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ABSTRACT

A method is described for the numerical solution of the non-linear magnetostatic equation for the vector potential over an arbitrary polygonal region using a distorted triangle mesh. The equations are solved by successive overrelaxation with automatic optimization of the overrelaxation parameter. Based on this method, a general purpose two-dimensional magnet code has been developed which will calculate magnetic fields in any plane arrangement of air, iron, and currents, with speed and accuracy comparable to that obtained with rectangular meshes.

I. INTRODUCTION

The use of an irregular triangle mesh for the numerical solution of partial differential equations in two dimensions was first proposed by Courant[1] in 1943. In 1953 MacNeal[2] employed it for the solution of the Poisson equation using an electrical mesh analogue computer, and in 1958 Leith[3] independently reinvented the method for a digital computer and used it to solve parabolic equations. Similar methods have been used by others.[4,5,6,7] This work, a revision of an earlier report,[8] is based on that of Leith.

In the following sections the basic assumptions are stated, the difference equations are derived, numerical methods of solution are discussed, and some examples of results are given.

II. DESCRIPTION OF THE METHOD

The equation to be solved is the generalized Poisson equation

$$\vec{\nabla}\cdot(\lambda\vec{\nabla}\phi) + S = 0 \qquad (1)$$

over a region R where λ is a function of the rectangular coordinates x,y and may also depend on ϕ, and S is a prescribed function of x,y. If λ depends on ϕ, it is considered to be a function of x,y when performing the differentiations indicated in (1); for example, if $\lambda = \lambda(\phi)$, then $(\lambda\phi_x)_x = \lambda'\phi_x^2 + \lambda\phi_{xx}$ and so on.

The boundary conditions are assumed to be of the form $a\phi + b\frac{\partial\phi}{\partial n} = 0$ where $\frac{\partial\phi}{\partial n}$ is the normal derivative and a,b are prescribed constants that may take on different values (not both zero) over different portions of the boundary. The dependent variable ϕ is assumed to be continuous. The quantities λ,S are assumed to be continuous over sub-regions of R, so that there may be internal interfaces at which λ and S are discontinuous. At such interfaces, $\lambda\frac{\partial\phi}{\partial n}$ is assumed to be continuous.

The basic assumptions of the finite difference method are:

(1) the boundaries and interfaces of the region R are approximated by straight line segments

(2) the region is triangulated

(3) the values of ϕ are defined at triangle vertices, and ϕ is assumed to vary linearly over each triangle

(4) λ and S are assumed to be constant over each triangle.

The type of triangulation used here is topologically regular; that is, it is topologically equivalent to an equilateral triangle array in which six triangles meet at every interior mesh point. The triangulation is carried out by an auxiliary calculation involving the numerical solution of another boundary value problem (see Appendix).

Since any polygonal region can be triangulated, the method can be applied to regions of any shape, and will produce a mesh in which boundaries and interfaces lie entirely on mesh lines. This causes a considerable simplification in the finite difference equations and in the specification of boundary conditions. It also permits a linear approximation to the dependent variable, and thereby makes it easier to derive the finite difference equations, to which we now turn.

III. DERIVATION OF THE DIFFERENCE EQUATIONS

Two derivations of the same finite difference equations, based on the assumptions of section II, will now be given, in order to establish different properties of the equations.

(1) Integral derivation [3]

Instead of equation (1), let us consider the non-linear diffusion equation

$$c \frac{\partial \phi}{\partial t} = \vec{\nabla} \cdot (\lambda \vec{\nabla} \phi) + S \qquad (2)$$

where the positive coefficient c, like λ, may be a function of ϕ. For a steady state, (2) reduces to (1).

Consider an interior mesh point in a triangle mesh in which the assumptions of section II hold. Associated with the primary triangle mesh we define a secondary mesh of 12-sided figures whose vertices are alternately the centroids of the six adjacent triangles and the midpoints of the six adjacent sides. This is shown in Fig. 1, in which a single such figure is shaded. The secondary mesh element surrounding a given vertex includes one-third of the area of each of the six primary mesh triangles sharing that vertex, so that each triangle of area A is divided into three equal quadrilaterals of area $a = \frac{A}{3}$.

Consider the triangle $i + 1/2$ defined by the two side vectors $\vec{s}_i$, $\vec{s}_{i+1}$, with values ϕ_i, ϕ_{i+1}, ϕ at the respective vertices as shown in Fig. 2. Since ϕ is assumed to be a linear function of position, each such triangle has a vector $\vec{\nabla}\phi_{i+1/2}$ associated with it, which satisfies the equation

$$\phi_j = \phi + \vec{s}_j \cdot \vec{\nabla}\phi_{i+1/2} \qquad j = i,\ i+1 \tag{3}$$

and is given by

$$\vec{\nabla}\phi_{i+1/2} = \frac{(\phi_i - \phi)\not{s}_{i+1} - (\phi_{i+1} - \phi)\not{s}_i}{\vec{s}_i \cdot \not{s}_{i+1}} \tag{4}$$

where the vector $\not{s}$ represents the vector $\vec{s}$ rotated clockwise by an angle $\pi/2$.

Within each triangle the flux of the diffusing quantity is given by

$$\vec{F}_{i+1/2} = -\lambda_{i+1/2} \vec{\nabla}\phi_{i+1/2} \ .$$

The conservation law can be expressed through Gauss' theorem by equating the surface integral of the left side of Eq. (2) over the secondary mesh element to the integral of the normal component of $\vec{F}$ over the boundary of the secondary mesh element, added to the surface integral of S.

The flux contribution $R_{i+1/2}$ from the triangle i+1/2 shown in Fig. 2 to the rate of change R in the secondary mesh element is

$$\begin{aligned} R_{i+1/2} &= \vec{F}_{i+1/2} \cdot (\not{m}_1 + \not{m}_2) \\ &= \vec{F}_{i+1/2} \cdot \left(\frac{\not{s}_{i+1}}{2} - \frac{\not{s}_i}{2} \right) \\ &= \frac{1}{2} \vec{F}_{i+1/2} \cdot (\not{s}_{i+1} - \not{s}_i) \ . \end{aligned}$$

Summing around the vertex and using the central vertex value ϕ as the average value over the dodecagon, we obtain

$$\sum_{i=1}^{6} c_{i+1/2} \frac{\Delta\phi_{i+1/2}}{\Delta t} a_{i+1/2} \doteq \frac{\Delta\phi}{\Delta t} \sum_{i=1}^{6} c_{i+1/2} a_{i+1/2} = \sum_{i=1}^{6} \vec{F}_{i+1/2} \cdot \frac{1}{2} (\not{s}_{i+1} - \not{s}_i)$$

$$+ \sum_{i=1}^{6} S_{i+1/2} a_{i+1/2}$$

or

$$\Delta\phi = \frac{\Delta t \left[\sum_{i=1}^{6} \vec{F}_{i+1/2} \cdot \frac{1}{2} (\not{s}_{i+1} - \not{s}_i) + S \right]}{G} \ , \tag{5}$$

where

$$G = \sum_{i=1}^{6} c_{i+1/2} a_{i+1/2}$$

$$S = \sum_{i=1}^{6} S_{i+1/2} a_{i+1/2} \cdot \tag{6}$$

Using (4), and letting $\psi_i = \phi_i - \phi$, we can express the flux sum in Eq. (5) as

$$F = \frac{1}{2} \sum_i \frac{\lambda_{i+1/2}(\psi_i \vec{\mathscr{s}}_{i+1} - \psi_{i+1} \vec{\mathscr{s}}_i) \cdot (\vec{\mathscr{s}}_{i+1} - \vec{\mathscr{s}}_i)}{\vec{\mathscr{s}}_i \cdot \vec{s}_{i+1}}$$

$$= \frac{1}{2} \sum_i \frac{\lambda_{i+1/2}\, \psi_i \vec{s}_{i+1} \cdot (\vec{s}_{i+1} - \vec{s}_i) + \psi_{i+1} \vec{s}_i \cdot (\vec{s}_i - \vec{s}_{i+1})}{\vec{\mathscr{s}}_i \cdot \vec{s}_{i+1}}$$

where we have made use of the relations

$$\vec{\mathscr{s}} \cdot \vec{\mathscr{s}} = \vec{u} \cdot \vec{v}$$

$$\vec{u} \cdot \vec{\mathscr{s}} = - \vec{\mathscr{s}} \cdot \vec{v} \, .$$

Since the sum is cyclic, we can reduce the index by one in the second term, obtaining

$$F = \frac{1}{2} \sum_i \left[\frac{\lambda_{i+1/2}}{\vec{\mathscr{s}}_i \cdot \vec{s}_{i+1}} \vec{s}_{i+1} \cdot (\vec{s}_{i+1} - \vec{s}_i) + \frac{\lambda_{i-1/2}}{\vec{\mathscr{s}}_{i-1} \cdot \vec{s}_i} \vec{s}_{i-1} \cdot (\vec{s}_{i-1} - \vec{s}_i) \right] \psi_i \tag{7}$$

$$= \sum_i w_i (\phi_i - \phi) \, .$$

The coefficient of ψ_i in (7) is called the coupling coefficient for the line joining the vertex i and the center. It depends only on the nature of the two triangles having this as a common side, and can be written

$$w_i = \frac{1}{2} \left(\lambda_{i+1/2} \cot \theta_{i+1/2} + \lambda_{i-1/2} \cot \theta_{i-1/2} \right) . \tag{8}$$

where the angles $\theta_{i \pm 1/2}$ lie opposite the side i.

We note that the coupling between two points x_1, y_1 and x_2, y_2 is symmetric, so that

$$w_{12} = w_{21} \, . \tag{9}$$

Moreover, w_i can be positive, negative, or zero, but the sum of the couplings around a given point is positive:

$$\sum_{i=1}^{6} w_i = \frac{1}{4} \sum_i \frac{\lambda_{i+1/2}}{A_{i+1/2}} (\vec{s}_{i+1} - \vec{s}_i)^2 , \tag{10}$$

since $\vec{s}_i \cdot \vec{s}_{i+1} = 2A_{i+1/2}$.

Thus, Eq. (2) can be written in finite difference approximation

$$\frac{\Delta\phi}{\Delta t} = \frac{1}{G}\left[\sum_i^6 w_i(\phi_i - \phi) + S\right] \tag{11}$$

so that for a steady state the finite difference analogue of (1) is

$$\sum_i w_i(\phi_i - \phi) + S = 0 \tag{12}$$

The continuity properties of the finite difference solution are: (a) $(\vec{\nabla}\phi)_t$ continuous, (b) $(\lambda\vec{\nabla}\phi)_n$ continuous, where t and n refer to tangential and normal components respectively. Statement (a) follows directly from the expression (4) for $\vec{\nabla}\phi$. Statement (b) is a consequence of our derivation by means of Gauss' theorem together with (7) which shows that the normal components of the fluxes have been replaced by mesh currents $w_i(\phi_i - \phi)$ flowing along mesh lines. Conservation of these currents is guaranteed by (9).

Boundary points are treated in the same manner as interior points except that the coefficient λ of material outside the boundary is set equal to zero. If the outward normal gradient $\partial\phi/\partial n$ is prescribed at a boundary, we add an external current I_j at each surface point j given by

$$I_j = \left(\frac{\partial\phi}{\partial n}\right)_j \frac{1}{2} (\lambda_{j-1/2} s_{j-1/2} + \lambda_{j+1/2} s_{j+1/2})$$

where $\frac{1}{2}(s_{j-1/2} + s_{i+1/2})$ is the length of boundary associated with the boundary point and $\lambda_{j\pm1/2}$ are the coefficients of the associated triangles.

(2) Variational derivation[4, 5, 8]

Consider the integral

$$I(\phi) = \frac{1}{2}\iint_R [\lambda(\vec{\nabla}\phi)^2 - 2S\phi]dxdy \tag{13}$$

where ϕ,λ and S satisfy the conditions of section II. Using a restricted variation[15] in which λ is held fixed, equation (1) is just the Euler equation[9] for (13), so that $I(\phi)$ will be minimized if ϕ satisfies (1).

Given a triangulation of the region over which ϕ is to be found, we can derive the finite difference equations from (13) by an adaption of the Ritz variational method. Let $\alpha(x,y)$ be a so-called pyramid function[6] which takes on the value 1 at the mesh point x,y, the value zero at the nearest neighbor mesh points, and varies linearly with position. Then if $u_i(x_i,y_i)$ is the value of u at the mesh point x_i,y_i

$$u = \sum_{\text{all } i} \alpha_i(x_i,y_i)u_i(x_i,y_i) \tag{14}$$

is a continuous, piecewise linear function which takes on the values u(x,y) at mesh points and satisfies the assumptions of section II. Substituting u for ϕ in (13), we minimize I(u) by setting $\frac{\partial I}{\partial u} = 0$ at each mesh point. This gives us a set of simultaneous equations for the unknown u_i, one equation for each mesh point.

From the first term in (13) we find

$$\frac{1}{2}\frac{\partial}{\partial u}(\vec{\nabla}u)^2 = \vec{\nabla}\alpha \cdot \vec{\nabla}u \ .$$

Making use of (4) and

$$\vec{\nabla}\alpha_{i+1/2} = \frac{1}{2A_{i+1/2}}(\vec{s}_i - \vec{s}_{i+1})$$

we get as the contribution to $\frac{\partial I}{\partial u}$ from the first term

$$\left(\frac{\partial I}{\partial u}\right)_1 = \sum_i \iint (\lambda \vec{\nabla}\alpha \cdot \vec{\nabla}u)_{i+1/2} dxdy = - \sum_{i=1}^{6} w_i (u_i - u)$$

where the w_i are the same coupling coefficients as before, and the last sum is over the nearest neighbors of a given mesh point.

From the second term in (13) we obtain

$$\left(\frac{\partial I}{\partial u}\right)_2 = - \sum_i S_{i+1/2} \iint \alpha_i dxdy = - \frac{1}{3} \sum_{i=1}^{6} S_{i+1/2} A_{i+1/2} \qquad (15)$$

Setting $\left(\frac{\partial I}{\partial u}\right)_1 + \left(\frac{\partial I}{\partial u}\right)_2 = 0$ we finally get

$$\sum_i w_i (u_i - u) + \frac{1}{3} \sum_i S_{i+1/2} A_{i+1/2} = 0 \qquad (16)$$

as before (Eq. (12)).

We note from (15) that the assignment of one-third of the area of a triangle to each source density $S_{i+1/2}$ is the consequence of our assumption of linearity for u, rather than appearing as a result of an arbitrary partitioning of each triangle as in the previous derivation.

From (13) we see that the matrix corresponding to the couplings w_i is positive definite, since the first term in (13) is > 0 when $\phi = u$. Since the matrix is symmetric as well, we would have the necessary and sufficient conditions for the convergence of the method of successive overrelaxation applied to the equations (16) if they were linear: that is, if the w_i were independent of u. In the next section we discuss the solution of (16) and the problems created by the non-linearity of the w_i.

IV. NUMERICAL SOLUTION OF THE DIFFERENCE EQUATIONS

We make use of the well-known iterative method of successive overrelaxation[10] to obtain a numerical solution of the equations (16). Two versions have been considered, which differ in the manner in which they deal with the non-linearity of the equations.

(1) Linearized overrelaxation

Let us first assume that the problem is linear, so that the method of successive overrelaxation will converge. Solving (16) for u at a given mesh point, we get for the (n+1)st iteration

$$u^{n+1} = \frac{\sum_i w_i u_i^n + S}{\sum_i w_i} \tag{17}$$

where the denominator is positive by (10). Introducing the overrelaxation parameter $\omega (0 < \omega < 2)$ we have

$$u^{n+1} = u^n + \omega\left(\frac{\sum_i w_i u_i^{n,n+1} + S}{\sum_i w_i} - u^n\right)$$

or

$$u^{n+1} = u^n + \frac{\omega}{\sum_i w_i}\left[\sum_i w_i (u_i^{n,n+1} - u^n) + S\right] \tag{18}$$

The nearest neighbor values $u_i^{n,n+1}$ represent u_i^{n+1} if it has already been calculated, otherwise u_i^n.

In reality, λ is often a function of u or of its derivatives, at least in parts of the region, so that the w_i are functions of u. Depending on the rate of change of λ with u, the equations (18) can become unstable.

Stability can be regained by underrelaxing the w_i and by using a smaller value of ω in non-linear regions. If w_i^{new} is a newly-calculated value, we let

$$w_i^{n+1} = \rho w_i^{new} + (1-\rho) w_i^n \tag{19}$$

where ρ is a positive fraction less than one. The value of ρ is obtained by trial; it may be less than 0.1 (see below).

The value of ω used in non-linear regions is usually close to one, for reasons of stability. In linear regions, however it is important to choose ω to optimize the convergence rate η, defined as

$$\eta^n = \sqrt{\frac{\sum_i (u_i^{n+1} - u_i^n)^2}{\sum_i (u_i^n - u_i^{n-1})^2}} \tag{20}$$

summed over the whole mesh. According to the theory of the method of successive overrelaxation, the optimum value ω_{opt} is given by

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\lambda^2}} \tag{21}$$

where, for a given ω and the resulting η, we can obtain λ from the relation

$$\lambda = \frac{\omega+\eta-1}{\omega\sqrt{\eta}} \tag{22}$$

We have been able to combine (20), (21), and (22) into a satisfactory automatic scheme for optimizing ω by recalculating it every cycle in the following way. Given ω^n and η^n, we have

$$\lambda = \frac{\omega^n+\eta^n-1}{\omega^n\sqrt{\eta^n}}$$

$$\omega'_{opt} = \frac{2}{1+\sqrt{1-\lambda^2}} - \omega_0 \tag{23}$$

$$\omega^{n+1} = \beta\,\omega'_{opt} + (1-\beta)\omega^n$$

The constant $\omega_0 \approx .01$ is useful in non-linear problems to prevent ω from becoming too large. The constant $\beta \approx .05$ underrelaxes ω so that its changes do not appreciably perturb η. Occasionally η may be >1; we then hold ω constant until η has been < 1 for a certain number ($\sim$5-15) of cycles, after which we resume the automatic optimization.

The dimensionless quantity

$$\varepsilon = \sqrt{\frac{\sum_i (u_i^{n+1}-u_i^n)^2}{\sum_i (u_i^{n+1})^2}},$$

summed over all i, is used to test for convergence. We require $\varepsilon < \varepsilon_0$, where usually $\varepsilon_0 = 10^{-6}$ or 10^{-7}.

(2) Non-linear overrelaxation[11]

The set of simultaneous equations (16)

$$F(u_1,u_2 \ldots) = \sum_i w_i(u_i-u) + S = 0$$

can be solved by iteration using Newton's method, which explicitly takes account of the non-linearity. Given an estimate $F^n = F(u_1^n,u_2^n, \ldots)$, we can improve it by writing, at a vertex $u(x,y)$,

$$F^{n+1} = F^n + (u^{n+1}-u^n)\left(\frac{\partial F}{\partial u}\right)^n + \sum_{i=1}^{6} (u_i^{n+1}-u_i^n)\left(\frac{\partial F}{\partial u_i}\right)^n + \ldots = 0$$

Neglecting off-diagonal terms we get

$$u^{n+1} = u^n - \frac{F^n}{\left(\frac{\partial F}{\partial u}\right)^n}, \qquad \left(\left(\frac{\partial F}{\partial u}\right)^n \neq 0\right)$$

and introducing the overrelaxation parameter ω in the usual way to accelerate convergence we have

$$u^{n+1} = u^n - \omega \frac{F^n}{\left(\frac{\partial F}{\partial u}\right)^n} \tag{24}$$

For the equations (16),

$$\frac{\partial F}{\partial u} = -\sum_i w_i + \sum_i \left(\frac{\partial w_i}{\partial u}\right)(u_i-u) \tag{25}$$

The method of non-linear overrelaxation is based on immediate replacement of new values u_i^{n+1} for u_i^n in the terms F^n and $\left(\frac{\partial F}{\partial u}\right)^n$ in (24). Thus we have

$$\begin{aligned} u_1^{n+1} &= u_1^n - \omega \frac{F(u_1^n,u_2^n, \ldots)}{\frac{\partial F}{\partial u}(u_1^n,u_2^n, \ldots)} \\ u_2^{n+1} &= u_2^n - \omega \frac{F(u_1^{n+1},u_2^n, \ldots)}{\frac{\partial F}{\partial u}(u_1^{n+1},u_2^n, \ldots)} \end{aligned} \tag{26}$$

$$u_3^{n+1} = u_3^n - \omega \frac{F(u_1^{n+1}, u_2^{n+1}, u_3^n, \ldots)}{\frac{\partial F}{\partial u}(u_1^{n+1}, u_2^{n+1}, u_3^n, \ldots)} \qquad (26)$$
$$\vdots \qquad \text{(con't.)}$$

Using (25), (24) becomes

$$u^{n+1} = u^n + \frac{\omega}{\sum_i \left[w_i + \frac{\partial w_i}{\partial u}(u-u_i)\right]^{n,n+1}} \left[\sum_i w_i^{n,n+1}(u_i^{n,n+1} - u^n) + S\right] \qquad (27)$$

For magnetostatic problems it can be shown[12, 13] that the added term in the denominator of (27) is positive, so that its effect is equivalent to reducing the overrelaxation parameter in non-linear regions, thus stabilizing the equations without the necessity of underrelaxing the w_i. Concus has also found[12] that equations (26) converge more rapidly than the linearized equations.

For the linear case, $\frac{\partial w_i}{\partial u} = 0$ and equations (26) or (27) reduce to (18).

V. APPLICATION TO MAGNETOSTATIC PROBLEMS

We now show that plane magnetostatic problems can be put in the form of equation (1). Consider an arbitrary distribution of infinite straight parallel conductors carrying constant currents parallel to the z axis. The magnetic field $\vec{H}(x,y)$ and the magnetic induction $\vec{B}(x,y)$ have components only in the x,y plane while the current density $\vec{J}$ and magnetic vector potential $\vec{A}$ have only z components which we label simply $J(x,y)$ and $A(x,y)$.

From the relations

$$\vec{B} = \mu\vec{H} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{H} = 4\pi\vec{J}$$

where $\mu(x,y,|\vec{B}|)$ is the magnetic permeability, we obtain

$$\vec{\nabla} \times \left(\frac{1}{\mu}\vec{\nabla} \times \vec{A}\right) = 4\pi\vec{J} \qquad (28)$$

Because of the single-component nature of $\vec{A}$ and $\vec{J}$, (28) reduces to

$$\vec{\nabla} \cdot (\gamma \vec{\nabla} A) = -4\pi j \tag{29}$$

where $\gamma \equiv \mu^{-1}$.

The dependence of γ on $|\vec{B}|$ can be expressed by considering it to be a function of B^2. Then equation (29) written out in full becomes

$$(\gamma+2\gamma' A_x^2)A_{xx} + 4\gamma' A_x A_y A_{xy} + (\gamma+2\gamma' A_y^2)A_{yy} + 4\pi j = 0 \ .$$

Equation (29) is equivalent to (1) with the current density playing the role of source term. Its finite difference approximation in a triangle mesh is therefore (16):

$$\sum_i w_i (A_i - A) + 4\pi I = 0 \tag{30}$$

where A_i, A now stand for the vector potential, and

$$I = \sum_{i=1}^{6} j_{i+1/2} a_{i+1/2} \tag{31}$$

is the total current through the secondary mesh dodecagon surrounding the vertex x,y at which A is defined. The coefficients w_i in (30) are calculated by (8) using γ in place of λ.

The boundary conditions for the magnetic field require that at an interface the normal component of $\vec{B}$, $\vec{B}_n$, and the tangential component of $\vec{H}$, $\vec{H}_t$, be continuous. Because $\vec{A}$ has only a z component, its gradient and curl are equal in magnitude and orthogonal to each other. Thus

$$\vec{H}_t = (\gamma \vec{B})_t = (\gamma \vec{\nabla} \times \vec{A})_t = (\gamma \vec{\nabla} A)_n$$

and

$$\vec{B}_n = (\vec{\nabla} \times \vec{A})_n = (\vec{\nabla} A)_t$$

which we have already shown to be continuous (section III).

On external boundaries we assume a condition of no leakage of magnetic flux, so that A=0. For magnets that are symmetric about a median plane, we need to calculate only one-half of the magnet and set the normal derivative of A equal to zero on the median plane. We can easily accomplish this by setting equal to zero all

external coupling coefficients on the median plane. For quadrupole magnets the same procedure can be followed on two boundaries.

VI. RESULTS

A code (named TRIM) has been written for the IBM 7094 which uses the equations developed above (with linearized overrelaxation) to solve plane magnetostatic problems. Starting values of the vector potential are taken to be zero everywhere. In Figures 3-16 we show some results obtained by the TRIM code for a CERN proton cynchroton C-magnet and for an H-magnet. Equipotentials (which are also lines of force) have been drawn in by linear interpolation.

The principal advantages of TRIM are simplicity and generality. Every mesh point is treated in the same way, so that the code is quite short. Any triangle can carry a current of any magnitude and can be composed of air, iron, or other materials, so that material interfaces may occur anywhere in the mesh. Permeabilities are stored as tables of $\gamma(B^2)$, and space is provided for several different kinds of iron in the same problem. It has proved possible to calculate a variety of two-dimensional magnets, such as C- and H-magnets with and without median plane symmetry, quadrupole, and sextupole magnets. With a 32768 word memory, the calculation is memory-contained for meshes up to 1600 points (3200 triangles).

Using linearized overrelaxation, the calculating time with an IBM 7094 is 3 milliseconds per mesh point per cycle for points in iron, and 0.6 ms for points not in iron. The optimized overrelaxation parameter usually lies between 1.90 and 1.96. The number of cycles required for convergence in a 40x40 mesh usually lies between 200 and 800; some problems may require somewhat more. Thus typical finite-mu problems with 1600 mesh points take 15-30 minutes to converge; infinite-mu problems run about five times faster. Non-linear overrelaxation would take two to three times more calculating time (in iron) but might reduce the number of iterations substantially.

Extreme distortion of the mesh has been found to slow up convergence considerably in one magnet, but so far this problem has been encountered only once, and was eliminated by rezoning. In some problems, a relatively long period is required for the vector potential to converge in the iron. Use of a perimeter line integral to accelerate convergence in such problems is being considered.

VII. ACCURACY OF THE NUMERICAL SOLUTION

As in any magnetostatic code based on (29), we are interested not in the solution itself, but in its derivatives, particularly the first and second derivatives, which are the components of the magnetic field and their gradients. We have found that, by requiring regular zoning in the region where the derivatives are to be obtained, we are able to get about 1% accuracy in both first and second derivatives in a 40x40 mesh, calculating second derivatives by using values at three adjacent collinear mesh points. This accuracy is comparable to that obtainable with rectangular meshes using about the same number of zones. A method for constructing a smooth interpolating polynomial in two dimensions to approximate the solution for purposes of numerical differentiation is now being developed which may make it possible to relax these zoning restrictions.

Use of an interpolating polynomial is equivalent to adding higher-order terms while retaining the simplicity of the difference equations. A related technique which might be used to increase the accuracy is the method of difference corrections[17], in which a higher order correction term based on the converged numerical solution is added to the difference equations at each mesh point and the equations are then solved again. Application of this method to a triangle mesh has not yet been investigated.

In the absence of exact solutions or experimental measurements, it would be desirable to have a priori estimates of the accuracy of the results based on the size and shape of the triangle zones and on the method of numerical differentiation

applied to the converged solution. At present we make use instead of a posteriori estimates based on observing the effect of varying the zone spacing and shape.

Mean square convergence of the numerical solution to the solution of the differential equation has been proved for certain special meshes[4,14] but not yet for an arbitrary triangle mesh. Failure of the difference equations to converge in general to the differential equation at a point has also been shown[2,4]. Since the method is based on integration over a linear approximating function rather than on a Taylor's series expansion, this behavior is perhaps to be expected. This error, which appears in the form of too large or too small an area of the secondary mesh element associated with a mesh point (see Fig. 1) cancels when summed over the whole mesh due to the conservation law from which the equations are derived. It leads to lack of convergence of the second derivatives as the mesh size is reduced. However since the solution itself converges, these derivatives can be obtained by numerical differentiation as described above.

Negative coupling coefficients, due to triangles with obtuse angles, may lead to difficulties in certain cases. It is well-known that a solution to Laplace's equation $\nabla^2 u = 0$ cannot have a maximum or minimum in the interior of the region, but only on the boundary. The numerical solution of the corresponding finite difference equation $\sum_i w_i(u_i-u) = 0$ can be written at each mesh point $u = \sum_i w_i u_i / \sum_i w_i$, and if all w_i are non-negative it is clear that $\min u_i \leq u \leq \max u_i$. However if some of the w_i are negative this may no longer be true, so that a finite difference solution with $u = 0$ on the boundary but $u \neq 0$ in the interior might conceivably exist. This non-zero solution of the finite difference Laplace equation could be added to the solution of the finite difference Poisson equation (12) with an amplitude which would depend on the function used to begin the iteration. It is not known at present whether such a non-zero solution can exist. Obtuse triangles have been used freely, and an error of this type has not been observed.

VIII. EXTENSION TO PROBLEMS WITH CYLINDRICAL SYMMETRY

The diffusion equation (2) and the magnetic field equation (29) with cylindrical symmetry can be treated in cylindrical coordinates r, z, θ by transforming to Cartesian coordinates.

(a) Diffusion Equation

Let $\phi = \phi(r,z)$, independent of θ, and let $\vec{\nabla}, \vec{\nabla}_c$ represent the del operator in Cartesian coordinates and cylindrical coordinates respectively. Then for any vector function $\vec{f}$ which is independent of θ we have

$$\vec{\nabla}_c \cdot \vec{f} = \frac{1}{r} \vec{\nabla} \cdot (r\vec{f}) .$$

Hence Eq. (2) becomes

$$\frac{1}{r} \vec{\nabla} \cdot (r\lambda\vec{\nabla}\phi) = c \frac{\partial \phi}{\partial t}$$

or

$$\vec{\nabla} \cdot (r\lambda\vec{\nabla}\phi) = rc \frac{\partial \phi}{\partial t} .$$

Thus we see that by replacing λ by rλ, and c by rc, we can treat the cylindrical coordinates z,r as if they were Cartesian coordinates x,y respectively.

In finite difference form we have in place of (6) and (8) respectively

$$G = \sum_i c_{i+1/2} \bar{\bar{r}}_{i+1/2} a_{i+1/2}$$

$$w_i = \frac{1}{2} \left(\lambda_{i+1/2} \bar{r}_{i+1/2} \cot \theta_{i+1} + \lambda_{i-1/2} \bar{r}_{i-1/2} \cot \theta_{i-1} \right) ,$$

where $\bar{r}_{i+1/2}$, the average radius of triangle i+1/2, is given by

$$\bar{r}_{i+1/2} = \frac{1}{3} (r + r_i + r_{i+1})$$

and $\bar{\bar{r}}_{i+1/2}$ is the average radius of a quadrilateral at vertex r given by

$$\bar{\bar{r}}_{i+1/2} = \frac{7}{12} \bar{r}_{i+1/2} + \frac{5}{12} r.$$

(b) Magnetic Field Equation

For a vector function $\vec{f}(r,z)$ which has only a θ component we have

$$\vec{\nabla}_c \times \vec{f} = \frac{1}{r} \vec{\nabla} \times (r\vec{f}) .$$

For a vector function $\vec{g}(r,z)$ which has only r and z components

$$\vec{\nabla}_c \times \vec{g} = \vec{\nabla} \times \vec{g} .$$

Since $\vec{A}(r,z)$ and $\vec{j}(r,z)$ have only θ components,

$$\vec{B} = \vec{\nabla}_c \times \vec{A} = \frac{1}{r} \vec{\nabla} \times (r\vec{A}) .$$

Since $\vec{B}$ has only r and z components,

$$\vec{\nabla}_c \times (\gamma\vec{B}) = \vec{\nabla} \times (\gamma\vec{B}) = \vec{\nabla} \times \left[\frac{\gamma}{r} \vec{\nabla} \times (r\vec{A}) \right] = 4\pi\vec{j} .$$

Thus (29) becomes

$$\vec{\nabla} \cdot \left[\frac{\gamma}{r} \vec{\nabla}(rA) \right] = -4\pi j . \qquad (32)$$

We see that γ/r replaces γ, $r\vec{A}$ replaces $\vec{A}$, and $r\vec{B}$ replaces $\vec{B}$.

For a current loop, $A \sim r$ for small r, $A \sim 1/r^2$ for large r. Therefore the boundary conditions on rA are $\lim_{r\to 0}(rA) = \lim_{r\to\infty}(rA) = 0$.

For computational purposes (32) may be written

$$\vec{\nabla} \cdot (\gamma\vec{\nabla}A) = -4\pi j - \frac{\partial}{\partial r}\left(\frac{\gamma A}{r}\right) .$$

A finite difference approximation to the second term on the right may be obtained by the line integral method.

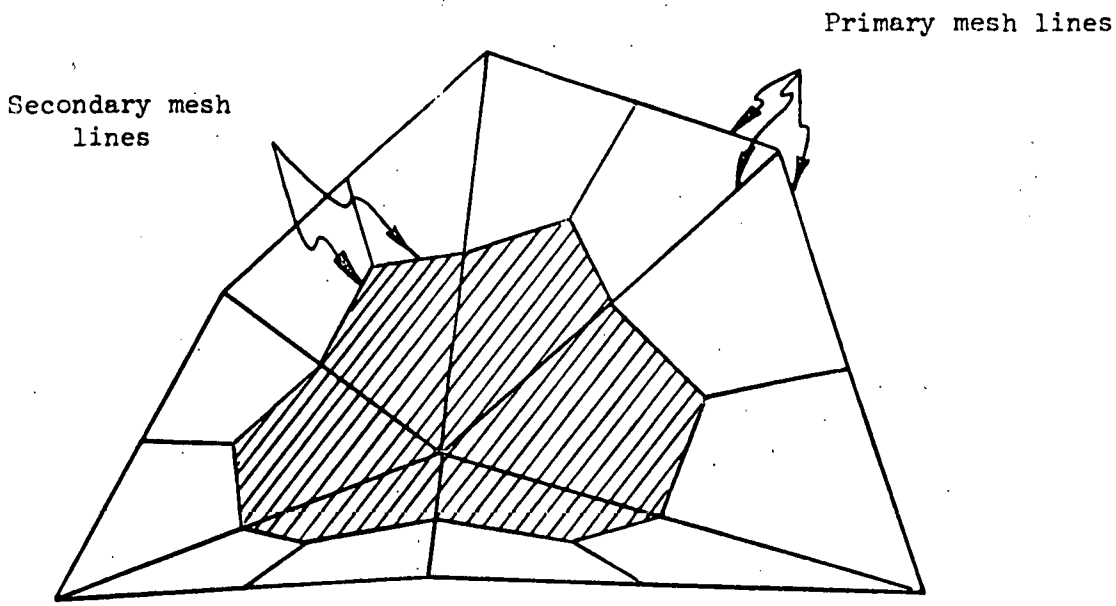

Fig. 1. Relation of secondary to primary mesh

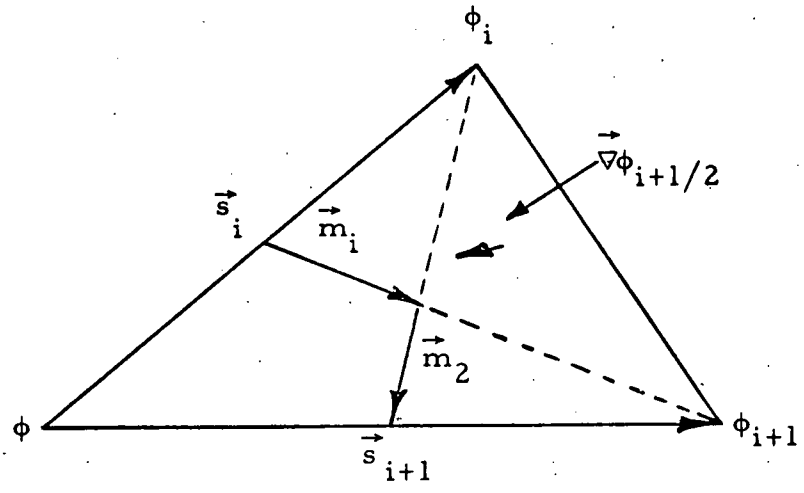

Fig. 2. Vectors used for flux calculation.

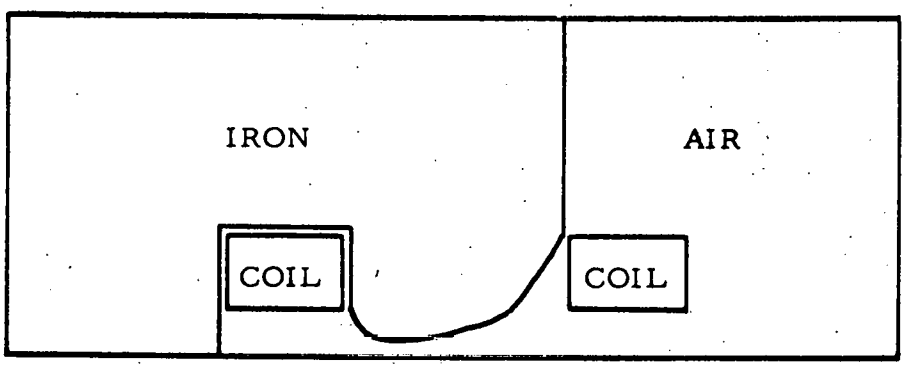

Fig. 3. C-magnet showing material interfaces.

° Points specified in problem input.

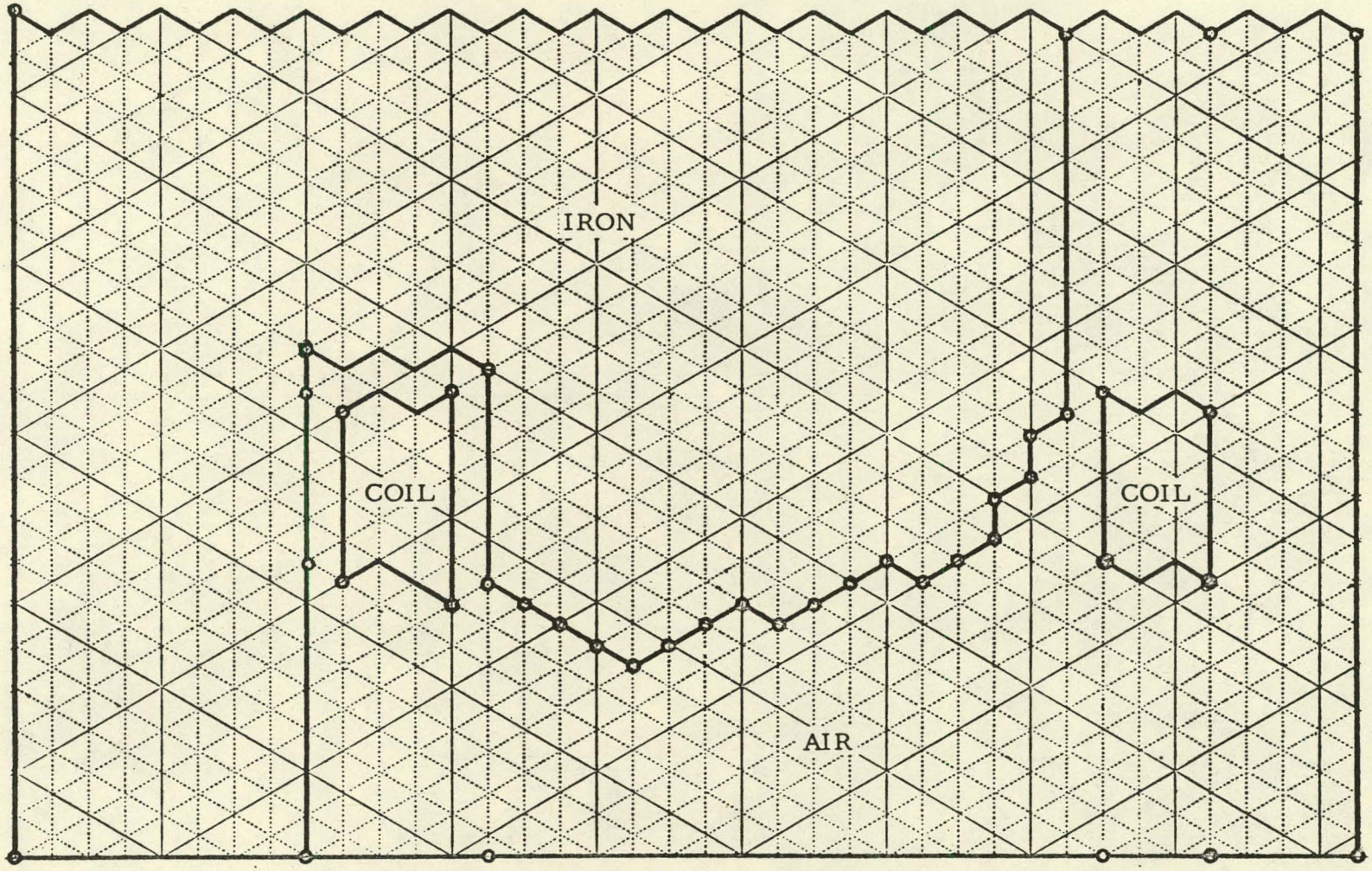

Fig. 4. Logical map of C-magnet mesh.

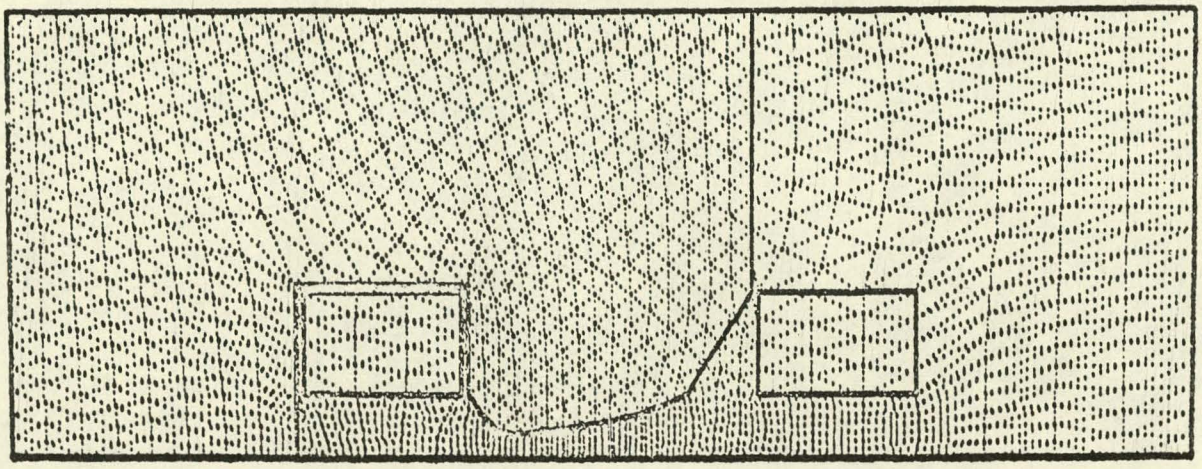

Fig. 5. C-magnet showing triangle zones.

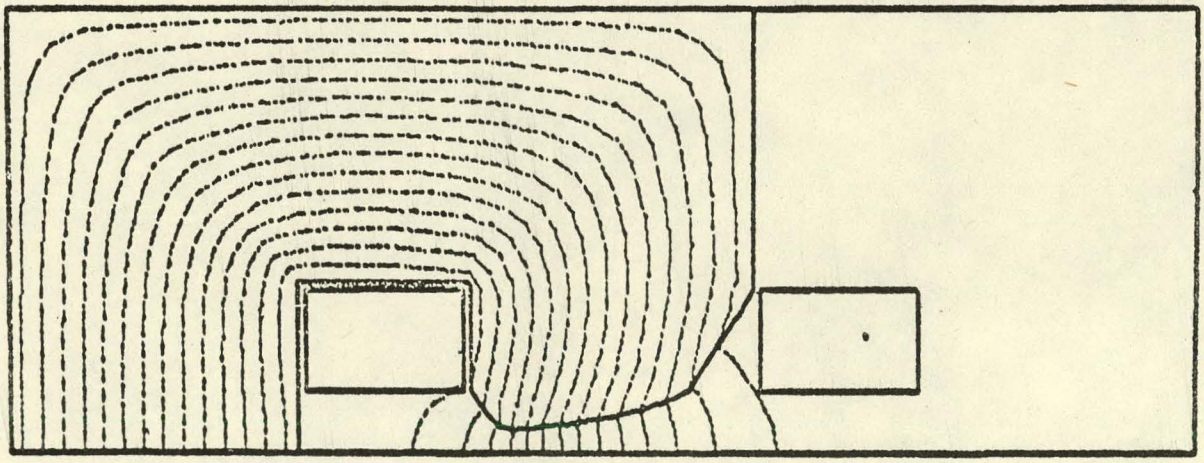

Fig. 6. C-magnet showing equipotentials.

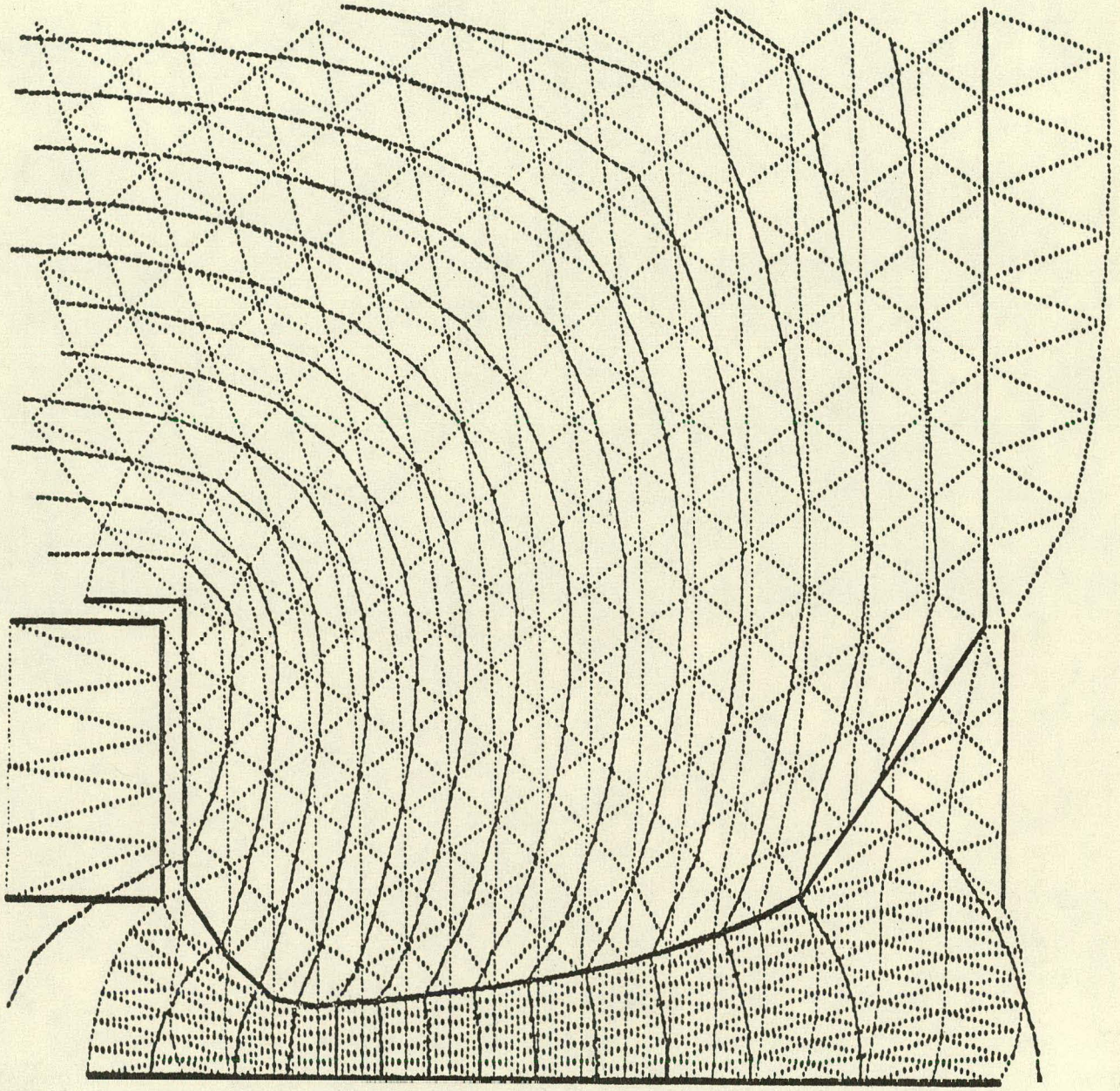

Fig. 7. Enlarged view of C-magnet.

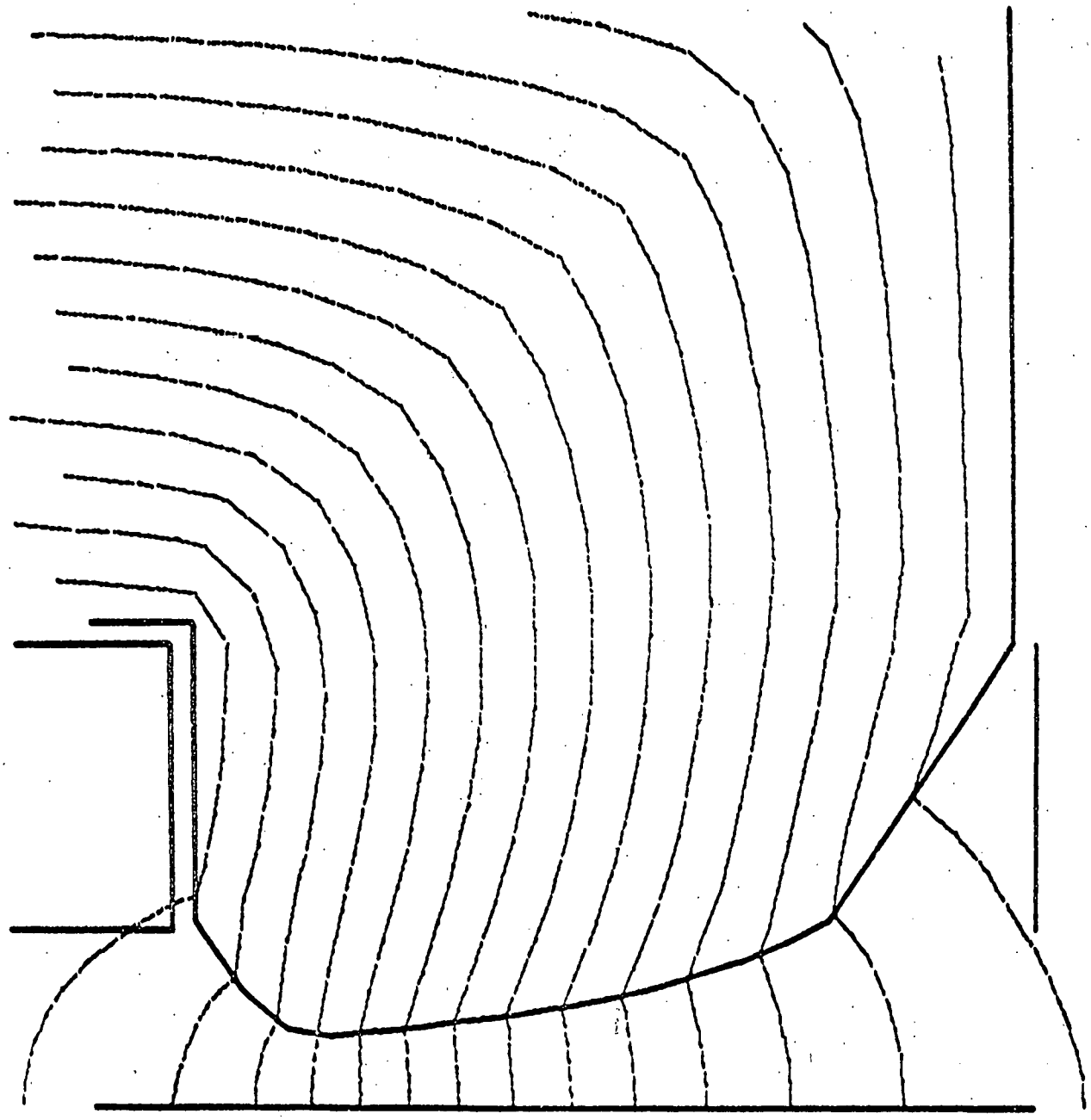

Fig. 8. Figure 7 without zone lines.

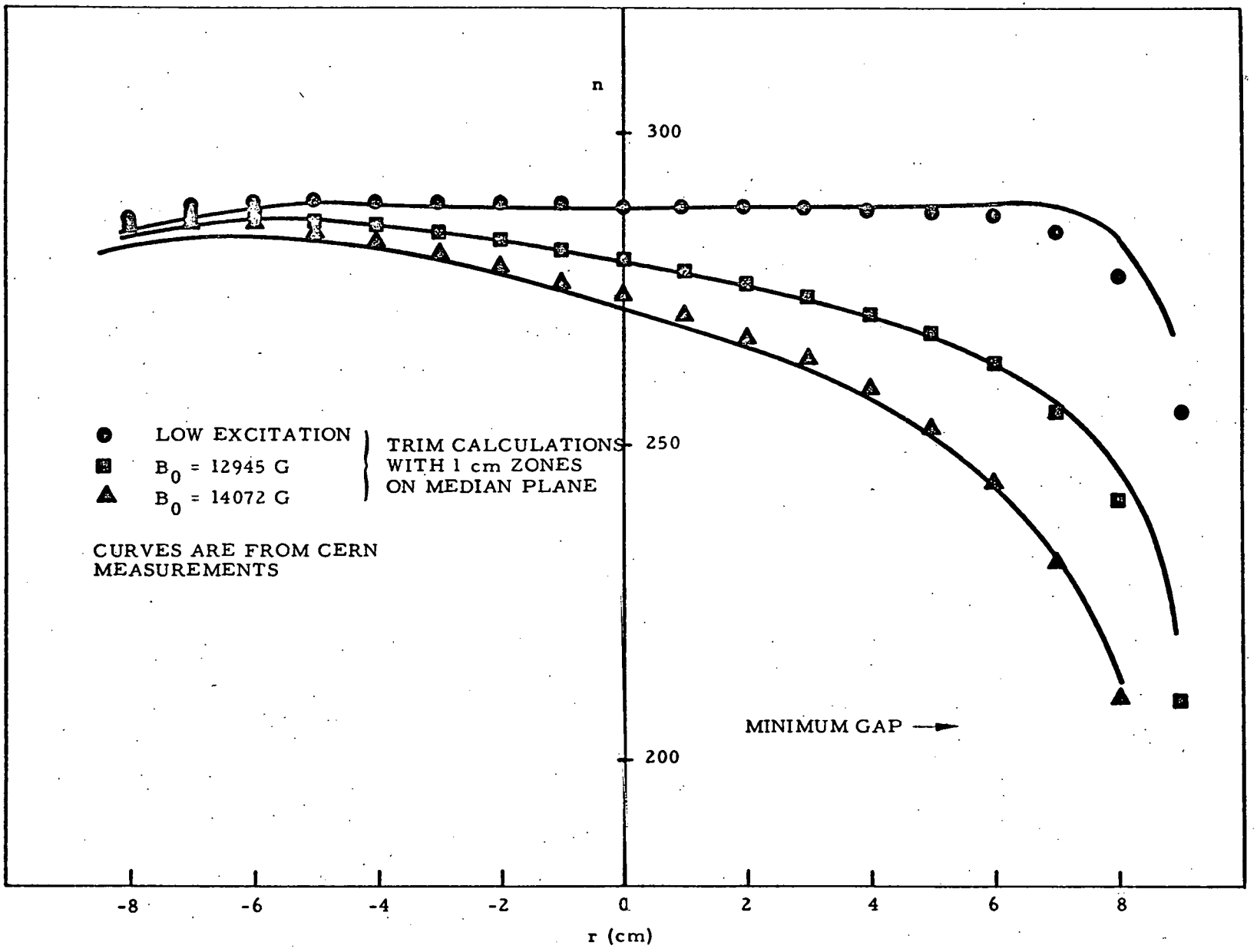

Fig. 9. n vs r for CERN proton synchrotron magnet (open sector). ($n = \left|R_0/B_0 \frac{\partial B}{\partial r}\right|$, R_0 = 70.079 m.)

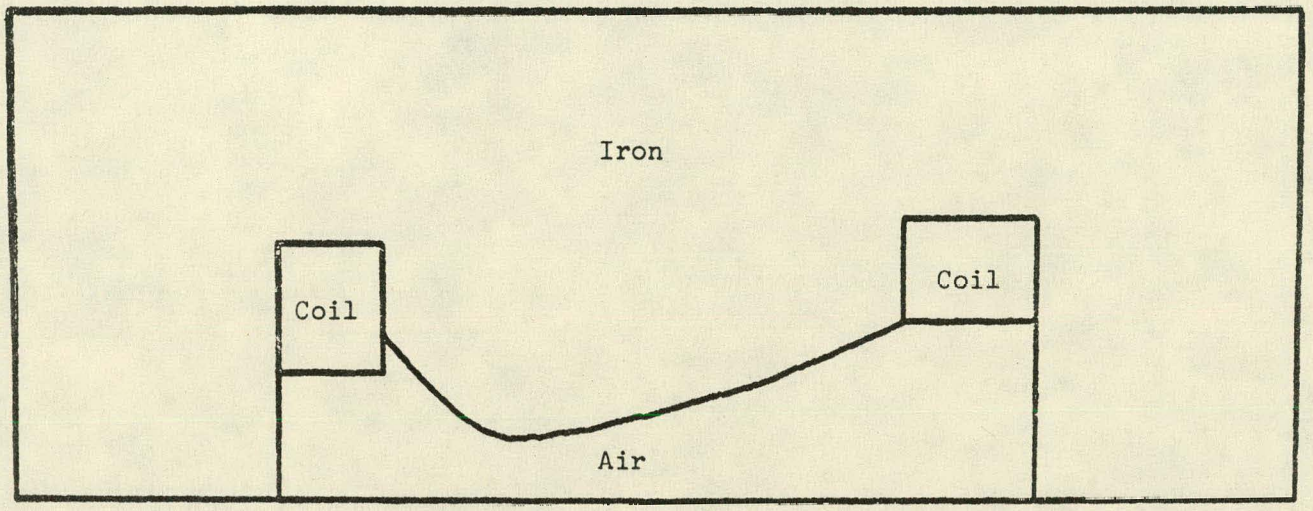

Fig. 11. H-magnet showing material interfaces.

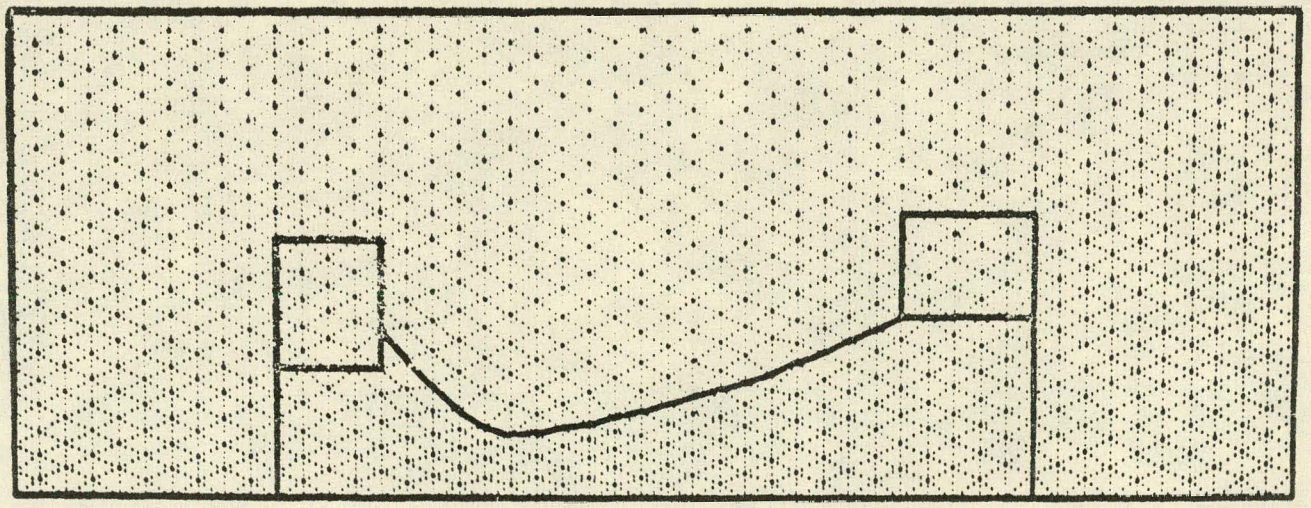

Fig. 12. H-magnet showing triangle zones.

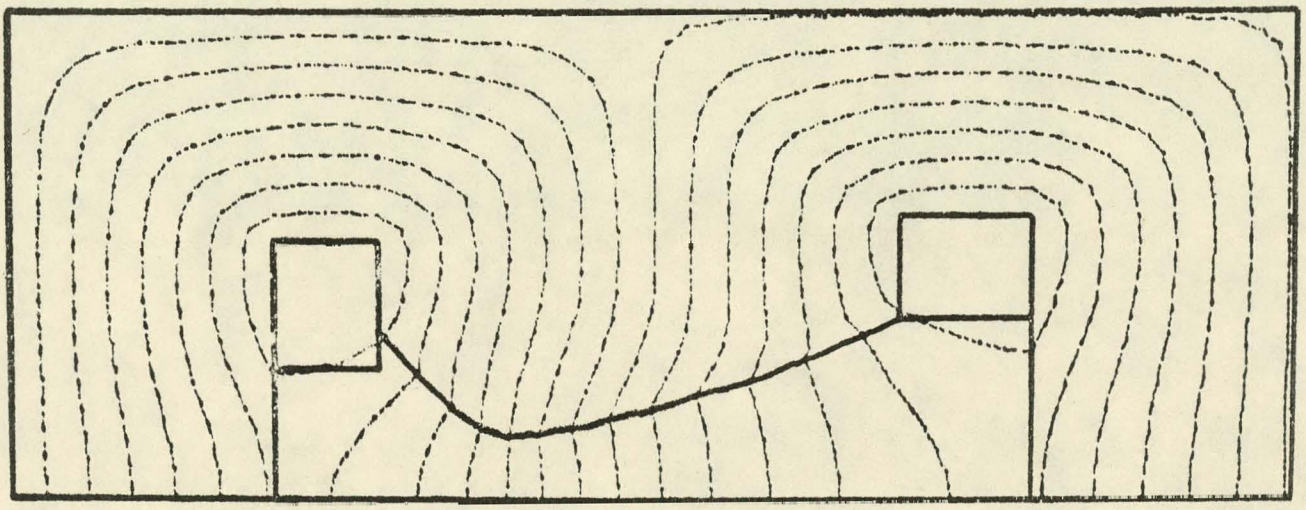

Fig. 13. H-magnet showing equipotentials.

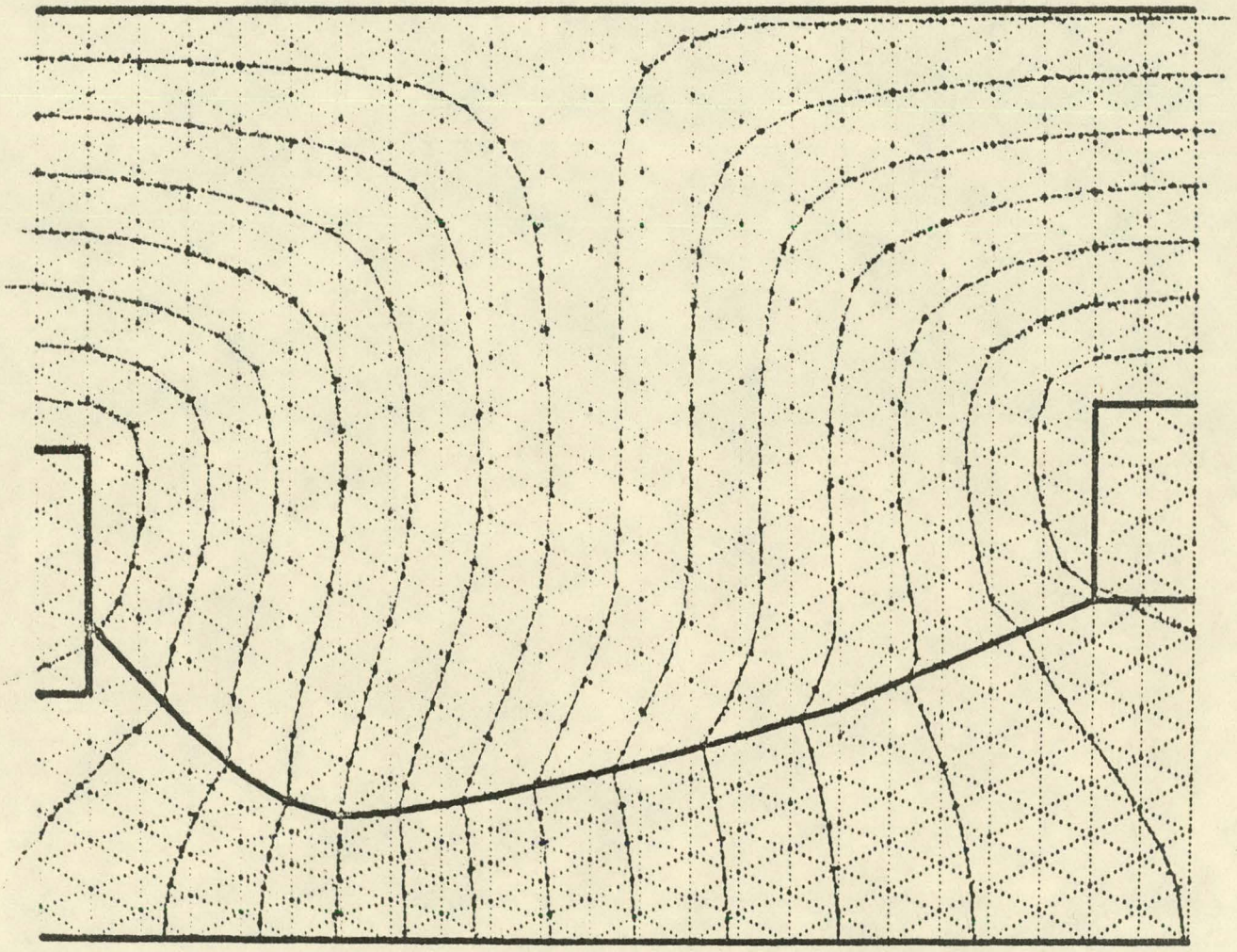

Fig. 14. Enlarged view of H-magnet.

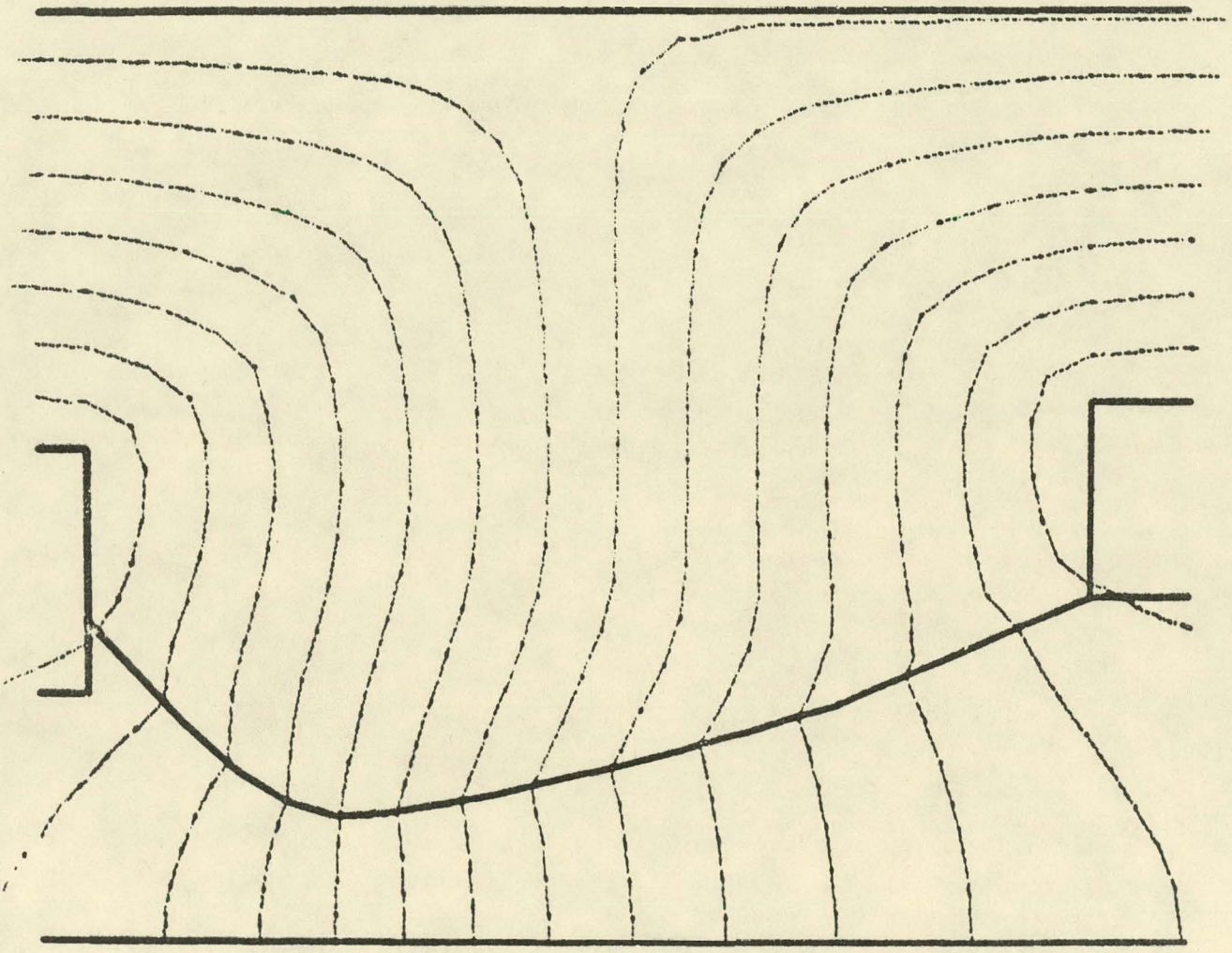

Fig. 15. Figure 14 without zone lines.

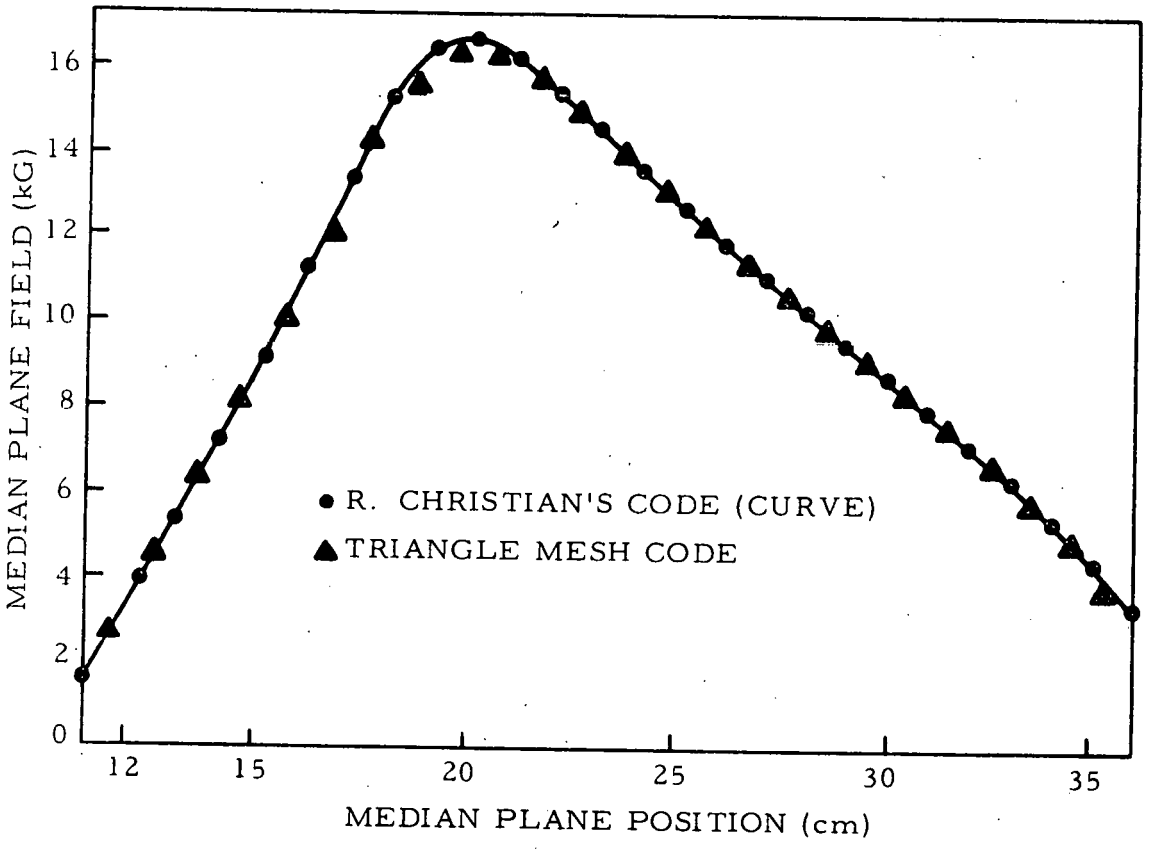

Fig. 16. Median plane field vs position for H-magnet.

APPENDIX

"Equipotential" Zoning[16]

The triangle mesh we have described can be mapped into a regular equilateral triangle array composed of three sets of straight lines intersecting each other at 60°. One of these sets is redundant, and we may consider the mesh to be defined by the remaining two sets.

We now regard each of these sets as the "equipotentials" of a boundary value problem. Let one set be associated with a function $\chi(x,y)$ and the other set with a function $\psi(x,y)$ which satisfy the equations

$$\begin{aligned} \nabla^2\chi &= 0 \\ \nabla^2\psi &= 0 \end{aligned} \tag{33}$$

over the region and boundary conditions determined by the boundary zoning. Then if we can solve (33), the intersecting lines χ = constant and ψ = constant together with the third set drawn through the intersection points, will be the desired triangle mesh. Because of the well-known averaging property of solutions to Laplace's equation, we might expect a mesh constructed in this way to be, in some sense, smooth. Of course, by using some other equations in place of (33) we could obtain a different mesh.

We first invert equations (33) and write them in terms of $x(\chi,\psi)$ and $y(\chi,\psi)$. Using the relations

$$\chi_x = -\frac{1}{J}\, y_\psi \qquad \psi_x = \frac{1}{J}\, y_\chi$$

$$\chi_y = \frac{x_\psi}{J} \qquad \psi_y = -\frac{1}{J}\, x_\chi$$

where the Jacobian $J = x_\psi y_\chi - x_\chi y_\psi$, we find that (33) are transformed into

$$\begin{aligned} \alpha x_{\chi\chi} - 2\beta x_{\chi\psi} + \gamma x_{\psi\psi} &= 0 \\ \alpha y_{\chi\chi} - 2\beta y_{\chi\psi} + \gamma y_{\psi\psi} &= 0 \end{aligned} \tag{34}$$

provided that $J \neq 0$. Here α,β,γ are the quadratic functions

$$\alpha = x_\psi^2 + y_\psi^2$$
$$\beta = x_\chi x_\psi + y_\chi y_\psi \qquad (35)$$
$$\gamma = x_\chi^2 + y_\chi^2$$

The solutions of equations (34) give the coordinates of a given equipotential directly.

Finite difference expressions for the derivatives in (34) and (35) can be obtained by the line integral method. We use a path of integration around a given vertex which passes through the six neighboring points for the first derivatives, and which follows the dodecagon in Fig. 1 for the second derivatives. In so doing we assign values to χ and ψ which differ by unity on adjacent lines, and vary linearly with position in χ,ψ space. In this way we find

$$x_\chi = \frac{1}{6}\left[(x_2+2x_1+x_6) - (x_3+2x_4+x_5)\right]$$
$$x_\psi = \frac{1}{6}\left[(x_1+2x_6+x_5) - (x_2+2x_3+x_4)\right]$$
$$x_{\chi\chi} = x_1-2x+x_4 \qquad (36)$$
$$x_{\chi\psi} = \frac{1}{2}\left[(x_1+x_6+x_3+x_4) - (x_2+x_5+2x)\right]$$
$$x_{\psi\psi} = x_6-2x+x_1$$

and similarly for the derivatives of y, where x,y is the center point and we have identified χ with the lines 2-1, 3-6, 4-5, etc. and ψ with the lines 2-3, 1-4, 6-5, etc. (Fig. 10).

We thus obtain for the finite difference analogues of (34)

$$\sum_{i=1}^{6} c_i(x_i-x) = 0$$
$$\qquad (37)$$
$$\sum_{i=1}^{6} c_i(y_i-y) = 0$$

where $c_1 = c_4 = \alpha-\beta$, $c_2 = c_5 = \beta$, and $c_3 = c_6 = \gamma-\beta$.

Although the coefficient matrix c_i in equations (37) is not symmetric, and we have not been able to prove positive definiteness, nevertheless we have found that the equations converge rapidly when solved by non-linear overrelaxation using automatic optimization as described in section IV with separate overrelaxation parameters for the x and y coordinates. Figures 5 and 12 show typical results obtained by this zoning method applied to sub-regions of the problem. By the use of linear interpolation on boundaries and interfaces, only a relatively few points need be specified in the problem input (see Fig. 4.).

The triangles produced by this method tend to be equilateral far away from boundaries. By redefining ψ to be a vertical zigzag line, a different set of weights c_i can be found which tend to produce right triangles[16].

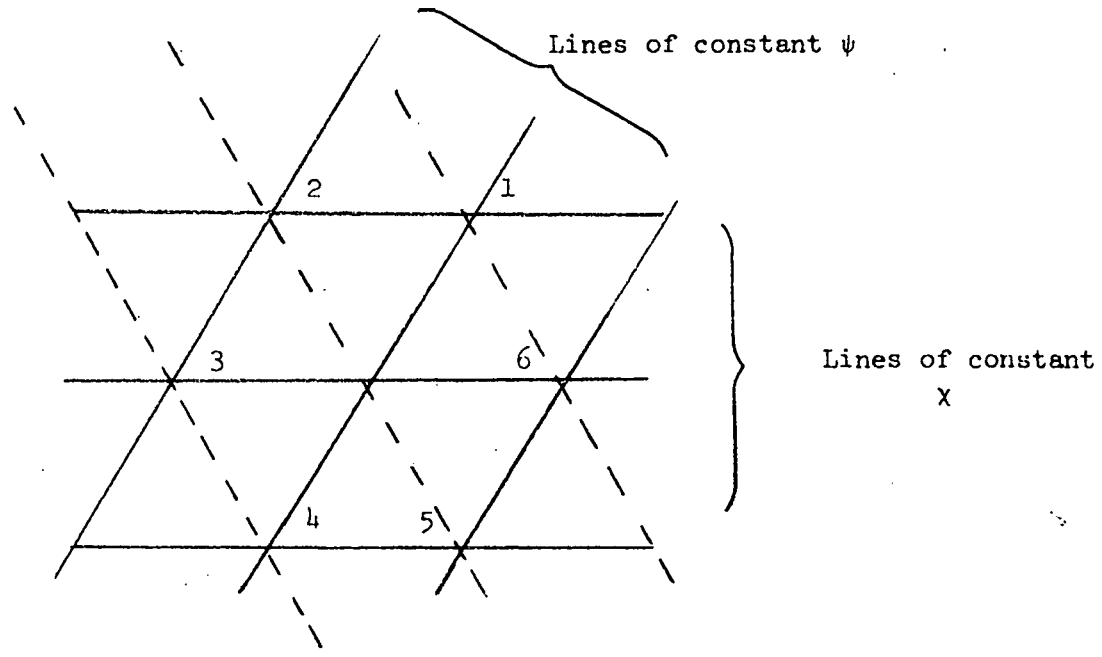

Fig. 10. A vertex and its six neighbors in ψ,χ space.

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