COO-3067(2)-14

MULTIPLICITY DISTRIBUTION IN THE MULTIPERIPHERAL MODEL AND A ONE-DIMENSIONAL GAS

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This research was supported in part by the U. S. Atomic Energy Commission.

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ABSTRACT

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An alternative systematic expansion method is developed, which is different from the virial series but appears to be of a greater practical value for this particular class of physical problems.

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1. Introduction

In this paper, we shall discuss the exact gas-analog problem in statistical mechanics that corresponds to the multiplicity distribution at high energy in the multiperipheral model of Amati, Fubini and Stanghellini[1] for the ϕ^3 theory (hereafter referred to either as the ϕ^3-multiperipheral model, or simply as the multiperipheral model). The interaction Lagrangian is assumed to be

$$(3!)^{-1} m g \phi^3 \tag{1}$$

where ϕ is a scalar field, m denotes its mass, and g is the dimensionless coupling constant. In the ϕ^3-multiperipheral model, the two-body elastic scattering is given simply by the sum of all t-channel ladder diagrams; the corresponding absorptive parts give then the multiplicity distribution. Such a sum of ladder diagrams is of interest since as is well known, it represents on the one hand the sum of all "leading" diagrams in a perturbation expansion of the ϕ^3 theory at high energy, and on the other hand, it gives the simplest prototype of field-theoretic models that exhibit the Regge behavior for elastic scattering[2], and a $\ln s$ dependence for multiplicity[1]. There exists already quite a sizable literature[3-6] which discusses the similarity between the meson distribution in a multiperipheral type model and the ensemble distribution of a gas system in statistical mechanics. However, as yet, the precise formulation and the explicit interaction of the gas-analog system have not been given. The purpose of this note is to provide this needed information in order to complete the connection between these two different types of physical problem.

In section 2, the ϕ^3-multiperipheral model is briefly reviewed. The equivalence between the multiperipheral model and its one-dimensional gas-analog is discussed in section 3 (and proved in Appendix A). As we shall see, in the gas-analog the potential energy U_N between N atoms of the gas is $\geqq 0$ everywhere; it can be decomposed into a sum of a two-body potential V_2 between only the nearest neighbors, a three-body potential V_3 between only the nearest and the next nearest neighbors, etc.,

$$U_N = V_2 + V_3 + V_4 + \cdots .$$

The explicit forms of V_2, V_3, $\cdots$, are given and the corresponding virial series discussed.

In section 4, an alternative systematic expansion method is developed, which is different from the virial series, but appears to be of a greater practical value, especially if the gas pressure p is not too small. In the first approximation of this new expansion method, we set the potential U_N to consist of only the two-body potentials

$$U_N \cong V_2 = \Sigma\, v(x_i)$$

where x_i is the distance between the i^{th} and the $(i+1)^{th}$ atoms. The corresponding equation of state is shown to be given by the simple formula

$$z^{-1} = \int_0^\infty dx \, \exp\left[-v(x) - p\,x\right] \tag{2}$$

where z is the fugacity. By using the functional dependence of p on z, one can readily determine the multiplicity distribution. The subsequent approximations of including also the three-body potential V_3, and then the four-body potential V_4, etc. are discussed in the same section (and also in Appendix C). The method developed is of a rather general character, not restricted to the specific ϕ^3-multiperipheral model. It is suggested that expressions such as (2) may be used to analyse phenomenologically the meson multiplicity problem in realistic cases of high energy collisions. From the observed multiplicity distributions (in the so-called "central region"), one can determine an effective "potential", which may be partly attractive and partly repulsive; by using the same potential, one can then calculate the m-body correlation functions and compare the results with measurements on various inclusive reactions.

2. Multiperipheral Model

For definiteness, let us consider the Lagrangian (1) and discuss the physical process of producing $(N+1)$ mesons

$$\phi(a) + \phi(b) \rightarrow (N+1)\,\phi$$

where N may vary from 1 to ∞, and a and b denote respectively the two initial 4-momenta. In the ϕ^3-multiperipheral model, the rate of this reaction is determined by only one Feynman diagram, which is given by (A) of Figure 1; in this diagram $q_1, q_2, \cdots q_{N+1}$ denote respectively the 4-momenta of the final mesons, and $k_1, k_2, \cdots k_N$ the 4-momenta of the N virtual meson lines. The Feynman amplitude of diagram (A) is

$$M_{N+1} = \prod_1^N (k_i^2 + m^2)^{-1} \quad , \qquad (3)$$

and the corresponding cross-section is

$$\sigma_{N+1} = 4\pi^4 (a_0\, b_0\, v)^{-1} (mg)^{2N+2} \int |\, M_{N+1}\,|^2 \prod_1^{N+1} \left[(16\pi^3)^{-1}\, d^3 q_j/(q_j)_0 \right]$$
$$\times \quad \delta^4 \left(\sum_1^{N+1} q_j - a - b \right) \qquad (4)$$

where the subscript 0 denotes the energy-component, v is the relative velocity between the two initial mesons, and s is the square of their center-of-mass energy. The sum of σ_{N+1} gives the total cross-section

$$\sigma_{total} = \sum_{N=1}^{\infty} \sigma_{N+1} \quad . \qquad (5)$$

Throughout the paper, we are interested in the meson multiplicity problem only in the limit $\ln s \to \infty$. In this limit, as is well-known[2] (and as will also be proved by using the gas-analog discussed in the next section), the total cross-section exhibits a Regge-pole behavior:

$$\sigma_{total} \sim s^{\alpha - 1} \quad . \tag{6}$$

It is convenient to use the laboratory frame Σ_{lab} in which $\phi(a)$ is at rest; i.e.,

$$a_0 = m \quad \text{and} \quad b_0 = (2m)^{-1} s - m \quad .$$

Let us introduce a set of N positive variables $x_1, x_2, \cdots x_N$, defined by

$$(k_1)_0 = b_0 e^{-x_1}$$

and for $1 < i \leqq N$

$$(k_i)_0 = (k_{i-1})_0 e^{-x_i} \tag{7}$$

where the subscript 0 denotes the energy-component of the relevant 4-momenta in Σ_{lab}. The energy of the final mesons in Σ_{lab} is then given by

$$(q_1)_0 = b_0(1 - e^{-x_1})$$

and for $1 < i \leqq N$

$$(q_i)_0 = b_0(1 - e^{-x_i}) \exp\left(- \sum_1^{i-1} x_j\right) \quad . \tag{8}$$

In the integration (4), one may trivially eliminate the $d^3 q_{N+1}$ integration by using the three-dimensional δ-function; the remaining δ-function of energy conservation becomes simply

$$\delta\left[(q_{N+1})_0 - m - b_0 \exp\left(-\sum_1^N x_i\right)\right] \quad . \tag{9}$$

Since $k_N = q_{N+1} - a$, one finds

$$(q_{N+1})_0 = (2m)^{-1}\left[k_N^2 + m^2 - q_{N+1}^2\right] \quad . \tag{10}$$

Because of the square of the Feynman propagator $(k_i^2 + m^2)^{-2}$, in the integral (4) the value of any k_i^2, including k_N^2, is on the average $O(m^2)$. Therefore, the on-mass-shell condition $q_{N+1}^2 + m^2 = 0$, together with (10), implies that the value of $(q_{N+1})_0$ is on the average $O(m)$.

To carry out the gas-analog, it is more convenient to impose the on-mass-shell condition

$$q_i^2 + m^2 = 0$$

only for $1 \leqq i \leqq N$. For the $(N+1)^{th}$ meson, we shall replace its on-mass-shell condition by

$$(q_{N+1})_0 = (1+\lambda)\, m \tag{11}$$

where λ is a positive constant, independent of s. In the integral (4), on account of (10), the important integration region now becomes one in which $q_{N+1}^2 \sim O(m^2)$ [instead of $(q_{N+1})_0 \sim O(m)$ and $q_{N+1}^2 = -m^2$]. The δ-function (9) becomes then

$$b_0^{-1} \exp\left(\sum_1^N x_i\right) \delta\left(\sum_1^N x_i - L\right) \tag{12}$$

where

$$L = \ln\left[b_0/(\lambda m)\right] \rightarrow \ln(s/m^2) \tag{13}$$

as $\ln s \rightarrow \infty$. The limiting value of L is therefore independent of λ. [If one wishes, one may choose λ so that $\langle q^2 + m^2 \rangle_{Av} = 0$ where $\langle\ \rangle_{Av}$ denotes some suitably defined average.]

It is clear that the above modification does not alter the multiplicity distribution in the $\ln s \rightarrow \infty$ limit.

3. A One-Dimensional Gas

Next, we consider a one-dimensional classical Boltzmann gas of N identical but distinguishable atoms on a ring of length L. Let U_N be the potential energy. For our purpose, we need only consider isotherms; therefore, the natural unit for energy is[7]

$$kT = 1$$

where k is the Boltzmann constant and T the absolute temperature. To evaluate the partition function Q_N, it is only necessary to consider an <u>ordered</u> set[8], say atoms 1, 2, ·· N distributed in a strictly sequential order with x_1 as the absolute distance between atoms 1 and 2, x_2 that between atoms 2 and 3, etc., as illustrated by diagram (B) of Figure 1. One has

$$Q_N = \int \delta\left(\sum_1^N x_i - L\right) \exp(-U_N) \prod_1^N dx_i \tag{14}$$

where $x_i \geq 0$. The grand partition function $\mathcal{Q}$ is given by

$$\mathcal{Q} = \sum_N z^N Q_N \tag{15}$$

where z denotes the fugacity. The thermodynamical pressure of the system is related to $\mathcal{Q}$ by

$$p = \lim_{L\to\infty} L^{-1} \ln \mathcal{Q} \quad .$$

By using the dependence of the pressure p on the fugacity z, one can compute directly the ensemble distribution of N as $L \to \infty$; e.g., the density ρ is given by

$$\frac{dp}{d\ln z} = \rho = \lim_{L\to\infty} L^{-1} \langle N \rangle \quad ,$$

the number fluctuation $\langle N^2 \rangle - \langle N \rangle^2$ is given by

$$\frac{d^2 p}{d(\ln z)^2} = \lim_{L\to\infty} L^{-1} \left[\langle N^2 \rangle - \langle N \rangle^2 \right] \quad ,$$

etc. Since the relative probability of finding N particles at a given length L is $z^N Q_N$, as L becomes large the relative probability approaches asymptotically the product of z^N multiplied by

$$(2\pi i)^{-1} \oint z^{-(N+1)} e^{pL} dz \tag{16}$$

where the contour can be any counter-clockwise small closed curve around the origin in the complex z-plane. [For a finite L, (16) holds if p is replaced by $p_L \equiv L^{-1} \ln \mathcal{Q}$, without taking the limit $L \to \infty$.]

As we shall prove in Appendix A, the multiplicity distribution in the ϕ^3-multiperipheral model is identical to the above number distribution in the grand canonical ensemble, provided that the potential U_N is of a specific form [determined by Eq. (A14) in Appendix A]. We may decompose

$$U_N = V_2 + V_3 + V_4 + \cdots \tag{17}$$

where V_2 is a sum of only two-body nearest neighbor potentials

$$V_2 = \sum_{i=1}^{N} v_2(x_i) \quad , \tag{18}$$

V_3 is a sum of three-body potentials between only the nearest and the next nearest neighbors

$$V_3 = \sum_{i=1}^{N} v_3(x_i, x_{i+1}) \tag{19}$$

and V_4 is a sum of four-body potentials between only the nearest and the next two nearest neighbors, etc. The explicit form for the two-body potential is given by

$$\exp\left[-v_2(x)\right] = \frac{1-e^{-x}}{1-e^{-x}+e^{-2x}}\left[1+\frac{e^{-x}-e^{-2x}}{1-e^{-x}+e^{-2x}}\ln\left(e^{-x}-e^{-2x}\right)\right] \tag{20}$$

where $x = x_i$ denotes the absolute value of the distance between any two nearest neighbor atoms, say i and $i+1$. The three-body potential between any three neighboring atoms, say i, $i+1$ and $i+2$, is given by

$$\exp\left[-v_3(x, x')\right] = \left[1+\epsilon\ln\frac{\epsilon}{1+\epsilon}\right]^{-1}\left[1+\epsilon'\ln\frac{\epsilon'}{1+\epsilon'}\right]^{-1}$$

$$\cdot\left[1+\lambda\ln\frac{(1+\epsilon)\lambda}{1+(1+\epsilon)\lambda} + \epsilon(1-\lambda)\ln\frac{\epsilon}{1+\epsilon} + \frac{\epsilon\lambda^2}{1+\lambda}\ln\frac{\epsilon\lambda}{1+(1+\epsilon)\lambda}\right] \tag{21}$$

where

$$\epsilon = \frac{e^{-x}-e^{-2x}}{1-e^{-x}+e^{-2x}},$$

$$\epsilon' = \frac{e^{-x'}-e^{-2x'}}{1-e^{-x'}+e^{-2x'}},$$

$$\lambda = (\epsilon'/\epsilon)\,e^{-x},$$

$x = x_i$ is the absolute value of the distance between atoms i and $i+1$, and $x' = x_{i+1}$ is that between atoms $i+1$ and $i+2$. The general expression of v_m is somewhat complicated, and will be discussed in Appendix A.

From (20) and (21), one can establish that both v_2 and v_3 are repulsive; i.e., $v_2(x) \geqq 0$ at arbitrary $x \geqq 0$, and $v_3(x, x') \geqq 0$ at arbitrary $x \geqq 0$ and $x' \geqq 0$. In Appendix A, it is shown that the total potential U_N is also repulsive; we find

$$U_N \geqq 0 \tag{22}$$

for arbitrary $x_i \geqq 0$, and $U_N \to \infty$ as any single $x_j \to 0$. From (20) and (21), one also sees that v_2 and v_3 represent short-range forces; i.e., $v_2(x) \to 0$ exponentially as $x \to \infty$, and $v_3(x, x') \to 0$ exponentially as either $x \to \infty$ or $x' \to \infty$. In Appendix A, as will be shown by Eq. (A22), similar short-range properties hold for other v_m's as well, provided that m is finite, independent of N. For $m \sim 0(N)$, the corresponding m-body force is clearly long range. As will also be shown in Appendix A [Eq. (A28)], for configurations near the average one:

$$x_1 = x_2 = \cdots = x_N = L/N ,$$

the total potential U_N has an upper bound given by

$$U_N = V_2 + V_3 + \cdots + V_N < N \cdot \text{constant} \tag{23}$$

where the constant denotes a finite function of the average interatomic distance L/N. Because this upper bound is linear in N, the presence of long-range interactions such as V_N, V_{N-1}, $\cdots$ does not jeopardize the thermodynamical limit. We note that due to the one-dimensional character of the gas and the pure repulsive nature of the forces, there should not be any phase transition for this particular system.

The detailed correspondence between this one-dimensional gas and the multiperipheral model is given in Table 1. For the gas system, the average number of particles is clearly proportional to the length L at large L. By using Table 1, or Eq. (13), one derives the familiar result[1] that the average meson multiplicity in the multiperipheral model increases linearly with ln s at large s. For the gas system, ln $\mathcal{Z}$ is also proportional to L at large L. By using Table 1, or Eq. (A13) in Appendix A, one obtains Eq. (6) which gives the Regge behavior for the multiperipheral model.

The explicit form of V_2, V_3, . . enables one to directly evaluate the virial series

$$p = \sum_{1}^{\infty} b_\ell z^\ell , \tag{24}$$

by using the well-known cluster expansion technique, developed by Mayer and Mayer[9]. For example,

$$b_1 = 1 ,$$

$$b_2 = \int_0^\infty f(x)\, dx \tag{25}$$

where

$$f(x) = e^{-V_2(x)} - 1 ,$$

$$b_3 = b_2^2 - \int_0^\infty x\, f(x)\, dx + \int_0^\infty dx \int_0^\infty dx'\, F(x, x') \tag{26}$$

where

$$F(x, x') = \left[1 + f(x)\right] \left[1 + f(x')\right] \left[e^{-V_3(x, x')} - 1\right] ,$$

and so on. For the specific $v_2(x)$ given by (20), one finds

$$b_2 = -\frac{8}{3\sqrt{3}} \sum_1^{\infty} m^{-2} \sin\left(\frac{m\pi}{3}\right) = -1.5626 \tag{27}$$

in which the sum equals the Clausen's integral[10] $f(\theta)$ at $\theta = \pi/3$. Similarly, b_3 can be evaluated, and the approximate numerical value is[11]

$$b_3 = 1.9573 \tag{28}$$

in which the ratio between the contribution of the three-body potential v_3 and that of the two-body potential v_2 is $\sim -1/4$.

From Table 1, one may derive the dependence of the Regge-pole power a on g^2 for the ϕ^3-multiperipheral model from the virial expansion:

$$a = -1 + \left(\frac{g}{4\pi}\right)^2 + b_2\left(\frac{g}{4\pi}\right)^4 + b_3\left(\frac{g}{4\pi}\right)^6 + \cdots . \tag{29}$$

The coefficients of $\left(\frac{g}{4\pi}\right)^2$ and $\left(\frac{g}{4\pi}\right)^4$ have been calculated in the literature[2, 12] and of course they agree with the above virial expansion results.

4. An Alternative Expansion Method

For most applications, since at high energy $\sigma_{total} \sim s^{\alpha-1} \sim$ constant, one is interested in multiplicity problems in which $\alpha \cong 1$; this corresponds to a gas system with its pressure $p \cong 2$. At such high pressure, the virial series $p = \Sigma\, b_\ell\, z^\ell$ does not offer the most practical method for evaluating the function $p = p(z)$. As we shall see, for the class of problems in which the potential U_N is a sum $U_N = V_2 + V_3 + \cdots$ where V_2 consists of only two-body interactions between the nearest neighbors and V_3 only three-body interactions between the nearest and the next nearest neighbors, $\cdots$, there exists an alternative new systematic expansion method, different from the virial series, but which appears to be more useful for practical applications. This new systematic expansion method is applicable to any one-dimensional gas with such a potential, not restricted to the specific multiperipheral model discussed in the previous two sections.

We observe that for large values of L and N, the function $\delta(\Sigma x_i - L)$ in (14) may be replaced by[13]

$$\exp\left[-\beta(\Sigma x_i - L)\right] \quad . \tag{30}$$

The partition function Q_N can then be written as

$$Q_N = e^{\beta L}\, h^N \tag{31}$$

where h is a function of β and N , given by

$$h^N = \int \exp\left[-U_N - \beta\, \Sigma x_i\right] \Pi\, dx_i \tag{32}$$

in which each x_i is integrated independently from 0 to ∞ . Correspondingly, the grand partition function becomes

$$\mathcal{Q} = \sum_{N} (z\,h)^{N} e^{\beta L} \qquad (33)$$

where β must be regarded as a function of N and L, determined by

$$\left(\frac{\partial \ln h}{\partial \beta}\right)_{N} = -N^{-1} L \; . \qquad (34)$$

In the grand canonical ensemble, the relative probability distribution $(zh)^{N} e^{\beta L}$ has a maximum at $N = \bar{N}(L)$, which can be obtained by setting the derivative of the relative probability with respect to N to be zero. One finds that at $N = \bar{N}(L)$,

$$h = z^{-1} \; . \qquad (35)$$

As $L \to \infty$, on account of (33) and (35), the value of β, evaluated at $N = \bar{N}(L)$, approaches the thermodynamical pressure p. Thus, by taking the logarithm of (35), we derive the basic equation

$$-\ln z = \lim_{N \to \infty} N^{-1} \ln \int \exp\left[-U_N - p \sum_{1}^{N} x_i\right] \prod dx_i \qquad (36)$$

in which, as in (32), all x_i are integrated independently from 0 to ∞.

The new expansion method consists of first neglecting all interactions, then including only V_2; then only $V_2 + V_3$, etc.:

1. In the zeroth approximation, we set

$$U_N = 0 \; .$$

The system satisfies the perfect gas law

$$p = z = \rho \; .$$

From (16), it follows that the number distribution is given by the familiar Poisson formula. For the ϕ^3-multiperipheral model, since U_N is positive, this zeroth approximation is also an upper bound; i.e., with the inclusion of $U_N \gtrless 0$

$$p \lesseqgtr z \tag{37}$$

or[14], by using Table 1,

$$\alpha \lesseqgtr -1 + \left(\frac{g}{4\pi}\right)^2 \tag{38}$$

where the equality holds only in the weak coupling limit.

2. The first approximation is to set

$$U_N = V_2 = \sum_1^N v_2(x_i) \quad . \tag{39}$$

By using (36), we find for arbitrary two-body potential $v_2(x)$

$$z^{-1} = \int_0^\infty dx \exp\left[-v_2(x) - px\right] \quad . \tag{40}$$

If one wishes, one may also expand p as a power series of z :

$$p = \sum b_\ell z^\ell \quad .$$

From the above closed expression (40), it follows directly that $b_1 = 1$, b_2 is given by (25) and b_3 is given by (26) with $F = 0$, etc. At large z, only the value of $\exp\left[-v_2(x)\right]$ near $x = 0$ is of importance. We may expand

$$\exp\left[-v_2(x)\right] = a_0 + a_1 x + \cdots \quad .$$

Equation (40) implies that

$$z^{-1} = a_0 p^{-1} + a_1 p^{-2} + \cdots \ .$$

If the potential is infinitely repulsive at $x = 0$, as is the case in the ϕ^3-multiperipheral model, then $a_0 = 0$. As $z \to \infty$, $p \to (a_1 z)^{\frac{1}{2}}$. For the ϕ^3-multiperipheral model, $v_2(x)$ is given by (20) which gives $a_1 = 1$, and therefore

$$p \to z^{\frac{1}{2}} \qquad \text{as} \qquad z \to \infty \ .$$

As will be shown in Appendix A [Eq. (A34)], for the ϕ^3-multiperipheral model the inclusion of all other V_3, V_4, $\cdots$ forces always increases the value of the repulsive potential, i.e.,

$$U_N \geqq V_2 \ .$$

Therefore the pressure p_1 determined by the first approximation (where the subscript 1 is added for clarity) also forms an upper bound for the rigorous pressure p, which is calculated with the entire U_N without any approximation; we derive then at any given $z \geqq 0$, the inequality

$$p \leqq p_1(z) \tag{41}$$

where, according to (20) and (40), $p_1(z)$ is given by

$$z^{-1} = \int_0^\infty \frac{1 - e^{-x}}{1 - e^{-x} + e^{-2x}} \left[1 + \frac{e^{-x} - e^{-2x}}{1 - e^{-x} + e^{-2x}} \ln\left(e^{-x} - e^{-2x}\right) \right] e^{-p_1 x}\, dx \ . \tag{42}$$

The inequality (41) is, of course, a better inequality than (37). As $z \to \infty$, (42) gives $p_1 \to z^{\frac{1}{2}}$, and therefore (41) implies

$$p \leqq z^{\frac{1}{2}} \qquad \text{as} \qquad z \to \infty \;, \tag{43}$$

or, by using Table 1,

$$a \leqq (4\pi)^{-1} g \quad \text{in the strong coupling limit,} \tag{44}$$

in agreement with the bound derived by Tiktopoulos and Treiman[14].

3. In the second approximation, we equate

$$U_N = V_2 + V_3 = \sum_{i=1}^{N} \left[v_2(x_i) + v_3(x_i\,,\, x_{i+1}) \right] \;. \tag{45}$$

It is convenient to consider a Hilbert space of base-vectors $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$, $\cdots$, which satisfy the usual orthonormal relation

$$\int_0^\infty \psi_i(x)\, \psi_j(x)\, dx = \delta_{ij} \;. \tag{46}$$

Among these, $\psi_1(x)$ is chosen to be

$$\psi_1(x) = (z_1)^{\frac{1}{2}} \exp\left\{ -\tfrac{1}{2} \left[v_2(x) + p x \right] \right\} \tag{47}$$

where z_1 is the normalization constant, defined by

$$z_1^{-1} = \int_0^\infty dx \exp\left[-v_2(x) - p x \right] \;; \tag{48}$$

the other base-vectors $\psi_2(x)$, $\psi_3(x)$, $\cdots$ can be arbitrary real functions that satisfy (46) and, together with $\psi_1(x)$, the completeness theorem. Let us define two real matrices ϵ and H_o in this Hilbert space:

$$\epsilon_{ij} \equiv \int_0^\infty dx \int_0^\infty dy\ \psi_i(x) < x | \epsilon | y > \psi_j(y) , \qquad (49)$$

$$< x | \epsilon | y > = \psi_1(x) \left[e^{-V_3(x,y)} - 1 \right] \psi_1(y) , \qquad (50)$$

$$(H_o)_{11} \equiv 1 \qquad (51)$$

and

$$(H_o)_{ij} \equiv 0 \quad \text{for all other } i \text{ and } j . \qquad (52)$$

Thus, the integral in (36) becomes

$$\int \exp \left[- U_N - p \sum_1^N x_i \right] \prod dx_i = z_1^{-N} \operatorname{trace} (H_o + \epsilon)^N . \qquad (53)$$

The logarithm of (53), at fixed N, can be readily evaluated as a power series in ϵ. By using (36) and taking the limit $N \to \infty$, one can verify directly that

$$- \ln z = - \ln z_1 + \epsilon_{11} + (\epsilon^2)_{11} - \frac{3}{2} (\epsilon_{11})^2$$
$$+ (\epsilon^3)_{11} - 4 (\epsilon^2)_{11} \epsilon_{11} + \frac{10}{3} (\epsilon_{11})^3 + \cdots . \qquad (54)$$

Since z_1 and

$$\epsilon_{11} = \int \psi_1(x) < x | \epsilon | y > \psi_1(y)\, dx\, dy ,$$

$$(\epsilon^2)_{11} = \int \psi_1(x) < x | \epsilon | y > < y | \epsilon | z > \psi_1(z)\, dx\, dy\, dz ,$$

etc., are functions of p, Eq. (54) determines $z = z(p)$.

Equation (54) can also be derived by a simpler method, without any direct calculations. We observe that the general form of the series (54) is independent of whether the matrix ϵ is symmetric or not. Thus, we may consider the special case of a symmetric three-body potential $v_3(x, y) = v_3(y, x)$, and therefore $\epsilon_{ij} = \epsilon_{ji}$. Let $\lambda = \lambda(p)$ be the largest eigenvalue of $H_o + \epsilon$. By using (36) and (53), we find the closed expression

$$\lambda(p) = z_1/z \quad . \tag{55}$$

The series expansion can then be obtained by noting that H_o has only one eigenvalue $= 1$, while all its other eigenvalues are 0. Thus, as $\epsilon \to 0$, $\lambda \to 1$ and (55) reduces to (40). For $\epsilon \neq 0$, the power series expansion of λ is given by the familiar perturbation formula

$$\lambda = 1 + \epsilon_{11} + \sum_{i \neq 1} \lambda^{-1} \epsilon_{1i}\epsilon_{i1} + \sum_{\substack{i \neq 1 \\ j \neq 1}} \lambda^{-2} \epsilon_{1i}\epsilon_{ij}\epsilon_{j1} + \cdots$$

which, together with (55), leads to (54).

The higher order approximations including V_4, V_5, $\cdots$ can be carried out in a similar manner. The details are given in Appendix C.

Remarks:

As noted earlier, the method developed in this section is of a rather general character, not restricted to the specific multiperipheral model discussed in the previous two sections. For practical applications, it seems reasonable to try first the approximation of only two-body nearest neighbor forces. Equation (40) can be used phenomenologically to determine an effective two-body potential $v_2(x)$ from the observed meson multiplicity distributions in high energy collisions, provided that $\ln(s/m^2)$ is sufficiently large and that the average multiplicity and its fluctuation are indeed linear in $\ln(s/m^2)$. Within this approximation, one may apply the same effective "potential" to evaluate the m-body correlation functions, which can then be compared with various inclusive reactions.

Acknowledgement

I wish to thank A. Mueller for giving me some lectures on topics related to the multiperipheral model. A major part of this work was done when I was visiting the Scuola Normale Superiore, Pisa, Italy in the early summer of 1972; I wish to thank G. Bernardini and L. A. Radicati for their kind hospitality.

Appendix A

To derive the explicit potential of the gas system, we start with Eq. (4) for the Feynman diagram (A) in Figure 1. In the laboratory frame, let $\hat{k}_{i-1}$ be the unit vector parallel to the three-momentum of the virtual meson line k_{i-1}. The three-momentum $\vec{q}_i$ of the i^{th} final meson can be written as

$$\vec{q}_i = \vec{\delta}_i + \left[(q_i)_0 - \tfrac{1}{2}(q_i)_0^{-1}(m^2 + \vec{\delta}_i^2)\right] \hat{k}_{i-1} \qquad \text{(A1)}$$

where $\vec{\delta}_i \perp \hat{k}_{i-1}$, and the energy $(q_i)_0$ is assumed to be $\gg m^2$ and $\vec{\delta}_j^2$. Since, as noted before, k_i^2 is of the order of m^2, $(k_i)_0$ may be assumed to be $\gg |k_i^2|^{\frac{1}{2}}$. By using $k_i = k_{i-1} - q_i$ and (7), one has

$$k_i^2 + m^2 = (1 - e^{-x_i})^{-1}\left[\vec{\delta}_i^2 + m^2 + k_{i-1}^2 e^{-x_i}(1 - e^{-x_i})\right] \qquad \text{(A2)}$$

where $k_i^2 \equiv \vec{k}_i^2 - (k_i)_0^2$. Through induction, (A2) can be written as

$$k_i^2 + m^2 = (1 - e^{-x_i})^{-1}\left[\sum_{a=1}^{i} A_{ia}\vec{\delta}_a^2 + m^2 B_i\right] \qquad \text{(A3)}$$

where

$$A_{ii} = 1 \ ,$$

$$A_{ia} = (1 - e^{-x_a})^{-1}(1 - e^{-x_i})\exp\left[-\sum_{j=a+i}^{i} x_j\right] \qquad \text{(A4)}$$

for $a < i$, and

$$B_i = \sum_{a=1}^{i} A_{ia}(1 - e^{-x_a} + e^{-2x_a}) \ .$$

The usual parametric form of Feynman propagator gives

$$\prod_i (k_i^2 + m^2)^{-2} = \int e^{-E} \prod_i (1 - e^{-x_i})^2 \, \ell_i \, d\ell_i \qquad \text{(A5)}$$

where each ℓ_i varies independently from 0 to ∞, and

$$E = \sum_{a=1}^{N} \vec{\delta}_a^{\,2} \left(\sum_{j=a}^{N} \ell_j A_{ja} \right) + m^2 \sum_{j=1}^{N} \ell_j B_j \quad .$$

The integration of (A5) over $\Pi\, d^2 \delta_i$ is an elementary one. One finds

$$\int \prod_{i=1}^{N} (k_i^2 + m^2)^{-2} d^2\delta_i = \pi^N \prod_{i=1}^{N} (1 - e^{-x_i})^2 \int \prod_{a=1}^{N} \left(\sum_{j=a}^{N} \ell_j A_{ja} \right)^{-1} \ell_a \, d\ell_a$$

$$\times \quad \exp\left(-m^2 \sum_{i=1}^{N} \ell_i B_i\right) \quad . \qquad \text{(A6)}$$

It is convenient to change the variables from $\ell_1, \ell_2, \cdots \ell_N$ to $\xi_1, \xi_2, \cdots \xi_N$, defined by

$$(1 - e^{-x_a} + e^{-2x_a})^{-1} \xi_a \equiv m^2 \sum_{j=a}^{N} \ell_j A_{ja} \quad . \qquad \text{(A7)}$$

Therefore,

$$m^2 \sum_i \ell_i B_i = \sum_i \xi_i \quad . \qquad \text{(A8)}$$

From (A4), one can readily establish $A_{ab} A_{bc} = A_{ac}$ for arbitrary a, b and c that satisfy $a \geq b \geq c$. The inverse transformation from ξ_i to ℓ_i is

$$m^2 \ell_N = (1 - e^{-x_N} + e^{-2x_N})^{-1} \xi_N$$

and

$$m^2 \ell_a = (1 - e^{-x_a} + e^{-2x_a})^{-1} (\xi_a - \lambda_{a+1} \xi_{a+1}) \qquad \text{(A9)}$$

for $a = 1, 2, \cdots, N-1$, where

$$\lambda_{a+1} = \left(\frac{e^{-x_{a+1}} - e^{-2x_{a+1}}}{1 - e^{-x_a}} \right) \left(\frac{1 - e^{-x_a} + e^{-2x_a}}{1 - e^{-x_{a+1}} + e^{-2x_{a+1}}} \right). \qquad \text{(A10)}$$

By using (4), (12), (A6) and

$$\prod_{j=1}^{N} (q_j)_0^{-1} d^3 q_j = \prod_i (1 - e^{-x_i})^{-1} e^{-x_i} dx_i \, d^2\delta_i \;,$$

one finds that, apart from a multiplicative constant <u>independent</u> of N and s, the cross-section σ_{N+1} in the ϕ^3-multiperipheral model is given by

$$\sigma_{N+1} \propto s^{-2} z^N Q_N = s^{-2} z^N \int e^{-\Delta} \delta\left(\sum_i^N x_i - L\right) \prod_{i=1}^{N} \frac{(1 - e^{-x_i})\, dx_i}{1 - e^{-x_i} + e^{-2x_i}} \tag{A11}$$

where $z = (4\pi)^{-2} g^2$ and Δ is a function of $x_1, x_2 \cdots x_N$, defined by the integral

$$e^{-\Delta} = \int (\xi_1 - \lambda_2 \xi_2)(\xi_2 - \lambda_3 \xi_3) \cdots (\xi_{N-1} - \lambda_N \xi_N)\, \xi_N \prod_{a=1}^{N} \xi_a^{-1} e^{-\xi_a} d\xi_a \tag{A12}$$

in which the integration domain extends over $\xi_1 \geqq \lambda_2 \xi_2$, $\xi_2 \geqq \lambda_3 \xi_3, \cdots$ $\xi_i \geqq \lambda_{i+1} \xi_{i+1}, \cdots$, and $\xi_N \geqq 0$. On account of (A11), as $\ln s \to \infty$, the total cross-section σ_{total} satisfies

$$\ln \sigma_{total} + 2 \ln s \to \ln \mathcal{Q} \tag{A13}$$

where

$$\mathcal{Q} = \sum_N z^N Q_N \;.$$

The function Q_N denotes the partition function of the gas-analog, and $\mathcal{Q}$ is the corresponding grand partition function. Upon comparing (A11) with (14), we find the potential U_N for the gas system to be

$$e^{-U_N} = e^{-\Delta} \prod_i \frac{1 - e^{-x_i}}{1 - e^{-x_i} + e^{-2x_i}} \quad . \tag{A14}$$

To derive the two-body potential $v_2(x_i)$, we need only consider, at a fixed x_i , the limit of all other $x_j \to \infty$. [To avoid problems with the boundary, we choose $i \neq 1$.] In this limit, all three-body forces, four-body forces, etc., are, by definition, zero; therefore,

$$U_N \to V_2 \to v_2(x_i) \quad . \tag{A15}$$

On the other hand, in the same limit, according to (A10),

$$\lambda_j \to 0 \qquad \text{for all} \quad j \neq i$$

and

$$\lambda_i \to \epsilon_i \equiv \frac{e^{-x_i} - e^{-2x_i}}{1 - e^{-x_i} + e^{-2x_i}} \quad . \tag{A16}$$

Thus,

$$\begin{aligned} e^{-\Delta} &\to \int_0^\infty d\xi_{i-1} \int_0^{\epsilon_i^{-1}\xi_{i-1}} d\xi_i \; \xi_{i-1}^{-1}(\xi_{i-1} - \epsilon_i \, \xi_i) \exp(-\xi_{i-1} - \xi_i) \\ &= 1 + \epsilon_i \ln \frac{\epsilon_i}{1+\epsilon_i} \end{aligned} \tag{A17}$$

which, together with (A14) and (A15), lead to the explicit two-body force given by (20).

To derive the three-body potential, we keep x_i and x_{i+1} fixed (where $i \neq 1$) , and then consider the limit of all other $x_j \to \infty$. In this limit, by definition,

$$U_N \to v_2(x_i) + v_2(x_{i+1}) + v_3(x_i\, , \, x_{i+1}) \quad . \tag{A18}$$

According to (A10), in the same limit λ_{i+1} remains fixed, $\lambda_i \to \epsilon_i$ which is given by (A16), and all other $\lambda_j \to 0$; therefore, the integral (A12) becomes

$$e^{-\Delta} \to \int (\xi_{i-1}\, \xi_i)^{-1} (\xi_{i-1} - \epsilon_i\, \xi_i)(\xi_i - \lambda_{i+1}\, \xi_{i+1}) \times \exp(-\xi_{i-1} - \xi_i - \xi_{i+1})\, d\xi_{i-1}\, d\xi_i\, d\xi_{i+1} \tag{A19}$$

where the integration domain is

$$\xi_{i-1} \geqq \epsilon_i\, \xi_i\ , \quad \xi_i \geqq \lambda_{i+1}\, \xi_{i+1} \quad \text{and} \quad \xi_{i+1} \geqq 0\ .$$

By using (A 14), (A 18) and (A 19), one obtains the explicit form (21) for $v_3(x_i, x_{i+1})$. In a similar manner, we can derive explicitly the four-body interaction, the five-body interaction, etc.

In general, to derive the m-body potential $v_m(x_i, x_{i+1}, \cdots, x_{i+m-2})$, we keep $x_i, x_{i+1}, \cdots, x_{i+m-2}$ fixed (where $i \neq 1$), and consider the limit of all other $x_j \to \infty$. In this limit, $\lambda_i \to \epsilon_i$ which is given by (A16), $\lambda_{i+1}, \cdots, \lambda_{i+m-2}$ remain unchanged, and all other $\lambda_j \to 0$. The potential U_N becomes

$$U_N \to U_m(x_i, x_{i+1}, \cdots, x_{i+m-2}) \equiv \sum_{k=i}^{i+m-2} v_2(x_k) + \sum_{k=i}^{i+m-3} v_3(x_k, x_{k+1}) + \cdots + v_m(x_i, x_{i+1}, \cdots, x_{i+m-2})\ , \tag{A20}$$

and

$$e^{-\Delta} \to \int (\xi_{i-1} - \epsilon_i\, \xi_i)(\xi_i - \lambda_{i+1}\, \xi_{i+1}) \cdots (\xi_{i+m-3} - \lambda_{i+m-2}\, \xi_{i+m-2})\, \xi_{i+m-2} \times \prod_{k=i-1}^{i+m-2} \xi_k^{-1}\, e^{-\xi_k}\, d\xi_k\ , \tag{A21}$$

where the integration extends over the domain

$$\xi_{i-1} \geqq \epsilon_i \xi_i \ , \cdot \quad \xi_i \geqq \lambda_{i+1} \xi_{i+1} , \cdot \cdot$$

$$\xi_{i+m-3} \geqq \lambda_{i+m-2} \xi_{i+m-2} \quad \text{and} \quad \xi_{i+m-2} \geqq 0 \ .$$

The explicit form of v_m can then be derived by using (A14), (A20) ,and (A21). We note that if $x_i \to \infty$, then $\epsilon_i \to 0$ exponentially; therefore

$$U_m(x_i , x_{i+1} , \cdot \cdot x_{i+m-2}) \to U_{m-1}(x_{i+1} , \cdot \cdot , x_{i+m-2}) \ . .$$

Similarly, if $x_{i+s} \to \infty$ $(s \geqq 1)$, then $\lambda_{i+s} \to 0$ exponentially, and therefore

$$U_m(x_i , x_{i+1} , \cdot \cdot x_{i+m-2}) \to U_{s+1}(x_i , \cdot \cdot x_{i+s-1}) U_{m-s-1}(x_{i+s+1} , \cdot \cdot , x_{i+m-2}) \ .$$

Together, these relations imply the short-range nature of v_m :

$$v_m(x_i , x_{i+1} , \cdot \cdot x_{i+m-2}) \to 0 \quad \text{exponentially} \qquad \text{(A22)}$$

as any single $x_k \to \infty$ where $i \geqq k \geqq i + m - 2$, provided that m is finite, independent of N.

<u>Inequalities:</u>

1. We note that while the range of v_m is short for any finite m, for m of the order of N (therefore, also of the order of L) the corresponding m-body force has to be long range. In order to establish a well-defined thermodynamical limit, we shall show that, at a constant density N/L and for regions near the average configuration

$$x_1 = x_2 = \cdots = x_N = L/N \;, \tag{A23}$$

the total potential energy U_N has an upper bound which increases linearly with N. For the configuration (A23), one has, on account of (A10),

$$\lambda_2 = \lambda_3 = \cdots = \lambda_N \equiv \lambda = \exp\left[-N^{-1}L\right] < 1 \;; \tag{A24}$$

furthermore, because of the inequality

$$\xi^{-1} > e^{-\xi} \;, \tag{A25}$$

$$e^{-\Delta} > \int (\xi_1 - \lambda\xi_2)(\xi_2 - \lambda\xi_3)\cdots(\xi_{N-1} - \lambda\xi_N)\,\xi_N \prod e^{-2\xi_i}\, d\xi_i \;. \tag{A26}$$

The right-hand side of (A26) can be readily evaluated. By taking the logarithm of (A26), one obtains

$$\frac{1}{N}\Delta < 2\ln 2 - 2\ln(1-\lambda) + \frac{2}{N}\sum_{m=1}^{N} \ln(1-\lambda^m) \;. \tag{A27}$$

Since $0 < \lambda < 1$, $\sum_{m=1}^{N} \ln(1-\lambda^m)$ is larger than $(1-\lambda)^{-1}\ln(1-\lambda)$ but less than 0. One finds, as $N \to \infty$,

$$\frac{1}{N}\sum_{1}^{N} \ln(1-\lambda^m) \to 0 \;.$$

Because of (A14), (A27) can be written as (for $N >> 1$)

$$\frac{1}{N} U_N < 2\ln 2 - 3\ln(1-\lambda) + \ln(1-\lambda+\lambda^2) \;. \tag{A28}$$

The inequality (A28) can be easily extended to any configuration $x_1, x_2, \cdots x_N$ in which x_i's can be unequal, but the maximum value λ_{max} of the corresponding $\lambda_2, \lambda_3, \cdots, \lambda_N$ is less than 1. From (A12), one can verify that, keeping $\lambda_{j \neq i}$ fixed,

$$\frac{\partial \Delta}{\partial \lambda_i} > 0 . \tag{A29}$$

Consequently, Δ satisfies the same inequality (A27), provided that λ is replaced by λ_{max}. We note that the inequality (A25) can be easily improved since $\xi^{-1} > e^{-\xi} \left[1 + \frac{1}{2!} \xi + \frac{1}{3!} \xi^2 + \cdots \right]$; therefore, the upper bound (A28) can also be improved.

2. Since $\xi_a^{-1} (\xi_a - \lambda_{a+1} \xi_{a+1}) \leqq 1$, (A12) implies the inequality

$$e^{-\Delta} \leqq 1 \tag{A30}$$

and therefore

$$e^{-U_N} \leqq \prod_i \frac{1 - e^{-x_i}}{1 - e^{-x_i} + e^{-2x_i}} . \tag{A31}$$

Because the right-hand side of (A31) is $\leqq 1$, we establish the repulsive nature of U_N; i.e., for arbitrary $x_i \geqq 0$,

$$U_N \geqq 0 ; \tag{A32}$$

Furthermore, as any single $x_i \to 0$, the right-hand side of (A31) approaches zero, and therefore $U_N \to \infty$.

3. The inequalities (A30) and (A32) can be readily improved. We shall show that

$$e^{-\Delta} \leqq \prod_i \left[1 + \frac{e^{-x_i} - e^{-2x_i}}{1 - e^{-x_i} + e^{-2x_i}} \ln (e^{-x_i} - e^{-2x_i}) \right] \qquad \text{(A33)}$$

and therefore

$$U_N \geqq V_2 \qquad \text{(A34)}$$

where V_2 is the two-body interaction, given by (18) and (20).

<u>Proof.</u> From (A10), it follows that

$$\lambda_a \geqq \epsilon_a \equiv \frac{e^{-x_a} - e^{-2x_a}}{1 - e^{-x_a} + e^{-2x_a}} \qquad \text{(A35)}$$

which, together with (A29), implies

$$e^{-\Delta} \leqq F \qquad \text{(A36)}$$

where

$$F \equiv \int_\Omega (\xi_1 - \epsilon_2 \xi_2)(\xi_2 - \epsilon_3 \xi_3) \cdot \cdot (\xi_{N-1} - \epsilon_N \xi_N) \xi_N \prod_{a=1}^{N} \xi_a^{-1} e^{-\xi_a} d\xi_a , \qquad \text{(A37)}$$

and the integration volume Ω extends over

$$\xi_2 \leqq \epsilon_2^{-1} \xi_1 , \quad \xi_3 \leqq \epsilon_3^{-1} \xi_2 , \cdot \cdot$$

$$\xi_{N-1} \leqq \epsilon_{N-1}^{-1} \xi_{N-2} \quad \text{and} \quad 0 \leqq \xi_N \leqq \epsilon_N^{-1} \xi_{N-1} . \qquad \text{(A38)}$$

It is useful to define

$$H(\epsilon_2, \epsilon_3, \ldots, \epsilon_N) \equiv \left[F - \prod_{i=2}^{N} f(\epsilon_i)\right] \prod_{j=2}^{N} \epsilon_j^{-1} \tag{A39}$$

where

$$f(\epsilon) \equiv 1 + \epsilon \ln \frac{\epsilon}{1+\epsilon} \quad . \tag{A40}$$

By a straightforward calculation, one can verify that

$$\frac{\partial}{\partial \epsilon^{-1}} \left[\epsilon^{-1} f(\epsilon)\right] = (1+\epsilon)^{-1}$$

and

$$\frac{\partial^{N-1}}{\prod \partial \epsilon_i^{-1}} \left[F \prod \epsilon_j^{-1}\right] = (1+\epsilon_2)^{-1}(1+\epsilon_3+\epsilon_3\epsilon_2)^{-1}(1+\epsilon_4+\epsilon_4\epsilon_3+\epsilon_4\epsilon_3\epsilon_2)^{-1}$$

$$\cdots (1+\epsilon_N+\epsilon_N\epsilon_{N-1}+\cdots+\epsilon_N\epsilon_{N-1}\cdots\epsilon_2)^{-1}$$

where the subscripts i and j vary independently from 2 to N. Consequently,

$$D_{N-1} H \equiv \frac{\partial^{N-1}}{(\partial \epsilon_2^{-1})(\partial \epsilon_3^{-1}) \cdots (\partial \epsilon_N^{-1})} H \leqq 0 \quad . \tag{A41}$$

in the entire physical region $\epsilon_i \geqq 0$. As $\epsilon_2^{-1} \to 0$, $f(\epsilon_2) \to 0$ and the integration volume (A38) $\to 0$; therefore,

$$D_{N-2} H \equiv \frac{\partial^{N-2}}{\partial(\epsilon_3^{-1})\,\partial(\epsilon_4^{-1}) \cdots \partial(\epsilon_N^{-1})} H = 0 \quad \text{at} \quad \epsilon_2^{-1} = 0$$

which, together with (A41), implies

$$D_{N-2} H \leqq 0$$

in the entire physical region. Similarly, one shows that

$$D_{N-3} H \equiv \frac{\partial^{N-3}}{\partial(\epsilon_4^{-1}) \cdots \partial(\epsilon_N^{-1})} H = 0 \quad \text{at} \quad \epsilon_3^{-1} = 0 ,$$

and therefore in the entire physical region

$$D_{N-3} H \leqq 0 \quad .$$

By induction, one finds, for arbitrary $\epsilon_i \gtrless 0$,

$$H \leqq 0 \quad , \tag{A42}$$

which, together with (A36) and (A39), leads to

$$e^{-\Delta} \leqq F \leqq \prod_{i=2}^{N} f(\epsilon_i) \quad . \tag{A43}$$

Thus, we complete the proof for (A33) and (A34). [The fact that the factor $f(\epsilon_1)$ is absent from the above product is relevant only to the boundary condition at $i = 1$; it has no effect on any of the thermodynamical properties that we are interested in.]

As noted in section 4, the inequality (A34) implies (41):

$$p \leqq p_1(z) \quad , \tag{A44}$$

where $p_1(z)$ is given by (42).

4. It may be of interest to compare the above inequality with one derived by Tiktopoulos and Treiman[14] for the multiperipheral model. We note that, on account of (20), the two-body potential V_2 satisfies

$$e^{-V_2} \leq \prod_i (1 - e^{-x_i}) \quad , \tag{A45}$$

which implies, via (A34),

$$e^{-U_N} \leqq \prod_i (1 - e^{-x_i}) . \tag{A46}$$

Now, let us consider a hypothetical case[15] in which the corresponding function e^{-U_N} is this upper bound $\prod (1 - e^{-x_i})$. By using (40), one finds that the gas pressure p' of the hypothetical case is related to the fugacity z by

$$z^{-1} = \int_0^\infty (1 - e^{-x}) e^{-p'x} dx ,$$

or

$$p'^2 + p' - z = 0 .$$

Thus, we obtain

$$p \leqq p'(z) = \tfrac{1}{2} \left[(1 + 4z)^{\frac{1}{2}} - 1 \right] . \tag{A47}$$

By using Table 1, one can also write (A47) in the form given by Tiktopoulos and Treiman[14]

$$a \leqq -\tfrac{3}{2} + \left[\tfrac{1}{4} + \left(\tfrac{g}{4\pi}\right)^2 \right]^{\frac{1}{2}} . \tag{A48}$$

Because of (A45), one has

$$p_1(z) \leqq p'(z) ; \tag{A49}$$

therefore, the inequality (A44) is a better one than (A47). As $z \to \infty$, both inequalities reduce to (43): $p \leqq z^{\frac{1}{2}}$. As $z \to 0$,

$$p_1(z) = z - 1.5626 z^2 + O(z^3)$$

while

$$p'(z) \simeq z - z^2 + O(z^3).$$

As expected, the second virial b_2 is correctly given by $p_1(z)$, but not by $p'(z)$.

Appendix B

In section 4, the partition function Q_N is evaluated after the replacement

$$\delta\left(\sum_1^N x_i - L\right) \rightarrow \exp\left[-\beta\left(\sum_1^N x_i - L\right)\right] . \tag{B1}$$

It is well-known that such a replacement leads to the correct thermodynamical limit; i.e., it gives the correct $O(N)$ term in $\ln Q_N$, as $N \rightarrow \infty$ at a fixed finite density N/L. In this Appendix, we shall show how this method may be extended to derive the correction term, which will turn out to be $O(\ln N)$. Since, on account of (A11), $s^{-2}\, Q_N$ is proportional to the cross-section σ_{N+1} in the multiperipheral model, this correction term is related to the question of $\ln s$ dependence of the cross-section.

For simplicity, we shall consider only two-body forces. The total potential U_N is assumed to be

$$U_N = \sum_1^N v(x_i) . \tag{B2}$$

The partition function $Q_N(L)$ is given by (14) :

$$Q_N(L) = \int e^{-U_N}\, \delta(\Sigma x_i - L) \prod dx_i \tag{B3}$$

where, as before, each x_i is $\geqq 0$. From (B3), one obtains the recursion formula

$$Q_{N+1}(L) = \int_0^\infty e^{-v(x)}\, Q_N(L - x)\, dx . \tag{B4}$$

Theorem For $N >> 1$, but $(N/L) \sim O(1)$,

$$\ln Q_N(L) = N \ln h(\beta) + L\beta - \tfrac{1}{2} \ln N + O(1) \tag{B5}$$

where

$$h(\beta) = \int_0^\infty e^{-v(x) - \beta x} dx \tag{B6}$$

and β is a function of L/N, determined by

$$-\frac{d \ln h(\beta)}{d\beta} = \frac{L}{N} . \tag{B7}$$

We note that the replacement (B1) leads directly to the first two terms on the right hand side of (B5), both of which are $O(N)$. The theorem gives also the correction term $-\frac{1}{2} \ln N$. As we shall see, by following the proof given below, one can systematically calculate the remaining $O(1)$ term as well.

<u>Proof</u> Let us define α and γ to be the solutions of β at

$$-\left[\frac{d \ln h(\beta)}{d\beta}\right]_{\beta = \alpha} = \frac{L}{N+1} \tag{B8}$$

and

$$-\left[\frac{d \ln h(\beta)}{d\beta}\right]_{\beta = \gamma} = \frac{L-x}{N} \equiv \eta \quad , \tag{B9}$$

and assume $Q_N(L)$ to be of the form

$$Q_N(L) = \left[h(\beta)\right]^N e^{\beta L} A_N \tag{B10}$$

where A_N is to be determined. It is convenient to introduce the Fourier transform of $e^{-v(x)}$ in the physical region $x \geqq 0$:

$$e^{-v(x)} = \int_{-\infty}^{\infty} C_\omega e^{i\omega x} d\omega . \tag{B11}$$

[The behavior of the Fourier integral in the unphysical region $x < 0$ is immaterial to our discussion.] Equation (B4) can then be written as

$$[h(\alpha)]^{N+1} e^{\alpha L} A_{N+1} = N A_N \int_{-\infty}^{\infty} C_\omega e^{i\omega L} d\omega \int_0^{N^{-1}L} [H_\omega(\eta)]^N d\eta \tag{B12}$$

where

$$H_\omega(\eta) = h(\gamma)\, e^{(\gamma - i\omega)\eta} \tag{B13}$$

and $\gamma = \gamma(\eta)$ is defined by (B9).

Let β be defined by (B7). It is straightforward to expand α around β :

$$\alpha = \beta + \left[\frac{d^2 \ln h(\beta)}{d\beta^2}\right]^{-1} \frac{L}{N(N+1)} + \cdots .$$

The logarithm of the left-hand side of (B12) is then given by

$$(N+1)\ln h(\beta) + \beta L + \ln A_{N+1} - \frac{1}{2N}\left(\frac{d^2 \ln h(\beta)}{d\beta^2}\right)^{-1}\left(\frac{L}{N}\right)^2 + O\left(\frac{1}{N^2}\right). \tag{B14}$$

To evaluate the right-hand side, we use the relation

$$\int H_\omega^N d\eta = \frac{H_\omega^{N+1}}{N+1}\frac{d\eta}{dH_\omega} - \frac{H_\omega^{N+2}}{(N+1)(N+2)}\frac{d}{dH_\omega}\left(\frac{d\eta}{dH_\omega}\right) + \cdots . \tag{B15}$$

Since $e^{-v(x)}$ is regular at the origin, as $\beta \to \infty$, both $h(\beta) \to 0$ and $\frac{d \ln h(\beta)}{d\beta} \to 0$. By using (B9), one sees that as $\eta \to 0$, $\gamma \to \infty$ and $H_\omega \to 0$. Let us consider the definite integral (B15) from $\eta = 0$ to $\eta = N^{-1} L$. The right-hand side of (B15) is zero at the lower limit $\eta = 0$; by substituting its value at the upper limit $\eta = N^{-1} L$ to the right-hand side of (B12), we find, e.g., the first term $(N+1)^{-1} H_\omega^{N+1} (d\eta/dH_\omega)$

in (B15) leads to

$$\frac{NA_N}{N+1}\left[h(\beta)\right]^N e^{\beta L}\int_{-\infty}^{\infty}\frac{C_\omega\, d\omega}{\beta - i\omega} = \frac{NA_N}{N+1}\left[h(\beta)\right]^{N+1} e^{\beta L} \ . \qquad (B16)$$

By carrying out the same operation for the remaining term, one can verify that the logarithm of the right-hand side of (B12) is

$$(N+1)\ln h(\beta) + \beta L + \ln A_N - \frac{1}{2N\, h(\beta)}\left(\frac{d^2 \ln h(\beta)}{d\beta^2}\right)^{-1}\frac{d^2 h(\beta)}{d\beta^2} + O\left(\frac{1}{N^2}\right) \ . \qquad (B17)$$

By using (B14) = (B17), we obtain, for $N >> 1$ and neglecting the $O(N^{-2})$ terms,

$$\frac{d \ln A_N}{dN} = -\frac{1}{2N}$$

or

$$A_N = N^{-\frac{1}{2}} \times \text{constant} \qquad (B18)$$

which completes the proof.

Remarks: In the sum for the grand partition function $\mathcal{Q} = \Sigma\, z^N Q_N$, the maximal value of $z^N Q_N$ occurs at $N = \bar{N}$, determined by

$$\frac{\partial \ln Q_N}{\partial N} + \ln z = 0 \qquad \text{at } N = \bar{N} \ . \qquad (B19)$$

Therefore, as $L \to \infty$ at a constant $\bar{N}/L$, (B19) reduces to (40):

$$z^{-1} = \int_0^{\infty} e^{-v(x) - px}\, dx$$

where

$$p = \beta \quad \text{evaluated at } L^{-1}N = L^{-1}\bar{N} \ . \qquad (B20)$$

In terms of the maximal value, one may write

$$z^N Q_N \cong z^{\bar{N}} Q_{\bar{N}} \exp\left[-(N-\bar{N})^2/\Lambda^2\right] \tag{B21}$$

where Λ^2 denotes the fluctuation. Thus,

$$\mathcal{Q} \cong \int z^N Q_N \, dN = z^{\bar{N}} Q_{\bar{N}} \Lambda \pi^{\frac{1}{2}}$$

or, since Λ^2 is proportional to $\bar{N}$,

$$\ln \mathcal{Q} = \bar{N} \ln z + \ln Q_{\bar{N}} + \tfrac{1}{2} \ln \bar{N} + O(1) \ . \tag{B22}$$

By using the theorem and (B20), one finds

$$\ln \mathcal{Q} = \rho L + O(1) \ . \tag{B23}$$

The $-\frac{1}{2} \ln N$ term in (B5) is cancelled by the $+\frac{1}{2} \ln \bar{N}$ term in (B22). From Table 1, one sees that the total cross-section σ_{total} is proportional to $s^{\alpha-1}$. [Without the $-\frac{1}{2} \ln N$ term in (B5), one would obtain an incorrect multiplicative factor $\ln s$ for σ_{total}.]

Appendix C

To illustrate how the method developed in section 4 can be generalized to include four-body, five-body, . . . , forces, let us assume the total potential energy U_N to be

$$U_N = V_2 + V_3 + V_4 \tag{C1}$$

where $V_2 + V_3$ is given by (45), and

$$V_4 = \sum_i v_4(x_i, x_{i+1}, x_{i+2}) . \tag{C2}$$

We define the matrix $\langle x_i, x_{i+1} | H | x_{i+2}, x_{i+3} \rangle$ by

$$\langle x_i, x_{i+1} | H | x_{i+2}, x_{i+3} \rangle \equiv \exp(-\phi_0 - \phi_1) \tag{C3}$$

where

$$\phi_0 = \tfrac{1}{3}(x_i + 2x_{i+1} + 2x_{i+2} + x_{i+3})\, p$$
$$+ \tfrac{1}{3}\left[v_2(x_i) + 2v_2(x_{i+1}) + 2v_2(x_{i+2}) + v_2(x_{i+3})\right] \tag{C4}$$

and

$$\phi_1 = \tfrac{1}{2}\left[v_3(x_i, x_{i+1}) + 2v_3(x_{i+1}, x_{i+2}) + v_3(x_{i+2}, x_{i+3})\right]$$
$$+ v_4(x_i, x_{i+1}, x_{i+2}) + v_4(x_{i+1}, x_{i+2}, x_{i+3}) . \tag{C5}$$

The integral in (36) then becomes

$$\int \exp\left[-U_N - p\sum_1^N x_i\right] \prod dx_i = \operatorname{trace} H^{N/2} \tag{C6}$$

where any power of the matrix H is defined according to the usual rule:

$$\langle x, y \mid H^m \mid s, t \rangle \equiv \int_0^\infty du \int_0^\infty dv \, \langle x, y \mid H^{m-1} \mid u, v \rangle \langle u, v \mid H \mid s, t \rangle \quad .$$

To evaluate the trace of $H^{N/2}$, we may separate

$$H = H_0 + \epsilon$$

where the matrix element of H_0 is equal to $\exp(-\phi_0)$ and that of ϵ is equal to $[\exp(-\phi_1) - 1] \exp(-\phi_0)$. With only minor changes, the discussion following Eq. (53) in section 4 can be directly applied to the present case. Similarly, one can extend the method to five-body, six-body, . . ., forces.

TABLE 1

Gas System		Multiperipheral Model (ϕ^3 theory)
fugacity z	=	$(\frac{g}{4\pi})^2$
pressure p	=	$\alpha + 1$
grand partition function $\mathcal{Z}$	~	$s^2 \sigma_{total}$
length L	~	$\ln \frac{s}{m^2}$
$L \frac{dp}{d \ln z}$	~	average multiplicity $\langle N \rangle$
$L \frac{d^2 p}{d \ln z^2}$	~	$\langle N^2 \rangle - \langle N \rangle^2$

Table 1. Correspondence between the one-dimensional gas and the multiperipheral model in the high energy limit [where s = (center-of-mass energy)^2 and α = Regge-pole power for forward elastic scattering].

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7. Usually, a thermodynamical system has three extensive variables: the number of particles N, the length (or volume) L and the energy E; corresponding to these three extensive variables, there are three intensive variables: the fugacity z, the pressure p and the temperature T. In the present case, there are only two extensive variables: N and $L \sim \ln(s/m^2)$; correspondingly, there are also only two intensive variables: z and p. Hence, kT can be chosen to be unity.

8. The permutation due to different ordering of these particles is cancelled by the usual $[(N-1)!]^{-1}$ factor due to Boltzmann statistics. [If the length L has ends, then the Boltzmann factor is $(N!)^{-1}$.]

9. See, e.g., J. E. Mayer and M. G. Mayer, Statistical Mechanics (John Wiley and Sons, New York, 1940).

10. See Table 27.8 in Handbook of Math Functions, Applied Math Series, vol. 55, ed. by M. Abramowitz and I. A. Stegun, National Bureau of Standards (1964).

11. I wish to thank M. Y. Chen for kindly checking the numerical integration for b_3.

12. See T. L. Trueman and T. Yao, Phys. Rev. 132, 2741 (1963), and S. J. Chang, T. M. Yan and Y. P. Yao, loc. cit.

13. That such a replacement gives the correct thermodynamical limit, as L and $N \to \infty$, is well known. In Appendix B, we shall show how this method can be extended to obtain also the correction term, when L and N are large but not infinite.

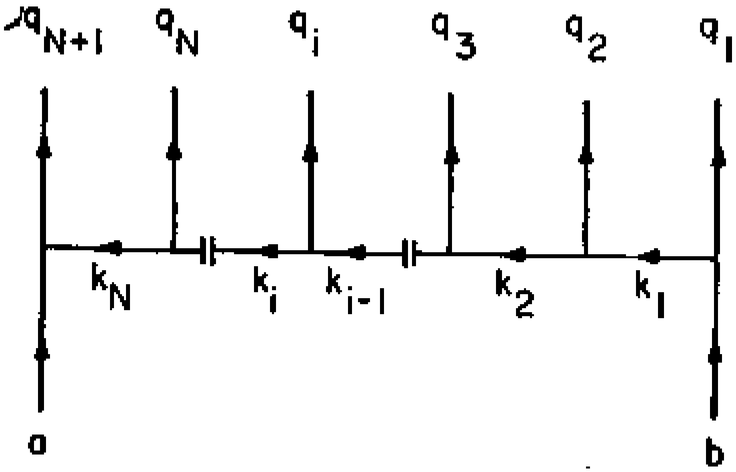
14. The upper bounds (38) and (A48) have been derived by G. Tiktopoulos and S. B. Treiman, Phys. Rev. 135 B, 711 (1964). For a still better upper bound, see (41) and (42). [See also G. Tiktopoulos and S. B. Treiman, Phys. Rev. 137 B, 1597 (1965) for discussions on lower bounds for models involving zero-mass particles.]

15. If there were only one space dimension, instead of three, in the ϕ^3-multiperipheral model, then the corresponding two-body potential would be $1 - e^{-x}$. This expression follows directly from the result of D. K. Campbell and S. J. Chang, loc. cit.; it can also be derived by using arguments similar to those given in Appendix A.

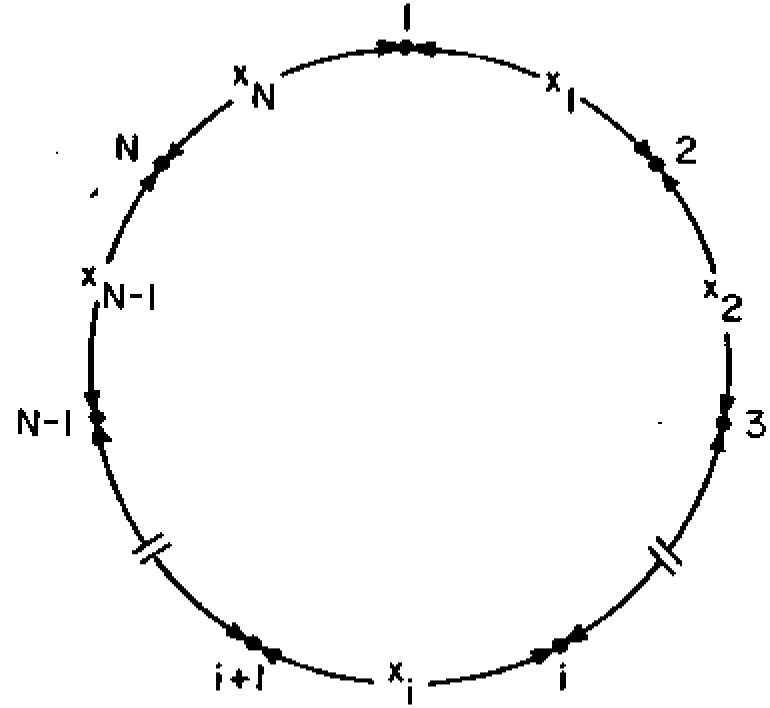
Figure Caption

Figure 1 (A) Feynman diagram for $N+1$ meson emission in the ϕ^3-multiperipheral model.

(B) Corresponding diagram for N atoms on a ring in the one-dimensional gas model.



(A)



(B)