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GENERAL CONDITIONS FOR ACHIEVING PHASE SHIFTS OF π AND 2π
BY MEANS OF QUADRUPOLE MULTIPLETS

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1. Introduction

The purpose of this paper is to formulate analytic conditions for achieving phase shifts of π and 2π by means of magnetic multiplets (mainly quadruplets, quintuplets and sextuplets) composed of quadrupoles.

Only the general theory will be given here; application to particular structures and optimization problems will be considered in forthcoming papers.

No use is made of the thin lens approximation; the thick lens approach has been applied throughout.

2. The Optical Problem

Consider¹ an optical system described by its transfer matrix

$$M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad (1)$$

We assume the matrix to be unimodular, i.e.,

$$ad - bc = 1 \quad (2)$$

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1. E. Regenstreif, "Basic Properties of Turn-Over Matrices," Lawrence Radiation Laboratory Internal Report AS/TH/02, August 10, 1966.

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Let x_o, x'_o be the displacement and the slope of an incoming ray at a distance p (taken positive toward the left) from the entrance of the system; similarly, let x, x' be the displacement and the slope of the outgoing ray at a distance q (taken positive toward the right) from the exit of the system.

The total transfer matrix is then

$$T = \begin{vmatrix} 1 & q \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} . \quad (3)$$

Putting

$$T = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (4)$$

one has

$$a_{11} = a + cq \quad (5)$$

$$a_{22} = d + cp \quad (6)$$

$$a_{12} = cpq + ap + dq + b \quad (7)$$

$$a_{21} = c \quad (8)$$

or in other words

$$x = (a + cq) x_o + (cpq + ap + dq + b) x'_o \quad (9)$$

$$x' = c x_o + (d + cp) x'_o . \quad (10)$$

We call the equation

$$cpq + ap + dq + b = 0 \quad (11)$$

the conjugation relation or the imaging condition because, if Eq. (11) is satisfied, p and q are the conventional object and image distances.

We call the equation

$$c = 0 \quad (12)$$

the afocality condition because, if Eq. (12) is satisfied, the focal distance of the optical system is infinite.

A. Phase Shift of π

For a phase shift of π , i.e.

$$x = -x_0 \quad \text{with arbitrary } x'_0 \quad (13)$$

$$x' = -x'_0 \quad \text{with arbitrary } x_0 \quad (14)$$

one must have

$$a + cq = -1 \quad cpq + ap + dq + b = 0 \quad (15)$$

$$c = 0 \quad d + cp = -1 \quad (16)$$

and these equations lead to

$$a = -1 \quad (17)$$

$$d = -1 \quad (18)$$

$$c = 0 \quad (19)$$

$$b = p + q \quad (20)$$

Equation (20) may be thought of as giving the image distance under "phase shift of π " conditions when the object distance is known. For real objects and images (p and q positive) b must be positive.

For achieving a phase shift of π the matrix of the optical system must therefore be of the form

$$M = \begin{vmatrix} -1 & b \\ 0 & -1 \end{vmatrix} \quad (21)$$

with b positive; the total transfer matrix is then the minus unity matrix

$$T = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \quad (22)$$

B. Phase Shift of 2π

For a phase shift of 2π , i.e.

$$x = x_0 \quad \text{with arbitrary } x'_0 \quad (23)$$

$$x' = x'_0 \quad \text{with arbitrary } x_0 \quad (24)$$

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one must have

$$a + cq = 1 \quad cpq + ap + dq + b = 0 \quad (25)$$

$$c = 0 \quad d + cp = 1 \quad (26)$$

and these equations lead to

$$a = 1 \quad (27)$$

$$d = 1 \quad (28)$$

$$c = 0 \quad (29)$$

$$b = - (p + q) \quad (30)$$

Equation (30) may be thought of as giving the image distance under "phase shift of 2π " conditions when the object distance is known. For real objects and images (p and q positive) b must be negative.

For achieving a phase shift of 2π the matrix of the optical system must therefore be of the form

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad (31)$$

with b negative; the total transfer matrix is then the unity matrix

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (32)$$

3. Use of a Simple Mathematical Device

We shall use a notation which greatly simplifies all calculations in connection with general multiplets.

Consider two consecutive quadrupoles of a multiplet, represented respectively by the transfer matrices

$$m_n = \begin{pmatrix} a_n & b_n \\ c_n & a_n \end{pmatrix} \quad (33)$$

and

$$m_{n+1} = \begin{vmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & a_{n+1} \end{vmatrix} \quad (34)$$

Let $L_{n,n+1}$ be the length of the drift space separating the two quadrupoles. We put

$$L_{n,n+1} + \frac{a_n}{c_n} + \frac{a_{n+1}}{c_{n+1}} = X_{n,n+1} \quad (35)$$

It so happens that in most of the calculations the drift lengths do not appear in isolated form but associated to the adjacent matrix elements as displayed by Eq. (35). This makes it possible to perform in a relatively simple manner a series of algebraic manipulations which would otherwise lead to very complicated expressions.

The property $X_{n,n+1} = X_{n+1,n}$ is obvious.

4. The Doublet

The doublet is of no value for achieving phase shifts of π or 2π but it is a convenient starting point to build up the theory.

The transfer matrix of a general doublet is given by

$$M_2 = \begin{vmatrix} A_2 & B_2 \\ C_2 & D_2 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ c_2 & a_2 \end{vmatrix} \times \begin{vmatrix} 1 & L_{12} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ c_1 & a_1 \end{vmatrix} \quad (36)$$

Using the notation

$$L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} = X_{12} \quad (37)$$

we find for the elements of the transfer matrix of the doublet

$$A_2 = c_1 a_2 \left(X_{12} - \frac{1}{a_2 c_2} \right) \quad (38)$$

$$D_2 = a_1 c_2 \left(X_{12} - \frac{1}{a_1 c_1} \right) \quad (39)$$

$$C_2 = c_1 c_2 X_{12} \quad (40)$$

$$B_2 = a_1 a_2 \left(X_{12} - \frac{1}{a_1 c_1} - \frac{1}{a_2 c_2} \right) \quad (41)$$

The elements A_2 and D_2 are symmetric with respect to each other, i.e., one obtains one from the other by interchanging the indices 1 and 2. The elements C_2 and B_2 are self-symmetric, i.e., interchanging the indices 1 and 2 does not alter these elements.

The focal distance of the doublet is given by

$$-\frac{1}{f_2} = c_1 c_2 X_{12} \quad (42)$$

Applying Eqs. (17) - (19) and (27) - (29) we can write the condition leading to a phase shift of π or 2π

$$X_{12} - \frac{1}{a_2 c_2} = \mp \frac{1}{c_1 a_2} \quad (43)$$

$$X_{12} - \frac{1}{a_1 c_1} = \mp \frac{1}{a_1 c_2} \quad (44)$$

$$X_{12} = 0 \quad (45)$$

where the upper signs apply to a phase shift of π and the lower signs to a phase shift of 2π .

From Eqs. (43) - (45) we find immediately

$$\boxed{c_1 = \pm c_2} \quad (46)$$

$$\boxed{L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} = 0} \quad (47)$$

and these are the conditions we were looking for.

With these conditions the element B_2 of the transfer matrix of the doublet can be written

$$\boxed{B_2 = \mp \left(\frac{a_1}{c_1} + \frac{a_2}{c_2} \right)} \quad (48)$$

We shall not discuss these equations here, the scope of the present paper being limited to the formulation of general conditions leading to phase shifts of π and 2π .

5. The Triplet

We can make use of the results obtained in the case of the doublet by writing the transfer matrix of the triplet in the form

$$M_3 = \begin{vmatrix} A_3 & B_3 \\ C_3 & D_3 \end{vmatrix} = \begin{vmatrix} a_3 & b_3 \\ c_3 & a_3 \end{vmatrix} \times \begin{vmatrix} 1 & L_{23} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} A_2 & B_2 \\ C_2 & D_2 \end{vmatrix} \quad (49)$$

Putting

$$L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} = X_{23} \quad (50)$$

and carrying out the calculations we find for the matrix elements of the triplet

$$A_3 = c_1 c_2 a_3 \left[x_{12} \left(x_{23} - \frac{1}{a_3 c_3} \right) - \frac{1}{c_2} \right] \quad (51)$$

$$D_3 = a_1 c_2 c_3 \left[x_{23} \left(x_{12} - \frac{1}{a_1 c_1} \right) - \frac{1}{c_2} \right] \quad (52)$$

$$C_3 = c_1 c_2 c_3 \left(x_{12} x_{23} - \frac{1}{c_2} \right) \quad (53)$$

$$B_3 = a_1 c_2 a_3 \left[\left(x_{12} - \frac{1}{a_1 c_1} \right) \left(x_{23} - \frac{1}{a_3 c_3} \right) - \frac{1}{c_2} \right] \quad (54)$$

The elements A_3 and D_3 are symmetric with respect to each other, i.e. one obtains one from the other by interchanging the indices 1 and 3, the index 2 remaining unchanged. The elements C_3 and B_3 are self-symmetric, i.e. interchanging the indices 1 and 3 does not alter these elements.

The focal distance of the triplet is given by

$$-\frac{1}{f_3} = c_1 c_2 c_3 \left(x_{12} x_{23} - \frac{1}{c_2} \right) \quad (55)$$

Applying Eqs. (17) - (19) and (27) - (29) we find for the conditions leading to a phase shift of π or 2π

$$x_{12} \left(x_{23} - \frac{1}{a_3 c_3} \right) = \frac{1}{c_2} \mp \frac{1}{c_1 c_2 a_3} \quad (56)$$

$$x_{23} \left(x_{12} - \frac{1}{a_1 c_1} \right) = \frac{1}{c_2} \mp \frac{1}{a_1 c_2 c_3} \quad (57)$$

$$x_{12} x_{23} = \frac{1}{c_2} \quad (58)$$

Solving for X_{12} and X_{23} one finds

$$X_{12} = \pm \frac{c_3}{c_1 c_2} \quad (59)$$

$$X_{23} = \pm \frac{c_1}{c_2 c_3} \quad (60)$$

The upper signs apply everywhere to a phase shift of π and the lower signs hold for a phase shift of 2π .

Reverting to the original matrix elements, Eqs. (59) and (60) can be written

$$L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} = \pm \frac{c_3}{c_1 c_2} \quad (61)$$

$$L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} = \pm \frac{c_1}{c_2 c_3} \quad (62)$$

These are therefore the conditions for achieving a phase shift of π or 2π by means of a triplet. With these conditions the element B_3 of the transfer matrix of the triplet becomes

$$B_3 = \frac{c_2}{c_1 c_3} \mp \left(\frac{a_1}{c_1} + \frac{a_3}{c_3} \right) \quad (63)$$

It should perhaps be emphasized at this point that if one desires a phase shift of π or 2π in both planes of an AG triplet, Eqs. (61) and (62) should be applied to both planes. If, in addition, one wants stigmatism, B_3 should be the same in both planes.

6. The Quadruplet

We again use the results obtained in the case of the triplet and write the matrix of the quadruplet in the form

$$M_4 = \begin{vmatrix} A_4 & B_4 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_4 & b_4 \\ c_4 & a_3 \end{vmatrix} \times \begin{vmatrix} 1 & L_{34} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} A_3 & B_3 \\ C_3 & D_3 \end{vmatrix} \quad (64)$$

According to our convention we put

$$L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} = X_{34} \quad (65)$$

Carrying out the calculations we find for the matrix elements of the quadruplet

$$A_4 = c_1 c_2 c_3 a_4 \left[\left(X_{12} X_{23} - \frac{1}{c_2} \right) \left(X_{34} - \frac{1}{a_4 c_4} \right) - \frac{X_{12}}{c_3} \right] \quad (66)$$

$$D_4 = a_1 c_2 c_3 c_4 \left[\left(X_{12} - \frac{1}{a_1 c_1} \right) \left(X_{23} X_{34} - \frac{1}{c_3} \right) - \frac{X_{34}}{c_2} \right] \quad (67)$$

$$C_4 = c_1 c_2 c_3 c_4 \left(X_{12} X_{23} X_{34} - \frac{X_{34}}{c_2} - \frac{X_{12}}{c_3} \right) \quad (68)$$

$$B_4 = a_1 c_2 c_3 a_4 \left[\left(X_{12} - \frac{1}{a_1 c_1} \right) \left(X_{34} - \frac{1}{a_4 c_4} \right) X_{23} - \frac{1}{c_2} \left(X_{34} - \frac{1}{a_4 c_4} \right) - \frac{1}{c_3} \left(X_{12} - \frac{1}{a_1 c_1} \right) \right] \quad (69)$$

We note again that the elements A_4 and D_4 are symmetric with respect to each other, i.e. one can obtain one from the other by interchanging respectively and simultaneously the indices 1 and 4, 2 and 3. The elements C_4 and B_4 are

self-symmetric, i.e. interchanging respectively and simultaneously the indices 1 and 4, 2 and 3 does not alter these elements.

For the focal distance of the quadruplet we find from Eq. (68)

$$-\frac{1}{f_4} = c_1 c_2 c_3 c_4 \left(X_{12} X_{23} X_{34} - \frac{X_{34}}{c_2} - \frac{X_{12}}{c_3} \right) \quad (70)$$

Applying again our basic equations we write the conditions for obtaining a phase shift of π or 2π

$$\left(X_{12} X_{23} - \frac{1}{c_2} \right) \left(X_{34} - \frac{1}{a_4 c_4} \right) - \frac{X_{12}}{c_3} = \mp \frac{1}{c_1 c_2 c_3 a_4} \quad (71)$$

$$\left(X_{12} - \frac{1}{a_1 c_1} \right) \left(X_{23} X_{34} - \frac{1}{c_3} \right) - \frac{X_{34}}{c_2} = \mp \frac{1}{a_1 c_2 c_3 c_4} \quad (72)$$

$$X_{12} X_{23} X_{34} = \frac{X_{34}}{c_2} + \frac{X_{12}}{c_3} \quad (73)$$

As before, the upper signs refer to a phase shift of π and the lower signs apply to a phase shift of 2π . Solving the set (71) - (73), we find

$$c_1 c_2 X_{12} = \pm c_3 c_4 X_{34} \quad (74)$$

$$X_{23} = \frac{1}{c_2} \frac{1}{X_{12}} + \frac{1}{c_3} \frac{1}{X_{34}} \quad (75)$$

Written out in full these equations become

$$c_1 c_2 \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) = \pm c_3 c_4 \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) \quad (76)$$

$$L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} = \frac{1}{c_2^2 \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right)} + \frac{1}{c_3^2 \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right)} \quad (77)$$

and the matrix element B_4 can then be written

$$B_4 = \frac{c_2 c_3}{c_1 c_4} \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \mp \left(\frac{a_1}{c_1} + \frac{a_4}{c_4} \right) \quad (78)$$

7. The Quintuplet

We use the same procedure as before, i.e. we write the matrix of the quintuplet in the form

$$M_5 = \begin{vmatrix} A_5 & B_5 \\ C_5 & D_5 \end{vmatrix} = \begin{vmatrix} a_5 & b_5 \\ c_5 & a_5 \end{vmatrix} \times \begin{vmatrix} 1 & L_{45} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} A_4 & B_4 \\ C_4 & D_4 \end{vmatrix} \quad (79)$$

and put

$$L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} = X_{45} \quad (80)$$

Carrying out the calculations we find for the matrix elements of the quintuplet

$$A_5 = c_1 c_2 c_3 c_4 a_5 \left[\left(X_{45} - \frac{1}{a_5 c_5} \right) \left(X_{12} X_{23} X_{34} - \frac{X_{34}}{c_2} - \frac{X_{12}}{c_3} \right) - \frac{1}{c_4} \left(X_{12} X_{23} - \frac{1}{c_2} \right) \right] \quad (81)$$

$$D_5 = a_1 c_2 c_3 c_4 c_5 \left[\left(X_{12} - \frac{1}{a_1 c_1} \right) \left(X_{23} X_{34} X_{45} - \frac{X_{45}}{c_3} - \frac{X_{23}}{c_4} \right) - \frac{1}{c_2} \left(X_{34} X_{45} - \frac{1}{c_4} \right) \right] \quad (82)$$

$$C_5 = c_1 c_2 c_3 c_4 c_5 \left[\left(X_{12} X_{23} - \frac{1}{c_2} \right) \left(X_{34} X_{45} - \frac{1}{c_4} \right) - \frac{X_{12} X_{45}}{c_3} \right] \quad (83)$$

$$B_5 = a_1 c_2 c_3 c_4 a_5 \left[\left(X_{12} - \frac{1}{a_1 c_1} \right) \left(X_{23} X_{34} - \frac{1}{c_3} \right) \left(X_{45} - \frac{1}{a_5 c_5} \right) - \frac{X_{34}}{c_2} \left(X_{45} - \frac{1}{a_5 c_5} \right) - \frac{X_{23}}{c_4} \left(X_{12} - \frac{1}{a_1 c_1} \right) + \frac{1}{c_2 c_4} \right] \quad (84)$$

As a check we note that the elements A_5 and D_5 are symmetric with respect to each other, i.e. one can obtain one from the other by interchanging respectively and simultaneously the indices 1 and 5, 2 and 4, and leaving 3 unchanged. The same operation performed on the elements C_5 and B_5 leaves them unchanged; C_5 and B_5 are therefore self-symmetric.

As before, the focal distance is given by the element C which is the simplest matrix element

$$-\frac{1}{f_5} = c_1 c_2 c_3 c_4 c_5 \left[\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{34} x_{45} - \frac{1}{c_4} \right) - \frac{x_{12} x_{45}}{c_3} \right] \quad (85)$$

We again apply Eqs. (17) - (19) and (27) - (29) to write the conditions for obtaining phase shifts of π and 2π , respectively

$$\left(x_{45} - \frac{1}{a_5 c_5} \right) \left(x_{12} x_{23} x_{34} - \frac{x_{34}}{c_2} - \frac{x_{12}}{c_3} \right) - \frac{1}{c_4} \left(x_{12} x_{23} - \frac{1}{c_2} \right) = \mp \frac{1}{c_1 c_2 c_3 c_4 a_5} \quad (86)$$

$$\left(x_{12} - \frac{1}{a_1 c_1} \right) \left(x_{23} x_{34} x_{45} - \frac{x_{45}}{c_3} - \frac{x_{23}}{c_4} \right) - \frac{1}{c_2} \left(x_{34} x_{45} - \frac{1}{c_4} \right) = \mp \frac{1}{a_1 c_2 c_3 c_4 c_5} \quad (87)$$

$$\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{34} x_{45} - \frac{1}{c_4} \right) = \frac{x_{12} x_{45}}{c_3} \quad (88)$$

The set of Eqs. (86) - (88) yields

$$x_{12} x_{23} x_{34} = \frac{x_{34}}{c_2} + \frac{x_{12}}{c_3} \pm \frac{c_5}{c_1 c_2 c_3 c_4} \quad (89)$$

$$x_{23} x_{34} x_{45} = \frac{x_{45}}{c_3} + \frac{x_{23}}{c_4} \pm \frac{c_1}{c_2 c_3 c_4 c_5} \quad (90)$$

or written out in full

$$\left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) = \frac{1}{c_2} \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) + \frac{1}{c_3} \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) \pm \frac{c_5}{c_1 c_2 c_3 c_4} \quad (91)$$

$$\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) = \frac{1}{c_3} \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) + \frac{1}{c_4} \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \pm \frac{c_1}{c_2 c_3 c_4 c_5} \quad (92)$$

These are therefore the conditions for obtaining phase shifts of π and 2π by means of a general quintuplet.

Taking into account these conditions, the matrix element B_5 can be written

$$B_5 = \frac{c_2 c_3 c_4}{c_1 c_5} \left[\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) - \frac{1}{c_3} \right] \mp \left(\frac{a_1}{c_1} + \frac{a_5}{c_5} \right) \quad (93)$$

8. The Sextuplet

We write the matrix of the sextuplet in the form

$$M_6 = \begin{vmatrix} A_6 & B_6 \\ C_6 & D_6 \end{vmatrix} = \begin{vmatrix} a_6 & b_6 \\ c_6 & a_6 \end{vmatrix} \times \begin{vmatrix} 1 & L_{56} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} A_5 & B_5 \\ C_5 & D_5 \end{vmatrix} \quad (94)$$

and put

$$L_{56} + \frac{a_5}{c_5} + \frac{a_6}{c_6} = X_{56} \quad (95)$$

Carrying out the calculations we find for the matrix elements

$$A_6 = c_1 c_2 c_3 c_4 c_5 a_6 \left\{ \left(x_{56} - \frac{1}{a_6 c_6} \right) \left[\left(x_{12} x_{23} - \frac{1}{c_2^2} \right) \left(x_{34} x_{45} - \frac{1}{c_4^2} \right) - \frac{x_{12} x_{45}}{c_3^2} \right] - \frac{1}{c_5^2} \left(x_{12} x_{23} x_{34} - \frac{x_{34}}{c_2^2} - \frac{x_{12}}{c_3^2} \right) \right\} \quad (96)$$

$$D_6 = a_1 c_2 c_3 c_4 c_5 c_6 \left\{ \left(x_{12} - \frac{1}{a_1 c_1} \right) \left[\left(x_{45} x_{56} - \frac{1}{c_5^2} \right) \left(x_{23} x_{34} - \frac{1}{c_3^2} \right) - \frac{x_{23} x_{56}}{c_4^2} \right] - \frac{1}{c_2^2} \left(x_{34} x_{45} x_{56} - \frac{x_{56}}{c_4^2} - \frac{x_{34}}{c_5^2} \right) \right\} \quad (97)$$

$$C_6 = c_1 c_2 c_3 c_4 c_5 c_6 \left[\left(x_{12} x_{23} - \frac{1}{c_2^2} \right) \left(x_{45} x_{56} - \frac{1}{c_5^2} \right) x_{34} - \frac{x_{12}}{c_3^2} \left(x_{45} x_{56} - \frac{1}{c_5^2} \right) - \frac{x_{56}}{c_4^2} \left(x_{12} x_{23} - \frac{1}{c_2^2} \right) \right] \quad (98)$$

$$B_6 = a_1 c_2 c_3 c_4 c_5 a_6 \left[\left(x_{12} - \frac{1}{a_1 c_1} \right) \left(x_{56} - \frac{1}{a_6 c_6} \right) \left(x_{23} x_{34} x_{45} - \frac{x_{45}}{c_3^2} - \frac{x_{23}}{c_4^2} \right) - \frac{1}{c_2^2} \left(x_{56} - \frac{1}{a_6 c_6} \right) \left(x_{34} x_{45} - \frac{1}{c_4^2} \right) - \frac{1}{c_5^2} \left(x_{12} - \frac{1}{a_1 c_1} \right) \left(x_{23} x_{34} - \frac{1}{c_3^2} \right) + \frac{x_{34}}{c_2^2 c_5^2} \right] \quad (99)$$

As before, the elements A_6 and C_6 are symmetric with respect to each other for one can obtain one from the other by interchanging respectively and simultaneously the indices 1 and 6, 2 and 4, 3 and 5. The elements C_6 and B_6 are self-symmetric, for the same operation performed on these elements leaves them unchanged.

From Eq. (98) we find for the focal distance of the sextuplet

$$-\frac{1}{f_6} = c_1 c_2 c_3 c_4 c_5 c_6 \left[\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{45} x_{56} - \frac{1}{c_5} \right) x_{34} - \frac{x_{12}}{c_3} \left(x_{45} x_{56} - \frac{1}{c_5} \right) - \frac{x_{56}}{c_4} \left(x_{12} x_{23} - \frac{1}{c_2} \right) \right] \quad (100)$$

Applying now our basic equations for obtaining phase shifts of π and 2π we have the conditions

$$\left(x_{56} - \frac{1}{a_6 c_6} \right) \left[\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{34} x_{45} - \frac{1}{c_4} \right) - \frac{x_{12} x_{45}}{c_3} - \frac{1}{c_5} \left(x_{12} x_{23} x_{34} - \frac{x_{34}}{c_2} - \frac{x_{12}}{c_3} \right) \right] = \mp \frac{1}{c_1 c_2 c_3 c_4 c_5 a_6} \quad (101)$$

$$\left(x_{12} - \frac{1}{a_1 c_1} \right) \left[\left(x_{45} x_{56} - \frac{1}{c_5} \right) \left(x_{23} x_{34} - \frac{1}{c_3} \right) - \frac{x_{23} x_{56}}{c_4} - \frac{1}{c_2} \left(x_{34} x_{45} x_{56} - \frac{x_{56}}{c_4} - \frac{x_{34}}{c_5} \right) \right] = \mp \frac{1}{a_1 c_2 c_3 c_4 c_5 c_6} \quad (102)$$

$$\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{45} x_{56} - \frac{1}{c_5} \right) x_{34} = \frac{x_{12}}{c_3} \left(x_{45} x_{56} - \frac{1}{c_5} \right) + \frac{x_{56}}{c_4} \left(x_{12} x_{23} - \frac{1}{c_2} \right) \quad (103)$$

This set which may seem complicated reduces in fact to

$$\left(x_{12} x_{23} - \frac{1}{c_2} \right) \left(x_{34} x_{45} - \frac{1}{c_4} \right) = \frac{x_{12} x_{45}}{c_3} \pm \frac{c_6}{c_1 c_2 c_3 c_4 c_5} \quad (104)$$

$$\left(x_{23} x_{34} - \frac{1}{c_3} \right) \left(x_{45} x_{56} - \frac{1}{c_5} \right) = \frac{x_{23} x_{56}}{c_4} \pm \frac{c_1}{c_2 c_3 c_4 c_5 c_6} \quad (105)$$

and written out in full the conditions for obtaining phase shifts of π and 2π by means of a sextuplet are

$$\left[\left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) - \frac{1}{c_2^2} \right] \left[\left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) - \frac{1}{c_4^2} \right] = \frac{1}{c_3^2} \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) \pm \frac{c_6}{c_1 c_2 c_3 c_4 c_5} \quad (106)$$

$$\left[\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) - \frac{1}{c_3^2} \right] \left[\left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) \left(L_{56} + \frac{a_5}{c_5} + \frac{a_6}{c_6} \right) - \frac{1}{c_5^2} \right] = \frac{1}{c_4^2} \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{56} + \frac{a_5}{c_5} + \frac{a_6}{c_6} \right) \pm \frac{c_1}{c_2 c_3 c_4 c_5 c_6} \quad (107)$$

When these conditions are satisfied the element B_6 of the transfer matrix of the sextuplet becomes

$$B_6 = c_2 c_3 c_4 c_5 \left(X_{23} X_{34} X_{45} - \frac{X_{45}}{c_3^2} - \frac{X_{23}}{c_4^2} \right) \mp \left(\frac{a_1}{c_1} + \frac{a_6}{c_6} \right) \quad (108)$$

or written out in full

$$B_6 = c_2 c_3 c_4 c_5 \left[\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) - \frac{1}{c_3^2} \left(L_{45} + \frac{a_4}{c_4} + \frac{a_5}{c_5} \right) - \frac{1}{c_4^2} \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \right] \mp \left(\frac{a_1}{c_1} + \frac{a_6}{c_6} \right) \quad (109)$$

9. Conclusion

It has been shown that by means of a simple mathematical device it is possible to formulate general analytic conditions for obtaining phase shifts of π or 2π by means of quadruplets, quintuplets and sextuplets composed of arbitrary quadrupoles.

In the case of the sextuplet the algebra becomes moderately complicated but the symmetric or antisymmetric structures which one is likely to use in practice introduce considerable simplification in the formulae given above.

Preliminary analytic work by A.A. Garren² and numerical computations by J.P. Blewett³ indicate that phase shifts of π and 2π can be obtained by means of particular quadruplets or quintuplets. Similar conclusions for particular cases have been reached by Dhuicq and Septier.⁴

A general comparison of quadruplets, quintuplets and sextuplets, based on the results obtained in the present paper, will be given in a forthcoming report. The case of the general AG quadruplet is elaborated in the appendix attached to this paper.

2. A.A. Garren, private communication.

3. J.P. Blewett, private communication.

4. D. Dhuicq and A. Septier, Comptes Rendus de l'Académie des Sciences, Paris, Vol. 263, August 1966.

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APPENDIX

Phase Shift of π and 2π by Means of a General AG Quadruplet

We start from Eqs. (76), (77) and (78)

$$c_1 c_2 \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right) = \pm c_3 c_4 \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right) \quad (A-1)$$

$$L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} = \frac{1}{c_2^2 \left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2} \right)} + \frac{1}{c_3^2 \left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4} \right)} \quad (A-2)$$

$$B_4 = \frac{c_2 c_3}{c_1 c_4} \left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3} \right) \mp \left(\frac{a_1}{c_1} + \frac{a_4}{c_4} \right) \quad (A-3)$$

Practical requirements in the case of an AG multiplet are:

- 1) Phase shift of π or 2π in both planes.
- 2) Stigmatism.

We first introduce the abbreviations

$$A = \frac{a_1}{c_1} + \frac{a_2}{c_2} \quad B = \frac{a_2}{c_2} + \frac{a_3}{c_3} \quad C = \frac{a_3}{c_3} + \frac{a_4}{c_4} \quad (A-4)$$

$$\lambda = c_3 c_4 \quad \mu = \frac{c_2 c_3}{c_1 c_4} \quad \nu = c_1 c_2 \quad (A-5)$$

Next we write out Eqs. (A-1) and (A-2) in the two basic planes of the quadruplet which we call x and y

$$v_x (L_{12} + A_x) = \pm \lambda_x (L_{34} + C_x) \quad (A-6)$$

$$v_y (L_{12} + A_y) = \pm \lambda_y (L_{34} + C_y) \quad (A-7)$$

$$L_{23} + B_x = \frac{1}{c_{2x}^2} \cdot \frac{1}{L_{12} + A_x} + \frac{1}{c_{3x}^2} \cdot \frac{1}{L_{34} + C_x} \quad (A-8)$$

$$L_{23} + B_y = \frac{1}{c_{2y}^2} \cdot \frac{1}{L_{12} + A_y} + \frac{1}{c_{3y}^2} \cdot \frac{1}{L_{34} + C_y} \quad (A-9)$$

Finally, we write the conditions of stigmatism $B_{4x} = B_{4y}$

$$\mu_x L_{23} + B_x (\mu_x \pm 1) \mp (A_x + C_x) = \mu_y L_{23} + B_y (\mu_y \pm 1) \mp (A_y + C_y) \quad (A-10)$$

We therefore have five equations for the three unknown quantities L_{12} , L_{23} , L_{34} . Solving for these we find

$$L_{12} = \frac{\pm \lambda_x \lambda_y (C_x - C_y) - \lambda_y v_x A_x + \lambda_x v_y A_y}{\lambda_y v_x - \lambda_x v_y} \quad (A-11)$$

$$L_{34} = \frac{\pm v_x v_y (A_x - A_y) - \lambda_x v_y C_x + \lambda_y v_x C_y}{\lambda_x v_y - \lambda_y v_x} \quad (A-12)$$

$$L_{23} = \frac{(\mu_y \pm 1) B_y - (\mu_x \pm 1) B_x \pm (A_x - A_y + C_x - C_y)}{\mu_x - \mu_y} \quad (A-13)$$

These relations allow the calculation of the interlens spacings in terms of the quadrupole parameters. Moreover, in order to comply with the imposed requirements (phase shift of π or 2π plus stigmatism), the quadrupole parameters must be related by the two equations

$$(\lambda_x v_y - \lambda_y v_x) (\mu_x - \mu_y) \left(\frac{1}{c_{2x}^2 \lambda_x} \pm \frac{1}{c_{3x}^2 v_x} \right) =$$

$$\left[\lambda_y (C_y - C_x) \pm v_y (A_x - A_y) \right] \left[A_x - A_y + C_x - C_y - (1 \pm \mu_y) (B_x - B_y) \right] \quad (A-14)$$

$$(\lambda_x \nu_y - \lambda_y \nu_x) (\mu_x - \mu_y) \left(\frac{1}{c_{2y}^2 \lambda_y} \pm \frac{1}{c_{3y}^2 \nu_y} \right) =$$

$$\left[\lambda_x (C_y - C_x) \pm \nu_x (A_x - A_y) \right] \left[A_x - A_y + C_x - C_y - (1 \pm \mu_x) (B_x - B_y) \right] .$$

(A-15)

These relations are quite general; they apply to a quadruplet composed of arbitrary quadrupoles. For instance, we may consider a structure of the type cddc (dccd) or cdcd (dcdc) or any other combination.

The formulae simplify considerably if some of the quadrupoles of the quadruplet are equal or if symmetric or antisymmetric structures are considered. Application of the formulae obtained to these cases will be made in a forthcoming paper.