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A CONVENIENT METHOD FOR OBTAINING WEIBULL'S MODULUS, m

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A CONVENIENT METHOD FOR OBTAINING WEIBULL'S MODULUS, m^*

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The analysis of strength data on brittle materials must rely on statistical techniques because of the wide variation of data even under very well controlled test conditions. The theory proposed by Weibull^{1,2} has received much acceptance and when applicable provides a satisfactory treatment of the strength behavior of brittle materials by accounting for size effects and stress distribution effects, and by giving quantitative predictions, which agree with experiment, of the probability of failure as a function of stress. The treatment of Weibull's theory is given with much detail elsewhere^{1,2} and only those aspects necessary for our specific purpose will be given here.

The fundamental and general form of Weibull's theory is

$$S = e^{-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m dV} \quad (1)$$

where S is the probability of a specimen surviving the stress σ . The integration is carried out over that volume where $\sigma - \sigma_u$ is positive. The parameter σ_u is the stress below which the survival probability is one, i.e., no specimens fail below σ_u . The parameters σ_0 and m are constants. m is com-

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monly called the Weibull modulus and is an index of variability in the data. High values of m correspond to data having low variability and conversely. The parameter σ_0 is essentially a scaling factor.

The form of equation (1) is cumbersome for many practical problems and it is found that satisfactory agreement between the theory and experimental results are frequently obtained with $\sigma_u = 0$, giving

$$S = e^{-\int_V (\sigma/\sigma_0)^m dV} \quad (2)$$

This form is also much more convenient from an analytical standpoint as it allows closed form solution of many common stress distributions.

A convenient method of completely defining the parameters of equation (2) will now be given. The equation will be reexpressed in such a way that the integration need not be performed, and graphs of the new expression will be used to evaluate m .

The stress $\sigma(V)$ (since it will generally vary within the specimen) is replaced by $\bar{\sigma}f(V)$ where $\bar{\sigma}$ is a reference stress, which we take as the maximum tensile stress, and $f(V)$ is an appropriate geometric function, giving:

$$S = e^{-(\bar{\sigma}/\sigma_0)^m \int_V [f(V)]^m dV} \quad (3)$$

If $\bar{\sigma}$ is set to the mean value, $\bar{\sigma}_a$, and the corresponding value of S is S_a , then

$$\frac{1}{\bar{\sigma}_a^m} \ln \frac{1}{S_a} = \frac{1}{\sigma_0^m} \int_V [f(V)]^m dV \quad (4)$$

For a given material and specimen geometry, the right side is constant.

If equations (3) and (4) are combined,

$$S = e^{\ln S_a (\bar{\sigma}/\bar{\sigma}_a)^m} \quad (5)$$

Equation (5) provides interesting insight into the strength behavior. The probability distribution may be obtained from data of various test specimens of the same material and from any stress distribution so long as the data is normalized to the mean value of each series of tests. Furthermore, the survival probability at the mean value must be independent of absolute value of the mean stress and hence be a function of m only. This relation between S_a and m will be obtained below.

It is evident that the mean value of strength is

$$\bar{\sigma}_a = \int_0^{\infty} S \, d\sigma$$

or, from (5),

$$\bar{\sigma}_a = \int_0^{\infty} e^{\ln S_a (\bar{\sigma}/\bar{\sigma}_a)^m} d\sigma$$

By setting

$$w = \ln S_a (\bar{\sigma}/\bar{\sigma}_a)^m$$

we get

$$\bar{\sigma}_a = \frac{\bar{\sigma}_a}{m \left(\ln \frac{1}{S_a} \right)^{1/m}} \int_0^{\infty} e^{-w} w^{\frac{1}{m}-1} dw$$

and, since the integral is $\Gamma(1/m)$,

$$\ln \frac{1}{S_a} = \left[\frac{\Gamma(1/m)}{m} \right]^m = \left[\Gamma\left(\frac{1}{m} + 1\right) \right]^m \quad (6)$$

The relation of equation (6) may be generalized to the survival probability at any given stress. If this stress is expressed as $k\bar{\sigma}_a$ (k may be less or greater than one), and the corresponding survival fraction as S_k , then equation (7) is obtained:

$$S_k = (S_a)^{k^m} \quad (7)$$

Equations (6) and (7) are used to plot Fig. 1. This graph may be used directly to obtain m from data of any series of tests. Furthermore, various portions of the data may be used to make separate evaluations of m . Regions of divergence from the distribution are therefore easily defined.

The method is to normalize the data, dividing by the mean strength, and enter the results into the frequency column of a table such as shown in Table 1. The survival fractions (cumulative distribution) are then computed and entered.

Table 1*

σ / σ_a	Frequency	Survival Fraction
.		.
.		.
.		.
0.91 - 0.95	. . .	0.69
0.96 - 1.0	. . .	0.54
1.01 - 1.05	. . .	etc.
.		.
.		.
.		.

* The entries are examples only.

The entries in the table constitute a differential and integral distribution of the data and give direct entry to Fig. 1 for any number of separate determinations of m . Ideally the locus of points in Fig. 1 will be a vertical line, showing that all tested portions of the data fit a single value of m . On the other hand the regions of deviation are manifested, where varying values of m are obtained, and the errors in the predicted survival probability, incurred by approximation, may be quantitatively evaluated.

For completeness and for purposes of comparison, one of the commonly used methods of obtaining m , as suggested by Weibull, will also be presented.

Following Weibull, we require that

$$\int (\sigma/\sigma_0)^m dV$$

from equation (2) be integrable to the form

$$CV(\sigma/\sigma_0)^m$$

This result obtains when the test is in uniform tension.

By taking logs twice of equation (2), obtain:

$$\ln \ln \frac{1}{S} = m \ln \sigma + \ln(CV/\sigma_0) \quad (8)$$

The condition on the integral is, however, superfluous as may be seen by combining equations (5) and (6), which gives:

$$S = e^{-\left[\frac{\sigma}{\sigma_a} \Gamma\left(\frac{1}{m} + 1\right)\right]^m} \quad (9)$$

Taking logs twice of equation (9) yields:

$$\ln \ln \frac{1}{S} = m \ln \frac{\sigma}{\sigma_a} + m \ln \Gamma\left(\frac{1}{m} + 1\right) \quad (10)$$

Equation (10), when plotted on appropriate coordinates yields a straight line of slope m if the data fit the assumed distribution function. If the data do not produce a straight line, the errors in predicting the survival probability are not immediately evident as they are on Fig. 1.

The author has found it convenient to use the graphs of Figs. 2 and 3 to find m when applying either equation (8) or (10).

Summary

a) The simplified form of the Weibull distribution function

$$S = e^{-\int_V (\sigma/\sigma_0)^m dV}$$

can, in general, be reexpressed as

$$S = e^{-\left[\frac{\sigma}{\sigma_a} \Gamma\left(\frac{1}{m} + 1\right)\right]^m}$$

where σ_a is the mean value of strength.

b) The relation between the survival probability at any stress $k\sigma_a$ (k may be greater or less than one) and m is plotted (Fig. 1) and provides a convenient method of finding m .

c) Figure 1 provides a perspective of survival probabilities for any range of m and can be used to directly test approximations of data in terms of survival probabilities.

d) Figure 1 allows several separate determinations of m which examine various portions of the data and locate stress regions within which deviation from the assumed distribution occur.

References

- 1 W. Weibull, "The Phenomenon of Rupture in Solids," Ingeniors vetenskaps Akademien, Stockholm Handlingar, NR 153, 1939 (in English).
- 2 W. Weibull, "A Statistical Theory of Strength," Ingeniors vetenskaps Akademien, Stockholm Handlingar, NR 151, 1938 (in English).

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Fig. 1. Survival fraction vs m for various values of the normalized stress k , where $k = \sigma/\sigma_a$.

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Fig. 2. Survival probability S vs $\ln \frac{1}{S}$.

Fig. 3. Survival probability as $\ln \frac{1}{S}$ vs normalized stress.