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Effects of Nonadiabatic Transitions on Invariants
of the Motion

R. E. Aamodt*

Department of Astro-Geophysics, University of Colorado
Boulder, Colorado 80302

and

MASTER

E. F. Jaeger†

Advanced Study Program, National Center for Atmospheric Research ‡
Boulder, Colorado 80302

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Abstract

A computer study is made of the change in adiabatic invariant for a single pendulum when a slow variation in the pendulum length forces the pendulum to change from a uniformly rotating motion to an oscillating state. The numerical results are shown to be in good agreement with a simple analytic approximation which itself is generalizable to calculations of nonadiabatic transitions in other, physically more interesting problems.

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†Present address: Joint Institute for Laboratory Astrophysics,
University of Colorado, Boulder, Colorado 80302

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The concept of adiabatic invariants provides one of the most powerful approximation methods in classical mechanics. For applicable systems the effects of large perturbations can be determined with negligible error, even for long times. The use of this property of invariants is commonplace, and in one dimensional systems for example, the formalism can be applied whenever the phase space motion is basically periodic and the perturbations restricted to those that induce only small changes in the basic motion per period. However, in a significant number of interesting cases the unperturbed orbits in neighboring regions of phase space exhibit quite different behavior, and then even extremely weak perturbations can cause motion across separating surfaces and hence invalidate the restrictions of the adiabatic theory. Examples of such states occur in the description of the longitudinal invariant for particles trapped in certain non-axisymmetric mirror geometrics or torodial magnetic fields;¹ the magnetic moment in the presence of short wavelength electric field fluctuations;² and the damping of a finite amplitude plasma wave.³ While these systems differ appreciably all three have the common feature that the relevant phase space is two dimensional with Hamiltonian dynamics, and in addition the Hamiltonian is an exact constant of the motion in the absence of the perturbing modulation. In this paper we numerically and analytically study the simplest dynamical model of a system which has all of these characteristics, namely that of a pendulum in a gravitational field wherein the length of the pendulum is varied slowly. Neglecting some small corrections and dimensional factors this problem corresponds to the calculation of the damping of a plasma wave.³

For a pendulum of prescribed length $\ell(t)$ and angle q with the vertical the Hamiltonian is

$$H' = \frac{p_2^2}{2m\ell^2} - \frac{m\dot{\ell}^2}{2} - 2mg\ell \cos^2(q/2). \quad (1a)$$

Changing to dimensionless time units measured in terms of the oscillation frequency at the bottom of the swing; $t = \int dt' \sqrt{2mg\ell(t')}$, Hamilton's equation follow from the effective Hamiltonian

$$H = \frac{p^2}{2a} - \cos^2(q/2), \quad (1b)$$

$$a^2(t) = 4m^2 g \ell^3(t).$$

For fixed a , the phase space motion described by (1b) is periodic with period $T(H, a)$, and the classical form of adiabatic invariance follows if

$$\frac{T}{a} \frac{da}{dt} \ll 1. \quad (2a)$$

In this case the lowest order invariant, J , is defined as

$$J = \oint p dq = a \oint dq \left[H + \cos^2(q/2) \right]^{1/2}, \quad (2b)$$

where for convenience of continuity of J when $H = 0$, the integral in (2b) is defined for rotational pendulum motion as twice the integral over a complete cycle, while for oscillatory motion the integral is over the usual cycle. With T as the period for these cycles, the time variation of J follows from (2b):

$$\frac{dJ}{dt} = \frac{d}{dt} \left(\frac{1}{a^2} \right) \cdot T \cdot \left[p^2 - \langle p^2 \rangle_t \right], \quad (3a)$$

$\langle p^2 \rangle_t$ being the time average of p^2 over a cycle,

$$\langle p^2 \rangle_t = \frac{1}{T(t)} \int_{t-T/2}^{t+T/2} p^2 dt \quad (3b)$$

The essence of the classical adiabatic theory is that because T is bounded the right side of Equation (3a) remains small and oscillates rapidly, hence any long time accumulated changes in J are essentially exponentially small. However, when the pendulum crosses a separatrix T goes to infinity and the change in J can be substantial. This is illustrated by computer calculations shown as the solid curves in Fig. 1 for different crossing points of the separatrix, q_s and different values of the slowness parameter $\frac{da}{adt} = \epsilon$.

To analytically determine this change in J , we shall assume, consistent with the simulation runs of Fig. 1, that the pendulum begins in a uniformly rotating state, after which time " a " is slowly increased until the pendulum falls into the oscillating state, whereafter $\frac{da}{dt}$ uniformly returns to zero. It is important to note that with this monotonic increase in a , crossing the separatrix defined by $H = 0$ can only happen once because

$$\frac{dH}{dt} = p^2 \frac{d}{dt} \left(\frac{1}{a^2} \right) \quad (4)$$

which is always less than zero both in the rotating state and on the separatrix, except at the unstable singular points, $p = 0$, $q = \pi(2N+1)$.

In other words, once H has become negative it cannot again become positive.

Also, as the unstable singular points are stationary under any variation of a , the change in J for the points $q_s = (2N+1)\pi$ are exactly calculated to be

$$\Delta J(\pi) = 8a_{\text{final}} - 8a_{\text{initial}} \quad (5)$$

For Fig. 1 as $a_{\text{initial}} = 1$, $a_{\text{final}} = 4$, $\Delta J = 24$. For any ϵ this change defines an upper bound for ΔJ , which in Fig. 1 is clearly achieved for $\epsilon_0 = 10^{-1}$, and obviously defines extreme singular variation of ΔJ as q_s approaches π for $\epsilon_0 = 10^{-2}, 10^{-3}$.

For other crossing points it is convenient to express the change in J over the entire interval of change in a , in the form

$$\Delta J = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' p^2(t') \left[A(t', t'') \frac{d}{dt'} \left(\frac{1}{a^2} \right) - A(t'', t') \frac{d}{dt''} \left(\frac{1}{a^2} \right) \right] \quad (6)$$

where $A(t', t'')$ is equal to 1 for $t' - \frac{T(t')}{2} < t'' < t' + \frac{T(t')}{2}$, $-\infty < t' < \infty$, and equal to zero outside of this area in t', t'' space. Fig. 2 shows the nonzero region of A for a typical orbit, where the separatrix crossing time, $t' = t_s$, has been shifted to $t' = 0$. In Fig. 2 the times t'_A, t'_B are defined by the equation

$$1 = \pm \frac{dT}{2dt'}(t'), \quad (7)$$

which imply T has to be large, or H small for which the asymptotic form of T reduces to

$$T \underset{|H| \rightarrow 0}{\approx} 8a(t) \ln \frac{4}{[|H|]^{1/2}} \quad (8)$$

Equations (7), (8), and (4) determine that the times t'_A, t'_B occur when $|H| \lesssim 4a\epsilon$, except in the vicinity of an unstable singular point, where

$|H| \lesssim \mathcal{O}\left[\exp\left(-\frac{1}{\epsilon}\right)\right]$. For $|t'_{A,B} - t'| \gg |t'_{A,B}|$, as $H(t') \approx \mathcal{O}(1)$, it is easy to show the integrand in (6) is $\mathcal{O}(\epsilon^2)$, and rapidly oscillating, yielding the usual exponentially small change in J over the adiabatic portion of the orbit. Only for times $|\Delta t'| \sim \mathcal{O}|t'_A|$, where $|\Delta H| \lesssim \mathcal{O}(a\epsilon)$, is there a significant contribution to ΔJ .

The essence of this particular analysis is that even though the pendulum changes its energy slowly, $\frac{dH}{dt} \lesssim \mathcal{O}(\epsilon)$, in fact the region in phase space over which J changes appreciably is equally thin. Defining this nonadiabatic region within an order of magnitude by (7), it follows that even in the vicinity of an unstable singular point, where $\frac{dH}{dt} \sim \mathcal{O}(e^{-1/\epsilon})$, the pendulum crosses these regions in a time of order (1), hence $t'_A, t'_B \sim \mathcal{O}(\epsilon^0)$. As this is a short time (6) approximates

$$\Delta J \approx \frac{d}{dt} \left(\frac{1}{a^2} \right)_{ts} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' p^2(t') [A(t', t'') - A(t'', t')]. \quad (9)$$

Integrating (9), over t'' , using Fig. 2 we find

$$\Delta J \approx \frac{d}{dt} \left(\frac{1}{a^2} \right)_{ts} \int_{-\infty}^{\infty} dt' p^2(t') \Delta(t'), \quad (10)$$

where the function, $\Delta(t')$, has zero total area, a positive logarithmic divergence at $t' = 0$, with a singular peak width of order $t'_B - t'_A$, then it goes negative on both sides of $t' = 0$, and finally for large $|t'|$ becomes $\mathcal{O}(\epsilon^2)$ and slowly varying. In other words, (10) is completely consistent with the previous discussion, $\Delta(t')$ explicitly determining the properties of the boundary layer.

In approximating (10) even further, some care must be taken, for although the pendulum tranverses this nonadiabatic region rapidly compared with

the basic period, given by (8) as $T \sim |\ln \epsilon|$, in fact throughout most of this cycle the pendulum moves rapidly, slowing down only as it approaches the unstable singular points. Therefore $p^2(t')$, in (10) needs to be reasonably approximately over times comparable to a single cycle. Away from the region exponentially close to an unstable singular point this can be best accomplished using approximate conservation of energy, Eqn. (10) reducing to

$$\Delta J \approx \epsilon(t_s) \int dt' \cos q(t') \Delta(t') \quad (11)$$

While an analytic expression for $q(t')$ is extremely complicated for times of the order of a period, in fact a Taylor series expansion of q about the separatrix crossing yields a reasonable approximation to $\cos q(t')$. Using

$$q(t') \approx q_s + \frac{2p_s}{a_s} t' + \dots, \quad (12a)$$

and the crudest possible form for $\Delta(t')$, which includes only the positive logarithmic divergence at $t' = 0$,

$$\Delta(t') \approx 8a_s \ln 4 \left| t' p_s \frac{d}{dt} \left(\frac{1}{a} \right) \right|_{ts}^{1/2}, \quad (12b)$$

(11) approximates

$$\Delta J \approx -2\pi a_s^2 \epsilon(t_s) \frac{\cos q_s}{|\cos(q_s/2)|}. \quad (13)$$

For all the simulation data shown in Fig. 1, $a(t_s)$ was held fixed, and the slowness parameter, $\epsilon(t)$ only changed by an overall scale factor ϵ_0 . With this choice of crossing times the coefficient of (13) simplified

to $156 \epsilon_0$, and this analytic form is plotted in Fig. 1 as the dashed curve.

The separatrix crossing is obviously a symmetric function under the interchange $(p_s, q_s) \rightarrow (-p_s, -q_s)$, but not with $p_s \rightarrow -p_s$ only. Therefore in Figs. (1a), (1b) the slight calculated asymmetry between $\pm p_s$ is also illustrated.

The comparison of the approximate analytic form (13) to the actual charge in J is rather revealing. Over a range of three orders of magnitude in ϵ , this crude approximation gives the correct result at 0° with less than an 8% error. Away from 0° the error increases, but for the smallest ϵ the agreement in ΔJ remains satisfactory out to $q_s \approx 3\pi/4$ where the error has only increased to 15%. For the biggest ϵ compared, it is surprising that there is any similarity because as $a(t)$ changes by a factor of four, when $\epsilon_0 = 10^{-1}$ the local $\epsilon(t)$ becomes equal to one, and in this interval J will certainly not be adiabatic even away from the separatrix region.

For any of the ϵ 's it is evident from Fig. 1 that as the pendulum crosses the separatrix near the unstable singular points, it is staying longer in the boundary region, and it is reasonable to expect the approximations (12) to begin to break down. The worst approximation of (12) is the form for $\Delta(t')$, because it is always positive whereas the actual $\Delta(t')$ is negative for $t' < t'_A$, and $t' > t'_B$. With this correction built into the analytic formula, ΔJ would be suppressed, especially in the region where $\cos(q_s/2)$ is small, and clearly this would reduce the analytic error in the vicinity of $q_s \approx (2N+1)\pi$.

Finally, the question of given some initially random distribution of phases of the pendulum, what are the corresponding values of q_s ; in particular is there an accumulation point, or concentrated region of crossing? To answer this, we first note that as phase space area is conserved during this motion, and the singular points are stationary, that there cannot be an accumulation point. To check the possibility of a concentrated crossing region for $\epsilon_0 = 10^{-2}$, eight different values of initial p 's, each with twelve different values of q 's were numerically followed through the trapping cycle. No accumulation was observed, and the summed total change in J was negative, indicating more crossings between $q_s = \pm \pi/2$, than in the singular region. For this class of initial conditions, and independently, those of Fig. 1, the average change in J was found to be of order $\epsilon(\ln \epsilon)^2$ in complete agreement with the Jahnke, Gardiner, Newcombe mean value theorem.⁵

References

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Figure Captions

Fig. Change in J vs. position of separatrix crossing, q_s for different values of the slowness parameter, ϵ_0 . Solid curves are computed, dashed curves are the lowest order analytic form.

Fig. 2 Region in $t' - t''$ space where the integrand of ΔJ , $A(t', t'')$, is non-zero.

FIG. 1 (a)

FIG. 1 (b)

FIG. 1(c)

FIG 2