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MATHEMATICAL AND EXPERIMENTAL ANALYSIS OF TEMPERATURE
DISTRIBUTIONS IN STORED IRRADIATED LAMPRE CAPSULES--
AN ITERATIVE EQUATION GENERATION PROCEDURE*

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ABSTRACT

Temperature profiles in stored irradiated LAMPRE (Los Alamos Molten Plutonium Reactor Experiment) fuel capsules were simulated both mathematically and experimentally. Each capsule which contains 90 atomic per cent plutonium and 10 atomic per cent iron fuel could produce up to 200 watts of fission product heat power after long term exposure in the reactor and could develop temperatures in storage that would accelerate the corrosion of the tantalum capsules. The experimental and mathematical simulations were made to investigate the temperature problems that could arise on storage of the capsules.

The mathematical analysis produced solution equations from the four coupled second-order differential equations by an iterative generation procedure. The generated equations were then used in an iterative computer program which produced the convergent temperature profiles by a calculation and correction technique. The iteratively generated equations, coupled with the computer program, produced temperatures which agreed very satisfactorily with the experimental simulation results. Because of this agreement, the iterative equation generation procedure can be recommended for similar complex longitudinal-radial heat transfer problems.

INTRODUCTION

Fuel capsules containing a plutonium-iron fuel from the Los Alamos Molten Plutonium Reactor Experiment (LAMPRE) after long term burnup in the reactor must be stored before reprocessing is started. Since as much as 200 watts of fission product heat may be generated per capsule, the temperature of the fuel in the capsule could reach levels that would produce accelerated attack on the tantalum. To determine the seriousness of the heating problem, temperature levels in the capsule and its stainless steel container were simulated as a function of heat generation level both experimentally and mathematically. The mathematical techniques and results of the mathematical and experimental simulations are reported here.

Since the heat sink is at the bottom of the capsule and its container, the mathematical analysis must consider both the radial and longitudinal heat transfer in the fuel, capsule, stainless steel pipe container, and in the helium and air surrounding the capsule and stainless steel pipe. To estimate the longitudinal profiles as a function of power generation, four coupled second-order differential equations were solved by an iterative equation generation procedure. The two coupled equations in each of the two regions above and below the fuel level were uncoupled by approximating the coupling temperature in one equation and then substituting the integrated results from the uncoupled equation into the second differential equation. This integrated result was then substituted back into the original equation to generate a second level iterated equation. The procedure could be continued to generate any order of approximating equation.

The first, second, and third level iterated equations were used in an iterative computer program to compute the required temperature profiles. The end conditions were selected to allow a prediction and self-correction type program in which initial estimates of temperatures and temperature gradients were successively corrected to final convergent values.

The iterative equations produced results which agreed quite satisfactorily with the experimental simulation of the capsule storage conditions. Both the first level and second level iterative equations gave satisfactory results when two arbitrary constants used in the initial temperature estimates in the equation generation procedure were adjusted to appropriate values.

DIFFERENTIAL EQUATIONS AND PROBLEM DESCRIPTION

Fuel capsules from the LAMPRE reactor are stored in water-cooled chambers set in a concrete floor. The capsule, stainless steel container, and storage chamber are shown in Figure 1. As shown in that figure, the 0.426" O.D. tantalum capsule is two-thirds full of fuel consisting of 90 atomic per cent plutonium and 10 atomic per cent iron. The capsule is contained inside a 1" O.D. stainless steel pipe which in turn is supported vertically in the 5.05" I.D. steel pipe storage chamber. On top of the assembly, and not shown in Figure 1, is a 7.5" high by 5" O.D. U-238 radiation shield plug. The chamber consists of a 5" Sch. 40 steel pipe with welded bottom. Beneath the chamber pipe is a cooling water channel, and surrounding the chamber pipe and cooling water chamber is concrete. Inside the stainless steel pipe is a capsule holder which is 0.4" wide. In the subsequent analysis this holder is ignored except as is necessary to describe the appropriate end conditions. Helium is contained in the annular space between the capsule and stainless steel container, and air is contained in the space outside the stainless steel container.

The purpose of the analysis is to describe the longitudinal temperature distributions in the fuel capsule and the stainless steel container at fission product power generation rates up to 200 watts. In the analysis described here, the following assumptions and simplifications have been made.

1. No radial temperature gradients exist in the tantalum capsule and its contained fuel and in the stainless steel container.
2. The physical properties and heat transfer coefficients are calculated at average temperatures determined for the capsule, stainless steel container, helium layer, air layer, and outside container wall.
3. Even though the air temperatures and outside wall temperatures do vary somewhat with position, for purposes of calculation average air temperatures and side wall temperatures are used. The air temperature is calculated by making a steady state conduction heat balance through the air phase. The average wall temperature is calculated from an equation derived by assuming a constant radial heat flux to the wall and longitudinal conduction along the length of the pipe wall with a maximum of temperature at the level of the fuel surface.

FIGURE 1

4. All free convection in the melted fuel and longitudinal conduction and convection in the helium layer are ignored.

With these assumptions, the following differential equations and boundary conditions describe the temperature profiles in the fuel capsule and stainless steel container.

1. Region I: Between bottom and top of fuel.

$$\frac{d^2 T_{tf}}{dx^2} = (T_{tf} - T_s)K_2 - K_3 \quad (1)$$

$$\frac{d^2 T_s}{dx^2} = - (T_{tf} - T_s)K_3 + (T_s - T_w)K_4 \quad (2)$$

2. Region II: Between top of fuel and top of capsule.

$$\frac{d^2 T_t}{dx^2} = (T_t - T_s)K_1 \quad (3)$$

$$\frac{d^2 T_s}{dx^2} = - (T_t - T_s)K_3 + (T_s - T_w)K_4 \quad (4)$$

3. Boundary conditions

Region I: $x = 0$ $\frac{dT_{tf}}{dx} = B$; $T_s = T_o$

$x = L_f$ $\frac{dT_{tf}}{dx} = C$; $\frac{dT_s}{dx} = A$

$T_{tf} = T_t$; $T_s = T_s$

Region II: $x = 0$ $\frac{dT_s}{dx} = C \left(\frac{A_t k_t + A_f k_f}{A_t k_t} \right)$; $\frac{dT_s}{dx} = \frac{dT_s}{dx}$

$T_f = T_{tf}$ $T_s = T_s$

$x = L_u$ $T_f = T_w$ $T_s = T_w$

The only temperature specified is that of the cooling water; therefore, supplementary equations are required to generate average air and outside wall temperatures and the value B for the fuel temperature gradient at the bottom of the fuel capsule. These equations are summarized below.

1. Fuel temperature gradient.

The fuel capsule rests on a support holder with almost a line contact area. The following equation was derived to describe the heat transfer through a very small area into a larger conducting volume as shown in Figure 2.

$$\frac{- \left(T_{tf} \Big|_{x=0} - T_o \right) k_s A_2}{\left[\frac{L_{12} \ln (A_1/A_2)}{(1 - A_1/A_2)} - L_2 b \right]} = B(k_t A_t + k_f A_f) \quad (5)$$

2. Average air temperature.

The average air temperature is calculated by equating the heat flow to the air by convection from the stainless steel container to the heat flow from the air by convection to the surrounding walls.

$$A_s h_{\text{air(conv)}} (T_{s(\text{avg})} - T_{\text{air}}) = A_w h_{(\text{walls})} (T_{\text{air}} - T_w) + A_b h_{(\text{base})} (T_{\text{air}} - T_o) \quad (6)$$

3. Average wall temperature.

The average wall temperature is estimated by the following equation which was derived by assuming that a uniform heat flux per unit area flows through the wall into the steel pipe container and that the heat is then conducted longitudinally along the pipe wall with the maximum in temperature occurring at the level of the fuel interface. The concrete conducts approximately 10% of the heat away from the pipe wall; however, this correction is neglected since it is small and it helps to compensate for the assumption of uniform heat flux.

$$T_w = K_9 \left(- \frac{L_w^2}{6} + \frac{L_f L_w}{2} \right) + T_o \quad (7)$$

FIGURE 2
HEAT TRANSFER THROUGH SMALL CONTACT AREA
AT BOTTOM OF CAPSULE

To calculate K_g , the heat transferred to the wall must be computed from equation (8).

$$Q_w = Q - B(A_t k_t + A_f k_f) - \frac{dT_s}{dx} \bigg|_{x=0} (A_s k_s) - A_b h_{(base)} (T_{air} - T_o) \quad (8)$$

METHOD OF SOLUTION

Computational Procedure

The set of equations, 1 through 4, can be solved exactly by standard simultaneous differential equation solution techniques¹. However, a complete solution with the boundary conditions as stated is extremely cumbersome since eight integration constants must be determined from eight simultaneous algebraic equations. Even when the problem is divided in half by solving the equations for Region I and Region II separately by assuming that the end conditions, constants A , B , C , T_o , are known for Region I and $T_{t(o)}$, $T_{s(o)}$, and T_w are known for Region II, the algebraic expressions for the integration constants in Region I are very complex. However, by the similarity of equations, the integration constants for the two regions are relatively simple when end conditions of known temperatures are assumed. This solution is presented as Section I of the Appendix.

The equations generated by the exact solutions with the known temperature end conditions do not lend themselves to an iterative type of technique for generating the solution required at a given heat generation level. If the equations in the Appendix were used, end temperatures would have to be assumed and then the temperature gradients at the top fuel interface would be matched. If the slopes did not check, then a new temperature guess would have to be made. However, the procedure does not give any good method for guessing the next set of temperatures. For this reason and the additional reason that the exact solution is very difficult to obtain for a system with more than two conducting layers, an iterative type method of approximating the actual solution was devised.

A set of equations was derived which approximate the exact solution but utilize the end conditions of temperature gradients and temperatures as listed above. This combination of end conditions allows the use of an iterative

scheme that corrects itself and converges toward a final solution from any reasonable set of initial guesses of temperatures and temperature gradients. The gradients A, B, C and temperatures $T_{s(avg)}$, $T_{tf(avg)}$, T_w , T_{air} are initially estimated. From these the temperature profiles in the fuel section are computed from the derived equations. From the calculated temperatures, T_s and T_{tf} at the fuel interface, the temperature distribution in the section above the fuel is calculated from the second set of derived equations. The gradients calculated for Region II are then used to adjust the gradients A and C. Also, new values of B, $T_{s(avg)}$, $T_{tf(avg)}$, T_w , and T_{air} can be calculated from the generated profiles and Equations 5, 6, 7, and 8. The values of slopes and temperatures utilized in the next iteration are obtained by taking the arithmetic average of the previously used the new calculated values. The procedure is then repeated. Usually five to six iterations are sufficient to obtain the solution. And as is shown below, the procedure with the derived equations is very easily programmed on digital computers.

Iteratively Generated Equations

The equations used in the above outlined iteration were themselves generated by an iteration procedure from the basic differential equations, Equations 1, 2, 3, and 4. Equations 1 and 2 are iteratively solved with the boundary conditions A, B, C, and T_o assumed to be known. First Equation 2 is used to generate an approximation to the T_s function in Region I by assuming some functional form for the $-(T_{tf} - T_s)K_s$ term which is not dependent upon T_{tf} . In the solutions presented here this term is approximated by $Q = (Q' / L_f) / A_s k_s$. Q' equals $Q - F_1(\Delta Q_{tf})$ which is approximately the total heat transferred to the stainless steel pipe from the capsule. The F_1 factor is chosen arbitrarily and is used to adjust the first T_s function so that it best approximates the solutions from higher level iterations. With this simplification, Equation 2 is uncoupled from Equation 1 and the $(T_s)_1$ function can be directly obtained by integration. Then $(T_{tf})_1$ function is obtained by substitution of $(T_s)_1$ into Equation 1 and direct integration. The procedure can then be continued to generate any number of approximations to the exact solution. The experience of this work indicates that only one to two iterations are necessary to generate a good approximation if a good choice of F_1 is made. After a trial computation F_1 can be adjusted

to make the $(T_s)_1$ and $(T_s)_2$ solutions to better agree.

The equations generated by three iterations are presented in section II of the Appendix. As seen by the form of the equations, all integration and particular solution constants are functions of the previously generated constants. Therefore, the programming of the constants for the equations is relatively straightforward and is easily checked by having the program generate the $(T_s)_i$ and $(T_{tf})_i$ functions at each level of generation.

The temperature profiles for Region II are similarly generated. Here the boundary conditions used are $T_{s(o)} = T_{s(L_f)}$, $T_{t(o)} = T_{tf(L_f)}$, $T_{s(L_u)} = T_w$, and $T_{t(L_u)} = T_w$. The uncoupling of the T_s and T_t equations is accomplished by letting T_t equal $K_7 T_s$ where K_7 equals T_{tf}/T_s at $x = L_f$ of Region I. When substitution is made in Equation 4, and an arbitrary adjustment constant is inserted, the equation takes the following form:

$$\frac{d^2 T_s}{dx^2} = (K_8 + K_4) T_s - T_w K_4 \quad (9)$$

where

$$K_8 = -F_2(K_7 - 1.0)K_5$$

The F_2 factor allows adjustment of the first integrated T_s equation to make the convergence of the generated equations more rapid. The procedure from this point is the same as outlined above. The $(T_s)_1$ equation is substituted into Equation 3 and the $(T_t)_1$ solution is obtained by integration. $(T_s)_2$ is obtained by substitution of $(T_t)_1$ into Equation 4 with subsequent integration. The equations generated by two complete iterations are presented in Section III of the Appendix.

EXPERIMENTAL SIMULATION OF FUEL STORAGE CONDITIONS ¹²

The temperatures to be expected in stored irradiated LAMPRE capsules were approximated by simulating experimentally the fuel storage conditions. A 4 inch resistance heater was inserted in a LAMPRE tantalum capsule which in turn was stored in the stainless steel pipe container, and the whole assembly

was placed in the 5" pipe chamber in the concrete floor. Temperatures were measured by thermocouples inset in the tantalum capsule wall and stainless steel pipe wall at a position of 8 cm from the bottom of the capsule. The heater was positioned so that its top was approximately 2 cm below the actual fuel level. The heater was surrounded with helium inside the tantalum capsule.

The experimental simulation approximated quite well the expected fuel storage problem. However, two conditions existed which made the experimental results differ slightly from the actual situation simulated. Since liquid metal was not contained in the capsule, the heat distribution in the capsule would be somewhat changed because of the absence of the longitudinal conduction in the liquid fuel. Also, the heater was only 4 inches long while the actual fuel length was approximately 6 inches long. The concentration of power input into the four inches would tend to make the observed temperatures in this region to be higher than those expected in the actual stored capsule.

Two experimental runs were made. In the first one, power levels up to 60 watts were successfully dissipated in the heater, and the helium was successfully contained between the capsule and the stainless steel container. In the second run power levels up to 100 watts were tested, but a seal failed and allowed air to displace a portion of the helium in the annular space. As a result, higher recorded temperatures were obtained in the second run. At all power levels adequate time was allowed for the temperatures to come to steady state. The experimental setup is shown in Figures 3 and 4, and the experimental results are presented in Figure 5.

Figure 3

Channel Configuration

Figure 4

EXPERIMENTAL TEMPERATURES OF CAPSULE & STAINLESS STEEL CONTAINER AT 8 CM FROM BOTTOM OF CAPSULE

FIGURE 5

- CAPSULE WITH HELIUM IN ANNULUS
- CAPSULE WITH HELIUM-AIR MIXTURE IN ANNULUS
- △ STAINLESS STEEL CONTAINER WITH HELIUM IN ANNULUS
- ◇ STAINLESS STEEL CONTAINER WITH AIR-HELIUM MIXTURE IN ANNULUS
- * TEMPERATURE ESTIMATES

For all temperatures the emissivities of tantalum and stainless steel were taken as 0.42⁹ and 0.40¹⁰. The conductivity of the outside steel wall was taken as 26 Btu/ hr ft °F.¹¹

(9)

(10)

(11)

(12)

(13)

(14)

(15)

(16)

RESULTS OF MATHEMATICAL SIMULATION

The questions to be answered by the simulation results are the following:

1. Do the constant coefficient differential equations adequately describe the temperatures in the capsule and stainless steel container?
2. How effective are the equations generated by the iteration procedure in approximating the analytical profiles?
3. Do the higher order iterative equations converge toward the analytical solutions?
4. What values of F_1 and F_2 are required to produce satisfactory results with the first level iterative equations?
5. Does the computer iteration procedure readily converge?

In investigating the properties of the iterated equations and the computer program the first level, second level, and third level equations were used. The results from each set of equations were compared with results from the other iterative equations and with the experimental simulation and exact analytical solution results. In view of the above questions, the mathematical simulations produced the following answers.

1. The constant coefficient equations produced results via the iterated solutions which very satisfactorily simulated the experimental data. In Figures 6 and 7 are shown the mathematical simulation results as compared to the experimental simulation data. Two sets of simulation data are shown in these figures. One set of curves was obtained with the capsule containing fuel and with helium or air in the annular space between the capsule and its container. The second set of curves was obtained by substituting the conductivities of helium or air for the fuel conductivity to more closely simulate the experimental conditions.

How accurate are the complete longitudinal profiles as generated by the constant coefficient equations is not exactly known. Since the radiation and convection coefficients and conductivities all increase with temperature, the use of constants calculated at an over-all average temperature causes the profiles to be depressed at temperatures lower than the average and increased at temperatures above the average. To determine exactly how much error is

involved, more comprehensive experiments would have to be run or a more complete numerical relaxation type of mathematical simulation would have to be made. However, for the purpose of the present investigation the constant coefficient equations with the iterated solutions gave very satisfactory results. The temperatures at the 8 cm level in Region I agreed quite well with the experimental temperatures. And this agreement indicates that the mathematical simulation does give a realistic estimation of the temperatures to be expected in the capsule while in storage.

This agreement is exceptionally good when the relative uncertainties of the physical parameters used in the equations are considered. As an example of this sensitivity, Figure 7 shows the very significant changes in capsule temperatures that occur when air is substituted for helium in the annulus. Also, the 20 to 30°C difference between simulated and experimental stainless steel container temperatures could be attributed to the uncertainty in the emissivities of the tantalum and stainless steel.

2. The above discussion partially answers the second question of the adequacy of iterated equations in that adequate simulation results were obtained. An additional estimate of the accuracy of the iterated solutions can be made by comparing the profiles from these equations with those from the exact solution equations as presented in the Appendix, Section I. Figures 8 and 9 show the exact solutions with the iterated solutions for Regions I and II at a power level of 200 watts. Both first level and second level solutions are shown. As can be seen, the solutions do not agree exactly; however, the agreement is quite satisfactory at both levels when the proper F_1 and F_2 factors are used.

3. The higher order iterative solutions converge quite rapidly toward the exact solutions. In Figures 9, 10, 11, and 12 are shown temperature profiles from various levels of the iterated equations for Regions I and II. Figure 9 shows the first and second level solutions in Region II with the arbitrary F_2 factor equal to 0.600. The agreement between the first and second level equations in Region I is shown in Figure 10. The optimum F_1 factor equal to 1.0 was utilized in these calculations. In Figures 11 and 12 are shown the convergence properties of the first, second, and third level equations when the F_1 factor is chosen to give a very poor $(T_s)_1$ solution. Even with these poor estimates, the

FIGURE 8

FIGURE 9

COMPARISON OF ITERATED SOLUTIONS WITH EXACT SOLUTIONS -

REGION II - 200 WATTS IN REGION I

$$F_2 = 0.600 \quad K_1 = 0.3026 \quad K_4 = 0.07613 \quad K_5 = 0.16696$$

○ EXACT SOLUTION

x 1ST LEVEL ITERATED SOLUTION

□ 2ND LEVEL ITERATED SOLUTION

COMPARISON OF 1ST & 2ND ITERATED SOLUTIONS FOR REGION I WITH CAPSULE CONTAINING FUEL

FIGURE 10

1000 REGION I TEMPERATURE PROFILES SHOWING CONVERGENCE RATES OF ITERATIVE SOLUTIONS - $F_1 = -1.0$ (T_3) , FUNCTION TOO LARGE

FIGURE 11

SECTION I TEMPERATURE PROFILES SHOWING CONVERGENCE RATES OF ITERATIVE SOLUTIONS — $F_1 = 3.0$ (T_5) , FUNCTION TOO SMALL
POWER LEVEL: 200 WATTS

equations converged fairly rapidly. However, Figures 11 and 12 show that with the F_1 factors used iterative equations above the third level are required for good agreement between the iterative and exact equations. Figures 13 and 14 present curves for intermediate values of F_1 . The convergence with these factors is much faster, and the third level equations provide satisfactory results.

Since the iterative equations contain possible discontinuities, convergent solutions may not be obtained for certain combinations of coefficient values. The particular solution constants contain factors with $(K_2 - K_4 - K_5)$, $(K_2 - K_4)$, and $(K_1 - K_4 - K_5)$ in the denominators. If these quantities were to equal zero, no solution could result. However, this investigation indicated that convergent results were obtained even though these quantities became quite small. Trouble was expected with the small denominators, but none was experienced with the iterated equations up to the third level.

The iterated solutions for Region II with fixed end temperatures have given results for at least three levels of iteration that converge quite satisfactorily to the exact solution. In one series of calculations in which F_2 was set equal to zero, the successive iterations gave $(dT_s/dx)_{x=0}$ values of -93.7 °C/cm, -63.2 °C/cm, -60.3 °C/cm for the three iterations. The exact solution was -59.2 °C/cm. Similarly, the $(dT_t/dx)_{x=0}$ values were -149.6 °C/cm and -139.2 °C/cm. The exact solution value was -137.1 °C/cm. In this calculation $K_1 = 0.2775$, $K_4 = 0.0717$, and $K_5 = 0.1505$ with $K_1 - K_4 - K_5 = 0.0553$. The above mentioned condition exists in these equations with the $K_1 - K_4 - K_5$ quantity appearing in the denominator of some of the particular solution constants. However, even when this quantity assumed small values, the solutions were quite stable and converged quite readily toward the exact solution.

4. The F_1 and F_2 arbitrary factors required to make the first and second level solutions agree as closely as possible and were found to be 1.0 and 0.5-0.6. These parameters probably change somewhat with varying physical conditions, but the present investigation indicated that the changes are not too great. By using the proper F values, only first level equations were needed for an adequate simulation of the temperature conditions.

5. The initial guesses for the temperatures and temperature gradients for starting the computer solution did not have to be very accurate for the computer iteration procedure to converge. For instance, the following set of

COMPARISON OF TEMPERATURE PROFILE CONVERGENCE FOR REGION I STAINLESS STEEL CONTAINER ITERATED EQUATIONS WITH

$F_1 = 0.0$ AND $F_1 = 2.0$
(OPTIMUM $F_1 = 1.0$)

FIGURE 13

FIGURE 14

TEMPERATURE - °C

COMPARISON OF TEMPERATURE PROFILE CONVERGENCE
FOR REGION I CAPSULE ITERATED EQUATIONS WITH $F_1 = 0.0$
AND $F_1 = 2.0$ (OPTIMUM $F_1 = 1.0$)

POWER LEVEL = 200 WATTS

$F_1 = 0.0$ $K_2 = 0.14892$ $K_3 = 47.186$ $K_4 = 0.07585$ $K_5 = 0.17065$ $K_6 = 54.070$

○ 1ST LEVEL SOLUTION □ 2ND LEVEL SOLUTION Δ 3RD LEVEL SOLUTION

$F_1 = 2.0$ $K_2 = 0.14109$ $K_3 = 47.796$ $K_4 = 0.07247$ $K_5 = 0.16464$ $K_6 = 32.392$

○ 1ST LEVEL SOLUTION □ 2ND LEVEL SOLUTION Δ 3RD LEVEL SOLUTION

--- CONVERGENT SOLUTION

FRACTION OF REGION I LENGTH

conditions for 200 watts of power generation converged within 7 iterations to the second set of conditions when the second level of iterated equations was used.

	$T_{tf(avg)}$	$T_{s(avg)}$	T_{air}	T_w	A	B	C
Initial	800.0	600.0	160	70	-75.0	200.0	-88.0
Final	662.9	419.3	108.6	40.4	-48.7	82.6	-55.9

Extremely poor estimates of the starting conditions do produce results that oscillate out. However, with poorer estimates, instability can be corrected by taking a weighted average of the previously used and newly calculated values which weights the previously used values more highly than the newly calculated values. With this procedure the number of iterations to converge may be larger, but poorer guesses are allowed.

The over-all results of the iterative generation procedure and the computed iteration technique were very satisfactory. The iteratively generated equations provide good approximations of the exact solutions which in themselves may be very difficult to obtain in explicit form. Under the conditions of this problem the exact solutions to the postulated differential equations were obtainable with known end condition temperatures. However, for a more complicated system with three or more longitudinally conducting layers, the sixth and higher order differential equations that are generated make an analytical solution almost impossible to obtain. However, the iterative procedure developed for this problem can be easily extended to generate profiles for any number of concentric layers. Also, with the proper choice of the initial function substituted in the initial equation, only one level of iterative equations need be generated. In the present example good results were obtained with the first level of equations when the arbitrary constants, F_1 and F_2 , were assigned appropriate values, 1.0 and 0.6. The iterative procedure provides a powerful tool for solving heat transfer problems involving complicated layered geometries.

NOMENCLATURE

A_t	Cross-sectional area of tantalum capsule, cm^2 .
A_f	Cross-sectional area of fuel in capsule, cm^2 .
A_s	Cross-sectional area of stainless steel pipe, cm^2 .
A	Temperature gradient in stainless steel pipe at top of fuel, $^{\circ}\text{C}/\text{cm}$.
A_1	Small contact area in Figure 2, cm^2
A_2	Large area in Figure 2, cm^2 .
A_w	Area of walls of container, cm^2 .
A_b	Area of base of container, cm^2 .
B	Temperature gradient in fuel and tantalum capsule at bottom of fuel, $^{\circ}\text{C}/\text{cm}$.
C	Temperature gradient in fuel and tantalum capsule at the top of fuel, $^{\circ}\text{C}/\text{cm}$.
D_{lm}	Log mean diameter, log mean of outside capsule diameter and inside stainless steel pipe diameter, cm .
D_s	Outside diameter of stainless steel pipe, cm .
D_t	Outside diameter of tantalum capsule.

F_1 Arbitrary factor used in adjusting estimate of heat flow to stainless steel container.

F_2 Arbitrary factor for adjustment of T_s equation in Region II in first iteration.

h_{he} Helium heat transfer coefficient, $\text{cal}/\text{cm}^2 \text{ } ^\circ\text{C sec}$

$$h_{he} = h_{he(\text{rad})} (D_t/D_{lm}) + k_{he}/\Delta r_{he}$$

$h_{he(\text{rad})}$ Radiation heat transfer coefficient through helium, $\text{cal}/\text{cm}^2 \text{ } ^\circ\text{C sec}$

$$h_{he(\text{rad})} = \epsilon_t \sigma \left((T_t + 273)^4 - (T_s + 273)^4 \right) / (T_t - T_s)$$

h_{air} Air heat transfer coefficient, $\text{cal}/(\text{cm}^2 \text{ sec } ^\circ\text{C})$

$$h_{air} = h_{air(\text{rad})} + h_{air(\text{conv})} (T_s - T_{air}) / (T_s - T_w)$$

$h_{air(\text{rad})}$ Radiation heat transfer coefficient through air outside stainless steel container, $\text{cal}/\text{cm}^2 \text{ } ^\circ\text{C sec}$,

$$h_{air(\text{rad})} = \epsilon_s \sigma \left((T_s + 273)^4 - (T_w + 273)^4 \right) / (T_s - T_w)$$

$h_{air(\text{conv})}$ Natural convection heat transfer coefficient from stainless steel pipe into air, $\text{cal}/\text{cm}^2 \text{ } ^\circ\text{C sec}$

$$h_{air(\text{conv})} = 0.27 \left((T_s - T_{air})(12)(1.80)/D_s \right)^{0.25} (0.0001355)$$

$h_{(\text{wall})}$ Natural convection heat transfer coefficient from air to vertical walls of container, $\text{cal}/\text{cm}^2 \text{ } ^\circ\text{C sec}$

$$h_{\text{wall}} = 0.27 \left((T_{air} - T_w)(1.80) \right)^{0.25} (0.0001355)$$

$h_{(base)}$	Natural convection heat transfer coefficient from air to base of container, cal/cm ² °C sec $h_{(base)} = 0.38 \left((T_{air} - T_o) 1.80 \right)^{0.25} (0.0001355)$
K_1	Heat transfer ratio, helium convection + radiation/ tantalum conduction; $K_1 = \pi D_{lm} h_{he} / k_t A_t$; cm ⁻²
K_2	Heat transfer ratio, helium convection + radiation/ tantalum + fuel conduction; $K_2 = \pi D_{lm} h_{he} / (k_t A_t + k_f A_f)$; cm ⁻²
K_3	Heat generation ratio, heat generated per unit length/ tantalum + fuel conduction; $K_3 = (Q / L_f) / (k_f A_f + k_t A_t)$; °C/cm ²
K_4	Heat transfer ratio, air convection + radiation/ stainless steel conduction; $K_4 = (\pi D_s h_{air}) / k_s A_s$; cm ⁻²
K_5	Heat transfer ratio, helium convection + radiation/ stainless steel conduction; $K_5 = (\pi D_{lm} h_{he}) / (k_s A_s)$; cm ⁻²
K_6	Modified heat generation ratio, heat generated per unit length/ stainless steel conduction; $K_6 = (Q' / L_f) / (A_s k_s)$; cm ⁻²
K_7	Temperature ratio, $(T_{tf} / T_s)_{x=L_f}$
K_8	T_s uncoupling constant for Region II; $K_8 = -F_2(K_7 - 1.0)K_5$
K_9	Heat transfer ratio, wall heat flux/ wall conduction, $K_9 = Q_w / (A_w \Delta r_w k_i)$
k_{He}	Thermal conductivity of helium, cal/ (cm °C sec)
k_t	Thermal conductivity of tantalum, cal/ (cm °C sec)
k_f	Thermal conductivity of fuel, cal/ (cm °C sec)
k_s	Thermal conductivity of stainless steel, cal/ (cm °C sec)
k_i	Thermal conductivity of iron, cal/ (cm °C sec)
k_{air}	Thermal conductivity of air, cal/ (cm °C sec)

L_f	Length of fuel in capsule, cm
L_u	Length of capsule above fuel, cm
L_{12}	Distance between areas 1 and 2 in Figure 2, cm
L_{2b}	Distance between area 2 and base in Figure 2, cm
L_w	Weight of exposed container wall surface, cm
Q	Total heat generation rate in fuel, cal/sec
Q'	Modified heat generation rate as transferred to stainless steel pipe, $Q' = Q - F_1(\Delta Q_{tf})$, cal/sec
ΔQ_{tf}	Heat transferred from fuel longitudinally to empty capsule at top and to capsule holder at bottom, cal/sec
Q_w	Heat transferred to wall of outside container by convection and radiation, cal/sec
Δr_{he}	Annular distance between tantalum capsule and stainless steel pipe, cm
Δr_w	Thickness of outside steel container wall, cm.
T_{tf}	Temperature of fuel and tantalum container at position x, Region I, °C
T_s	Temperature of stainless steel container at position x, Region I or II, °C
T_{air}	Average temperature of air around stainless steel pipe, °C
T_w	Average temperature of outside container wall, °C
T_o	Temperature of cooling water, °C
x	Height above bottom of capsule in Region I or above fuel level in Region II, cm.

Greek Symbols: ϵ_t Emissivity of tantalum
 σ Stefan-Boltzmann constant, cal/ (sec cm² °C⁴)
 ϵ_s Emissivity of stainless steel

Iteration Notation: $(T_i)_j$ jth iteration equation of T_i temperature profile

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APPENDIX.

SECTION I - EXACT SOLUTIONS

REGION I.

SOLUTION OF: $\frac{d^2 T_{tf}}{dx^2} = K_2(T_{tf} - T_3) - K_3$

$$\frac{d^2 T_3}{dx^2} = -K_5(T_{tf} - T_3) + (T_3 - T_w)K_4$$

BOUNDARY CONDITIONS: $x=0 \quad T_3 = T_{30r} \quad T_{tf} = T_{tfo}$
 $x=L_f \quad T_3 = T_{3L} \quad T_{tf} = T_{tfl}$

SOLUTION EQUATIONS:

$$T_3 = \alpha_{H_1} e^{-m_1 x} + \beta_{H_1} e^{m_1 x} + \gamma_{H_1} e^{-m_2 x} + \theta_{H_1} e^{m_2 x} + T_w + \frac{K_5 K_3}{K_2 K_4}$$

$$T_{tf} = \alpha_{H_2} e^{-m_1 x} + \beta_{H_2} e^{m_1 x} + \gamma_{H_2} e^{-m_2 x} + \theta_{H_2} e^{m_2 x} + T_w + \frac{K_3(K_4 + K_5)}{K_2 K_4}$$

RELATIONSHIP OF CONSTANTS.

$$\alpha_{H_2} = \alpha_{H_1} \left(\frac{K_4 + K_5 - m_1^2}{K_5} \right); \quad \beta_{H_2} = \beta_{H_1} \left(\frac{K_4 + K_5 - m_1^2}{K_5} \right)$$

$$\gamma_{H_2} = \gamma_{H_1} \left(\frac{K_4 + K_5 - m_2^2}{K_5} \right); \quad \theta_{H_2} = \theta_{H_1} \left(\frac{K_4 + K_5 - m_2^2}{K_5} \right)$$

EXPONENT FORMULAS:

$$m_1 = \left(\frac{(K_2 + K_4 + K_5) + ((K_2 + K_4 + K_5)^2 - 4K_2 K_4)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}$$

$$m_2 = \left(\frac{(K_2 + K_4 + K_5) - ((K_2 + K_4 + K_5)^2 - 4K_2 K_4)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}$$

INTEGRATION CONSTANTS.

$$\theta_{H_1} = \frac{M_{H_1} - M_{H_2} e^{m_2 L_f}}{(K_{13} - K_{12})(1.0 - e^{2m_2 L_f})}$$

$$\gamma_{H_1} = \frac{M_{H_1} - M_{H_2} e^{-m_2 L_f}}{(K_{13} - K_{12})(1.0 - e^{-2m_2 L_f})}$$

$$\beta_{H_1} = \frac{M_{H_3} - M_{H_4} e^{m_1 L_f}}{(K_{12} - K_{13})(1.0 - e^{-2m_1 L_f})}$$

$$M_{H_1} = T_{+f_0} - T_{so_I} K_{12} + T_w (K_{12} - 1.0) + \frac{K_3}{K_2 K_4} (K_{12} K_5 - K_4 - K_5)$$

$$M_{H_2} = T_{+f_L} - T_{s_L} K_{12} + T_w (K_{12} - 1.0) + \frac{K_3}{K_2 K_4} (K_{12} K_5 - K_4 - K_5)$$

$$M_{H_3} = T_{+f_0} - T_{so_I} K_{13} + T_w (K_{13} - 1.0) + \frac{K_3}{K_2 K_4} (K_{13} K_5 - K_4 - K_5)$$

$$M_{H_4} = T_{+f_L} - T_{s_L} K_{13} + T_w (K_{13} - 1.0) + \frac{K_3}{K_2 K_4} (K_{13} K_5 - K_4 - K_5)$$

$$K_{12} = \frac{K_4 + K_5 - m_1^2}{K_5} \quad ; \quad K_{13} = \frac{K_4 + K_5 - m_2^2}{K_5}$$

REGION II.

DIFFERENTIAL EQUATIONS:

$$\frac{d^2 T_+}{dx^2} = K_1 (T_+ - T_s)$$

$$\frac{d^2 T_s}{dx^2} = -K_5 (T_+ - T_s) + (T_s - T_w) K_4$$

BOUNDARY CONDITIONS:

$$x=0 \quad T_s = T_{so_{II}} \quad T_+ = T_{+0}$$

$$x=L_u \quad T_s = T_w \quad T_+ = T_w$$

SOLUTION EQUATIONS:

$$T_s = \alpha_{N_1} e^{-\eta_1 x} + \beta_{N_1} e^{\eta_1 x} + \gamma_{N_1} e^{-\eta_2 x} + \theta_{N_1} e^{\eta_2 x} + T_w$$

$$T_+ = \alpha_{N_2} e^{-\eta_1 x} + \beta_{N_2} e^{\eta_1 x} + \gamma_{N_2} e^{-\eta_2 x} + \theta_{N_2} e^{\eta_2 x} + T_w$$

RELATIONSHIP OF INTEGRATION CONSTANTS.

$$\alpha_{N_2} = \alpha_{N_1} K_{10} ; \beta_{N_2} = \beta_{N_1} K_{10}$$

$$\gamma_{N_2} = \gamma_{N_1} K_{11} ; \theta_{N_2} = \theta_{N_1} K_{11}$$

$$K_{10} = \frac{K_5 + K_4 - n_1^2}{K_5} ; K_{11} = \frac{K_5 + K_4 - n_2^2}{K_5}$$

EXPONENT FORMULAS.

$$n_1 = \left(\frac{(K_1 + K_4 + K_5) + ((K_1 + K_4 + K_5)^2 - 4K_1K_4)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}$$

$$n_2 = \left(\frac{(K_1 + K_2 + K_5) - ((K_1 + K_4 + K_5)^2 - 4K_1K_4)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}$$

INTEGRATION CONSTANTS

$$\theta_{N_1} = \frac{(T_{f0} - T_w) - K_{10}(T_{so_{II}} - T_w)}{(K_{11} - K_{10})(1.0 - e^{-2n_2 L_u})}$$

$$\gamma_{N_1} = \frac{(T_{f0} - T_w) - K_{10}(T_{so_{II}} - T_w)}{(K_{11} - K_{10})(1.0 - e^{-2n_2 L_u})}$$

$$\beta_{N_1} = \frac{(T_{f0} - T_w) - K_{11}(T_{so_{II}} - T_w)}{(K_{10} - K_{11})(1.0 - e^{-2n_1 L_u})}$$

$$\alpha_{N_1} = \frac{(T_{f0} - T_w) - K_{11}(T_{so_{II}} - T_w)}{(K_{10} - K_{11})(1.0 - e^{-2n_1 L_u})}$$

APPENDIX

SECTION II - ITERATED SOLUTIONS

REGION I

DIFFERENTIAL EQUATIONS:

$$\frac{d^2 T_{tf}}{dx^2} = (T_{tf} - T_s) K_2 - K_3$$

$$\frac{d^2 T_s}{dx^2} = -(T_{tf} - T_s) K_5 + (T_s - T_w) K_4$$

BOUNDARY CONDITIONS:

$$x=0 \quad T_s = T_{s0I} \quad \frac{dT_{tf}}{dx} = B$$

$$x=L_f \quad \frac{dT_s}{dx} = A \quad \frac{dT_{tf}}{dx} = C$$

UNCOUPLED EQUATION FOR FIRST ITERATION

$$\frac{d^2 T_s}{dx^2} - K_4 T_s = -K_6 - K_4 T_w$$

$$\text{WHERE } K_6 = \frac{(Q - F_1 \Delta Q)/L_f}{A_s k_s}$$

FIRST ITERATED EQUATION FOR T_s .

$$(T_s)_1 = \alpha_{I1} e^{-\sqrt{K_4} x} + \beta_{I1} e^{\sqrt{K_4} x} + D_1$$

$$D_1 = T_w + K_6 / K_4$$

$$\alpha_{I1} = \frac{\sqrt{K_4} e^{\sqrt{K_4} L_f} (T_{s0I} - Z_{11}) - A}{Z_{12}}$$

$$\beta_{I1} = \frac{\sqrt{K_4} e^{-\sqrt{K_4} L_f} (T_{s0I} - Z_{11}) + A}{Z_{12}}$$

$$Z_{11} = D_1$$

$$Z_{12} = \sqrt{K_4} (e^{\sqrt{K_4} L_f} + e^{-\sqrt{K_4} L_f})$$

FIRST ITERATED EQUATION FOR T_{ff}

$$(T_{ff})_1 = \gamma_{I_1} e^{-\sqrt{K_2}x} + \theta_{I_1} e^{\sqrt{K_2}x} + M_1 e^{-\sqrt{K_4}x} + N_1 e^{\sqrt{K_4}x} + P_1$$

$$M_1 = \alpha_{I_1} K_2 / (K_2 - K_4) ; N_1 = \beta_{I_1} K_2 / (K_2 - K_4)$$

$$P_1 = K_3 / K_2 + D_1$$

$$\gamma_{I_1} = \frac{e^{\sqrt{K_2} L_f} (B - Z_{13}) + Z_{14} - C}{Z_{15}}$$

$$\theta_{I_1} = \frac{e^{-\sqrt{K_2} L_f} (B - Z_{13}) + Z_{14} - C}{Z_{15}}$$

$$Z_{13} = -\sqrt{K_4} M_1 + \sqrt{K_4} N_1$$

$$Z_{14} = \sqrt{K_4} (-M_1 e^{-\sqrt{K_4} L_f} + N_1 e^{\sqrt{K_4} L_f})$$

$$Z_{15} = \sqrt{K_2} (e^{-\sqrt{K_2} L_f} - e^{\sqrt{K_2} L_f})$$

SECOND ITERATED EQUATION FOR T_3 .

$$(T_3)_2 = \alpha_{I_2} e^{-\sqrt{K_4+K_5}x} + \beta_{I_2} e^{\sqrt{K_4+K_5}x} + D_2 e^{-\sqrt{K_2}x} + E_2 e^{\sqrt{K_2}x} + F_{2I} e^{-\sqrt{K_4}x} + G_2 e^{\sqrt{K_4}x} + H_2$$

$$D_2 = \gamma_{I_1} K_5 / (K_4 + K_5 - K_2) ; E_2 = \theta_{I_1} K_5 / (K_5 + K_4 - K_2)$$

$$F_{2I} = M_1 ; G_2 = N_1 ; H_2 = \left(\frac{K_5}{K_4 + K_5} \right) P_1 + \left(\frac{K_4}{K_4 + K_5} \right) T_w$$

$$\alpha_{I_2} = \frac{\sqrt{K_4+K_5} e^{\sqrt{K_4+K_5} L_f} (T_{30I} - Z_{21}) + Z_{22} - A}{Z_{23}}$$

$$\beta_{I_2} = \frac{\sqrt{K_4+K_5} e^{-\sqrt{K_4+K_5} L_f} (T_{30I} - Z_{21}) - Z_{22} + A}{Z_{23}}$$

$$Z_{21} = D_2 + E_2 + F_2 + G_2 + H_2$$

$$Z_{22} = +\sqrt{K_2} (-D_2 e^{-\sqrt{K_2} L_f} + E_2 e^{\sqrt{K_2} L_f}) + \sqrt{K_4} (-F_2 e^{-\sqrt{K_4} L_f} + G_2 e^{\sqrt{K_4} L_f})$$

$$Z_{23} = \sqrt{K_4+K_5} (e^{\sqrt{K_4+K_5} L_f} + e^{-\sqrt{K_4+K_5} L_f})$$

SECOND ITERATED EQUATION FOR T_{ff}

$$(T_{ff})_2 = \gamma_{I_2} e^{-\sqrt{K_2} x} + \theta_{I_2} e^{\sqrt{K_2} x} + M_2 e^{-\sqrt{K_4+K_5} x} + N_2 e^{\sqrt{K_4+K_5} x} + P_2 x e^{-\sqrt{K_2} x} + Q_2 x e^{\sqrt{K_2} x} \\ + R_2 e^{-\sqrt{K_4} x} + S_2 e^{\sqrt{K_4} x} + U_2$$

$$M_2 = \alpha_{I_2} K_2 / (K_2 - K_4 - K_5) ; N_2 = \beta_{I_2} K_2 / (K_2 - K_4 - K_5)$$

$$P_2 = \sqrt{K_2} D_2 / 2.0 ; Q_2 = -\sqrt{K_2} E_2 / 2.0 ; R_2 = F_2 K_2 / (K_2 - K_4)$$

$$S_2 = G_2 K_2 / (K_2 - K_4) ; U_2 = H_2 + K_3 / K_2$$

$$\gamma_{I_2} = \frac{e^{\sqrt{K_2} L_f} (B - Z_{24}) + Z_{25} - C}{Z_{26}}$$

$$\theta_{I_2} = \frac{e^{-\sqrt{K_2} L_f} (B - Z_{24}) + Z_{25} - C}{Z_{26}}$$

$$Z_{24} = \sqrt{K_4+K_5} (-M_2 + N_2) + P_2 + Q_2 + \sqrt{K_4} (-R_2 + S_2)$$

$$Z_{25} = \sqrt{K_4+K_5} (-M_2 e^{-\sqrt{K_4+K_5} L_f} + N_2 e^{\sqrt{K_4+K_5} L_f}) + P_2 e^{-\sqrt{K_2} L_f} (1.0 - L_f \sqrt{K_2}) \\ + Q_2 e^{\sqrt{K_2} L_f} (1.0 + L_f \sqrt{K_2}) + \sqrt{K_4} (-R_2 e^{-\sqrt{K_4} L_f} + S_2 e^{\sqrt{K_4} L_f})$$

$$Z_{26} = \sqrt{K_2} (e^{-\sqrt{K_2} L_f} - e^{\sqrt{K_2} L_f})$$

THIRD ITERATED EQUATION FOR T_3 .

$$(T_3) = \alpha_{I_3} e^{-\sqrt{K_4+K_5} x} + \beta_{I_3} e^{\sqrt{K_4+K_5} x} + e^{-\sqrt{K_2}} (C_3 + G_3 x) + e^{\sqrt{K_2}} (D_3 + H_3 x) \\ + E_3 x e^{-\sqrt{K_4+K_5} x} + F_3 x e^{\sqrt{K_4+K_5} x} + I_3 e^{-\sqrt{K_4} x} + J_3 e^{\sqrt{K_4} x} + L_3$$

$$L_3 = (K_5 U_2 + T_w K_4) / (K_4 + K_5) ; J_3 = S_2 ; I_3 = R_2$$

$$H_3 = K_5 Q_2 / (K_4 + K_5 - K_2) ; G_3 = K_5 P_2 / (K_4 + K_5 - K_2)$$

$$F_3 = -K_5 N_2 / (2\sqrt{K_4+K_5}) ; E_3 = K_5 M_2 / (2\sqrt{K_4+K_5})$$

$$D_3 = (K_5 \theta_{I_2} + 2\sqrt{K_2} H_3) / (K_4 + K_5 - K_2)$$

$$C_3 = (K_5 \gamma_{I_2} - 2\sqrt{K_2} G_3) / (K_4 + K_5 - K_2)$$

$$\alpha_{I_3} = \frac{\sqrt{K_4+K_5} e^{\sqrt{K_4+K_5} L_f} (T_{30I} - Z_{31}) + Z_{32} - A}{Z_{33}}$$

$$\beta_{I_3} = \frac{\sqrt{K_4+K_5} e^{-\sqrt{K_4+K_5} L_f} (T_{30I} - Z_{31}) - Z_{32} + A}{Z_{33}}$$

$$Z_{31} = C_3 + D_3 + I_3 + J_3 + L_3$$

$$\begin{aligned} Z_{32} = & \sqrt{K_2} e^{-\sqrt{K_2} L_f} (-C_3 + G_3 (1/\sqrt{K_2} - L_f)) + \sqrt{K_2} e^{\sqrt{K_2} L_f} (+D_3 + H_3 (\frac{1}{\sqrt{K_2}} + L_f)) \\ & + E_3 e^{-\sqrt{K_4+K_5} L_f} (1.0 - L_f \sqrt{K_4+K_5}) + F_3 e^{\sqrt{K_4+K_5} L_f} (1.0 + L_f \sqrt{K_4+K_5}) \\ & + \sqrt{K_4} (-I_3 e^{-\sqrt{K_4} L_f} + J_3 e^{\sqrt{K_4} L_f}) \end{aligned}$$

$$Z_{33} = \sqrt{K_4+K_5} (e^{\sqrt{K_4+K_5} L_f} + e^{-\sqrt{K_4+K_5} L_f})$$

THIRD ITERATED EQUATION FOR T_{ff} .

$$\begin{aligned} (T_{ff})_3 = & \gamma_{I_3} e^{-\sqrt{K_2} x} + \theta_{I_3} e^{\sqrt{K_2} x} + e^{-\sqrt{K_4+K_5} x} (M_3 + P_3 x) + e^{\sqrt{K_4+K_5} x} (N_3 + Q_3 x) \\ & + e^{-\sqrt{K_2} x} (x^2 R_3 + x S_3) + e^{\sqrt{K_2} x} (T_3 x^2 + U_3 x) + V_3 e^{-\sqrt{K_4} x} + W_3 e^{\sqrt{K_4} x} + Y_3 \end{aligned}$$

$$Y_3 = L_3 + K_3/K_2 ; W_3 = J_3 K_2 / (K_2 - K_4) ; V_3 = I_3 K_2 / (K_2 - K_4)$$

$$T_3 = -\sqrt{K_2} H_3 / 4 ; R_3 = \sqrt{K_2} G_3 / 4 ; Q_3 = F_3 K_2 / (K_2 - K_4 - K_5)$$

$$P_3 = E_3 K_2 / (K_2 - K_4 - K_5) ; U_3 = -\sqrt{K_2} D_3 / 2 + H_3 / 4$$

$$S_3 = \sqrt{K_2} C_3 / 2 + G_3 / 4 ; N_3 = (K_2 \beta_{I_3} + 2 Q_3 \sqrt{K_4+K_5}) / (K_2 - K_4 - K_5)$$

$$M_3 = (K_2 \alpha_{I_3} - 2 P_3 \sqrt{K_4+K_5}) / (K_2 - K_4 - K_5)$$

$$\gamma_{I_3} = \frac{e^{\sqrt{K_2} L_f} (B - Z_{34}) + Z_{35} - C}{Z_{36}}$$

$$\theta_{I_3} = \frac{e^{-\sqrt{K_2} L_f} (B - Z_{34}) + Z_{35} - C}{Z_{36}}$$

$$Z_{34} = \sqrt{K_4+K_5} (-M_3 + N_3) + P_3 + Q_3 + S_3 + U_3 + \sqrt{K_4} (-V_3 + W_3)$$

$$\begin{aligned}
 Z_{35} = & \sqrt{K_4 + K_5} e^{-\sqrt{K_4 + K_5} L_f} (-M_3 + P_3 (1.0 / \sqrt{K_4 + K_5} - L_f)) \\
 & + \sqrt{K_4 + K_5} e^{\sqrt{K_4 + K_5} L_f} (N_3 + Q_3 (1.0 / \sqrt{K_4 + K_5} + L_f)) \\
 & + e^{-\sqrt{K_2} L_f} (R_3 (2L_f - L_f^2 \sqrt{K_2}) + S_3 (1.0 - L_f \sqrt{K_2})) \\
 & + e^{\sqrt{K_2} L_f} (T_3 (2L_f + L_f^2 \sqrt{K_2}) + U_3 (1.0 + L_f \sqrt{K_2})) \\
 & + \sqrt{K_4} (-V_3 e^{-\sqrt{K_4} L_f} + W_3 e^{\sqrt{K_4} L_f})
 \end{aligned}$$

$$Z_{36} = \sqrt{K_2} (e^{-\sqrt{K_2} L_f} - e^{\sqrt{K_2} L_f})$$

APPENDIX

SECTION III

ITERATED SOLUTIONS - REGION II

DIFFERENTIAL EQUATIONS:

$$\frac{d^2 T_+}{d x^2} = (T_+ - T_s) K_1$$

$$\frac{d^2 T_s}{d x^2} = -(T_+ - T_s) K_5 + (T_s - T_w) K_4$$

BOUNDARY CONDITIONS:

$$x=0 \quad T_s = T_{s0II} \quad T_+ = T_{+0}$$

$$x=L_u \quad T_s = T_w \quad T_+ = T_w$$

UNCOUPLED EQUATION FOR FIRST ITERATION

$$\frac{d^2 T_s}{d x^2} - (K_8 + K_4) T_s = -K_4 T_w$$

$$\text{WHERE } K_8 = F_2 K_5 (1.0 - K_7) \text{ AND } K_7 = T_{+0}/T_{s0II}$$

$$F_2 = \text{ARBITRARY CONSTANT}$$

FIRST ITERATED EQUATION FOR T_s .

$$(T_s)_1 = \gamma_{II_1} e^{-\sqrt{K_8+K_4}x} + \theta_{II_1} e^{\sqrt{K_8+K_4}x} + D_{II_1}$$

$$D_{II_1} = T_w K_4 / (K_4 + K_8)$$

$$\gamma_{II_1} = \frac{e^{\sqrt{K_8+K_4}L_u} (T_{s0II} - \gamma_{II_1}) + \gamma_{12} - T_w}{\gamma_{13}}$$

$$\theta_{II_1} = \frac{e^{-\sqrt{K_8+K_4}L_u} (T_{s0II} - \gamma_{II_1}) + \gamma_{12} - T_w}{-\gamma_{13}}$$

$$\gamma_{II_1} = \gamma_{12} = D_{II_1}$$

$$\gamma_{13} = e^{\sqrt{K_8+K_4}L_u} - e^{-\sqrt{K_8+K_4}L_u}$$

FIRST ITERATED EQUATION FOR T_+ .

$$(T_+)_1 = \alpha_{II_1} e^{-\sqrt{K_1} x} + \beta_{II_1} e^{\sqrt{K_1} x} + M_{II_1} e^{-\sqrt{K_4+K_8} x} + N_{II_1} e^{\sqrt{K_4+K_8} x} + P_{II_1}$$

$$M_{II_1} = \gamma_{II_1} K_1 / (K_1 - K_4 - K_8) ; N_{II_1} = \theta_{II_1} K_1 / (K_1 - K_4 - K_8)$$

$$P_{II_1} = D_{II_1}$$

$$\alpha_{II_1} = \frac{e^{\sqrt{K_1} L_4} (T_{+0} - Y_{14}) + Y_{15} - T_w}{Y_{16}}$$

$$\beta_{II_1} = \frac{e^{-\sqrt{K_1} L_4} (T_{+0} - Y_{14}) + Y_{15} - T_w}{-Y_{16}}$$

$$Y_{14} = M_{II_1} + N_{II_1} + P_{II_1}$$

$$Y_{15} = M_{II_1} e^{-\sqrt{K_4+K_8} L_4} + N_{II_1} e^{\sqrt{K_4+K_8} L_4} + P_{II_1}$$

$$Y_{16} = e^{\sqrt{K_1} L_4} - e^{-\sqrt{K_1} L_4}$$

SECOND ITERATED EQUATION FOR T_s

$$(T_s)_2 = \gamma_{II_2} e^{-\sqrt{K_4+K_5} x} + \theta_{II_2} e^{\sqrt{K_4+K_5} x} + D_{II_2} e^{-\sqrt{K_1} x} + E_{II_2} e^{\sqrt{K_1} x} + G_{II_2} e^{-\sqrt{K_4+K_8} x} + H_{II_2} e^{\sqrt{K_4+K_8} x} + I_{II_2}$$

$$D_{II_2} = \alpha_{II_1} K_5 / (K_4 + K_5 - K_1) ; E_{II_2} = \beta_{II_1} K_5 / (K_4 + K_5 - K_1)$$

$$G_{II_2} = M_{II_1} K_5 / (K_5 - K_8) ; H_{II_2} = N_{II_1} K_5 / (K_5 - K_8)$$

$$I_{II_2} = (T_w K_4 + P_{II_1} K_5) / (K_4 + K_5)$$

$$\gamma_{II_2} = \frac{e^{\sqrt{K_4+K_5} L_4} (T_{s0II} - Y_{21}) + Y_{22} - T_w}{Y_{23}}$$

$$\theta_{II_2} = \frac{e^{-\sqrt{K_4+K_5} L_4} (T_{s0II} - Y_{21}) + Y_{22} - T_w}{-Y_{23}}$$

$$Y_{21} = D_{II_2} + E_{II_2} + G_{II_2} + H_{II_2} + I_{II_2}$$

$$Y_{22} = D_{II_2} e^{-\sqrt{K_1} L_4} + E_{II_2} e^{\sqrt{K_1} L_4} + G_{II_2} e^{-\sqrt{K_4+K_8} L_4} + H_{II_2} e^{\sqrt{K_4+K_8} L_4} + I_{II_2}$$

$$Y_{23} = e^{\sqrt{K_4+K_5} L_4} - e^{-\sqrt{K_4+K_5} L_4}$$

SECOND ITERATED EQUATION FOR T_+

$$(T_+)_2 = \alpha_{II_2} e^{-TK_1 x} + \beta_{II_2} e^{TK_1 x} + M_{II_2} e^{-TK_4+K_5 x} + N_{II_2} e^{TK_4+K_5 x} + P_{II_2} x e^{-TK_1 x} \\ + Q_{II_2} x e^{TK_1 x} + R_{II_2} e^{-TK_4+K_8 x} + S_{II_2} e^{TK_4+K_8 x} + T_{II_2}$$

$$M_{II_2} = \delta_{II_2} K_1 / (K_1 - K_4 - K_5); \quad N_{II_2} = \theta_{II_2} K_1 / (K_1 - K_4 - K_5)$$

$$P_{II_2} = D_{II_2} TK_1 / 2 \quad ; \quad Q_{II_2} = -E_{II_2} TK_1 / 2$$

$$R_{II_2} = G_{II_2} K_1 / (K_1 - K_4 - K_8); \quad S_{II_2} = H_{II_2} K_1 / (K_1 - K_4 - K_8)$$

$$T_{II_2} = I_{II_2}$$

$$\alpha_{II_2} = \frac{e^{TK_1 L_4} (T_{+0} - Y_{24}) + Y_{25} - T_w}{Y_{26}}$$

$$\beta_{II_2} = \frac{e^{-TK_1 L_4} (T_{+0} - Y_{24}) + Y_{25} - T_w}{-Y_{26}}$$

$$Y_{24} = M_{II_2} + N_{II_2} + R_{II_2} + S_{II_2} + T_{II_2}$$

$$Y_{25} = M_{II_2} e^{-TK_4+K_5 L_4} + N_{II_2} e^{TK_4+K_5 L_4} + P_{II_2} L_4 e^{-TK_1 L_4} \\ + Q_{II_2} L_4 e^{TK_1 L_4} + R_{II_2} e^{-TK_4+K_8 L_4} + S_{II_2} e^{TK_4+K_8 L_4} + T_{II_2}$$

$$Y_{26} = e^{TK_1 L_4} - e^{-TK_1 L_4}$$