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A NOTE ON INVERTING THE R-MATRIX

T. Watanabe

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By

T. Watanabe

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A Note on Inverting the R-Matrix

T. Watanabe

Abstract

The procedure of rank annihilation is applied to the inversion of the R-matrix. It is shown that the method inverts the R-matrix in a way similar to the method of Thomas but that it leads to a different form. The advantage of this particular method and some approximations are discussed.

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A Note on Inverting the R-Matrix

T. Watanabe

I. Introduction

Within the framework of the R-matrix formalism of Wigner and Eisenbud⁽¹⁾, the transformation from the R-matrix to the collision (S-matrix) requires the inversion of a matrix expression containing the R-matrix. (For sake of brevity we will use the term inverting the R-matrix to mean the same thing.) The inversion of the R-matrix is usually carried out by a method introduced by Wigner or by a slightly modified version of it. Lane and Thomas⁽²⁾ in their review article give a generalized version of this method. In both methods, the R-matrix is transformed from a channel matrix to what is termed a level matrix and the R-matrix inversion is obtained by inverting the level matrix. The purpose of this paper is to describe the application of a matrix inversion termed method of rank annihilation to invert the R-matrix, to mention what might be considered favorable aspects of this particular method, and to point out some approximations which can be made with this method of matrix inversion.

II. Collision Matrix of Wigner and Eisenbud

According to the Wigner-Eisenbud R-matrix formalism, the R-matrix is transformed to the collision matrix by the following expression

$$U = \omega \frac{I + iC + iBRB}{I - iC - iBRB} \omega \quad (1)$$

where I = unit matrix

C = real, diagonal matrix

B = diagonal matrix

R = R-matrix-symmetric, real (Hermitean)

ω = diagonal unitary matrix

U = symmetric unitary matrix

The components of the R-matrix have the following form

$$R_{s\ell, s'\ell'} = \sum_{\lambda} \frac{\gamma_{\lambda s\ell} \gamma_{\lambda s'\ell'}}{E_{\lambda} - E} \quad (2)$$

where s = channel

ℓ = channel spin

λ = level

In matrix notation the R-matrix becomes

$$R = \sum_{\lambda} \frac{\gamma_{\lambda} \times \gamma_{\lambda}}{E_{\lambda} - E} \quad (3)$$

where $\gamma \times \gamma \equiv \gamma \tilde{\gamma}$

$\tilde{\gamma}$ = transpose

where γ is considered a column vector.

As pointed out by Wigner, the matrix which is multiplied on the left and the right by ω

$$\omega^{-1} U \omega^{-1} = \frac{I + iC + iBRB}{I - iC - iBRB} \quad (4)$$

is both unitary and symmetric because both numerator and denominator are expressed in terms of a unit matrix and a symmetric Hermitean matrix. For ease in manipulating Equation (4), the following changes will be made:

a. Since I and C are both real diagonal matrices, we define a new diagonal matrix

$$D = I - iC \quad (5)$$

and

$$D^* = I + iC \quad (6)$$

b. Since B is a diagonal matrix and since the (matrix) product of a diagonal matrix with a vector is symmetric, the matrix BRB will be written as

$$BRB = B \frac{\sum \gamma_\lambda x \gamma_\lambda}{E_\lambda - E} B = \sum_\lambda \beta_\lambda x \alpha_\lambda \quad (7)$$

where

$$\frac{B\gamma_\lambda}{E_\lambda - E} = \beta_\lambda$$

$$\gamma_\lambda B = \alpha_\lambda$$

With these changes and definitions, Equation (4) can be written

$$\omega^{-1} U \omega^{-1} = \frac{D^* + i \sum_\lambda \beta_\lambda x \alpha_\lambda}{D - i \sum_\lambda \beta_\lambda x \alpha_\lambda} \quad (8)$$

and the matrix to be inverted is the denominator

$$D - i \sum_\lambda \beta_\lambda x \alpha_\lambda \quad (9)$$

III. R-Matrix Inversion

The matrix inversion of Equation (9) proceeds as follows: Since D is a diagonal matrix, its inverse is easily found. We let

$$C^1 = D^{-1} \quad (10)$$

We next define the following matrix

$$C^2 = [D - i \beta_1 \times \alpha_1]^{-1} = [(C^1)^{-1} - i \beta_1 \times \alpha_1]^{-1} \quad (11)$$

To invert this matrix Equation (11), the matrix inversion method of rank annihilation⁽³⁾ is applied which states that

$$[A + u \times v]^{-1} = A^{-1} - \frac{(A^{-1}u) \times (vA^{-1})}{1 + (vA^{-1},u)} \quad (12)$$

where A is a matrix whose inverse is known and u, v are column vectors.

Applying this method to the matrix Equation (11) we obtain

$$C^2 = C^1 + i \frac{(C^1 \beta_1) \times (\alpha_1 C^1)}{1 - i(\alpha_1 C^1, \beta_1)} \quad (13)$$

where $(\alpha_1 C^1, \beta_1)$ = scalar product

next we define

$$C^3 = [D - i \beta_1 \times \alpha_1 - i \beta_2 \times \alpha_2]^{-1} = [(C^2)^{-1} - i \beta_2 \times \alpha_2]^{-1} \quad (14)$$

and again applying Equation (12) to the matrix Equation (14), C^3 becomes

$$C^3 = C^2 + i \frac{(C^2 \beta_2) \times (\alpha_2 C^2)}{1 - i(\alpha_2 C^2, \beta_2)} \quad (15)$$

Repeating this process, we find for $\lambda = n$

$$\begin{aligned} C^{n+1} &= [(C^n)^{-1} - i \beta_n \times \alpha_n]^{-1} \\ &= C^n + i \frac{(C^n \beta_n) \times (\alpha_n C^n)}{1 - i(\alpha_n C^n, \beta_n)} \end{aligned}$$

$$= [D - i \sum_{\lambda}^n \beta_{\lambda} \times \alpha_{\lambda}]^{-1} \quad (16)$$

It is to be noted that in each inversion that is carried out, the condition that A be a matrix having a known inverse is satisfied.

IV. Expression for the Collision Matrix

In Equation (8), we can write the numerator

$$(C^{n+1*})^{-1} = D^* + i \sum_{\lambda} \beta_{\lambda} \times \alpha_{\lambda} \quad (17)$$

where the denominator has already been defined. Since we have obtained the inverse of the denominator for Equation (8), we will proceed to obtain Equation (4) for the following cases.

$$\lambda = 0$$

$$(C^{1*})^{-1} = D^*$$

$$C^1 = D$$

$$\omega^{-1} U_1 \omega^{-1} = (C^{1*})^{-1} C^1 = D^* D^{-1} \quad (18)$$

$$\lambda = 1$$

$$(C^{2*})^{-1} = D^* + i \beta_1 \times \alpha_1 = (C^{1*})^{-1} + i \beta_1 \times \alpha_1$$

$$C^2 = C^1 + i \frac{(C^1 \beta_1) \times (\alpha_1 C^1)}{1 - i(\alpha_1 C^1, \beta_1)}$$

$$\omega^{-1} U_2 \omega^{-1} = (C^{2*})^{-1} C^2 = (C^{1*})^{-1} C^1 + 2i \frac{(C^1 \beta_1) \times (\alpha_1 C^1)}{1 - i(\alpha_1 C^1, \beta_1)} \quad (19)$$

$$\lambda = 2$$

$$(C^{3*})^{-1} = (C^{2*})^{-1} + i \beta_2 \times \alpha_2$$

$$C^3 = C^2 + i \frac{(C^2 \beta_2) \times (\alpha_2 C^2)}{1 - i(\alpha_2 C^2, \beta_2)}$$

$$\omega^{-1} U_3 \omega^{-1} = (C^{3*})^{-1} C^3 = (C^{2*})^{-1} C^2 + 2i \frac{(C^2 \beta_2) \times (\alpha_2 C^2)}{1 - i(\alpha_2 C^2, \beta_2)}$$

$$\omega^{-1} U_3 \omega^{-1} = \omega^{-1} U_2 \omega^{-1} + 2i \frac{(C^2 \beta_2) \times (\alpha_2 C^2)}{1 - i(\alpha_2 C^2, \beta_2)} \quad (20)$$

By extension, we obtain the following expression for Equation (4) in

its general form for $\lambda = n$

$$\omega^{-1} U_n \omega^{-1} = (C^{n+1*})^{-1} C^{n+1} = \omega^{-1} U_{n-1} \omega^{-1} + 2i \frac{(C^n \beta_n) \times (\alpha_n C^n)}{1 - i(\alpha_n C^n, \beta_n)} \quad (21)$$

Since we can write

$$C^{n+1} = C^n + i \frac{(C^n \beta_n) \times (\alpha_n C^n)}{1 - i(\alpha_n C^n, \beta_n)}$$

$$= C^1 + i \sum_{i=1}^n \frac{(C^i \beta_i) \times (\alpha_i C^i)}{1 - i(\alpha_i C^i, \beta_i)}$$

$$C^{n+1} - C^1 = i \sum_{i=1}^n \frac{(C^i \beta_i) \times (\alpha_i C^i)}{1 - i(\alpha_i C^i, \beta_i)}$$

Equation (21) can therefore be written

$$\begin{aligned}\omega^{-1}U_n\omega^{-1} &= (C^{1*})^{-1} C^1 + 2i \sum \frac{(C^1\beta_i) \times (\alpha_i C^1)}{1 - i(\alpha_i C^1, \beta_i)} \\ &= (C^{1*})^{-1} C^1 + 2 (C^{n+1} - C^1)\end{aligned}\quad (22)$$

V. Discussion

In Appendix A, matrix inversion of the R-matrix for the case of one, two and three levels have been obtained and the expression for $\omega^{-1}U_n\omega^{-1}$ (Eq. 22) takes the form:

$$\omega^{-1}U_n\omega^{-1} = (C^{1*})^{-1} C^1 + 2 \sum_{\mu\nu} \sum A_{\mu\nu}^n C^1 \beta_\mu \times \alpha_\nu C^1$$

This form of U_n is somewhat similar to the expression obtained by Lane and Thomas. A notable difference in the quantities $A_{\mu\nu}^n$ as derived by the two methods is the number terms associated with each $A_{\mu\nu}^n$. In the present method, after rearrangement of the matrix to a form as shown in Equation (22), the number of terms contained in each $A_{\mu\nu}^n$ have the following sequence. If levels are sequentially numbered 1, 2, 3, etc., the number of terms contained in each coefficients $A_{\mu\nu}^n$ follow this sequence:

$$A_{11}^n > A_{12}^n > A_{13}^n$$

$$A_{21}^n > A_{22}^n > A_{23}^n$$

$$A_{12}^n > A_{21}^n \text{ or } A_{\mu\nu}^n = A_{\nu\mu}^n$$

that is, the number of terms in A_{11}^n is greater than in A_{12}^n etc. The effect of having a preponderance of terms associated with a level

designated as the first level has not been investigated. Although the level quantities $A_{\mu\nu}^n$ from this derivation are different from that of Lane and Thomas, since the form of Equation (22) are alike the two expressions should be equivalent.

On comparing the method of R-matrix inversion of Thomas and the present method, the most notable difference between the two methods is the procedure for obtaining the elements of the level matrix required for the inversion. In order to obtain the level matrix in the Thomas method for n-levels, it is necessary to solve a set of n by n linear equations for the elements which make up the level matrix. The present method avoids the necessity of solving for the entire level matrix for all levels. Instead, the level matrix is obtained level by level using the preceding level matrix as the basis for obtaining the level matrix for the next level to be included in the level expansion.

It is interesting that this particular matrix inversion method which was attempted because the special form of the R-matrix fitted particularly well to the rank annihilation scheme results in an inverse which has a level matrix form although no attempt was made to transform to a level matrix. In contrast, the method of Lane and Thomas or Wigner for inverting the R-matrix was to obtain a transformation to a level matrix based on the special form of the R-matrix.

This level by level expansion or inversion of the R-matrix has certain favorable aspects which are:

(a) By avoiding determinants, this method allows calculations to be more straightforward.

(b) Because this method generates the required level matrix for any given level expansion from a recursion formula, it is in principle

a simple matter to extend the inversion from n to $n + 1$ level. This feature also makes this method more flexible which may be desirable.

(c) Since the elements of the level matrix may be obtained in a more direct fashion, certain approximations can be made after the R-matrix has been inverted. This should mean that the approximations are made directly to the collision matrix rather than by way of the R-matrix.

In Appendix B, C and D, approximations which might be made are discussed. The simplest case of course being the single level approximation or the case of non-interference between levels. This approximation appears to be most applicable for obtaining the collision matrix element for a many channel process in which the correlation of channels between levels are considered vanishingly small.

The approximation discussed in Appendix D has certain merit because of the simplification which is obtained while still retaining the interference between levels. This approximation however removes the correlation between all channels except the two channels in question however interference between levels remains. The effect of reducing the correlation between channels except the two in question is not fully known at present.

In conclusion, comparison of this method with the Thomas' method indicates similarity except for procedure. However calculations are more straightforward by the present method because sets of linear equations are avoided. This however does not imply that there is a decrease in the amount of necessary detailed calculations that are required. The amount of detailed calculation that is required for this method may be seen by referring to Appendix A for the case of 1, 2 or 3 level inversion.

References

1. E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947).
2. A. M. Lane and R. G. Thomas, Revs. Modern Phys. 30, 257 (1958).
3. H. S. Wilf, J. Soc. Indust. Appl. Math. 7, 149 (1959).

APPENDIX A

The expression for C^i for 3 cases are as follows:

(a) 1 level

$$C^2 = C^1 + A_{11}^1 C^1 \beta_1 \times \alpha_1 C^1$$

(b) 2 level

$$C^3 = C^1 + A_{11}^2 C^1 \beta_1 \times \alpha_1 C^1 + A_{12}^2 C^1 \beta_1 \times \alpha_2 C^1 \\ + A_{21}^2 C^1 \beta_2 \times \alpha_1 C^1 + A_{22}^2 C^1 \beta_2 \times \alpha_2 C^1$$

(c) 3 level

$$C^4 = C^1 + A_{11}^3 C^1 \beta_1 \times \alpha_1 C^1 + A_{12}^3 C^1 \beta_1 \times \alpha_2 C^1 + A_{13}^3 C^1 \beta_1 \times \alpha_3 C^1 \\ + A_{21}^3 C^1 \beta_2 \times \alpha_1 C^1 + A_{22}^3 C^1 \beta_2 \times \alpha_2 C^1 + A_{23}^3 C^1 \beta_2 \times \alpha_3 C^1 \\ + A_{31}^3 C^1 \beta_3 \times \alpha_1 C^1 + A_{32}^3 C^1 \beta_3 \times \alpha_2 C^1 + A_{33}^3 C^1 \beta_3 \times \alpha_3 C^1$$

The coefficients $A_{\mu\nu}^n$ are given in the next section. The expression for $\omega^{-1} U \omega^{-1}$ (Eqn. 22) take on the following form for the above 3 cases.

(a) 1 level ($n = 1$)

$$\omega^{-1} U_1 \omega^{-1} = (C^{1*})^{-1} C^1 + 2(C^2 - C^1) = (C^{1*})^{-1} C^1 + 2 A_{11}^1 C^1 \beta_1 \times \alpha_1 C^1$$

(b) 2 level ($n = 2$)

$$\omega^{-1} U_2 \omega^{-1} = (C^{1*})^{-1} C^1 + 2(C^3 - C^1) = (C^{1*})^{-1} C^1 + 2 \sum_{\mu\nu}^{22} A_{\mu\nu}^2 C^1 \beta_\mu \times \alpha_\nu C^1$$

(c) 3 level ($n = 3$)

$$\omega^{-1} U_3 \omega^{-1} = (C^{1*})^{-1} C^1 + 2(C^4 - C^1) = (C^{1*})^{-1} C^1 + 2 \sum_{\mu\nu}^{33} A_{\mu\nu}^3 C^1 \beta_\mu \times \alpha_\nu C^1$$

The expressions for $A_{\mu\nu}^n$ are as follows:

(a) 1 level ($n = 1$)

$$C^2 = C^1 + iA_1 C^1 \beta_1 \times \alpha_1 C^1$$

$$\text{where } A_1 = \frac{1}{1 - i(\alpha_1 C^1, \beta_1)}$$

where $(\alpha_1 C^1, \beta_1) \equiv$ scalar product

$$\text{where } A_{11}^1 = iA_1$$

(b) 2 level ($n = 2$)

$$C^3 = C^1 + iA_1 C^1 \beta_1 \times \alpha_1 C^1$$

$$+ iA_2 [iA_1 iA_1 (\alpha_2, C^1 \beta_1) (\alpha_1 C^1, \beta_2)] C^1 \beta_1 \times \alpha_1 C^1$$

$$+ iA_2 [iA_1 (\alpha_1 C^1, \beta_2)] C^1 \beta_1 \times \alpha_2 C^1$$

$$+ iA_2 [iA_1 (\alpha_2, C^1 \beta_1)] C^1 \beta_2 \times \alpha_1 C^1$$

$$+ iA_2 [1] C^1 \beta_2 \times \alpha_2 C^1$$

$$\text{where } A_2 = \frac{1}{1 - i(\alpha_2 C^2, \beta_2)}$$

where $(\alpha C, \beta) \equiv$ scalar product

The coefficients $A_{\mu\nu}^2$ are as follows:

$$A_{11}^2 = iA_1 + iA_2 [iA_1 iA_1 (\alpha_2 C^1, \beta_1) (\alpha_1 C^1, \beta_2)]$$

$$A_{12}^2 = iA_2 [iA_1 (\alpha_1 C^1, \beta_2)]$$

$$A_{21}^2 = iA_2 [iA_1 (\alpha_2 C^1, \beta_1)]$$

$$A_{22}^2 = iA_2$$

(c) 3 level (n = 3)

$$c^4 = c^1$$

$$+ iA_1 c^1 \beta_1 \times \alpha_1 c^1$$

$$+ iA_2 [iA_1 iA_1 (\alpha_1 c^1, \beta_2) (\alpha_2, c^1 \beta_1)] c^1 \beta_1 \times \alpha_1 c^1$$

$$+ iA_2 [iA_1 (\alpha_1 c^1, \beta_2)] c^1 \beta_1 \times \alpha_2 c^1$$

$$+ iA_2 [iA_1 (\alpha_2, c^1 \beta_1)] c^1 \beta_2 \times \alpha_1 c^1$$

$$+ iA_2 [1] c^1 \beta_2 \times \alpha_2 c^1$$

$$+ iA_3 [iA_1 iA_1 iA_2 iA_2 (\alpha_1 c^1, \beta_2) (\alpha_2, c^1 \beta_1) (\alpha_2 c^2, \beta_3) (\alpha_3, c^2 \beta_2)] c^1 \beta_1 \times \alpha_1 c^1$$

$$+ iA_3 [iA_1 iA_1 iA_2 (\alpha_1 c^1, \beta_2) (\alpha_2 c^2, \beta_3) (\alpha_3, c^1 \beta_1) + iA_1 iA_1 iA_2 (\alpha_1 c^1, \beta_3) (\alpha_2, c^1 \beta_1) (\alpha_3, c^2 \beta_2)] c^1 \beta_1 \times \alpha_1 c^1$$

$$+ iA_3 [iA_1 iA_1 (\alpha_3, c^1 \beta_1) (\alpha_1 c^1, \beta_3)] c^1 \beta_1 \times \alpha_1 c^1$$

$$+ iA_3 [iA_1 iA_2 iA_2 (\alpha_1 c^1, \beta_2) (\alpha_2 c^2, \beta_1) (\alpha_3, c^2 \beta_2) + iA_1 iA_2 (\alpha_1 c^1, \beta_3) (\alpha_3, c^2 \beta_2)] c^1 \beta_1 \times \alpha_2 c^1$$

$$+ iA_3 [iA_1 (\alpha_1 c^1, \beta_3) + iA_1 iA_2 (\alpha_1 c^1, \beta_2) (\alpha_2 c^2, \beta_3)] c^1 \beta_1 \times \alpha_3 c^1$$

$$+ iA_3 [iA_2 iA_2 (\alpha_2 c^2, \beta_3) (\alpha_3, c^2 \beta_2)] c^1 \beta_2 \times \alpha_2 c^1$$

$$+ iA_3 [iA_1 iA_2 (\alpha_2 c^2, \beta_3) (\alpha_3, c^1 \beta_1) + iA_1 iA_2 (\alpha_2, c^1 \beta_1) (\alpha_2 c^2, \beta_3) (\alpha_3, c^2 \beta_2)] c^1 \beta_2 \times \alpha_1 c^1$$

$$+ iA_3 [iA_2 (\alpha_2 c^2, \beta_3)] c^1 \beta_2 \times \alpha_3 c^1$$

$$+ iA_3[iA_2(\alpha_3, c^2\beta_2)]c^1\beta_3 \times \alpha_2 c^1$$

$$+ iA_3[iA_1(\alpha_3, c^1\beta_1) + iA_1 iA_2(\alpha_2, c^1\beta_1)(\alpha_3, c^2\beta_2)]c^1\beta_3 \times \alpha_1 c^1$$

$$+ iA_3[1]c^1\beta_3 \times \alpha_3 c^1$$

$$\text{where } A_1 = \frac{1}{1 - i(\alpha_1 c^1, \beta_1)}$$

$$\text{where } (\alpha_i c^j, \beta_k) = \text{scalar product}$$

$$A_2 = \frac{1}{1 - i(\alpha_2 c^2, \beta_2)}$$

$$A_3 = \frac{1}{1 - i(\alpha_3 c^3, \beta_3)}$$

The coefficients A_{uvw}^3 are as follows:

$$A_{11}^3 = iA_1$$

$$+ iA_2[iA_1 iA_1(\alpha_1 c^1, \beta_2)(\alpha_2, c^1\beta_1)]$$

$$+ iA_3[iA_1 iA_1 iA_2(\alpha_1 c^1, \beta_2)(\alpha_2, c^1\beta_1)(\alpha_2 c^2, \beta_3)(\alpha_3, c^2\beta_3)]$$

$$+ iA_3[iA_1 iA_1 iA_2(\alpha_1 c^1, \beta_2)(\alpha_2 c^2, \beta_3)(\alpha_3, c^1\beta_1)$$

$$+ iA_1 iA_1 iA_2(\alpha_1 c^1, \beta_3)(\alpha_2, c^1\beta_1)(\alpha_3, c^2\beta_2)]$$

$$+ iA_3[iA_1 iA_1(\alpha_3, c^1\beta_1)(\alpha_1 c^1, \beta_3)]$$

$$A_{12}^3 = iA_2[iA_1(\alpha_1 c^1, \beta_2)]$$

$$+ iA_3[iA_1 iA_2 iA_2(\alpha_1 c^1, \beta_2)(\alpha_2 c^2, \beta_2)(\alpha_3, c^2\beta_2)$$

$$+ iA_1 iA_2(\alpha_1 c^1, \beta_3)(\alpha_3, c^2\beta_2)]$$

$$A_{13}^3 = iA_3[iA_1(\alpha_1 c^1, \beta_3) + iA_2 iA_2(\alpha_1 c^1, \beta_2)(\alpha_2 c^2, \beta_3)]$$

$$A_{21}^3 = iA_2[iA_1(\alpha_2 c^1, \beta_1)]$$

$$+ iA_3[iA_1 iA_2(\alpha_2 c^2, \beta_3)(\alpha_3, c^1 \beta_1)]$$

$$+ iA_1 iA_2 iA_2(\alpha_2, c^1 \beta_1)(\alpha_2 c^2, \beta_3)(\alpha_3, c^2 \beta_2)]$$

$$A_{22}^3 = iA_2 + iA_3[iA_2 iA_2(\alpha_2 c^2, \beta_3)(\alpha_3, c^2 \beta_2)]$$

$$A_{23}^3 = iA_3[iA_2(\alpha_2 c^2, \beta_3)]$$

$$A_{31}^3 = iA_3[iA_1(\alpha_3, c^1 \beta_1) + iA_2 iA_2(\alpha_2, c^1 \beta_1)(\alpha_3, c^2 \beta_2)]$$

$$A_{32}^3 = iA_3[iA_2(\alpha_3, c^2 \beta_2)]$$

$$A_{33}^3 = iA_3$$

APPENDIX B

Single level approximation

The simplest approximation to make is to consider only the single level terms. To drop the cross terms $C^1_{\beta_{\mu}} \times \alpha_{\nu} C^1$ ($\mu \neq \nu$), all that is necessary to require is

$$(\alpha_{\nu} C^1, \beta_{\mu}) = 0 \quad \text{for } \mu \neq \nu$$

This is equivalent to stating that $\alpha_{\nu} C^1$ and β_{μ} are orthogonal vectors.

For the 3 level ($n = 3$) case, the single level approximation for

$$C^4 = C^1 + iA_1 C^1_{\beta_1} \times \alpha_1 C^1 + iA_2 C^1_{\beta_2} \times \alpha_2 C^1 + iA_3 C^1_{\beta_3} \times \alpha_3 C^1$$

$$\omega^{-1} U_3 \omega^{-1} = (C^{1*})^{-1} C^1 + 2(iA_1 C^1_{\beta_1} \times \alpha_1 C^1 + iA_2 C^1_{\beta_2} \times \alpha_2 C^1 + iA_3 C^1_{\beta_3} \times \alpha_3 C^1)$$

$$\text{where } A_1 = \frac{1}{1 - i(\alpha_1 C^1, \beta_1)}$$

$$A_2 = \frac{1}{1 - i(\alpha_2 C^1, \beta_2)}$$

$$A_3 = \frac{1}{1 - i(\alpha_3 C^1, \beta_3)}$$

APPENDIX C

Another approximation which can be made is to do the following: Suppose there is only one neutron and one fission channel and the remainder are capture channel. Then, a possible approximation would be:

Consider the case for $n = 3$. For reaction in which level interference is important write:

$$[U_3]_{12} = [(C^{1*})^{-1}C^1]_{12} + [2 \sum_{\mu\nu}^3 A_{\mu\nu}^3 C_{\beta_\mu}^1 \times \alpha_\nu C^1]_{12}$$

where 1, 2 refers to possibly (n,f) reactions.

For reactions in which level interference are considered negligible write:

$$[U_3]_{13} = [(C^{1*})^{-1}C^1]_{13} + [2 \sum_{\mu}^3 A_{\mu\mu}^3 C_{\beta_\mu}^1 \times \alpha_\mu C^1]_{13}$$

where 1, 3 refers to possibly to a (n, γ) reaction. This last expression is the single level approximation considered in Appendix B.

This method of approximation is essentially the one made by Reich and Moore¹ for the capture channels. The one difference is that the total width appears in the denominator of A_i instead of only the radiation width.

¹Reich, C. W. and Moore, M. S., Phys. Rev. 111, 929 (1958).

APPENDIX D

Another modification which reduces the number of terms in the coefficients $A_{\mu\nu}^n$ but which still retains the interference between levels is to do the following: In the text it is shown that the matrix C^{n+1} is developed from C^n and C^n is obtained from C^{n-1} , that is, the C^n is developed sequentially from C^1 ; as an example consider $n+1 = 4$

$$C^4 = C^1 + iA_1 C^1 \beta_1 \times \alpha_1 C^1 + iA_2 C^2 \beta_2 \times \alpha_2 C^2 + iA_3 C^3 \beta_3 \times \alpha_3 C^3$$

$$C^3 = C^1 + iA_1 C^1 \beta_1 \times \alpha_1 C^1 + iA_2 C^2 \beta_2 \times \alpha_2 C^2$$

$$C^2 = C^1 + iA_1 C^1 \beta_1 \times \alpha_1 C^1$$

The modification to make is to diagonalize the various C^n which go to make up C^{n+1} . The diagonalization is started with C^2 which is inserted in C^3 and then C^3 is made diagonal before being used in C^4 . Of course the C^2 used in C^4 is also diagonalized. In addition the coefficient

$$A_i = \frac{1}{1 - i(\alpha_i C^1, \beta_i)} \quad \text{instead of} \quad A_i = \frac{1}{1 - i(\alpha_i C^i, \beta_i)} \quad \text{is also used in}$$

this approximation.

The effect of diagonalizing the matrix C^i simplifies the expression $C^i \beta_i$ or $\alpha_i C^i$. Instead of a column or row vector

$$C^i \beta_i = \begin{bmatrix} C_{11}^i \beta_1^i + C_{12}^i \beta_2^i + C_{13}^i \beta_3^i + \dots \\ C_{21}^i \beta_1^i + C_{22}^i \beta_2^i + C_{23}^i \beta_3^i + \dots \\ C_{31}^i \beta_1^i + \dots \end{bmatrix}$$

The diagonalization reduces the preceding expression to

$$C^i \beta_i = \begin{bmatrix} C_{11}^i \beta_1^i \\ C_{22}^i \beta_2^i \\ C_{33}^i \beta_3^i \\ \vdots \end{bmatrix}$$

APPENDIX E

For a 2 level expansion in which the complex diagonal matrix C^1 is assumed to be a unit diagonal matrix

$$C^1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

and in which 3 channels are considered, neutron, fission, gamma; the expression for σ_{nf} becomes:

$$\begin{aligned} \sigma_{nf} &= 4\pi\chi_1^2 \left| [U_2]_{12} \right|^2 = 4\pi\chi_1^2 \left| [U_2]_{nf} \right|^2 \\ [U_2]_{12} &= [U_2]_{nf} = 2i \frac{\left[\frac{1}{2} \Gamma_{1n}^{\frac{1}{2}} \Gamma_{1f}^{\frac{1}{2}} \right]}{E_1 - E - i \frac{\Gamma_1}{2}} - 2i \frac{\left[\xi^2 \right] \left[\frac{1}{2} \Gamma_{1n}^{\frac{1}{2}} \Gamma_{1f}^{\frac{1}{2}} \right]}{(E_1 - E - i \frac{\Gamma_1}{2}) \left[(E_1 - E - i \frac{\Gamma_1}{2})(E_2 - E - i \frac{\Gamma_2}{2}) + \xi^2 \right]} \\ &\quad - \frac{2(\xi) \left[\frac{1}{2} \Gamma_{1n}^{\frac{1}{2}} \Gamma_{2f}^{\frac{1}{2}} + \frac{1}{2} \Gamma_{2n}^{\frac{1}{2}} \Gamma_{1f}^{\frac{1}{2}} \right]}{(E_1 - E - i \frac{\Gamma_1}{2})(E_2 - E - i \frac{\Gamma_2}{2}) + \xi^2} \\ &\quad + \frac{2i \left[(E_1 - E - i \frac{\Gamma_1}{2}) \frac{1}{2} \Gamma_{2n}^{\frac{1}{2}} \Gamma_{2f}^{\frac{1}{2}} \right]}{(E_1 - E - i \frac{\Gamma_1}{2})(E_2 - E - i \frac{\Gamma_2}{2}) + \xi^2} \end{aligned}$$

where

$$\Gamma_1 = \Gamma_{1n} + \Gamma_{1f} + \Gamma_{1\gamma} \quad \text{where} \quad \frac{\Gamma_{1n}}{\sqrt{2}} = k_n \frac{1}{2} \gamma_n^1 \quad \frac{\Gamma_{1f}}{\sqrt{2}} = k_f \frac{1}{2} \gamma_f^1 \quad \frac{\Gamma_{1\gamma}}{\sqrt{2}} = k_\gamma \frac{1}{2} \gamma_\gamma^1$$

$$\Gamma_2 = \Gamma_{2n} + \Gamma_{2f} + \Gamma_{2\gamma} \text{ where } \frac{\Gamma_{2n}^{\frac{1}{2}}}{\sqrt{2}} = k_n^{\frac{1}{2}} \gamma_n^2 \quad \frac{\Gamma_{2f}^{\frac{1}{2}}}{\sqrt{2}} = k_f^{\frac{1}{2}} \gamma_f^2 \frac{\Gamma_{2\gamma}^{\frac{1}{2}}}{\sqrt{2}} = k_\gamma^{\frac{1}{2}} \gamma_\gamma^2$$

$$k_n^{\frac{1}{2}} = \left(\frac{1}{\lambda}\right)^{\frac{1}{2}} \text{ varies with rel-energy between neutron and target}$$

$$k_f^{\frac{1}{2}} = k_\gamma^{\frac{1}{2}} = \text{constant, is assumed}$$

$$\xi = (\alpha_2, \alpha_1) = (\alpha_1 \alpha_2) = \left(\frac{\Gamma_{2n}^{\frac{1}{2}}}{\sqrt{2}} \frac{\Gamma_{1n}^{\frac{1}{2}}}{\sqrt{2}} + \frac{\Gamma_{2f}^{\frac{1}{2}}}{\sqrt{2}} \frac{\Gamma_{1f}^{\frac{1}{2}}}{\sqrt{2}} + \frac{\Gamma_{2\gamma}^{\frac{1}{2}}}{\sqrt{2}} \frac{\Gamma_{1\gamma}^{\frac{1}{2}}}{\sqrt{2}} \right)$$

AUG 12 1966

Errata for IDO-17142

A NOTE ON INVERTING THE R-MATRIX

Page 1, paragraph 1, line 2: insert "matrix" after "collision"

Page 6, next to last equation: insert plus sign after C^1

Page 7, tenth line up from the bottom: insert "of" after "number"

Page 8, line 3: insert phrase "is similar to that obtained by Thomas"
in place of "are alike".

Page 20, line 5: insert after " σ_{nf} ", the phrase "for the $\ell = 0$ case"

Page 20, line 6: in the expression for σ_{nf} delete the number 4.

Page 21, line 2: read " $k_n^{1/2} = (\frac{1}{\lambda})^{1/2}$ " instead of " $k_n^{1/2} = (\frac{1}{\lambda})^{1/2}$ ".

Page 21, line 3: read " $k_f^{1/2}$ and $k_\gamma^{1/2}$ are both assumed to be a
constant" instead of " $k_f^{1/2} = k_\gamma^{1/2} = \text{constant}$,
is assumed".

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