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Estimation of Weibull Parameters
from Failure Data

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MASTER

(1) Introduction

In a 1951 paper, Weibull [1] popularized a density function now bearing his name. He suggested a wide variety of situations in which this particular function provided a good description of the data. His examples included such random variables as the yield strength of a Bofors steel, the size distribution of fly ash, and the fiber strength of Indian cotton.

In recent years, the Weibull function has indeed been applied to a variety of problems. This is especially true in the field of life testing. Although much of the early work in life-testing was based on the exponential model, more recent attention has been directed toward using the more general Weibull model to describe failure data, since the exponential model has proven to be unrealistic in a number of situations.

This emphasis on the Weibull model has not extended over very many years. A bibliography compiled by Splitstone [2] in his 1967 Iowa State University Master's thesis on estimation of the Weibull parameters demonstrates that the emphasis is recent. Of the 25 references he lists, only two were dated before 1959 and only 7 before 1963. Fourteen were dated 1965 or later.

Judging by the dates of these publications, application of the Weibull model to reactor fuel element failures at Hanford represents somewhat of a pioneering effort. This model was first applied to such failure data in 1954, and has been used with success ever since. In addition to this application to fuel elements from plutonium production reactors, similar success has been

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achieved when describing fuel failure data from commercial power reactors.

Methods of estimating the parameters of this distribution have been developed over the 15 years since the model was first applied at Hanford. These improvements are largely a result of increased computer capabilities which permits the use of estimation methods that were literally impossible to apply without such computational assistance. No doubt, increased computer capabilities have also contributed to other recent interest in the Weibull function as mentioned earlier, since many of the estimation procedures suggested by the various authors require large-scale computers.

In this paper, a method used to estimate the shape and scale parameters of the Weibull distribution as applied to reactor fuel element failures is discussed. The location parameter is assumed to be zero. The method of estimation may, of course, be used on other kinds of failure data as well.

(2) The Data

The data for which the estimation method was developed are grouped data. These are normally the way in which reactor fuel element failure data are made available. The data are characterized by large numbers of elements, and small numbers of failures.

The data appear as in Table (2.1)

Table (2.1)
Failure Data

<u>Time</u>	<u>Number of elements</u>	<u>Number of losses</u>	<u>Number of failures</u>
$0-T_1$	N_1	L_1	r_1
T_1-T_2	N_2	L_2	r_2
T_2-T_3	N_3	L_3	r_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T_{k-1}-T_k$	N_k	L_k	r_k

In this table N_1 represents the total number of elements under study. In the i^{th} time interval, L_i elements are "lost", while r_i failures are observed. A "lost" element, in the context of this example, is one which has either been discharged from the reactor at some time between T_{i-1} and T_i , or else is still in the reactor with an accumulated life which lies in this i^{th} interval. N_2 , the number of elements present at the beginning of the second time interval, is clearly $(N_1 - L_1)$, and, in general, $N_i = (N_{i-1} - L_{i-1})$. (The number of losses, L_i , is defined to include the number of failures, r_i).

(3) Non-Parametric Estimation of the Failure Rate

For grouped failure data such as these, Kaplan and Meier [3] gave non-parametric procedures for estimating the failure rate. A simple procedure, called the actuarial method, consists of calculating the failure rate in each interval as

$$\begin{aligned} p_i &= r_i / (N_i - 0.5L_i) \\ &= r_i / m_i \end{aligned} \quad (3.1)$$

The probability that an element survives to time T_j is

$$1 - F(T_j) = \prod_{i=1}^j (1 - p_i) \quad (3.2)$$

The cumulative failure rate at time T_j is then:

$$F(T_j) = 1 - \prod_{i=1}^j (1 - p_i) \quad (3.3)$$

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For small p_i , this may be written approximately as:

$$F(T_j) \approx \sum_{i=1}^j p_i \quad (3.4)$$

A plot of $F(T_j)$ versus T_j is the failure rate curve for the non-parametric case in which no failure model is assumed. The problem is to use these basic data to estimate the shape and scale parameters of the two-parameter Weibull model assuming that this is the model.

(4) Estimation of Weibull Parameters

The Weibull cumulative distribution function may be written:

$$F(t) = 1 - \exp(-t^a/b) \quad (4.1)$$

where $F(t)$ = cumulative failure rate at time t

a = shape parameter

b = scale parameter

An obvious method of estimating the two parameters is to first linearize the model (4.1).

$$\begin{aligned} \ln[1-F(t)] &= -t^a/b \\ \ln \left\{ -\ln[1-F(t)] \right\} &= a \ln t - \ln b \end{aligned} \quad (4.2)$$

This suggests that, with $F(T_j)$ defined as in (3.3), one could plot $\ln \left\{ -\ln[1-F(t_j)] \right\}$ versus $\ln T_j$. This plot should be a straight line with a slope equal to the shape parameter a , and an intercept of $\ln b$.

Point estimates of the two parameters can be found by a simple procedure such as "eye-balling" a straight line through the data, and reading off the slope and intercept. Normally, however, one would like more than just the point estimates, and would also prefer that the estimation procedure be more objective.

An improvement on this approach consists in using Aitken's generalized least squares method to estimate the slope and intercept [4]. This idea was used by Krane [5] when fitting a polynomial through failure data, which is a generalization of this problem. It has also been used by the author on fuel element failure data for a number of years with good success.

However, there is a feature of this approach that is disturbing. To avoid singularity in the matrix, it is necessary to have $r_i > 0$ in each interval. This necessitates combining intervals. This practice is disturbing when the last couple intervals, for example, both have $r_i = 0$. One worries about combining these intervals with the last one in which a failure occurred. Perhaps the fact that $r_i = 0$ in both intervals is evidence that the Weibull failure model does not provide a satisfactory fit to the data, a situation that may be disguised if the intervals are combined to too large an extent.

To circumvent this difficulty, an estimation procedure was devised which permits estimation even when $r_i = 0$ in some of the intervals. Further, the procedure allows one to check on the adequacy of the assumption that the underlying model is the Weibull model. The procedure is described in the next section.

(5) Proposed Estimation Procedure

The estimation procedure consists in finding that combination of shape and scale parameters which gives the "best" overall agreement between the observed numbers of failures in each interval and those predicted by the

model. For a given value of the shape parameter a , the maximum likelihood estimate of the scale parameter, b , is found by summing the element life-times raised to the a^{th} power for both the failed and non-failed elements, and dividing by the number of failures. For the grouped data as given in Table (2.1), the estimate of b for given a may be written

$$b_a = \frac{\sum_{i=1}^k L_i \left(\frac{x_{i-1}^a + x_i^a}{2} \right)}{\sum_{i=1}^k r_i} \quad (5.1)$$

Values are assigned the shape parameter and b_a is computed for each a . Each combination of the shape and scale parameters is then used to find the predicted number of failures which will occur in each time interval. For the grouped data, the number predicted in the i^{th} interval is

$$E_i = m_i \left\{ \exp\left(-\frac{x_{i-1}^a}{b_a}\right) - \exp\left(-\frac{x_i^a}{b_a}\right) \right\} \quad (5.2)$$

The E_i are to be compared with the r_i . The value of the shape parameter, a , and the corresponding maximum likelihood estimate, b_a , which result in the "best" agreement between the E_i and the r_i will be the estimates derived from this procedure.

A problem is how to measure this agreement. The E_i values will generally be too small to permit valid application of a Chi-square procedure, i.e., one which regards the statistic:

$$\sum \frac{(E_i - r_i)^2}{E_i}$$

A search was made for an alternate method of measuring the extent of this agreement.

It was decided to associate a probability, P_i , with each combination (E_i, r_i) . When r_i is equal to E_i , one should like to have P_i equal to one, while as r_i deviates from E_i either on the low side or on the high side, P_i should become progressively smaller, approaching zero as the discrepancy becomes larger and larger. This criterion has intuitive appeal, but will be more firmly motivated later in the paper.

To define P_i specifically, assume that the number of failures, r_i , observed in interval i is a random variable following the Poisson distribution with parameter E_i . With the large sample sizes and small probabilities encountered in the application under discussion, this is a reasonable assumption. Then, one can define a probability P_i' as follows:

$$\left. \begin{aligned} \text{If } r_i \leq E_i, \quad P_i' &= \sum_{j=0}^{r_i} e^{-E_i} (E_i)^j / j! \\ \text{And if } r_i > E_i, \quad P_i' &= \sum_{j=r_i}^{\infty} e^{-E_i} (E_i)^j / j! \end{aligned} \right\} \quad (5.3)$$

Equations (5.3) define P_i' such that if less failures (r_i) are observed than expected (E_i) from the model, then the probability of observing this number (r_i) or less given a mean of E_i is calculated. If more failures are observed (r_i) than expected (E_i), then the probability of observing r_i or more failures given E_i is calculated.

With the definition given by (5.3), large discrepancies between E_i and r_i will result in small values of P_i' , while small discrepancies result in large values of P_i' . However, P_i' will not range from 0 to 1 as desired; the maximum value will depend on r_i . One can normalize these probabilities by dividing each P_i' by some factor, F_i , which is a function of r_i . Specifically,

$$F_i = \sum_{j=0}^{r_i} e^{-r_i} (r_i)^j / j! , r_i > 0 \quad (5.4)$$

$$= 1 \text{ for } r_i = 0$$

Then, defining $P_i = P_i' / F_i$ gives a random variable with range from zero to one.

Having defined P_i for each i , the problem is then to define a statistic which will combine the P_i 's in some fashion to give an overall measure of agreement between the E_i and the r_i . A statistic that suggests itself is

$$S = -2 \sum_{i=1}^k \ln P_i \quad (5.5)$$

The motivation for choosing S comes from the fact that if q_i is a random variable uniformly distributed on $(0,1)$, and if q_i and q_j are independent for all (i,j) , then $-2 \sum_{i=1}^k \ln q_i$ follows the Chi-square distribution with $2k$ degrees of freedom [6]. This fact is used, as one application, when combining independent tests of significance to obtain an overall significance level. This is the application that suggested the use of S in (5.5).

It is recognized that even though P_i is distributed over the interval $(0,1)$, it is not uniformly distributed over this interval. Thus, the S statistic will not be distributed as Chi-square except, perhaps, as the E_i become larger. More is said on this point in the next section. From an estimation viewpoint, the only interest at this point is to define S , and establish some sort of motivation for the definition.

Having defined S , the estimation procedure suggested is summarized in the following steps.

- (1) Choose a series of values for the shape parameter, a , defined over some selected interval (a_1, a_2) .
- (2) For each value of a , calculate b_a from (5.1)
- (3) For each combination of values (a, b_a) , calculate the E_i from (5.2)
- (4) For each i , calculate P_i from (5.4), (5.3), and $P_i = P_i' / F_i$.
- (5) Calculate S from (5.5)
- (6) The estimates of the shape and scale parameters are the combination which results in a minimum S value.

(6) Limits of Uncertainty

Having estimated the two parameters, it is important to obtain a confidence region for them. Further, one would like to know the uncertainty associated with a predicted failure rate at some time t . This portion of the paper addresses itself to this question.

The approach to finding a confidence region for the parameters will be to find those combinations of values which result in S values equal to some critical value S_c . This critical value will be one which would be exceeded by chance less than $\alpha\%$ of the time if, in fact, the Weibull model with given parameters were the true model. The resulting combinations of parameters will bound the required confidence region.

If S were distributed approximately as the Chi-square distribution with $2k$ degrees of freedom, it would be a simple matter to determine S_c from an appropriate table of this distribution. However, for small values of the E_i , the assumption that the P_i are approximately uniformly distributed over $(0,1)$ is not valid, and hence, this approach cannot be used.

An investigation was made to find how S_c may be determined, especially for small E_1 . The approach was to calculate an average S value and its variance for each of a series of E_1 values by listing the possible values for r_1 and their associated probabilities of occurrence. An example will serve to illustrate the approach. E_1 is set equal to 0.5 in this example

Table (6.1)

r_1 equals	With probability q	P_1' [from (5.3)]	P_1 [from (5.4)]	$S_1 = -2 \ln P_1$
0	.6065	.6065	.6065	1.00
1	.3033	.3935	.5346	1.25
2	.0758	.0902	.1332	4.03
3	.0126	.0144	.0223	7.61
4	.00158	.00175	.00278	11.77
5	.000158	.000172	.000279	16.37
6	.000013	.000014	.000023	21.36
≥ 7	0			

Columns 2 and 5 give the distribution of S_1 for one of the k intervals. Operating on these two columns gives the first two moments for S at $E_1 = 0.5$.

$$\text{Average } S = 1.408$$

$$\text{Variance } S = 1.33$$

Note that if S were truly distributed as Chi-square with two degrees of freedom, its mean would be 2 and its variance 4. An inspection of columns 2 and 4 in Table (6.1) illustrates how grossly the assumption that P_1 is uniformly distributed on (0,1) is violated for $E_1 = 0.5$. It is not surprising in this case that the average and variance of S are quite different from what would be expected from the Chi-square distribution.

The two moments were calculated for values of E_1 ranging from 0.1 to 10. A plot of the mean and variance of S versus E is given in figure (6.1). Note

that the mean and variance gradually approach the limiting values of two and four respectively as E increases in value.

Having determined these moments as a function of E , one can obtain the moments for the sum of the S_i over the k intervals, recognizing that the S_i are independently distributed. An appeal can then be made to the Central Limit Theorem to assert that the sum, S , is approximately normally distributed, and hence obtain a value for S_c from a table of this distribution once α is specified.

This procedure was considered sufficiently precise for the application under discussion. If greater precision were required, one could find the higher moments of S and appeal to some family of curves such as the Pearson or Johnson curves to determine S_c . This work is under consideration at present.

Once the confidence region for the shape and scale parameters is found, the limits on a predicted failure rate at a given time t are found by sweeping around the extremities of this region and locating the combinations of the parameters which yield the minimum and maximum values for the failure rate. This is done for a series of time values to give a confidence region for the failure rate curve.

It should be noted that the size of the confidence region depends not only on the number of failures observed, but it is also affected by the reasonableness of the underlying assumption that the Weibull function is applicable. If this assumption is not valid, it may well be that even the minimum value for S is larger than the critical value, suggesting that no combination of the Weibull parameters will provide an adequate fit.

(7) Numerical Example

A numerical example of the estimation procedure is given to illustrate the method. The failure data are listed.

Table (7.1)
Failure data for example

<u>Time*</u>	<u>N_i</u>	<u>L_i</u>	<u>r_i</u>
0-500	6056	514	0
500-600	5542	968	2
600-700	4574	1310	2
700-800	3264	1154	2
800-900	2110	818	3
900-1000	1292	565	0
1000-1100	727	412	0
1100-1200	315	315	0

* units of exposure

The computer program varied the shape parameter $\underline{a}$ from 1.0 to 10.0 in increments of 0.2. The minimum S value (5.5) was found to occur at $a = 4.0$. Values of E_i and P_i are given in Table (7.2) for this value of $\underline{a}$ and the associated maximum likelihood value of $b = 2.906 \times 10^{14}$.

Table (7.2)

<u>Time</u>	<u>E_i</u>	<u>r_i</u>	<u>P_i</u>
0-500	1.25	0	0.29
500-600	1.16	2	0.48
600-700	1.48	2	0.64
700-800	1.55	2	0.68
800-900	1.43	3	0.27
900-1000	1.18	0	0.31
1000-1100	0.82	0	0.44
1100-1200	0.33	0	0.72

$$S = -2 \sum \ln P_i = 12.93$$

Reference to figure (6.1) gives the first two moments for S for the E_1 values in Table (7.2).

Mean of S = 13.05
Standard Deviation = 4.02

From a table of the normal distribution, S exceeds $13.05 + (1.282)(4.02) = 18.20$ with a probability of 0.10. Therefore, the 90% limits on the shape parameter are given by finding the smallest and largest values of the parameter which give values of S equal to 18.20. These are:

lower limit: $a = 2.1$
upper limit: $a = 6.5$

Finally, 90% limits of uncertainty for the predicted failure rate are reproduced in Table (7.3).

Table (7.3)
Confidence limits on
predicted failure rates

<u>t</u>	<u>$F(t) \times 10^{-4}$</u>	<u>Lower Limit $\times 10^{-4}$</u>	<u>Upper Limit $\times 10^{-4}$</u>
500	2.15	0.44	6.57
600	4.46	1.41	10.94
700	8.26	3.78	16.84
800	14.09	8.89	25.63
900	22.55	16.70	40.86
1000	34.35	24.16	63.52
1100	50.26	31.57	99.22
1200	71.10	38.22	152.89

(8) Acknowledgement

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FIGURE 1