SEP 23 1964

BNL 8293

MASTER

CONF-640606-19

BINDING ENERGIES OF NUCLEAR TWO-, THREE- AND FOUR-BODY SYSTEMS

Y. C. Tang

Brookhaven National Laboratory, Upton, New York

E. W. Schmid[†]

Florida State University, Tallahassee, Florida

R. C. Herndon

Lawrence Radiation Laboratory, Livermore, California^{††}

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

~~NOT FOR PUBLIC RELEASE - OFFICIAL
DISTRIBUTION MAY BE MADE. OFFICIAL
REQUESTS MAY BE FILLED. REPORT
CONTAINS NOTHING OF PATENT INTEREST.
PROCEDURES ON FILE IN RECEIVING
SECTION.~~

[†] On leave from Max-Planck-Institut, Munich, Germany

^{††} This work was supported by the U. S. Atomic Energy Commission.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

ABSTRACT

The binding energies of the nuclear three- and four-body problems are calculated using a variational method with a hard-core, central potential of various core radii, which fits the two-nucleon low-energy scattering data. Both the upper and the lower bound are computed. The results show that when the core radius has a value between 0.35 and 0.55 fm, reasonable agreement with experimental data on the binding energies and rms radii of the triton and the alpha particle, and the Coulomb energy of He^3 can be obtained.

1. Introduction

A recent investigation on the ground state of Li^6 ¹⁾ indicates that with the cluster model²⁾, certain features of this nucleus can be explained quantitatively, when a variational calculation is performed with a two-body central potential containing a hard core, which explains the low-energy two-nucleon scattering data. This shows that some of the properties of the light nuclei can be explained with a potential which is considerably simpler than the phenomenological nucleon-nucleon potentials³⁾ presently accepted. As a next target, it seems desirable to extend the calculation to other light nuclei with the same potential used in the calculation of Li^6 . Before this is attempted, one should, however, first examine closely the properties of the deuteron-, triton-, He^3 - and alpha-cluster obtained with this two-body potential, since these clusters are the basic units in the cluster model. It is quite clear that for the calculation to yield meaningful results, one must demand that the two-body potential should at least reproduce fairly well the binding energies and rms radii of the deuteron, triton and alpha particle, and the Coulomb energy of He^3 . We should point out here that with a central potential containing no repulsive component, it is not possible to meet these requirements. It was shown by Irving⁴⁾ that a purely attractive central potential which fits the low-energy two-body scattering data yielded binding energies for triton and alpha particle far exceeding the experimental values.

With the two-body potential used in the calculation of Li^6 ¹⁾, the binding energy of the alpha particle has already been calculated by the present authors⁵⁾. In this investigation, we reconsider this problem with a better trial wave function⁶⁾, since it is our desire also to obtain a reliable estimate of the rms radius. Using a very similar potential, Kikuta, et al.⁷⁾ have computed the binding energy

of the triton. The trial wave function which they used has, however, rather limited flexibility. In a calculation on the binding energy of the hypertriton, we have found that the upper bound obtained with the type of trial function employed in their investigation has a value quite different from that achieved with a more flexible trial function⁸⁾.

The nuclear three- and four-body problems have also been examined by Mang and Wild with the independent pair model⁹⁾. The agreement between their calculated value and the experimental value of the binding energies was fairly good, but the two-body potential of a square-well shape used in their calculation was not too realistic. Also, their results do not have a high degree of accuracy. It was found by Folk¹⁰⁾ that with a simple Gaussian potential, the Mang-Wild method yields a result of -9.2 MeV as the energy of the triton, which is already larger than the upper bound of -9.74 MeV obtained by a variational calculation¹¹⁾. For this reason, we feel that in a systematic investigation on the properties of the light nuclei with a more realistic force, it is perhaps better to adopt a standard variational procedure rather than the method of Mang and Wild.

To complete the historical review, we should also mention here the calculation on the alpha particle with a hard core, two-body potential by Cohen and Willis¹²⁾. Unfortunately, this calculation contains a programming error which makes the results invalid¹³⁾.

The main part of this work will be concerned with the investigation of the properties of the deuteron, triton, He^3 and alpha particle with the potential used in the calculation of Li^6 ¹⁾, namely, a potential containing a hard core of radius 0.35fm. However, to see if there is any freedom in the choice of the radius of the hard core to meet the requirements specified in the previous paragraph, we shall also compute with two-body potentials of hard-core radii

equal to 0.2, 0.4 and 0.6fm. For this latter purpose, the potentials used by Kikuta, et al.⁷⁾ will be adopted.

In the next section, the variational method to be used in this investigation will be briefly outlined. Both the upper and lower bound to the eigenvalue will be computed. For the evaluation of the various expectation values, we shall use a Monte-Carlo method. As this method has already been discussed in detail elsewhere¹⁴⁾, it will not be further described here.

The two-body potentials and the form of the trial wave function will be discussed in sect. 3. The potential used in the calculation of Li^6 ¹⁾ will be called type A, while the potentials of Kikuta, et al.⁷⁾ will be called type B. These two types are quite similar to each other. Both have a hard core and an attractive part of an exponential shape. They also fit the low energy nucleon-nucleon scattering data equally well. The only difference is that type A favors the fit to the proton-proton data, while type B favors the fit to the neutron-proton data. For the trial wave function, we adopt a form originally proposed by Austern and Iano¹⁵⁾. In this form, the trial function is written as a product of functions, each depending individually on the interparticle distance. For each of these functions, the solution of the two-body Schrödinger equation is used up to a certain inter-nucleon separation, which is then connected to a variational function for larger distances.

The results obtained with potential A and B will be separately discussed in sects. 4 and 5. In sect. 6, we shall give a review of the results and a discussion of the conclusions.

In the appendix, a brief study on the adequacy of the trial wave function will be presented. From this study, it will be possible to gain some information about the regions in the configuration space where the trial function differs considerably from the eigenfunction.

2. Variational Principles

The upper bound will be computed by the usual Rayleigh-Ritz method, i.e.,

$$E_0 \leq E_U = \langle \mathcal{H} \rangle, \quad (1)$$

where E_0 is the lowest eigenvalue of the Hamiltonian. For the lower bound, we adopt a method of Temple¹⁶⁾ which gives

$$E_0 \geq E_L = \langle \mathcal{H} \rangle - \frac{\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2}{E_1 - \langle \mathcal{H} \rangle}, \quad (2)$$

subject to the condition that $\langle \mathcal{H} \rangle$ is less than E_1 , the energy of the first excited state having the same symmetry property as the ground state. In the case of the triton, E_1 is equal to the energy of the deuteron, which may be obtained quite easily by a numerical solution of the two-body Schrödinger equation. For the alpha particle, one should in principle use the ground-state eigenvalue of the triton. Although this latter quantity is not rigorously known, a good approximation to it may be obtained from a study of the upper and lower bound of the energy of the triton. Since, in this particular case, $E_1 - \langle \mathcal{H} \rangle$ has a rather large value (around 20 MeV), such an approximation for E_1 will be quite sufficient.

The lower-bound method of Temple has been applied extensively in the calculation of the ground-state energy of the helium atom¹⁷⁾. There it was found that the lower bound produced by this method lies much farther from the eigenvalue than the upper bound does. The quantity η defined by

$$\eta = \frac{E_0 - E_L}{E_U - E_0}, \quad (3)$$

has, in this case, a large value around 50. Similar experience has also been gained in the case of the deuteron⁵⁾. The question thus arises whether there is a better lower bound than the one given by

the formula of Temple. As we shall see, there is no better one if $\langle \mathcal{H} \rangle$, $\langle \mathcal{H}^2 \rangle$ and E_1 are the only available information about the system. In principle, the quantity η can be equal to zero, although this possibility can be excluded in most instances. Usually, some additional information about the system is available in the form of qualitative arguments which can be used to get an estimate of the quantity η which, in general, turns out to be rather large. Such an estimation of η will be very helpful in our calculation, since with hard-core potential, the gap between the upper and lower bound will be quite large such that the result would not be too useful unless one knows that η is a large number.

In order to estimate η , we first define another quantity ϵ from the equation

$$E_0 = \langle \mathcal{H} \rangle - \frac{\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2}{\epsilon - \langle \mathcal{H} \rangle} \quad (4)$$

In terms of this quantity ϵ , the value of η is

$$\eta = \frac{\epsilon - \langle \mathcal{H} \rangle}{E_1 - \langle \mathcal{H} \rangle} - 1 \quad (5)$$

To get an expression for ϵ , we expand the trial function Ψ in terms of the eigenfunctions Φ_n of the Hamiltonian $\mathcal{H}$, i.e.,

$$\Psi = \sqrt{1 - \sum_{n=1}^{\infty} |a_n|^2} \Phi_0 + \sum_{n=1}^{\infty} a_n \Phi_n \quad (6)$$

With this expansion, we get

$$\langle \mathcal{H} \rangle = E_0 + \sum_{n=1}^{\infty} |a_n|^2 (E_n - E_0) \quad (7)$$

and

$$\langle \mathcal{H}^2 \rangle = E_0^2 + \sum_{n=1}^{\infty} |a_n|^2 (E_n^2 - E_0^2) \quad (8)$$

Inserting eqs. (7) and (8) into eq. (4), we obtain

$$\epsilon = \frac{\sum_{n=1}^{\infty} |a_n|^2 (E_n - E_0) E_n}{\sum_{n=1}^{\infty} |a_n|^2 (E_n - E_0)} \quad (9)$$

As is seen, ϵ has the form of an average excitation energy of the eigenfunctions of the excited states which the trial function mixes into the ground-state eigenfunction; this average is taken with the excitation energies $(E_n - E_0)$ as weighting factors, which causes ϵ to become rather large in most cases.

The smallest possible value of ϵ is E_1 , which occurs when Φ_1 is the only function by which the trial function ψ differs from the ground-state eigenfunction. In this particular instance, η is equal to zero and E_L is equal to E_0 . In our case, since the inadequacy of the trial function will be mainly in the short range behavior, the possibility of η equal to zero can be safely excluded.

It should be noted that because of the appearance of the normalizing denominator, ϵ depends only on the relative magnitudes of the expansion coefficients a_n . Thus, when the gap between the upper and lower bound becomes smaller during the variational process, the quantity ϵ and accordingly, η does not necessarily have to become smaller. In fact, they even tend to become larger, since it is usually easier to reduce the magnitudes of the lower-order expansion coefficients than those of the higher-order coefficients. In our calculation with hard-core potentials, for instance, the behavior of the short-range three-nucleon correlations will not be described too well by our type of trial function, which means that expansion coefficients of rather high order will have an important influence on ϵ .

From eqs. (4) and (9), Temple's formula can be easily derived. We would like to point out that in the cases where Temple's formula cannot be applied, for instance, when $\langle H \rangle > E_1$, eqs. (4) and (9) can

still be used to estimate E_0 .

In the appendix, we shall estimate ϵ approximately using the information gained about the adequacy of the trial wave function and calculate η with eq. (5). In this way, it will be possible to get an estimate of the eigenvalue by eq. (3) using the values of the upper and lower bound obtained in this calculation.

3. Two-Body Potentials and Trial Wave Function

The nuclear two-body potentials to be used are assumed to be charge-independent and have the form

$$V_{ik} = \frac{1 + P_{ik}^{\sigma}}{2} V_t(r_{ik}) + \frac{1 - P_{ik}^{\sigma}}{2} V_s(r_{ik}), \quad (10)$$

where P_{ik}^{σ} is the spin-exchange operator. The quantities $V_t(r_{ik})$ and $V_s(r_{ik})$ are the triplet and singlet potentials in the even states and are chosen to be of the following exponential type:

$$\begin{aligned} V_t(r) &= \infty, & (r < r_c) \\ &= -V_{0t} \exp[-\kappa_t(r - r_c)], & (r > r_c) \\ V_s(r) &= \infty, & (r < r_c) \\ &= -V_{0s} \exp[-\kappa_s(r - r_c)], & (r > r_c) \end{aligned} \quad (11)$$

The potential in the odd states does not need to be specified, since, in the ^{is} investigation, we shall use the approximation of a totally space-symmetric trial wave function to calculate the upper and lower bound. As is mentioned in the introduction, two types of potentials will be considered. In Table 1, we list the values of the parameters.

Potential A is the potential used in the calculation of Li^6 . The value of 0.35fm for r_c is chosen so that this potential fits also the p-p 1S_0 phase shifts up to about 300 MeV. Potentials B are the potentials used by Kikuta, et al. in the investigation of the binding energy of the triton. The fit of these potentials to the

n-p triplet, n-p singlet and p-p singlet effective range data is shown in Table 2. In this table, the experimental values are taken from the compilation of Gamel and Thaler¹⁸⁾, and of MacGregor, et al.¹⁹⁾.

From Table 2, it is seen that potential A sacrifices some fit to the n-p singlet data for the fit to the p-p singlet data and vice versa for potential B. This is necessary, since it is known that the nuclear forces are not strictly charge-independent. If we interpolate from Table 1 to get a two-body potential of type B with r_c equal to 0.35fm, this potential will give a larger value for the binding energy of the triton and the alpha particle than potential A. From the results to be given in sects. 4 and 5, we shall be able to get a quantitative estimate of this difference.

For the trial wave function, we use, for simplicity, a totally space-symmetric function of the form

$$\Psi = \psi \chi, \quad (12)$$

with ψ and χ being the spatial and the charge-spin function, respectively. The function ψ will be chosen as

$$\psi = \prod_{i,j=1}^N f(r_{ij}), \quad (13)$$

where N represents the total number of particles, i.e., three for the triton and four for the alpha particle. Due to the spin-dependence of the two-body potential, the eigenfunction is, of course, not totally space-symmetric, but contains also less symmetric parts. We can estimate the amount of binding energy which is lost due to this simplifying assumption by comparing the lower bound achieved with the actual two-body potential with that achieved with an effective two-body potential[†],

[†] With the assumption of a totally space-symmetric trial function, the upper bounds obtained using the actual two-body potential and the effective two-body potential are the same.

$$V_e(r) = \frac{1}{2} [V_t(r) + V_s(r)], \quad (14)$$

which has no spin-dependence. This will be discussed below.

The Hamiltonian of the system has the form

$$\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^N \nabla_i^2 + \sum_{i>j=1}^N V_{ij} + \sum_{i>j=1}^N V_c(r_{ij}) \epsilon_{ij}, \quad (15)$$

where ϵ_{ij} is equal to one if i and j are protons and zero otherwise. The Coulomb potential V_c is chosen as

$$V_c(r) = \frac{e^2}{r} \left[1 - \left(1 + \frac{11\lambda r}{16} + \frac{3\lambda^2 r^2}{16} + \frac{\lambda^3 r^3}{48} \right) \exp(-\lambda r) \right], \quad (16)$$

which is the potential between two protons of exponential charge distribution²⁰⁾. The constant λ is related to the rms radius of the proton; for a rms radius of 0.8fm ²¹⁾, λ is equal to 4.32fm^{-1} .

After summing over the spin and isobaric spin coordinates, the expectation value of the Hamiltonian $\mathcal{H}$ and of the square of the Hamiltonian $\mathcal{H}^2$ have the following expressions:

$$\langle \mathcal{H} \rangle = \frac{1}{\mathcal{N}} \int \psi^* \mathcal{H} \psi d\tau, \quad (17)$$

$$\langle \mathcal{H}^2 \rangle = \frac{1}{\mathcal{N}} \left[\int \psi^* \mathcal{H}^2 \psi d\tau + Q_N \right], \quad (18)$$

where $\mathcal{N}$ is a normalization factor and has the form

$$\mathcal{N} = \int \psi^* \psi d\tau. \quad (19)$$

In eqs. (17) and (18), the definitions are

$$\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^N \nabla_i^2 + \sum_{i>j=1}^N [V_e(r_{ij}) + V_c(r_{ij}) \epsilon_{ij}], \quad (20)$$

$$Q_3 = \int \psi^* [3f^2(r_{12}) - 3f(r_{12})f(r_{13})] \psi d\tau, \quad (21)$$

and

$$Q_4 = \int \psi^* [6f^2(r_{12}) - 12f(r_{12})f(r_{13}) + 6f(r_{12})f(r_{34})] \psi d\tau, \quad (22)$$

with

$$g(r) = \frac{1}{2} [V_t(r) - V_s(r)] \quad (23)$$

From eqs. (21)-(23), it is seen that the term Q_N enters into eq. (18) as a consequence of the spin-dependence of the two-body potential. The lower bound calculated with this value of $\langle \mathcal{H}^2 \rangle$, therefore, reflects also the inadequacy of our trial wave function in its omission of the less symmetric parts which are present in the eigenfunction.

To assess how good our trial wave function is against the best possible form of a space-symmetric trial function, we shall also compute the lower bound with the potential $V_e(r)$ acting between all pairs of the particles[†]. In this case, the expression for the lower bound is

$$E_{oe} \geq E_{Le} = \langle \mathcal{H} \rangle - \frac{\langle \mathcal{H}^2 \rangle_e - \langle \mathcal{H} \rangle^2}{E_{Le} - \langle \mathcal{H} \rangle}, \quad (24)$$

with $\langle \mathcal{H}^2 \rangle_e$ given by

$$\langle \mathcal{H}^2 \rangle_e = \frac{1}{N} \int \psi^* H^2 \psi d\tau, \quad (25)$$

and the subscript e denoting that these quantities are appropriate to the system with the effective potential V_e .

With the upper bound E_U and the lower bound E_{Le} , we can also make an approximate determination of the eigenvalue E_{oe} . For this purpose, it will be necessary to have the values of η_e and ϵ_e which are defined in an analogous way as η and ϵ in eqs. (3) and (9). In the appendix, we shall make a brief study of the behavior of $H\psi/\psi$ in the configuration space. From this study, ϵ_e can be estimated.

For the function $f(r)$, we use a form originally proposed by Austern and Iano,¹⁵⁾ i.e.,

[†] Since the potential $V_e(r)$ does not possess any spin-dependence, the ground state eigenfunction is totally space-symmetric.

$$f(r) = u(r)/r, \quad (r < d)$$

$$= A r^{-\frac{1}{(N-1)}} [\exp(-\alpha r) + B \exp(-\beta r)], \quad (r > d) \quad (26)$$

where $u(r)$ is a solution of the equation

$$-\frac{\hbar^2}{m} \frac{d^2 u}{dr^2} + [V_e(r) - e] u = 0 \quad (27)$$

The constants A and B are adjusted such that the function $f(r)$ and its first derivative are continuous at the separation distance d . There are a total number of four variational parameters in this function, namely, e , d , α and β .

The factor $r^{1/(N-1)}$ in the function $f(r)$ is chosen such that the behavior in some parts of the asymptotic region is given properly. For instance, when one nucleon is far away from the other nucleons, the wave function ψ takes on the asymptotic form

$$\psi \sim \left[\prod_{i,j=1}^{N-1} f(r_{ij}) \right] \exp[-(N-1)\alpha R] / R, \quad (28)$$

where R is the distance from this nucleon to the center of mass of the rest of the system. The parameter α is thus related to the separation energy of the nucleon. If we have some idea about the magnitude of this separation energy, a fairly good guess for the value of α to start the variational process can then be obtained.

The type of trial function given here has been previously used to calculate the binding energy of a three-body system with a purely attractive two-body potential of the form

$$V_t(r) = V_s(r) = -V_0 \exp(-\kappa r), \quad (29)$$

with $V_0 = 96.95$ MeV and $\kappa = 1.156 \text{ fm}^{-1}$ (11). There, it was found that the upper and lower bound are rather close to each other, being

-7.65 and -7.84 MeV, respectively. This shows that when the calculation is done with the effective potential, the upper bound yielded by the type of space-symmetric trial function used here can be remarkably close to the eigenvalue E_{oe} . In this particular case, we believe that the difference between the upper bound and the eigenvalue is less than 0.05 MeV.

As is mentioned in the introduction, all the integrals which arise from the computation of the expectation value of the Hamiltonian and of the square of the Hamiltonian are done by a Monte-Carlo method which has been discussed in detail in our previous publications^{5, 14}). For the numerical calculation, an IBM-7094 computer was used. To insure a high degree of accuracy, double-precision arithmetic has been used whenever it was felt necessary.

4. Calculation with Potential A

4.1. Results for H^3 and He^3

With potential A, the results obtained for H^3 with the optimum parameters of the upper and lower bound, respectively, are as follows:

$$\begin{aligned}
 \text{upper bound: } & \alpha = 0.275 \text{ fm}^{-1}, \quad \beta = 3.0 \text{ fm}^{-1}, \\
 & d = 1.2 \text{ fm}, \quad e = -6.0 \text{ MeV}, \\
 & E_U = -7.63 \pm 0.05 \text{ MeV}, \\
 & E_L = -60.14 \pm 2.57 \text{ MeV}, \\
 & E_{Le} = -17.24 \pm 1.72 \text{ MeV}, \\
 & \text{rms radius} = 1.62 \text{ fm}, \\
 & \text{number of estimates} = 50,000, \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 \text{lower bound: } & \alpha = 0.245 \text{ fm}^{-1}, \quad \beta = 2.8 \text{ fm}^{-1}, \\
 & d = 1.3 \text{ fm}, \quad e = -4.8 \text{ MeV}, \\
 & E_U = -7.36 \pm 0.05 \text{ MeV}, \\
 & E_L = -54.37 \pm 1.95 \text{ MeV}, \\
 & E_{Le} = -14.52 \pm 1.20 \text{ MeV}, \\
 & \text{rms radius} = 1.77 \text{ fm}, \\
 & \text{number of estimates} = 60,000. \quad (31)
 \end{aligned}$$

In eqs. (30) and (31), the lower bounds are calculated with $E_1 = -2.23$ MeV and $E_{1e} = -0.24$ MeV, which are the eigenvalues of the corresponding two-body problem with the potentials V_t and V_e , respectively.

The value of Δ and Δ_e defined as

$$\begin{aligned}\Delta &= (E_U)_{\min} - (E_L)_{\max}, \\ \Delta_e &= (E_U)_{\min} - (E_{Le})_{\max},\end{aligned}\tag{32}$$

is equal to 46.74 and 6.89 MeV, respectively. Since these values are rather large comparing with the absolute value of $(E_U)_{\min}$, the statement that the eigenvalue lies between the upper and lower bound is not a very useful one. Hence, we must resort to the use of the quantities η and η_e to get an estimate of the value of E_0 and E_{oe} . Using $\eta = \eta_e = 100$ as discussed in the appendix, we get with the values of E_U , E_L and E_{Le} of eq. (30)[†]

$$\begin{aligned}E_0 &= -8.15 \pm 0.30 \text{ MeV}, \\ E_{oe} &= -7.72 \pm 0.10 \text{ MeV},\end{aligned}\tag{33}$$

where the error quoted is that due to the uncertainty in the magnitude of η or η_e and the statistical uncertainty arising from the Monte-Carlo calculation. It is necessary to assign a large error to E_0 , since we have not specified the nature of the two-body potential in the odd orbital-angular-momentum states, which alone can cause an uncertainty of about 0.1 MeV in E_0 ¹¹⁾.

From the values of E_U , E_{Le} and E_{oe} , we can also get a measure of the accuracy of the trial function Ψ . It is easy to show that the overlapping integral satisfies the inequality

[†] When η or η_e is used to estimate the eigenvalue, the value of the upper and lower bound obtained with the same set of variational parameters must be employed.

$$\langle \Psi | \Phi_{oe} \rangle \geq \left[(E_{le} - E_0) / (E_{le} - E_{oe}) \right]^{1/2}, \quad (34)$$

where Φ_{oe} is the ground-state eigenfunction of the three-body system with the two-body interaction given by $V_e(r)$. Thus, using eqs. (30) and (33), we obtain

$$\langle \Psi | \Phi_{oe} \rangle \gtrsim 99.4 \% , \quad (35)$$

which clearly demonstrates that the function Ψ with the optimum parameters of the upper bound is a good representation of the eigenfunction Φ_{oe} .

Also, we wish to point out that our calculation is internally consistent to a significant degree. From eq. (28), it is seen that if Ψ resembles Φ_{oe} closely, the variationally determined value of α should satisfy approximately the relation

$$\alpha = \left[\frac{2m}{N(N-1)\hbar^2} (E_{le} - E_{oe}) \right]^{1/2}, \quad (36)$$

where $(E_{le} - E_{oe})$ represents the separation energy of a single nucleon from the rest of the system. Using E_{oe} of eq. (33), the right hand side of eq. (36) is equal to 0.25fm^{-1} which is only 10% smaller than the optimum value of α at the upper bound.

The Coulomb energy of He^3 is calculated by a perturbation method, using the potential $V_c(r)$ of eq. (16) and the optimum wave function of H^3 at the upper bound. The value obtained is 0.71 MeV, which is 0.04 MeV smaller than the value which would be obtained if the proton were treated as a point charge.

4.2. Results for He^4

The results obtained for He^4 with the optimum parameters of the upper and lower bound, respectively, are as follows:

$$\begin{aligned}
 \text{upper bound: } \alpha &= 0.310 \text{ fm}^{-1} , & \beta &= 3.2 \text{ fm}^{-1} , \\
 d &= 1.2 \text{ fm} , & e &= -16.0 \text{ MeV} , \\
 E_U &= -30.28 \pm 0.16 \text{ MeV} , \\
 E_L &= -77.62 \pm 1.90 \text{ MeV} , \\
 E_{Le} &= -54.20 \pm 1.79 \text{ MeV} , \\
 \text{rms radius} &= 1.39 \text{ fm} , \\
 \text{number of estimates} &= 45,000 , & (37)
 \end{aligned}$$

$$\begin{aligned}
 \text{lower bound: } \alpha &= 0.285 \text{ fm}^{-1} , & \beta &= 2.6 \text{ fm}^{-1} , \\
 d &= 1.1 \text{ fm} , & e &= -14.0 \text{ MeV} , \\
 E_U &= -29.73 \pm 0.18 \text{ MeV} , \\
 E_L &= -69.53 \pm 1.90 \text{ MeV} , \\
 E_{Le} &= -46.62 \pm 1.70 \text{ MeV} , \\
 \text{rms radius} &= 1.47 \text{ fm} , \\
 \text{number of estimates} &= 35,000 . & (38)
 \end{aligned}$$

In the calculation of the lower bounds, E_L and E_{Le} are taken as -8.15 and -7.72 MeV, respectively, which are the estimated eigenvalues for the triton given in eq. (33).

The values of Δ and Δ_e , being 39.25 and 16.34 MeV, are again too large to be of significant use. Hence, we have to resort to the use of η and η_e in this case also to get an estimate of E_o and E_{oe} . Using $\eta = \eta_e = 50$ as estimated in the appendix, we obtain

$$\begin{aligned}
 E_o &= -31.23 \pm 0.60 \text{ MeV} , \\
 E_{oe} &= -30.76 \pm 0.40 \text{ MeV} , & (39)
 \end{aligned}$$

where a generous allowance in the uncertainty of η and η_e has been made to get the quoted error.

The results obtained here can be compared with those of a previous investigation⁵⁾ where E_U and E_L were found after 60,000 estimates to be -28.76 ± 0.16 and -118.68 ± 2.60 MeV with the optimum parameters of the upper bound[†]. The improvement is 1.52 and 41.06 MeV for E_U and E_L , respectively. The ratio of these improvements is 27 which is somewhat smaller than the value of η used in this calculation. This is, however, to be expected, since the main difference between the trial function used here and that in the previous investigation⁵⁾ is in the long range part⁶⁾. As the major defect of the present trial function is in the region where the particles are close to each other (see the appendix), we should anticipate a value of η greater than 27. This constitutes, therefore, some additional support for using η equal to about 50 in this calculation.

The overlapping integral $\langle \Psi | \Phi_{oe} \rangle$, with Φ_{oe} being the eigenfunction of the four-body system with the two-body potential $V_e(r)$, can also be calculated from eq. (34). This turns out to be larger than 0.99, which indicates that the function Ψ is a good approximation to the eigenfunction Φ_{oe} .

As in the triton case, the self-consistency condition (36) is quite well satisfied. The value of α calculated from eq. (36) is 0.304 fm^{-1} which is very close to the value of 0.310 fm^{-1} obtained from the variational calculation.

4.3. Comparison with Experiment

The results of this calculation are compared with the experimental data in Table 3. In this table, the symbols E_{ot} , $(\text{rms})_t$, E_c , E_{α} and $(\text{rms})_{\alpha}$ represent the ground-state energy of the triton, the rms radius of the nucleon distribution in the triton, the Coulomb energy of He^3 , the ground-state energy of the alpha particle and the rms radius of the nucleon distribution in the alpha particle, respectively. The experimental value for the rms radius of

[†] In reference 5, the results quoted were obtained with only 4,000 estimates in the Monte-Carlo calculation.

the triton is obtained from the bare form factor tabulated by Srivastava²²⁾ for the three-nucleon system using the experimental data of Collard, et al.²³⁾ and an expression derived by Schiff²⁴⁾. From this table, it is seen that the results obtained with potential A compare favorably with those from experimental analyses. This supports the validity of the result which we obtained for Li^6 and justifies the use of this two-body potential in the future investigation on the nuclear properties of other light nuclei.

As is mentioned in sect. 3, potential A fits the p-p singlet data better than the n-p singlet data. If we had used for the n-p pairs in the triton and the alpha particle a potential which has the same core radius as potential A but a better fit to the n-p singlet data, E_{ot} and $E_{o\alpha}$ would be decreased by about 0.13 and 0.32 MeV, respectively[†]. Since these corrections are rather small compared with the uncertainties in E_{ot} and $E_{o\alpha}$ listed in Table 3, we feel that it is not necessary to take this extra complication into account in any future calculation on the binding energies of the light nuclei.

5. Calculation with Potential B

5.1. Results for H^3 and He^3

The results obtained for H^3 with potential B of core radii equal to 0.2, 0.4 and 0.6 fm are given in Table 4. Only the optimum parameters for the upper bound have been determined, since we learn from the results obtained with potential A that the quantities Δ and Δ_e have rather large values when the calculation is performed with a hard-core potential using the type of space-symmetric trial wave function employed in this investigation.

In the calculation for E_L and E_{Le} , the value of E_1 used is -2.23 MeV, corresponding to the energy of the deuteron, and the value of

[†] These estimates are obtained by a comparison between the results of potential A and potential B.

E_{le} used is -0.34, -0.28 and -0.19 MeV for the case with r_c equal to 0.2, 0.4 and 0.6fm, respectively.

Using the same two-body potential, Ohmura⁷⁾ has obtained -8.81, -7.52 and -5.79 MeV as the upper bound of H^3 for the cases with $r_c = 0.2, 0.4$ and 0.6fm , respectively. Comparing with the value of E_U listed in the table, we find that there is an improvement of 0.21, 0.26 and 0.48 MeV when our trial wave function is used. Also, the value of E_U for the case with r_c equal to 0.4fm can be compared with that obtained by Müller²⁵⁾. With a four-parameter trial function, he obtained -7.60 MeV which is 0.18 MeV higher than our value. This indicates that with the same number of variational parameters, our type of trial function is more efficient than the type employed in the calculation of Müller.

To get the values of E_0 and E_{oe} given in the table, we have used $\eta = \eta_e = 70, 110$ and 180 for the three cases with $r_c = 0.2, 0.4$ and 0.6fm .

The self-consistency condition is also well satisfied. The value of α computed from eq. (36) is $0.280, 0.248$ and 0.227fm^{-1} for r_c equal to $0.2, 0.4$ and 0.6fm , which is rather close to the variationally determined value of α listed in Table 4.

The Coulomb energy of He^3 computed for the case with $r_c = 0.2, 0.4$ and 0.6fm is equal to $0.76, 0.70$ and 0.64 MeV, respectively. If the proton were treated as a point charge, the corresponding value would be increased to $0.83, 0.73$ and 0.66 MeV.

5.2. Results for He^4

The results obtained for He^4 are given in Table 5. In this table, E_L and E_{Le} are calculated with E_1 and E_{le} set equal to the values of E_0 and E_{oe} of H^3 given in Table 4. For the calculation of E_0 and E_{oe} of the alpha particle, $\eta = \eta_e = 35, 55$ and 90 for the cases with $r_c = 0.2, 0.4$ and 0.6fm have been used.

Here again, we observe that the self-consistency condition is nicely fulfilled. The value of α computed from eq. (36) is 0.330, 0.300 and 0.271fm^{-1} for the case with $r_c = 0.2, 0.4$ and 0.6fm , which is nearly equal to the variationally determined value of α listed in Table 5.

5.3. Comparison with Experiment

The results obtained with potential B are compared with the experimental data in Table 6[†]. From this table, it is seen that reasonable agreement with experiment can be obtained if the two-body potential contains a hard core of radius in the range from 0.35 to 0.55fm . The best fit is obtained when the radius of the core is equal to 0.45fm , at which value the average deviation from the experimental values is only about 5%.

From Table 6, it can also be deduced that with a potential containing no hard core, the agreement with the experimental data will be very poor. In fact, already for the case with r_c equal to 0.2fm , the binding energy of the alpha particle exceeds the experimental value by more than 25%.

6. Conclusions

The results of this investigation show that with the hard-core potential of radius 0.35fm used in the calculation of Li^6 ¹⁾, it is also possible to get reasonable agreement with the experimentally determined values of the binding energy and rms radius of the triton and the alpha particle, and the Coulomb energy of He^3 . This means that it will now be a worthwhile project to investigate systematically the nuclear properties of other p-shell nuclei with this potential. Using the Monte-Carlo technique of this calculation, we estimate that with the computers presently available, it will be feasible to go on to a nucleus in the neighborhood of C^{12} .

[†] If we had used for the like-nucleon pairs in H^3 and He^4 a potential which fits better the p-p singlet data, E_{0t} and $E_{0\alpha}$ would be increased by about 0.28 and 0.68 MeV, respectively.

We would like to point out that the results which will be obtained with this hard-core potential in the calculation on the p-shell nuclei will be more reliable than those obtained previously using the cluster model²⁶⁾. This is so, since in these previous calculations, it was necessary, due to the use of a non-saturating two-body potential, to take into account the compressibility of the deuteron-, triton- and alpha-cluster in a way which was not too satisfactory. With the hard-core potential used here, this unsatisfactory feature can be completely avoided.

To see if there is any freedom in the choice of the radius of the hard core to satisfy the requirement specified in the above paragraph, we have also calculated with core radii equal to 0.2, 0.4 and 0.6fm. The results indicate that as long as the core radius is within 0.35 and 0.55fm, reasonable agreement with experiment can be obtained. This is interesting, since the phenomenological hard-core nucleon-nucleon potentials obtained from analyses of the two-nucleon scattering experiments all have a core radius lying within this range.

Finally, we should mention that relativistic corrections have not been considered in this calculation. If they are included, our conclusion about the acceptable range of hard-core radii may have to be slightly modified.

* * *

The authors wish to thank Dr. C. E. Porter for his interest in this work.

Appendix

Study of Trial Wave Function

In this appendix, we make a brief study of the behavior of the function $H\psi/\psi$ to gain some information about the regions in the configuration space where the trial function differs appreciably from the eigenfunction Φ_{oe} . From this study, it will be possible to make a crude estimate of the magnitude of the quantity ϵ_e defined in sect. 3. For the sake of brevity, only the study on the triton problem will be presented.

In fig. A1, $H\psi/\psi$ is plotted against $(r_{12} + r_{13} + r_{23})$ for the three-body case with potential A using the optimum parameters given by eq. (30). The meaning of this graph is that over 95% of the points with $(r_{12} + r_{13} + r_{23})$ greater than 4 fm lie within the blocked area. For $(r_{12} + r_{13} + r_{23})$ smaller than 4 fm, the points are widely scattered, with a majority taking on a positive value for $H\psi/\psi$. Also, the shaded area represents the region in which points with at least one of the r_{ij} 's having a value within about 0.1 fm of the matching distance d are located. The dot in each block represents the weighted average of $H\psi/\psi$ of all the points lying within that particular block. The dashed line gives the position of the upper bound, $(E_U)_{\min}$, given in sect. 4.

From fig. A1, it is seen that there are at least two regions in the configuration space where our trial function with the optimum parameters does not give a proper description of the ground-state eigenfunction. These two regions are:

- (i) the region where points with at least one of the r_{ij} 's having a value within about 0.1 fm of the matching distance d are located,

- (ii) the region that includes these points which describe the close-packing of three particles. From our brief study, it may be concluded that in this particular case, our trial function does not resemble the eigenfunction too closely when all three particles are within a distance of about 1.4 fm to each other.

To see if there is any other region where our wave function might also be improper, we have studied, for the sake of comparison, the three-body case where the two-nucleon potential is given by eq. (29) of sect. 3. The reason why we choose this particular example is that in this case, we believe the upper bound of $-7.65 \text{ MeV}^{11)}$ obtained with a trial function of the type described by eqs. (13) and (26) is very close to the eigenvalue and hence, the study of $H\psi/\psi$ can give us a clear conception of how $H\psi/\psi$ should behave as a function of $(r_{12} + r_{13} + r_{23})$ in the case where the trial function describes the eigenfunction properly.

The behavior of $H\psi/\psi$ for the above-mentioned example is depicted in fig. A2. To obtain this figure, we have used the optimum parameters of the upper bound given in a previous publication¹¹⁾. We note that in this case, the close resemblance between the trial function and the eigenfunction is demonstrated by the fact that all of the points take on a value of $H\psi/\psi$ in the narrow range of -5.0 and -10.0 MeV , and the dots are very close to the line representing the position of $(E_U)_{\min}$, which in our opinion, is almost equal to the eigenvalue E_{oe} .

We would like to point out that the behavior of $H\psi/\psi$ is a rather sensitive indicator of the adequacy of the trial wave function. For example, if the trial function

$$\psi = \exp [-\lambda (r_{12} + r_{13} + r_{23})], \quad (\text{A1})$$

with $\lambda = 0.39 \text{ fm}^{-1}$ is used, the behavior of $H\psi/\psi$ as shown in fig. A3

looks much worse[†]. This is rather striking, since the upper bound obtained with this trial function (-7.47 MeV) is only 0.18 MeV higher than that obtained with the more elaborate trial function which yields fig. A2.

By comparing fig. A1 with figs. A2 and A3, we believe that, besides the two regions mentioned above, there is no other region in the configuration space where our trial function for the case with potential A can be sufficiently different from the eigenfunction as to produce a possibly bad upper bound.

A.1. Improving the Trial Function Near the Matching Distance

In the calculation with potential A and B, we have used a trial function which contains the function $f(r)$ given by eqs. (26) and (27). This function has a continuous first derivative at the matching distance d , but not necessarily a continuous second derivative. To see what the influence this discontinuity in the second derivative has on the value of $(E_U)_{\min}$ and $(E_{Le})_{\max}$, we shall now perform a calculation with a more flexible function $g(r)$ given by

$$\begin{aligned} g(r) &= u(r)/r, & (r < d) \\ &= Ar^{-\frac{1}{N-1}} [\exp(-\alpha r) + B \exp(-\beta r) + C \exp(-\gamma r)], & (r > d) \end{aligned} \quad (A2)$$

where the constants A, B, C are adjusted such that the function $g(r)$ and its first and second derivative are continuous at the matching distance d . There are now a total number of five variational parameters, namely, α , β , γ , d and e .

[†] The fact that the dots deviate quite strongly from the line representing $(E_U)_{\min}$ indicates that this simple exponential function is not the best trial function with the functional form $f(r_{12}+r_{13}+r_{23})$. Indeed, Feshbach and Rubinow²⁷⁾ showed that the best trial function with the form $f(r_{12}+r_{13}+r_{23})$ would yield a value for $(E_U)_{\min}$ which is about 1% deeper than the value that is obtained with the exponential trial function.

For simplicity, we shall present here only the results for the triton problem with potential A. In this case, the values of $(E_U)_{\min}$ and $(E_{Le})_{\max}$ obtained with the function $g(r)$ are -7.64 ± 0.05 MeV and -14.22 ± 1.10 MeV, respectively. Comparing with the corresponding values of -7.63 ± 0.05 MeV and -14.52 ± 1.20 MeV obtained with the function $f(r)$, we note that the improvement in either $(E_U)_{\min}$ or $(E_{Le})_{\max}$ is so small compared to the statistical uncertainty as to have only minor significance[†]. The reason for this small improvement is as follows. During the variational procedure, the parameters α , β , d and e in $f(r)$ try to adjust themselves in such a way that the second derivative at the matching distance does not become overly discontinuous. This means that in the configuration space, the region where the trial function differs appreciably from the eigenfunction due to this source of discontinuous second derivative is, in this case, a rather small one.

From this study with the function $g(r)$, we conclude that it is quite sufficient to use $f(r)$ to calculate both the upper and the lower bound. This is an important point, since with our computational method, the amount of computing time saved would be considerable if there could be a fewer number of variational parameters.

A.2. Estimation of η_e and η

Having established that the relatively large gap found in our calculation is not a result of the discontinuous second derivative in the function $f(r)$, we are now confident that it is due to the improper behavior of the three-nucleon correlations contained in our trial wave function. To see how this latter source affects the

[†] When the two-body potential used is the exponential potential of eq. (29) or the Gaussian potential considered in the investigation of Baker, et al.²⁸⁾, we have reached the same conclusion that improvement in either $(E_U)_{\min}$ or $(E_{Le})_{\max}$ is quite insignificant.

upper and lower bound, we should, in principle, perform a calculation with a wave function improved in this respect. This is, however, not at all a simple matter. In a previous calculation with potential A on the binding energy of the alpha particle⁵⁾, we have found that with a multiplicative three-nucleon correlation function of the form

$$\prod_{\substack{i,k,l=1 \\ k,l \neq i \\ k > l}}^4 \left\{ 1 + \lambda_1 \exp[-\lambda_2 (r_{ik} + r_{il} - 2r_c)] \right\} \quad (A3)$$

in the trial wave function, only very small improvement in either $(E_U)_{\min}$ or $(E_{Le})_{\max}$ can be obtained²⁹⁾. The reason for this is not yet clear. At the present moment, we are not even certain whether it is due to a lack of a sufficient number of variational parameters or that the form of this three-nucleon function is entirely inadequate. In any case, it seems to us that before a major study on the nature of the three-nucleon correlations contained in the eigenfunction Φ_{oe} is undertaken, the next best course is to use the information gained here from the study of the function $H\psi/\psi$ to estimate the magnitude of the quantity ϵ_e and consequently, η_e . With η_e estimated, we can then determine the difference between the value of $(E_U)_{\min}$ found in our calculation and the eigenvalue E_{oe} .

For the estimation of ϵ_e , we use the information that our trial function does not describe the eigenfunction properly when all three nucleons are within a distance of 1.4 fm to each other. Approximately, we can then identify ϵ_e as the expectation value of the Hamiltonian for a wave function which has an appreciable value only when all three inter-nucleon distances are less than 1.4 fm. Even with this approximation, it is, however, still difficult to compute ϵ_e without making further simplification. In this study, we shall make the additional approximation of setting ϵ_e equal to three times the ground state eigenvalue of a two-body system in which the two nucleons interact through the potential $V_e(r)$ and there is a

boundary condition that the wave function vanishes at $r \geq 1.4$ fm. This latter eigenvalue is approximately equal to $\langle \phi | h | \phi \rangle$, where h is the two-body Hamiltonian and ϕ is a normalized wave function of the form

$$\phi = \frac{1}{[2\pi(\ell - r_c)]^{\frac{1}{2}}} \frac{\sin \pi \frac{r - r_c}{\ell - r_c}}{r}, \quad (r \leq 1.4 \text{ fm}) \quad (\text{A4})$$

with $\ell = 1.4$ fm. A simple computation yields a value of 250 MeV for this expression, which means that ϵ_e is approximately equal to 750 MeV. Using a formula for η_e analogous to eq. (5), we then get η_e to be approximately equal to 100. We realize, of course, that this way of estimating ϵ_e and η_e is quite crude. However, the gap between the upper and lower bound is still small enough such that even a large error in η_e would produce only a relatively small error in E_{oe} .

Using a similar analysis, we have also estimated η_e for the cases where potential B is used. These turn out to be equal to 70, 110 and 180 when the core radius is equal to 0.2, 0.4 and 0.6 fm, respectively.

The estimation of the magnitude of ϵ is even harder, since comparing with the eigenfunction Φ_0 , our trial function also lacks a part which is not totally space-symmetric. It is reasonable to expect, however, that ϵ is less than $\epsilon_e^\dagger$, since the less-symmetric part is not likely to have an excitation energy as large as 750 MeV²⁴⁾. This does not mean that η is less than η_e . Although the value of the numerator in eq. (5) is now smaller than that which enters into the corresponding equation for η_e , the value of the denominator also

[†] One should not expect, however, that ϵ is much less than ϵ_e . Due to the appearance of the weighting factor $(E_n - E_0)$ in eq. (9)^e, the major contribution to ϵ comes from the function which describes the close-packing of three nucleons.

becomes smaller. These two compensating factors probably lead to the result that η and η_e are almost equal. In this calculation, we shall, for simplicity, take η equal to η_e .

Similar analyses have also been performed on the alpha-particle problem. With potential A, η_e is found to be approximately equal to 50. With potential B, η_e is approximately equal to 35, 55 and 90 for the three cases with r_c equal to 0.2, 0.4 and 0.6 fm, respectively.

References

1. E. W. Schmid, Y. C. Tang and K. Wildermuth, Phys. Letters 7 (1963) 263
2. K. Wildermuth and T. Kanellopoulos, Nuclear Physics 7 (1958) 150;
Y. C. Tang, K. Wildermuth and L. D. Pearlstein, Phys. Rev. 123 (1961) 548
3. See, for example, B. P. Nigam, Revs. Modern Phys. 35 (1963) 117
4. J. Irving, Phil. Mag. 42 (1951) 338
5. E. W. Schmid, Y. C. Tang and R. C. Herndon, Nuclear Physics 42 (1963) 95
6. E. W. Schmid, Y. C. Tang and R. C. Herndon, Nuclear Physics 42 (1963) 247
7. T. Kikuta (Ohmura), M. Morita and M. Yamada, Progr. Theor. Phys. 15 (1956) 222; T. Ohmura, Progr. Theor. Phys. 22 (1959) 34
8. R. C. Herndon, Y. C. Tang and E. W. Schmid, Nuovo Cimento (to be published)
9. H. J. Mang and W. Wild, Z. Phys. 154 (1959) 182
10. R. Folk, Bull. Am. Phys. Soc. 8 (1963) 56
11. Y. C. Tang, R. C. Herndon and E. W. Schmid, Phys. Rev. 134 (1964) B743
12. L. Cohen and J. B. Willis, in Proc. London Conf. on nuclear forces and the few-nucleon problem (Pergamon Press, London 1960) Vol. 2, p. 399
13. L. Cohen (private communication); see, also, reference 5
14. E. W. Schmid, Nuclear Physics 32 (1962) 82; see, also, reference 5
15. N. Austern and P. Iano, Nuclear Physics 18 (1960) 672
16. G. Temple, Proc. Roy. Soc. 119 (1928) 276
17. T. Kinoshita, Phys. Rev. 105 (1957) 1490; 115 (1959) 366
18. J. L. Gammel and R. H. Thaler, Prog. in Elementary Particle and Cosmic Ray Physics 5 (1960) 99
19. M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Ann. Rev. Nuclear Sci. 10 (1960) 291

20. H. Ohmura and T. Ohmura, Phys. Rev. 128 (1962) 729
21. L. R. B. Elton, Nuclear Sizes (Oxford University Press, 1961) p. 19
22. B. K. Srivastava, Phys. Rev. 133 (1964) B545
23. H. Collard, R. Hofstadter, A. Johansson, R. Parks, M. Ryneveld, A. Walker, M. R. Yearian, R. B. Day and R. T. Wagner, Phys. Rev. Letters 11 (1963) 132
24. L. I. Schiff, Phys. Rev. 133 (1964) B802
25. D. Müller, Z. Phys. 171 (1963) 371
26. K. Wildermuth, Y. C. Tang, F. C. Khanna and L. D. Pearlstein, in Proc. Rutherford Jubilee Int. Conf. (Haywood and Co., London 1961) p. 259; Y. C. Tang, F. C. Khanna, R. C. Herndon and K. Wildermuth, Nuclear Physics 35 (1962) 421
27. H. Feshbach and S. I. Rubinow, Phys. Rev. 98 (1955) 188
28. G. A. Baker, Jr., J. L. Gammel, B. J. Hill and J. G. Wills, Phys. Rev. 125 (1962) 1754
29. E. W. Schmid, Y. C. Tang and R. C. Herndon (unpublished)

Table 1
Parameters of the two-body potentials

Potential type	r_c (fm)	V_{ot} (MeV)	K_t (fm ⁻¹)	V_{os} (MeV)	K_s (fm ⁻¹)
A	0.35	434.0	2.40	216.0	1.97
B	0.2	286.246	1.8946	155.046	1.6608
B	0.4	475.044	2.5214	235.414	2.0344
B	0.6	947.023	3.6765	397.307	2.6272

Table 2
Fit to low-energy data

	n-p triplet		n-p singlet		p-p singlet	
	scattering length(fm)	effective range(fm)	scattering length(fm)	effective range(fm)	scattering length(fm)	effective range(fm)
Potential A	5.38	1.70	-16.69	2.73	-7.70	2.61
Potential B	5.38	1.70	-23.69	2.70	-8.80	2.59
Experiment	5.39	1.704	-23.74	2.67	-7.68	2.65

Table 3
Comparison of results obtained using
potential A with experimental values

	Calculated with potential A	Experimental analysis
E_{ot} (MeV)	- 8.15±0.30	-8.48
$(rms)_t$ (fm)	1.62	1.70±0.10
E_c (MeV)	0.71	0.76
$E_{o\alpha}$ (MeV)	-31.23±0.60	-28.3
$(rms)_\alpha$ (fm)	1.39	1.44±0.07

Table 4

Results obtained for H^3 with potential B

r_c (fm)	α (fm ⁻¹)	β (fm ⁻¹)	d (fm)	e (MeV)	E_U^* (MeV)	E_L^* (MeV)	E_{Le}^* (MeV)	E_o (MeV)	E_{oe} (MeV)	rms radius (fm)
0.2	0.300	2.30	1.16	- 2.0	-9.02±0.04	- 28.16± 0.61	-10.93±0.33	-9.29±0.18	-9.05±0.05	1.51
0.4	0.272	3.35	1.20	- 8.0	-7.78±0.05	- 70.18± 3.08	-27.30±1.94	-8.35±0.33	-7.96±0.14	1.65
0.6	0.249	4.50	1.26	-15.4	-6.27±0.07	-253.80±14.50	-61.20±6.60	-7.64±0.76	-6.58±0.23	1.79

* The statistical accuracy is achieved with 60,000 estimates.

Table 5

Results obtained for He^4 with potential B

r_c (fm)	α (fm ⁻¹)	β (fm ⁻¹)	d (fm)	e (MeV)	E_U^* (MeV)	E_L^* (MeV)	E_{Le}^* (MeV)	E_o (MeV)	E_{oe} (MeV)	rms radius (fm)
0.2	0.345	2.80	1.16	-12.6	-35.93±0.16	- 54.48±1.05	- 44.83±0.97	-36.46±0.43	-36.18±0.29	1.25
0.4	0.315	3.40	1.20	-20.0	-29.75±0.18	- 96.59±2.80	- 65.14±2.46	-30.97±0.79	-30.39±0.50	1.40
0.6	0.285	3.80	1.26	-25.0	-23.88±0.21	-228.43±7.78	-113.88±5.60	-26.15±1.35	-24.88±0.71	1.55

* The statistical accuracy is achieved with 40,000 estimates.

Table 6

Comparison of results obtained using potential B with
experimental values

		Calculated with potential B			Experimental analysis
r_c	(fm)	0.2	0.4	0.6	
E_{ot}	(MeV)	- 9.29±0.18	- 8.35±0.33	- 7.64±0.76	- 8.48
$(rms)_t$	(fm)	1.51	1.65	1.79	1.70±0.10
E_c	(MeV)	0.76	0.70	0.64	0.76
$E_{o\alpha}$	(MeV)	-36.46±0.43	-30.97±0.79	-26.15±1.35	-28.3
$(rms)_\alpha$	(fm)	1.25	1.40	1.55	1.44±0.07

Captions of the figures:

fig. A1: $H\psi/\psi$ as a function of $(r_{12}+r_{13}+r_{23})$ for the three-body case with potential A.

fig. A2: $H\psi/\psi$ as a function of $(r_{12}+r_{13}+r_{23})$ for the three-body case with the exponential potential of eq. (29). The wave function ψ used has the form given by eqs. (13) and (26).

fig. A3: $H\psi/\psi$ as a function of $(r_{12}+r_{13}+r_{23})$ for the three-body case with the exponential potential of eq. (29). The wave function ψ used is the simple exponential function given by eq. (A1).

FIGURE 1a.

FIGURE 2a.

FIGURE 3 a.