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Abstract

It is shown in this note that some results of water waves on a sloping beach based on the linearized shallow water theory [Stoker (1947)] can be derived from the integral representations of the complex velocity potentials by Friedrichs (1948).

1. Introduction

K. O. Friedrichs (1948) derived integral representations of the complex velocity potentials for water waves on a shallow sloping beach. He also gave asymptotic estimates for these integrals when the slope of the beach is small. Based on the linearized shallow water theory, Stoker (1947) derived an expression for the velocity potential at the equilibrium level. It is the purpose of this note to show that Stoker's result can be obtained from Friedrichs asymptotic estimate by noting the integral representations for the Hankel functions from Watson (1922). The region of validity of the linearized shallow water theory can be ascertained from the region where the representations of the Hankel functions can approximate the Friedrichs estimates.

2. Friedrichs' Results and Stoker's Results

The two-dimensional linear theory of surfaces in water over a sloping beach consists of finding a velocity potential $\phi(x, y, z, t)$ satisfying (see Fig. 1)

- i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ for $(x, y) \in \Gamma$,
- ii) $\frac{\partial \phi}{\partial \theta} = 0$ for $\theta = -\omega$,
- iii) $g\phi_y + \phi_{tt} = 0$ for $y = 0$.

If the time dependence enters harmonically as $\sin \sigma t$ or $\cos \sigma t$ and if we introduce the dimensionless quantities (and then dropping the subscripts)

$$x_1 = mx, \quad y_1 = my, \quad t_1 = \sigma t$$

where

$$mg = \sigma^2,$$

we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{for } (x, y) \in \Gamma, \quad (1)$$

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{for } \theta = -\omega, \quad (2)$$

$$\frac{\partial \phi}{\partial y} - \phi = 0 \quad \text{for } y = 0. \quad (3)$$

When we write $\phi(x, y)$ as the real part of an analytic function $\chi(z) = \phi(x, y) + i\psi(x, y)$ of the complex variable $z = x + iy$, the problem we have to solve reduces to finding $\chi(z)$ in the sector $z = re^{i\theta}$, where $-\omega \leq \theta \leq 0$, $r \geq 0$, and satisfying the conditions

$$\operatorname{Im} \left(\frac{d}{dz} + i \right) \chi = 0, \quad \theta = 0,$$

$$\operatorname{Im} \left(e^{-i\omega} \frac{d}{dz} \chi \right) = 0, \quad \theta = -\omega.$$

For $\omega = \pi/2n$, $n = 1, 2, \dots$, Friedrichs showed that there exist two solutions

denoted by χ^S and χ^R ,

Fig. 2

$$\chi^S = \chi_+ + \chi_-$$

$$\chi^R = i\chi_+ - i\chi_-$$

where (see Fig. 2)

$$\chi_{\pm} = \int_{P_{\pm}} e^{\xi z} G_n^{-1}(\xi) \xi^{n-1} d\xi,$$

$$G_n(\xi) = (\xi - \xi_1)(\xi - \xi_2) \dots (\xi - \xi_n), \quad \xi_{\nu} = i e^{2i\nu\omega} = i e^{i\pi\nu/n}.$$

The function χ^S is singular at the origin and χ^R is regular. The solution to our problem will be

$$\phi(x, 0, t) = \alpha \left[\phi^S(x, \sigma) \sin(\sigma t + \epsilon_1) + \phi^R(x, 0) \cos(\sigma t + \epsilon_2) \right]$$

where α , ϵ_1 , and ϵ_2 are real constants.

Stoker showed, based on the linearized shallow water theory, that

$$\phi^R(x, 0) \approx 2\pi J_0(2\sqrt{x/\omega}),$$

$$\phi^S(x, 0) \approx 2\pi Y_0(2\sqrt{x/\omega}).$$

Using the asymptotic estimates for χ_+ and χ_- for small values of ωx , Friedrichs was able to identify the asymptotic estimates for ϕ^R and ϕ^S as those for the Bessel and the Neumann functions.

3. Identification of Friedrichs Integral for Small Values of ωx

For ω small, it is shown by Friedrichs that

$$\chi_{\pm} \sim -i \int_{P_{\pm}} \exp(\xi z) \exp \left[\omega^{-1} H(\xi) \right] \frac{(\xi^{-1} + i)^{\frac{1}{2}}}{(\xi^{-1} - i)^{\frac{1}{2}}} \frac{d\xi}{\xi}$$

where the phase of $(\xi^{-1} \pm i)^{\frac{1}{2}}$ will be the principal value, i. e.,

$$\text{phase } (\xi^{-1} + i)^{-\frac{1}{2}} = \pi/4, \quad |\xi| \rightarrow \infty,$$

$$\text{phase } (\xi^{-1} - i)^{-\frac{1}{2}} = -\pi/4, \quad |\xi| \rightarrow \infty,$$

$$H(\xi) = \frac{\pi}{2} \int_0^1 \log \left(\xi^{-1} + i e^{-i\pi\sigma} \right) d\sigma.$$

For z positively real and finite, $\omega z = \omega x$ will be small.

If we try to approximate the above integral by the saddle point method, we see that the significant contributions come from the neighborhoods of the point ξ satisfying

$$\omega x = \frac{d}{d\xi} H(\xi).$$

Friedrichs' work shows that if $H'(\xi)$ is small, ξ is large and $H(\xi)$ has the representation

$$H(\xi) \sim \xi^{-1} - \frac{1}{g} \xi^{-3} + \dots$$

Furthermore

$$\frac{(\xi^{-1} + i)^{\frac{1}{2}}}{(\xi^{-1} - i)^{\frac{1}{2}}} \sim i.$$

Therefore for ωx small we can approximate

$$\begin{aligned} \chi_{\pm} &\sim \int_{P_{\pm}} \exp\left(\xi x - \omega^{-1} \frac{1}{\xi}\right) \frac{d\xi}{\xi} \\ &= \int_{P_{\pm}} \exp\left[\sqrt{\frac{x}{\omega}} \left(\sqrt{\omega x} \xi - \frac{1}{\sqrt{\omega x} \xi}\right)\right] \frac{d\xi}{\xi}. \end{aligned}$$

From Watson (p. 178) we can identify

$$\chi_{\pm} \sim \int_{P_{\pm}} \exp\left(\xi x - \omega^{-1} \frac{1}{\xi}\right) \frac{d\xi}{\xi} = \mp i\pi H_0^{(1,2)}\left(2\sqrt{\frac{x}{\omega}}\right).$$

These approximations give

$$\begin{aligned} \chi^S &= \chi_+ + \chi_- \approx 2\pi Y_0\left(2\sqrt{\frac{x}{\omega}}\right), \\ \chi^R &= i\chi_+ - i\chi_- \approx 2\pi J_0\left(2\sqrt{\frac{x}{\omega}}\right), \end{aligned}$$

for $\omega x \ll 1$, x small. These two results have been obtained by Stoker (1947) in his linearized shallow water theory.

In Stoker's work ω is not restricted to $\pi/2n$. For an unrestricted ω we go to the work of Peters (1952). Peters has shown that two solutions are

$$\phi_1(x, y) = \operatorname{Re} \left[i\lambda_1 \int_{P_1} \frac{e^{z\xi} g(\xi, 1)}{\xi + i} d\xi \right]$$

and

$$\phi_2(x, y) = \operatorname{Re} \left[\lambda_2 \int_{P_2} \frac{e^{z\xi} g(\xi, 1)}{\xi + i} d\xi \right]$$

where

λ_1 and λ_2 are real constants,

$$g(\xi, 1) = \exp \left[-\frac{1}{\pi} \int_0^\infty \frac{\xi}{v^2 + \xi^2} \ln \left(\frac{v^\mu - 1}{v^\mu} \frac{v^2}{v^2 - 1} \right) dv \right],$$

$$-\frac{\pi}{2} - 2\omega < \arg \xi < \frac{\pi}{2} + 2\omega,$$

$$\mu = \frac{\pi}{\omega}; \quad P_1 = P_- - P_+, \quad P_2 = P_+ + P_-.$$

Furthermore Peters also shows that

$$g(\xi, 1) \sim 1 - \frac{1}{\xi} \cot \omega \sim \exp \left(- \frac{1}{\xi} \cot \omega \right).$$

For $x\omega$ small and x small, we therefore see that

$$\phi_1(x, 0) \approx \operatorname{Re} \left[i\lambda_1 \left(\int_{P_-} - \int_{P_+} \right) \exp \left(x\xi - \frac{1}{\xi \tan \omega} \right) d\xi \right] \approx \tilde{\lambda}_1 J_0 \left(2\sqrt{\frac{x}{\omega}} \right)$$

and

$$\phi_2(x, 0) \approx \operatorname{Re} \left[\lambda_2 \left(\int_{P_+} + \int_{P_-} \right) \exp \left(x\xi - \frac{1}{\xi \tan \omega} \right) d\xi \right] \approx \tilde{\lambda}_2 Y_0 \left(2\sqrt{\frac{x}{\omega}} \right).$$

Hence we see that Friedrichs' results hold also for unrestricted ω .

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