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SINGULAR INTEGRALS AND SINGULAR INTEGRAL EQUATIONS WITH A CAUCHY KERNEL AND THE METHOD OF SYMMETRIC PAIRING

by

Erwin H. Bareiss and Charles P. Neuman

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SINGULAR INTEGRALS AND SINGULAR INTEGRAL EQUATIONS WITH A CAUCHY KERNEL AND THE METHOD OF SYMMETRIC PAIRING

by

Erwin H. Bareiss and Charles P. Neuman

ABSTRACT

The method of symmetric pairing, combined with midpoint integration, is introduced for the numerical evaluation of Hilbert transforms (and the Kramers-Krönig relations) and singular integral equations with a Cauchy kernel. The method, for use on high-speed computers, is simple, reliable, and inexpensive. Numerical examples are presented.

I. INTRODUCTION

Many problems in mathematical physics and engineering require the solution of the singular integral equation^(17,19)

$$a(x) \phi(x) - \lambda \int_a^b \frac{\phi(y)}{(y-x)} dy = f(x), \quad (1.1)$$

where the principal value should be taken for the integral.⁽¹⁵⁾ Equation (1.1) is a Fredholm integral equation of the second kind with a Cauchy kernel.

Through the introduction of Hilbert transforms, the solution of the integral equation (1.1) may be written explicitly in terms of the known $f(x)$. The computational procedure for solving equation (1.1) requires the evaluation of the Hilbert transform of the given function $f(x)$ or other singular integrals with a Cauchy kernel.

This paper discusses some analytic solutions of equation (1.1) and gives computational schemes for evaluating Hilbert transforms of known functions. It discusses also the numerical solution of singular integral equations of the type (1.1). The method of "symmetric pairing" is introduced and shown to be preferable to other methods. Numerical examples are given, and the results of computer calculations are compared with known analytic results. Examples of problems are given from classical electric network theory, radiative transfer, and neutron transport theory.

II. MATHEMATICAL PRELIMINARIES

A. Definitions and Basic Theorems

In this section, the Cauchy principal value of an integral is defined and the Hilbert transformation is introduced.

Definition

Cauchy's principal value of the integral of a function $f(x)$ that becomes infinite at an interior point $x=x_0$ of the interval of integration (a, b) is the limit

$$P \int_a^b f(x) dx \equiv \lim_{\epsilon \rightarrow 0} \left(\int_a^{x_0 - \epsilon} + \int_{x_0 + \epsilon}^b \right) f(x) dx, \quad (2.1)$$

where

$$0 < \epsilon \leq \min(x_0 - a, b - x_0).$$

We shall denote this integral by placing P in front of the usual integration sign.

If $f(x) = g(x)/(x - x_0)$, where $g(x)$ is any integrable function (in the sense of Lebesgue), then the above limit exists and is finite for almost every x_0 in (a, b) ; and if $g(x)$ belongs to the class L_p with $p > 1$, the principal integral also belongs to L_p .⁽¹⁸⁾

The Cauchy principal value of a singular integral coincides with the ordinary Riemann integral (definite or indefinite) if the latter exists.⁽¹³⁾

Definition

The Hilbert transformation of a function $\phi(t)$ is defined by

$$f(x) = H_x[\phi(t)] \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi(t)}{t - x} dt. \quad (2.2)$$

If $\phi(t)$ is a sufficiently regular function, as explained below in Theorems 1 to 3, we will have

$$f(x) = H_x[\phi(t)], \quad (2.3)$$

and

$$\phi(x) = -H_x[f(t)]. \quad (2.4)$$

The two functions so connected are called Hilbert transforms.⁽¹⁴⁾ The functions $f(x)$ and $g(x)$ are also called a pair of conjugate functions [Titchmarsh,⁽¹⁸⁾ p.120]. Equations (2.3) and (2.4) show that f and g are skew reciprocal, i.e., reciprocal, apart from a minus sign.⁽¹²⁾

The following three theorems [Tricomi,⁽¹⁹⁾ p.168] are basic for infinite intervals, and proved in Ref. 18.

Theorem 1 (Reciprocity theorem)

If the function $\phi(x)$ belongs to the class L_p ($p > 1$) in the basic interval $(-\infty, \infty)$, then formula (2.2) defines almost everywhere a function $f(x)$, which also belongs to L_p , whose Hilbert transform $H[f]$ coincides almost everywhere with $-\phi(x)$.

That is, for any L_p -function,

$$H\{H[\phi]\} = -\phi. \quad (2.5)$$

Theorem 2 (Generalized Parseval's formula)

Let the functions $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L_{p_1} and L_{p_2} , respectively. Then if

$$1/p_1 + 1/p_2 = 1, \quad (2.6)$$

we have

$$\int_{-\infty}^{\infty} \phi_1(x) \phi_2(x) dx = \int_{-\infty}^{\infty} H_x[\phi_1(y)] H_x[\phi_2(y)] dx. \quad (2.7)$$

Theorem 3

Let $\Phi(x+iy)$ be an analytic function, regular for $y > 0$, which, for all values of y , satisfies the condition

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^p dx < K, \quad (p > 1) \quad (2.8)$$

where K is a positive constant. Then as $y \rightarrow +0$, $\Phi(x+iy)$ converges for almost all x to a limit function,

$$\Phi(x+i0) = u(x) + iv(x),$$

whose real and imaginary parts $u(x)$ and $v(x)$ are two L_p -functions connected by the reciprocity formulas (2.3) and (2.4).

Hence, in particular, we have, almost everywhere,

$$\begin{aligned} \operatorname{Re} \Phi(x+i0) &= H_x[\operatorname{Im} \Phi(\xi+i0)]; \\ \operatorname{Im} \Phi(x+i0) &= H_x[\operatorname{Re} \Phi(\xi+i0)]. \end{aligned} \quad (2.9)$$

Conversely, given any real function $v(x)$ of the class L_p , if we put

$$u(x) = H_x[v(t)], \quad (2.10)$$

then the analytic function $\Phi(z)$ corresponding to the pair (u, v) can be calculated by means of the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t) + iv(t)}{t - z} dt \quad (\operatorname{Im} z > 0), \quad (2.11)$$

and it satisfies condition (2.8). Hence, we can be sure that an analytic function $\Phi(z)$, regular for $\operatorname{Im} z > 0$, satisfies condition (2.8) whenever its limit values $\Phi(x+i0) = u(x) + iv(x)$ on the real axis exist almost everywhere and the functions $u(x)$ and $v(x)$ both belong to the class L_p with $p > 1$. In other words, under the previous hypothesis, it is not necessary, a priori, that u and v be connected by equations (2.3) and (2.4).

B. A Typical Singular Integral Equation

The introduction of Hilbert transforms permits an elegant treatment of singular integral equations of the second kind with Cauchy kernel,

$$\phi(x) - \lambda P \int_{-\infty}^{\infty} K(x, t) \phi(t) dt = f(x), \quad (2.12)$$

in L_p -space. In fact, consider the following equation of the second kind,

$$\phi(x) - \lambda P \int_{-\infty}^{\infty} \frac{\phi(t)}{t - x} dt = f(x), \quad (2.13)$$

or symbolically,

$$\phi(x) - \lambda \pi H_x \{\phi(t)\} = f(x). \quad (2.14)$$

If we apply the Hilbert transform to both sides of the equation and use the skew-symmetry property (2.5), equation (2.14) becomes

$$H_x \{\phi(t)\} + \lambda \pi \phi(x) = H_x \{f(t)\}. \quad (2.15)$$

Consequently, multiplying equation (2.15) by $\lambda\pi$ and adding the result to (2.14), yields

$$(1 + \lambda^2 \pi^2) \phi(x) = f(x) + \lambda\pi H_x \{f(t)\}. \quad (2.16)$$

or explicitly,

$$\phi(x) = \frac{1}{1 + \lambda^2 \pi^2} \left\{ f(x) + \lambda P \int_{-\infty}^{\infty} \frac{f(t)}{t - x} dt \right\}. \quad (2.17)$$

Equation (2.17) is the explicit solution of the singular integral equation (2.13) written in terms of the known forcing function $f(x)$ and its Hilbert transform.

This discussion shows that to solve an integral equation of the second kind with a Cauchy kernel, it is sufficient to evaluate the Hilbert transform of the known forcing function.

Moreover, if the kernel is of the type

$$K(x, y) = \frac{1}{y - x} + K^*(x, y),$$

where K^* is bounded (or, at least, integrable), then [see Tricomi,⁽¹⁹⁾ p. 173]

$$\begin{aligned} (1 + \lambda^2 \pi^2) \phi(x) = & f(x) + \lambda P \int_{-\infty}^{\infty} \frac{f(y)}{y - x} dy + \lambda \int_{-\infty}^{\infty} K^*(x, y) \phi(y) dy \\ & + \lambda^2 P \int_{-\infty}^{\infty} \frac{dz}{z - x} \int_{-\infty}^{\infty} K^*(x, y) \phi(y) dy. \end{aligned} \quad (2.18)$$

If the order of the two successive integrations can be interchanged, this is an ordinary Fredholm equation of the second kind with kernel

$$K^*(x, y) + \lambda P \int_{-\infty}^{\infty} \frac{K^*(z, y)}{z - x} dz. \quad (2.19)$$

"Ordinary" means that the integral in (2.19) is an ordinary one, not a Cauchy principal integral; however, the basic interval is still infinite. To obtain the regular kernel, we have to evaluate a Hilbert transformation of K^* .

C. Finite Hilbert Transformations

Cauchy's principal integrals over a finite interval have applications in aerodynamics, in the theory of radiative transfer and neutron transport. One defines [Tricomi,⁽¹⁹⁾ p. 173] the finite Hilbert Transformation of $g(t)$ by

$$f(x) = \frac{1}{\pi} P \int_{-1}^{+1} \frac{g(t)}{t - x} dy \equiv T_x[g(t)].$$

This transformation occurs in singular integral equations of Carleman type, in equations of radiative and neutron transfer, in aerodynamics, etc.^(3,5,19) Again, in many cases the explicit solutions can be given in forms that require the evaluation of finite Hilbert transformations, or the equations can be transformed into ordinary Fredholm equations whose kernel contains finite Hilbert transforms.

It is, therefore, desirable to have practical methods for the numerical evaluation of Hilbert transformations.

III. GENERAL METHODS FOR THE EVALUATION OF HILBERT TRANSFORMATIONS

A. Subtraction of Singularity

The numerical evaluation of Hilbert transformations requires an approximation for the principal value integral (2.1). A general method for numerically evaluating Cauchy principal values is that of removing the singularity as follows:

1. Since the integrand in the Hilbert transformation is of the form

$$F(x,t) = f(t)/(t-x), \quad (3.1)$$

subtract from this integrand $F(x,t)$ a function $G(x,t)$ which can be integrated analytically and has a singularity of the same kind as $F(x,t)$.

2. Next, evaluate the integral

$$\int_a^b G(x,t) dt \quad (3.2)$$

analytically, and evaluate

$$\int_a^b \{F(x,t) - G(x,t)\} dt \quad (3.3)$$

numerically.

This method of subtracting out the singularity has been used widely.⁽⁸⁾ The best-known example of this method is as follows. Assume $f(t)$ to be continuous at x . That is,

$$\begin{aligned} P \int_a^b \frac{f(t)}{t-x} dt &= \int_a^b \frac{f(t) - f(x)}{t-x} dt + f(x) P \int_a^b \frac{dt}{t-x} \\ &= \int_a^b \frac{f(t) - f(x)}{t-x} dy + f(x) \log \frac{b-x}{x-a}^* \end{aligned} \quad (3.4)$$

If $f(t)$ also has a continuous derivative at x , the function

$$\phi_x(t) = [f(t) - f(x)]/(t-x) \quad (3.5)$$

*Note that for $a = -b$, $\lim_{b \rightarrow \infty} \log [(b-x)/(x-a)] = 0$.
($a = -b$)

is defined at $t = x$ by

$$\phi_x(x) = \left[\frac{df}{dt} \right]_{t=x} \equiv f'(x) \quad (3.6)$$

and is indeed regular in the neighborhood of x . Hence,

$$\int_a^b \phi_x(t) dt \quad (3.7)$$

can be evaluated by any standard method for numerical integration.

It will be of interest to know the contribution of the neighborhood $x - h \leq t \leq x + h$ to the integral (3.7) when h is small. We assume that as many derivatives of $f(t)$ exist as may be needed in this interval. Shifting the singularity to the origin, and expanding $f(x+t)$ into a Taylor series, yields

$$\left. \begin{aligned} \int_{x-h}^{x+h} \frac{f(t) - f(x)}{t - x} dt &= \int_{-h}^{+h} \frac{f(x+t) - f(x)}{t} dt \\ &= \int_{-h}^{+h} \left\{ f'(x) + \frac{1}{2} t f''(x) + \frac{1}{6} t^2 f'''(x) + \dots \right\} dt \\ &= 2hf'(x) + (h^3/9) f'''(x) + \dots \end{aligned} \right\} \quad (3.8)$$

This is an odd function of h and contains only the odd derivatives of f evaluated at x .

B. The Method of Symmetric Pairing

We assume the same integrand with the same properties as in the previous section. However, we start the integration at the singularity x and integrate simultaneously to the left and to the right; i.e., we "pair" the function values of the integrand for the arguments that are symmetric with respect to the singularity x . We obtain explicitly

$$\left. \begin{aligned} \int_a^b \frac{f(t)}{t - x} dt &= \lim_{\epsilon \rightarrow 0} \left[- \int_{\epsilon}^{x-a} \frac{f(x-t)}{t} dt + \int_{\epsilon}^{b-x} \frac{f(x+t)}{t} dt \right] \\ &= \int_0^M \frac{f(x+t) - f(x-t)}{t} dt + \sigma \int_m^M \frac{f(x+\sigma t)}{t} dt \\ &= \int_0^M \psi_x(t) dt, \end{aligned} \right\} \quad (3.9)$$

where

$$m = \min(x - a; b - x), \quad M = \max(x - a; b - x);$$

$$\sigma = \begin{cases} +1 & \text{if } x - a < b - x, \\ -1 & \text{if } x - a > b - x; \end{cases} \quad (3.10)$$

$$\psi_x(t) = \begin{cases} 2f'(x), & \text{for } t = 0; \\ [f(x+t) - f(x-t)]/t, & \text{for } 0 < t \leq m; \\ \sigma f(x+\sigma t)/t, & \text{for } m < t \leq M. \end{cases} \quad (3.11)$$

The contribution of the neighborhood of x to the integral (3.9) is given by

$$\begin{aligned} P \int_{x-h}^{x+h} \frac{f(y)}{y-x} dy &= \int_0^h \frac{f(x+t) - f(x-t)}{t} dt \\ &= \int_0^h 2 \left\{ f'(x) + \frac{1}{6} t^2 f'''(x) + \dots \right\} dt \\ &= 2hf'(x) + (h^3/9) f'''(x) + \dots, \end{aligned} \quad (3.12)$$

which is the same as in the previous case.

C. Discussion

To be sure, both methods are suitable for the numerical evaluation of the finite Hilbert transformation. However, we should like to point out the following advantages of the method of symmetric pairing in connection with its use on high-speed computers:

1. There is no need to call a logarithm subroutine to evaluate a second term. This saves time and an additional error evaluation.
2. The actual number of the operations $f(x_1) - f(x_2)$ is always less than in Method A and can be as few as one-half.
3. The integrand at the singularity x behaves like $c_1 + t^2 c_2$; i.e., we have an extremum.
4. In the infinite Hilbert transformation, the integrand approaches zero faster than in Method A for many practical applications. Only one cut-off procedure in the numerical integration routine will be necessary.

To illustrate these points, for $f(t) = (1+t^2)^{-1}$ and $x = 1$, Figure 1 shows the integrand of the Hilbert transformation, Figure 2 shows the integrand after removing the singularity, and Figure 3 shows the integrand for the method of symmetric pairing.

The numerical values for Figures 1, 2, and 3 are listed in Tables I, II, and III, respectively.

Fig. 1

Integrand of Hilbert
Transformation for
 $f(t) = 1/(1+t^2)$ and $x = 1$

Table I

NUMERICAL VALUES FOR FIGURE 1

$$f(t) = 1/(1+t^2) \text{ and } x = 1$$

t	f(t)	$\frac{f(t)}{t-x}$	t	f(t)	$\frac{f(t)}{t-x}$
-10	0.0099	-0.0009	1.0	0.5	$\pm\infty$
-5	0.0385	-0.0064	1.1	0.452	4.52
-4	0.0588	-0.0118	1.25	0.39	1.56
-3	0.1	-0.025	1.5	0.308	0.616
-2	0.2	-0.067	2	0.2	0.2
-1	0.5	-0.25	3	0.1	0.05
0	1.0	-1.0	4	0.0588	0.0196
0.5	0.8	-1.6	5	0.0385	0.0096
0.75	0.64	-2.56	10	0.0099	0.0011
0.9	0.554	-5.54

Fig. 2. Integrand of Hilbert Transformation
for $f(t) = 1/(1+t^2)$ and $x = 1$ after
Removal of Singularity (Method A)

Table II

NUMERICAL VALUES FOR FIGURE 2

(Removal of Singularity) $f(t) = 1/(1+t^2)$ and $x = 1$

t	$f(t)$	$f(x) - f(t)$	$\frac{f(t) - f(x)}{t - x}$
-10	0.0099	-0.4901	0.0455
-5	0.0385	-0.4615	0.0769
-4	0.0588	-0.4412	0.0882
-3	0.1	-0.4	0.1
-2	0.2	-0.3	0.1
-1	0.5	0	0
0	1.0	0.5	-0.5
0.5	0.8	0.3	-0.6
0.75	0.64	0.14	-0.56
0.9	0.554	0.054	-0.54
1.0	0.5	0	*
1.1	0.452	-0.048	-0.48
1.25	0.39	-0.11	-0.44
1.5	0.308	-0.182	-0.364
2	0.2	-0.3	-0.3
3	0.1	-0.4	-0.2
4	0.0588	-0.4412	-0.147
5	0.0385	-0.4615	-0.1154
10	0.0099	-0.4901	-0.0544

* $f(1) = 0.500$.

Fig. 3

Integrand of Hilbert Transformation
for $f(t) = 1/(1+t^2)$ and $x = 1$ after
Symmetric Pairing (Method B)

Table III

NUMERICAL VALUES FOR FIGURE 3

$$f(t) = 1/(1+t^2) \text{ and } x = 1$$

t	α $f(x+t)$	$\frac{f(x+t)}{t}$	β $f(x-t)$	$\frac{-f(x-t)}{t}$	$\frac{\alpha - \beta}{t}$
10	0.0082	0.0008	0.0122	-0.00122	-0.0004
5	0.027	0.0054	0.0588	-0.0118	-0.0064
4	0.0385	0.0096	0.1	-0.025	-0.0154
3	0.0588	0.0196	0.2	-0.067	-0.0471
2	0.1	0.05	0.5	-0.25	-0.2
1	0.2	0.2	1.0	-1.0	-0.8
0.5	0.308	0.616	0.8	-1.6	-0.984
0.25	0.39	1.56	0.64	-2.56	-1.00
0.10	0.453	4.53	0.553	-5.53	-1.00
0.01	0.495	49.5	0.505	-50.5	-1.00
0					*

$$* \lim_{t \rightarrow 0} \frac{f(x+t) - f(x-t)}{t} \Big|_{x=1} = 2f'(x) = -1.$$

D. Finite Hilbert Transformations of Polynomials

If

$$f(x) = \sum_{i=0}^N c_i x^i, \quad (3.13)$$

then

$$\begin{aligned} \frac{f(t) - f(x)}{t - x} &= \sum_{i=1}^N c_i \sum_{k=1}^i x^{k-1} t^{i-k} \\ &= \sum_{k=1}^N x^{k-1} \sum_{i=k}^N c_i t^{i-k} \\ &= \sum_{j=0}^{N-1} x^j \sum_{\ell=1}^{N-j} c_{j+\ell} t^{\ell-1}. \end{aligned} \quad (3.14)$$

Hence, using equation (3.4), we obtain

$$\int_a^b \frac{f(t)}{t - x} dt = \sum_{j=0}^{N-1} d_j x^j + f(x) \log \frac{b - x}{x - a}, \quad (3.15)$$

where

$$d_j = \sum_{k=1}^{N-j} c_{j+k} \frac{b^k - a^k}{k}. \quad (3.16)$$

In particular, if

$$f(x) = \sum_{n=0}^N c_n P_n(x), \quad (N \geq 1), \quad (3.17)$$

where the $P_n(x)$ are Legendre polynomials, and $b = -a = 1$, then

$$\int_{-1}^{+1} \frac{f(t)}{t - x} dt = f(x) \log \frac{1 - x}{1 + x} + 2 \sum_{n=1}^N c_n W_{n-1}(x), \quad (N \geq 1), \quad (3.18)$$

where

$$W_{n-1}(x) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t) - P_n(x)}{t - x} dx = \sum_{m=1}^n \frac{P_{m-1}(x) P_{n-m}(x)}{m} \quad (3.19)$$

are the polynomials that occur in connection with the Legendre functions of second kind, $Q_n(x)$. The first few values for $W_{n-1}(x)$ are*

$$\left. \begin{aligned} W_{-1} &= 0; \\ W_0 &= 1; \\ W_1 &= \frac{3}{2}x; \\ W_2 &= \frac{5}{2}x^2 - \frac{2}{3}; \\ W_3 &= \frac{35}{8}x^3 - \frac{55}{24}x; \\ W_4 &= \frac{63}{8}x^4 - \frac{49}{8}x^2 + \frac{8}{15}. \end{aligned} \right\} \quad (3.20)$$

These formulas are of practical use when $f(x)$ can be approximated by a low-order polynomial.

* $W_n = \frac{2n+1}{n+1} x W_{n-1} - \frac{n}{n+1} W_{n-2}$

IV. NUMERICAL PROCEDURE FOR THE METHOD OF SYMMETRIC PAIRING

A. Linear Endpoint Approximation

In practical applications, the Hilbert transformation is generally needed for many values of x in (2.3). We divide (a,b) into N equal intervals

$$h = \frac{b - a}{N}, \quad (4.1)$$

compute the

$$f_k = f(a + kh), \quad k = 0, 1, \dots, N, \quad (4.2)$$

and store them in the fast-access memory of the computer, where they can be used over and over again. Hence, it is for practical reasons that we do not consider Gaussian-type integration.

To compute $\psi_x(0)$, we recall that the derivatives of $f(x)$ can be represented as

$$f'(x) = [f(x+h) - f(x-h)]/(2h) - (h^2/6) f'''(x + \theta_1 h), \quad (4.3)$$

or

$$f'(x) = (1/12h) \{8[f(x+h) - f(x-h)] - [f(x+2h) - f(x-2h)]\} + (h^4/30) f^{(5)}(x + \theta_2 h). \quad (4.4)$$

Here, and in what follows,

$$-1 \leq \theta_i \leq 1, \quad i = 1, 2, \dots \quad (4.5)$$

From the definition (3.11) of $\psi_x(t)$ we have, by (4.3),

$$\psi_x(0) = \psi_x(h) - (h^2/3) f'''(x + \theta_1 h), \quad (4.6)$$

or, by (4.4),

$$\psi_x(0) = (1/3) \{4\psi_x(h) - \psi_x(2h)\} + (h^4/15) f^{(5)}(x + \theta_2 h). \quad (4.7)$$

First we compute (3.9) by the trapezoidal rule. (2,9) The contribution of the first interval is

$$\begin{aligned}
\int_0^h \psi_x(t) dt &= \frac{h}{2} [\psi_x(0) + \psi_x(h)] - \frac{h^3}{12} \psi''(\theta_3 h) \\
&= h \psi_x(h) - \frac{h}{2} \cdot \frac{h^2}{3} f'''(x + \theta_1 h) - \frac{h^3}{12} \psi''(\theta_3 h). \quad (4.8)
\end{aligned}$$

The estimation of the second error term is rather cumbersome. On the other hand, from (3.12), (3.11), and (4.6),

$$\begin{aligned}
\int_0^h \psi_x(t) dt &= h \psi_x(0) + (h^3/9) f'''(x + \theta_4 h) \\
&= h \psi_x(h) - (h^3/3) f'''(x + \theta_1 h) + (h^3/9) f'''(x + \theta_4 h) \\
&= h \psi_x(h) - (2/9) h^3 f'''(x + \theta_5 h). \quad (4.9)
\end{aligned}$$

Comparing (4.8) and (4.9), we see that the truncation error of the trapezoidal rule in the first interval $(0, h)$ is about $(2/9) h^3 f'''(x + \theta_5 h)$.

The contribution of the interval (h, m) is

$$\int_h^m \psi_x(t) dt = h \left\{ \frac{1}{2} \psi_x(h) + \sum_{\ell=1}^{\frac{m}{h}-1} \psi_x(\ell h) + \frac{1}{2} \frac{f(x+m) - f(x-h)}{m} \right\}, \quad (4.10)$$

and the contribution of the interval (m, M) is

$$\int_m^M \psi_x(t) dt = \sigma h \left\{ \frac{1}{2} \frac{f(x+\sigma m)}{m} + \sum_{\ell=\frac{m}{h}+1}^{\frac{M}{h}-1} \frac{f(x+\sigma \ell h)}{\ell} + \frac{1}{2} \frac{f(x+\sigma M)}{M} \right\} \quad (4.11)$$

if

$$x = a + kh, \quad (0 < k < n).$$

Combining these three parts, we obtain, for

$$x = a + kh, \quad (0 < k < N),$$

$$\int_a^b \frac{f(t)}{t-x} dt = \int_0^M \psi_x(t) dt \doteq$$

$$\frac{3}{2} (f_{k+1} - f_{k-1}) + \sum_{\ell=2}^{\min(k, N-k)-1} \frac{f_{k+\ell} - f_{k-\ell}}{\ell} + \begin{cases} \frac{f_{2k} - \frac{1}{2}f_0}{k} + \sum_{\ell=k+1}^{N-k-1} \frac{f_{k+\ell}}{\ell} + \frac{1}{2} \frac{f_n}{N-k}, & (0 < 2k < N); \\ \frac{f_N - f_0}{N}, & (2k = N); \\ \frac{1}{2} \frac{f_N - f_{N-2k}}{N-k} - \sum_{\ell=k+1}^{k-1} \frac{f_{k-\ell}}{\ell} - \frac{1}{2} \frac{f_0}{k}, & (2n > 2k > N). \end{cases} \quad (4.12)$$

If $x = a$ (or $x = b$), we know from (3.4) that $f(a)$ [or $f(b)$] must be zero in order that (3.9) exists. Hence, assume we have, asymptotically, for small t ,

$$f(t) = c_1(t-a)^\alpha + c_2(t-a)^{\alpha+1} + \dots, \quad (4.13)$$

i.e.,

$$f(a+t) = c_1 t^\alpha + c_2 t^{\alpha+1} + \dots$$

Then the contribution of the first interval will be

$$\begin{aligned} \int_a^{a+h} \frac{f(t)}{t-a} dt &= \int_0^h \frac{f(a+t)}{t} dt = \int_0^h (c_1 t^{\alpha-1} + c_2 t^\alpha + \dots) dt \\ &= \begin{cases} (c_1/\alpha) h^\alpha + [c_2/(\alpha+1)] h^{\alpha+1} + \dots, & \text{for } \alpha > 0; \\ \infty, & \text{for } \alpha \leq 0. \end{cases} \end{aligned} \quad (4.14)$$

Hence, if we can calculate the transform at its end point a , we have

$$\int_a^b \frac{f(t)}{t-a} dt \doteq \frac{c_1}{\alpha} h^\alpha + \frac{c_2}{\alpha+1} h^{\alpha+1} + \dots + \frac{1}{2} f_1 + \sum_{\ell=2}^{n-1} \frac{f_\ell}{\ell} + \frac{1}{2} \frac{f_n}{n}, \quad \text{for } \alpha > 0. \quad (4.15)$$

Similarly, if we can calculate the transform at b , we have

$$f(b-t) = b_1 t^\alpha + b_2 t^{\alpha+1} + \dots, \quad (\alpha > 0) \quad (4.16)$$

and

$$\left. \begin{aligned} \int_a^b \frac{f(t)}{t-b} dt &= - \left\{ \int_0^h \frac{f(b-t)}{t} dt + \int_h^{b-a} \frac{f(b-t)}{t} dt \right\} \\ &\doteq - \left\{ (b_1/\alpha) h^\alpha + [b_2/(\alpha+1)] h^{\alpha+1} + \dots + \frac{1}{2} f_{n-1} + \sum_{\ell=1}^{n-1} \frac{f_{n-\ell}}{\ell} + \frac{1}{2} \frac{f_0}{n} \right\}, \quad \text{for } \alpha > 0. \end{aligned} \right\} \quad (4.17)$$

B. Midpoint Approximation

We obtain an even simpler and more accurate approximation when we replace the trapezoidal rule by the midpoint formula.

Assume the singularity to be at $x_{k+1/2} = a + (k+1/2)h$. Then

$$J_k = P \int_a^b \frac{f(t)}{t - x_{k+1/2}} dt = \int_0^m \frac{f(x_{k+1/2} + t) - f(x_{k+1/2} - t)}{t} dt + \sigma \int_m^M \frac{f(x_{k+1/2} + \sigma t)}{t} dt \quad (4.18)$$

is approximated by

$$J_k \doteq \begin{cases} F_0 + \sum_{\ell=1}^{N-1} \frac{f_{1/2+\ell}}{\ell}, & \text{for } k = 0; \\ \frac{3}{2} (f_{k+3/2} - f_{k-1/2}) + \sum_{\ell=2}^k \frac{f_{k+1/2+\ell} - f_{k+1/2-\ell}}{\ell} + \sum_{\ell=k+1}^{N-(k+1)} \frac{f_{k+1/2+\ell}}{\ell}, & \text{for } 1 < k+1/2 < N/2; \\ \frac{3}{2} (f_{k+3/2} - f_{k-1/2}) + \sum_{\ell=2}^{N/2} \frac{f_{k+1/2+\ell} - f_{k+1/2-\ell}}{\ell} + 0, & \text{for } k+1/2 = N/2; \\ \frac{3}{2} (f_{k+3/2} - f_{k-1/2}) + \sum_{\ell=2}^{N-(k+1)} \frac{f_{k+1/2+\ell} - f_{k+1/2-\ell}}{\ell} - \sum_{\ell=N-k}^k \frac{f_{k+1/2-\ell}}{\ell}, & \text{for } N/2 < k+1/2 < N; \\ F_N - \sum_{\ell=1}^{N-1} \frac{f_{N-1/2-\ell}}{\ell}, & \text{for } k = N-1; \end{cases} \quad (4.19)$$

where, from (3.12),

$$\left. \begin{aligned} F_0 &= P \int_{-h/2}^{+h/2} \frac{f(a+h/2+t)}{t} dt \doteq hf'_{1/2} + O(h^3); \\ F_N &= P \int_{-h/2}^{+h/2} \frac{f(b-h/2+t)}{t} dt \doteq hf'_{N-1/2} + O(h^3). \end{aligned} \right\} \quad (4.20)$$

If we know the value of $f(t)$ at a , i.e., $f(a) = f_0$, then

$$F_0 \doteq hf'_{1/2} = (1/3)(f_{3/2} + 3f_{1/2} - 4f_0) + O(h^3). \quad (4.21)$$

Often we know that $f_0 = 0$. Similarly,

$$F_N \doteq hf'_{N-1/2} = (1/3)(4f_N - 3f_{N-1/2} - f_{N-3/2}) + O(h^3). \quad (4.22)$$

If $f(t)$ behaves like the function $f(t)$ in (4.13) or (4.16), then the following integral will be helpful to estimate F_0 or F_N :

$$P \int_0^h \frac{t^\alpha}{t - h/2} dt = 2^{(1-\alpha)} h^\alpha \left\{ \alpha + \frac{1}{3} \binom{\alpha}{3} + \frac{1}{5} \binom{\alpha}{5} + \dots \right\}$$

$$= \begin{cases} 0, & \text{for } \alpha = 0; \\ h, & \text{for } \alpha = 1; \\ h^2, & \text{for } \alpha = 2. \end{cases} \quad (4.23)$$

The term in $\{ \}$ is always positive for $\alpha > 0$, and one can show that

$$\alpha < \{ \} < 1, \quad \text{for } 0 < \alpha < 1. \quad (4.24)$$

The first terms in the remaining formulas of (4.19) are obtained from

$$\left(\int_0^{h/2} + \int_{h/2}^{3h/2} \right) \left(\frac{f(x_{k+1/2} + t) - f(x_{k+1/2} - t)}{t} dt \right) =$$

$$hf'(x_{k+1/2}) + (h^3/72) f'''(x_{k+1/2}) + \dots$$

$$+ h \frac{f_{k+1/2+1} - f_{k+1/2-1}}{h} + (h^3/24) \psi''_{x_{k+1/2}}(h) + \dots \quad (4.25)$$

Approximating $f'(x_{k+1/2})$ by $(1/2h)(f_{k+1/2+1} - f_{k+1/2-1}) + O(h^2)$, and combining these terms yields $(3/2)(f_{k+3/2} - f_{k-1/2})$. Q.E.D.

We give the matrix representation of (4.19). Define the column vectors with the N elements $f_{k+1/2}$ and J_k ($k = 0, 1, \dots, N-1$) by f and J , respectively. Define further the $N \times N$ matrix A_N by

$$A_N = \begin{bmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N-1} \\ -\frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N-2} \\ -\frac{1}{2} & -\frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2} & \cdots & \frac{1}{N-3} \\ -\frac{1}{3} & -\frac{1}{2} & -\frac{3}{2} & 0 & \frac{3}{2} & \cdots & \frac{1}{N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N-3} & \cdots & -\frac{1}{2} & -\frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{N-2} & \cdots & -\frac{1}{3} & -\frac{1}{2} & -\frac{3}{2} & 0 & \frac{3}{2} \\ -\frac{1}{N-1} & \cdots & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 \end{bmatrix} \quad (4.26)$$

Then, (4.19) has the representation, for $F_0 = F_N = 0$,

$$J = Af. \quad (4.27)$$

If we neglect the first part of the integral in (4.25), viz.,

$$\int_0^{h/2} \frac{f(x_{k+1/2} + t) - f(x_{k+1/2} - t)}{t} dt = 0, \quad (4.28)$$

then the matrix A_N becomes the skew-symmetric matrix B_N ,

$$B_N = \begin{bmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N-1} \\ -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N-2} \\ -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N-3} \\ -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \cdots & \frac{1}{N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N-1} & -\frac{1}{N-2} & -\frac{1}{N-3} & -\frac{1}{N-4} & -\frac{1}{N-5} & -\frac{1}{N-6} & \cdots & 0 \end{bmatrix} \quad (4.29)$$

If the integration limits in (4.18) are at infinity, (4.19) takes the following simple form for a singularity at x :

$$P \int_{-\infty}^{+\infty} \frac{f(t)}{t-x} dt = \frac{3}{2} [f(x+h) - f(x-h)] + \sum_{\ell=2}^N \frac{f(x+h\ell) - f(x-h\ell)}{\ell} + R(x), \quad (4.30)$$

where N is chosen so large that the remainder $R(x)$ can be neglected.

We should like to point out that (4.30), in spite of its great simplicity, is not trivial. Take, for example, the integral

$$\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{dt}{x-t} = 0. \quad (4.31)$$

This result is indeed obtained by the use of (4.30). If, however, we evaluate (4.31) by the midpoint formula without the method of symmetric pairing, we obtain the series representation for $\cot \xi$ ($\xi = x/h$) as is known in analysis,

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{x-t} = \frac{h}{\pi} \sum_{\ell=-\infty}^{+\infty} \frac{1}{x-\ell h} = \frac{1}{\pi} \sum_{\ell=-\infty}^{+\infty} \frac{1}{\xi-\ell} = \cot \pi \xi. \quad (4.32)$$

C. Higher-order Approximations

To obtain formulas of higher accuracy, one proceeds similarly. If the singular point x_k is an interior point such that $2 \leq k \leq N-2$, we may use (4.7) to calculate $\psi_x(0)$, and then apply Simpson's rule for the integration. For the singularities at x_1 and x_{N-1} , and x_0 and x_N , one may use, respectively,

$$h\psi_1(0) = \frac{1}{6} \{-3f_0 - 10f_1 + 18f_2 - 6f_3 + f_4\} - \frac{h^4}{20} f^{(v)}(h+\theta\xi);$$

$$h\psi_{N-1}(0) = \frac{1}{6} \{-f_{N-4} + 6f_{N-3} - 18f_{N-2} + 10f_{N-1} + 3f_N\} - \frac{h^4}{20} f^{(v)}(h+\theta\xi);$$

$$h\psi_0(0) = \frac{1}{12} \{-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4\} + \frac{h^4}{5} f^{(v)}(h+\theta\xi);$$

$$h\psi_N(0) = \frac{1}{12} \{3f_{N-4} - 16f_{N-3} + 36f_{N-2} - 48f_{N-1} + 25f_N\} + \frac{h^4}{5} f^{(v)}(h+\theta\xi).$$

One inserts these values in the numerical integration formulas as before. For $0 < k < n$, there will be a discontinuity at $\min(k, n - k)$. Hence, each of these intervals must be treated separately. Furthermore, special formulas (say higher-order Newton-Cotes formulas) must be applied near the end of each interval to adjust for the actual number of mesh points. For these reasons, and the possibility of numerical instability, higher-order approximations are less desirable.

V. EXAMPLES OF HILBERT TRANSFORMS

A. Physical Interpretation in Network Theory

The intimate connection provided by the Hilbert transform between the real and imaginary parts of an analytic function is of physical interest. In the electrostatic case, for example, the Hilbert transform yields a connection between the electrostatic potential and the total charge. In vibration theory, the Hilbert transform yields a relation between the real and imaginary parts of the impedance, that is, a relation between the resistive and reactive parts. The complex variable in this case is the complex frequency. The physical situations introduced in this section will be from vibration theory and, in particular, from electric network theory.

Consider a positive, real, minimum-reactance, driving-point impedance $Z(p)$.⁽²⁰⁾ In general, this impedance is a complex-valued function of the real frequency variable ω ; that is,

$$Z(\omega) = R(\omega) + iX(\omega). \quad (5.1)$$

If $Z(\omega)$ is analytic in the upper complex half-plane, then we have, in accordance with Theorem 3,

$$R(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{X(\xi)}{\xi - \omega} d\xi; \quad (5.2)$$

$$X(\omega) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{R(\xi)}{\xi - \omega} d\xi.$$

If, on the other hand, as is often the case in practical applications, $Z(\omega)$ is only analytic in the lower half of the complex plane, then one can show easily that

$$\left. \begin{aligned} R(\omega) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\xi)}{\xi - \omega} d\xi; \\ X(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R(\xi)}{\xi - \omega} d\xi. \end{aligned} \right\} \quad (5.3)$$

These equations state that with a knowledge of the reactance X at all frequencies, the resistance R may be determined at all frequencies, and that with a knowledge of the resistance R at all frequencies, the reactance X may be determined as a function of frequency.

The reciprocal relations (2.3) and (2.4) show that the resistance and reactance of an impedance of a network are Hilbert transforms or conjugate integrals of each other.

An important relation is that between the modulus and the phase of an impedance function.⁽¹⁰⁾ To establish this, consider the logarithm of the impedance. That is,

$$\ln z(\omega) = (1/2) \ln[R^2(\omega) + X^2(\omega)] + i \tan^{-1} (X/R). \quad (5.4)$$

Then, the logarithm of the modulus of an impedance function and its phase are Hilbert transforms of each other.

If we let

$$\left. \begin{aligned} \alpha(\omega) &= \ln|z(\omega)| = (1/2) \ln(R^2(\omega) + X^2(\omega)), \\ \beta(\omega) &= \tan^{-1} [X(\omega)/R(\omega)], \end{aligned} \right\} \quad (5.5)$$

and use equations (2.3) and (2.4), the relation between the logarithm of the modulus and the phase is the Hilbert transform pair,

$$\alpha(\omega) = \begin{pmatrix} + \\ - \end{pmatrix} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)}{\xi - \omega} d\xi, \quad (5.6)$$

and

$$\beta(\omega) = \begin{pmatrix} - \\ + \end{pmatrix} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\xi)}{\xi - \omega} d\xi, \quad (5.7)$$

where the upper (lower) sign is used if $\ln z(\omega)$ is analytic in the upper (lower) half of the complex plane.

Similar to the Parseval theorem for Fourier transforms, there is the Parseval theorem for Hilbert transforms (Theorem 2), which, when applied to networks, states that if two impedance functions are

$$Z_1(\omega) = R_1(\omega) + i X_1(\omega), \quad (5.8)$$

and

$$Z_2(\omega) = R_2(\omega) + i X_2(\omega), \quad (5.9)$$

then

$$\int_{-\infty}^{\infty} R_1(\omega) R_2(\omega) d\omega = \int_{-\infty}^{\infty} X_1(\omega) X_2(\omega) d\omega. \quad (5.10)$$

If $R_1(\omega) = R_2(\omega)$ and $X_1(\omega) = X_2(\omega)$, equation (5.10) becomes

$$\int_{-\infty}^{\infty} R^2(\omega) d\omega = \int_{-\infty}^{\infty} X^2(\omega) d\omega. \quad (5.11)$$

B. Numerical Examples

A few specific simple examples for which the analytic solutions are known will illustrate the type of problem for which the numerical approach, in contrast with the analytic procedure, is appropriate. Primarily, the numerical approach is appropriate when the given function is specified graphically, or by some function, or by several of these representations pieced together over confluent frequency (ω) intervals.

1. The first example to be discussed is shown in Figure 4. The given function $R(\omega)$ is a rectangular pulse centered at the origin, that is,

$$R(\omega) = \begin{cases} 1, & |\omega| < 1; \\ 0, & |\omega| \geq 1. \end{cases} \quad (5.20)$$

Then, by (5.2),

$$X(\omega) = -\frac{1}{\pi} P \int_{-1}^1 \frac{d\xi}{\xi - \omega}, \quad (5.21)$$

Fig. 4

A Typical Real Part for Which the Determination of the Conjugate Mate Is Accomplished by Means of the Hilbert Transform (5.2) as Shown in Equations (5.21) and (5.22)

which yields

$$X(\omega) = \frac{1}{\pi} \ln \left| \frac{\omega + 1}{\omega - 1} \right|. \quad (5.22)$$

The absolute-value signs are needed here to preserve the real character of $X(\omega)$.

The lower half of Figure 4 shows that a plot of this resulting $X(\omega)$ function exhibits logarithmic singularities at the discontinuities of $R(\omega)$ and has a slope at $\omega = 0$ inversely proportional to the width of the pulselike $R(\omega)$ function.

The Hilbert transform (5.21) was evaluated numerically on the computer by the midpoint quadrature with the interval $(-1,1)$ divided into 100 parts. The analytic values (5.22) and the computer values of (5.21) as a function of the frequency are compared in Table IV.

Table IV

NUMERICAL EVALUATION OF (5.21) BY MIDPOINT
FORMULA AND COMPARISON WITH VALUE OF
ANALYTIC EXPRESSION ($h = 2/100$)

ω	Calculated by (5.21)	Analytic Values (5.22)
0	0	0
0.1	0.06388	0.06387
0.2	0.12906	0.12906
0.3	0.19705	0.19705
0.4	0.26970	0.26969
0.5	0.34970	0.34968
0.6	0.44127	0.44124
0.7	0.55214	0.55208
0.8	0.69940	0.69926
1.25	0.69940	0.69931
1.50	0.51230	0.51228
1.75	0.41357	0.41357
2	0.34970	0.34969
3	0.22064	0.22063
4	0.16260	0.16260
6	0.10710	0.10710
10	0.06388	0.06388
25	0.02548	0.02548

2. To illustrate the application of Hilbert transforms to electric network theory, consider the positive, real, minimum-reactance, driving-point impedance function of the network of Figure 5. At real frequencies, the impedance function is

$$Z(i\omega) = 1/(1+i\omega) = (1-i\omega)/(1+\omega^2), \quad (5.23)$$

which has a singularity at $\omega = i$, but is analytic in the lower half of the complex plane. Hence, we must use (5.3).

Fig. 5. Example of Simple Network

From equation (5.23), the resistance function is

$$R(\omega) = 1/(1 + \omega^2). \quad (5.24)$$

The reactance, as determined from $R(\omega)$ by (5.3), is

$$x(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{1}{1 + \xi^2} \frac{d\xi}{\xi - \omega}, \quad (5.25)$$

which is

$$x(\omega) = -\omega/(1 + \omega^2), \quad (5.26)$$

as is given by equations (5.1) and (5.23).

Equation (5.25) was evaluated numerically for 12 values of ω by (4.30) using $h = 1$, $1/2$, and $1/10$, but such that $Nh = 100$, and compared with the numerical values of (5.26). The results are presented in Table V. It can be seen that $h = 1/2$ gives results that are accurate to from three to five decimal places. Because the integration is cut off at $\omega \pm 100$, not more than an accuracy of 10^{-4} can be expected.

The influence of the cut-off point in the integration can be seen from Table VI, where $h = 1/2$, and $Nh = 500$. All calculations for Tables V and VI took only a small fraction of a second on the CDC-3600 computer.

Table V

EVALUATION OF (5.25) BY (4.30) FOR $Nh = 100$

ω	$\frac{-i\omega}{1+\omega^2}$ Analytic Value	$h = 1, N = 100$		$h = 1/2, N = 200$		$h = 1/10, N = 1000$	
		Computed Value	Error	Computed Value	Error	Computed Value	Error
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	-0.50000	-0.47004	0.02996	-0.49878	0.00122	-0.49999	0.00000
5	-0.19231	-0.19338	-0.00106	-0.19235	-0.00004	-0.19231	0.00000
10	-0.09901	-0.09939	-0.00037	-0.09901	0.00000	-0.09901	0.00000
15	-0.06637	-0.06662	-0.00024	-0.06636	0.00001	-0.06637	0.00000
20	-0.04988	-0.05005	-0.00017	-0.04987	0.00001	-0.04987	0.00000
25	-0.03994	-0.04007	-0.00013	-0.03993	0.00001	-0.03992	0.00001
30	-0.03330	-0.03341	-0.00011	-0.03328	0.00001	-0.03328	0.00001
35	-0.02855	-0.02864	-0.00008	-0.02853	0.00002	-0.02853	0.00001
40	-0.02498	-0.02506	-0.00007	-0.02496	0.00002	-0.02496	0.00002
45	-0.02221	-0.02227	-0.00005	-0.02218	0.00002	-0.02219	0.00002
50	-0.01999	-0.02004	-0.00004	-0.01996	0.00003	-0.01996	0.00003

Table VI

EVALUATION OF (5.25) BY (4.30) FOR $h = 1/2, N = 1000$ ($hN = 500$)

ω	Analytic Value	Computed Value	Error
0	0.00000000	0.00000000	0.00000000
1	-0.50000000	-0.49877922	0.00122078
5	-0.19230769	-0.19235134	-0.00004365
10	-0.09900990	-0.09901208	-0.00000218
15	-0.06637168	-0.06637230	-0.00000062
20	-0.04987531	-0.04987564	-0.00000033
25	-0.03993610	-0.03993631	-0.00000021
30	-0.03329634	-0.03329647	-0.00000014
35	-0.02854812	-0.02854821	-0.00000008
40	-0.02498438	-0.02498442	-0.00000004
45	-0.02221125	-0.02221126	-0.00000000
50	-0.01999200	-0.01999197	0.00000003

3. From equation (5.23), the reactance function of the network of Figure 5 is

$$X(\omega) = -\omega/(1+\omega^2). \quad (5.27)$$

The resistance, as determined from $X(\omega)$ by (5.3) is then

$$R(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{-\xi}{1 + \xi^2} \frac{d\xi}{\xi - \omega}, \quad (5.28)$$

which is

$$R(\omega) = 1/(1 + \omega^2), \quad (5.29)$$

as is given by equation (5.23).

As a final consideration of examples 2 and 3, substitute equations (5.24) and (5.27) into Parseval's equation. Thus,

$$\int_{-\infty}^{\infty} R^2(\omega) d\omega = \int_{-\infty}^{\infty} \frac{d\omega}{(1 + \omega^2)^2} = \int_{-\pi/2}^{\pi/2} \cos^2 x dx = \frac{\pi}{2},$$

and

$$\int_{-\infty}^{\infty} X^2(\omega) d\omega = \int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{(1 + \omega^2)^2} = \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \frac{\pi}{2},$$

which shows that Parseval's theorem is satisfied.

4. Given the function

$$\alpha(t) = \sin t,$$

its Hilbert transform

$$\beta(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \xi}{\xi - t} d\xi = \cos t \quad (5.30)$$

is evaluated numerically by (4.30). A comparison of the analytical and numerical values, as a function of ω , is given in Table VII.

Table VII
NUMERICAL EVALUATIONS OF (5.30) BY (4.30)

t	cos t	h = 1, N = 100		h = 1/2, N = 200		h = 1/10, N = 1000	
		Computed Value	Error	Computed Value	Error	Computed Value	Error
0	1.000	0.943	-0.057	0.987	-0.013	0.994	-0.006
1	0.540	0.509	-0.031	0.533	-0.007	0.537	-0.003
5	0.284	0.267	-0.016	0.280	-0.004	0.282	-0.001
10	-0.839	-0.791	0.048	-0.828	0.011	-0.834	0.005
15	-0.760	-0.716	0.043	-0.750	0.010	-0.755	0.004
20	0.408	0.385	-0.023	0.403	-0.005	0.406	-0.002
25	0.991	0.935	-0.057	0.979	-0.012	0.986	-0.005
30	0.154	0.145	-0.009	0.152	-0.001	0.153	-0.001
35	-0.904	-0.852	0.052	-0.892	0.011	-0.899	0.005
40	-0.667	-0.629	0.038	-0.658	0.008	-0.663	0.004
45	0.525	0.495	-0.030	0.519	-0.006	0.522	-0.003
50	0.965	0.910	-0.055	0.953	-0.012	0.959	-0.005

5. Finally, given the function

$$\beta(t) = -\cos t,$$

its Hilbert transform is

$$\alpha(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-\cos \xi}{\xi - t} d\xi = \sin t.$$

C. The Krönig-Kramers Transforms

In most physically attainable cases, the nature of the impedance is such that if we reverse the sign of ω in (5.1), the resistance does not change but the reactance changes sign; i.e.,

$$\left. \begin{aligned} Z(-\omega) &= \bar{Z}(\omega); \\ R(-\omega) &= R(\omega); \\ X(-\omega) &= -X(\omega). \end{aligned} \right\} \quad (5.31)$$

Under this assumption, (5.3) can be written as

$$\left. \begin{aligned} R(\omega) &= -\frac{2}{\pi} \int_0^{\infty} \frac{\xi X(\xi)}{\xi^2 - \omega^2} d\xi; \\ X(\omega) &= +\frac{2}{\pi} \omega \int_0^{\infty} \frac{R(\xi)}{\xi^2 - \omega^2} d\xi. \end{aligned} \right\} \quad (5.32)$$

These equations are known in physics as the Krönig-Kramers transforms [MacDonald and Brachman,⁽¹¹⁾ p. 415]. They are quite general and, under the conditions discussed earlier in this paper, enable us to find the real part of the response of a system if we know the imaginary part as a function of frequency, and vice versa. Besides the electrical impedance Z , which has been used as an example in this section (and is defined as the ratio of voltage to current at a given point in a circuit), it can be applied to a mechanical impedance (complex ratio between force and velocity), wave impedance, and other response functions such as the electric conductivity σ , the magnetic susceptibility χ , and the dielectric constant ϵ . Often (5.1) is written for convenience in the form

$$Z(\omega) = R(\omega) - iX(\omega). \quad (5.33)$$

Then (5.32) is equivalent to (5.2). We discuss as an example the susceptibility

$$M = \chi H,$$

where H is the magnetic field and M the magnetization. Let us make the following assumptions: (7)

- (a) The system is linear, so that $M_1 + M_2 = \chi(H_1 + H_2)$.
- (b) The composition of the system does not change with time. If $M(t) = \chi H(t)$, then $M(t - t_0) = \chi H(t - t_0)$.
- (c) The system obeys the causality principle; that is, the effect must not precede the cause. If $H(t) = 0$ for $t < t_0$, then $M(t) = 0$ for $t < t_0$.
- (d) $\chi = \chi' - i\chi''$.

Then the Krönig-Kramers transforms are

$$\chi'(\omega) - \chi'(\infty) = \frac{2}{\pi} P \int_0^{\infty} \frac{u \chi''(u)}{u^2 - \omega^2} du;$$

$$\chi''(\omega) = -\frac{2\omega}{\pi} \int_0^{\infty} \frac{\chi'(u) - \chi'(\infty)}{u^2 - \omega^2} du.$$

In plasma physics, the simplest type of a so-called dispersion relation is given by the Kramers-Krönig relation [Balescu, (1) p. 98]. Let $\epsilon_+(z)$ be the plus-dielectric constant with complex frequency z such that

$$\epsilon_+(\omega) = \epsilon_1(\omega) - i\epsilon_2(\omega).$$

Then,

$$\epsilon_1(\omega) - 1 = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\epsilon_2(\nu)}{\nu - \omega} d\nu \quad (\omega \text{ real}).$$

For all of these examples, the method of numerical evaluation outlined in the previous sections is recommended.

VI. AN APPLICATION IN TRANSPORT THEORY

We shall treat Milne's problem here because it is well-known in radiative transfer and neutron transport theory. Given a homogeneous source-free medium occupying the half-space $x > 0$, the half-space $x < 0$ being a vacuum, and $x = 0$ being a free surface. The photon or neutron supply is at infinity. Following Galloni and Ghilardotti,⁽⁵⁾ we obtain for the flux distribution ψ at $x = 0$ the following singular integral equation:

$$\left[1 - \frac{c\mu}{2} \ln \frac{1+\mu}{1-\mu}\right] \mu\psi(\mu) = \frac{c\mu}{2} P \int_{-1}^0 \frac{\nu\psi(\nu)d\nu}{\nu - \mu}, \quad (\mu < 0). \quad (6.1)$$

If the singularity under the integral sign is removed in this equation, we obtain

$$\psi(\mu) = \frac{c}{2} \left\{ -\mu\psi(\mu) \ln \left(1 - \frac{1}{\mu}\right) + \int_{-1}^0 \frac{\nu\psi(\nu) - \mu\psi(\mu)}{\nu - \mu} \right\}, \quad (\mu < 0), \quad (6.2)$$

which corresponds to equation (11), page 132 of Ref. 3. The analytic solution of these equations was obtained by Wiener and Hopf and is given in Ref. 3 by equations (12) and (13), page 133. After appropriate substitutions and normalization, we have

$$\int_{-1}^0 \mu\psi(\mu) d\mu = 1,$$

$$\psi(\mu) = \frac{c}{2} \sqrt{\frac{\kappa_0^2}{1-c}} \frac{1}{1-\kappa_0^2\mu} \exp \left\{ \frac{c}{2} \int_0^1 \frac{\log \left(1 - \frac{\mu}{\nu}\right)}{(1 - c\nu \tanh^{-1}\nu)^2 + \left(\frac{\pi}{2}c\nu\right)^2} \left[1 + \frac{c\nu^2}{1-\nu^2}\right] d\nu \right\},$$

$$(\mu < 0), \quad (6.3)$$

where $\pm\kappa_0$ are the solutions of

$$\frac{c}{\kappa_0} \tanh^{-1}\kappa_0 = 1. \quad (6.4)$$

We show that (6.1) can be solved directly by using the midpoint formulas. Multiply (6.1) by $1/\mu$, and rearrange the terms so that

$$\psi(\mu) = \frac{c}{2} \left\{ \mu\psi(\mu) \ln \frac{1+\mu}{1-\mu} + P \int_{-1}^0 \frac{\nu\psi(\nu)d\nu}{\nu - \mu} \right\}, \quad (\mu < 0).$$

Let

$$\left. \begin{aligned} \mu_k &= \frac{k + 1/2}{N} - 1; \\ \psi_k &\doteq \psi(\mu_k). \end{aligned} \right\} \quad (k = 0, 1, \dots, N-1) \quad (6.6)$$

If we use (4.19) to evaluate the integral, the discretization of (6.5) takes the form, for N even and $F_0 = F_N = 0$,

$$\psi_k = \frac{c}{2} \left\{ \mu_k \ln \frac{1 + \mu_k}{1 - \mu_k} \psi_k + J_k(\psi) \right\}, \quad (k = 0, 1, \dots, N-1); \quad (6.7)$$

$$J_k = \left\{ \begin{aligned} &\sum_{\ell=1}^{N-1} \frac{\mu_\ell \psi_\ell}{\ell}, \quad \text{for } k = 0; \\ &\frac{3}{2} (\mu_{k+1} \psi_{k+1} - \mu_{k-1} \psi_{k-1}) + \sum_{\ell=2}^k \frac{\mu_{k+\ell} \psi_{k+\ell} - \mu_{k-\ell} \psi_{k-\ell}}{\ell} + \sum_{\ell=k+1}^{N-k-1} \frac{\mu_{k+\ell} \psi_{k+\ell}}{\ell}, \\ &\hspace{15em} \text{for } 1 \leq k \leq N/2; \\ &\frac{3}{2} (\mu_{k+1} \psi_{k+1} - \mu_{k-1} \psi_{k-1}) + \sum_{\ell=2}^{N-k-1} \frac{\mu_{k+\ell} \psi_{k+\ell} - \mu_{k-\ell} \psi_{k-\ell}}{\ell} - \sum_{\ell=N-k}^k \frac{\mu_{k-\ell} \psi_{k-\ell}}{\ell}, \\ &\hspace{15em} \text{for } \frac{N}{2} < k \leq N-2; \\ &-\sum_{\ell=1}^{N-1} \frac{\mu_{N-\ell-1} \psi_{N-\ell-1}}{\ell}, \quad \text{for } k = N-1. \end{aligned} \right. \quad (6.8)$$

We give the matrix representation of (6.7). Let

$$D = \begin{bmatrix} \mu_0 & & & 0 \\ & \mu_1 & & \\ & & \ddots & \\ 0 & & & \mu_{N-1} \end{bmatrix} \quad (6.9)$$

and

$$L = \begin{bmatrix} \ell n \frac{1 + \mu_0}{1 - \mu_0} & & 0 \\ & \ell n \frac{1 + \mu_1}{1 - \mu_1} & \\ 0 & & \ell n \frac{1 + \mu_{N-1}}{1 - \mu_{N-1}} \end{bmatrix} \quad (6.10)$$

be diagonal matrices, and

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{N-1} \end{bmatrix} \quad (6.11)$$

be a column vector. Then (6.7) is given by

$$\psi = (c/2)(A + L)D\psi, \quad (6.12)$$

where A is defined by (4.26), letting $A_N = A$. Furthermore, let

$$M = (A + L)D, \quad (6.13)$$

and

$$\lambda = 2/c; \quad (6.14)$$

then (6.12) becomes

$$M\psi = \lambda\psi. \quad (6.15)$$

This is an eigenvalue problem for a linear system of equations and therefore has a finite set of eigenvalues and eigenvectors. From the physical problem and from the analytical solution (6.3), we conclude, however, that every positive value of λ must be an eigenvalue of (6.1) and (6.2). Hence, we must conclude that (6.7) has in general no solution, except when $c/2$ coincides with the reciprocal value of an eigenvalue λ of M .

The reader may have the intuition that (6.15) may yield a good approximation to (6.1) even if λ is no eigenvalue. That this is the case is demonstrated by the numerical examples below.

Calculations were performed for $N = 10, 50$, and 100 . To get a general idea of the elements in M , the entries of M for $N = 10$ are given to three decimal places only. To make the matrix more readable, the elements are stated in units of 10^{-3} . Thus,

$$M = 10^{-3} M_1, \quad (6.16)$$

where

$$M_1 = \begin{bmatrix} 3480 & -850 & -375 & -217 & -138 & -90 & -58 & -36 & -19 & -6 \\ 1425 & 2135 & -1125 & -325 & -183 & -112 & -70 & -42 & -21 & -6 \\ 475 & 1275 & 1454 & -975 & -275 & -150 & -88 & -50 & -25 & -7 \\ 317 & 425 & 1125 & 1008 & -825 & -225 & -117 & -63 & -30 & -8 \\ 238 & 283 & 375 & 975 & 680 & -675 & -175 & -83 & -38 & -10 \\ 190 & 213 & 250 & 325 & 825 & 536 & -525 & -125 & -50 & -12 \\ 158 & 170 & 188 & 217 & 275 & 675 & 256 & -375 & -75 & -16 \\ 136 & 147 & 150 & 162 & 183 & 225 & 525 & 128 & -225 & -25 \\ 119 & 121 & 125 & 130 & 138 & 150 & 175 & 375 & 5 & -75 \\ 106 & 106 & 107 & 108 & 110 & 112 & 117 & 125 & 150 & 5 \end{bmatrix} \quad (6.17)$$

The eigenvalues of this matrix (but with 12-digit accuracy) were calculated on the CDC-3600 using QR-transformations.⁽⁶⁾ The results are two real eigenvalues,

$$\lambda_0 = 0.8740,$$

and

$$\lambda_1 = 1.7112;$$

and four pairs of complex eigenvalues:

$$0.527 \pm 0.1189i,$$

$$0.720 \pm 0.412i,$$

$$1.030 \pm 0.806i,$$

and

$$1.640 \pm 1.269i.$$

The same calculation for the matrix M of order $N = 100$ yielded only the four real eigenvalues,

$$0.00953,$$

$$0.17469,$$

$$0.71022,$$

and

$$3.19663,$$

and additional four pairs of absolutely small eigenvalues with imaginary parts smaller than $(1/2)10^{-2}$. Therefore, one would normally expect (6.15) to yield a good approximation for these few real eigenvalues only. That this is not so can be anticipated by looking at (6.17). The first element of the first row dominates all others. Hence, small variations of μ_0 ($=0.95$) will affect the eigenvalues significantly (see 6.10), but not necessarily the eigenvectors. Our intuition therefore leads us to impressing the desired eigenvalue λ on the system (6.15) and to adjusting the first element of M (if not actually, at least theoretically). This is done simply by deleting the first row in M and solving the remaining $(N-1)$ homogeneous equations of (6.15) for the given value λ . To normalize the solutions we arbitrarily (but conveniently) set $\psi_0 = 1$.

The numerical calculations summarized below were carried out for $c = 1$ because the Milne problem as stated in (6.1) was solved previously and tabulated for this value of c [Chandrasekhar,⁽⁴⁾ Shure⁽¹⁶⁾]. The eigenvalue λ for $c = 1$ is $\lambda = 2$. The components of the corresponding eigenvector obtained by Gaussian elimination are listed in Tables VIII and IX for $N = 10$ and $N = 50$, respectively.

Table VIII

EIGENVECTOR OF $(M-\lambda)\psi = 0$ FOR $\lambda = 2$ AND $N = 10$

k	ψ_k	k	ψ_k
0	1.000000000000	5	0.687124577179
1	0.946080909246	6	0.620735829732
2	0.882525213620	7	0.552935565317
3	0.817802021052	8	0.483005411011
4	0.752705068184	9	0.382614411280

Table IX

EIGENVECTOR OF $(M-\lambda)\psi = 0$ FOR $\lambda = 2$ AND $N = 50$

k	ψ_k	k	ψ_k
0	1.000000000000	25	0.696561455777
1	0.996659007145	26	0.683790856153
2	0.985376280948	27	0.670987796502
3	0.973071635392	28	0.658149457398
4	0.960589989825	29	0.645272689582
5	0.948080004119	30	0.632353955109
6	0.935571006393	31	0.619389255517
7	0.923067014375	32	0.606374043647
8	0.910566312294	33	0.593303113767
9	0.898066127037	34	0.580170462913
10	0.885563779694	35	0.566969113640
11	0.873056915028	36	0.553690884019
12	0.860543490648	37	0.540326084060
13	0.848021709794	38	0.526863107370
14	0.835489952890	39	0.513287869945
15	0.822946718193	40	0.499583018873
16	0.810390574231	41	0.485726783197
17	0.797820121120	42	0.471691243554
18	0.785233958333	43	0.457439610235
19	0.772630657569	44	0.442921696997
20	0.760008738021	45	0.428065834522
21	0.747366643410	46	0.412762940900
22	0.734702719326	47	0.396830273536
23	0.722015190162	48	0.379919064439
24	0.709302134025	49	0.357234785967

In Table X, the values of ψ which correspond to $-\mu = 0.05(30)0.95$ are tabulated for $N = 10$, $N = 50$, and Chandrasekhar's values.

Table X

CALCULATED FLUX $\psi(\mu)$ FOR DIFFERENT APPROXIMATIONS

$-\mu$	N = 10	N = 50	Chandrasekhar
0.05	0.382 614 4	0.396 830	0.390 94
0.35	0.620 735 8	0.606 374	0.597 15
0.65	0.817 802 0	0.797 820	0.785 54
0.95	1.000 000 0	0.985 376	0.969 55

In Table XI, these values of ψ are normalized so that $\psi(-0.95) = 1$.

Table XI

CALCULATED FLUX $\psi(\mu)$ FOR DIFFERENT APPROXIMATIONS; NORMALIZED

$-\mu$	N = 10	N = 50	Chandrasekhar
0.05	0.382 6	0.402 7	0.403 2
0.35	0.620 7	0.615 4	0.615 9
0.65	0.817 8	0.809 7	0.810 2
0.95	1.000 0	1.000 0	1.000 0

In Table XII, the ratios of consecutive values of Table X are taken to permit a better overall comparison.

Table XII

FINAL COMPARISON OF CALCULATED FLUX FOR DIFFERENT APPROXIMATIONS

	N = 10	N = 50	Chandrasekhar
$\psi(-0.05)/\psi(-0.35)$	0.616 4	0.654 4	0.654 7
$\psi(-0.35)/\psi(-0.65)$	0.759 0	0.760 0	0.760 2
$\psi(-0.65)/\psi(-0.95)$	0.817 8	0.809 7	0.810 2

We conclude from Tables XI and XII that the calculations with an M of order N = 10 produce good results. The values for N = 50 agree with Chandrasekhar's results to three decimal figures.

A comparison of the results given by Chandrasekhar⁽⁴⁾ with those given by Shure⁽¹⁶⁾ shows that their results agree within one unit in the third decimal place. Hence the results computed here for N = 50 are at least as accurate as either Chandrasekhar's or Shure's.

The total computation time for all examples in this section, including calculation of the elements of M and processing the code (compiling), took less than a minute on the CDC-3600.

VII. CONCLUSIONS

We conclude that the use of the midpoint formulas for the direct evaluation of singular integrals and integral equations with a Cauchy kernel on high-speed computers is most convenient, reliable, and inexpensive. Further study of the method presented in this paper is justified. Its additional advantage is its great simplicity, compared with the analytical methods used so far.

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