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MASTER

ANGULAR CORRELATION THEORY IN HIGH ENERGY PHYSICS

Ph. D. Thesis Submitted to Iowa State University, August 1971

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Angular correlation theory in high energy physics

by

Christopher Karl Schmidt

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ABSTRACT

The angular distribution for double resonance production is derived by separating the S-matrix into individually Lorentz invariant production and decay factors. Each factor is treated separately using helicity states. Direct and crossed channel amplitudes are used to obtain two expressions for the direct channel angular distribution. Kinematic constraints, parity predictions and expectation functions for linear combinations of production density matrix elements are obtained using moment analysis. A Regge-pole model for the production reaction is presented. The results of this dissertation are applied to the reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$.

1. INTRODUCTION

Whenever a one-step process occurs in empty space, such as the natural decay of a nucleus to a daughter nucleus and one decay product, isotropy of space predicts that the process will exhibit spherical symmetry. However, if the nucleus is struck by a photon causing the nucleus to decay, a correlation between the directions of the incident photon and decay products is in general expected. Angular correlation theories describing this and other multistep processes exist in nuclear physics literature (1-3). These theories use angular momentum expansions which are useful in nuclear physics because the relative angular momentum quantum number is limited to only the few lowest possible values by the low energy of the nuclear processes. The high energies of relativistic processes in the same theories involve many values of the relative angular momentum quantum number. Pion nucleon scattering at an incident pion lab momentum of 10 GeV/c requires approximately 50 partial waves. This large number of partial waves in high energy processes makes the low energy theories too complicated to be useful for relativistic reactions. Since angular correlations exist at all energies, it is desirable to have an angular correlation theory for high energy reactions.

The purpose of the research described in this dissertation is to construct a feasible high energy angular correlation theory. The most complicated case considered a double resonance production although

resonance particle and double particle production can also be handled by the theory. The production reaction is described by Regge trajectory exchange in the crossed channel and the decays are described using partial wave analysis. Helicity states are used. The theory assumes that all particles and resonances are known and are different at any step of the reaction and that no interference exists between outgoing particles.

In Chapter II the basic ideas and kinematics of the complete reaction are established. In Chapter III the basic formalism is expanded to include spin and the reaction is formulated in terms of density matrices. There is also a section on a model independent calculation of production density matrix elements for nucleon pion type final states. The Regge-pole formalism for particles with spin is the subject of Chapter IV. The general results of this dissertation and some specific results for the reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$ are presented in Chapter V. Original work in this dissertation consists of extensions of ideas already in existence to double resonance production and is contained in Sections B-E of Chapter III.

II. GENERAL THEORY

A. Kinematics and the S-matrix

The first step in the two step process considered in this dissertation is the production reaction which is illustrated in Figure 1. The decays of unstable particles c and d form the second step. The particles are labeled by letters and are described by their 4-momenta denoted by P_i where i is the particle labeled. The decay products of c and d are labeled by numbers 1-4 rather than letters. The 4-momenta satisfy

$$P_i^2 = M_i^2 \quad (2.1)$$

where M_i is the rest mass of particle i in units where $\hbar = c = 1$. The small arrows labeled s and t in Figure 1 indicate the direction of time in the direct or s-channel and the crossed or t-channel reactions. The direct channel reaction is

$$a + b \rightarrow c + d \quad (2.2)$$

and the crossed channel reaction is defined to be

$$a + \bar{c} \rightarrow \bar{b} + d \quad (2.3)$$

where a bar over a particle label means antiparticle. Particles travel

Figure 1. Production reaction.

Figure 2. Double Resonance Production.

in the directions indicated by the arrows on the particle lines. Antiparticles travel in the direction opposite to the arrows on the particle lines. The direct and crossed channel scattering angles are the angles between the directions of particles a and c and between a and $\bar{b}$ respectively. The Lorentz invariants for the production are defined by

$$s = (P_a + P_b)^2 = (P_c + P_d)^2 \quad (2.4)$$

and

$$t = (P_a - P_c)^2 = (P_b - P_d)^2 \quad (2.5)$$

$\sqrt{s}$ is called the invariant mass of the system of particles a and b and is equal to the sum of their center of momentum energies. In terms of these invariants, the cosine of the crossed channel scattering angle which is used in the crossed channel Regge-pole formalism in Chapter IV is

$$x = \cos \theta_t = \frac{2t(s - M_a^2 - M_b^2) + (t + M_a^2 - M_c^2)(t + M_b^2 - M_d^2)}{[(t - M_a^2 - M_c^2)^2 - 4M_a^2 M_c^2]^{\frac{1}{2}} [(t - M_b^2 - M_d^2)^2 - 4M_b^2 M_d^2]^{\frac{1}{2}}} \quad (2.6)$$

The function $\lambda(u, v, w)$ (4) is defined by

$$\lambda(u,v,w) = u^2 + v^2 + w^2 - 2uv - 2uw - 2vw . \quad (2.7)$$

This function will always be written in its functional form and should not be confused with helicities which are introduced in the next chapter.

The initial and final free particle states long before and long after the production respectively are related by the S-matrix (5, 6). S-matrix theory postulates the existence of an analytic, unitary S-matrix, the Lorentz invariance of the S-matrix, and the disconnectedness of the S-matrix. The disconnectedness is due to the short range of strong interaction forces and is the basis for the factorization of the complete S-matrix into production and decay steps. The factorization is described in detail in the next section. The S-matrix preserves isospin and is invariant under the operations of parity, charge conjugation, time reversal, G parity, rotations and identical particle exchange for strong reactions. The transition or T-matrix is defined from the S-matrix by

$$i(2\pi)^4 \delta^4(P_f - P_i)T = S - 1 \quad (2.8)$$

where the δ -function guarantees the required conservation of energy and momentum between the initial and final states.

B. General Formalism

For double resonance production, a Lorentz invariant description of the reaction

is needed. c and d are resonances that decay into particles 1-4. The reaction is illustrated in Figure 2 where the lines indicate particles and the circles indicate interactions. The Lorentz invariant description is obtained by using Lorentz invariant phase space, which is described in the next section, and Lorentz invariant matrix elements.

In the physical description of the reaction, stable particles a and b interact to produce resonances c and d which subsequently decay to stable particles $1 + 2$ and $3 + 4$ respectively. Particles 1-4 are assumed to leave the decay region without interfering. Since the double resonance production is a strong interaction, the forces involved are short range. It is therefore reasonable to expect that the short range forces instrumental in the production reaction do not extend to the resonance decays. Similarly, the decay forces have no effect on the production. When this separation of forces exists, the resonance production and decays can be described independently. They

are, of course, connected by the polarizations of the resonances. For relativistic energies, this approximation should be very accurate except perhaps for resonances with extremely short lifetimes. When the production and decays occur separately, the transition matrix for the complete reaction factors into production and decay parts. The resonances form the final state of the production and the initial states of the decays.

Since c and d are resonances, their amplitudes have poles associated with their energy behavior. These poles are written out explicitly as relativistic Breit-Wigner amplitudes normalized to unit total probability. Then the complete transition matrix for double resonance production is

$$\begin{aligned}
 T(ab \rightarrow cd \rightarrow 1234) = & T(ab \rightarrow cd) \sqrt{\frac{M_c \Gamma_c}{\pi}} \frac{T(c \rightarrow 12)}{M_c^2 - s_c - i M_c \Gamma_c} \\
 & \times \sqrt{\frac{M_d \Gamma_d}{\pi}} \frac{T(d \rightarrow 34)}{M_d^2 - s_d - i M_d \Gamma_d}
 \end{aligned}
 \tag{2.10}$$

where Γ_c (Γ_d) is the width of resonance c (d) and s_c (s_d) is the square of the invariant mass of resonance c (d) obtained from the decay products by

$$s_c = P_c^2 = (P_1 + P_2)^2 \tag{2.11a}$$

and

$$s_d = P_d^2 = (P_3 + P_4)^2 . \quad (2.11b)$$

$T(ab \rightarrow cd)$ describes the production and depends on the Lorentz invariants s and t . Actually, the production transition matrix depends on the polar angles θ and ϕ between incident and outgoing particles in the center of momentum system but θ and t are related for fixed s and there is no ϕ dependence for unpolarized beam and target particles. The symbol α is used to stand for these polar angles in the production process, even though t could be used. The decay transition matrices $T(c \rightarrow 12)$ and $T(d \rightarrow 34)$ depend on the polar angles of particles $1 + 2$ and $3 + 4$ respectively.

Since each of the three transition matrices depends on different variables and since the complete transition matrix is Lorentz invariant, it follows that each of the three parts must separately be Lorentz invariant. If this were not the case, a Lorentz transformation could be found that changed $T(c \rightarrow 12)$ but left $T(d \rightarrow 34)$ unchanged. Since the rest of $T(ab \rightarrow cd \rightarrow 1234)$ depends only on Lorentz invariants, this violates the requirement that the complete T-matrix be Lorentz invariant.

Because all three factors of $T(ab \rightarrow cd \rightarrow 1234)$ are Lorentz invariant, they can be considered in different Lorentz frames. Therefore, the production is considered in the ab center of momentum frame and each decay is considered in the rest frame of the decaying particle. These Lorentz frames are related by Lorentz boosts along the directions of motion of the resonances c and d . This causes no difficulty when

particles with spin are included provided helicity states are used. However, before discussing spin it is necessary to consider the Lorentz invariant density of states.

C. Lorentz Invariant Phase Space

For the reaction $ab \rightarrow cd \rightarrow 1234$, there are four particles in the final state. The number of states available to these particles is called the density of states. A Lorentz invariant form for the density of states in momentum space is Lorentz invariant phase space (abbreviated Lips) (4). Actually, restricted Lips is used because it includes the factor $(2\pi)^4 \delta^2(P_a + P_b - P_1 - P_2 - P_3 - P_4)$ to insure conservation of energy and momentum. The restricted Lips element for four particles in the final state is denoted by $dLips(s; P_1, P_2, P_3, P_4)$.

When there are many particles in the final state, the following recurrence relation can be used to reduce one restricted Lips element to two restricted Lips elements each involving fewer final state particles than the original.

$$\begin{aligned} dLips(s; P_1, \dots, P_j, P_{j+1}, \dots, P_n) \\ = \frac{1}{2\pi} dLips(s; P_c, P_{j+1}, \dots, P_n) dLips(s_c; P_1, \dots, P_j) ds_c \end{aligned} \quad (2.12)$$

where

$$P_c = \sum_{i=1}^j P_i \quad (2.12a)$$

and

$$s_c = P_c^2. \quad (2.12b)$$

This has the advantage that P_c can be interpreted as the four momentum of an intermediate particle which decays into particles 1-j.

For double resonance production with two intermediate particles and four final state particles, the recurrence relation is used twice to give

$$\begin{aligned} dLips(s; P_1, P_2, P_3, P_4) \\ = (2\pi)^{-2} dLips(s; P_c, P_d) dLips(s_c; P_1, P_2) \\ \times dLips(s_d; P_3, P_4) ds_c ds_d \end{aligned} \quad (2.13)$$

where s_c and s_d are given in Equation 2.11.

The restricted Lips element for two particles in the final state is

$$dLips(s; P_a, P_b) = \frac{\sqrt{\lambda(s, M_a^2, M_b^2)}}{32\pi^2 s} d\Omega \quad (2.14)$$

where Ω stands for polar angles θ and ϕ of particle a in the ab center of momentum frame. For double resonance production,

$$dLips(s; P_1, P_2, P_3, P_4) = \frac{\sqrt{\lambda(s, s_c, s_d) \lambda(s_c, M_1^2, M_2^2) \lambda(s_d, M_3^2, M_4^2)}}{2(4\pi)^8 s_c s_d} \times d\Omega_1 d\Omega_3 ds_c ds_d \quad (2.15)$$

where Ω , Ω_1 , and Ω_3 stand for the polar angles of particles c , 1 , and 3 respectively.

The total cross section for double resonance production is

$$\sigma = [2\sqrt{\lambda(s, M_a^2, M_b^2)}]^{-1} \int dLips(s; P_1, P_2, P_3, P_4) |T(ab \rightarrow cd \rightarrow 1234)|^2. \quad (2.16)$$

The total cross section, written this way, is a Lorentz invariant. The angular distribution obtained from this equation is

$$\frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} = \int ds_c ds_d k(s, s_c, s_d) |T T^c T^d|^2 \quad (2.17)$$

where the abbreviations include

$$T = T(ab \rightarrow cd), \quad (2.17a)$$

$$T^c = T(c \rightarrow 12), \quad (2.17b)$$

$$T^d = T(d \rightarrow 34) \quad (2.17c)$$

and

$$k(s, s_c, s_d) = \sqrt{\frac{\lambda(s, s_c, s_d) \lambda(s_c, M_1^2, M_2^2) \lambda(s_d, M_3^2, M_4^2)}{\lambda(s, M_a^2, M_b^2)}} \\ \times \frac{M_c M_d \Gamma_c \Gamma_d}{\pi (4\pi)^9 s s_c s_d [(M_c^2 - s_c)^2 + M_c^2 \Gamma_c^2] [(M_d^2 - s_d)^2 + M_d^2 \Gamma_d^2]}.$$

(2.17d)

The limits on the s_c and s_d intergrations correspond to the experimental cutoff points of the resonances c and d .

III. FORMALISM FOR PARTICLES WITH SPIN

A. Helicity States

In this section helicity states are introduced to extend the theory to include particles with spin. Helicity states have the advantage that pure rotations and Lorentz boosts along the direction of motion of a particle do not change the helicity (7). Therefore, the helicity of resonance c (d) is not changed by the Lorentz boost that connects the overall center of momentum frame to the rest frame of resonance c (d). Helicity states were introduced in 1959 by Jacob and Wick (8) and have been treated by many others since that time. Several are listed in the bibliography (7, 9-11).

It is necessary to consider two kinds of helicity states and the connection between them. The first of these is the linear momentum helicity state which is important because linear momentum is the measured quantity in most experiments. The other is the angular momentum helicity state which is used in theoretical analysis. One and two particle states are considered. Because all the particles are known in double resonance production, the fixed quantum numbers, such as the mass and spin of each particle, will not be written out explicitly in the state vector. It should nevertheless be remembered that these quantum numbers are necessary to completely specify a state.

Helicity is the component of angular momentum along the direction of motion. The helicity operator is written as $\hat{J} \cdot \hat{p}$ where $\hat{p}$ is the unit

vector in the direction of motion of a particle. The helicity eigenvalues are denoted by the symbols λ and μ .

For a single particle the linear momentum helicity state is defined by

$$|p\theta\varphi\lambda\rangle = R(\varphi,\theta,0)L_z(p)|000\lambda\rangle \quad (3.1)$$

where $L_z(p)$ is a Lorentz boost along the z-axis that takes the linear momentum from 0 to p and $R(\varphi,\theta,0)$ is a rotation from the z-direction to an orientation with polar angles θ and φ . The ranges of θ and φ are

$$0 \leq \theta \leq \pi \quad (3.1a)$$

$$-\pi \leq \varphi \leq \pi. \quad (3.1b)$$

The state $|000\lambda\rangle$ is the rest state of the particle and is an eigenstate of the operators P , J^2 , and J_z with eigenvalues $(m, \underline{p}=0)$, $S(S+1)$, and λ respectively. λ takes on values for $-S$ to S in integer steps. The phase of these states is fixed by requiring

$$(J_{x\pm} + iJ_{y\pm})|000\lambda\rangle = \sqrt{(S\pm\lambda)(S\pm\lambda+1)}|000\lambda\pm 1\rangle \quad (3.2)$$

so the rotation is given by

$$R(\varphi,\theta,0)|000\lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}^S(\varphi,\theta,0)|000\lambda'\rangle \quad (3.3)$$

where the D matrix is the $(2S+1)$ dimensional representation of the rotation group (3).

For a system of two particles, the linear momentum helicity state is defined in the center of momentum frame by a direct product of single particle states.

$$|p00\lambda_1\lambda_2\rangle = |p00\lambda_1\rangle |-p00\lambda_2\rangle . \quad (3.4)$$

Here, particle 1 moves along the z-axis in the positive direction with absolute linear momentum p and particle 2 moves along the z-axis in the negative direction with the same absolute linear momentum. The state $|-p00\lambda_2\rangle$ is defined by

$$|-p00\lambda_2\rangle = (-1)^{S_2-\lambda_2} e^{-i\pi J_y} |p00\lambda_2\rangle \quad (3.5)$$

where the phase is chosen so $|-p00\lambda_2\rangle$ reduces to $|000-\lambda_2\rangle$ for particle 2 at rest.

The general two particle linear momentum helicity state is obtained by rotating to a final configuration with polar angles θ and φ .

$$|p\theta\varphi\lambda_1\lambda_2\rangle = R(\varphi,\theta,0) |p00\lambda_1\lambda_2\rangle . \quad (3.6)$$

Here particle 1 moves in the direction specified by the polar angles and particle 2 moves in the opposite direction. Because helicity is the component of angular momentum along the direction of motion and the

particles are moving in opposite direction,

$$\hat{J} \cdot \hat{p} |p\theta\phi\lambda_1\lambda_2\rangle = (\lambda_1 - \lambda_2) |p\theta\phi\lambda_1\lambda_2\rangle \quad (3.7)$$

where $\hat{p}$ is arbitrarily fixed to be $\hat{p}_1$.

The angular momentum helicity states are denoted by $|Jm\lambda_1\lambda_2\rangle$ for two particles. J and m are the total angular momentum and z component of the angular momentum quantum numbers respectively. These states are treated in the references indicated at the beginning of this section.

To find the connection between linear momentum and angular momentum helicity states, the definition of the rotated linear momentum helicity state and the unitarity of the rotation matrices are used. This gives

$$\begin{aligned} \langle p\theta\phi\lambda_1\lambda_2 | Jm\lambda_1\lambda_2 \rangle \\ = \langle p00\lambda_1\lambda_2 | R^{-1}(\phi, \theta, 0) | Jm\lambda_1\lambda_2 \rangle \end{aligned} \quad (3.8a)$$

$$= \sum_{m'} \langle p00\lambda_1\lambda_2 | D_{m'm}^{J\dagger}(\phi, \theta, 0) | Jm'\lambda_1\lambda_2 \rangle \quad (3.8b)$$

$$= \sum_{m'} D_{mm'}^{J*}(\phi, \theta, 0) \langle p00\lambda_1\lambda_2 | Jm'\lambda_1\lambda_2 \rangle \delta_{m', \lambda_1 - \lambda_2} \quad (3.8c)$$

$$= D_{m, \lambda_1 - \lambda_2}^{J*}(\phi, \theta, 0) \langle p00\lambda_1\lambda_2 | J, \lambda_1 - \lambda_2, \lambda_1\lambda_2 \rangle \quad (3.8d)$$

where the $\dagger$ indicates Hermitian conjugation and the $*$ indicates complex

conjugation. The δ -function in Equation 3.8c results from equating the z components of angular momentum. This can be done because the orbital angular momentum does not contribute along the direction of motion.

The normalizations for these states are

$$\langle Jm\lambda_1\lambda_2 | J'm'\lambda_1'\lambda_2' \rangle = \delta_{JJ'} \delta_{mm'} \delta_{\lambda_1\lambda_1'} \delta_{\lambda_2\lambda_2'} \quad (3.9)$$

and

$$\langle p\theta\varphi\lambda_1\lambda_2 | p\theta'\varphi'\lambda_1'\lambda_2' \rangle = \delta(\varphi-\varphi') \delta(\cos\theta-\cos\theta') \delta_{\lambda_1\lambda_1'} \delta_{\lambda_2\lambda_2'} \quad (3.10)$$

Equations 3.8-10, A7, and A9 and insertion of a complete set give

$$1 = \langle Jm\lambda_1\lambda_2 | Jm\lambda_1\lambda_2 \rangle \quad (3.11a)$$

$$= \langle Jm\lambda_1\lambda_2 | \sum_{\lambda_1'\lambda_2'} \int d\cos\theta d\varphi | p\theta\varphi\lambda_1'\lambda_2' \rangle \quad (3.11b)$$

$$= \int d\cos\theta d\varphi [d_{m,\lambda_1-\lambda_2}^J(\theta)]^2 |\langle J,\lambda_1-\lambda_2,\lambda_1\lambda_2 | p00\lambda_1\lambda_2 \rangle|^2 \quad (3.11c)$$

$$= \frac{4\pi}{2J+1} |\langle J,\lambda_1-\lambda_2,\lambda_1\lambda_2 | p00\lambda_1\lambda_2 \rangle|^2 \quad (3.11d)$$

With this result, Equation 3.8 becomes

$$\begin{aligned}
\langle p\theta\phi\lambda_1\lambda_2 | Jm\lambda_1'\lambda_2' \rangle \\
= \delta_{\lambda_1\lambda_1'} \delta_{\lambda_2\lambda_2'} \frac{2J+1}{4\pi} D_{m,\lambda_1-\lambda_2}^{J*}(\phi,\theta,0) .
\end{aligned} \tag{3.12}$$

A partial wave expansion in angular momentum helicity states for the two particle linear momentum helicity states is now possible. Insertion of a complete set gives

$$|p\theta\phi\lambda_1\lambda_2\rangle = \sum_{\lambda_1'\lambda_2'} \sum_{Jm} |Jm\lambda_1'\lambda_2'\rangle \langle Jm\lambda_1'\lambda_2' | p\theta\phi\lambda_1\lambda_2 \rangle \tag{3.13a}$$

$$= \sum_{Jm} D_{m,\lambda_1-\lambda_2}^J(\phi,\theta,0) \frac{2J+1}{4\pi} |Jm\lambda_1\lambda_2\rangle . \tag{3.13b}$$

This expansion will be used in Section D after symmetries of the transition matrix and density matrices are discussed.

B. Symmetries

For inelastic scattering processes, like double resonance production, the S-matrix and T-matrix have the same symmetries. The invariance of the T-matrix under parity, charge conjugation, time reversal, and rotations is considered for linear momentum helicity states. These symmetries reduce the total number of independent density matrix elements needed to describe a reaction.

Parity and a rotation of π about an axis normal to the production plane are considered simultaneously. With axes picked so the initial particles are incident along the z-axis and the y-axis is normal to

the production plane, the rotation mentioned above is $R(0,\pi,0) = e^{-i\pi J_y}$.

P denotes the parity operator and Y denotes the operator that is the combination of parity and $R(0,\pi,0)$. The effect of the parity operation on a state is to invert all linear momenta and change the signs of the helicities. When parity is combined with the rotation $R(0,\pi,0)$, the linear momenta are returned to their original directions so the physical picture is unchanged by the Y operation. The helicities are unaffected by the rotation so they change sign because of the parity operation.

The operator Y is defined by

$$Y = R(0,\pi,0)P = e^{-i\pi J_y}P. \quad (3.14)$$

This definition is valid only when the particles are in the production plane and perpendicular to the y -axis.

The parity of a particle at rest is the intrinsic parity which is denoted by η .

$$P|000\lambda\rangle = \eta|000\lambda\rangle. \quad (3.15)$$

A Lorentz boost in the production plane and Y commute so

$$Y |p00\lambda\rangle = L_z(p) e^{-i\pi J_y} |000\lambda\rangle \quad (3.16a)$$

$$= L_z(p) \sum_{\lambda'} d_{\lambda'\lambda}^S(\pi) \eta |000\lambda'\rangle \quad (3.16b)$$

$$= L_z(p) \eta (-1)^{S-\lambda} |000-\lambda\rangle \quad (3.16c)$$

$$= \eta (-1)^{S-\lambda} |p00-\lambda\rangle \quad (3.16d)$$

where Equation A5 has been used in Equation 3.16b. Since Y commutes with rotations about the y -axis,

$$Y |p\theta 0\lambda\rangle = R(0, \theta, 0) Y |p00\lambda\rangle \quad (3.17a)$$

$$= \eta (-1)^{S-\lambda} |p\theta 0-\lambda\rangle. \quad (3.17b)$$

Since p commutes with all rotations,

$$p |p\theta\varphi\lambda\rangle = R(\varphi, 0, 0) R^{-1}(0, \pi, 0) Y |p00\lambda\rangle \quad (3.18a)$$

$$= R(\pi+\varphi, \pi-\theta, -\pi) \eta (-1)^{S-\lambda} |p00-\lambda\rangle \quad (3.18b)$$

$$= R(\pi+\varphi, \pi-\theta, 0) \times e^{-i(-\pi)J_z} \eta (-1)^{S-\lambda} |p00-\lambda\rangle \quad (3.18c)$$

$$= R(\pi+\varphi, \pi-\theta, 0) \eta (-1)^{S-2\lambda} |p00-\lambda\rangle \quad (3.18d)$$

$$= \eta (-1)^{-S} |p, \pi-\theta, \pi+\varphi, -\lambda\rangle \quad (3.18e)$$

where the fact that $S-\lambda$ is an integer has been used to write

$(-1)^{S-2\lambda} = (-1)^{-S}$ in the last step. The product of rotations in Equation 3.18a is explained in the Appendix. (-1) raised to a non-integer power α is defined to be $(-1)^\alpha = e^{i\pi\alpha}$. Y operating on a state $|-p00\lambda\rangle$ is

$$Y|-p00\lambda\rangle = Y(-1)^{S-\lambda} e^{-i\pi J} Y|p00\lambda\rangle \quad (3.19a)$$

$$= (-1)^{S-\lambda} e^{-i\pi J} Y \eta(-1)^{S-\lambda} |p00-\lambda\rangle \quad (3.19b)$$

$$= \eta(-1)^{S+\lambda} (-1)^{S+\lambda} e^{-i\pi J} Y|p00-\lambda\rangle \quad (3.19c)$$

$$= \eta(-1)^{S+\lambda} |-p00-\lambda\rangle \quad (3.19d)$$

where $(-1)^{-2\lambda} = (-1)^{2\lambda}$ has been used in Equation 3.19b.

For two particle states in their center of momentum frame

$$Y|p\theta 0\lambda_1\lambda_2\rangle = R(0,\theta,0)Y(|p00\lambda_1\rangle |-p00\lambda_2\rangle) \quad (3.20a)$$

$$= R(0,\theta,0)\eta_1\eta_2(-1)^{S_1-\lambda_1+S_2+\lambda_2}(|p00-\lambda_1\rangle |-p00\lambda_2\rangle) \quad (3.20b)$$

$$= \eta_1\eta_2(-1)^{S_1-\lambda_1+S_2+\lambda_2}|p\theta 0-\lambda_1-\lambda_2\rangle \quad (3.20c)$$

and

$$P |p\theta\phi\lambda_1\lambda_2\rangle = R(\phi, \theta, 0) R^{-1}(0, \pi, 0) Y |p00\lambda_1\lambda_2\rangle \quad (3.21a)$$

$$= R(\pi+\phi, \pi-\theta, -\pi) \eta_1 \eta_2 (-1)^{S_1-\lambda_1+S_2+\lambda_2} |p00-\lambda_1-\lambda_2\rangle \quad (3.21b)$$

$$= R(\pi+\phi, \pi-\theta, 0) \eta_1 \eta_2 (-1)^{S_1-2\lambda_1+S_2+2\lambda_2} |p00-\lambda_1-\lambda_2\rangle \quad (3.21c)$$

$$= \eta_1 \eta_2 (-1)^{-S_1-S_2} |p, \pi-\theta, \pi+\phi, -\lambda_1-\lambda_2\rangle. \quad (3.21d)$$

The effect of the parity operation on a two particle angular momentum helicity state is calculated in the references indicated in the last section and is

$$P |Jm\lambda_1\lambda_2\rangle = \eta_1 \eta_2 (-1)^{J-S_1-S_2} |Jm-\lambda_1-\lambda_2\rangle. \quad (3.22)$$

Charge conjugation symmetry gives a reduction in the number of independent T-matrix elements whenever the initial and final particles in the production each form a particle antiparticle pair. For a particle antiparticle state, the effect of the charge conjugation operation is identical with the effect of the exchange operation. Let C denote the charge conjugation operator and P_{12} denote the operator that exchanges particles 1 and 2. Then, abbreviating the intrinsic quantum numbers by 1 and 2 ($\bar{1}$ and $\bar{2}$ for antiparticles),

$$P_{12} |12p\theta\phi\lambda_1\lambda_2\rangle = |21p\theta\phi\lambda_1\lambda_2\rangle \quad (3.23)$$

and

$$C|12p\theta\varphi\lambda_1\lambda_2\rangle = |\bar{1}\bar{2}p\theta\varphi\lambda_1\lambda_2\rangle \quad (3.24)$$

which are equal for particle antiparticle pairs because $1=\bar{2}$ and $2=\bar{1}$.

The effect of C on a two particle linear momentum state is

$$C|12p\theta\varphi\lambda_1\lambda_2\rangle = R(\varphi,\theta,0)P_{12}(|1p00\lambda_1\rangle|2p00\lambda_2\rangle) \quad (3.25a)$$

$$= R(\varphi,\theta,0)|2p00\lambda_1\rangle|1p00\lambda_2\rangle \quad (3.25b)$$

$$= R(\varphi,\theta,0)|2p00\lambda_1\rangle(-1)^{S-\lambda_2}e^{-i\pi J_y}|1p00\lambda_2\rangle \quad (3.25c)$$

$$= R(\varphi,\theta,0)e^{-i\pi J_y}(-1)^{2S}[e^{-i\pi J_y}|2p00\lambda_1\rangle] \\ \times (-1)^{S-\lambda_2}|1p00\lambda_2\rangle \quad (3.25d)$$

$$= R(\varphi,\theta,0)R(0,\pi,0)(-1)^{2S-\lambda_2+\lambda_1}|1p00\lambda_2\rangle \\ \times (-1)^{S-\lambda_1}e^{-i\pi J_y}|2p00\lambda_2\rangle \quad (3.25e)$$

$$= R(\pi+\varphi,\pi-\theta,\pi)(-1)^{2S-\lambda_2+\lambda_1}|12p00\lambda_2\lambda_1\rangle \quad (3.25f)$$

$$= R(\pi+\varphi,\pi-\theta,0)(-1)^{2S}|12p00\lambda_2\lambda_1\rangle \quad (3.25g)$$

$$= (-1)^{2S}|12p,\pi-\theta,\pi+\varphi,\lambda_2\lambda_1\rangle. \quad (3.25h)$$

Time reversal τ is an antiunitary transformation so the effect on the T-matrix is not invariance but rather

$$\tau T \tau^{-1} = T^\dagger. \quad (3.26)$$

Therefore, time reversal symmetry will reduce the total number of independent T-matrix elements only in the case of elastic scattering.

In all other cases, time reversal relates different reactions.

In coordinate space the time reversal operator is

$$\tau = e^{-i\pi J_y} \kappa = R(0, \pi, 0) \kappa \quad (3.27)$$

where κ is the complex conjugation operator. For two particle linear momentum helicity states

$$\tau |p\theta\varphi\lambda_1\lambda_2\rangle = R(\varphi, \theta, 0) R(0, \pi, 0) \kappa |p00\lambda_1\lambda_2\rangle \quad (3.28a)$$

$$= R(\pi+\varphi, \pi-\theta, 0) (-1)^{-\lambda_1+\lambda_2} \kappa |p00\lambda_1\lambda_2\rangle \quad (3.28b)$$

$$= R(\pi+\varphi, \pi-\theta, 0) (-1)^{-\lambda_1+\lambda_2} \kappa |p00\lambda_1\lambda_2\rangle \quad (3.28c)$$

$$= (-1)^{-\lambda_1+\lambda_2} |p, \pi-\theta, \pi+\varphi, \lambda_1\lambda_2\rangle$$

$$\times (\text{phase independent of } \underline{p} \text{ and } \lambda\text{'s}) . \quad (3.28d)$$

The phase that is independent of $\underline{p}$ and the helicities disappears when matrix elements are considered.

These symmetry relations are used in the next section. It is important to observe that while some of the symmetry operations change the direction of linear momentum, changing both the initial and final linear momenta in the same way preserves the relative orientation so the Lorentz invariants are not changed. In particular, the center of momentum scattering angle is unchanged by any of the symmetry operations.

C. Density Matrices

The natural decay of an unstable particle into two distinguishable different particles can be described by a density matrix. The density matrices for both resonances have the same form so only the c decay matrix is treated here. For resonance c the decay density matrix is defined by

$$\rho_{mm'}^c = \frac{2S_c + 1}{4\pi} [\sum_{\lambda_1, \lambda_2} |T_{\lambda_1, \lambda_2}^c|^2]^{-1} \sum_{\lambda_1, \lambda_2} D_{m, \lambda_1 - \lambda_2}^{S_c*}(\varphi_1, \theta_1, 0) \times D_{m', \lambda_1 - \lambda_2}^{S_c}(\varphi_1, \theta_1, 0) |T_{\lambda_1, \lambda_2}^c|^2 \quad (3.29)$$

where $T_{\lambda_1 \lambda_2}^c$ is a reduced matrix element that depends only on the helicities of the decay products and S_c is the spin of c . The sum over the decay product helicities λ_1 and λ_2 results from the sum over final helicities for unobserved polarizations. The polar angles θ_1 and φ_1

take into account the recent history of resonance c in the production reaction and are carefully defined in the next section.

It follows from the definition that ρ^c is hermitian. ρ^c also has the properties that

$$\int \rho_{mm'}^c d\Omega_1 = \delta_{mm'} \quad (3.30)$$

and

$$\sum_m \rho_{mm}^c = \frac{2S_c + 1}{4\pi} \quad (3.31)$$

where $d\Omega_1 = d\cos\theta_1 d\phi_1$ and the orthogonality of the D matrices have been used.

The reduced matrix element is defined by

$$T_{\lambda_1 \lambda_2}^c = \langle S_c^m \lambda_1 \lambda_2 | T | S_c^m \rangle. \quad (3.32)$$

Therefore, for parity conserving decays

$$\begin{aligned} & \langle S_c^m \lambda_1 \lambda_2 | T^c | S_c^m \rangle \\ &= (-1)^{S_1 + S_2 - S_c} \eta_1 \eta_2 \eta_c \langle S_c^{m - \lambda_1 - \lambda_2} | T^c | S_c^m \rangle \end{aligned} \quad (3.33a)$$

or

$$T_{-\lambda_1 -\lambda_2}^c = \eta_1 \eta_2 \eta_c (-1)^{s_1 + s_2 - s} T_{\lambda_1 \lambda_2}^c \quad (3.33b)$$

where Equation 3.22 and

$$P |S_c m\rangle = \eta_c |S_c m\rangle \quad (3.34)$$

have been used. The decay density matrix for resonance d is obtained from Equation 3.29 by replacing the set $c, 1, 2, m, m'$ by $d, 3, 4, n, n'$ respectively. The d decay density matrix has the same properties as the c density matrix.

Density matrices can also be used to describe the production reaction. Both direct and crossed channel production density matrices can be defined. Since the direct channel corresponds to the real world, the applicable symmetries of any reaction must be contained in the direct channel production density matrix. The effect of these symmetries on the crossed channel production density matrix can then be obtained by using the crossing relation given in the next section. The production density matrices are defined in terms of linear momentum helicity states. For convenience, from now on these states are denoted by their helicities only. In terms of these states, the direct channel production density matrix is defined by

$$\rho_{\lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = \sum_{\lambda_a \lambda_b} \langle \lambda_c \lambda_d | T | \lambda_a \lambda_b \rangle \langle \lambda_c' \lambda_d' | T | \lambda_a \lambda_b \rangle^* . \quad (3.35)$$

Using the symmetry relations derived in the last section, the direct channel production density matrix can be shown to have the following symmetries. Under P

$$\rho_{\lambda_c \lambda_c' \lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = \rho_{-\lambda_c -\lambda_c' -\lambda_d -\lambda_d'}^{-\lambda_c -\lambda_c'} \quad (3.36)$$

Under Y

$$\rho_{\lambda_c \lambda_c' \lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = (-1)^{\lambda_c - \lambda_c' - \lambda_d + \lambda_d'} \rho_{-\lambda_c -\lambda_c' -\lambda_d -\lambda_d'}^{-\lambda_c -\lambda_c'} \quad (3.37)$$

Under C , for particle antiparticle pairs in both the initial and final states,

$$\rho_{\lambda_c \lambda_c' \lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = \rho_{\lambda_d' \lambda_d \lambda_c' \lambda_c}^{\lambda_d \lambda_d'} \quad (3.38)$$

Under τ for elastic scattering

$$\rho_{\lambda_c \lambda_c' \lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = (-1)^{\lambda_c - \lambda_c' - \lambda_d + \lambda_d'} \rho_{\lambda_d \lambda_d' \lambda_c \lambda_c'}^{\lambda_c \lambda_c'} \quad (3.39)$$

Also, since ρ is hermitian,

$$\rho_{\lambda_c \lambda_c' \lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = \rho_{\lambda_d' \lambda_d \lambda_c' \lambda_c}^{\lambda_d \lambda_d'} \quad (3.40)$$

Equations 3.36 and 3.37 for P and Y invariance taken together give

$$\rho_{\lambda_d \lambda_d'}^{\lambda_c \lambda_c'} = \begin{cases} \rho_{-\lambda_d' -\lambda_d}^{-\lambda_c' -\lambda_c} & \text{if } \lambda_c - \lambda_c' - \lambda_d + \lambda_d' = \text{an even integer} \\ 0 & \text{if } \lambda_c - \lambda_c' - \lambda_d + \lambda_d' = \text{an odd integer} . \end{cases} \quad (3.41)$$

The crossed channel production density matrix is defined by

$$\rho_{nn'}^{mm'} = \sum_{\mu_a \mu_b} \langle \mu_b n | T | \mu_a m \rangle \langle \mu_b n' | T | \mu_a m' \rangle^* \quad (3.42)$$

where the $m, m', n,$ and n' are magnetic quantum numbers. The crossed and direct channel production density matrices are related by

$$\begin{aligned} \rho_{nn'}^{mm'} &= \sum_{\lambda_c \lambda_c' \lambda_d \lambda_d'} d_{\lambda_c m}^{S_c}(\alpha_c) d_{\lambda_c' m'}^{S_c}(\alpha_c) d_{\lambda_d n}^{S_d}(\alpha_d) \\ &\quad \times d_{\lambda_d' n'}^{S_d}(\alpha_d) \rho_{\lambda_d \lambda_d'}^{\lambda_c \lambda_c'} . \end{aligned} \quad (3.43)$$

This relation is derived from information presented in the next section. Using the symmetries of the d matrices listed in Appendix A, the symmetries of the crossed channel production matrix can be derived.

Under Y

$$\rho_{nn'}^{mm'} = (-1)^{m-m'-n+n'} \rho_{-n-n'}^{-m-m'} . \quad (3.44)$$

Under C for particle antiparticle pairs in the initial and final states in the direct channel

$$\rho_{nn'}^{mm'} = \rho_{mm'}^{nn'} \quad (3.45)$$

Under τ for elastic scattering in the direct channel

$$\rho_{nn'}^{mm'} = (-1)^{m-m'-n+n'} \rho_{nn'}^{mm'} \quad (3.46)$$

Since ρ is hermitian

$$\rho_{nn'}^{mm'} = \rho_{n'n}^{m'm*} \quad (3.47)$$

ρ invariance gives the same result as Y invariance.

D. Angular Distribution

In this section the angular distribution for the reaction

$$\begin{array}{c}
 a + b \rightarrow c + d \\
 \begin{array}{l}
 \downarrow \quad \downarrow \\
 \rightarrow 3 + 4 \\
 \rightarrow 1 + 2
 \end{array}
 \end{array} \quad (3.48)$$

is derived for particles with fixed but arbitrary spins. The initial and final state particles are unpolarized or no polarization measurements are made so there must be an average over initial helicities and a sum over final helicities. The average over initial helicities is accomplished by summing over initial helicities and dividing by the multiplicity. The T-matrices are now considered in terms of the

physical states which are linear momentum helicity states. For convenience, the states are denoted by their helicities only. No problems are encountered deleting the linear momentum quantum numbers because the Lorentz invariants s and t that describe the production reaction are unchanged by any of the symmetry operations.

The physical states in the ab (cd) center of momentum frame for the direct channel production reaction are $|\lambda_a \lambda_b\rangle$ and $|\lambda_c \lambda_d\rangle$. For the decays the physical states are $|\lambda_c\rangle$ & $|\lambda_d\rangle$ and $|\lambda_1 \lambda_2\rangle$ & $|\lambda_3 \lambda_4\rangle$ in the c and d rest frames respectively. Axes for the production reaction are picked in the ab center of momentum frame so the y -axis is normal to the production plane and the z -axis is in the direction in which particle a is incident. These axes were used in the section on symmetries. In the direct channel description, the axes for the decays are picked so the y -axis is normal to the production plane and the z -axis is the helicity quantization axis for the decaying particle in the rest frame of the decaying particle. For resonance c (d) the z -axis is in the opposite direction of the linear momentum of resonance d (c) in the c (d) rest frame. The y -axes are in the same direction for both the production and the decays because the y -axis and the Lorentz boost that connects the two frames are perpendicular. The polar angles are obtained from the coordinate axes in the usual way. θ is measured from the z -axis and ϕ is measured from the xz plane. This choice of axes for the direct channel description results in the least complicated mathematical formalism.

Using Equation 2.17 and the physical states just described, the direct channel angular distribution, including the average over initial helicities and the sum over final helicities, can be written as

$$\frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} = K \sum_{\lambda_a \lambda_b \lambda_1 \lambda_2 \lambda_3 \lambda_4} \sum_{\lambda_c \lambda_d} \langle \lambda_c \lambda_d | T | \lambda_a \lambda_b \rangle \times \langle \lambda_1 \lambda_2 | T^c | \lambda_c \rangle \langle \lambda_3 \lambda_4 | T^d | \lambda_d \rangle^2 \quad (3.49)$$

$$\text{where } K = [(2S_a + 1)(2S_b + 1)]^{-1} \int ds_c ds_d k(s, s_c, s_d) \quad (3.49a)$$

S_a and S_b are the spins of particles a and b respectively. There is a sum over intermediate helicities λ_c and λ_d because it is possible at least in principle to go from an initial state to a final state by any of the intermediate states.

The decay amplitudes can be simplified by writing the final two particle states in the partial wave expansion of Equation 3.13 and using the Wigner-Eckart theorem (12).

$$\langle \lambda_1 \lambda_2 | T^c | \lambda_c \rangle = \sum_{Jm} \sqrt{\frac{2J+1}{4\pi}} D_{m, \lambda_1 - \lambda_2}^{J*}(\varphi_1, \theta_1, 0) \langle Jm \lambda_1 \lambda_2 | T^c | \lambda_c \rangle \quad (3.50a)$$

$$= \sum_{Jm} \sqrt{\frac{2J+1}{4\pi}} D_{m, \lambda_1 - \lambda_2}^{J*}(\varphi_1, \theta_1, 0) \delta_{m\lambda_c} \delta_{JS_c} \langle S_c \lambda_c \lambda_1 \lambda_2 | T^c | \lambda_c \rangle \quad (3.50b)$$

$$= \sqrt{\frac{2S_c + 1}{4\pi}} D_{\lambda_c, \lambda_1 - \lambda_2}^{S_c*}(\varphi_1, \theta_1, 0) T_{\lambda_1 \lambda_2}^c \quad (3.50c)$$

where $T_{\lambda_1 \lambda_2}^c$ is a reduced matrix element that depends on the helicities of the decay products and θ_1' and φ_1 are the polar angles of particle 1 in the 1-2 center of momentum frame (c decay rest frame). Use has been made in Equation 3.50b of the fact that helicity and angular momentum states are identical when the helicity quantization axis coincides with the z-axis for a single particle at rest. Similarly,

$$\langle \lambda_3 \lambda_4 | T^d | \lambda_d \rangle = \sqrt{\frac{2S_d + 1}{4\pi}} D_{\lambda_d, \lambda_3 - \lambda_4}^{S_d*}(\varphi_3, \theta_3', 0) T_{\lambda_3 \lambda_4}^d. \quad (3.51)$$

With this simplification, the direct channel angular distribution can be written in terms of the density matrices defined in the last section.

$$\begin{aligned} \frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} &= K \sum_{\lambda_a \lambda_b \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_c \lambda_d \lambda_c' \lambda_d'} \langle \lambda_c \lambda_d | T | \lambda_a \lambda_b \rangle \\ &\times \langle \lambda_c' \lambda_d' | T | \lambda_a \lambda_b \rangle^* \langle \lambda_1 \lambda_2 | T^c | \lambda_c \rangle \langle \lambda_1 \lambda_2 | T^c | \lambda_c' \rangle^* \\ &\times \langle \lambda_3 \lambda_4 | T^d | \lambda_d \rangle \langle \lambda_3 \lambda_4 | T^d | \lambda_d' \rangle^* \end{aligned} \quad (3.52a)$$

$$= NK \sum_{\lambda_c \lambda_c' \lambda_d \lambda_d'} \rho_{\lambda_c \lambda_c'}^c \rho_{\lambda_d \lambda_d'}^d \quad (3.52b)$$

where

$$N = \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} |T_{\lambda_1 \lambda_2}^c|^2 |T_{\lambda_3 \lambda_4}^d|^2. \quad (3.52c)$$

Another description of the direct channel angular distribution, called the crossed channel angular distribution, uses crossed channel amplitudes instead of direct channel amplitudes. Since the amplitudes are assumed to be analytic, these two descriptions are equivalent. Since the crossed channel description may be mathematically less complicated than the direct channel description, it is worth investigating.

Trueman and Wick (13) and Muzinich (14) have formulated crossing relations relating direct and crossed channel helicity amplitudes. In the crossed channel the physical states are $|\mu_a \mu_c\rangle$ and $|\mu_b \mu_d\rangle$. The crossing relation between these states and the direct channel physical states is

$$\begin{aligned} \langle \lambda_c \lambda_d | T | \lambda_a \lambda_b \rangle = & \sum_{\mu_a \mu_b \mu_c \mu_d} d_{\lambda_a \mu_a}^{s_a}(\alpha_a) d_{\lambda_b \mu_b}^{s_b}(\alpha_b) \\ & \times d_{\lambda_c \mu_c}^{s_c}(\alpha_c) d_{\lambda_d \mu_d}^{s_d}(\alpha_d) \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle \end{aligned} \quad (3.53)$$

where the angles α are given in the Reference (13). It is sufficient for this discussion to know the geometric interpretation of the angles which is given in Figure 3. In this velocity space diagram, the points labeled indicate rest frames for the system or particle labeled. For instance, s is the rest frame for the center of momentum of the direct channel production reaction and a is the rest frame of particle a . As seen from point labeled s , particles b and d are moving in the sb and sd directions respectively.

Figure 3. Velocity space diagram for angles α .

Figure 4. Axes and polar angles for resonance c in the crossed channel description.

Now the axes for the crossed channel description of the decays can be specified. These axes are picked to obtain the least complicated mathematical expression. Following Gottfried and Jackson (15), the z-axis is picked to point opposite to the direction of the 3 momentum transfer of the production reaction in the rest frame of the decaying particle. Specifically, the z-axis for resonance c is in the direction opposite to the linear momentum of particle a in the c rest frame. For resonance d the z-axis is in the direction of the linear momentum of particle b in the d rest frame. As before, the y-axis is picked to be normal to the production plane. The axes and polar angles for the decay of resonance c are pictured in Figure 4. From Figure 3 it can be seen that the angles between the helicity quantization axes, which are the direct channel description decay z-axes, and the crossed channel description decay z-axes are just the angles α_c and α_d for resonances c and d respectively. In fact, the rotation in the Trueman and Wick crossing relation for resonance c (d) rotates the state $|\lambda_c\rangle$ ($|\lambda_d\rangle$) into a linear combination of single particle angular momentum states quantized along the crossed channel description decay z-axis for resonance c (d) or equivalently

$$|\lambda_c\rangle = \sum_m d_{\lambda_c m}^S(\alpha_c) |m\rangle \quad (3.54)$$

and

$$|\lambda_d\rangle = \sum_n d_{\lambda_d n}^S(\alpha_d) |n\rangle . \quad (3.55)$$

Here m and n are magnetic quantum numbers with respect to the crossed channel description decay z -axes for resonances c and d respectively.

The decay matrices can be simplified by using a partial wave expansion for the final states, this time along the crossed channel description decay z -axes, and the Wigner-Eckart theorem. The results are

$$\langle \lambda_1 \lambda_2 | T^c | \lambda_c \rangle = \sum_m \langle \lambda_1 \lambda_2 | T^c | m \rangle d_{\lambda_c m}^{S_c}(\alpha_c) \quad (3.56a)$$

$$= \sum_m \sqrt{\frac{2S_c + 1}{4\pi}} D_{m, \lambda_1 - \lambda_2}^{S_c*}(\varpi_1, \theta_1, 0) T_{\lambda_1 \lambda_2}^c d_{\lambda_c m}^{S_c}(\alpha_c) \quad (3.56b)$$

and

$$\langle \lambda_3 \lambda_4 | T^d | \lambda_d \rangle = \sum_n \langle \lambda_3 \lambda_4 | T^d | n \rangle d_{\lambda_d n}^{S_d}(\alpha_d) \quad (3.57a)$$

$$= \sum_n \sqrt{\frac{2S_d + 1}{4\pi}} D_{n, \lambda_3 - \lambda_4}^{S_d*}(\varpi_3, \theta_3, 0) T_{\lambda_3 \lambda_4}^d d_{\lambda_d n}^{S_d}(\alpha_d) \quad (3.57b)$$

It should be noted that the decay angles θ_1 and θ_3 are not the same in the direct and crossed channel descriptions. The difference between θ_1 (θ_3) and θ_1 (θ_3) is the angle α_c (α_d). Since the xz plane and y axis are the same in both descriptions, the angle ϖ is the same.

Substitution into Equation 3.49 of the equivalent expressions in terms of the crossed channel description given in Equation 3.53, 3.56, and 3.57 results in the crossed channel angular distribution.

$$\begin{aligned}
\frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} &= K \sum_{\lambda_a \lambda_b \lambda_1 \lambda_2 \lambda_3 \lambda_4} \sum_{\lambda_c \lambda_d \mu_a \mu_b \mu_c \mu_d} \sum_{mn} \\
&\times d_{\lambda_a \mu_a}^{S_a}(\alpha_a) d_{\lambda_b \mu_b}^{S_b}(\alpha_b) d_{\lambda_c \mu_c}^{S_c}(\alpha_c) d_{\lambda_d \mu_d}^{S_d}(\alpha_d) \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle \\
&\times \langle \lambda_1 \lambda_2 | T^c | m \rangle d_{\lambda_c m}^{S_c}(\alpha_c) \langle \lambda_3 \lambda_4 | T^d | n \rangle d_{\lambda_d n}^{S_d}(\alpha_d) \quad (3.58)
\end{aligned}$$

When the absolute square is expanded and the orthogonality of the d matrices is used, this expression simplifies to

$$\begin{aligned}
\frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} &= K \sum_{mm' nn'} \sum_{\mu_a \mu_b} \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \langle \mu_b n | T | \mu_a m \rangle \\
&\times \langle \mu_b n' | T | \mu_a m' \rangle^* \langle \lambda_1 \lambda_2 | T^c | m \rangle \langle \lambda_1 \lambda_2 | T^c | m' \rangle^* \\
&\times \langle \lambda_3 \lambda_4 | T^d | n \rangle \langle \lambda_3 \lambda_4 | T^d | n' \rangle^* \quad (3.59a)
\end{aligned}$$

$$= NK \sum_{mm' nn'} \rho_{nn'}^{nm'} \rho_{mm'}^{cn} \rho_{nn'}^d \quad (3.59b)$$

The angular correlation between any of the possible combinations of angles is obtained from Equations 3.52 and 3.59 by integrating over the physical regions of the other angles.

In the next section a prescription is given for calculating production density matrices from the experimental data and in Chapter IV a theoretical Regge-pole model for the production matrix is described.

The prescription and model can be used for both direct and crossed channel production matrices.

E. Calculation of Production Density Matrix Elements

Whenever the absolute square of the reduced matrix element in the decay density matrix is independent of the helicities of the decay products, the procedure described in this section can be used to calculate simple linear combinations of several production density matrix elements from experimental data. The most important use of this procedure is its application to a strong decay into a nucleon and a pion.

When the absolute square of the reduced matrix element is independent of the helicities of the decay products, the c density matrix (on temporarily dropping the subscripts on the polar angles and the spin of c) becomes

$$\rho_{mm'}^c = \frac{2S+1}{4\pi} \left[\sum_{\lambda_1 \lambda_2} |T_{\lambda_1 \lambda_2}^c|^2 \right]^{-1} \sum_{\lambda_1 \lambda_2} D_{m, \lambda_1 - \lambda_2}^{S*}(\varphi, \theta, 0) \times D_{m', \lambda_1 - \lambda_2}^S(\varphi, \theta, 0) |T_{\lambda_1 \lambda_2}^c|^2 \quad (3.60a)$$

$$= \frac{2S+1}{4\pi} [(2S_1+1)(2S_2+1)]^{-1} \sum_{\lambda_1 \lambda_2} D_{m, \lambda_1 - \lambda_2}^{S*}(\varphi, \theta, 0) \times D_{m', \lambda_1 - \lambda_2}^S(\varphi, \theta, 0) \quad (3.60b)$$

Now the c density matrix is multiplied by $D_{\gamma 0}^J(\varphi, \theta, 0)$ and the result is integrated over $d\Omega = d\cos\theta d\varphi$.

$$\int d\Omega D_{\gamma 0}^J(\varphi, \theta, 0) \rho_{mm}^c$$

$$= \frac{2S+1}{4\pi(2S_1+1)(2S_2+1)} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\varphi D_{\gamma 0}^J(\varphi, \theta, 0) \\ \times \sum_{\lambda_1 \lambda_2} D_{m, \lambda_1 - \lambda_2}^{S*}(\varphi, \theta, 0) D_{m', \lambda_1 - \lambda_2}^S(\varphi, \theta, 0) \quad (3.61a)$$

$$= \frac{2S+1}{4\pi(2S_1+1)(2S_2+1)} \sum_{\lambda_1 \lambda_2} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\varphi \sum_j \\ \times \langle JS_j, \gamma+m' | J\gamma S m' \rangle \langle JS_j, \lambda_1 - \lambda_2 | JOS, \lambda_1 - \lambda_2 \rangle \\ \times D_{\gamma+m', \lambda_1 - \lambda_2}^j(\varphi, \theta, 0) D_{m, \lambda_1 - \lambda_2}^{S*}(\varphi, \theta, 0) \quad (3.61b)$$

$$= \frac{2S+1}{4\pi(2S_1+1)(2S_2+1)} \sum_{\lambda_1 \lambda_2 j} \langle JS_j, \gamma+m' | J\gamma S m' \rangle \\ \times \langle JS_j, \lambda_1 - \lambda_2 | JOS, \lambda_1 - \lambda_2 \rangle \int_{-1}^1 d\cos\theta d_{\gamma+m', \lambda_1 - \lambda_2}^j(\theta) \\ \times d_{m, \lambda_1 - \lambda_2}^S(\theta) \int_0^{2\pi} d\varphi e^{-i(\gamma+m'-m)\varphi} \quad (3.61c)$$

$$= \frac{2S+1}{4\pi(2S_1+1)(2S_2+1)} \sum_{\lambda_1 \lambda_2 j} \langle JS_j, \gamma+m' | J\gamma S m' \rangle \\ \times \langle JS_j, \lambda_1 - \lambda_2 | JOS, \lambda_1 - \lambda_2 \rangle \frac{2\delta_{Sj}}{2S+1} 2\pi \delta_{\gamma+m', m} \quad (3.61d)$$

$$= [(2S_1+1)(2S_2+1)]^{-1} \sum_{\lambda_1 \lambda_2} \langle JSS, \gamma+m' | J\gamma S m' \rangle \\ \times \langle JSS, \lambda_1 - \lambda_2 | JOS, \lambda_1 - \lambda_2 \rangle \delta_{\gamma+m', m} \quad (3.61e)$$

where Equations A6 - A9 have been used and $\langle \begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix} | \begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix} \rangle$ is a Clebsch-Gordon coefficient in the form $\langle j_1 j_2 j_3 | j_1 m_1 j_2 m_2 \rangle$.

It is worthwhile to note that the same result is obtained using both common phase conventions for helicity states. The two conventions involve rotations of $R(\varphi, \theta, 0)$, used in this dissertation, and $R(\varphi, \theta, -\varphi)$. Because

$$D_{m, \lambda}^{S*}(\varphi, \theta, -\varphi) D_{m', \lambda}^S(\varphi, \theta, -\varphi) = D_{m, \lambda}^{S*}(\varphi, \theta, 0) D_{m', \lambda}^S(\varphi, \theta, 0), \quad (3.62)$$

the calculations leading to Equation 3.61e give the same result.

For the d density matrix this calculation gives

$$\begin{aligned} & \int d\Omega_3 D_{\gamma, 0}^{J'}(\omega_3, \theta_3, 0) \rho_{nn'}^d \\ &= [(2S_3+1)(2S_4+1)]^{-1} \sum_{\lambda_3 \lambda_4} \langle J' S_d S_d, \gamma' + n' | J' \gamma' S_d n' \rangle \\ & \times \langle J' S_d S_d, \lambda_3 - \lambda_4 | J' 0 S_d, \lambda_3 - \lambda_4 \rangle \delta_{\gamma' + n', n'} \end{aligned} \quad (3.63)$$

Multiplying the angular distribution in Equation 3.52 by $D_{\gamma 0}^J(\omega_1, \theta_1, 0)$ and integrating the result over $d\Omega_1 = d\cos\theta_1 d\phi_1$ gives

$$\begin{aligned}
& \int d\Omega_1 {}^1D_{\gamma 0}^J(\varphi_1, \theta_1', 0) \frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} \\
&= NK \sum_{\lambda_c \lambda_c' \lambda_d \lambda_d' \lambda_1 \lambda_2} \rho_{\lambda_d \lambda_d'}^{\lambda_c \lambda_c'} \rho_{\lambda_d \lambda_d'}^d [(2S_1+1)(2S_2+1)]^{-1} \\
&\quad \times \langle JS_c S_c, \gamma + \lambda_c' | J\gamma S_c \lambda_c' \rangle \langle JS_c S_c, \lambda_1 - \lambda_2 | J0 S_c, \lambda_1 - \lambda_2 \rangle \\
&\quad \times \delta_{\gamma + \lambda_c', \lambda_c} \quad (3.64a)
\end{aligned}$$

$$\begin{aligned}
&= \frac{NK}{(2S_1+1)(2S_2+1)} \sum_{\lambda_c' \lambda_d \lambda_d' \lambda_1 \lambda_2} \rho_{\lambda_d \lambda_d'}^{\gamma + \lambda_c' \lambda_c'} \rho_{\lambda_d \lambda_d'}^d \\
&\quad \times \langle JS_c S_c, \gamma + \lambda_c' | J\gamma S_c \lambda_c' \rangle \langle JS_c S_c, \lambda_1 - \lambda_2 | J0 S_c, \lambda_1 - \lambda_2 \rangle. \quad (3.64b)
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int d\Omega_1 {}^1D_{\gamma 0}^J(\varphi_1, \theta_1', 0) {}^1D_{\gamma' 0}^{J'}(\varphi_3, \theta_3', 0) \frac{d\sigma}{d\Omega d\Omega_1 d\Omega_3} \\
&= \frac{NK}{(2S_1+1)(2S_2+1)(2S_3+1)(2S_4+1)} \sum_{\lambda_c' \lambda_d' \lambda_1 \lambda_2 \lambda_3 \lambda_4} \rho_{\gamma' + \lambda_d' \lambda_d'}^{\gamma + \lambda_c' \lambda_c'} \\
&\quad \times \langle JS_c S_c, \gamma + \lambda_c' | J\gamma S_c \lambda_c' \rangle \langle JS_c S_c, \lambda_1 - \lambda_2 | J0 S_c, \lambda_1 - \lambda_2 \rangle \\
&\quad \times \langle J'S_d S_d, \gamma' + \lambda_d' | J'\gamma' S_d \lambda_d' \rangle \langle J'S_d S_d, \lambda_3 - \lambda_4 | J'0 S_d, \lambda_3 - \lambda_4 \rangle. \quad (3.65)
\end{aligned}$$

Equations 3.64 and 3.65 are written in terms of the direct channel production density matrix but are exactly correct for the crossed

channel production density matrix providing the set $\lambda_c, \lambda_c', \lambda_c, \lambda_c', \theta_1', \theta_3'$ is replaced by $m, m', n, n', \theta_1, \theta_3$ respectively. The functions $D_{\gamma 0}^j(\varphi, \theta, 0)$ are spherical harmonics and are listed in the Appendix.

The integrals in Equations 3.64 and 3.65 can be calculated easily from experimental data. The expectation or average value of any function $f(n_1, n_3)$ can be written as

$$\begin{aligned} & \int d n_1 d n_3 f(n_1, n_3) \times (\text{probability of } n_1, n_3 \text{ configuration}) \\ &= \int d n_1 d n_3 f(n_1, n_3) \frac{d\sigma}{d n d n_1, d n_3} \end{aligned} \quad (3.66a)$$

$$= \langle f(n_1, n_3) \rangle . \quad (3.66b)$$

The average value of $f(n_1, n_3)$ is calculated from the data by

$$\langle f(n_1, n_3) \rangle = \frac{1}{Q} \sum_{i=1}^Q f(n_{1i}, n_{3i}) \quad (3.67)$$

where Q is the total number of data points and the sum is over all data points.

The most common decay products for a strong decay are a nucleon and a pion. To see that in this case the absolute square of the reduced density matrix element is independent of the helicities of the decay products, consider the c decay and take particle 1 to be the nucleon. Then $\lambda_1 = \pm \frac{1}{2}$ and $\lambda_2 = 0$. For a strong decay, parity is conserved so

$$T_{\lambda,0}^c = -\eta_c (-1)^{\frac{1}{2}-S} c T_{-\lambda,0} = \pm T_{-\lambda,0} \quad (3.68)$$

and

$$|T_{\frac{1}{2}0}^c| = |T_{-\frac{1}{2}0}^c|^2. \quad (3.69)$$

Therefore, the absolute square of the reduced matrix element is independent of the helicities of the nucleon and pion. If c and d each decay into a nucleon and a pion, Equations 3.65 and 3.66 with $S_1=S_2=\frac{1}{2}$ and $S_2=S_4=0$ give

$$\begin{aligned} & \langle D_{\gamma 0}^J(\varphi_1, \theta_1', 0) D_{\gamma' 0}^{J'}(\varphi_3, \theta_3', 0) \rangle \\ &= \frac{1}{4} N K \sum_{\lambda_c' \lambda_d'} \rho_{\gamma' + \lambda_d' \lambda_d'}^{\gamma + \lambda_c' \lambda_c'} \langle J S_c S_c, \gamma + \lambda_c' | J \gamma S_c \lambda_c' \rangle \\ & \quad \times \langle J' S_d S_d, \gamma' + \lambda_d' | J' \gamma' S_d \lambda_d' \rangle \sum_{\lambda_1 \lambda_3} \langle J S_c S_c \lambda_1 | J O S_c \lambda_1 \rangle \\ & \quad \times \langle J' S_d S_d \lambda_3 | J' O S_d \lambda_3 \rangle \end{aligned} \quad (3.70)$$

for the direct channel production matrix and

$$\begin{aligned}
& \langle D_{\gamma 0}^J(\varphi_1, \theta_1, 0) D_{\gamma' 0}^{J'}(\varphi_3, \theta_3, 0) \rangle \\
&= \frac{1}{4} N K \sum_{m' n'} \rho_{n' + \gamma'}^{m' + \gamma} \frac{m'}{n'} \langle J S_c S_c, \gamma + m' | J \gamma S_c m' \rangle \\
&\quad \times \langle J' S_d S_d, \gamma' + n' | J' \gamma' S_d n' \rangle \sum_{\lambda_1 \lambda_3} \langle J S_c S_c \lambda_1 | J O S_c \lambda_1 \rangle \\
&\quad \times \langle J' S_d S_d \lambda_3 | J' O S_d \lambda_3 \rangle
\end{aligned} \tag{3.71}$$

for the crossed channel production matrix where Equation 3.59 has been used. The nucleon helicities λ_1 and λ_3 take the values $\pm \frac{1}{2}$. The values of J , J' , γ , and γ' are arbitrary but, because of the Clebsch-Gordon coefficients, only certain choices give useful information. For instance, non-zero results are possible only for J in the range

$$0 \leq J \leq 2S_c \tag{3.72a}$$

and γ in the range

$$-J \leq \gamma \leq J. \tag{3.72b}$$

The sum over nucleon helicities gives zero results for odd values of J , since on using the symmetries of the Clebsch-Gordon coefficients (16), the sum over nucleon helicities becomes

$$\begin{aligned}
& \langle JS_c S_c \frac{1}{2} | JS_c \frac{1}{2} \rangle + \langle JS_c S_c -\frac{1}{2} | JS_c -\frac{1}{2} \rangle \\
& = [1 + (-1)^J] \langle JS_c S_c \frac{1}{2} | JS_c \frac{1}{2} \rangle
\end{aligned} \tag{3.73}$$

which is zero for J equal to an odd integer. The same remarks can be made about J' and γ' .

Inserting Equation 3.73 in Equation 3.71 for nucleon pion final states

$$\begin{aligned}
& \langle D_{\gamma 0}^J(\varphi_1, \theta_1, 0) D_{\gamma' 0}^{J'}(\varphi_3, \theta_3, 0) \rangle \\
& = \frac{1}{4} N K \sum_{m' n' p_{n'+\gamma}}^{m'+\gamma \ m'} \langle JS_c S_c, \gamma+m' | J\gamma S_c^{m'} \rangle \\
& \quad \times \langle J'S_d S_d, \gamma'+n' \rangle [1 + (-1)^J] [1 + (-1)^{J'}] \\
& \quad \times \langle JS_c S_c \frac{1}{2} | JS_c \frac{1}{2} \rangle \langle J'S_d S_d \frac{1}{2} | J'\gamma S_d \frac{1}{2} \rangle .
\end{aligned} \tag{3.74}$$

The left hand side of this equation is calculated from experimental data for each choice of J, J' , γ , and γ' and the right hand side is a linear combination of production density matrix elements. The exact linear combination is determined by calculating a few Clebsch-Gordon coefficients.

Actually, the zeros resulting from odd values of J in Equation 3.73 allow a systematic check on the data. Since this result follows from the kinematics only, all data must give zero expectation values whenever

$J=1$ and/or $J'=1$. Parity conservation also requires some zero expectation values which can be used either to check the data or to check for parity nonconservation. Some examples are included in Chapter V.

The procedure presented in this section uses only the kinematics of unstable particle decay. It is therefore a model independent determination of production density matrix elements. Moment analysis can be used for other reactions (25).

IV. REGGE MODEL FOR THE PRODUCTION

When the data is strongly peaked in the forward direction for the production reaction, the reaction is called peripheral. In this case, a Regge-pole model might be expected to work. The Regge-pole model for particles with arbitrary spin presented in this chapter was developed first by Calogero, Charap, and Squires (17) and described later by Thews (18) and Collins and Squires (5).

The general procedure starts with a partial wave expansion of the crossed channel production amplitude. The partial waves are analytically continued to complex angular momentum by means of a single dispersion relation. After signature is introduced, the crossed channel amplitude is written as an integral transform. At high energies the resulting expression is dominated by the Regge poles.

In this model the following four assumptions are made.

1. Single dispersion relations exist in either energy or momentum transfer with a finite number of subtractions for helicity amplitudes.
2. The partial wave amplitudes have only simple poles that move with energy in the complex angular momentum plane.
3. Cuts either do not exist or can be approximated by simple poles.
4. The partial wave amplitudes satisfy the extended Mandelstam symmetry for particles with spin (19).

The direct channel high energy behavior is obtained from the crossed channel production amplitude. The partial wave expansion of the crossed channel production amplitude with the ϕ dependence removed is

$$\langle \mu_b \mu_d | T | \mu_a \mu_c \rangle = \sum_{J=\max(|\mu|, |\mu'|)}^{\infty} (J+\frac{1}{2}) \langle \mu_b \mu_d | T^J(t) | \mu_a \mu_c \rangle \times d_{\mu\mu'}^J(x) . \quad (4.1)$$

where $\mu = \mu_a - \mu_c$ and $\mu' = \mu_b - \mu_d$.

$\langle \mu_b \mu_d | T^J(t) | \mu_a \mu_c \rangle$ is the partial wave amplitude which depends only on the direct channel four momentum transfer t . $d_{\mu\mu'}^J(x)$ is a rotation matrix defined in the Appendix and x is the crossed channel scattering angle given in Equation 2.6. Using the orthogonality of the d - matrices, Equation 4.1 can be inverted to give

$$\langle \mu_b \mu_d | T^J(t) | \mu_a \mu_c \rangle = \int_{-1}^1 dx \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle d_{\mu\mu'}^J(x) . \quad (4.2)$$

The d - matrix is continued to complex angular momentum by means of the hypergeometric function (A1). The partial wave amplitude is continued to complex angular momentum by means of Equation 4.2 and the following dispersion relation for the crossed channel amplitude (5).

$$\begin{aligned} \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle &= \left(\frac{1+x}{2} \right)^{\frac{1}{2}(\mu+\mu')} \left(\frac{1-x}{2} \right)^{\frac{1}{2}(\mu-\mu')} \frac{1}{\pi} \\ &\times \left(\int \frac{\langle \mu_b \mu_d | A^S | \mu_a \mu_c \rangle}{z-x} dz + \int \frac{\langle \mu_b \mu_d | A^U | \mu_a \mu_c \rangle}{z+x} dz \right) . \end{aligned} \quad (4.3)$$

Substitution of this dispersion relation in Equation 4.2 gives

$$\begin{aligned}
\langle \mu_b \mu_d | T^J(t) | \mu_a \mu_c \rangle &= \frac{2}{\pi} \int \langle \mu_b \mu_d | A^S | \mu_a \mu_c \rangle \left(\frac{1+z}{2} \right)^{\frac{1}{2}(\mu+\mu')} \times \\
&\times \left(\frac{1-z}{2} \right)^{\frac{1}{2}(\mu-\mu')} e_{\mu\mu'}^J(z) dz + (-1)^{J-\mu} \frac{2}{\pi} \times \\
&\times \int \langle \mu_b \mu_d | A^U | \mu_a \mu_c \rangle \left(\frac{1+z}{2} \right)^{\frac{1}{2}(\mu-\mu')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(\mu+\mu')} e_{\mu, -\mu'}^J(z) dz . \quad (4.4)
\end{aligned}$$

The $e_{\mu\mu'}^J(z)$ which result from performing the x integration are rotation coefficients of the second kind (20).

The factor $(-1)^{J-\mu}$ in the second term is not suitable for analytic continuation so signature is introduced by defining two partial wave amplitudes with $(-1)^{J-\mu}$ replaced by $+$ or $-$ for $J-\mu$ an even or odd integer respectively.

$$\langle \mu_b \mu_d | T^{J\pm}(t) | \mu_a \mu_c \rangle = \frac{1}{2} [1 \pm (-1)^{J-\mu}] \langle \mu_b \mu_d | T^J(t) | \mu_a \mu_c \rangle . \quad (4.5)$$

Then the partial wave expansion becomes

$$\begin{aligned}
\langle \mu_b \mu_d | T | \mu_a \mu_c \rangle &= \sum_{J=\max(|\mu|, |\mu'|)}^{(J+\frac{1}{2})} [\langle \mu_b \mu_d | T^{J+} | \mu_a \mu_c \rangle d_{\mu\mu'}^{J+}(x) \\
&+ \langle \mu_b \mu_d | T^{J-} | \mu_a \mu_c \rangle d_{\mu\mu'}^{J-}(x)] \quad (4.6)
\end{aligned}$$

where

$$d_{\mu\mu'}^{J+}(x) = \frac{1}{2} [d_{\mu\mu'}^J(x) \pm d_{\mu, -\mu'}^J(-x)] . \quad (4.7)$$

The partial wave amplitudes with signature can now be continued to complex angular momentum and the complete amplitude can be written as a Sommerfeld-Watson transformation. Apart from subtraction the amplitude is

$$\begin{aligned} \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle &= \frac{1}{2i} \int_C \frac{dJ (J + \frac{1}{2})}{\sin \pi (J - \mu)} [\langle \mu_b \mu_d | T^{J+} | \mu_a \mu_c \rangle \\ &\times d_{\mu, -\mu'}^{J+}(-x) + \langle \mu_b \mu_d | T^{J-} | \mu_a \mu_c \rangle d_{\mu, -\mu'}^{J-}(-x)] \end{aligned} \quad (4.8)$$

where the contour C is a large semicircle in the right half complex angular momentum plane closed along the line $\text{Re } J = -\frac{1}{2}$. The most important subtractions are the Regge-pole terms. The direct channel high energy region corresponds to small negative t and large positive s or equivalently large negative x . In this region the Regge-pole terms dominate the amplitude. Then the crossed channel helicity amplitude in the physical high energy region of the direct channel production reaction is given by

$$\begin{aligned} \langle \mu_b \mu_d | T | \mu_a \mu_c \rangle &= -\pi \sum_{\text{Regge poles}} (\alpha + \frac{1}{2}) \frac{\langle \mu_b \mu_d | \beta^+(t) | \mu_a \mu_c \rangle}{\sin \pi (\alpha - \mu)} d_{\mu, -\mu'}^{\alpha+}(-x) \\ &+ \text{Background} \end{aligned} \quad (4.9)$$

for $\alpha > 0$. Here $\alpha = \alpha(t)$ is the Regge trajectory which equals the value of the angular momentum J at the Regge poles. The $\beta^\pm(t)$ are the residues of the partial wave amplitudes $T_{\mu\mu'}^{\pm}(t)$ at the Regge poles. The sum is over Regge poles. Each Regge pole has either + or - signature.

Difficulties arise if the Regge trajectory is negative in the physical region. The Regge poles may not dominate the contour integration if the contour in Equation 4.8 is extended to the left to include the Regge trajectory. Mandelstam solved this problem for spinless particles. His method, extended to particles with spin, uses $e_{-\mu,\mu'}^{-J-\frac{1}{2}}(-x)$ in place of $d_{\mu,-\mu'}^{J+\frac{1}{2}}(-x)$ in Equation 4.8. When the contour is extended to the left and all subtractions made, the Regge poles and other subtractions dominate at high energy. However, when the partial wave amplitudes satisfy the extended Mandelstam symmetry,

$$T_{\mu\mu'}^{J+\frac{1}{2}\pm} = (-1)^{\mu-\mu'} \begin{cases} T_{\mu\mu'}^{-J-\frac{3}{2}\pm} & \text{for } J, \mu, \mu' \text{ integer} \\ T_{\mu\mu'}^{-J-\frac{3}{2}\mp} & \text{for } J, \mu, \mu' \text{ half-integer,} \end{cases} \quad (4.10)$$

the other subtractions cancel leaving the Regge poles. In this case the amplitude can be written as

$$\langle \mu_b \mu_d | T | \mu_a \mu_c \rangle = \sum_{\text{Regge}} (\alpha + \frac{1}{2}) \frac{\langle \mu_b \mu_d | \beta^\pm(t) | \mu_a \mu_c \rangle}{\cos \pi(\alpha - \mu)} e_{-\mu,\mu'}^{-\alpha-\frac{1}{2}}(-x) + \text{Background} \quad (4.11)$$

for $\alpha < 0$.

When the $d^\pm$ and $e^\pm$ functions are expanded in hypergeometric functions, Equations 4.9 and 4.11 can be written as one equation.

$$\langle \mu_b \mu_d | T | \mu_a \mu_c \rangle = -\sqrt{\pi} \sum_{\substack{\text{Regge} \\ \text{poles}}} \frac{\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 1)} \frac{[1 + e^{-i\pi(\alpha - \mu)}]}{2 \sin \pi(\alpha - \mu)} \times$$

$$\langle \mu_b \mu_d | \beta^\pm(t) | \mu_a \mu_c \rangle (-2x)^{\alpha(t)} F_{\mu\mu'}(\alpha, x) \quad (4.12)$$

where the factor in brackets, $[]$, is the signature term and $(-2x)^{\alpha(t)}$ gives the high energy Regge behavior $(S-S_0)^{\alpha(t)}$. The functions $F_{\mu\mu'}(\alpha, x)$ are listed in the appendix of Thews' paper (18).

At sufficiently high energy the background can be ignored. Using the form of the amplitude in Equation 4.9 and ignoring the background, the crossed channel production matrix can be written as

$$\rho_n^{m m'} = \pi^2 \sum_{\mu_a \mu_b} \sum_{\substack{\text{Regge} \\ \text{poles} \\ \alpha}} \sum_{\substack{\text{Regge} \\ \text{poles} \\ \alpha'}} (\alpha + \frac{1}{2}) (\alpha' + \frac{1}{2}) \frac{\langle \mu_b n | \beta^\pm(t) | \mu_a m \rangle}{\sin \pi(\alpha - \mu_a + m)}$$

$$\times \frac{\langle \mu_b n' | \beta^\pm(t) | \mu_a m' \rangle}{\sin \pi(\alpha' - \mu_a + m')} d_{\mu_a - m, n - \mu_b}^{\alpha+}(-x) d_{\mu_a - m', n' - \mu_b}^{\alpha'+}(-x). \quad (4.13)$$

For one Regge pole this becomes

$$\rho_{n n'}^{m m'} = \pi^2 \sum_{\mu_a \mu_b} (\alpha + \frac{1}{2})^2 \frac{\langle \mu_b n | \beta(t) | \mu_a m \rangle \langle \mu_b n' | \beta(t) | \mu_a m' \rangle}{\sin \pi (\alpha - \mu_a + m) \sin \pi (\alpha - \mu_a + m')} \\ \times d_{\mu_a - m, n - \mu_b}^{\alpha}(-x) d_{\mu_a - m', n' - \mu_b}^{\alpha}(-x) \quad (4.14)$$

where the residues and d^{α} functions all have the same signature.

Each Regge trajectory has only one signature associated with it. In addition, the residues are real because they are considered for t in the region below the crossed channel threshold. Consequently, the density matrix element predictions in Equations 4.13 - 14 are real. Since this model predicts only the existence of the residues and not their values, the residues are usually treated as parameters for fits to experimental data.

V. RESULTS

A. General Results

In this dissertation double resonance production for particles with spin has been handled by separating the reaction into production and decay parts and treating each part separately. The production and decays are connected only by the direction of the resonances and their polarizations. Angular distributions for both direct channel and crossed channel helicity amplitudes have been derived and are given in Equations 3.52 and 3.59. The angular correlation between any set of angles is obtained by integrating over the physical regions of the other angles. The symmetries of the production density matrices are given in Equations 3.36 - 40 and 3.44 - 47.

It is important to note that the angles appearing in the decay density matrices are different for the direct channel and crossed channel angular distributions but in both cases the angles are measured in the rest frame of the decaying particle. These angles are related to the angles measured in the laboratory by a Lorentz transformation (21).

For the case of particle-resonance production or double resonance production with one resonance unobserved, the decay density matrix of the particle or unobserved resonance is written as the unit matrix. For double particle production both decay density matrices are replaced by unit matrices and the angular distributions simplify to the form of a production reaction as they must.

The prediction of a simple Regge-pole model for the crossed channel amplitude is given in Equation 4.9. In this model the production density

matrix elements are all real and are given in Equation 4.13 where the background has been ignored.

From a practical point of view, the most important contribution of this dissertation is the moment analysis for determining production density matrix elements from experimental data. This procedure, described in the last section of Chapter III, is model independent and can be used whenever the square of the reduced matrix element is independent of the helicities of the decay products (pion nucleon type final states).

These results are applied to the double resonance production reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$ in the next section.

B. Application

In this section the results of this dissertation are applied to the double resonance production reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$. Studies of the reaction $\bar{p}p \rightarrow \bar{p}\pi^- p\pi^+$ at antiproton laboratory momentum of 2.7 (22), 2.4 and 2.9 (23) GeV/c have been made at Iowa State University where it was determined that 90 - 100% of the reaction proceeded by double resonance production $\bar{\Delta}\Delta$. Δ is the doubly charged baryon resonance of mass 1236 MeV. A Regge-pole model would not be expected to work well at these energies but the predictions of the model are indicated for completeness. Perhaps much higher energy data will be available in the future.

For the reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$

$$s_a = s_b = s_1 = s_3 = \frac{1}{2},$$

$$\lambda_a = \mu_a = \lambda_b = \mu_b = \lambda_1 = \lambda_3 = \pm \frac{1}{2},$$

$$s_c = s_d = \frac{3}{2}, \quad (5.1)$$

$$m = m' = n = n' = \lambda_c = \lambda'_c = \lambda_d = \lambda'_d = \pm \frac{1}{2}, \pm \frac{3}{2},$$

$$s_2 = s_4 = 0,$$

$$\text{and } \lambda_2 = \lambda_4 = 0$$

To avoid fractional superscripts and subscripts, the magnetic quantum numbers and helicities are denoted by twice their actual values in this section.

For each production density matrix there are $4^4 = 256$ complex matrix elements. This number is reduced to 28 (16 of these real) independent matrix elements for the direct channel production density matrix by the symmetries given in Equations 3.36 - 38 and 3.40. There are more than 28 independent matrix elements for the crossed channel production density matrix because Y and P do not give distinct symmetries for the crossed channel production density matrix elements. However, since the crossed channel production reaction is $\bar{p}\Delta$ elastic scattering, time reversal may be applicable. If it is, there are exactly 28 independent matrix elements

in each of the production density matrices. Twelve of these are complex so there are 40 independent parameters of which less than half are accessible from cross section data. Polarization data is needed for the others.

The Regge-pole model prediction is obtained by inserting Equation 4.13 or 4.14 into 3.59. Since the residues are real, this model predicts 28 independent production matrix elements. In the crossed channel production reaction the π , ρ , and R trajectories can be exchanged. For each trajectory there are a minimum of 12 independent residues. Therefore, the angular distribution expression resulting from the Regge-pole model is very complicated even if one Regge trajectory dominates. Because of the large number of parameters and the low energy of the reactions, no attempt was made to analyze the ISU data using the Regge-pole model. Nevertheless, the formalism using the model has been presented and might work very well for high energy resonances with lower spins.

Since the Δ decays into a p and a π^+ and $\bar{\Delta}$ decays into a $\bar{p}$ and π^- , the equation derived in the last section of Chapter III for nucleon pion type final states can be used to calculate linear combinations of production density matrix elements from experimental data. Setting

$S_c = S_d = \frac{3}{2}$ in Equation 3.74 gives

$$\begin{aligned}
& \langle D_{\gamma 0}^J(\varphi_1, \theta_1, 0) D_{\gamma' 0}^{J'}(\varphi_3, \theta_3, 0) \rangle \\
&= \frac{1}{4} NK \sum_{mn} \rho_{n+\gamma, m}^{m+\gamma} \langle J \frac{3}{2} \frac{3}{2}, \gamma+m | J \frac{3}{2} m \rangle \\
&\quad \times \langle J' \frac{3}{2} \frac{3}{2}, \gamma'+n | J' \frac{3}{2} n \rangle [1+(-1)^J] [1+(-1)^{J'}] \\
&\quad \times \langle J \frac{3}{2} \frac{3}{2} \frac{1}{2} | J 0 \frac{3}{2} \frac{1}{2} \rangle \langle J' \frac{3}{2} \frac{3}{2} \frac{1}{2} | J' 0 \frac{3}{2} \frac{1}{2} \rangle. \tag{5.2}
\end{aligned}$$

As an example of the ease with which the linear combinations of production matrix elements are calculated, consider the case

$J = \gamma = J' = \gamma' = 2$. Then using the spherical harmonics listed in the Appendix and calculating several Clebsch-Gordan coefficients, Equation 5.2 becomes

$$\begin{aligned}
& \frac{3}{8} \langle \sin^2 \theta_1 \sin^2 \theta_3 e^{-2i(\varphi_1 + \varphi_3)} \rangle \\
&= \frac{NK}{4} \left(\frac{2}{5} \rho_{3-1}^{3-1} + \frac{2}{5} \rho_{3-1}^{1-3} + \frac{2}{5} \rho_{1-3}^{3-1} + \frac{2}{5} \rho_{1-3}^{1-3} \right) \frac{4}{5} \tag{5.3a}
\end{aligned}$$

$$= \frac{2}{25} NK (\rho_{3-1}^{3-1} + \rho_{3-1}^{1-3} + \rho_{3-1}^{1-3*} + \rho_{3-1}^{3-1*}) \tag{5.3b}$$

$$= \frac{4}{25} NK \operatorname{Re}(\rho_{3-1}^{3-1} + \rho_{3-1}^{1-3}) \tag{5.3c}$$

where the Y and hermitian symmetries of the density matrix elements have

been used. Equating real and imaginary parts of Equation 5.3 gives

$$\text{Re}(\rho_{3-1}^{3-1} + \rho_{3-1}^{1-3}) = \frac{75}{32NK} \langle \sin^2 \theta_1 \sin^2 \theta_3 \cos 2(\varphi_1 + \varphi_3) \rangle \quad (5.4)$$

and

$$0 = \langle \sin^2 \theta_1 \sin^2 \theta_3 \sin 2(\varphi_1 + \varphi_3) \rangle. \quad (5.5)$$

Equation 5.5 is a prediction of parity conservation (actually Υ which includes parity in its definition) because the symmetry of the density matrix elements under parity was used to show that the right hand side of Equation 5.3 is real. Other predictions of parity are listed later. They are obtained in the same way Equation 5.5 was obtained.

When all 24 relevant Clebsch-Gordan coefficients are calculated, the combinations and expectation functions can be derived. These and their corresponding J , J' , γ , and γ' values are listed in Table I. The following definitions have been used in the table.

$$\rho_{\underline{\quad}}^{m \ m'} = \rho_{3 \ 3}^{m \ m'} + \rho_{-3 \ -3}^{m \ m'} - \rho_{1 \ 1}^{m \ m'} - \rho_{-1 \ -1}^{m \ m'}. \quad (5.6)$$

$$\rho^{m \ m'} = \rho_{3 \ 3}^{m \ m'} + \rho_{1 \ 1}^{m \ m'} + \rho_{-1 \ -1}^{m \ m'} + \rho_{-3 \ -3}^{m \ m'}. \quad (5.7)$$

The definitions for m and m' subscripts are the same but with all superscripts and subscripts exchanged. The factors NK are the same for all choices of J , γ , J' , and γ' so they can be determined from Equation 5.8m which gives the differential cross section for the production reaction.

Table 1. Production density matrix elements.

$J \gamma$	$J' \gamma'$		
2 2	2 2	$\text{Re}(\rho_{3-1}^{3-1} + \rho_{3-1}^{1-3}) = \frac{75}{32NK} \langle \sin^2 \theta_1 \sin^2 \theta_3 \cos 2(\varphi_1 + \varphi_3) \rangle$	(5.8a)
2 2	2 1	$\text{Re}(\rho_{3-1}^{3-1} + \rho_{3-1}^{1-3}) = \frac{75}{32NK} \langle \sin^2 \theta_1 \sin 2\theta_3 \cos(2\varphi_1 + \varphi_3) \rangle$	(5.8b)
2 2	2 0	$\text{Re} \rho_{3-1}^{3-1} = \frac{25\sqrt{3}}{16NK} \langle \sin^2 \theta_1 \cos 2\varphi_1 (3\cos^2 \theta_3 - 1) \rangle$	(5.8c)
2 2	2 -1	$\text{Re}(\rho_{1-3}^{3-1} + \rho_{1-3}^{1-3}) = \frac{75}{32NK} \langle \sin^2 \theta_1 \sin 2\theta_3 \cos(2\varphi_1 - \varphi_3) \rangle$	(5.8d)
2 2	2 -2	$\text{Re}(\rho_{1-3}^{3-1} + \rho_{1-3}^{1-3}) = \frac{75}{32NK} \langle \sin^2 \theta_1 \sin^2 \theta_3 \cos 2(\varphi_1 - \varphi_3) \rangle$	(5.8e)
2 1	2 1	$\text{Re}(\rho_{3-1}^{3-1} - \rho_{3-1}^{1-3}) = \frac{75}{32NK} \langle \sin 2\theta_1 \sin 2\theta_3 \cos(\varphi_1 + \varphi_3) \rangle$	(5.8f)
2 1	2 0	$\text{Re} \rho_{3-1}^{3-1} = \frac{25\sqrt{3}}{16NK} \langle \sin 2\theta_1 \cos \varphi_1 (3\cos^2 \theta_3 - 1) \rangle$	(5.8g)
2 1	2 -1	$\text{Re}(\rho_{1-3}^{3-1} - \rho_{1-3}^{1-3}) = \frac{75}{32NK} \langle \sin 2\theta_1 \sin 2\theta_3 \cos(\varphi_1 - \varphi_3) \rangle$	(5.8h)
2 0	2 0	$\rho_{3-1}^{3-1} - \rho_{3-1}^{1-1} = \frac{25}{32NK} \langle (3\cos^2 \theta_1 - 1)(3\cos^2 \theta_3 - 1) \rangle$	(5.8i)
2 2	0 0	$\text{Re} \rho_{3-1}^{3-1} = -\frac{5\sqrt{3}}{8NK} \langle \sin^2 \theta_1 \cos 2\varphi_1 \rangle$	(5.8j)
2 1	0 0	$\text{Re} \rho_{3-1}^{3-1} = -\frac{5\sqrt{3}}{8NK} \langle \sin 2\theta_1 \cos \varphi_1 \rangle$	(5.8k)
2 0	0 0	$\rho_{3-1}^{3-1} + \rho_{1-1}^{1-1} = \rho_{3-1}^{3-3} + \rho_{1-1}^{1-1} = -\frac{5}{16NK} \langle (3\cos^2 \theta_1 - 1) \rangle$	(5.8l)
0 0	0 0	$\rho_{1-1}^{1-1} + \bar{\rho}_{1-1}^{1-1} + \rho_{3-3}^{3-3} + \bar{\rho}_{3-3}^{3-3} = 2(\rho_{3-3}^{3-3} + \rho_{1-1}^{1-1}) = \frac{1}{NK} \langle 1 \rangle$	(5.8m)

All choices of J , γ , J' , and γ' not included in Table I are related to choices included in the table by charge conjugation symmetry. The charge conjugate relations are obtained from the table by simultaneously exchanging the superscripts and subscripts on the density matrix elements and ϕ_1 , θ_1 with ϕ_3 , θ_3 in the expectation functions.

Equation 5.81 can be written as

$$\rho_{\underline{3} \underline{3}} + \rho_{\underline{1} \underline{1}} = \rho_{\underline{3} \underline{3}} - \rho_{\underline{1} \underline{1}} \quad (5.9)$$

using charge conjugation symmetry. Then Equations 5.8i, 5.8l, 5.8m and 5.9 can be rearranged to give the results listed in Table II.

Table II. Rearranged density matrix elements.

$J \gamma \quad J' \gamma'$		
$\begin{matrix} 2 \ 0 & 2 \ 0 \\ & + \\ 2 \ 0 & 0 \ 0 \end{matrix}$	$\rho_{\underline{1} \underline{1}} = -\frac{15}{16NK} \langle (3\cos^2\theta_1 - 1)(5\cos^2\theta_3 - 1) \rangle$	(5.10a)
$\begin{matrix} 2 \ 0 & 2 \ 0 \\ & - \\ 2 \ 0 & 0 \ 0 \end{matrix}$	$\rho_{\underline{3} \underline{3}} = \frac{5}{16NK} \langle (3\cos^2\theta_1 - 1)(15\cos^2\theta_3 - 7) \rangle$	(5.10b)
$\begin{matrix} 0 \ 0 & 0 \ 0 \\ & - \\ 2 \ 0 & 0 \ 0 \end{matrix}$	$\rho_{\underline{3} \underline{3}} = -\frac{1}{8NK} \langle (15\cos^2\theta_1 - 7) \rangle$	(5.10c)
$\begin{matrix} 0 \ 0 & 0 \ 0 \\ & + \\ 2 \ 0 & 0 \ 0 \end{matrix}$	$\rho_{\underline{1} \underline{1}} = \frac{3}{8NK} \langle (5\cos^2\theta_1 - 1) \rangle$	(5.10d)

The equations in Tables I and II agree with those listed by Jespersen, Kernan, and Leacock (23) (except for an error in their Equation 13b) providing $\langle \sin^2 \theta_i f(\Omega) \rangle$ is replaced by $\frac{4}{5} \langle f(\Omega) \rangle$ in Equation 5.8. Here $i = 1, 3$ and Ω stands for the other angles. This discrepancy can be understood by realizing that the angular distribution contains a factor of $\sin^2 \theta_i$, so integrating by parts gives

$$\int_{-1}^1 \sin^2 \theta_i d\cos \theta_i \sin^2 \theta_i = \frac{4}{5} \int_{-1}^1 \sin^2 \theta_i d\cos \theta_i \quad (5.11a)$$

or equivalently

$$\langle \sin^2 \theta_i f(\Omega) \rangle = \frac{4}{5} \langle f(\Omega) \rangle . \quad (5.11b)$$

As noted earlier, parity conservation predicts that some functions should have zero expectation values. These are listed in Table III. The parity predictions listed in Table III can be used to check conservation of parity or to check the experimental data. Results contrary to Equation 5.12 could indicate a systematic error or a poor cut on the data to isolate double resonance production.

Another check on the data results from the kinematic constraints. When $J = 1$ and/or $J' = 1$, the kinematics require a zero expectation value. This was explained in Section E of Chapter III. Higher values of J and J' are not as useful because the expectation functions become too complicated to give decisive results. A few of the kinematic constraints are listed in Table IV.

Table III. Parity predictions.

$J \gamma$	$J' \gamma'$		
2 2	2 2	$0 = \langle \sin 2(\varphi_1 + \varphi_3) \rangle$	(5.12a)
2 2	2 1	$0 = \langle \sin 2\theta_3 \sin(2\varphi_1 + \varphi_3) \rangle$	(5.12b)
2 2	2-1	$0 = \langle \sin 2\theta_3 \sin(2\varphi_1 - \varphi_3) \rangle$	(5.12c)
2 2	2-2	$0 = \langle \sin 2(\varphi_1 - \varphi_3) \rangle$	(5.12d)
2 1	2 1	$0 = \langle \sin 2\theta_1 \sin 2\theta_3 \sin(\varphi_1 + \varphi_3) \rangle$	(5.12e)
2-1	2-1	$0 = \langle \sin 2\theta_1 \sin 2\theta_3 \sin(\varphi_1 - \varphi_3) \rangle$	(5.12f)
2 2	0 0	$0 = \langle \sin 2\varphi_1 \rangle$	(5.12g)
2 1	0 0	$0 = \langle \sin 2\theta_1 \sin \varphi_1 \rangle$	(5.12h)

+ charge conjugation relations.

Table IV. Kinematic constraints.

$J \gamma$	$J' \gamma'$		
1 1	$1_{\pm} 1$	$0 = \langle \sin \theta_1 \sin \theta_3 \cos(\varphi_1 \pm \varphi_3) \rangle$	(5.13a)
1 1	$1_{\pm} 1$	$0 = \langle \sin \theta_1 \sin \theta_3 \sin(\varphi_1 \pm \varphi_3) \rangle$	(5.13b)
1 1	1 0	$0 = \langle \sin \theta_1 \cos \varphi_1 \cos \theta_3 \rangle$	(5.13c)
1 1	1 0	$0 = \langle \sin \theta_1 \sin \varphi_1 \cos \theta_3 \rangle$	(5.13d)
1 0	1 0	$0 = \langle \cos \theta_1 \cos \theta_3 \rangle$	(5.13e)
1 0	0 0	$0 = \langle \cos \theta_1 \rangle$	(5.13f)
$1_{\pm} 1$	0 0	$0 = \langle \sin \theta_1 \cos \varphi_1 \rangle$	(5.13g)
$1_{\pm} 1$	0 0	$0 = \langle \sin \theta_1 \sin \varphi_1 \rangle$	(5.13h)
1 0	2 0	$0 = \langle \cos \theta_1 (3 \cos^2 \theta_3 - 1) \rangle$	(5.13i)

+ charge conjugation relations.

Two generalizations can be made as a result of the application of the methods of this dissertation to the reaction $\bar{p}p \rightarrow \bar{\Delta}\Delta \rightarrow \bar{p}\pi^- p\pi^+$. The production reaction is the most difficult to describe. (Specialized theoretical predictions for the production reaction are often complicated.) Use of moment analysis on the decays produces many predictions with a minimum effort. Moment analysis also has the advantage that maximum information attainable from experimental data is clearly defined.

VI. APPENDIX

The d- matrix is real and following Rose (3) can be written in terms of a hypergeometric function (16) as

$$d_{mn}^j(x) = (-1)^{m-n} \sqrt{\frac{\Gamma(j+m+1)\Gamma(j-n+1)}{\Gamma(j-m+1)\Gamma(j+n+1)}} \left(\frac{1+x}{2}\right)^{\frac{1}{2}(m+n)} \left(\frac{1-x}{2}\right)^{\frac{1}{2}(m-n)} \\ \times \frac{F(m-j, j+m+1; m-n+1; \frac{1-x}{2})}{\Gamma(m-n+1)} \quad (A1)$$

for $m \geq n$. The symmetries of the d- matrices are

$$d_{mn}^j(x) = (-1)^{m-n} d_{nm}^j(x) , \quad (A2)$$

$$d_{mn}^j(x) = (-1)^{m-n} d_{-m-n}^j(x) , \quad (A3)$$

$$\text{and } d_{mn}^j(x) = d_{n-m}^j(x) . \quad (A4)$$

The following formulae, found in physics textbooks (3, 7, 24) are used in this dissertation.

$$d_{mn}^j(\pi) = (-1)^{j-n} \delta_{m, -n} . \quad (A5)$$

$$D_{m' n'}^j(\varphi, \theta, 0) D_{m'' n''}^{j'}(\varphi, \theta, 0) = \sum_j \langle j j' J, m+m' | j m j' m' \rangle \\ \times \langle j j' J, n+n' | j n j' n' \rangle D_{m+m', n+n'}^J(\varphi, \theta, 0) . \quad (A6)$$

$$D_{mn}^j(\varphi, \theta, 0) = e^{-im\varphi} d_{mn}^j(\theta) . \quad (A7)$$

$$\int_0^{2\pi} d\varphi e^{-i(m-n)\varphi} = 2\pi \delta_{mn} . \quad (A8)$$

$$\int_{-1}^1 d\cos\theta d_{mn}^j(\theta) d_{mn}^{j'}(\theta) = \frac{2\delta_{jj'}}{2j+1} . \quad (A9)$$

$$D_{m0}^j(\varphi, \theta, 0) = \sqrt{\frac{4\pi}{2j+1}} Y_{jm}^*(\theta, \varphi) \quad (A10)$$

where Y_{jm} is a spherical harmonic.

$$Y_{j-m}(\theta, \varphi) = (-1)^m Y_{jm}^*(\theta, \varphi) . \quad (A11)$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} . \quad (A12a)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} . \quad (A12b)$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta . \quad (A12c)$$

$$Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\varphi} . \quad (A12d)$$

$$Y_{21} = -\frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin 2\theta e^{i\varphi} . \quad (A12e)$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) . \quad (A12f)$$

$$R(\varphi, \theta, 0)R(0, \pi, 0) = R(\pi + \varphi, \pi - \theta, \pi) . \quad (A13)$$

$$R(\varphi, \theta, 0)R^{-1}(0, \pi, 0) = R(\pi + \varphi, \pi - \theta, -\pi) . \quad (A14)$$

It is worthwhile to verify Equation A13. In terms of rotation matrices the product of rotations is

$$\begin{aligned} \sum_n D_{mn}^j(\varphi, \theta, 0) D_{nm'}^j(0, \pi, 0) \\ &= \sum_n e^{-im\varphi} d_{mn}^j(\theta) d_{nm'}^j(\pi) \\ &= e^{-im\varphi} d_{m, -m'}^j(\theta) (-1)^{j-m'} \\ &= e^{-im\varphi} (-1)^{j+m} d_{mm'}^j(\pi - \theta) (-1)^{-j+m'} \\ &= e^{-i(\pi + \varphi)m} d_{mm'}^j(\pi - \theta) e^{-i\pi m'} \\ &= D_{mm'}^j(\pi + \varphi, \pi - \theta, \pi) \end{aligned}$$

where $d_{m, -m'}^j(\theta) = (-1)^{j+m} d_{mm'}^j(\pi - \theta)$ was used in the second step. Verification of Equation A14 is similar after $[D_{nm'}^j(0, \pi, 0)]^{-1} = (-1)^{2j} D_{nm'}^j(0, \pi, 0)$ is used.

VII. LITERATURE CITED

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