synthesis technique has been performed by Westinghouse Bettis, and the results of their analyses indicate that the synthesis method will treat multi-dimensional kinetics problems very successfully. Synthesis techniques enable one to construct three-dimensional solutions from one- and two-dimensional solutions.

Kaplan⁽⁵⁵⁾ has written a report which outlines the history and development of synthesis techniques up to about 1965. Synthesis methods were first applied to multi-dimensional static problems and later to time-dependent situations. Early versions of synthesis approximations were summarized by Meyer⁽¹⁰⁹⁾ in a paper for the 1958 Geneva Conference. To illustrate Meyer's method, consider a cylindrical reactor which is

divided into two axial zones by a control rod bank (Fig. 1). The basic idea is the assumption that the flux is separable within each axial zone. That is, if $H_1^g(x,y)$ and $H_2^g(x,y)$ represent x-y flux distributions appropriate to the rods-out and rods-in zones, respectively, then the method constructs an approximation in the form

$$\phi^g(x,y,z) \approx Z^g(z) \begin{cases} H_1^g(x,y), & 0 \leq z \leq z_R \\ H_2^g(x,y), & z_R \leq z \leq L \end{cases}$$

The functions H_1^g , H_2^g are found from two-dimensional calculations using guessed values of axial bucklings, B_z^2 . To find the $Z^g(z)$ Meyer inserted this assumed form into the three-dimensional diffusion equations and then integrated over the fuel bearing portion of the x,y cross section. The result of this integration is a set of one-dimensional diffusion equations in the functions $Z^g(z)$. The major shortcoming of the method stems from the fact that the assumed form of solution is inherently discontinuous.

An improvement on the conventional (Meyer's) synthesis method was reported by Wachspress⁽⁴⁴⁾ in 1962. The method was discussed briefly in this report in the nodal analysis section. Wachspress divided the reactor into vertical "channels" as shown in Fig. 2. Within each channel he assumed the flux to be in the form assumed by Meyer, where the H_i are the same two-dimensional solutions used by Meyer. However, each channel now has its own Z function. To determine these Z functions, Wachspress integrated separately over each channel area and represented the leakage between channels using an idea which he called the "gross coupling concept". He viewed this coupling between channels as a generalization of the usual

FIGURE 1
Cylindrical Reactor With Two Axial Zones

FIGURE 2
Channels in the Wachspress Synthesis

finite difference coupling between mesh points. The effect of this "multichannel" synthesis procedure is to reduce the axial discontinuities at the cost of introducing hopefully small horizontal discontinuities at the channel interfaces.

Kaplan⁽⁵⁶⁾ then introduced a new synthesis idea which has been the basis for most synthesis investigations since 1962. Instead of the form suggested by Meyer, Kaplan suggested expanding the flux in the form

$$\phi^g(x,y,z) \approx Z_1^g(z) H_1^g(x,y) + Z_2^g(z) H_2^g(x,y).$$

In this form the x,y shape of the flux is not required to switch abruptly from the rods-in shape to the rods-out shape. Rather, at each elevation, z, the x,y shape is a mixture of these two, the proportions being determined by the values of the mixing functions, Z_1, Z_2 at that z. If more axial zones are present, the set of H functions (or "trial" functions as they are called) may be enlarged to include any functions that intuition or experience suggests would improve the approximation, i.e.,

$$\phi^g(x,y,z) \approx \sum_{k=1}^{K^g} Z_k^g(z) H_k^g(x,y).$$

If the trial functions are well chosen, and enough of them are used, the above form has the capability of giving an exceedingly good approximation to the exact solution. The procedure employed to determine the functions Z_k is the "weighted residual" procedure,⁽¹¹⁰⁾ of which a variational principle is a specific case.

To illustrate the weighted residual procedure, suppose the function $\phi(x)$ describing the state of a physical system is determined by a

differential equation which can be written symbolically

$$L[\phi(x)] = R[\phi(x)], \quad x_0 \leq x \leq x_1. \quad (1)$$

Multiplying the system equation by an arbitrary function $W(x)$ and integrating yields

$$\int_{x_0}^{x_1} W(x) L[\phi(x)] dx = \int_{x_0}^{x_1} W(x) R[\phi(x)] dx. \quad (2)$$

Equations (1) and (2) may be regarded as two alternate formulations of the physical problem; the first states that the equality (1) must hold for every point x , the second that equality (2) must hold for every function $W(x)$. The two formulations imply each other. That Eq. (1) implies Eq. (2) is obvious; that Eq. (2) implies Eq. (1) is essentially the fundamental lemma of the calculus of variations.⁽¹¹⁰⁾ The two formulations are therefore entirely equivalent, yet the second is more suggestive of approximation procedures. Thus, suppose it is desired to construct an approximate solution in the form of a linear combination of known functions $H_k(x)$, i.e.,

$$\phi(x) \approx \sum_{k=1}^K C_k H_k(x). \quad (3)$$

Then the quantity

$$L \left[\sum_{k=1}^K C_k H_k(x) \right] - R \left[\sum_{k=1}^K C_k H_k(x) \right]$$

may be regarded as a residual error when the form of Eq. (3) is substituted into Eq. (1). If the exact solution were of the form of

Eq. (3), one could expect to find C_k such that the weighted integral of the residual

$$\int_{x_0}^{x_1} W(x) \left\{ L \left[\sum_{k=1}^K C_k H_k(x) \right] - R \left[\sum_{k=1}^K C_k H_k(x) \right] \right\} dx \quad (4)$$

would vanish for all weight functions $W(x)$. However, since the form of Eq. (3) is only approximate, the best one can do is to make the integral in Eq. (4) vanish for some functions $W(x)$. Choosing K such functions yields K conditions which can be solved for the C_k , completing the approximation.

In choosing the weight functions there is, in principle, great latitude. Indeed it can be shown that corresponding to various rules for choosing the weight functions the method of weighted residuals contains as special cases many standard approximation methods. For example, it contains the method of collocation, the method of least squares, and the method of moments. Two particular choices have been primarily used in synthesis methods. The first is Galerkin's method,⁽¹¹¹⁾ where the weight functions are chosen to be the same as the trial functions. In the second case the weight functions are chosen to represent the adjoint flux in each axial zone in the same way that the trial functions represent the ordinary flux. This latter method is really an application of the variational principle originally suggested for use in reactor physics by Selengut.⁽¹¹²⁾ The Galerkin method is more economical because adjoint calculations are not required, but it is not clear that it gives results superior to the variational method.

Kaplan⁽⁵⁶⁾ utilized the synthesis approach, with the variational method, to obtain solutions to the three-dimensional static diffusion equations. This "space-synthesis" idea was applied to a two-dimensional model reactor using two energy groups and three trial functions. The results of the synthesis method were compared with a two-dimensional finite difference solution PDQ,⁽¹¹³⁾ and the results were excellent. In principle, arbitrarily complicated configurations may be treated by synthesis methods if enough trial functions are used. As an illustration of how far one may go in this direction and at what cost, Kaplan⁽⁵⁵⁾ has reported the results of a calculation performed by Hecker of Bettis Laboratory. The reactor model used was a cylindrical, seed-blanket type with ten distinct axial regions. To synthesize this r-z problem, eight r-dependent trial functions were chosen for the thermal group, and seven trial functions were used in each of the three fast groups. The weight functions were chosen to be the same as the trial functions. Again comparisons with the PDQ solution indicated the accuracy of the synthesis method and the reduced cost.

A reactor designer must often optimize particular nuclear parameters such as poison and fuel loading to obtain desired values of such quantities as eigenvalues and peak power. Solutions of two- and three-dimensional few-group neutron-diffusion equations to obtain eigenvalue and peak power as a function of nuclear parameters are expensive to obtain, especially for cores requiring a detailed mesh. Curlee⁽⁵⁹⁾ performed several test analyses of some one- and two-dimensional cells to obtain various parameters by exact methods and synthesis methods. His results indicate

that for a 10,000-mesh-point, three-energy-group, two-dimensional problem, the computing time can be reduced from about one hour for a conventional finite-difference solution to about 0.12 hours for a synthesis solution using four trial functions at each point of interest. In addition, there is no significant loss of accuracy when the synthesis method is used.

Flanagan⁽⁶⁰⁾ has reported some results obtained using the weighted-residual, flux-synthesis scheme to determine core eigenvalue, rod position for criticality, power distribution, and differential control-rod reactivity worth for various critical mockups of a PWR reactor. The synthesis results indicate excellent agreement with detailed three-dimensional diffusion theory calculations. Synthesis calculations employing considerable spatial detail ($\approx 14,000$ xy and ≈ 120 z points) were performed for 15 critical configurations. The results indicate that the synthesis scheme required about two orders of magnitude less computer time than the direct solution. Comparisons of the synthesis results with some experimental results were in good agreement. Thus the synthesis scheme seems to provide a practical tool for inexpensive reactor design studies.

A key idea in the synthesis process just described is the blending of trial functions that are appropriate to asymptotic ranges of the z variable. This blending idea may also be applied to other variables such as time. For example, consider two-dimensional lifetime studies. In performing neutron-flux depletion calculations, the practice has been⁽¹¹⁴⁾ to divide the lifetime of the reactor into time steps, assuming the flux shape is constant over each time step, and depleting according to this

shape for the period of that time step. This process is costly primarily because a new diffusion calculation has to be done at each time step. The synthesis method can be used to reduce this cost in the following way. First, an ordinary depletion calculation is done, depleting to end-of-life using a small number (e.g., two or three) of large time steps and saving the flux shape obtained at each step. Then, using these steps as trial functions, one returns to the beginning-of-life and depletes in small time steps, at each step synthesizing a flux shape in the form of a linear combination of these trial functions, i.e.,

$$\phi_{\text{approx}}^g(x,y,t) = T_1^g(t)H_1^g(x,y) + T_2^g(t)H_2^g(x,y) + T_3^g(t) \cdot H_3^g(x,y).$$

A test of this time synthesis has been performed in two dimensions by Flanagan⁽¹¹⁵⁾ using a cell configuration representing a portion of the second core of the Shippingport seed-and-blanket reactor. This configuration was depleted "exactly" using nine time steps to represent 10,000 h of reactor life. The calculation was performed using the TURBO program⁽¹¹⁴⁾ and the cell was represented by 1200 mesh points. The exact calculation was then simulated by repeating the calculation using the synthesis program TNT02.⁽¹¹⁶⁾ For each of the nine time steps, the spatial solution flux was synthesized from trial-function flux distributions obtained from a three time-step depletion calculation. The synthesis calculation used about half the computing time of the TURBO calculation, gave very good results and required much less computer memory capacity. Thus three-dimensional depletion-calculations could be performed using synthesis techniques where exact calculations would not be possible or practical.

The ideas of space and time synthesis can be combined to study a flux tilt transient resulting from a sudden insertion of reactivity on one side of a reactor. Intuitively it may be expected that any time during the transient the actual spatial shape of the flux can be well represented by some linear combination of the initial unperturbed shape and the final asymptotic shape. Thus the synthesized solution takes the form:

$$\phi_{\text{approx}}^g(\vec{r}, t) = T_1^g(t) \cdot H_1^g(\vec{r}) + T_2^g(t) H_2^g(\vec{r}).$$

To find the mixing coefficients, $T_1^g(t)$, in the synthesized solution one may use a time dependent variational principle. Kaplan⁽⁵⁷⁾ has evaluated this procedure by performing a series of one-dimensional test problems. The model reactor is essentially an array of cells arranged into a ring. Each cell in the model contains a rod-in region and a rod-out region. The flux tilt transient induced by decreasing the thermal absorption cross section in one region of the reactor was calculated exactly by WIGLE and approximately by two modal methods (lambda and omega) and by a synthesis expression. For small perturbations all three approximate methods gave very good results. For large perturbations the synthesis method was superior to the modal methods. In general, the synthesis approach has two major advantages over the modal analysis method. First, the trial functions in the synthesis approach can usually be found easily whereas it is often difficult to calculate the appropriate higher eigenfunctions for a modal analysis. Second, the trial functions may be specifically tailored to the problem at hand so that only a few trial functions are needed.

Recently, in connection with the large power reactor program, certain reactor designs have been considered that involve a great deal of complexity in the axial direction. In principle, the synthesis procedure can be extended to treat such designs simply by choosing enough trial functions. However, if it is necessary to use 20 or 30 trial functions in each energy group, the computing time required would become very large. Yasinsky and Kaplan⁽⁶¹⁾ have suggested a method of extending the flux synthesis technique in a systematic way to allow the possibility of using different sets of trial functions in different axial zones. That is, for z near the bottom of the core, one would like to use only those trial functions relevant to the lower regions. Then, as z increases, one would like to discard the lower region trial functions. In this way, one would use only a few functions at each elevation and this would decrease the required computing time. In other words, the flux is expressed in the form

$$\phi^g(x,y,z) \approx \sum_{k=1}^K H_k^g(x,y,z) Z_k^g(z);$$

The $H_k^g(x,y,z)$ are discontinuous functions of z . However, the variational principle used previously to determine the mixing coefficients Z_k^g will admit only continuous trial functions. Selengut and Wachspress⁽¹⁰²⁾ have now developed a variational principle of diffusion theory which includes discontinuous trial functions within its domain. Using this principle, Yasinsky and Kaplan⁽⁶¹⁾ have performed a series of numerical experiments to test the performance of this "space-discontinuous" synthesis technique. Two energy groups and two dimensions were used, and the trial functions were also used as weight functions (Galerkin's method). The model reactors

used had fairly complicated axial structures. Both continuous-and discontinuous-synthesis methods were used, and the results of both were compared to a direct finite difference solution of the two-dimensional diffusion equations, PDQ-5. The results illustrate the reduction in computation time resulting from the discontinuous-synthesis technique while retaining very good accuracy.

Yasinsky⁽⁶²⁾ has described a "time-discontinuous" synthesis approximation for the time-dependent group diffusion equations which permits the use of different sets of spatial trial functions in different time intervals. The synthesis equations, derived from a variational principle extended to include time-discontinuous trial functions, are shown to reduce to the continuous time synthesis equations when the same set of trial functions is used for the entire transient interval. The numerical experiment performed by Yasinsky and Henry⁽²¹⁾ to determine the space-time behavior of a one-dimensional slab reactor by various approximate techniques has been mentioned previously. In that analysis the conventional time-continuous synthesis technique yielded results that were the most comparable to WIGLE. Yasinsky⁽⁶²⁾ investigated this same model using a time-discontinuous synthesis technique. The results of this analysis indicate that by using time-discontinuous synthesis, one can obtain a solution as accurate as the conventional synthesis result, but at a reduced cost. The success of the method is based upon a reasonable knowledge of the nature of the transient being investigated.

The time-discontinuous synthesis approximation is used as the basis of a computer code, called PART2, currently being developed at

V. BIBLIOGRAPHY

1. H. Hurwitz, Jr. "The Derivation and Integration of the Pile-Kinetic Equations," Nucleonics, 5, pp. 61-67. 1949.
2. S. Glasstone and M. C. Edlund. The Elements of Nuclear Reactor Theory. D. Van Nostrand Co., Inc., New York, N.Y., 1952.
3. A. M. Weinberg and E. P. Wigner. The Physical Theory of Neutron Chain Reactors. The University of Chicago Press, Chicago, Ill., 1958.
4. A. F. Henry. "The Application of Reactor Kinetics to the Analysis of Experiments," Nuc. Sci. and Eng., 3, pp. 52-70. 1958.
5. J. Lewins. "Variational Representations in Reactor Physics Derived from a Physical Principle," Nuc. Sci. and Eng., 8, pp. 95-104. 1960.
6. T. Gozani. "The Concept of Reactivity and its Application to Kinetic Measurements," Nukleonik, 5, pp. 55-62. 1963.
7. A. F. Henry. "Definition and General Considerations Concerning Large Changes in Reactivity," Trans. Am. Nucl. Soc., 9, pp. 235-236. 1966.
8. H. Hurwitz, Jr. Note on the Theory of Coefficients, KAPL-98. Knolls Atomic Power Laboratory, Schenectady, N.Y., 1948.
9. L. N. Ussachoff. "Equation for the Importance of Neutrons, Reactor Kinetics and the Theory of Perturbation," paper 656, First United Nations International Conference on the Peaceful Uses of Atomic Energy. 1956.
10. D. S. Selengut. Variational Analysis of Multi-dimensional Systems, HW-59126. General Electric, Richland, Washington, 1958.
11. B. B. Kadomtsev. "A Physical Interpretation of the Adjoint Function," (translated title). Doklady Akad. Nauk S.S.S.R. 113: 3. 1957.
12. J. Lewins. "The Time-dependent Importance of Neutrons and Precursors," Nuc. Sci. and Eng., 7, pp. 268-274. 1960.

13. H. P. Flatt. "Collocation Method for the Numerical Solution of the Reactor Kinetics Equation," IBM Nucl. Comp. Bull., 5. 1962.
14. J. J. Kaganove. "Numerical Solution of the One-Group Reactor Kinetics Equations," ANL-6132, Argonne National Laboratory, Argonne, Ill. 1960.
15. K. F. Hansen, B. V. Koen and W. W. Little, Jr. "Stable Numerical Solutions of the Reactor Kinetics Equations," Nuc. Sci. and Eng., 22, pp. 51-59. 1965.
16. G. R. Keepin and C. W. Cox. "General Solution of the Reactor Kinetic Equations," Nuc. Sci. and Eng., 8, pp. 670-690. 1960.
17. E. R. Cohn. "Some Topics in Reactor Kinetics," 11: 302-309. Second United Nations International Conference on the Peaceful Uses of Atomic Energy. 1958.
18. S. Kaplan, A. F. Henry, S. G. Marglois, and J. J. Taylor. "Space-Time Reactor Dynamics," 3:41. Third United Nations International Conference on the Peaceful Uses of Atomic Energy. 1964.
19. W. R. Cadwell, A. F. Henry and A. J. Vigilotti. "WIGLE - A Program for the Solution of the Two-Group Space-Time Diffusion Equations in Slab Geometry," WAPD-TM-416, Westinghouse Electric Corp. 1964.
20. A. F. Henry and A. V. Vota. "WIGL2 - A Program for the Solution of the One-Dimensional, Two-Group, Space-Time Diffusion Equations Accounting for Temperature, Xenon, and Control Feedback," WAPD-TM-532, Westinghouse Electric Corp. 1965.
21. J. B. Yasinsky and A. F. Henry. "Some Numerical Experiments Concerning Space-Time Reactor Kinetics Behavior," Nuc. Sci. and Eng., 22, pp. 171-181. 1965.
22. S. O. Johnson, E. P. Gyftopoulos and M. E. Radd. "Effects of Flux-Shape Changes on Power-Excursion Behavior," Trans. Am. Nucl. Soc., 8, p. 221. 1965.

23. J. P. Hamric. "Spatial Kinetics in Large Reactors," Trans. Am. Nucl. Soc. 8, pp. 499-500. 1965.
24. R. E. Tiller and C. A. Oster. CLUMSY-I - A Time and Space Dependent Reactor Kinetics Code for the IBM-7090 Computer, HW-77221. General Electric, Richland, Washington. 1963.
25. P. T. Hansson and L. R. Foulke. "Investigations in Spatial Reactor Kinetics," Nuc. Sci. and Eng., 17, pp. 528-533. 1963.
26. T. W. Kerlin. "Status of Space-Time Analysis," Nuc. Safety, Vol. 6, No. 4, pp. 395-398. 1965.
27. J. V. Wilson. "Xenon Instabilities," Nuc. Safety, Vol. 5, No. 4, pp. 345-354. 1964.
28. M. E. Radd. WIGLE-40, A Two-Group, Time-Dependent Diffusion Theory Program for the IBM-5040 Computer, IDO-17125. Phillips Petroleum Company, Idaho Falls, Idaho. 1965.
29. J. W. Riese. VARIQUIR-A Two-Dimensional Time Dependent Multigroup Diffusion Code, WANL-TNR-133. 1963.
30. J. W. Riese. "A Variational Technique for Space-Time Neutron Diffusion," Trans. Am. Nucl. Soc., 7, pp. 22-23. 1964.
31. K. F. Hansen. A Comparative Review of Two-Dimensional Kinetics Methods, GA-8169. 1967.
32. K. F. Hansen and S. R. Hohnson. An Approximate Method for Multi-Dimentional Diffusion Theory Problems, GA-7544. 1967
33. J. B. Andrews, II, and K. F. Hansen. GAKIN--A Program for the Solution of the One-Dimensional, Multigroup, Space-Time Dependent Diffusion Equations, GA-7543. 1967.

34. A. F. Henry, and N. J. Curlee, "Verification of a Method for Treating Neutron Space-Time Problems," Nuc. Sci. and Eng., 4, pp. 727-744. 1958.
35. W. M. Stacey, Jr., and C. H. Adams. "The Time-Integrated Method: A Quasistatic Neutron Space-Time Approximation," Trans. Am. Nucl. Soc., 10, pp. 251-252. 1967.
36. J. T. Madell, and K. Ott. "Quasistatic Treatment of Spatial Phenomena in Reactor Dynamics," Nuc. Sci. and Eng., 26, pp. 563-565. 1966.
37. N. J. Curlee, Jr. "Nonseparable Space-Time Transients Resulting from Changes in Inlet Coolant Temperature," Nuc. Sci. and Eng., 6, pp. 1-10. 1959.
38. D. L. West, M. P. Kalman, J. E. Boyden, R. C. Stirn, and J. J. Barth. "Use of the Adiabatic Approximation in Prompt Critical, Large-Core Reactor-Excursion Analyses," Trans. Am. Nucl. Soc., 8, pp. 497-498. 1965.
39. H. Nishihara. "Application of Adiabatic Approximation to the Space-Dependent Boiling-Water Reactor Kinetics," Trans. Am. Nucl. Soc., 7, pp. 222-223. 1964.
40. R. Avery. "Theory of Coupled Reactors," 12: 182. Second United Nations International Conference on the Peaceful Uses of Atomic Energy. 1958.
41. P. R. Pluta. "Coupled Core kinetic behavior," AEC Symposium Series #7, Neutron Dynamics and Control, CONF-650413. (U.S. Atomic Energy Commission Div. Tech. Information, Springfield, Virginia.) 1966.
42. D. C. Wade, and H. H. Rubin. "A Nodal Calculation of a Space-Time Transient Using Coupling Coefficients which Account for Changing Internodal Leakages," Trans. Am. Nucl. Soc., 10, pp 250-251. 1967.
43. H. Plaza, and W. H. Kohler. "Coupled-Reactors Kinetics Equations," Nuc. Sci. and Eng., 26, pp 419-422. 1966.

44. E. L. Wachspress, R. D. Burgess, and S. Baron. "Multichannel Flux Synthesis," Nuc. Sci. and Eng., 12, pp. 381-389. 1962.
45. E. L. Wachspress, and M. Becker. Variational Multichannel Synthesis with Discontinuous Trial Functions, KAPL-3095. 1965.
46. A. Radkowsky, (Ed.). Selected Basic Techniques, Naval Reactors Physics Handbook, Vol. 1. U. S. Atomic Energy Commission. 1964.
47. H. L. Garabedian, and G. B. Leffert. "A Time-Dependent Analysis of Spatial Flux Distributions," Nuc. Sci. and Eng., 6, pp 26-32. 1959.
48. A. Foderaro, and H. L. Garabedian. "Two Group Reactor Kinetics, Nuc. Sci. and Eng., 14, pp. 22-29. 1962.
49. H. L. Garabedian, and R. E. Lynch. "Nonlinear Reactor Kinetics Analysis," Nuc. Sci. and Eng., 21, pp. 550-564. 1965.
50. R. L. Erven. MULE-A Fortran Program for the Calculation of Three Types of Overtone Modes. WAPD-Tm-369. 1963.
51. D. E. Dougherty, and C. N. Shen. "The Space-Time Neutron Kinetic Equations Obtained by the Semidirect Variational Method," Nuc. Sci. and Eng., 13, pp. 141-148. 1962.
52. D. E. Dougherty. The Space-Time Neutron Kinetics by a Variational Method with Application to Power Reactor Dynamics. KAPL-2217. 1962.
53. N. E. Carter. Solution of Space-Time Kinetic Equations for Coupled-Core Nuclear Reactors. Iowa State University, Ames, Iowa, 1967. (Ph.D. thesis)
54. J. C. Robinson. Approximate Solution to the Time-Dependent Multigroup Neutron Diffusion Equations Using a Restricted variational Principle. University of Tennessee, Knoxville, Tenn., 1966. (Ph.D. thesis)

55. S. Kaplan. "Synthesis Methods in Reactor Analysis," Advances in Nuclear Science and Technology, edited by P. R. Greebler and E. J. Henley. Academic Press, New York, 1966. pp. 233-266.
56. S. Kaplan. "Some New Methods of Flux Synthesis," Nuc. Sci. and Eng., 13, pp. 22-31, 1962.
57. S. Kaplan, O. J. Marlowe, and J. Bewick. "Application of Synthesis Techniques to Problems Involving Time Dependence," Nuc. Sci. and Eng., 18, pp. 163-176. 1964.
58. S. Kaplan, and J. A. Bewick. Space and Time Synthesis by the Variational Method, WAPD-BT-28. Westinghouse Electric Corp. Atomic Power Division, Pittsburgh, Pa. 1963.
59. N. J. Curlee, Jr. "Application of Flux Synthesis to Parametric-Optimization Studies," Trans. Am. Nucl. Soc., 7, p. 12. 1964.
60. C. A. Flanagan, F. E. Smith, G. F. Bogar, and C. H. Rutherford. "An Evaluation of a Three-Dimensional Flux Synthesis Method as a Nuclear Design Tool," Trans. Am. Nucl. Soc., 7, pp. 289-290. 1964
61. J. B. Yasinsky, and S. Kaplan. "Synthesis of Three-Dimensional Flux Shapes Using Discontinuous Sets of Trial Functions," Nuc. Sci. and Eng., 28, pp. 426-437. 1967.
62. J. B. Yasinsky. "The Solution of the Space-Time Neutron Group Diffusion Equations by a Time-Discontinuous Synthesis Method," Nuc. Sci. and Eng., 29, pp. 381-391. 1967.
63. J. Chernick, G. Lellouche, and W. Woolman. "The Effect of Temperature on Xenon Instability," Nuc. Sci. Eng., 10, pp 120-131. 1961.
64. R. R. Haefner, "Flux Oscillations Caused by Xenon Instability," Nucl. Sci. Technol., 2(3), pp 291. December 1956.

65. J. W. Simpson and H. G. Rickover, "Shippingport Atomic Power Station (PWR)," Nucleonics, 16(9), p. 72. September 1958.
66. G. C. Fullmer. Xenon Instability in Graphite Reactors, HW-69084. General Electric Co., Richland, Washington, February 1961
67. R. I. Vaughan. A Method of Analyzing Transients in the Flux Distribution in a Thermal Reactor, British DEG-Report-149, Feb. 19, 1960.
68. A. Hitchcock. Nuclear Reactor Stability, Chap. 4. Temple Press, London, 1960.
69. G. S. Lellouche. Control of Xenon Spatial Oscillations, BNL-6330. Brookhaven National Laboratory, August 1962.
70. S. G. Margolis, and N. J. Curlee, Jr. "Control-Induced Xenon Oscillations," Trans. Am. Nucl. Soc., 7, pp. 220-221. 1964.
71. A. G. Ward. The Problem of Flux Instability in Large Power Reactors, Canadian Report CRRP-657 (AECL-345), July 26, 1956.
72. A. F. Henry, and J. D. Germann. "Oscillations in the Power Distribution Within a Reactor," Nuc. Sci. and Eng., 2, pp. 469-480
73. D. Randall, and D. S. St. John. "Xenon Spatial Oscillations," Nucleonics, 16(3), pp. 82-86 (March 1958).
74. R. S. Wick. Space- and Time-Dependent Flux Oscillations (and instability) in Thermal Reactors Due to Nonuniform Formation and Depletion of Xenon, WAPD-TM-138. Westinghouse Electric Corp., Atomic Power Division, August 1958.
75. D. R. Harris and P. S. Lacy. "A Simple Approximate Test for Spatial Xenon Stability," Trans. Am. Nucl. Soc., 3, pp. 437-438. 1960.
76. S. Kaplan. "The Property of Finality and the Analysis of Problems in Reactor Space-Time Kinetics by Various Modal Expansions," Nuc. Sci. and Eng., 9(3), pp. 357-361. March 1961.
77. D. R. Harris and P. S. Lacy. "A Simple Approximate Test for Spatial Xenon Stability," Trans. Am. Nucl. Soc., 3(2), p. 437. December 1960.

78. G. L. Gyorey. "The Effect of Modal Interaction in the Xenon Instability Problem," Nuc. Sci. and Eng., 13, pp. 338-344. 1962.
79. R. M. Pearce. "Method of Studying Xenon Spatial Instability with an Analog Computer," Nuc. Sci. and Eng., 11, p. 328. 1961.
80. H. B. Smets. "The Effect of Burnable Fission Products in Power Reactor Kinetics," Nuc. Sci. and Eng., 11, pp. 133-141. 1961.
81. E. P. Gyftopoulos. "Applications of Geometric Theory to Nonlinear Reactor Dynamics," Nuc. Sci. and Eng., 10, pp. 254-268. 1961.
82. W. M. Barss. Digital Simulation of Xenon Instability in Reactors, Canadian Report CRRP-1002(AECL-1226), March 1961.
83. R. M. Pearce and R. E. Roth. A G-20 Program for Studying Xenon Oscillations and Application to CANDU, Canadian Report CRRP-1151 (AECL-1803), July 1963.
84. A. R. Curtis, J. G. Tyror, and H. E. Wrigley. STAB: A Kinetic, Three-Dimensional, One-Group Digital Computer Program, British Report AEEW-R-77. October 1961.
85. A. R. Curtis, J. G. Tyror, and H. E. Wrigley. Data Requirements for the 1960 Version of STAB, British Report AERE-M-898, June 1961.
86. T. R. England, G. L. Hartfield, and R. K. Deremer. Xenon Stability in Large Seed-Blanket reactors, WAPD-TM-606. Westinghouse Electric Corporation, Bettis Atomic Power Laboratory, Pittsburgh, Pa., 1967.
87. D. J. DiLeonardo. X WAVE - A Nodal Analysis Program for Calculating the Space-Time Reactor Power Behavior in the Presence of Xenon and Temperature Feedback, WAPD-TM-679. Westinghouse Electric Corporation, Bettis Atomic Power Laboratory, Pittsburgh, Pa., 1967.

88. K. J. Chock, S. Kaplan, S. G. Margolis, and C. M. Lane. "Analog Studies of Xenon Oscillations-The Effect of Coolant Recirculation on Spatial Stability," Trans. Am. Nucl. Soc., 8, p 237. 1965.
89. S. Kaplan and J. B. Yasinsky. "Natural Modes of the Xenon Problem with Flow Feedback-An example," Nuc. Sci. and Eng., 25, pp. 430-438. 1966.
90. W. M. Stacey, Jr. "Xenon Spatial Oscillations," Trans. Am. Nucl. Soc., 10, p. 254. 1967.
91. J. V. Wilson. Unpublished Data. Oak Ridge National Laboratory. Personal Communication.
92. W. M. Stacey, Jr. "Optimal Control of Spatial Xenon Transients," Trans. Am. Nucl. Soc., 10, pp. 572-573. 1967.
93. A. A. Feldbaum. Optimal Control Systems. Academic Press, New York, 1965.
94. R. Bellman. Dynamic Programming. Princeton University Press, Princeton, New Jersey. 1957.
95. D. M. Wiberg. Optimal Feedback Control of Spatial Xenon Oscillations in a Nuclear Reactor. California Institute of Technology, 1965. (Ph.D. thesis)
96. D. M. Wiberg. "Controllability of the Spatial Flux Shape," Nuc. Sci. and Eng., 27, p. 600. 1967.
97. J. B. Andrews, II. Numerical Solution of the Space-Dependent Reactor Kinetics Equations. Massachusetts Institute of Technology, 1967. (Ph.D. thesis)
98. J. W. Smiley. STREAK-A Numerical Solution for the Space-Time Neutron Diffusion Equations, Engineering Research Bulletin B-95. Pennsylvania State University, University Park, Pa. 1966.
99. S. Kaplan and S. G. Margolis. "Delayed Neutron Effects During Flux Tilt Transients," Nuc. Sci. and Eng., 7, pp. 276-277. 1960.

100. E. S. Beckjord. "Dresden Reactor Stability Tests," Trans. Am. Nucl. Soc., 3, p 433. 1960.
101. P. R. Pluta, W. G. Hubschmann, and G. W. Lill. "Kinetics of Coupled Thermal-Fast Spectrum Power Reactors," Proceedings of the National Topical Meeting of American Nuclear Society for Coupled Reactor Kinetics. Texas A&M University, College Station, Texas. 1967.
102. E. L. Wachspress and M. Becker. Variational Synthesis with Discontinuous Trial Functions, ANL-7050. Argonne National Laboratory, 1965.
103. B. Friedman. Principles and Techniques of Applied Mathematics. John Wiley and Sons, Inc., New York, N. Y. 1956.
104. H. L. Garabedian and D. H. Thomas. "An Analytic Approach to Two-Dimensional Reactor Theory," Nuc. Sci. and Eng., 14, pp 266-271. 1962.
105. R. L. Ewen. Calculation of Complex Natural Modes for Spatial Xenon Oscillations and Comparison with a Simple Approximation, WAPD-BT-27. Westinghouse Electric Corporation, 1962.
106. S. Kaplan and J. Bewick. Space and Time Synthesis by the Variational Method, WAPD-BT-28. Westinghouse Electric Corp., 1963.
107. G. J. Habetler and M. A. Martino. "Existence Theorems and Spectral Theory for the Multigroup Diffusion Model," American Mathematical Society Symposia in Applied Mathematics-Nuclear Reactor Theory, Proceedings. 1961
108. D. Okrent, et al. The Reactor Kinetics of the Transient Reactor Test Facility (TREAT), ANL-6174. Argonne National Laboratory, 1960.
109. J. E. Meyer. "Synthesis of Three-Dimensional Power Shapes-A Flux-Weighting Synthesis Technique." paper 11, pp. 519-522, Second United Nations International Conference on the Peaceful Uses of Atomic Energy. 1958.

110. S. H. Crandall. Engineering Analysis. McGraw-Hill, New York, N. Y., 1956.
111. L. V. Kantorovich and V. I. Kryloff. Approximate Methods of Higher Analysis. Wiley (Interscience), New York, N. Y., 1959.
112. D. S. Selengut. Variational Analysis of Multidimensional Systems, HW-59126. General Electric Company, Richland, Washington, 1958.
113. W. R. Cadwell. PDQ-4--A Programm for the Solution of the Neutron Diffusion Equations in Two Dimensions on the Philco-2000, WAPD-TM-230. Westinghouse Electric Corporation, 1960.
114. O. J. Marlowe. Nuclear Reactor Depletion Programs for the Philco-2000 Computer, WAPD-TM-221. Westinghouse Electric Corporation, 1961.
115. C. A. Flanagan. "Few Long-Time-Increment Depletion Calculations for Approximating by Synthesis a Many Short-Time-Increment Depletion Calculation," Trans. Am. Nucl. Soc. 8, p. 518. 1965.
116. C. A. Flanagan, et al. TNT02, A Three-Dimensional Neutron Flux Synthesis and Depletion Calculation Computer Program, WAPD-TM-468. Westinghouse Electric Corporation, 1964.
117. A. F. Henry. Bettis Atomic Power Laboratory. Private communication.
118. E. L. Wachspress. Iterative Solution of Elliptic Systems. Prentice-Hall, Inc. Englewood Cliffs, N. J. 1966.
119. J. B. Yasinsky. Bettis Atomic Power Laboratory, Private communication.
120. W. G. Clarke and S. G. Margolis. The Multimode Synthesis Approximation for Space-Time Reactor Dynamics: An Appraisal of Finite Differencing Methods, WAPD-TM-635. 1967.
121. K. Ott, D. A. Meneley and E. S. Wiener. Quasistatic Treatment of Fast Reactor Excursions. ANL-7210. 1966.
122. K. Ott, D. A. Meneley and E. S. Wiener. Space-Time Kinetics, The QX 1 Code. ANL-7310. 1967.
123. S. Kaplan, R. Ewen, N. J. Curlee, and S. Margolis. "Space-Time Kinetics, Section 5.4," Naval Reactors Physics Handbook, A. Radkowsky, editor. 1962.

DISTRIBUTION

<u>No. of Copies</u>		<u>No. of Copies</u>	
1	<u>AEC Chicago Operations Office</u> 9800 South Cass Avenue Argonne, Illinois 60439 G. H. Lee, Director Patent Division		AEC Library, Wash. (continued)
2	<u>AEC Chicago Patent Group</u> Federal Bldg., Area 700 Richland, Washington 99352 R. K. Sharp		M. A. Rosen, Asst. Dir. Plant Engineering D. E. Pollock, Asst. Dir. Program Analysis A. Giambusso, Asst. Dir. Project Management E. A. Kinter, Asst. Dir. Reactor Engineering I. F. Zartman, Chief Reactor Physics Branch E. E. Sinclair, Asst. Dir. Reactor Technology N. Grossman, Chief Special Technology Branch E. E. Erb, Acting Chief Water Projects Branch
5	<u>AEC Division of Technical Information Extension</u> P. O. Box E Oak Ridge, Tennessee 37830 Head, Division of Technical Information Extension		
18	AEC Library, Washington 20545 Division of Reactor Develop- ment and Technology A. J. Pressesky, Chief Analysis and Evaluation Board E. E. Kinter, Chief Core Design Branch W. A. Williams, Chief Desalting Branch J. W. Crawford, Asst. Dir. Engineering Standards E. E. Kinter, Chief Fuel Fabrication Branch J. M. Simmons, Chief Fuels and Materials Branch A. N. Tardiff, Chief (2) Heavy Water Reactors Branch R. W. Bean, Chief Instruments and Control Branch J. A. Lieberman, Asst. Dir. Nuclear Safety M. A. Rosen, Asst. Dir. Plant Engineering	1	<u>AEC Library, Washington 20545</u> Division of Technical Information, MS149 W. A. Kee, Technical Utiliza- tion Officer
		1	<u>Office of General Counsel</u> R. A. Anderson, Asst. General Counsel for Patent
		2	<u>AEC RDT Senior Site Representative</u> P. O. Box 550, Area 300, Bldg., 3760, Richland, Washington 99352 W. E. Fry P. G. Holsted/A. S. Waterhouse
		1	<u>AEC RDT Senior Site Representative</u> OROO, P. O. Box X Oak Ridge, Tennessee 37830 D. F. Cope

No. of
CopiesNo. of
Copies

<p>1 <u>AEC RDT Site Representative</u>
 P. O. Box 2108
 Idaho Falls, Idaho 83401</p> <p> K. A. Trickett</p> <p>1 <u>AEC RDT Site Representative</u>
 P.O. Box 500
 Windsor, Connecticut 06095</p> <p> G. U. Merckel</p> <p>1 <u>AEC RDT Site Office</u>
 P.O. Box 591
 Canoga Park, California 91305</p> <p> Robert L. Morgan,
 Senior Site Rep., AI</p> <p>2 <u>AEC Richland, Operation Office</u>
 Federal Bldg.
 Richland, Washington 99352</p> <p> C. L. Robinson, Director
 Research and Development
 Division
 Technical Library</p> <p>3 <u>USAEC Scientific Representative</u>
 c/o Atomic Energy of Canada Ltd.
 Chalk River, Ontario, Canada</p> <p> R. W. Ramsey</p> <p>1 <u>USAEC Technical Representative</u>
 c/o Atomic Energy of Canada Ltd.
 Sheridan Park
 Ontario, Canada</p> <p> H. J. Reynolds</p> <p> USAEC/AECL COOPERATIVE PROGRAM
 TAC AND SUBCOMMITTEES</p> <p>5 <u>Technical Advisory Committee</u>
 (TAC) - United States</p> <p> A. N. Tardiff, U. S. Chairman
 of TAC
 AEC Headquarters
 Washington, D. C. 20545</p> | <p>H. J. Reynolds, U.S. Secretary
 of TAC
 c/o Atomic Energy of Canada Ltd.
 Sheridan Park, Ontario, Canada.</p> <p>D. F. Babcock, U.S. Member
 of TAC
 E. I. duPont de Nemours & Co.
 Wilmington, Delaware</p> <p>H. Harty, U.S. Member of TAC
 Heavy Water Reactor Program
 Office
 Pacific Northwest Laboratory
 P.O. Box 999, Area 300,
 Bldg. 3702
 Richland, Washington 99352</p> <p>E. A. Mason, U.S. Member of TAC
 Massachusetts Institute of
 Technology
 Cambridge, Massachusetts, 02139</p> <p>5 <u>Technical Advisory Committee</u>
 (TAC) - Canadian</p> <p>D. D. Stewart, Canadian
 Chairman of TAC
 Atomic Energy of Canada Ltd.
 Chalk River Nuclear Labs.
 Chalk River, Ontario, Canada</p> <p>W. R. Livingston, Canadian
 Secretary of TAC
 Atomic Energy of Canada Ltd.
 Chalk River Nuclear Labs.
 Chalk River, Ontario, Canada</p> <p>L. Pease, Canadian Member of
 TAC
 Atomic Energy of Canada Ltd.
 Toronto, Ontario, Canada</p> <p>R.F.S. Robertson, Canadian
 Member of TAC
 Atomic Energy of Canada Ltd.
 Whiteshell Nuclear Research
 Establishment
 Pinawa, Manitoba, Canada</p> |
|---|--|

No. of
CopiesNo. of
Copies

7	<p>W. R. Thomas, Canadian Member of TAC Atomic Energy of Canada Ltd. Chalk River Nuclear Labs. Chalk River, Ontario, Canada</p> <p><u>TAC Subcommittee - REACTOR STABILITY AND CONTROL</u></p> <p>R. Feit Division of Reactor Development & Technology U.S. Atomic Energy Commission Washington, D.C. 20545</p> <p>A. Pearson Atomic Energy of Canada Ltd., Chalk River Nuclear Labs. Chalk River, Ontario, Canada</p> <p>F. Bevilacqua Combustion Engineering, Inc. P.O. Box 500 Windsor, Connecticut 06095</p> <p>G. Dessauer E. I. duPont de Nemours & Co. Savannah River Laboratory P.O. Box 117 Aiken, South Carolina 29801</p> <p>L. Pease Atomic Energy of Canada Ltd. Toronto, Ontario, Canada</p> <p>D. B. Primeau Physics & Analysis Branch Atomic Energy of Canada Ltd. Sheridan Park, Ontario, Canada</p> <p>J. R. Worden Engineering Development Dept. Pacific Northwest Laboratory P.O. Box 999, Area 300, Bldg., 328 Richland, Washington 99352</p>	4	<p><u>TAC Subcommittee - HEAT TRANSFER AND FLUID DYNAMICS</u></p> <p>R. M. Scroggins Division of Reactor Development and Technology U.S. Atomic Energy Commission Washington D. C. 20545</p> <p>G. A. Wikhammer Advance Engineering Branch Advanced Project & Reactor Physics Division Atomic Energy of Canada Ltd. Chalk River, Ontario, Canada</p> <p>J. M. Batch Engineering Development Dept. Pacific Northwest Laboratory Battelle Memorial Institute P.O. Box 999 Richland, Washington 99352</p> <p>J. S. Neill E. I. duPont de Nemours & Co. 1007 Market Street Wilmington, Delaware 19801</p>	9	<p><u>TAC Subcommittee - SAFETY</u></p> <p>A. J. Pressessky Analysis & Evaluation Branch Division of Reactor Development And Technology U.S. Atomic Energy Commission Washington D. C. 20545</p> <p>G. Hake Atomic Energy of Canada Ltd. Chalk River Nuclear Laboratory Chalk River, Ontario, Canada</p> <p>R. Vollmer HWOCR Office Atomics International P.O. Box 309 Canoga Park, California 91304</p>
---	--	---	---	---	--

No. of
CopiesNo. of
Copies

R. K. Hilliard
Battelle-Northwest
Richland, Washington 99352

F. C. Boyd
Atomic Energy Control Board
Chalk River, Ontario, Canada

S. Davies
Canadian General Electric Co.,
Ltd.
107 Park Street, N.
Petersborough, Ontario, Canada

F. Schroeder
Water Reactor Safety Program
Office
Phillips Petroleum Co.
Idaho Falls, Idaho 83401

J. J. Billington
Engineering & Physics
Dilworth, Second, Meagher
& Associates
4195 Dundas St., West
Toronto, Ontario, Canada

J. D. Sainsbury
Atomic Energy of Canada Ltd.
Sheridan Park, Ontario, Canada

6 TAC Subcommittee - FUELS AND
MATERIALS

A. Van Echo
Division of Reactor Development
& Technology
U.S. Atomic Energy Commission
Washington, D. C. 20545

J. A. L. Robertson
Atomic Energy of Canada Ltd.
Chalk River Nuclear Laboratories
Chalk River, Ontario, Canada

R. G. Wheeler
Engineering Materials and
Mechanics Section
Battelle-Northwest
Richland, Washington 99352

W. R. Thomas
Atomic Energy of Canada Ltd.
Chalk River Nuclear Labs.
Chalk River, Ontario, Canada

M. F. Lyons
Fuels Development and Design
Materials
Nuclear Technology Department
General Electric Company
Pleasanton, California 94566

W. B. DeLong
Atomic Energy Division
E. I. du Pont de Nemours & Co.
Wilmington, Delaware 19898

6 TAC Subcommittee - PHYSICS

J. W. Lewellen,
Reactor Physics Branch
Division of Reactor Development
& Technology
U.S. Atomic Energy Commission
Washington, D. C. 20545

C. H. Millar
Atomic Energy of Canada Ltd.
Chalk River Nuclear Laboratory
Chalk River, Ontario, Canada

J. L. Crandall
Experimental Physics Division
Savannah River Laboratory
E. I. duPont de Nemours & Co.
Aiken, South Carolina

S. Visner
Combustion Engineering
P.O. Box 500
Windsor, Connecticut 06095

<u>No. of Copies</u>		<u>No. of Copies</u>	
	L. C. Schmid Battelle-Northwest Richland, Washington 99352	1	<u>E. I. duPont de Nemour & Co.</u> Wilmington, Delaware 19898 Reactor Engineering Section
	A. Pasanen Atomic Energy of Canada Ltd. Sheridan Park, Ontario, Canada	1	<u>Iowa State University</u> Ames Iowa R. Danofsky
1	<u>AAEC Research Establishment</u> Lucas Heights, Australia M. M Yarosh (To be opened by addressee only)	1	<u>Massachusetts Institute of Technology</u> 77 Massachusetts Avenue Room 24-109 Cambridge, Massachusetts 02139
1	<u>Babcock and Wilcox Company</u> P. O. Box 1260 Lynchburg, Virginia 24505 M. W. Croft		Department of Nuclear Engineering
1	<u>Columbia University</u> 632 West 125th Street New York, N. Y. 10027 J. F. Casterline Engineering Research Lab. Dept. of Chem. Engineering	2	<u>Oak Ridge National Laboratory</u> P.O. Box X Oak Ridge, Tennessee 37830 J. A. Lane, Studies and Evaluation Reactor Program
1	<u>Combustion Engineering</u> P.O. Box 500 Windsor, Connecticut W. P. Chernock	1	<u>Phillips Petroleum Co.</u> P.O. Box 2067 Idaho Falls, Idaho 83401 F. Schroeder, Water Reactors Safety Program Office
6	<u>Douglas United Nuclear, Inc.</u> Federal Building Richland, Washington 99352 L. Agee W. S. Nechodom G. F. Owsley File - 3	1	<u>Savannah River Laboratory</u> Aiken, South Carolina 29801 S. W. O'Rear, Supervisor Technical Information Services

No. of
Copies

42

Battelle-Northwest

F. W. Albaugh
S. R. Bierman
G. J. Busselman
J. L. Carter
N. E. Carter - 10
E. D. Clayton
E. C. Davis
F. G. Dawson
B. H. Duane
A. G. Gibbs
R. A. Harvey
R. E. Heineman
P. L. Hofmann
U. P. Jenquin
B. B. Lane
D. D. Lanning
P. Loizzo
W. R. Lewis
R. C. Liikala
C. W. Lindenmeier
W. W. Little
R. E. Peterson
D. L. Prezbindowski
G. D. Seybold
R. I. Smith
R. E. Turley
V. O. Uotinen
H. S. Zwibel
Technical Information Files (5)