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CALCULATIONS FOR ZPR-VII
UNIFORM LATTICE CRITICAL EXPERIMENTS
WITH THORIA-URANIA FUEL IN HEAVY WATER

by

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CALCULATIONS FOR ZPR-VII UNIFORM LATTICE CRITICAL EXPERIMENTS WITH THORIA-URANIA FUEL IN HEAVY WATER

by

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ABSTRACT

Calculations are reported relating to the uniform lattice critical experiments performed in the ZPR-VII critical assembly with THUD or Babcock and Wilcox thoria-urania fuel and heavy water as moderator. Constants appearing in the criticality equation and material bucklings were computed for lattice pitches used in the experiments. Reflector savings were calculated and used in conjunction with experimental critical dimensions to calculate geometric bucklings. Analyses were carried out for various experiments, such as measurements of U^{235} and thorium cadmium ratios, water worth, and fuel-substitution experiments. Theoretical and experimental values are compared whenever possible.

I. INTRODUCTION

Experiments have been performed with the ZPR-VII critical assembly⁽¹⁾ with both THUD and Babcock and Wilcox (B&W) fuel and heavy water as moderator. The THUD fuel consists of pellets of thorium oxide and highly enriched uranium oxide with an atom ratio of thorium to U^{235} of 25:1, whereas the B&W fuel consists of similar pellets with an atom ratio of thorium to U^{235} of 15:1. Both THUD and B&W pellets are clad in aluminum tubes. The properties of both types of fuel are described in more detail below. Experiments were done with both types of fuel in uniform triangular lattices having pitches of $2a_0$, $3a_0$, $4a_0$, and $6a_0$, where a_0 is 0.9525 cm ($\frac{3}{8}$ in.). The results of experiments for both uniform and clustered lattices have been given in Ref. 2.

Only uniform lattices are treated in this report. First, the critical equation and calculation of the parameters appearing therein are discussed, and numerical values of these parameters are presented for the various lattices. Then, calculated reflector savings are combined with experimental dimensions to yield geometric bucklings, and, finally, analyses are made of various experimental measurements.

II. THE CRITICALITY EQUATION AND CALCULATIONAL METHODS

A. The Criticality Equation

The criticality equation used for the calculation of material bucklings is essentially based on that discussed by Thie in Chapter X of Ref. 3, although some changes have been made. The following equations describe the slowing down and thermal diffusion of the neutrons:

$$D_f \nabla^2 \Phi_f - \Sigma_{slf} \Phi_f + \epsilon \eta f \Sigma_{at} \Phi_t + \epsilon \eta_e f_e (1 - p_e) q_e(0) e^{-\bar{\tau}_e B^2} = 0 \quad (1)$$

$$\nabla^2 q_r = \frac{\partial q_r}{\partial \tau} \quad \text{where } q_r(0) = \Sigma_{slf} \Phi_f \quad (2)$$

$$\nabla^2 q_e - \frac{\Sigma_a}{D} q_e - \frac{\xi \Sigma}{D} \frac{\partial q_e}{\partial u} = 0 \quad \text{where } q_e(0) = p q_r(\tau_r) \quad (3)$$

$$D_t \nabla^2 \Phi_t - \Sigma_{at} \Phi_t + q_e(\tau_e) = 0 \quad (4)$$

Fission neutrons from fissions in both thermal (t) and epithermal (e) groups are produced in the fast group (f) where slowing down is described by a diffusion kernel. The slowing down in resonance (r) and epithermal groups is described by Fermi age kernels with the sharp resonance absorption in thorium taking place at the bottom of the resonance group, and $1/v$ -absorption in thorium and nonthermal absorption in other materials taking place in the epithermal group. Lower energy limits of the fast, resonance, and epithermal groups are 180 kev, 40 ev, and 5 kT_n, respectively, where T_n is the effective core temperature. The corresponding lethargies are denoted by u_f, u_r and u_t.

On replacing ∇^2 by $-B^2$, we get from Eqs. (2) through (4)

$$q_e = \left(\frac{\Sigma_{at}(1 + L^2 B^2) e^{\tau_e B^2}}{p_e} e^{-\int_{u_r}^u \frac{\Sigma_a}{\xi \Sigma} du} e^{-B^2 \int_{u_r}^u \frac{D}{\xi \Sigma} du} \right) \Phi_t$$

$$q_r = \left(\frac{\Sigma_{at}(1 + L^2 B^2) e^{\tau_e B^2} e^{\tau_r B^2}}{p p_e} e^{-\tau B^2} \right) \Phi_t$$

$$\Phi_f = \left(\frac{\Sigma_{at}(1 + L^2 B^2) e^{\tau_e B^2} e^{\tau_r B^2}}{p p_e \Sigma_{slf}} \right) \Phi_t$$

where

$$p_e = \exp \left[- \int_{u_r}^{u_t} \frac{\Sigma_a}{\xi \Sigma} (u) du \right] \text{ and } \tau_e = \int_{u_r}^{u_t} \frac{D}{\xi \Sigma} (u) du \quad .$$

Now, on substituting these results for q_e , q_r , and Φ_f in Eq. (1), there results

$$\frac{p_e k_{\infty} + (1 - p_e) k_{\infty e} (1 + L^2 B^2) e^{(\tau_e - \bar{\tau}_e) B^2}}{(1 + L^2 B^2)(1 + \tau_f B^2) e^{(\tau_e + \tau_r) B^2}} = 1 \quad , \quad (5)$$

which is the criticality equation where $\tau_f = D_f / \Sigma_s l_f$, $k_{\infty} = \epsilon \eta p f$, and $k_{\infty e} = \epsilon \eta_e p f_e$. The quantity $\bar{\tau}_e$ is defined by writing $(1 - p_e) q_e(0) e^{-B^2 \bar{\tau}_e} = \int_{u_r}^{u_t} \frac{\Sigma_a}{\xi \Sigma} q_e du$ and considering only first-order terms in B^2 for an absorption cross section of $1/v$ energy dependence and small compared with the constant scattering cross section. This gives

$$\bar{\tau}_e = \tau_e \left[1 - \frac{2}{u_t - u_r} + \frac{e^{(1/2)u_r}}{e^{(1/2)u_t} - e^{(1/2)u_r}} \right] \quad . \quad (6)$$

The numerator of Eq. (5) may be written as

$$\epsilon p f \left[p_e \eta + (1 - p_e) \eta_e \frac{f_e}{f} (1 + L^2 B^2) e^{(\tau_e - \bar{\tau}_e) B^2} \right] \quad ,$$

which demonstrates that the effective η is an average of thermal and epithermal values. Although η_e is smaller than η , it is weighted by the ratio of f_e to f , which is larger than unity and also the inverse of the leakage between epithermal and thermal absorptions.

B. Calculation of Criticality Equation Constants

1. Thermal and Epithermal Cross Sections

In calculating thermal-group cross sections, a hardened Maxwellian flux with an arbitrary cutoff of $5kT_n$ was used, where T_n is the effective temperature of the core. This is the same cutoff as used by Westcott⁽⁴⁾ in his computations of g and s factors. There are other possible methods for defining a cutoff energy, but they are mutually inconsistent. Deutsch⁽⁵⁾ gives a formula in which the cutoff energy decreases with increasing absorption but is infinite for zero absorption. The formula

of Cohen⁽⁶⁾ also gives a cutoff energy decreasing with increasing absorption but finite for zero absorption. On the other hand, Hurwitz et al.⁽⁷⁾ find a cutoff energy approximately equal to that of Cohen for zero absorption but which increases with increasing absorption.

The value of T_n was found from the equation

$$T_n = T_0 \left(1 + 1.92 \frac{\bar{\Sigma}_a(k T_0)}{\Sigma_s} \right)$$

given in Ref. 8, where T_0 is the physical temperature of the core and $\bar{\Sigma}_a(k T_0)$ is its absorption cross section at T_0 , which was taken to be 293°K in this work.

The average thermal absorption or fission cross section is given by

$$\begin{aligned} \frac{\bar{\sigma}}{\sigma_0} &= \frac{\int_0^{E_c} \sigma(E) \phi(E) dE}{\int_0^{E_c} \phi(E) dE} = \frac{\int_0^{E_c} \left(\sqrt{\frac{E_0}{E}} + S(E) \right) \frac{E}{(k T_n)^2} e^{-E/k T_n} dE}{\int_0^{E_c} \frac{E}{(k T_n)^2} e^{-E/k T_n} dE} \\ &= \left(\frac{k T_0}{k T_n} \right)^{1/2} \frac{\int_0^{\frac{E_c}{k T_n}} X^{1/2} e^{-X} dX}{\int_0^{\frac{E_c}{k T_n}} X e^{-X} dX} \\ &\quad \left[1 + \left(\frac{k T_n}{k T_0} \right)^{1/2} \frac{\int_0^{\frac{E_c}{k T_n}} S(X) X e^{-X} dX}{\int_0^{\frac{E_c}{k T_n}} X^{1/2} e^{-X} dX} \right] \end{aligned}$$

where $S(E)$ gives the deviation of the cross section from a $1/v$ form. Thus,

$$\frac{\bar{\sigma}}{\sigma_0} = \frac{g(k T_n) \left(\frac{k T_0}{k T_n} \right)^{1/2} \int_0^{\frac{E_c}{k T_n}} X^{1/2} e^{-X} dX}{1 - \left(1 + \frac{E_c}{k T_n} \right) e^{-E_c/k T_n}},$$

where $g(kT_n)$ is the same as Westcott's g factor except that Westcott took the upper limits of the integrals involved as infinity. However, this makes little difference, and we will consider Westcott's g factor to be the same as ours. For a thermal cutoff at $5kT_n$,

$$\frac{\bar{\sigma}}{\sigma_0} = (0.9063) \left(\frac{kT_0}{kT_n} \right)^{1/2} g(kT_n) \quad .$$

The thermal values of σ_s and $\sigma_{tr} = (1 - \bar{\mu}) \sigma_s$ were taken as constant over the thermal region for all materials involved.

On assuming a $1/E$ epithermal flux above the thermal cutoff and extending the upper limit of the epithermal energy range to infinity, we may take the epithermal absorption or fission cross section to be

$$\begin{aligned} \frac{\bar{\sigma}}{\sigma_0} &= \int_{\mu kT_n}^{\infty} \frac{\sigma}{\sigma_0} \frac{dE}{E} = \int_{\mu kT_n}^{\infty} \left[\sqrt{\frac{E_0}{E}} + S(E) \right] \frac{dE}{E} \\ &= 2\sqrt{\frac{kT_0}{\mu kT_n}} \left(g(kT_n) + \frac{\sqrt{\pi\mu}}{4} s(kT_n) \right) \quad , \end{aligned}$$

where the Westcott g and s factors are given by

$$g(kT_n) = 1 + \frac{2}{\sqrt{\pi}} \sqrt{\frac{kT_n}{kT_0}} \int_0^{\infty} \frac{E}{(kT_n)^2} e^{-E/kT_n} S(E) dE$$

and

$$s(kT_n) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{kT_n}{kT_0}} \int_{\mu kT_n}^{\infty} \left[\frac{\sigma(E)}{\sigma_0} - g\sqrt{\frac{E_0}{E}} \right] \frac{dE}{E} \quad .$$

For a $5kT_n$ cutoff,

$$\frac{\bar{\sigma}}{\sigma_0} = 0.8944 \left(\frac{kT_0}{kT_n} \right)^{1/2} \left(g(kT_n) + \frac{s(kT_n)}{1.01} \right) \quad .$$

2. Thermal Group Constants

The thermal utilization f is calculated from the equation

$$f = \frac{\sum_{i(\text{fuel})} \Sigma_{a_i} V_i \bar{\Phi}_i}{\sum_i \Sigma_{a_i} V_i \bar{\Phi}_i} \quad ,$$

where Σ_{a_i} denotes the macroscopic thermal-absorption cross section of material i , V_i is the volume of material i present in the unit cell, and $\bar{\Phi}_i$ is the average thermal flux in material i . In the above equation the absorption in the numerator includes all materials present in the fuel rod. We define f^{25} in the same manner as f except that the absorption in the numerator includes that in U^{235} only.

The value of η is calculated from the equation

$$\eta = \frac{\nu \Sigma_f^{25} V_{25}}{\sum_{i(\text{fuel})} \Sigma_{a_i} V_i} ,$$

where ν is the number of neutrons produced per thermal fission in U^{235} and Σ_f^{25} is the macroscopic thermal-fission cross section of U^{235} . The quantity η^{25} is defined similarly except the denominator includes absorption in U^{235} only. Therefore, the product ηf equals $\eta^{25} f^{25}$.

The average thermal-absorption cross section in the core is defined by

$$\bar{\Sigma}_a = \frac{\sum_i \Sigma_{a_i} V_i \bar{\Phi}_i}{\sum_i V_i \bar{\Phi}_i} .$$

Average core-transport and scattering cross sections are defined in a similar manner. The thermal diffusion area is given by $L^2 = 1/3 \bar{\Sigma}_a \bar{\Sigma}_{tr}$ and the diffusion constant by $D = 1/3 \bar{\Sigma}_{tr}$.

In order to compute the thermal constants it is necessary to know the average thermal fluxes in the various regions. One would not expect diffusion theory to give an adequate value for the disadvantage factor, i.e., the ratio of the average flux in the moderator to that in the fuel, because of the small radii of the fuel rods involved. However, P_3 ⁽⁹⁾ or higher P_n approximations⁽¹⁰⁾ do not yield adequate values either. This is because of a boundary condition difficulty at the outer radius of the unit cell in cylindrical geometry. The usual condition of the P_n approximation, that the angular flux in a given direction at the outer radius of the cell is equal to the angular flux in the mirror image direction,⁽⁹⁾ does not ensure that the derivative of the total flux is zero at the outer boundary (except in the P_1 or diffusion-theory approximation). A similar difficulty occurs in the S_n method. In fact, for a fuel rod of given radius, the disadvantage factor calculated by P_n (except P_1) or S_n methods may actually increase as the outer radius of the unit cell is decreased because of the increase in the derivative of the total flux at the outer cell boundary. Examples of this type of behavior are given in Ref. 11. A more complete

discussion of the boundary condition difficulty is given in Ref. 12. Incidentally, this anomalous behavior of the disadvantage factor does not occur in slab geometry as the slab boundary conditions ensure that the slope of the total flux is zero at the outer boundary.

A method for the calculation of the disadvantage factor which does not have boundary condition difficulty is the integral transport theory method of Amouyal et al.⁽¹³⁾ This method uses transport theory in the fuel with diffusion theory in the moderator and provides a simple formula for the disadvantage factor. In this report, the liquid moderator, aluminum clad, and thin void region between clad and fuel were homogenized in calculating moderator constants for the two-region integral-transport-theory method.

Another suitable method for calculations of the disadvantage factor is the Monte Carlo method. Calculations of average fluxes in the various regions of the unit cell were carried out with the use of the TUT-T5 code,⁽¹⁴⁾ which solves the transport equation in x-y geometry. The histories of about 1000 neutrons are followed in each "experiment," and the sources may be chosen arbitrarily in each region. In the present application the source densities in each region were taken proportional to $\xi \Sigma_g$. Of course, we do not have x-y geometry in our lattices, so it was necessary to change the annular regions of the unit cell into regions of equal area in x-y geometry. The output of the code gives capture probabilities in each region from which average fluxes can be found. Monte Carlo calculations were done only for lattices of $1a_0$, $2a_0$, and $3a_0$ pitch because of the rapid increase in machine time necessary to get reasonable accuracy with increasing lattice pitch.

The average fluxes used in the calculations of thermal constants were based on a judicious combination of the results from the method of Amouyal et al., and the Monte Carlo results.

3. Epithermal Group Constants

The epithermal utilization f_e is defined in a manner similar to the thermal utilization except that epithermal cross sections are used and the epithermal fluxes are taken to be the same in all materials. Likewise, f_e^{25} , η_e , and η_e^{25} are defined in a manner similar to the corresponding thermal quantities.

The epithermal absorption escape probability p_e is given by the equation

$$p_e = \exp \left[\frac{-\sum_i \Sigma_{a_i} V_i}{\sum_i (\xi \Sigma_s)_i V_i} \right],$$

where the Σ_{ai} are the macroscopic epithermal absorption cross sections and the $(\xi \Sigma_s)_i$ are the epithermal slowing-down cross sections. For thorium, only $1/v$ -absorption is included here since the non- $1/v$ -absorption is taken account of in the resonance escape probability.

4. Resonance Escape Probability

The probability of escaping capture in the thorium resonances is

$$p = \exp \left[- \frac{\sigma_{\text{eff}} N_{\text{ThO}_2} V_{\text{ThO}_2}}{\sum_{i(\text{non-fuel})} (\xi \Sigma_s)_i V_i} \right] ,$$

where σ_{eff} is the effective resonance integral of thorium oxide. This is given by the empirical form

$$\sigma_{\text{eff}} = A + B \left(\frac{S}{M} \right)_{\text{ThO}_2} (1 - C) ,$$

where A and B are constants, $(S/M)_{\text{ThO}_2}$ is the ratio of the fuel lump surface to the mass of thorium oxide contained in it, and C is the Dancoff correction.⁽¹⁵⁾ The $1/v$ -absorption is not contained in σ_{eff} since it is included in calculating p_e . The Dancoff correction is a function of the radius of the fuel rod, the distance between centers of neighboring fuel rods, and the scattering mean free path in the moderator. In the original paper by Dancoff and Ginsburg⁽¹⁵⁾ curves are given enabling one to find C , but most values for heavy-water-moderated reactors lie outside the range of the curves. Thie⁽¹⁶⁾ has given a simple approximate expression which can be used for the Dancoff correction:

$$C = \frac{4}{\pi^2} K_{i3} \left[\delta \left(\frac{d}{\rho} \right) \frac{\rho}{\lambda} \right] \sin^{-1} \left(\frac{\rho}{d} \right) ,$$

where ρ is the fuel-rod radius, d the rod separation, λ the moderator mean free path, and K_{i3} is a function defined and tabulated in Ref. 17. The function $\delta(d/\rho)$ lies between $(d/\rho) - 2$ and d/ρ . Dancoff corrections were calculated for these two extreme values and the results averaged to give the values used in this report.

5. Fast Fission Factor

The fast fission factor ϵ was taken to be unity for all lattices considered. Calculations show that ϵ differs little from unity even for the $2a_0$ lattices because the fuel rods are of small radius and the fast fission cross section of Th^{232} is considerably smaller than that of U^{238} .

6. Age

As is seen from the criticality equation, the age has been broken up into three parts: τ_f , τ_r , and τ_e . In calculating ages the volume fraction occupied by D_2O plus ThO_2 was considered to be all D_2O , since calculations⁽¹⁸⁾ show that the addition of ThO_2 to D_2O in concentrations as used here will not affect the age much. Then the method of equivalence factors^(5,19) was used to find the age of a mixture of D_2O and aluminum. The formulae are

$$\tau_f = 49 \frac{1}{(VF_{D_2O})^2 \left(1 + 0.620 \frac{VF_{Al}}{VF_{D_2O}} \right)} ;$$

$$\tau_r = 43.58 \frac{1}{(VF_{D_2O})^2 \left(1 + 0.357 \frac{VF_{Al}}{VF_{D_2O}} \right)} ;$$

$$\tau_e = 4.94 \ln \left(\frac{40}{Ec} \right) \frac{1}{(VF_{D_2O})^2 \left(1 + 0.357 \frac{VF_{Al}}{VF_{D_2O}} \right)} ,$$

where VF denotes volume fraction and the volume fraction of fuel is added to that of D_2O . For pure D_2O an age to indium resonance at 1.44 ev of 109 cm^2 was obtained, which agrees with the experimental value.⁽²⁰⁾ For experimental lattices with small H_2O impurities, a small negative correction was made to the ages by use of the experimental results of Ref. 20.

III. LATTICE, FUEL, AND MICROSCOPIC CROSS-SECTION DATA

A. Lattice Dimensions and Fuel Composition

Since the ZPR-VII cores were constructed of fuel rods in a triangular array, the outer radii of the equivalent circular unit cells are given by

$$r = \sqrt{\frac{\sqrt{3}}{2\pi}} p$$

where p is the pitch. The pitches used for both THUD and B&W fuel were $2a_0$, $3a_0$, $4a_0$, and $6a_0$, where $a_0 = 0.9525$ cm and is the separation of the centers of adjacent holes in the aluminum grids used for spacing the rods.

Table I gives the radii of the components of the unit cell for both THUD and B&W fuel.

Volume fractions of the unit cell components are listed in Table II.

Table I

RADII OF UNIT CELL COMPONENTS

	<u>Fuel</u>	
	<u>THUD</u>	<u>B&W</u>
Fuel Rod Radius (cm)	0.29718	0.33020
Aluminum Clad, Inner Radius (cm)	0.30861	0.35687
Aluminum Clad, Outer Radius (cm)	0.39243	0.39370
Outer Radius of Unit Cell (n^* cm)	0.50010	0.50010

*Pitch is na_0 cm.

Table II

VOLUME FRACTIONS OF UNIT CELL COMPONENTS

<u>Pitch</u>	<u>THUD Fuel</u>				<u>B&W Fuel</u>			
	<u>Fuel</u>	<u>Void</u>	<u>Al</u>	<u>D₂O</u>	<u>Fuel</u>	<u>Void</u>	<u>Al</u>	<u>D₂O</u>
$2a_0$	0.08828	0.00692	0.05874	0.84606	0.10899	0.01832	0.02763	0.84506
$3a_0$	0.03924	0.00308	0.02611	0.93157	0.04844	0.00814	0.01228	0.93114
$4a_0$	0.02207	0.00173	0.01468	0.96152	0.02725	0.00458	0.00691	0.96126
$6a_0$	0.00981	0.00077	0.00653	0.98289	0.01211	0.00204	0.00307	0.98278

The THUD fuel has 25 atoms of Th^{232} to 1 atom of U^{235} to 0.061 atom of U^{238} , and a density of 8.434 gm/cm^3 . Also, there is another THUD fuel consisting of 50 atoms of Th^{232} to 1 of U^{235} to 0.061 of U^{238} , with a density of 8.272 gm/cm^3 , which was used in fuel-substitution experiments. The dimensions of the fuel and cladding are the same for the two types of THUD fuel. The B&W fuel was taken to consist of 15 atoms of Th^{232} to 1 atom of U^{235} , with U^{238} being ignored. Its density is 8.2 gm/cm^3 . In the following, the various fuels will often be referred to by the ratio of the number of Th^{232} atoms to U^{235} atoms, that is, as 25:1 fuel, 50:1 fuel, or 15:1 fuel.

B. Microscopic Cross-section Data

Microscopic thermal cross sections for the various materials are given in Table III. Most of this data were obtained from BNL-325 and its supplement.⁽²¹⁾

Table III

MICROSCOPIC THERMAL CROSS SECTIONS

<u>Material</u>	<u>Density (gm/cm³)</u>	<u>σ_s (b)</u>	<u>σ_{tr} (b)</u>	<u>$\sigma_a(2200)$ (b)</u>	<u>$\sigma_f(2200)$ (b)</u>
U^{235}O_2		17.5	17.1	694	582
U^{238}O_2		15.8	15.4	2.71	0
ThO_2		19.9	19.5	7.45	0
Al	2.706	1.4	1.37	0.23	0
D_2O	1.109	14.0	11.9	0.0012	0
H_2O	1.00	79.9	63.0	0.664	0

The Westcott g factors were taken as unity for all materials except U^{235} , for which the g factor was found as a function of T_n from Ref. 4. In the case of the clad on the 15:1 fuel, the value of $\sigma_a(2200)$ for Al was increased to 0.3055 to allow for boron impurity. The clad on the THUD fuel contains negligible impurities.

Table IV gives the values of σ_s and $\xi\sigma_s$ used in resonance and epithermal calculations. The Westcott g and s factors used in calculating epithermal absorption and fission cross sections were taken from Ref. 4.

The value used for the effective resonance integral for ThO_2 is

$$\sigma_{\text{eff}} \text{ (in barns)} = 6.7 + 12.3 \left(\frac{S}{M} \right)_{\text{ThO}_2} (1 - C) \quad ,$$

where $1/v$ -absorption is not included. This value is quoted on p. 133 of Ref. 3 and, as explained there, is an average value found from several measurements whose analysis is generally rather difficult. Measurements by Dayton and Pettus(22) give much higher values for σ_{eff} , but such values would lead to resonance escape probabilities which would be much too small.

Table IV

RESONANCE AND EPITHERMAL VALUES
OF σ_s AND $\xi \sigma_s$

<u>Material</u>	<u>σ_s (b)</u>	<u>$\xi \sigma_s$ (b)</u>
U ²³⁵ O ₂	17.5	0.987
U ²³⁸ O ₂	15.8	0.971
ThO ₂	19.9	1.010
Al	1.4	0.1012
D ₂ O	10.49	5.35
H ₂ O	44.5	41.2

IV. VALUES OF PARAMETERS INVOLVED IN THE CRITICALITY EQUATION

A. Calculated Thermal Fluxes

Values of disadvantage factors calculated by the integral transport theory method⁽¹³⁾ are given in Table V, in which the disadvantage factor is defined as the average flux in the region outside the fuel divided by the average flux in the fuel.

Table VI lists average thermal fluxes calculated by the TUT-T5 Monte Carlo code. The notation $\Delta\phi(50\%)$ means that there is a 50% chance that the error in the flux is less than $\Delta\phi(50\%)$; the quantity $\Delta\phi(95\%)$ has a similar definition. No flux is quoted for the small void region inside the clad as the TUT code does not allow voids. Consequently, the clad for the purposes of the TUT calculations was considered to be so shrunk as to fit the fuel with the clad area being preserved. The column labelled "Experiments" refers to the number of times that the histories of approximately 1000 neutrons were followed. Note that calculations were done for a $1a_0$ pitch for the 15:1 and 25:1 fuels in order to check the trend of the values.

The fluxes used in the calculation of the thermal parameters were based on a combination of Amouyal approximation and Monte Carlo results, and are given in Table VII. These values are in fair agreement with the experimental disadvantage factors quoted in Table XXVI of Ref. 2, although the agreement is not good for the $2a_0$ lattices, for which the theoretical values are higher than the experimental ones.

B. Values of Criticality Equation Parameters

Tables VIII through XII give values of the quantities involved in the criticality equation for all the lattices studied. Table VIII presents the results for 25:1 fuel and standard D_2O (99.8% D_2O + 0.2% H_2O), whereas Tables IX and X present the 25:1 fuel results for the D_2O purities used in the experiments. The values used in Table IX are 98.5% D_2O + 1.5% H_2O for the $2a_0$, $3a_0$, and $4a_0$ lattices, and 98.0% D_2O + 2.0% H_2O for the $6a_0$ lattice. Values in Table X are based on a D_2O purity of 97.1%, as some of the 25:1 lattices were rebuilt roughly 21 months after the original lattices in order to check the D_2O purity effect. In Table XI the calculations for 15:1 fuel are reported for standard D_2O , whereas Table XII gives the corresponding calculations for the actual D_2O purity of 97.2%. In addition to the material bucklings B_m^2 , calculated from the criticality Eq. (5), approximate values denoted by $B_m'^2$ are given which are calculated from the equation

$$\frac{k_{\infty}}{(1 + L^2 B_m'^2) e^{\tau B_m'^2}} = 1 \quad \text{where } \tau = \tau_f + \tau_r + \tau_e$$

Table V

INTEGRAL TRANSPORT-THEORY
DISADVANTAGE FACTORS

<u>Pitch</u>	<u>15:1 Fuel</u>	<u>25:1 Fuel</u>	<u>50:1 Fuel</u>
2a ₀	1.330	1.202	1.122
3a ₀	1.351	1.215	1.130
4a ₀	1.368	1.224	1.136
6a ₀	1.393	1.239	1.144

Table VI

MONTE CARLO AVERAGE FLUXES

<u>Fuel</u>	<u>Pitch</u>	<u>Experiments</u>	<u>Region</u>	<u>ϕ</u>	<u>$\Delta\phi(50\%)$</u>	<u>$\Delta\phi(95\%)$</u>
15:1	1a ₀	24	fuel	1.000	±0.005	±0.014
			clad	1.185	±0.007	±0.021
			D ₂ O	1.284	±0.006	±0.019
15:1	2a ₀	10	fuel	1.000	±0.009	±0.030
			clad	1.148	±0.010	±0.034
			D ₂ O	1.303	±0.012	±0.038
15:1	3a ₀	5	fuel	1.000	±0.006	±0.022
			clad	1.131	±0.005	±0.020
			D ₂ O	1.298	±0.009	±0.034
25:1	1a ₀	17	fuel	1.000	±0.007	±0.021
			clad	1.120	±0.007	±0.022
			D ₂ O	1.177	±0.008	±0.025
25:1	2a ₀	10	fuel	1.000	±0.006	±0.019
			clad	1.105	±0.009	±0.030
			D ₂ O	1.189	±0.007	±0.024
25:1	3a ₀	5	fuel	1.000	±0.009	±0.032
			clad	1.093	±0.008	±0.030
			D ₂ O	1.202	±0.012	±0.046
50:1	2a ₀	5	fuel	1.000	±0.007	±0.025
			clad	1.059	±0.009	±0.034
			D ₂ O	1.093	±0.007	±0.027

Table VII

FLUXES USED IN CALCULATING THERMAL PARAMETERS

<u>Fuel</u>	<u>Region</u>	<u>2a₀</u>	<u>3a₀</u>	<u>4a₀</u>	<u>6a₀</u>
15:1	fuel	1.00	1.00	1.00	1.00
	void & aluminum	1.15	1.13	1.12	1.11
	D ₂ O	1.29	1.31	1.33	1.36
25:1	fuel	1.00	1.00	1.00	1.00
	void & aluminum	1.11	1.10	1.09	1.08
	D ₂ O	1.19	1.20	1.21	1.23
50:1	fuel	1.00	1.00	1.00	1.00
	void & aluminum	1.06	1.05	1.05	1.04
	D ₂ O	1.10	1.11	1.12	1.13

Table VIII

CONSTANTS FOR 25:1 LATTICES WITH 99.8% D₂O

<u>Pitch</u>	<u>$\bar{\Sigma}_a$ (cm⁻¹)</u>	<u>$\bar{\Sigma}_{tr}$ (cm⁻¹)</u>	<u>L² (cm²)</u>	<u>f</u>	<u>η</u>	<u>f²⁵</u>	<u>η^{25}</u>
2a ₀	0.03973	0.3785	22.17	0.9824	1.627	0.7675	2.082
3a ₀	0.01849	0.3913	46.07	0.9805	1.628	0.7675	2.080
4a ₀	0.01053	0.3956	80.02	0.9776	1.628	0.7658	2.078
6a ₀	0.004713	0.3987	177.4	0.9690	1.629	0.7595	2.078
<u>Pitch</u>	<u>f_e</u>	<u>η_e</u>	<u>f_e²⁵</u>	<u>η_e^{25}</u>	<u>p_e</u>	<u>p</u>	<u>k_{∞e}</u>
2a ₀	0.9850	1.504	0.7788	1.902	0.7352	0.8583	1.271
3a ₀	0.9831	1.509	0.7767	1.909	0.8763	0.9340	1.386
4a ₀	0.9806	1.509	0.7745	1.911	0.9290	0.9622	1.424
6a ₀	0.9737	1.511	0.7690	1.913	0.9678	0.9831	1.446
<u>Pitch</u>	<u>k_∞</u>	<u>τ_f (cm²)</u>	<u>τ_r (cm²)</u>	<u>τ_e (cm²)</u>	<u>$\bar{\tau}_e$ (cm²)</u>	<u>B_m² (cm⁻² × 10⁻³)</u>	<u>B_m¹² (cm⁻² × 10⁻³)</u>
2a ₀	1.372	54.02	48.83	30.77	21.72	2.050	2.037
3a ₀	1.491	51.14	45.80	29.44	20.87	2.432	2.348
4a ₀	1.531	50.18	44.81	28.98	20.60	2.226	2.155
6a ₀	1.552	49.52	44.12	28.69	20.43	1.615	1.578

Table IX

CONSTANTS FOR 25:1 LATTICES WITH ORIGINAL D₂O PURITIES

Pitch	$\bar{\Sigma}_a$ (cm ⁻¹)	$\bar{\Sigma}_{tr}$ (cm ⁻¹)	L^2 (cm ²)	f	η	f^{25}	η^{25}
2a ₀	0.04017	0.3978	20.86	0.9769	1.627	0.7633	2.082
3a ₀	0.01880	0.4123	43.01	0.9670	1.627	0.7570	2.078
4a ₀	0.01081	0.4173	73.91	0.9531	1.628	0.7466	2.078
6a ₀	0.00506	0.4290	153.6	0.9014	1.629	0.7065	2.078
Pitch	f_e	η_e	f_e^{25}	η_e^{25}	p_e	p	$k_{\infty e}$
2a ₀	0.9805	1.504	0.7754	1.902	0.7501	0.8690	1.282
3a ₀	0.9721	1.509	0.7679	1.910	0.8840	0.9392	1.378
4a ₀	0.9607	1.511	0.7588	1.913	0.9331	0.9653	1.401
6a ₀	0.9189	1.512	0.7256	1.915	0.9704	0.9850	1.369
Pitch	k_{∞}	τ_f (cm ²)	τ_r (cm ²)	τ_e (cm ²)	$\bar{\tau}_e$ (cm ²)	B_m^2 (cm ⁻² x 10 ⁻³)	$B_m'^2$ (cm ⁻² x 10 ⁻³)
2a ₀	1.381	52.73	47.67	30.04	21.21	2.162	2.141
3a ₀	1.478	49.73	44.54	28.63	20.30	2.464	2.383
4a ₀	1.498	48.80	43.58	28.18	20.04	2.199	2.135
6a ₀	1.446	47.57	42.38	27.56	19.62	1.469	1.439

Table X

CONSTANTS FOR 25:1 LATTICES WITH 97.1% D₂O

Pitch	$\bar{\Sigma}_a$ (cm ⁻¹)	$\bar{\Sigma}_{tr}$ (cm ⁻¹)	L^2 (cm ²)	f	η	f^{25}	η^{25}
2a ₀	0.04056	0.4183	19.64	0.9721	1.627	0.7597	2.082
3a ₀	0.01906	0.4347	40.26	0.9554	1.627	0.7480	2.078
4a ₀	0.01108	0.4403	68.30	0.9325	1.628	0.7308	2.078
Pitch	f_e	η_e	f_e^{25}	η_e^{25}	p_e	p	$k_{\infty e}$
2a ₀	0.9766	1.504	0.7723	1.902	0.7662	0.8786	1.291
3a ₀	0.9626	1.509	0.7604	1.910	0.8916	0.9438	1.371
4a ₀	0.9438	1.511	0.7457	1.913	0.9369	0.9679	1.381
Pitch	k_{∞}	τ_f (cm ²)	τ_r (cm ²)	τ_e (cm ²)	$\bar{\tau}_e$ (cm ²)	B_m^2 (cm ⁻² x 10 ⁻³)	$B_m'^2$ (cm ⁻² x 10 ⁻³)
2a ₀	1.390	51.15	46.23	29.13	20.57	2.287	2.258
3a ₀	1.467	47.85	42.86	27.55	19.53	2.529	2.446
4a ₀	1.470	46.73	41.73	26.99	19.19	2.214	2.149

Table XI

CONSTANTS FOR 15:1 LATTICES WITH 99.8% D₂O

Pitch	$\bar{\Sigma}_a$ (cm ⁻¹)	$\bar{\Sigma}_{tr}$ (cm ⁻¹)	L ² (cm ²)	f	η	f ²⁵	η^{25}
2a ₀	0.06150	0.3828	14.16	0.9931	1.782	0.8493	2.084
3a ₀	0.02919	0.3933	29.04	0.9919	1.782	0.8499	2.080
4a ₀	0.01668	0.3968	50.38	0.9901	1.782	0.8490	2.078
6a ₀	0.007439	0.3992	112.2	0.9845	1.783	0.8446	2.078
Pitch	f _e	η_e	f _e ²⁵	η_e^{25}	Pe	p	k _{∞e}
2a ₀	0.9945	1.636	0.8591	1.894	0.5943	0.8428	1.371
3a ₀	0.9934	1.645	0.8562	1.909	0.7956	0.9259	1.513
4a ₀	0.9920	1.648	0.8556	1.911	0.8793	0.9574	1.565
6a ₀	0.9880	1.650	0.8521	1.914	0.9443	0.9809	1.599
Pitch	k _∞	τ_f (cm ²)	τ_r (cm ²)	τ_e (cm ²)	$\bar{\tau}_e$ (cm ²)	B _m ² (cm ⁻² × 10 ⁻³)	B _m ^{1/2} (cm ⁻² × 10 ⁻³)
2a ₀	1.492	52.88	47.39	29.31	20.61	2.797	2.790
3a ₀	1.637	50.67	45.21	28.81	20.40	3.380	3.233
4a ₀	1.689	49.93	44.49	28.65	20.34	3.240	3.085
6a ₀	1.721	49.41	43.98	28.53	20.29	2.550	2.457

Table XII

CONSTANTS FOR 15:1 LATTICES WITH 97.2% D₂O

Pitch	$\bar{\Sigma}_a$ (cm ⁻¹)	$\bar{\Sigma}_{tr}$ (cm ⁻¹)	L ² (cm ²)	f	η	f ²⁵	η^{25}
2a ₀	0.06281	0.4215	12.59	0.9869	1.782	0.8442	2.084
3a ₀	0.02985	0.4353	25.65	0.9766	1.782	0.8369	2.080
4a ₀	0.01724	0.4399	43.96	0.9620	1.782	0.8250	2.078
6a ₀	0.00795	0.4432	94.61	0.9216	1.783	0.7907	2.078
Pitch	f _e	η_e	f _e ²⁵	η_e^{25}	Pe	p	k _{∞e}
2a ₀	0.9898	1.637	0.8549	1.896	0.6365	0.8655	1.403
3a ₀	0.9821	1.646	0.8473	1.908	0.8200	0.9366	1.514
4a ₀	0.9714	1.648	0.8378	1.911	0.8938	0.9636	1.543
6a ₀	0.9421	1.651	0.8124	1.914	0.9500	0.9838	1.530
Pitch	k _∞	τ_f (cm ²)	τ_r (cm ²)	τ_e (cm ²)	$\bar{\tau}_e$ (cm ²)	B _m ² (cm ⁻² × 10 ⁻³)	B _m ^{1/2} (cm ⁻² × 10 ⁻³)
2a ₀	1.522	50.22	45.00	27.83	19.57	3.146	3.109
3a ₀	1.630	47.65	42.52	27.09	19.18	3.595	3.445
4a ₀	1.652	46.84	41.75	26.88	19.09	3.346	3.205
6a ₀	1.616	46.24	41.16	26.70	18.99	2.491	2.410

V. REFLECTOR SAVINGS CALCULATIONS AND GEOMETRIC BUCKLINGS

A. Reflector Savings Calculations

The cores constructed in ZPR-VII have a bottom reflector consisting of an inhomogeneous mixture of heavy water and aluminum structure with some stainless steel and void. The radial reflector is composed of the heavy water between the core and inner radius of the ZPR-VII tank and contains aluminum control rod guides. There was no top reflector except for the forest of fuel rods above the critical level of the heavy water in the core and radial reflector. This forest of fuel rods was always ignored in considerations of experimental results. More detailed descriptions of the reflectors may be found in Ref. 2.

Since the inner radius of the ZPR-VII tank is 102.9 cm, whereas the core radii are less than 40 cm, there is a thick radial reflector which produces a large distortion of the flux from the J_0 shape of a bare core near the core-reflector boundary. This distortion is so large that it is not possible to obtain reasonably accurate values of radial reflector savings by fitting a J_0 Bessel function to the "unperturbed" central portion of an experimental flux plot. Experimental axial reflector savings were determined for some cores by fitting cosine curves to measured axial flux plots and are reported in Ref. 2. However, the lack of experimental values of radial reflector savings makes it impossible to determine strictly experimental geometric bucklings. It might be assumed that constant values of λ_R and λ_H (the radial and axial reflector savings, respectively) could be found which would combine with experimental values of R and H to yield an approximately constant geometric buckling for various pairs of values of R and H . However, this assumption of constant λ_R and λ_H is not valid, and so this procedure cannot legitimately be used. In the following work, experimental values of R and H are combined with calculated values of λ_R and λ_H in order to determine what will be loosely referred to as experimental bucklings.

The reflector savings calculations were performed by means of Fortran codes RE-169 and RE-188 on the IBM-704. Code RE-169 is a two-group, two-region axial reflector savings code which requires as input the core and reflector material constants, reflector thickness, and radial buckling, and gives as output the buckling and axial reflector savings. The reflector savings are calculated from an analytical formula that is derived with only one approximation. This approximation is well justified except for cores of much smaller height and radial buckling than those considered here. For the radial calculations, RE-188 requires as input the two-group core and reflector material constants, axial buckling, outer radius of the reflector, and two guesses of the core critical radius. It gives as output the core buckling and the critical radius R , found by evaluating the critical determinant for the input guesses for R and several interpolated values.

In order to adopt the constants of the criticality Eq. (5) to two-group reflector savings calculations, it is necessary to cast Eq. (5) into the form of the two-group criticality equation. This is done by writing

$$\eta_{2G} = \eta \left[p_e + (1 - p_e) \frac{\eta_e f_e}{\eta_f} (1 + L^2 B^2) e^{(\tau_e - \bar{\tau}_e) B^2} \right] ; k_{\infty}^{2G} = \epsilon p f \eta_{2G}$$

and

$$1 + \tau_{2G} B^2 = (1 + \tau_f B^2) e^{(\tau_e + \tau_r) B^2} ,$$

which gives

$$\frac{\epsilon p f \eta_{2G}}{(1 + L^2 B^2)(1 + \tau_{2G} B^2)} = 1 .$$

In determining radial reflector savings, it is difficult to know what volume fraction of aluminum to use to allow for the control rod guides, which represent a marked heterogeneity in the reflector. Calculations were performed in which the reflector was treated as having aluminum volume fractions of 0.005, 0.01, and 0.02. The remainder of the reflector is heavy water plus the light water impurity. A value of 200 cm² was used for the age in the reflector⁽²³⁾ as well as values calculated in the same manner as the sum of τ_f , τ_r , and τ_e in the core. For a given lattice and D₂O purity, radial calculations were done for six sets of reflector constants corresponding to the three aluminum volume fractions and the two ages. For each set of constants several values of axial buckling were used.

Constants for the bottom reflector were estimated by use of the volume fractions of the various materials present, with the materials near the core being given a higher weight. Because of the great heterogeneity, the values of these constants are quite rough, and no account was taken of the different light water impurities present at various times. Axial calculations were performed with four sets of constants and combinations of L^2 and τ found from $L^2 = 200$ cm² or 250 cm² and $\tau = 150$ cm² or 225 cm². Several values of radial buckling were used with a given set of axial constants.

Values of the radial and axial reflector savings for the 2a₀ lattice with 25:1 fuel are given in Tables XIII and XIV. In Table XIV, the axial reflector savings is the sum of the bottom reflector savings and the top reflector savings of 0.71 λ_{tr} . These tables are presented to show the effects of reflector constants and heavy water purity on the results. Complete results for the other lattices are not given here, but the trend of the variation of reflector savings with different core constants may be seen in Tables XV and XVI.

Table XIII

RADIAL REFLECTOR SAVINGS FOR $2a_0$ LATTICES WITH 25:1 FUEL

Al (%)	r_r (cm ²)	B_z^2 (cm ⁻² × 10 ⁻³)	R (cm)	λ_R (cm)	Al (%)	r_r (cm ²)	B_z^2 (cm ⁻² × 10 ⁻³)	R (cm)	λ_R (cm)
99.8% D ₂ O					98.5% D ₂ O				
0.5	200	0.44	24.505	35.429	0.5	200	0.44	25.976	31.994
0.5		0.84	35.023	34.111			0.61	29.487	31.577
1.0		0.39	24.121	34.903			0.75	32.768	31.254
1.0		0.80	34.405	33.614			0.87	35.950	30.982
2.0		0.33	24.181	33.805	1.0		0.38	25.365	31.620
2.0		0.76	34.379	32.577			0.56	28.912	31.191
0.5	123	0.59	25.232	37.706			0.71	32.277	30.856
0.5		0.93	34.565	37.293			0.83	35.323	30.595
1.0		0.55	25.081	37.012	2.0		0.32	25.267	30.782
1.0		0.90	34.321	36.594			0.50	28.624	30.383
2.0		0.50	25.388	35.695			0.66	32.022	30.051
2.0		0.87	34.738	35.270			0.79	35.161	29.788
97.1% D ₂ O					0.5	120	0.58	26.460	34.022
							0.73	29.644	33.879
							0.85	32.645	33.774
0.5	200	0.38	25.904	29.224			0.97	36.016	33.669
0.5		0.88	35.854	28.350	1.0		0.53	26.057	33.491
1.0		0.38	26.304	28.824			0.68	29.144	33.347
1.0		0.88	36.222	27.982			0.81	32.195	33.233
2.0		0.30	25.789	28.215			0.93	35.413	33.130
2.0		0.83	35.725	27.365	2.0		0.47	26.017	32.465
0.5	114	0.44	24.970	31.048			0.63	29.148	32.314
0.5		0.92	34.407	30.732			0.77	32.287	32.194
1.0		0.44	25.430	30.588			0.89	35.363	32.093
1.0		0.92	34.862	30.277					
2.0		0.38	25.358	29.770					
2.0		0.88	34.761	29.443					

Table XIV

AXIAL REFLECTOR SAVINGS FOR $2a_0$ LATTICES WITH 25:1 FUEL

L_r^2 (cm ²)	r_r (cm ²)	B_r^2 (cm ⁻² × 10 ⁻³)	λ_H (cm)	L_r^2 (cm ²)	r_r (cm ²)	B_r^2 (cm ⁻² × 10 ⁻³)	λ_H (cm)
99.8% D ₂ O				98.5% D ₂ O			
200	225	1.32	21.402	200	225	1.43	20.728
200		1.73	21.164			1.63	20.605
250		1.32	22.297			1.76	20.525
250		1.73	22.042			1.84	20.476
200	150	1.32	21.839	250		1.43	21.567
200		1.73	21.663			1.63	21.434
250		1.32	22.952			1.76	21.348
250		1.73	22.767			1.84	21.295
97.1% D ₂ O				200	150	1.43	21.109
						1.63	21.012
						1.76	20.949
200	225	1.75	19.924			1.84	20.910
200		1.97	19.785	250		1.43	22.151
250		1.75	20.698			1.63	22.047
250		1.97	20.547			1.76	21.979
200	150	1.75	20.277			1.84	21.937
200		1.97	20.162				
250		1.75	21.239				
250		1.97	21.113				

Table XV

CRITICALITY DATA FOR 25:1 LATTICES

Pitch	Loading	D ₂ O (%)	R (cm)	λ_R (cm)	B_F^2 (cm ⁻² x 10 ⁻³)	H (cm)	λ_H (cm)	B_Z^2 (cm ⁻² x 10 ⁻³)	B^2 (cm ⁻² x 10 ⁻³)	$\overline{B^2}$ (cm ⁻² x 10 ⁻³)
2a ₀	90	98.5	25.801	30.719	1.8103	150.88	21.31	0.3329	2.1432	
	89		26.187	30.673	1.7887	145.40	21.32	0.3551	2.1438	
	91		27.310	30.539	1.7281	132.84	21.36	0.4151	2.1432	
	92		28.390	30.411	1.6726	123.53	21.40	0.4699	2.1425	
	93		29.461	30.301	1.6193	115.95	21.43	0.5229	2.1422	
	94		30.447	30.205	1.5720	110.03	21.46	0.5708	2.1428	
	95		31.733	30.074	1.5114	103.30	21.51	0.6336	2.1450	2.143
2a ₀	456	97.1	27.310	28.085	1.8846	139.09	20.61	0.3870	2.2716	2.272
3a ₀	103	98.5	27.056	25.968	2.0569	148.22	18.19	0.3564	2.4133	
	104		28.199	25.797	1.9835	133.79	18.23	0.4271	2.4106	
	105		29.063	25.667	1.9306	125.20	18.27	0.4795	2.4101	
	106		30.203	25.501	1.8637	115.81	18.31	0.5487	2.4124	
	107		30.795	25.429	1.8294	111.85	18.33	0.5824	2.4118	
	108		32.225	25.255	1.7504	103.70	18.39	0.6621	2.4125	2.412
4a ₀	118	98.5	32.636	24.890	1.7475	147.61	17.31	0.3629	2.1104	
	119		33.665	24.758	1.6943	137.08	17.34	0.4139	2.1082	
	120		34.602	24.637	1.6479	129.35	17.37	0.4585	2.1064	2.108
4a ₀	463	97.1	36.394	22.901	1.6449	148.74	16.79	0.3602	2.0051	
	464		36.831	22.849	1.6237	143.66	16.80	0.3833	2.0070	2.006

Table XVI

CRITICALITY DATA FOR 15:1 LATTICES

Pitch	Loading	D ₂ O (%)	R (cm)	λ_R (cm)	B_F^2 (cm ⁻² x 10 ⁻³)	H (cm)	λ_H (cm)	B_Z^2 (cm ⁻² x 10 ⁻³)	B^2 (cm ⁻² x 10 ⁻³)	$\overline{B^2}$ (cm ⁻² x 10 ⁻³)
2a ₀	388	97.2	18.578	26.854	2.8018	132.95	19.91	0.4224	3.2242	
	385		19.135	26.773	2.7440	123.02	19.94	0.4829	3.2269	
	381		20.522	26.570	2.6077	104.56	20.06	0.6355	3.2432	
	382		22.253	26.320	2.4511	90.14	20.16	0.8112	3.2623	
	384		24.356	26.105	2.2712	78.88	20.33	1.0027	3.2739	
	383		26.216	25.915	2.1280	71.77	20.44	1.1608	3.2888	3.253
3a ₀	403	97.2	19.154	23.182	3.2266	150.79	16.98	0.3506	3.5772	
	402		20.680	22.885	3.0470	115.45	17.11	0.5616	3.6086	
	401		26.880	21.991	2.4213	72.36	17.61	1.2193	3.6406	
	400		31.075	21.555	2.0879	59.70	17.89	1.6394	3.7273	3.638
4a ₀	407	97.2	23.584	21.838	2.8031	127.30	16.21	0.4792	3.2823	
	405		25.539	21.573	2.6055	103.03	16.35	0.6925	3.2980	
	404		31.439	20.862	2.1142	72.77	16.72	1.2324	3.3466	3.309
6a ₀	411	97.2	34.734	21.494	1.8292	137.13	15.80	0.4220	2.2512	
	410		36.380	21.334	1.7362	122.08	15.86	0.5187	2.2549	
	409		39.807	21.041	1.5619	103.56	15.97	0.6908	2.2527	2.253

B. Bucklings Derived from Experimental Critical Dimensions and Calculated Reflector Savings

There are twenty-four (6 x 4) possible combinations of axial and radial constants which can be used with experimental values of R and H to give experimental bucklings. All combinations were used with the critical dimensions of the tallest and shortest cores for each lattice to find bucklings. In general, the differences between the bucklings found for the two extreme dimensions of each lattice were smallest for combinations in which the radial set with $\tau_r = 200 \text{ cm}^2$ and 2% aluminum in the reflector was used. This set has the recommended high value for the two-group age and what is probably the most realistic value for the aluminum volume fraction in view of the fact that the aluminum control rod guides are near

the core in a region of high flux. The bucklings are much less sensitive to the axial reflector savings than to the radial, since the radial buckling is generally considerably larger than the axial and λ_R is a much larger fraction of $R + \lambda_R$ than is λ_H of $H + \lambda_H$. In calculating bucklings for all critical dimensions of a lattice, the above-mentioned radial set was used along with the axial set having $L^2 = 250 \text{ cm}^2$ and $\tau = 225 \text{ cm}^2$.

Tables XV and XVI give values of R , λ_R , B_R^2 , H , λ_H , B_H^2 , and B^2 for all critical dimensions of all the lattices. Values of R and H used in these tables have been obtained from the data of Table V of Ref. 2. The axial reflector savings in these tables are in reasonable agreement with experimental values reported in Table XVI of Ref. 2.

It will be noticed in Table XV that no values are presented in the case of the $6a_0$ lattice with 25:1 fuel. This is because the $6a_0$ experimental lattices have considerably larger core radii than the other lattices so that the four control rod guides are entirely in the core region. Thus, a significant complication is introduced into the analysis, and the derivation of an experimental buckling to compare with the calculated value would be difficult.

It is seen from Table XV that values of B^2 for various dimensions of a given 25:1 fuel lattice are essentially constant as desired. On the other hand, Table XVI gives values of B^2 for 15:1 lattices which increase with decreasing core height except for the case of the $6a_0$ lattice, for which the values are sensibly constant. It may be noted that the variations in B^2 in Table XVI occur for lattices which have shorter cores or smaller core radii than the lattices for which B^2 is essentially constant. Much of the difficulty may be due to the fact that the axial and radial fluxes are not exactly separable, and thus the separation of the geometric buckling into radial and axial components, i.e.,

$$B_g^2 = \left(\frac{2.4048}{R + \lambda_R} \right)^2 + \left(\frac{\pi}{H + \lambda_H} \right)^2 ,$$

is not strictly valid. A comparison of one- and two-dimensional calculations was made for a lattice which was similar to the 25:1 fuel, $2a_0$ lattice, although it contained five aluminum control rod guides in the core which were later removed. According to one-dimensional calculations, this core should have been critical with a core radius of 34 cm and a height of 84.6 cm. A PDQ calculation⁽²⁴⁾ gave a value of k_{eff} of 0.993 for this core, whereas a CURE⁽²⁵⁾ calculation yielded $k_{\text{eff}} = 0.996$. This core was not as short as some of the 15:1 fuel cores, and the radius was not small. It is possible that two-dimensional effects are more significant for very short cores of small radius.

As mentioned previously, the forest of fuel rods above the critical water level was ignored. This seems to be a reasonable approximation since the rods are not nearly as close together as for many light water reactors. If the parts of the rods above the water level had a significant effect on reactivity, one might expect a variation of buckling with height for all lattices.

Comparison of Tables IX and XV shows that the average experimental values of B^2 and the theoretical values are in fair agreement for the 25:1 lattices with 98.5% D_2O , the experimental values being a little lower. From Tables IX and X we see that a decrease in heavy water purity from 98.5% to 97.1% produced a theoretical change in buckling of +0.125 millibuck for the $2a_0$ lattice and of +0.015 millibuck for the $4a_0$ lattice. The experimental values derived from Table XV are +0.129 millibuck for the $2a_0$ lattice and -0.102 millibuck for the $4a_0$ lattice. Thus, the agreement is good in the case of the $2a_0$ lattice, but poor for the $4a_0$ lattice. The experimental changes are quite dependent on changes in reflector savings. To illustrate this, we note that if the 98.5% D_2O reflector savings were used for the 97.1% D_2O lattices, the resulting fictitious experimental buckling changes would be -0.031 millibuck and -0.185 millibuck for the $2a_0$ and $4a_0$ lattices, respectively.

A possible explanation for some of the discrepancy in the experimental and theoretical changes in buckling with D_2O purity for the $4a_0$ lattices is the values used for the heavy water purity. The values of 98.5% and 97.1% are based on mass-spectrographic analyses of water samples from ZPR-VII.⁽²⁾ However, as mentioned in Ref. 2, the analysis of the heavy water content on receipt at Savannah River gave a result 1.6% below the average of three samples withdrawn during the final unloading operation. Thus, there may be a systematic error in the mass-spectrographic determinations.

The theoretical values of buckling for the 15:1 lattices in Table XII and the experimental values in Table XVI are in reasonable agreement, with the experimental values being higher than the theoretical ones for the $2a_0$ and $3a_0$ lattices and lower for the $4a_0$ and $6a_0$ lattices. It is difficult to estimate how much error is present in the experimental values because of the necessity of using calculated reflector savings with experimental dimensions to give experimental bucklings. The trend of the variation in B^2 for the various dimensions of the $2a_0$, $3a_0$, and $4a_0$ lattices makes one wonder if the average of the values is the best experimental value. Also, the uncertainty in D_2O purity introduces some doubt as to the validity of the comparison between theory and experiment.

VI. CADMIUM RATIOS

A. Cadmium Ratios of Th²³² and Derived Values of Resonance Escape Probability

The cadmium ratio CR of a thorium foil in the fuel rod is given by

$$CR - 1 = \frac{fF_t \Sigma_{at} \Phi_t + f_e F_e \int_{\mu_{Cd}}^{\mu_t} \frac{\Sigma_a}{\xi \Sigma} q du}{(1-p)q_r(\tau_r) + f_e F_e \int_{\mu_r}^{\mu_{Cd}} \frac{\Sigma_a}{\xi \Sigma} q du},$$

where F_t and F_e are the fractions of the neutrons absorbed in the fuel rod which are absorbed in thorium for the thermal and epithermal groups. On using the formulae of Section II, we get

$$\begin{aligned} CR - 1 &= \frac{fF_t \Sigma_{at} \Phi_t + f_e F_e q_e(0) p_e^+ e^{-\tau_e^+ B^2} (1 - p_e^-) e^{-\bar{\tau}_e^- B^2}}{(1-p)q_r(\tau_r) + f_e F_e q_e(0) (1 - p_e^+) e^{-\bar{\tau}_e^+ B^2}} \\ &= \frac{p p_e f F_t \frac{e^{-\tau_e B^2}}{1 + L^2 B^2} + p p_e^+ (1 - p_e^-) f_e F_e e^{-\tau_e^+ B^2} e^{-\bar{\tau}_e^- B^2}}{(1-p) + p(1 - p_e^+) f_e F_e e^{-\bar{\tau}_e^+ B^2}}. \end{aligned} \quad (7)$$

In Eq. (7),

$$p_e^+ = e^{-\int_{\mu_r}^{\mu_{Cd}} \frac{\Sigma_a}{\xi \Sigma} du}; \quad p_e^- = e^{-\int_{\mu_{Cd}}^{\mu_t} \frac{\Sigma_a}{\xi \Sigma} du};$$

$$\tau_e^+ = \int_{\mu_r}^{\mu_{Cd}} \frac{D}{\xi \Sigma} du; \quad \tau_e^- = \int_{\mu_{Cd}}^{\mu_t} \frac{D}{\xi \Sigma} du,$$

and $\bar{\tau}_e^+$ and $\bar{\tau}_e^-$ are defined in a manner similar to $\bar{\tau}_e$ in Eq. (6), so that

$$\bar{\tau}_e^+ = \tau_e^+ \left(1 - \frac{2}{\mu_{Cd} - \mu_r} + \frac{e^{\frac{1}{2} \mu_r}}{e^{\frac{1}{2} \mu_{Cd}} - e^{\frac{1}{2} \mu_r}} \right)$$

and

$$\bar{\tau}_e^- = \tau_e^- \left(1 - \frac{2}{\mu_t - \mu_{Cd}} + \frac{e^{\frac{1}{2} \mu_{Cd}}}{e^{\frac{1}{2} \mu_t} - e^{\frac{1}{2} \mu_{Cd}}} \right) .$$

Equation (7) may be rewritten as

$$p = \left[1 + \frac{1}{(CR - 1)} \left(\frac{fF_t e^{-\tau_e B^2}}{1 + L^2 B^2} , p_e + f_e F_e p_e^+ (1 - p_e^-) e^{-\tau_e^+ B^2} e^{-\bar{\tau}_e^- B^2} \right. \right. \\ \left. \left. - f_e F_e (1 - p_e^+) e^{-\bar{\tau}_e^+ B^2} \right)^{-1} \right] , \quad (8)$$

which gives the experimental value of p in terms of the measured cadmium ratio and calculated constants.

Table XVII gives the experimental cadmium ratios and derived values of p for all the lattices. Superscripts E and C refer to experimental and calculated values, respectively. Experimental values of the effective resonance integral are given which are calculated from

$$\sigma_{eff}^E = \sigma_{eff}^C \left(\frac{\ln p^E}{\ln p^C} \right) .$$

The calculated cadmium ratios are those which would yield the calculated values of p . Calculations were done with the use of cadmium cutoff energies E_{Cd} equal to the thermal cutoff energy E_C and 0.4 ev for the 25:1 lattices, and 0.4 ev and 0.5 ev for the 15:1 lattices. The cadmium covers on the thorium foils were 0.028 cm thick and 0.051 cm thick in the 25:1 and 15:1 experiments, respectively. Cadmium cutoff energies have been computed by Dayton and Pettus⁽²⁶⁾ for foils in slab geometry, but they considered only a $1/E$ spectrum with no Maxwellian distribution. For thin foils the neglect of the Maxwellian distribution is serious. Stoughton et al.,⁽²⁷⁾ have calculated effective cutoff energies for a point $1/v$ absorber inside spherical cadmium filters. In their Fig. 2 they present cutoff energies as a function of the effective Maxwellian temperature

for foils of various thicknesses and for various ratios of the Maxwellian flux to the epithermal flux per unit lethargy. Their figure shows that cutoff energies for 0.028-cm (≈ 11 mils) cadmium filters are of the order of the thermal cutoff energy of $5kT_n$. Thus, for the 25:1 lattices, the cutoff energy $E_{Cd} = E_c$ appears to be more realistic than the value of 0.4 ev. Certainly the experimental values of p and σ_{eff} are in much better agreement with the theoretical values for a cutoff energy of E_c than for a cutoff energy of 0.4 ev.

Table XVII

THORIUM CADMIUM RATIOS AND RESONANCE ESCAPE PROBABILITIES

<u>25:1 Fuel</u>									
<u>Pitch</u>	<u>p^C</u>	<u>σ_{eff}^C</u>	<u>CR^E</u>	<u>$E_{Cd} = E_c$</u>			<u>$E_{Cd} = 0.4$ ev</u>		
				<u>p^E</u>	<u>σ_{eff}^E</u>	<u>CR^C</u>	<u>p^E</u>	<u>σ_{eff}^E</u>	<u>CR^C</u>
$2a_0$	0.8690	14.39	1.717	0.8687	14.42	1.719	0.8393	17.95	1.876
$3a_0$	0.9392	15.92	2.835	0.9411	15.41	2.791	0.9291	18.66	3.104
$4a_0$	0.9653	16.46	4.09	0.9639	17.14	4.182	0.9574	20.29	4.698

<u>15:1 Fuel</u>									
<u>Pitch</u>	<u>p^C</u>	<u>σ_{eff}^C</u>	<u>CR^E</u>	<u>$E_{Cd} = 0.4$ ev</u>			<u>$E_{Cd} = 0.5$ ev</u>		
				<u>p^E</u>	<u>σ_{eff}^E</u>	<u>CR^C</u>	<u>p^E</u>	<u>σ_{eff}^E</u>	<u>CR^C</u>
$2a_0$	0.8655	13.72	1.486	0.8617	14.14	1.499	0.8551	14.87	1.523
$3a_0$	0.9366	15.33	2.239	0.9353	15.65	2.262	0.9330	16.23	2.301
$4a_0$	0.9636	15.90	3.127	0.9613	16.92	3.246	0.9602	17.41	3.306
$6a_0$	0.9838	16.23	5.471	0.9823	17.75	5.830	0.9818	18.26	5.949

For the 15:1 lattices the cadmium thickness of about 20 mils gives a cutoff energy of the order of 0.4 ev or a little less. The experimental and theoretical values of p and σ_{eff} for 15:1 lattices are in rather good agreement for a cutoff energy of 0.4 ev and would be in better agreement for a slightly lower value.

B. Cadmium Ratios of U^{235}

The cadmium ratio of fissions for U^{235} is given by $1/\delta$, where δ is equal to the ratio of fissions above cadmium cutoff to total fissions. On using the notation of Sections II and VIA, we have

$$\begin{aligned} \delta &= \frac{\int_{\mu_r}^{\mu_{Cd}} \frac{\epsilon f_e \eta_e}{\nu_e} \frac{\Sigma_a}{\xi \Sigma} q du}{\frac{\epsilon \eta f \Sigma_{at} \Phi_t}{\nu_{th}} + \int_{\mu_r}^{\mu_t} \frac{\epsilon f_e \eta_e}{\nu_e} \frac{\Sigma_a}{\xi \Sigma} q du} \\ &= \frac{\frac{\epsilon f_e \eta_e}{\nu_e} \frac{(1 + L^2 B^2)}{P_e} e^{\tau_e B^2} (1 - p_e^+) e^{-\bar{\tau}_e^+ B^2}}{\frac{\epsilon \eta f}{\nu_{th}} + \frac{\epsilon \eta_e f_e}{\nu_e} \frac{(1 + L^2 B^2)}{P_e} e^{\tau_e B^2} (1 - p_e) e^{-\bar{\tau}_e B^2}} \quad (9) \end{aligned}$$

Now, on writing $\nu_e = \nu_{th} = \nu$ and making use of the criticality Eq. (5), there results

$$\begin{aligned} \delta &= \frac{1}{CR} = \frac{k_{\infty e} (1 + L^2 B^2) e^{\tau_e B^2} e^{-\bar{\tau}_e^+ B^2} (1 - p_e^+)}{k_{\infty p_e} + k_{\infty e} (1 + L^2 B^2) (1 - p_e) e^{(\tau_e - \bar{\tau}_e) B^2}} \\ &= \frac{k_{\infty e} (1 - p_e^+) e^{-\bar{\tau}_e^+ B^2}}{(1 + \tau_f B^2) e^{\tau_r B^2}} \quad (10) \end{aligned}$$

Equation (10) gives the inverse of the cadmium ratio of fissions for foils placed between pins in a fuel rod. For cases where a fission counter is used in the moderator, the term $\epsilon \eta f \Sigma_{at} \Phi_t / \nu_{th}$ in Eq. (9) must be multiplied by the ratio D of the thermal flux in the moderator to that in the fuel. If we denote the cadmium ratio in the moderator and its inverse by CR_M and δ_M , respectively, we have in place of Eq. (10)

$$\delta_M = \frac{1}{CR_M} = \frac{k_{\infty e} (1 + L^2 B^2) e^{\tau_e B^2} e^{-\bar{\tau}_e^+ B^2} (1 - p_e^+)}{D k_{\infty p_e} + k_{\infty e} (1 - p_e) (1 + L^2 B^2) e^{(\tau_e - \bar{\tau}_e) B^2}} \quad (11)$$

Values of the cadmium ratio of fissions in the fuel and in the moderator for 25:1 lattices were calculated for cadmium cutoffs of 0.3 ev and 0.4 ev. Results are listed in Table XVIII. Experimental values were measured for the $2a_0$ and $3a_0$ lattices only, by use of a U^{235} fission counter in the moderator with its cadmium cover being 0.041 cm (≈ 1.6 mils) thick. The measured values of CR_M were 6.9 and 14.2 for the $2a_0$ and $3a_0$ lattices, respectively (Ref. 2, Table XXXVI). One might expect a cadmium cutoff of the order of 0.3 ev for the cover thickness used from inspection of Fig. 2 of Ref. 27.

Table XVIII

CADMIUM RATIOS OF FISSION FOR 25:1 LATTICES

<u>Pitch</u>	<u>$E_{Cd} = 0.3 \text{ ev.}$</u>		<u>$E_{Cd} = 0.4 \text{ ev.}$</u>	
	<u>CR</u>	<u>CR_M</u>	<u>CR</u>	<u>CR_M</u>
$2a_0$	5.405	6.177	6.143	7.018
$3a_0$	11.85	13.93	13.56	15.94
$4a_0$	20.10	24.01	23.00	27.50

VII. WATER-WORTH EXPERIMENTS

The purpose of water-worth experiments is to determine information concerning the neutron migration area. In performing a water-worth experiment, the reactor is first made critical with all control rods withdrawn. Then, the water level is raised a small amount and a control rod is inserted to achieve criticality at the new height. The reactor is then held critical for a while in order to achieve equilibrium with the delayed-neutron emitters. Finally, the control rod is withdrawn as rapidly as possible and the power level is observed over several doubling times in order to determine the asymptotic period.

The change in reactivity, $\Delta\rho$, is related to the increase in height, ΔH , by the equation

$$\begin{aligned} \frac{\Delta\rho}{\Delta H} &= \frac{\Delta}{\Delta H} \left(1 - \frac{1}{k_{\text{eff}}} \right) \\ &= M'^2 \frac{2\pi^2}{(H + \lambda_H)^3} \left(1 + \frac{\Delta\lambda_H}{\Delta H} + \frac{B_r^2}{B_z^2} \frac{(H + \lambda_H)}{(R + \lambda_R)} \frac{\Delta\lambda_R}{\Delta H} \right) \quad , \quad (12) \end{aligned}$$

where M'^2 , the modified migration area, equals $\Delta\rho/\Delta B^2$ and depends on the form of the criticality equation used. For example, in two-group theory

$$M'^2 = \frac{L^2}{1 + L^2 B^2} + \frac{\tau}{1 + \tau B^2} \quad .$$

In Eq. (12), $\Delta\rho$ is assumed to be known from the inhour equation which gives the reactivity in terms of the asymptotic period and delayed-neutron parameters. Also, ΔH , H , R , λ_H , and λ_R are known experimentally or, perhaps, in the case of λ_H and λ_R , theoretically. However, in order to find M'^2 , it is necessary to calculate

$$C = C_H + C_R \quad ,$$

where

$$C_H = \Delta\lambda_H/\Delta H$$

and

$$C_R = \frac{B_r^2}{B_z^2} \frac{(H + \lambda_H)}{(R + \lambda_R)} \frac{\Delta\lambda_R}{\Delta H} \quad .$$

Calculation of C by a modified one-group theory has been considered in Ref. 28. A somewhat different approach by means of two-group theory and computer calculations has been mentioned and applied in Ref. 29. In the following two sections the one-group treatment is presented as well as a two-group treatment involving computer calculations. A demonstration of the equivalence of the methods of Refs. 28 and 29 is given in the Appendix.

A. One-group Treatment

We consider a reactor having a cylindrical core, and radial and bottom reflectors. The diffusion equations for the core and reflectors and the delayed-neutron precursor density equation are

$$D_c \nabla^2 \Phi_c - \Sigma_{ac} \Phi_c + (1 - \beta) k_\infty \Sigma_{ac} \Phi_c + \sum_i \frac{C_i}{T_i} = \frac{1}{V} \frac{\partial \Phi_c}{\partial t} \quad ; \quad (13)$$

$$\frac{\partial C_i}{\partial t} = \Sigma_{ac} k_\infty \beta_i \Phi_c - \frac{C_i}{T_i} \quad ; \quad (14)$$

$$D_r \nabla^2 \Phi_r - \Sigma_{ar} \Phi_r = \frac{1}{V} \frac{\partial \Phi_r}{\partial t} \quad ; \quad (15)$$

$$D_b \nabla^2 \Phi_b - \Sigma_{ab} \Phi_b = \frac{1}{V} \frac{\partial \Phi_b}{\partial t} \quad , \quad (16)$$

where notation is standard and subscripts c , r , and b refer to core, radial reflector, and bottom reflector. We assume a time dependence for the fluxes and delayed-neutron precursor densities proportional to $e^{t/T}$, where T is the asymptotic period. Equations (13) and (14) may be combined to give

$$(\nabla^2 + B_g^2) \Phi_c = 0 \quad , \quad (13')$$

where the following definitions are used:

$$L_c^2 = \frac{D_c}{\Sigma_{ac}} \quad ; \quad B_m^2 = \frac{k_\infty - 1}{L_c^2} \quad ; \quad l_{oc} = \frac{1}{\Sigma_{ac} V} \quad ,$$

and

$$B_m^2 - B_g^2 = \frac{k_\infty}{L_c^2} \left[\frac{l_{oc}}{k_\infty T} + \sum_i \frac{\beta_i T_i}{T + T_i} \right] \quad .$$

On substituting

$$\ell_c = \frac{\ell_{oc}}{1 + L_c^2 B_g^2} ,$$

$$k_{eff} = \frac{k_{\infty}}{1 + L_c^2 B_g^2} ,$$

and

$$\rho = 1 - \frac{1}{k_{eff}} ,$$

there is obtained

$$B_m^2 - B_g^2 = \frac{k_{\infty} \rho}{L_c^2} = \frac{k_{\infty}}{L_c^2} \left[\frac{1}{k_{eff}} \left(\frac{\ell_c}{T} + k_{eff} \sum_i \frac{\beta_i T_i}{T + T_i} \right) \right] , \quad (17)$$

where the expression for the reactivity ρ in the square brackets in Eq. (17) agrees with the usual definition of reactivity in terms of the asymptotic period, delayed-neutron parameters, and prompt-neutron lifetime (Ref. 30, p. 301).

Equations (15) and (16) may be rewritten as

$$(\nabla^2 - K_{rg}^2) \Phi_r = 0 \quad ; \quad (15')$$

$$(\nabla^2 - K_{bg}^2) \Phi_b = 0 \quad , \quad (16')$$

where

$$K_{rg}^2 = K_r^2 \left(1 + \frac{\ell_{or}}{T} \right) \text{ with } K_r^2 = \frac{1}{L_r^2} = \frac{\Sigma_{ar}}{D_r} \quad ; \quad \ell_{or} = \frac{1}{\Sigma_{ar} V} \quad ,$$

and similar definitions hold for the bottom reflector.

The solutions of Eqs. (13'), (15'), and (16') for a reactor with core radius R and height H in the supercritical state are

$$\Phi_c = A \sin(B_{gz}z) J_0(B_{gr}r) \quad ; \quad (18)$$

$$\Phi_r = C \sin(B_{gz}z) K_0(p_r r) \quad ; \quad (19)$$

$$\Phi_b = E \sinh p_b (t + H - z) J_0(B_{gr}r) \quad , \quad (20)$$

where

$$B_g^2 = B_{gr}^2 + B_{gz}^2 \quad ; \quad K_{rg}^2 = p_r^2 - B_{gz}^2 \quad ; \quad K_{bg}^2 = p_b^2 - B_{gr}^2 \quad ,$$

A, C, and E are constants, t is the thickness of the bottom axial reflector, and the radial reflector has been taken as infinitely thick. The origin of coordinates is at the center of the top surface of the core and z is measured downwards. It has been assumed that the geometry is separable, although this is not strictly true when radial and bottom reflectors are present.

From the equation

$$\left. \frac{D_c}{\Phi_c} \frac{d\Phi_c}{dr} \right|_R = \left. \frac{D_r}{\Phi_r} \frac{d\Phi_r}{dr} \right|_R \quad ,$$

which expresses the continuity of current and flux at the core-radial reflector interface, we have

$$D_c B_{gr} \frac{J_1(B_{gr}R)}{J_0(B_{gr}R)} = D_r p_r \frac{K_1(p_r R)}{K_0(p_r R)} \quad . \quad (21)$$

Likewise, the core-axial reflector interface continuity condition yields

$$D_c B_{gz} \cot(B_{gz}H) = -D_b p_b \coth(p_b t) \quad . \quad (22)$$

Now, consider a critical reactor identical with the supercritical one except that the water level corresponds to a core height H_0 , where $H = H_0 + \Delta H$. Equation (21) will still be valid for the critical reactor except that B_{gr} and p_r must be replaced by B_{gr_0} and p_{r_0} , where the zero subscripts denote criticality values. On writing

$$X = B_{gr_0}R + \Delta X = X_0 + \Delta X \quad ; \quad y = p_{r_0}R + \Delta y = y_0 + \Delta y,$$

and using Eq. (21) with and without the zero subscripts, we get

$$\begin{aligned} \frac{X_0 \Delta X}{y_0 \Delta y} &= \frac{\Delta B_{gr}^2}{\Delta p_r^2} = \frac{-\frac{2B_{gr_0}^3}{j_{01}} \Delta \lambda_R}{K_r^2 \frac{\ell_{or}}{T} - \frac{2B_{gz_0}^3}{\pi} (\Delta H + \Delta \lambda_H)} \\ &= \frac{X_0 J_0(X_0) J_1(X_0)}{y_0 K_0(y_0) K_1(y_0)} \left(\frac{K_1^2(y_0) - K_0^2(y_0)}{J_1^2(X_0) + J_0^2(X_0)} \right) = e \quad , \end{aligned}$$

where $B_{gr_0} = \frac{j_{01}}{R + \lambda_{R_0}}$ and $B_{gz_0} = \frac{\pi}{H_0 + \lambda_{H_0}}$. This may be written as

$$\left(\frac{B_{gr_0}^3}{B_{gz_0}^3} \frac{\pi}{j_{01}} \frac{\Delta \lambda_R}{\Delta H} \right) - e \left(\frac{\Delta \lambda_H}{\Delta H} \right) = e \left(1 - \frac{K_r^2 \ell_{or}}{T} \frac{\pi}{2B_{gz_0}^3 \Delta H} \right) . \quad (23)$$

From Eq. (22) with and without zero subscripts on B_{gz} , p_b , and H , we obtain

$$(e') \left(\frac{B_{gr_0}^3}{B_{gz_0}^3} \frac{\pi}{j_{01}} \frac{\Delta \lambda_R}{\Delta H} \right) - \left(\frac{\Delta \lambda_H}{\Delta H} \right) = -e'' + \frac{K_b^2 \ell_{ob}}{T} \frac{\pi}{2B_{gz_0}^3 \Delta H} e' \quad , \quad (24)$$

where

$$e' = \frac{B_{gz_0}^2}{p_{b_0}^2} \frac{\left(1 - \frac{p_{b_0} t}{\sinh(p_{b_0} t) \cosh(p_{b_0} t)} \right)}{\left(1 - \frac{B_{gz_0} H_0}{\sin(B_{gz_0} H_0) \cos(B_{gz_0} H_0)} \right)}$$

and

$$1 + e'' = \frac{\pi}{B_{gz_0} H_0 - \sin(B_{gz_0} H_0) \cos(B_{gz_0} H_0)} .$$

Equation (17) is equivalent to

$$\left(\frac{B_{gr_0}^3}{B_{gz_0}^3} \frac{\pi}{j_{01}} \frac{\Delta \lambda_R}{\Delta H} \right) + \left(\frac{\Delta \lambda_H}{\Delta H} \right) = -1 + \frac{\rho}{L'^2} \frac{\pi}{2B_{gz_0}^3 \Delta H} \quad , \quad (25)$$

where $L'^2 = L^2/(1 + L^2 B_g^2)$ and we have used $B_m^2 = B_{g_0}^2 = B_{gz_0}^2 + B_{gr_0}^2$.

Equations (23) and (24) may now be solved for $C_H = \Delta\lambda_H/\Delta H$ and $C_R = (B_{gr0}^3/B_{gz0}^3) (\pi/j_{01}) (\Delta\lambda_R/\Delta H)$, and the results substituted in Eq. (25) to give

$$\left[1 + C_0 + C_1 (\Delta H, T)\right] L'^2 = \frac{\rho}{2B_{gz0}^3} \frac{\pi}{\Delta H} \quad (26)$$

Here

$$C = C_0 + C_1 \quad ; \quad C_0 = \frac{e + ee' + ee'' + e''}{1 - ee'} \quad ,$$

and

$$C_1 = \frac{-\pi}{2B_{gz0}^3 \Delta H} \frac{\left(\frac{K_r^2 \ell_{or}}{T} e(1 + e') + \frac{K_b^2 \ell_{ob}}{T} e'(1 + e) \right)}{1 - ee'}$$

Equation (26) may be rewritten as

$$(1 + C_0) L'^2 \left(\frac{2B_{gz0}^3}{\pi} \right) \Delta H = \rho + \frac{\ell_c}{T} \left(\frac{D_c}{D_r} \frac{e(1 + e')}{1 - ee'} + \frac{D_c}{D_b} \frac{e'(1 + e)}{1 - ee'} \right) \quad (27)$$

where, from (17),

$$\rho = \frac{1}{k_{eff}} \left(\frac{\ell_c}{T} + k_{eff} \sum_i \frac{\beta_i T_i}{T + T_i} \right)$$

The term $\ell_c/k_{eff}T$ is only a small fraction of ρ for periods of the magnitudes of those used in water-worth experiments. Also, the second term on the right hand side of Eq. (27) is roughly of the order of magnitude of $\ell_c/k_{eff}T$ and so can be ignored in comparison with ρ . Thus, we write approximately that $C = C_0$ and

$$\frac{\rho}{\Delta H} = L'^2 \left(\frac{2B_{gz0}^3}{\pi} \right) (1 + C) \quad (28)$$

Illustrative one-group calculations have been done with the use of Eqs. (21), (22), (23), and (24) with the $1/T$ terms in the reflectors being ignored. In these calculations, the same constants were used as for the two-group reflector savings calculations, with modified one-group constants

being defined by replacing L^2 with $L^2 + \tau_{2G}$ and D with $(\tau_{2G} D_1 + L^2 D_2) / (\tau_{2G} + L^2)$. Results for 15:1 lattices are given in Table XIX. The value of C increases with increasing critical height because the increase in $(B_{gr0}^3 / B_{gz0}^3) \pi / j_{01}$ produces an increase in C_R , although both $\Delta\lambda_R / \Delta H$ and $\Delta\lambda_H / \Delta H$ decrease with increasing critical height. A check of the approximation made by ignoring the second term on the right-hand side of Eq. (27) was carried out by finding the ratio of this term to $\ell_c / (k_{eff} T)$ for two cases with $2a_0$ pitch. For the cases corresponding to $B_r^2 / B^2 = 0.509$ and 0.755 , the values of this ratio are 0.516 and 0.600 , respectively, which shows that neglecting the second term is satisfactory.

Table XIX

ONE-GROUP CALCULATIONS OF C FOR 15:1 LATTICES

Pitch	H_0 (cm)	λH_0 (cm)	R (cm)	λR_0 (cm)	B_r^2 / B^2	C_H	$\frac{\Delta\lambda_R}{\Delta H}$	C_R	C
$2a_0$	63.12	15.42	36.83	22.18	0.509	0.0594	0.1336	0.1846	0.2440
	68.59	15.37	33.54	22.21	0.571	0.0502	0.1132	0.2266	0.2768
	75.37	15.32	30.73	22.24	0.632	0.0412	0.0934	0.2746	0.3158
	84.07	15.28	28.26	22.32	0.693	0.0328	0.0747	0.3317	0.3645
	95.85	15.22	26.09	22.39	0.755	0.0245	0.0566	0.3987	0.4232
$3a_0$	59.65	14.40	32.40	20.40	0.535	0.0602	0.1332	0.2153	0.2755
	64.20	14.34	29.99	20.43	0.587	0.0519	0.1154	0.2556	0.3075
	72.87	14.26	26.90	20.51	0.664	0.0402	0.0899	0.3271	0.3673
	85.16	14.19	24.22	20.63	0.742	0.0287	0.0656	0.4179	0.4466
	96.93	14.14	22.64	20.73	0.794	0.0217	0.0500	0.4919	0.5135
$4a_0$	59.96	14.00	34.90	20.00	0.516	0.0561	0.1339	0.1925	0.2486
	67.13	13.99	30.95	20.10	0.597	0.0446	0.1078	0.2533	0.2979
	76.81	13.88	27.68	20.24	0.677	0.0337	0.0826	0.3280	0.3617
	90.94	13.78	24.90	20.40	0.758	0.0233	0.0585	0.4236	0.4469
$6a_0$	69.87	14.09	42.98	20.61	0.505	0.0380	0.1139	0.1556	0.1916
	80.80	13.92	36.98	20.83	0.611	0.0277	0.0857	0.2208	0.2485
	97.31	13.76	32.27	21.10	0.717	0.0184	0.0583	0.3080	0.3264
	126.88	13.61	28.39	21.43	0.823	0.0100	0.0325	0.4269	0.4369

It should be mentioned that values of C for light water reactors are in general considerably smaller than for heavy water reactors. This is essentially because the migration area, $M^2 = L^2 + \tau$, in the reflectors is much larger for heavy water reactors.

B. Two-group Treatment

The two-group time-dependent diffusion equations in the core and the delayed-neutron precursor density equation are

$$D_1 \nabla^2 \Phi_1 - \Sigma_1 \Phi_1 + (1 - \beta) \frac{k_{\infty}}{p} \Sigma_2 \Phi_2 + \sum_i \frac{C_i}{T_i} = \frac{1}{v_1} \frac{\partial \Phi_1}{\partial t} ; \quad (29)$$

$$D_2 \nabla^2 \Phi_2 - \Sigma_2 \Phi_2 + p \Sigma_1 \Phi_1 = \frac{1}{V_2} \frac{\partial \Phi_2}{\partial t} \quad ; \quad (30)$$

$$\frac{\partial C_i}{\partial t} = - \frac{C_i}{T_i} + \beta_i \frac{k_\infty}{p} \Sigma_2 \Phi_2 \quad , \quad (31)$$

where the subscripts 1 and 2 refer to fast and thermal constants, respectively. On assuming time dependencies proportional to $e^{t/T}$ for Φ_1 , Φ_2 , and C_i , we obtain

$$\nabla^2 \Phi_1 - \frac{1}{\tau} \Phi_1 + \frac{\Sigma_2}{D_1} \frac{k_\infty}{p} \Phi_2 = \frac{1}{D_1 V_1 T} \Phi_1 + \frac{\Sigma_2 k_\infty}{D_1 p} \sum_i \frac{\beta_i T_i}{T + T_i} \Phi_2 \quad (29')$$

and

$$\nabla^2 \Phi_2 - \frac{1}{L^2} \Phi_2 + p \frac{\Sigma_1}{D_2} \Phi_1 = \frac{1}{D_2 V_2 T} \Phi_2 \quad . \quad (30')$$

If we define ρ by writing

$$\nabla^2 \Phi_i = -B_g^2 \Phi_i$$

for $i = 1$ or 2 ,

$$\rho = 1 - \frac{1}{k_{\text{eff}}} \quad ; \quad k_{\text{eff}} = \frac{k_\infty}{(1 + L^2 B_g^2) (1 + \tau B_g^2)} \quad ;$$

$$\text{and } 1 = \frac{k_\infty}{(1 + L^2 B_m^2) (1 + \tau B_m^2)} \quad ,$$

there results

$$\rho = \frac{1}{k_{\text{eff}}} \left[\frac{\ell_1 + \ell_2}{T} + \frac{\ell_1 \ell_2}{T^2} + k_{\text{eff}} \sum_i \frac{\beta_i T_i}{T + T_i} \right] \quad ,$$

where

$$\ell_1 = \frac{1}{1 + \tau B_g^2} \frac{1}{\Sigma_1 V_1 T}$$

and

$$\ell_2 = \frac{1}{1 + L^2 B_g^2} \frac{1}{\Sigma_2 V_2 T} \quad .$$

Since $\ell_1 \ll \ell_2$, this is almost the same as the one-group expression for ρ in Eq. (17).

If the terms in $1/T$ in Eqs. (29') and (30') are ignored, these equations become

$$\nabla^2 \Phi_1 - \frac{1}{\tau} \Phi_1 + \frac{\Sigma_2 k_\infty}{D_1 P} \left(1 - \sum_i \frac{\beta_i T_i}{T + T_i} \right) \Phi_2 = 0 \quad ; \quad (29'')$$

$$\nabla^2 \Phi_2 - \frac{1}{L^2} \Phi_2 + p \frac{\Sigma_1}{D_2} \Phi_1 = 0 \quad . \quad (30'')$$

The only difference between Eqs. (29'') and (30'') and the time-independent equations is that k_∞ is replaced by

$$k_\infty' = k_\infty \left(1 - \sum_i \frac{\beta_i T_i}{T + T_i} \right) \approx k_\infty (1 - \rho) \quad .$$

Also, the time-dependent equations in the reflectors reduce to the time-independent ones if the terms in $1/T$ are neglected. Thus, one can use reflector savings codes RE-169 and RE-188 to make calculations for the critical reactor and for the supercritical reactor treated as a critical reactor with a different value of k_∞ , namely $k_\infty' = k_\infty (1 - \rho)$, and consequently a different B^2 . Then, making use of the fact that R is constant, one can find values for $\Delta\lambda_R$, $\Delta\lambda_H$, ΔH , and, consequently, C . This procedure was used in making two-group calculations for both 15:1 and 25:1 lattices. The constants used in the RE-169 and RE-188 inputs were the same as those used for the reflector savings calculations of Section V which were combined with experimental dimensions to yield geometric bucklings. A value of $k_\infty' = k_\infty - 0.003$ was used in the supercritical cases.

Table XX gives the dimensions and reflector savings involved in the calculation of C for one of the 25:1 lattice $2a_0$ cases. Values of λ_H in Table XX are smaller than in Table XIV because those in Table XIV include the top reflector savings of $0.71 \lambda_{tr}$ which is included in H in Table XX. From Table XX we have

$$\Delta H = 1.604 \text{ cm} \quad ; \quad \Delta\lambda_R = 0.091 \text{ cm} \quad ; \quad \Delta\lambda_H = 0.026 \text{ cm} \quad ;$$

$$C_H = \frac{\Delta\lambda_H}{\Delta H} = 0.0162 \quad ; \quad C_R = \frac{B_R^2}{B_Z^2} \left(\frac{H + \lambda_H}{R + \lambda_R} \right) \frac{\Delta\lambda_R}{\Delta H} = 0.4493 \quad ,$$

$$\text{and } C = C_H + C_R = 0.4655 \quad .$$

Results of the two-group calculations of C for all the lattices are given in Tables XXI and XXII. Comparison of the two-group values for 15:1 lattices in Table XXII and the one-group values in Table XIX shows that for some cases the two-group values are larger, whereas for other

cases the one-group values are larger. For a given value of B_r^2/B^2 , the variation of C for different pitches is larger in the two-group than in the one-group case.

Table XX

DIMENSIONS AND REFLECTOR SAVINGS INVOLVED IN A TYPICAL CALCULATION OF C

B_z^2 ($\text{cm}^{-2} \times 10^{-3}$)	$R+\lambda_R$ (cm)	R (cm)	λ_R (cm)	B_r^2 ($\text{cm}^{-2} \times 10^{-3}$)	$H+\lambda_H$ (cm)	H (cm)	λ_H (cm)
$B^2 = 2.1609 \times 10^{-3} \text{ cm}^{-2} \text{ (critical)}$							
0.5000	59.007	28.624	30.383	1.6609	140.496	120.868	19.628
$B^2 = 2.1444 \times 10^{-3} \text{ cm}^{-2} \text{ (supercritical)}$							
0.4919	59.157	28.690	30.467	1.6593	142.638	122.986	19.652
0.4871	59.071	28.594	30.477	1.6560	142.155	122.501	19.654
0.4823	58.987	28.499	30.488	1.6527	141.677	122.021	19.656
	R (cm)	λ_R (cm)	B_r^2 ($\text{cm}^{-2} \times 10^{-3}$)	H (cm)	λ_H (cm)	B_z^2 ($\text{cm}^{-2} \times 10^{-3}$)	B^2 ($\text{cm}^{-2} \times 10^{-3}$)
critical	28.624	30.383	1.6609	120.868	19.628	0.5000	2.1609
supercritical	28.624	30.474	1.6558	122.472	19.654	0.4886	2.1444

Table XXI

TWO-GROUP CALCULATIONS OF C FOR 25:1 LATTICES

Pitch	R (cm)	λ_R (cm)	H (cm)	λ_H (cm)	B_r^2/B^2	C_H	C_R	C
$2a_0$	35.161	29.788	91.952	19.821	0.634	0.0280	0.2980	0.3260
	32.022	30.051	102.551	19.735	0.695	0.0223	0.3554	0.3777
	28.624	30.383	120.868	19.628	0.769	0.0162	0.4493	0.4655
	25.267	30.782	156.110	19.510	0.852	0.0090	0.5777	0.5867
$3a_0$	33.294	25.125	96.483	16.732	0.688	0.0192	0.3226	0.3418
	30.020	25.523	112.725	16.612	0.761	0.0144	0.3996	0.4140
	26.493	26.053	146.856	16.468	0.850	0.0077	0.5170	0.5247
	22.664	26.791	297.864	16.295	0.959	0.0012	0.7393	0.7405
$4a_0$	37.093	24.347	105.565	15.805	0.696	0.0131	0.2796	0.2927
	34.934	24.594	115.849	15.738	0.741	0.0115	0.3228	0.3343
	32.552	24.901	132.436	15.660	0.796	0.0084	0.3806	0.3890
	30.181	25.253	160.044	15.576	0.855	0.0048	0.4405	0.4453

Table XXII

TWO-GROUP CALCULATIONS OF C FOR 15:1 LATTICES

<u>Pitch</u>	<u>R</u> (cm)	<u>λ_R</u> (cm)	<u>H</u> (cm)	<u>λ_H</u> (cm)	<u>B_r^2/B^2</u>	<u>C_H</u>	<u>C_R</u>	<u>C</u>
$2a_0$	30.881	25.548	67.338	18.806	0.577	0.0482	0.2773	0.3255
	26.655	25.870	78.329	18.623	0.666	0.0357	0.3691	0.4048
	22.197	26.326	101.207	18.391	0.781	0.0229	0.5325	0.5554
	17.524	27.008	189.050	18.101	0.927	0.0051	0.8141	0.8192
$3a_0$	35.910	21.179	57.387	16.253	0.494	0.0549	0.1923	0.2472
	29.294	21.694	68.914	15.963	0.619	0.0383	0.2954	0.3337
	22.477	22.534	99.912	15.575	0.794	0.0175	0.5209	0.5384
	17.463	23.512	241.280	15.230	0.958	0.0020	0.8551	0.8571
$4a_0$	34.973	20.540	66.751	15.188	0.561	0.0361	0.2101	0.2462
	29.471	21.041	80.652	14.944	0.677	0.0243	0.3163	0.3406
	23.644	21.830	119.332	14.626	0.836	0.0100	0.5179	0.5279
	19.904	22.564	251.122	14.391	0.958	0.0016	0.7332	0.7348
$6a_0$	42.030	20.869	83.305	14.583	0.587	0.0187	0.1651	0.1838
	37.796	21.196	94.594	14.452	0.667	0.0139	0.2124	0.2263
	33.373	21.627	116.156	14.292	0.767	0.0098	0.2964	0.3062
	28.800	22.219	177.091	14.100	0.892	0.0033	0.4107	0.4140

C. Comparison of Theory and Experiment

The criticality equation, Eq. (5), corresponds to

$$k_{\text{eff}} = \frac{p_e k_{\infty} + (1 - p_e) k_{\infty e} (1 + L^2 B^2) e^{(\tau_e - \bar{\tau}_e) B^2}}{(1 + L^2 B^2) (1 + \tau_f B^2) e^{(\tau_e + \tau_r) B^2}},$$

which gives

$$M^2 = (\tau_e + \tau_r + \tau_f) + \frac{L^2}{1 + L^2 B^2} - \tau_f \left(\frac{\tau_f B^2}{1 + \tau_f B^2} \right) - (1 - p_e) \frac{k_{\infty e} e^{-(\tau_r + \bar{\tau}_e) B^2}}{1 + \tau_f B^2} \left(\frac{L^2}{1 + L^2 B^2} + \tau_e - \bar{\tau}_e \right) \quad (32)$$

Table XXIII gives theoretical values of the terms involved in the modified migration areas for all lattices. In this table,

$$T_1 = \tau = \tau_e + \tau_r + \tau_f \quad ; \quad T_2 = \frac{L^2}{1 + L^2 B^2} \quad ; \quad T_3 = \frac{\tau_f (\tau_f B^2)}{1 + \tau_f B^2} \quad ; \quad \text{and}$$

$$T_4 = (1 - p_e) \frac{k_{\infty} e^{-(\tau_r + \bar{\tau}_e) B^2}}{1 + \tau_f B^2} \left(\frac{L^2}{1 + L^2 B^2} + \tau_e - \bar{\tau}_e \right) .$$

Table XXIII

VALUES OF TERMS INVOLVED IN MODIFIED MIGRATION AREAS

<u>Fuel</u>	<u>Pitch</u>	<u>T₁</u> (cm ²)	<u>T₂</u> (cm ²)	<u>T₃</u> (cm ²)	<u>T₄</u> (cm ²)	<u>T₂-T₃-T₄</u> (cm ²)	<u>M'²</u> (cm ²)
25:1	2a ₀	130.44	19.96	5.40	7.13	7.43	137.87
	3a ₀	122.90	38.89	5.43	5.73	27.73	150.63
	4a ₀	120.56	63.58	4.73	5.28	53.57	174.13
15:1	2a ₀	123.05	12.11	6.85	7.32	-2.06	120.99
	3a ₀	117.26	23.48	6.97	5.85	10.66	127.92
	4a ₀	115.47	38.32	6.35	5.33	26.64	142.11
	6a ₀	114.10	76.56	4.78	4.98	66.80	180.90

Values of the reactivity were calculated for the experimental periods by means of the equation

$$\rho = \frac{\ell}{T} + E_D \sum_{i=1}^6 \frac{\beta_i}{1 + \lambda_i T} + E_P \sum_{j=1}^9 \frac{\beta_j}{1 + \lambda_j T}$$

$$= L(T) + E_D D(T) + E_P P(T) \quad . \quad (33)$$

The yields and transformation constants of the six delayed-neutron groups and nine photoneutron groups were taken from Table XXXVIII of Ref. 2 except that the yield of the photoneutron group of longest half-life was taken as $\beta = 5 \times 10^{-7}$ rather than as 4.07×10^{-7} . As mentioned in Ref. (2), the delayed-neutron constants were obtained from the data of Keepin and Wimett,⁽³¹⁾ whereas the photoneutron constants are those of Bernstein et al.⁽³²⁾ In Eq. (33) the factor E_D allows for the fact that delayed neutrons are born with less energy than prompt neutrons and consequently have less probability of leakage during slowing down. Its value was taken to be $E_D = \exp \left[\left(\frac{1}{3} \right) \tau B^2 \right]$. The quantity E_P takes care of the reduced leakage

probability of photoneutrons and the absorption by fuel elements of some gamma rays which would otherwise produce photoneutrons. The first of these situations tends to make $E_P > 1$, whereas the second tends to make it less than one. Values of the ratio $E = E_P/E_D$ have been measured for some lattices and are recorded in Table XXXIX of Ref. 2. Based on this table, the values $E_P/E_D = 0.7, 0.8$, and 1.0 were used for the 25:1 lattices with pitches $2a_0, 3a_0$, and $4a_0$, respectively; the values $0.6, 0.7, 0.9$, and 1.0 were used for the $2a_0, 3a_0, 4a_0$, and $6a_0$ lattices with 15:1 fuel. Results of the calculations of L, D, P , and ρ for experimental values of T and ΔH are given in Tables XXIV and XXV. The quantities $D(T)$ and $P(T)$ were obtained from a tabulation by K. Plumlee⁽³³⁾ who calculated them by means of an IBM-704 code.

Table XXIV

EXPERIMENTAL VALUES OF REACTIVITIES FOR 25:1 LATTICES

Pitch	Loading	ΔH (cm)	T (sec)	L ($\times 10^{-3}$)	D ($\times 10^{-3}$)	P ($\times 10^{-3}$)	ρ ($\times 10^{-3}$)	$\rho/\Delta H$ ($\times 10^{-4}\text{cm}^{-1}$)	Pitch	Loading	ΔH (cm)	T (sec)	L ($\times 10^{-3}$)	D ($\times 10^{-3}$)	P ($\times 10^{-3}$)	ρ ($\times 10^{-3}$)	$\rho/\Delta H$ ($\times 10^{-4}\text{cm}^{-1}$)
2a ₀	90	1.524	68.53	0.00127	0.83407	0.25761	1.1161	7.323	3a ₀	105	0.762	74.80	0.00248	0.78229	0.24910	1.0881	14.28
	90	2.794	28.02	0.00312	1.4997	0.35430	1.9238	6.886		105	1.143	41.75	0.00445	1.1725	0.30891	1.5745	13.78
	Average = 7.105									105	1.524	26.06	0.00712	1.5643	0.36301	2.0584	13.51
	89	1.270	77.33	0.00113	0.76325	0.24591	1.0291	8.103		Average = 13.86							
	89	2.032	42.07	0.00208	1.1666	0.30807	1.5212	7.486		106	0.762	56.74	0.00327	0.95387	0.27652	1.3029	17.10
	89	2.794	22.35	0.00391	1.7061	0.38200	2.1728	7.777		106	0.508	112.4	0.00165	0.57226	0.21180	0.82197	16.18
	Average = 7.789									106	1.016	38.72	0.00532	1.2306	0.31719	1.6471	16.21
	92	0.762	83.63	0.00104	0.71976	0.23851	0.97555	12.80		Average = 16.50							
	92	1.524	28.95	0.00332	1.4712	0.35045	1.8898	12.40		107	0.762	51.11	0.00363	1.0250	0.28732	1.3915	18.26
	92	1.143	47.72	0.00183	1.0736	0.29454	1.4083	12.32		107	0.508	101.5	0.00183	0.62023	0.22078	0.88315	17.38
	Average = 12.51									107	1.016	34.91	0.00532	1.3134	0.32880	1.7488	17.21
	93	1.270	32.65	0.00267	1.3686	0.33643	1.7656	13.90		Average = 17.62							
	95	0.762	51.50	0.00170	1.0197	0.28652	1.3428	17.62		108	0.508	74.14	0.00250	0.78742	0.24994	1.0945	21.55
	95	1.016	31.31	0.00279	1.4039	0.34128	1.8082	17.80		108	0.889	31.88	0.00582	1.3896	0.33919	1.8418	20.72
	95	1.270	20.63	0.00423	1.7825	0.39217	2.2649	17.83		108	0.711	46.95	0.00395	1.0853	0.29626	1.4664	20.62
	Average = 17.75									Average = 20.96							
3a ₀	103	1.524	49.40	0.00376	1.0490	0.29088	1.4213	9.326	4a ₀	118	1.270	53.51	0.00585	0.99335	0.28255	1.4004	11.03
	103	2.286	25.81	0.00719	1.5731	0.36420	2.0693	9.052		118	0.762	125.5	0.00249	0.52377	0.20238	0.79617	10.45
	103	0.762	153.2	0.00121	0.44452	0.18607	0.65749	8.628		118	1.778	33.95	0.00922	1.3362	0.33196	1.8326	10.31
	Average = 9.002									Average = 10.60							
	104	1.016	61.00	0.00304	0.90658	0.26917	1.2349	12.24									
	104	1.778	23.89	0.00777	1.6437	0.37367	2.1563	12.13									
	Average = 12.19																

Table XXV

EXPERIMENTAL VALUES OF REACTIVITIES FOR 15:1 LATTICES

Pitch	Loading	ΔH (cm)	T (sec)	L ($\times 10^{-3}$)	D ($\times 10^{-3}$)	P ($\times 10^{-3}$)	ρ ($\times 10^{-3}$)	$\rho/\Delta H$ ($\times 10^{-4}\text{cm}^{-1}$)	Pitch	Loading	H (cm)	T (sec)	L ($\times 10^{-3}$)	D ($\times 10^{-3}$)	P ($\times 10^{-3}$)	ρ ($\times 10^{-3}$)	$\rho/\Delta H$ ($\times 10^{-4}\text{cm}^{-1}$)
2a ₀	385	1.016	63.91	0.00084	0.87704	0.26451	1.1795	11.61	4a ₀	407	1.270	42.80	0.00460	1.1537	0.30622	1.6297	12.83
	381	0.635	78.48	0.00068	0.75491	0.24451	1.0267	16.17	405	1.270	15.41	0.01277	2.0749	0.43113	2.8131	22.15	
	382	0.635	45.40	0.00118	1.1098	0.29845	1.4679	23.12	404	0.284	54.84	0.00359	0.97666	0.28001	1.4006	49.32	
	383	0.381	52.73	0.00102	1.0034	0.23407	1.3368	35.09	6a ₀	409	1.270	14.92	0.02703	2.1081	0.43557	2.8226	22.23
3a ₀	402	1.524	23.05	0.00503	1.6769	0.37810	2.2398	14.70									
	401	0.191	113.0	0.00103	0.56984	0.21133	0.82718	43.31									
	400	0.142	115.5	0.00100	0.55997	0.20945	0.81429	57.34									

Tables XXVI and XXVII compare the experimental and theoretical values of M^{12} for 25:1 and 15:1 lattices, respectively. In these tables C was found from graphs of C vs B_T^2/B^2 drawn from use of the data of Tables XXI

and XXII, $(d\rho/dH)_C$ is the product of $(2B_Z^3/\pi)(1+C)$ and the $M_C'^2$ values taken from Table XXIII, $(d\rho/dH)_E$ is from Tables XXIV and XXV, $M_E'^2$ is the ratio of $(d\rho/dH)_E$ to $(2B_Z^3/\pi)(1+C)$, C_E is the value of C which would make $M_E'^2$ equal to $M_C'^2$ and Δ is $100 \left(\frac{M_E'^2 - M_C'^2}{M_C'^2} \right)$.

Table XXVI

CALCULATED AND EXPERIMENTAL VALUES OF M'^2 FOR 25:1 LATTICES

Pitch	Loading	B_F^2/B^2	C	$\frac{2B_Z^3}{\pi}(1+C)$ (cm ⁻³ x 10 ⁻⁶)	$M_C'^2$ (cm ²)	$\left(\frac{d\rho}{dH}\right)_C$ (cm ⁻¹ x 10 ⁻⁴)	$\left(\frac{d\rho}{dH}\right)_E$ (cm ⁻¹ x 10 ⁻⁴)	$M_E'^2$ (cm ²)	C_E	Δ
2a ₀	90	0.845	0.572	6.079	137.87	8.381	7.105	116.9	0.333	-15.2
	89	0.834	0.557	6.632		9.144	7.789	117.4	0.326	-14.8
	92	0.781	0.481	9.605		13.24	12.51	130.2	0.399	-5.6
	93	0.756	0.448	11.023		15.20	13.90	126.1	0.324	-8.5
	95	0.705	0.388	14.095		19.43	17.75	125.9	0.268	-8.7
Average = 123.3									Average	-10.6
3a ₀	103	0.852	0.527	6.541	150.63	9.853	9.002	137.6	0.395	-8.7
	104	0.823	0.488	8.359		12.59	12.19	145.8	0.441	-3.2
	105	0.801	0.457	9.737		14.67	13.86	142.3	0.377	-5.5
	106	0.773	0.426	11.665		17.57	16.50	141.4	0.339	-6.1
	107	0.759	0.412	12.632		19.03	17.62	139.5	0.307	-7.4
	108	0.726	0.377	14.936		22.50	20.96	140.3	0.283	-6.9
Average = 141.2									Average	-6.3
4a ₀	118	0.828	0.417	6.236	174.13	10.86	10.60	170.0	0.383	-2.4

Table XXVII

CALCULATED AND EXPERIMENTAL VALUES OF M'^2 FOR 15:1 LATTICES

Pitch	Loading	B_F^2/B^2	C	$\frac{2B_Z^3}{\pi}(1+C)$ (cm ⁻³ x 10 ⁻⁶)	$M_C'^2$ (cm ²)	$\left(\frac{d\rho}{dH}\right)_C$ (cm ⁻¹ x 10 ⁻⁴)	$\left(\frac{d\rho}{dH}\right)_E$ (cm ⁻¹ x 10 ⁻⁴)	$M_E'^2$ (cm ²)	C_E	Δ
2a ₀	385	0.850	0.675	11.316	120.99	13.69	11.61	102.6	0.421	-15.2
	381	0.804	0.595	16.267		19.68	16.17	99.4	0.311	-17.8
	382	0.751	0.508	22.181		26.84	23.12	104.2	0.299	-13.9
	383	0.647	0.385	34.871		42.19	35.09	102.7	0.152	-15.1
Average = 102.2									Average	-15.5
3a ₀	402	0.844	0.621	13.735	127.92	17.57	14.70	107.0	0.356	-16.4
	401	0.665	0.378	37.350		47.78	43.31	116.0	0.249	-9.3
	400	0.560	0.285	54.302		69.46	57.34	105.6	0.061	-17.4
Average = 109.5									Average	-14.4
4a ₀	407	0.854	0.553	10.371	142.11	14.74	12.83	123.7	0.352	-13.0
	405	0.790	0.460	16.938		24.07	22.15	130.8	0.344	-8.0
	404	0.632	0.301	35.833		50.92	49.32	137.6	0.260	-3.2
Average = 130.7									Average	-8.0
6a ₀	409	0.693	0.243	14.367	180.90	25.99	22.23	154.7	0.063	-14.5

It is seen from Tables XXVI and XXVII that experimental values of M'^2 are always less than theoretical values, with the disagreement generally being greatest for the more closely packed lattices, although this trend was reversed for the 6a₀ lattice with 15:1 fuel. However, only one value was measured for this lattice, and so this reversal may be due to experimental error. There is a fair amount of scatter in the experimental values of M'^2 for different loadings of the same lattice, most of which is the result of errors in measuring the small increment of height, ΔH . Girard *et al.*,⁽²⁹⁾

in their results for natural uranium-D₂O lattices, also found experimental values of M^{i2} to be systematically smaller than theoretical values with the disagreement being worse for more closely packed lattices. They suggest that the values of $1 + C$ (E in notation of Ref. 29) calculated by means of two-group theory may be too high. Their highest value of C was 0.48, with most values being about 0.1 or 0.2, since their D₂O reflectors in some cases were not very thick. Since the ZPR-VII lattices generally have thicker reflectors than those of Ref. 29, errors in C can produce larger errors in M^{i2} . Another possible source of error is the value of E_P/E_D used in the inhour equation. It is difficult to estimate the experimental error involved in measuring this quantity. Higher values of E_P/E_D would give higher experimental values of ρ and consequently higher values of M^{i2} .

VIII. FUEL-SUBSTITUTION EXPERIMENTS

The technique of fuel-substitution experiments starts with a critical core containing only 25:1 fuel. Then 50:1 fuel is substituted for part of the 25:1 fuel as uniformly as possible throughout the core and the water level is raised to re-establish criticality. This may be repeated several times until the critical water level is almost at the top of the fuel rods.

The purpose of fuel-substitution experiments is to obtain information on the neutron age. The criticality Eq. (5) may be rewritten as

$$y = \ln \left(\frac{1 + L^2 B^2}{f^{25}} \right) + \ln \left(\frac{1 + \tau_f B^2}{e \tau_f B^2} \right) + \ln \left(\frac{1 - \gamma}{p_e} \right) = \ln(\epsilon \eta^{25} p) - \tau B^2 \quad , \quad (34)$$

where

$$\gamma = \frac{k_{\infty e} (1 - p_e) e^{-(\tau_r + \bar{\tau}_e) B^2}}{1 + \tau_f B^2} \quad .$$

It is assumed that ϵ , η^{25} , p , η_e^{25} , the various ages, and the effective neutron temperature are not changed appreciably by the substitution of fuel. Thus, if the quantity y on the left-hand side of Eq. (34) is plotted against B^2 , a straight line with slope $-\tau$ and x-intercept $\ln(\epsilon \eta^{25} p)$ results.

Data on fuel-substitution experiments are reported in Table XXIV of Ref. (2). It is seen from this table that several lattices having the same total number of rods but different numbers of 50:1 rods were constructed for each pitch. In the calculations reported here, only the data for the lattices with all 25:1 rods and the maximum number of 50:1 rods were used in applying Eq. (34). The parameters in the criticality equation for the mixed fuel lattices were calculated in essentially the same manner as for the lattices with all 25:1 fuel. Thermal fluxes were obtained from Table VII, with the 50:1 values renormalized to give the same fluxes in the D_2O associated with both types of fuel. Reflector savings calculations by means of codes RE-169 and RE-188 were performed to provide values to use with the experimental critical dimensions in order to obtain bucklings for the two cores considered for each pitch.

The criticality equation constants for the loadings with the maximum number of 50:1 fuel rods are given in Table XXVIII; the constants for the loadings with all 25:1 fuel were given in Table IX. In Table XXIX, experimental dimensions combined with calculated reflector savings give experimental bucklings. The loadings marked with an asterisk are loadings including 50:1 fuel for which bucklings were calculated by means of reflector savings for cores with all 25:1 fuel in order to demonstrate the effects of reflector savings changes.

Table XXVIII

CONSTANTS FOR LATTICES WITH MAXIMUM NUMBERS OF 50:1 RODS

Pitch	Loading	25:1 Rods	50:1 Rods	$\bar{\Sigma}_a$ (cm ⁻¹)	$\bar{\Sigma}_{tr}$ (cm ⁻¹)	L^2 (cm ²)	f	η
2a ₀	99	720	289	0.03626	0.3974	23.13	0.9743	1.566
3a ₀	114	327	134	0.01697	0.4121	47.66	0.9633	1.566
4a ₀	121	265	34	0.01042	0.4172	76.68	0.9512	1.605
Pitch	f^{25}	η^{25}	f_e	η_e	f_e^{25}	η_e^{25}	P_e	p
2a ₀	0.7327	2.082	0.9780	1.453	0.7473	1.902	0.7753	0.8690
3a ₀	0.7257	2.078	0.9686	1.457	0.7388	1.910	0.8963	0.9392
4a ₀	0.7348	2.078	0.9589	1.492	0.7479	1.913	0.9359	0.9653
Pitch	$k_{\infty e}$	k_{∞}	τ_f (cm ²)	τ_r (cm ²)	τ_e (cm ²)	$\bar{\tau}_e$ (cm ²)	B_m^2 (cm ⁻² x 10 ⁻³)	
2a ₀	1.235	1.326	52.73	47.67	30.04	21.21	1.851	
3a ₀	1.325	1.416	49.73	44.54	28.63	20.30	2.130	
4a ₀	1.381	1.474	48.80	43.58	28.18	20.04	2.083	

Table XXIX

EXPERIMENTAL BUCKLINGS INVOLVED IN FUEL-SUBSTITUTION EXPERIMENTS

Pitch	Loading	R (cm)	λ_R (cm)	B_R^2 (cm ⁻² x 10 ⁻³)	H (cm)	λ_H (cm)	B_Z^2 (cm ⁻² x 10 ⁻³)	B^2 (cm ⁻² x 10 ⁻³)
2a ₀	95	31.783	30.079	1.5112	103.30	21.51	0.6336	2.1448
	99	31.783	31.573	1.4407	141.95	21.73	0.3684	1.8091
	99*	31.783	30.079	1.5112	141.95	21.51	0.3694	1.8806
3a ₀	110	32.225	25.268	1.7496	103.67	18.39	0.6624	2.4120
	114	32.225	26.536	1.6749	143.18	18.52	0.3775	2.0524
	114*	32.225	25.268	1.7496	143.18	18.39	0.3781	2.1277
4a ₀	120	34.602	24.651	1.6471	129.35	17.37	0.4585	2.1056
	121	34.602	25.121	1.6214	148.90	17.41	0.3568	1.9782
	121*	34.602	24.651	1.6471	148.90	17.37	0.3570	2.0041

Table XXX lists values of quantities involved in finding the neutron age from Eq. (34) along with the ages so found. The quantities y_1 , y_2 , and y_3 are the terms on the left-hand side of Eq. (34) in order of their appearance, whereas y is their sum. Under the heading "Method," "calculated" denotes $\tau = \tau_f + \tau_r + \tau_e$, whereas the following two items are the combinations of loadings used to find τ from Eq. (34).

One could also find an experimental value of p from Eq. (34) by dividing $\exp(y + \tau B^2)$ by $\epsilon \eta^{25}$. However, the experimental error in p produced by experimental error in τ is of the order of $\Delta p = 0.002 \Delta \tau$. Thus, this is not a good method for finding p , especially for large pitches for which p is close to unity.

Table XXX

VALUES OF NEUTRON AGE FROM FUEL-SUBSTITUTION EXPERIMENTS

Pitch	Loading	B^2 ($\text{cm}^{-2} \times 10^{-3}$)	y_1	y_2	y_3	y	Method	τ (cm^2)
$2a_0$	95	2.1448	0.31379	-0.00598	0.00240	0.31021	calculated	130.4
	99	1.8091	0.35199	-0.00429	0.00140	0.34910	95-99	115.8
	99*	1.8806	0.35354	-0.00469	0.00379	0.35264	95-99*	160.6
$3a_0$	110	2.4120	0.37700	-0.00668	-0.00678	0.36354	calculated	122.9
	114	2.0524	0.41396	-0.00489	-0.00608	0.40299	110-114	109.7
	114*	2.1277	0.41726	-0.00529	-0.00509	0.40688	110-114*	152.4
$4a_0$	120	2.1056	0.43664	-0.00499	-0.00797	0.42368	calculated	120.6
	121	1.9782	0.44935	-0.00439	-0.00757	0.43739	120-121	107.6
	121*	2.0041	0.45108	-0.00459	-0.00738	0.43911	120-121*	152.0

The experimental values of τ in Table XXX are about 10 to 15 cm^2 lower than the calculated values. However, if the same reflector savings are used for the lattices with and without 50:1 fuel, the values of τ are around 30 cm^2 too high. Thus, τ is strongly dependent on the changes of reflector savings with substitution of fuel. This is analogous to the dependence of the modified migration areas in the water with experiments on the factor C which involved changes in reflector savings.

IX. DISCUSSION

In general, the agreement between theory and experiment is satisfactory, although the methods of analysis used were rather crude. However, such crude methods should be fairly good for D_2O lattices as long as they are not greatly undermoderated. The thick radial reflector produced difficulties in comparing theory and experiment. The strong distortion of the flux near the interface made it impossible to determine experimental radial reflector savings. Also, the radial reflector savings are of the same order as the radii and so affect the radial bucklings strongly. Thus, the so-called experimental geometric bucklings depend greatly on calculated radial savings, and so comparison with calculated material bucklings does not yield a direct comparison between theory and experiment. Also, changes in reflector savings introduced a significant complication into the analysis of the differential experiments; namely, the water-worth and fuel-substitution experiments.

APPENDIX

Equivalence of Independent Water Worth Treatments

In this appendix the equivalence between the method used here for the calculation of the water-worth correction factor C and that of Girard et al.,⁽²⁹⁾ will be demonstrated. The boundary conditions at the core-radial reflector and core-axial reflector interfaces are written in Ref. 29 as

$$f(R, \alpha^2, \beta^2) = 0 \quad (\text{radial}) \quad (A1)$$

and

$$g(H, \alpha^2, \beta^2) = 0 \quad (\text{axial}) \quad , \quad (A2)$$

where $\alpha^2 = B_z^2$ and $\beta^2 = B_r^2$. These equations are equivalent to Eqs. (21) and (22) for the one-group model if the terms involving $1/T$ in the reflectors are ignored, as was done in replacing Eq. (27) with Eq. (28). We may consider α^2 and β^2 as functions of R and H in Eqs. (A1) and (A2), and write

$$df = 0 = \frac{\partial f}{\partial \alpha^2} \frac{\partial \alpha^2}{\partial H} + \frac{\partial f}{\partial \beta^2} \frac{\partial \beta^2}{\partial H} \quad ; \quad (A3)$$

$$dg = 0 = \frac{\partial g}{\partial H} + \frac{\partial g}{\partial \alpha^2} \frac{\partial \alpha^2}{\partial H} + \frac{\partial g}{\partial \beta^2} \frac{\partial \beta^2}{\partial H} \quad . \quad (A4)$$

From Eqs. (A3) and (A4) we obtain

$$\frac{\partial B_g^2}{\partial H} = \frac{\partial}{\partial H} (\alpha^2 + \beta^2) = \frac{-2\pi^2}{(H + \lambda_H)^3} \frac{(1 + e'')(1 + e)}{1 - ee'} \quad , \quad (A5)$$

where

$$e = \frac{-\frac{\partial f}{\partial \alpha^2}}{\frac{\partial f}{\partial \beta^2}} \quad ; \quad e' = \frac{-\frac{\partial g}{\partial \beta^2}}{\frac{\partial g}{\partial \alpha^2}} \quad ; \quad 1 + e'' = \frac{\frac{\partial g}{\partial H}}{\frac{2\pi^2}{(H + \lambda_H)^3} \frac{\partial g}{\partial \alpha^2}} \quad .$$

Thus, from Eq. (A5),

$$C = \frac{e + ee' + ee'' + e''}{1 - ee'} \quad .$$

On using Eqs. (21) and (22) to evaluate e , e' , e'' , and C , we have

$$f = D_c (\beta^2)^{1/2} \frac{J_1((\beta^2)^{1/2} R)}{J_0((\beta^2)^{1/2} R)} - D_r (K_r^2 + \alpha^2)^{1/2} \frac{K_1[(K_r^2 + \alpha^2)^{1/2} R]}{K_0[(K_r^2 + \alpha^2)^{1/2} R]} = 0 \quad ;$$

$$\frac{\partial f}{\partial \alpha^2} = \frac{-D_r R}{2} \left(\frac{K_1^2 - K_0^2}{K_0^2} \right) \quad ;$$

$$\frac{\partial f}{\partial \beta^2} = \frac{D_c R}{2} \left(\frac{J_0^2 + J_1^2}{J_0^2} \right) \quad ;$$

$$e = \frac{\beta}{p_r} \frac{J_0 J_1}{K_0 K_1} \left(\frac{K_1^2 - K_0^2}{J_1^2 + J_0^2} \right) \quad ;$$

$$g = D_c (\alpha^2)^{1/2} \cot \left((\alpha^2)^{1/2} H \right) + D_b (\beta^2 + K_b^2)^{1/2} \coth \left[(\beta^2 + K_b^2)^{1/2} t \right] = 0 \quad ;$$

$$\frac{\partial g}{\partial \beta^2} = \frac{D_b}{2 p_b} \coth(p_b t) \left(1 - \frac{p_b t}{\sinh(p_b t) \cosh(p_b t)} \right) \quad ;$$

$$\frac{\partial g}{\partial \alpha^2} = \frac{D_c}{2 \alpha} \cot(\alpha H) \left(1 - \frac{\alpha H}{\sin(\alpha H) \cos(\alpha H)} \right) \quad ;$$

$$\frac{\partial g}{\partial H} = \frac{-D_c \alpha^2}{\sin^2(\alpha H)} \quad ;$$

$$e' = \frac{\alpha^2}{p_b^2} \frac{\left(1 - \frac{p_b t}{\sinh(p_b t) \cosh(p_b t)} \right)}{\left(1 - \frac{\alpha H}{\sin(\alpha H) \cos(\alpha H)} \right)} \quad ;$$

$$1 + e'' = \frac{\pi}{\alpha H - \sin(\alpha H) \cos(\alpha H)} \quad .$$

The expressions for e , e' , e'' , and, therefore, C are the same as those derived in Section VIIA.

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