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DEVELOPMENT OF PULSED NEUTRON APPLICATION TO POWER REACTOR START-UP PROCEDURES

First Quarterly Progress Report, April 1-June 30, 1962

By
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July 1, 1962

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TO
POWER REACTOR START-UP PROCEDURES

FIRST QUARTERLY PROGRESS REPORT
APRIL 1 - JUNE 30, 1962

by

Paul Meyer
Edward Garelis

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SECTION I

INTRODUCTION

The purpose of this work is to develop a technique to use pulsed neutron sources to measure the shutdown of large, thermal power reactors. A theoretical investigation to identify an experimental observable is conducted.

The experimental work presently centers around the problem of detecting neutrons in the presence of a high gamma field of about 10^7 R/hr. The latter fact renders conventional techniques with scintillation crystal and photomultiplier tube mounted in-core impractical, because of gamma pulse stack-up.

SECTION II

EXPERIMENTAL

The experimental effort centered so far around the neutron detection technique. Conventional methods with in-core scintillation crystal and photomultiplier are impractical in large power reactors, because of the high gamma background. In the Dresden reactor, for instance, the latter is about 10^7 R/hr after shutdown.

The possibility of lucite or quartz lightpipes to transmit scintillation signals from an incore, thin Li^6 -crystal to an out-of-core photomultiplier was first examined in the Nuclear Test Reactor of our laboratory. Although signal transmission was observed below 1 KW-thermal through a lucite rod of 3/4 inch dia. and 10 ft. length, above this power level all signals were blanked out in a sea of noise. After exposing the lightpipe in the NTR at 30 KW for about 30 minutes, it turned yellow and lost its transparency and thus its usefulness. A similar pipe of quartz instead of lucite was unavailable. Manufacturers of quartz pointed out that the optical qualities of such a rod would be severely affected, if the boron content were to be very minute, as it indeed had to be required for use in a thermal reactor core.

Presently, the feasibility of solid state detectors with a thin U-235 coating is being studied. Low gamma sensitivity, high speed and small physical size are advantages of these devices; relatively low efficiency of the order of 0.004 percent, and low-level signals (about 1 mv/Mev) are disadvantages. Highly sensitive preamplifiers complicate the electronics and leakage currents from Compton-scattering of gammas in the detector crystal may heat up the latter and require additional cooling. Experiments in the NTR with these components are expected to start about the middle of August.

SECTION III

THEORETICAL

The effect of a multi-region assembly on the fundamental decay constant of an injected burst of neutrons is presently under investigation.

A two-region slab problem is first considered with a delta function in space and time representing the source. Laplace transform techniques via a spatial Green's function yield the solution to the original diffusion equation. This solution depends upon the roots of a complicated transcendental equation and these roots are now being examined. A memorandum by E. Garelis follows, giving a perturbation theory solution of the pulsed-source time-dependent neutron thermalization problem of a bare, homogeneous, multiplying medium. The physical model uses diffusion theory with the heavy gas scattering kernel, assuming $1/v$ cross sections with constant $D/(\xi\sigma_0)$. Infinite sums involving Laguerre polynomials express the fundamental eigenfunction and eigenvalue. Numerical results are given.

PART II

TIME DEPENDENT NEUTRON THERMALIZATION FOR A
BARE MULTIPLYING MEDIUM

by

Edward Garelis

I. INTRODUCTION

Many investigators have considered the time-dependent neutron thermalization problem for non-multiplying media. In this study, we consider the time-dependent neutron thermalization for bare multiplying media. In particular we consider the exponential time decay for a pulsed bare multiplying homogeneous assembly. This leads to an eigenvalue problem which can be solved by the use of standard perturbation theory. Some simplifying assumptions are used. In particular, the absorption and fission cross sections are assumed to be $1/v$ and the transport cross section is assumed to be constant. The scattering kernel is based on the heavy gas model, i.e., the Wilkins kernel, and the fission source distribution is assumed to be a delta function at the mean fission spectrum energy. Although the perturbation calculation is straight-forward, certain sums over the Laguerre polynomials of large arguments appear. The evaluation of these sums is somewhat involved. Numerical expressions are given for the fundamental eigenfunction and eigenvalue to the first and second order, respectively. In a previous note, (1), the writer considered a perturbation solution for a non-multiplying medium for the case of the heavy gas scattering model with $1/v$ absorption and gave numerical results for the eigenvalues and for the fundamental eigenfunction. The eigenfunction was expressed in an infinite series of Laguerre polynomials. In this study an explicit analytical expression is given for the eigenfunction.

II. GENERAL FORMULATION

Consider the time dependent-diffusion equation for a homogeneous bare multiplying medium: that is,

$$\left[\sigma_a(E) - D\nabla^2 + (1/v)\partial/\partial t \right] \phi(\vec{r}, E, t) = \partial q/\partial E + \chi(E)v \int_0^\infty \sigma_f(E') \phi(\vec{r}, E', t) dE' \quad (1)$$

where $\chi(E)$ represents the energy distribution of the fission neutrons and the remaining symbols have the usual meaning. For the heavy gas model

$$\partial q/\partial E = \xi \sigma_0 \left[\phi(\vec{r}, E, t) + E(\partial \phi/\partial E) + ET(\partial^2 \phi/\partial E^2) \right] \quad (2)$$

where ξ represents the average logarithmic energy decrement, σ_0 the bound atom cross section, and T is the temperature in units of kT . We seek solutions of Eq. (1) in the form of

$$\phi(\vec{r}, E, t) = \phi(\vec{r})\phi(E)e^{-\lambda t} \quad (3)$$

where $\phi(\vec{r})$ satisfies the usual Helmholtz equation,

$$\nabla^2 \phi(\vec{r}) + B^2 \phi(\vec{r}) = 0. \quad (4)$$

For $1/v$ cross sections, $\sigma(E) = \sigma(T)/\sqrt{\epsilon}$ where $\epsilon = E/T$ and the velocity can be expressed as $v = v_0 \sqrt{\epsilon}$ where v_0 is the velocity of a neutron at energy T , i.e., $v_0 = \sqrt{2T/m}$. Defining the absorption and fission parameters as $\Delta = 4\sigma_a(T)/(\xi\sigma_0)$ and $\Delta_f = 4\sigma_f(T)/\xi\sigma_0$, and using the above, Eq. (1) becomes

$$\epsilon \phi'' + \epsilon \phi' + \phi - \frac{\Delta \phi}{4\sqrt{\epsilon}} + \frac{\chi(\epsilon) v \Delta_f}{4} \int_0^\infty \frac{\phi(\epsilon')}{\sqrt{\epsilon'}} d\epsilon' = \alpha \phi \quad (5)$$

where the primes indicate differentiation with respect to ϵ , $\Delta' = \Delta - (4\lambda)/(\xi\sigma_0 v_0)$ and $\alpha = (DB^2)/(\xi\sigma_0)$. We now consider Eq. (5) as an eigenvalue problem with α representing the eigenvalue, i.e.,

$$H\phi = \alpha\phi \quad (6)$$

where $H\phi$ represents the l.h.s. of Eq. (5).

We consider a perturbation theory solution (see for example Courant and Hilbert (2)); that is, let

$$\begin{aligned} H &= H_0 + \tau H_1 \\ \phi &= \phi_0 + \tau \phi_1 + \tau^2 \phi_2 + \dots \\ \alpha &= \alpha_0 + \tau \alpha_1 + \tau^2 \alpha_2 + \dots \end{aligned} \quad (7)$$

where τ is the usual expansion parameter. The unperturbed operator H_0 is given by

$$H_0 = \epsilon(d^2/d\epsilon^2) + \epsilon(d/d\epsilon) + 1 \quad (8)$$

and

$$H_1 = -\frac{\Delta'}{4\sqrt{\epsilon}} + \frac{\chi(\epsilon) v \Delta_f}{4} \int_0^\infty \frac{d\epsilon'}{\sqrt{\epsilon'}} \quad (9)$$

Substituting (7) into (6) and equating like powers of τ gives directly

$$H_0 \phi_0 = \alpha_0 \phi_0 \quad (10)$$

$$H_0 \phi_1 + H_1 \phi_0 = \alpha_0 \phi_1 + \alpha_1 \phi_0 \quad (11)$$

$$H_0 \phi_2 + H_1 \phi_1 = \alpha_0 \phi_2 + \alpha_1 \phi_1 + \alpha_2 \phi_0. \quad (12)$$

The zeroth order equation, Eq. (10), gives

$$\epsilon \phi_0'' + \epsilon \phi_0' + \phi_0 = \alpha_0 \phi_0. \quad (13)$$

Physically valid solutions of the above are given by

$$\phi_{0n} = A_n e^{-\epsilon} L_n^{(1)}(\epsilon) \quad (14)$$

for

$$\alpha_{0n} = -n, \quad n = 0, 1, 2, \dots \quad (15)$$

where the A_n are arbitrary and the $L_n^{(1)}(\epsilon)$ are the associated Laguerre polynomials as defined in Appendix A, see Eq. (A-1).

An extra subscript has been added to the eigenfunction to denote the corresponding eigenvalue. Now consider the first order perturbation, Eq. (11). Following the usual procedure, the perturbed eigenfunction, ϕ_{1n} , is expanded in terms of the ϕ_{0n} ; that is,

$$\phi_{1n} = \sum_{j=0}^{\infty} a_{nj} \phi_{0j} \quad (16)$$

where the a_{nj} are to be determined. Substituting Eq. (16) into (11) gives

$$\sum_{j=0}^{\infty} (n-j) a_{nj} \phi_{0j} + H_1 \phi_{0n} = \alpha_{1n} \phi_{0n}. \quad (17)$$

Now multiplying the resulting equation by $L_k^{(1)}(\epsilon)$ and integrating over all energy, we obtain by virtue of Eq. (A-2)

$$A_k (n-k)(1+k) a_{nk} + A_n \int_0^{\infty} L_k^{(1)}(\epsilon) H_1 e^{-\epsilon} L_n^{(1)}(\epsilon) d\epsilon = \alpha_{1n} A_n (1+k) \delta_{nk}. \quad (18)$$

Before proceeding any further with the formalism of perturbation theory, consider the integrand in Eq. (18). Using the expression for H_1 , Eq. (9), we have

$$H_1 e^{-\epsilon} L_n^{(1)}(\epsilon) = -\frac{\Delta'}{4} e^{-\epsilon} \frac{1}{\sqrt{\epsilon}} L_n^{(1)}(\epsilon) + \frac{\chi(\epsilon) v \Delta_f}{4} \int_0^{\infty} \epsilon' e^{-\epsilon'} \frac{1}{\sqrt{\epsilon'}} L_n^{(1)}(\epsilon') d\epsilon'. \quad (19)$$

Integrals of the type appearing above in Eq. (19) have been previously considered (1). We define the symmetrical constant v_{kj}^{-1} as

$$v_{kj}^{-1} = \int_0^{\infty} x e^{-x} L_k^{(1)}(x) (1/\sqrt{x}) L_j^{(1)}(x) dx \quad (20)$$

and it can be shown (1) that

$$v_{kj}^{-1} = \frac{(j+1)! \Gamma(1/2)}{2^{2k} k!} \sum_{n=0}^{\infty} \frac{(2n+1)! (2k-2n)! (2n)! 2^{-(2n+1)}}{(j-n)! (1+n)(n!)^4 (k-n)!} \quad k > j \quad (21)$$

where $\Gamma(1/2) = \sqrt{\pi}$ represents the usual gamma function. Using Stirling's approximation to the factorials we obtain

$$v_{k0}^{-1} = 1/\sqrt{4k}, \quad k \gg 1. \quad (22)$$

Therefore, using the above, and assuming $\chi(\epsilon)$ is given by a delta function,

$$\chi(\epsilon) = \delta(\epsilon - \epsilon_f)$$

where ϵ_f is the mean energy of the fission spectrum, Eq. (18) becomes

$$A_k(n-k)(1+k)a_{nk} - \frac{1}{4} A_n \Delta' v_{nk}^{-1} + \frac{1}{4} A_n v_{f n 0} \Delta' v_{n 0}^{-1} L_k^{(1)}(\epsilon_f) = \alpha_{ln} A_n (1+k) \delta_{nk}. \quad (23)$$

Eq. (23) gives

$$a_{nk} = \frac{A_n [\Delta' v_{nk}^{-1} - v_{f n 0} \Delta' v_{n 0}^{-1} L_k^{(1)}(\epsilon_f)]}{4 A_k (n-k)(1+k)}, \quad n \neq k \quad (24)$$

and

$$\alpha_{ln} = \frac{v_{f n 0} \Delta' v_{n 0}^{-1} L_n^{(1)}(\epsilon_f) - \Delta' v_{nn}^{-1}}{4(1+n)} \quad (25)$$

where a_{nn} is determined by normalization and for the first order perturbation $a_{nn} = 0$.

The second order perturbation is derived in a similar fashion, carrying out the calculation we find that

$$\begin{aligned} \alpha_{2n} = & -\frac{\Delta'^2}{16(1+n)} \sum_{j=0}^{\infty} \frac{(v_{nj}^{-1})^2}{(n-j)(1+j)} + \frac{v_{f n 0} \Delta' v_{n 0}^{-1}}{16(1+n)} \sum_{j=0}^n \frac{L_j^{(1)}(\epsilon_f) v_{jn}^{-1}}{(n-j)(1+j)} \\ & + \frac{\Delta' v_{f n} L_n^{(1)}(\epsilon_f)}{16(1+n)} \sum_{j=0}^{\infty} \frac{v_{j 0}^{-1} v_{nj}^{-1}}{(n-j)(1+j)} - \frac{v_{f n 0}^2 \Delta' v_{n 0}^{-1} L_n^{(1)}(\epsilon_f)}{16(1+n)} \sum_{j=0}^{\infty} \frac{L_j^{(1)}(\epsilon_f) v_{j 0}^{-1}}{(n-j)(1+j)} \end{aligned} \quad (26)$$

where the primes on the summation mean $j \neq n$.

The fundamental eigenvalue ($n = 0$) to the second order, is given by Eqs. (25) and (26); that is,

$$\alpha_0 = (v\Delta_F - \Delta') \left\{ \frac{v_{00}^{-1}}{4} - \frac{\Delta'}{16} \sum_{j=1}^{\infty} \frac{(v_{j0}^{-1})^2}{j(1+j)} + \frac{v\Delta_F v_{00}^{-1}}{16} \sum_{j=1}^{\infty} \frac{L_j^{(1)}(\epsilon_F) v_{j0}^{-1}}{j(1+j)} \right\} \quad (27)$$

where we have used the relation that $L_0^{(1)}(x) = 1$. Similarly, the fundamental eigenfunction to the first order is given by

$$\phi_0(\epsilon) = \epsilon e^{-\epsilon} - \frac{\Delta'}{4} \sum_{j=1}^{\infty} \frac{\epsilon e^{-\epsilon} v_{j0}^{-1} L_j^{(1)}(\epsilon)}{j(1+j)} + \frac{v\Delta_F v_{00}^{-1}}{4} \sum_{j=1}^{\infty} \frac{\epsilon e^{-\epsilon} L_j^{(1)}(\epsilon) L_j^{(1)}(\epsilon_F)}{j(1+j)}. \quad (28)$$

Assuming perturbation theory to be valid, the problem is now formally solved. Eq. (27) represents a quadratic in Δ' , which in principle can be solved for the time decay constant, λ . Before a useful solution can be obtained, the sums involving the $L_j^{(1)}(\epsilon_F)$ must be evaluated. Since ϵ_F is of the order of 10^8 (in units of kT), a straight-forward summation of these sums is out of the question. The evaluation of these sums will be considered in the following section.

III. EVALUATION OF SUMS

Let us first consider Eq. (27). The numerical value of the first sum has been evaluated (1) and is given by

$$\frac{1}{16} \sum_{j=1}^{\infty} \frac{(v_{j0}^{-1})^2}{j(1+j)} = 81912 \pm 4 \times 10^{-7}. \quad (29)$$

Now consider the second sum involving the Laguerre polynomials evaluated at ϵ_F . The approach will be to form a differential equation for the sum

$$u(x) = \sum_{j=1}^{\infty} \frac{L_j^{(1)}(x) v_{j0}^{-1}}{j(1+j)} \quad (30)$$

and then to proceed by solving the resulting equation. First it should be pointed out that the above sum is absolutely convergent. Using the inequality given in Eq. (A-3), the above sum becomes

$$|u(x)| \leq e^{x/2} \sum_{j=1}^{\infty} v_{j0}^{-1}/j$$

and using Eq. (22) we see that the sum is absolutely convergent.

Consider the differential operator

$$M = x(d^2/dx^2) + x(d/dx) \quad (31)$$

then

$$MU_n = -(1+n)U_n, \quad n = 0, 1, 2, \dots \quad (32)$$

where

$$U_n = xe^{-x} L_n^{(1)}(x).$$

Let

$$S(x) = xe^{-x} \sum_{j=1}^{\infty} \frac{L_j^{(1)}(x) v_{j0}^{-1}}{j(1+j)} \quad (33)$$

then

$$MS(x) = S(x) + xe^{-x} v_{00}^{-1} - \sum_{j=0}^{\infty} \frac{L_j^{(1)}(x) v_{j0}^{-1}}{(1+j)} \quad (34)$$

By expanding $1/\sqrt{x}$ in terms of Laguerre polynomials we have

$$1/\sqrt{x} = \sum_{j=0}^{\infty} \frac{L_j^{(1)}(x) v_{j0}^{-1}}{(1+j)} \quad (35)$$

and hence using the above, Eq. (34) becomes

$$xS'' + xS' + S = xe^{-x}(v_{00}^{-1} - 1/\sqrt{x}). \quad (36)$$

A solution of the homogeneous part of Eq. (36) is given by xe^{-x} ; therefore, to generate the total solution let

$$S(x) = xe^{-x} U(x). \quad (37)$$

Substituting the above into (36) gives

$$U'' + \left(\frac{2}{x} - 1\right)U' = (v_{00}^{-1} - 1/\sqrt{x})/x,$$

a linear first order equation in U' . The integrating factor is $x^2 e^{-x}$, therefore by quadrature we obtain

$$U(x) = C_2 + C_1 \int_a^x \frac{e^z}{z^2} dz + v_{00}^{-1} \int_a^x \frac{e^z}{z^2} dz \int_0^z te^{-t} dt - \int_a^x \frac{e^z}{z^2} dz \int_0^z \sqrt{t} e^{-t} dt \quad (38)$$

where C_2 and C_1 are arbitrary and a is some non zero constant. In the limit as x goes to zero the last two integrals on the r.h.s.

of Eq. (38) remain finite and the first integral,

$$\int_a^x \frac{e^z}{z^2} dz,$$

becomes infinite, hence, C_1 must be set equal to zero. Now set a equal to zero and further note that

$$\begin{aligned} C_2 &= U(0) \\ &= \sum_{j=1}^{\infty} v_{j0}^{-1} / j. \end{aligned} \quad (39)$$

Therefore Eq. (38) becomes:

$$U(x) = U(0) + v_{00}^{-1} \int_0^x \left(\frac{e^z}{z^2} - \frac{1+z}{z^2} \right) dz - \int_0^x \frac{e^z}{z^2} dz \int_0^z \sqrt{t} e^{-t} dt. \quad (40)$$

Eq. (40) represents a convenient formulation for calculating the fundamental eigenfunction. For small values of x , the integrals can be evaluated by expanding the exponentials and for intermediate values machine computation can be used. The constant, $U(0)$, will be considered later. However, for the eigenvalue calculation, x is large (i.e. $x \sim 10^8$), and hence Eq. (40) must be transformed to a more suitable form. This transformation is carried out in Appendix B; the result is

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(x) v_{n0}^{-1}}{n(1+n)} = \sum_{n=1}^{\infty} \frac{(3n+2) v_{n0}^{-1}}{n(1+n)} + \sqrt{\frac{\pi}{2}} \left(\frac{1}{x} - \log 4x - \gamma \right) - \int_x^{\infty} \frac{e^z}{z^2} dz \int_z^{\infty} \sqrt{t} e^{-t} dt \quad (41)$$

where

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(3n+2) v_{n0}^{-1}}{n(1+n)} &= \sqrt{\frac{\pi}{2}} (1 + 4 \log 2) \\ &= 3.343370 \end{aligned} \quad (42)$$

and γ represents the Euler-Macheroni constant, $\gamma = .577215\dots$

For large values of x , the integral appearing in Eq. (41) can be evaluated by an asymptotic expansion giving

$$\int_x^\infty \frac{e^z}{z^2} dz \int_z^\infty \sqrt{t} e^{-t} dt = \frac{2}{\sqrt{x}} \left\{ 1 + \frac{1}{2 \cdot 3} \frac{1}{x} - \frac{1}{2^2 \cdot 5} \frac{1}{x^2} + \frac{1 \cdot 3}{2^3 \cdot 7} \frac{1}{x^3} - \dots \right\}. \quad (43)$$

For $x \sim 10^8$, Eq. (41) may well be approximated by

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(x) v_{n0}^{-1}}{n(1+n)} = 3.343370 - \sqrt{\frac{\pi}{2}} (\log 4x + \gamma). \quad (44)$$

Note that Eq. (41) is suitable for the calculation of the asymptotic eigenfunction.

To evaluate the sum appearing in the fundamental eigenfunction, Eq. (28),

$$\sum_{n=1}^{\infty} \frac{e e^{-\epsilon} L_n^{(1)}(\epsilon) L_n^{(1)}(\epsilon_f)}{n(1+n)} \quad (45)$$

we could proceed in the same fashion as we did in the previous case; that is, by recognizing that a representation of the delta function is given by

$$\delta(\epsilon - \epsilon_f) = \sum_{n=0}^{\infty} \frac{e e^{-\epsilon} L_n^{(1)}(\epsilon) L_n^{(1)}(\epsilon_f)}{(1+n)} \quad (46)$$

and by defining a suitable operator to yield a given differential equation. However, we will consider an alternate method, a method which could have been applied equally as well to the sum considered in the eigenvalue expansion. To evaluate the sum given in (45) we integrate Eq. (46); thus,

$$\int_0^\epsilon \delta(z - \epsilon_f) dz = \int_0^\epsilon z e^{-z} dz + \sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f)}{n(1+n)} \int_0^\epsilon n z e^{-z} L_n^{(1)}(z) dz. \quad (47)$$

Using Eq. (A-6), the above becomes

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f) L_{n-1}^{(2)}(\epsilon)}{n(1+n)} = H(\epsilon - \epsilon_f) \frac{e^\epsilon}{\epsilon^2} + \frac{1+\epsilon}{\epsilon^2} - \frac{e^\epsilon}{\epsilon^2} \quad (48)$$

where $H(x)$ is the Heaviside function,

$$H(x) = 1 \quad x > 0 \\ = 0 \quad x < 0.$$

Using the integration formula given by Eq. (A-7), Eq. (48) is again integrated from 0 to ϵ to yield:

$$-\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f) L_n^{(1)}(\epsilon)}{n(1+n)} + \sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f)}{n} = H(\epsilon - \epsilon_f) \int_{\epsilon_f}^{\epsilon} \frac{e^z}{z^2} dz \\ + \int_0^{\epsilon} \left(\frac{1+z}{z^2} - \frac{e^z}{z^2} \right) dz \quad (49)$$

which is the desired expression. The constant appearing in Eq. (49),

$$\text{Const.} = \sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f)}{n}, \quad (50)$$

can be evaluated as follows: Eq. (49) is multiplied by $\epsilon e^{-\epsilon} d\epsilon$ and integrated from 0 to ∞ to give:

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f)}{n} = \int_0^{\infty} H(t - \epsilon_f) t e^{-t} dt \int_{\epsilon_f}^t \frac{e^z}{z^2} dz + \int_0^{\infty} t e^{-t} dt \int_0^t \left(\frac{1+z}{z^2} - \frac{e^z}{z^2} \right) dz. \quad (51)$$

In Appendix C the above expression is evaluated and yields

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f)}{n} = \frac{1}{\epsilon_f} - \log \epsilon_f - \gamma \quad (52)$$

hence, our original sum is given by:

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f) L_n^{(1)}(\epsilon)}{n(1+n)} = \frac{1}{\epsilon_f} - \log \epsilon_f - \gamma - H(\epsilon - \epsilon_f) \int_{\epsilon_f}^{\epsilon} \frac{e^z}{z^2} dz \\ + \int_0^{\epsilon} \left[\frac{e^z}{z^2} - \frac{1+z}{z^2} \right] dz, \quad (53)$$

a form reasonably convenient for computation for $\epsilon < \epsilon_f$.

The integral expression can readily be evaluated by expanding the exponential and integrating term by term. Therefore, the final expression for the sum in (45) is given by

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f) L_n^{(1)}(\epsilon)}{n(1+n)} = \frac{1}{\epsilon_f} - \log \epsilon_f - \gamma + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n(1+n)!}, \quad \epsilon < \epsilon_f. \quad (54)$$

IV. EVALUATION OF THE TIME CONSTANT, λ

To make some qualitative arguments concerning the time constant, λ , it is convenient to resort to typical numerical values. Assuming the average fission neutron to have an energy of about 1.6 Mev and further assuming a temperature of 20 degrees C, then

$$\epsilon_f = 1.6 \times 10^6 / .0235 = .632 \times 10^8$$

and Eq. (44) gives

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(\epsilon_f) v_0^{-1}}{n(1+n)} = -14.316.$$

Using the above and Eq. (29), Eq. (27) can be written as

$$\alpha_0 / (v\Delta_f - \Delta') = \sqrt{\pi}/8 - .0081912 \Delta' - .79297 v\Delta_f. \quad (55)$$

For convenience let us rewrite the above as

$$\alpha_0 / (v\Delta_f - \Delta') = \sqrt{\pi}/8 - a \Delta' - S v\Delta_f \quad (56)$$

where the values of a and S are given in Eq. (55).

We recall that $\sqrt{\pi}/8$ is the first order perturbation term and $(a \Delta' + S v\Delta_f)$ corresponds to the second order perturbation. For perturbation theory to be valid, one would expect the second order term to be small compared to the first order term; that is,

$$\sqrt{\pi}/8 \gg (a \Delta' + S v\Delta_f), \quad (57)$$

and hence the r.h.s. of Eq. (56) should be positive. Since

$\alpha_0 = DB^2/(\xi \sigma_0)$ is some positive quantity, it follows that $(v\Delta_f - \Delta') > 0$. This last inequality is equivalent to

$$v\Delta_f - \Delta + 4\lambda/(\xi \sigma_0 v_0) > 0$$

or

$$4\lambda/(\xi\sigma_0 v_0) > \Delta(1-k) \quad (58)$$

where $v\Delta_F/\Delta$ has been replaced by k , the infinite medium multiplication constant. It is interesting to note that this inequality corresponds to the form that one would expect from a one-group model. For a one-group model

$$\lambda\ell = 1 - k + M^2 B^2 \quad (59)$$

where ℓ is the prompt neutron infinite medium lifetime, $\ell = (\sigma v)^{-1}$, and M^2 is some appropriately averaged migration area. Since $M^2 B^2 > 0$ then it follows that $\lambda\ell > (1 - k)$ and hence an inequality of the form given in (58) follows. Also it can be shown that the inequality, $(v\Delta_F - \Delta') < 0$, leads to a contradiction. For if $(v\Delta_F - \Delta') < 0$ then $4\lambda/(\xi\sigma_0 v_0) < \Delta(1 - k)$; however, it is possible to have a subcritical assembly, $\lambda > 0$, with $k > 1$, a possibility which this latter inequality does not permit. The inequality given in (57) is equivalent to:

$$4\lambda/(\xi\sigma_0 v_0) >> \Delta - (1/a) (\sqrt{\pi}/8 - Sv\Delta_F).$$

If $(\sqrt{\pi}/8 - Sv\Delta_F) < 0$ then the r.h.s. of the above is always greater than zero. For a supercritical reactor, $\lambda < 0$, this leads to a contradiction. Therefore we conclude that for our particular model to be valid, we require: $\sqrt{\pi}/8 > Sv\Delta_F$.

We now proceed to solve Eq. (56) for λ ; expanding Eq. (56) in powers of Δ' gives

$$a\Delta'^2 - (F + av\Delta_F)\Delta' + (v\Delta_F F - \alpha_0) = 0 \quad (60)$$

where

$$F = \sqrt{\pi}/8 - Sv\Delta_F. \quad (61)$$

Assuming:

$$4a|v\Delta_F F - \alpha_0| \ll (F + av\Delta_F)^2, \quad (62)$$

the quadratic in Eq. (60) gives:

$$\Delta' = \frac{v\Delta_F F - \alpha_0}{F + av\Delta_F} + \frac{a(v\Delta_F F - \alpha_0)^2}{(F + av\Delta_F)^3} \quad (63)$$

Expanding the above by assuming that:

$$a\nu\Delta_f/F \ll 1$$

and solving the resulting expression for λ gives:

$$4\lambda/(\xi \sigma_o \nu_o) = \Delta(1 - k) + \frac{\alpha_o}{F} \left[1 - \frac{a}{F} \left(\frac{\alpha_o}{F} - \nu\Delta_f \right) \right]. \quad (64)$$

For a non-multiplying medium, $\nu\Delta_f = 0$ and, hence, Eq. (64) reduces to

$$\lambda = \bar{\nu}_o \left[\bar{\sigma}_a + DB^2 (1 - cB^2) \right], \quad (65)$$

where

$$\bar{\nu}_o = (2/\sqrt{\pi}) \nu_o \quad \bar{\sigma}_a = (\sqrt{\pi}/2) \sigma_a$$

and

$$c = (64 a D)/(\pi \xi \sigma_o).$$

The quantity c represents the diffusion cooling constant. Numerically $c = .16687 (D/\xi \sigma_o)$, the constant .16687 corresponds to the value of $(.148)(2/\sqrt{\pi})$ as calculated by Hurwitz and Nelkin (3) in their work on diffusion cooling.

Equation (64) can be further simplified. Neglecting the quantity $(a/F)[(\alpha_o/F) - \nu\Delta_f]$ as compared to unity in the r.h.s. of Eq. (64), we have

$$\lambda \ell = 1 - k + L^2 B^2 / [1 - (64 S \nu \bar{\sigma}_f)/(\pi \xi \sigma_o)] \quad (66)$$

where $\ell = (\bar{\nu}_o \bar{\sigma}_a)^{-1}$ and $L^2 = D/\bar{\sigma}_a$. Comparing the above with the one-group model, Eq. (59), we see that an equivalent M^2 is given by

$$M^2 = \frac{L^2}{1 - (64 S \nu \bar{\sigma}_f)/(\pi \xi \sigma_o)}. \quad (67)$$

V. FUNDAMENTAL EIGENFUNCTION

It is of interest to consider the fundamental mode of the flux solution. To the first order

$$\Delta' = v\Delta_F - 8\alpha_0/\sqrt{\pi} \quad (68)$$

by Eq. (55) and substituting this value into Eq. (28) gives, after some rearrangement,

$$\begin{aligned} \phi_0 = \epsilon e^{-\epsilon} \left[1 - v\Delta_F \frac{\sqrt{\pi}}{8} \log 4 + \alpha_0 \log 4 + v\Delta_F \frac{\sqrt{\pi}}{8} \left(\frac{1}{\epsilon_F} - \log \epsilon_F - \gamma \right) \right. \\ \left. + \left(\frac{v\Delta_F}{4} - \frac{2}{\sqrt{\pi}} \right) \int_0^\epsilon \frac{e^z}{z^2} dz \int_0^z \sqrt{t} e^{-t} dt + \alpha_0 \int_0^\epsilon \left(\frac{e^z}{z^2} - \frac{1+z}{z^2} \right) dz \right]. \end{aligned} \quad (69)$$

For small values of ϵ

$$\int_0^\epsilon \frac{e^z}{z^2} dz \int_0^z \sqrt{t} e^{-t} dt = (4/3)\sqrt{\epsilon} + O(\epsilon^{3/2})$$

and

$$\int_0^\epsilon \left(\frac{e^z}{z^2} - \frac{1+z}{z^2} \right) dz = O(\epsilon),$$

therefore Eq. (69) becomes

$$\phi_0 = \epsilon e^{-\epsilon} \left\{ 1 - v\Delta_F (\sqrt{\pi}/8) \log 4 \epsilon_F C + (1/3) [v\Delta_F - (8/\sqrt{\pi})\alpha_0] \sqrt{\epsilon} + O(\epsilon) \right\} \quad (70)$$

where the terms $\alpha_0 \log 4$ and $1/\epsilon_F$ have been neglected and $C = \exp(\gamma)$. Since the flux is everywhere positive we note that we must have:

$$1 - v\Delta_F (\sqrt{\pi}/8) \log 4 \epsilon_F C > 0 \quad (71)$$

or

$$v\Delta_F < (8/\sqrt{\pi}) \frac{1}{\log 4 \epsilon_F C}. \quad (72)$$

Using the value of ϵ_F given in Section IV, we have $v\Delta_F < .226$ or $Sv\Delta_F < .181$. This is in agreement with the value obtained by using physical arguments in Section IV.

VI. CONCLUSIONS

Using perturbation theory and a reasonable physical model it is possible to calculate the time-dependent neutron thermalization problem for a multiplying medium for the heavy gas case. It should be noted that the expression for the eigenvalue is probably not valid near criticality, $\lambda = 0$, since the higher order terms in the expansion were neglected. The higher order terms can be interpreted as leakage effects. For example in Eq. (56), the constant a is proportional to the diffusion cooling constant and hence represents the leakage effects of the high energy thermal neutrons. Similarly the coefficient S can be interpreted as the effect of the leakage during the slowing down of the fission neutrons.

It should be noted that, to the first order perturbation, the flux solution is solved. This is particularly useful in the interpretation of thermalization problems involving non-multiplying media for which perturbation theory gives very satisfactory results. For example, see the comparison of Michael's (4) work with the earlier perturbation calculations (1).

The third order term for the fundamental eigenvalue expansion was investigated. The coefficients of Δ'^2 and $v\Delta_p\Delta'$ were readily summable in closed form; however, an analytic closure of the sum representing the coefficient of $(v\Delta_p)^2$ was not found. A complete third order expansion for the eigenvalue would yield a cubic in Δ' . In this case it would be difficult to report the result with λ as the dependent variable and hence to make comparisons with the one group diffusion theory model.

Appendix A

The following formulae were taken from the Bateman Compilations (5) with the exception of Eqs. (A-6) and (A-7) which were given by Morse and Feshbach (6).

$$L_n^{(\alpha)}(x) = \sum_{m=0}^n \binom{n+\alpha}{m+\alpha} \frac{(-x)^m}{m!} \quad (\text{A-1})$$

$$\int_0^\infty x e^{-x} L_n^{(1)}(x) L_m^{(1)}(x) dx = (1+n) \delta_{nm} \quad (\text{A-2})$$

$$|L_j^{(1)}(x)| \leq (j+1) e^{x/2} \quad (\text{A-3})$$

$$\int_0^\infty e^{-pt} t^{n-1/2} \log t \, dt = \frac{\sqrt{\pi} \, 1 \cdot 3 \cdot 5 \dots (2n-1)}{p^{n+1/2} 2^n} \left[2\left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) - \log 4Cp \right] \quad (\text{A-4})$$

for $n \geq 1$ where $C = e^\gamma$ and $\gamma = .5772156649\dots$, the Euler-Mascheroni constant.

$$\int_0^\infty x^{\beta-1} e^{-x} L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha-\beta+n+1)\Gamma(\beta)}{n! \Gamma(\alpha-\beta+1)}, \quad \text{Re } \beta > 0 \quad (\text{A-5})$$

$$\frac{d}{dz} [z^\alpha e^{-z} L_n^{(\alpha)}(z)] = (n+1) z^{\alpha-1} e^{-z} L_{n+1}^{(\alpha-1)}(z) \quad (\text{A-6})$$

$$\frac{dL_n^{(\alpha)}(z)}{dz} = -L_{n+1}^{(\alpha+1)}(z) \quad (\text{A-7})$$

Appendix B

To derive Eq. (41) we proceed in the following manner. Recalling that $v_{00}^{-1} = \sqrt{\pi}/2$ and expressing this in terms of the gamma function

$$v_{00}^{-1} = \int_0^{\infty} \sqrt{t} e^{-t} dt \quad (B-1)$$

and changing orders of integration in the last integral on the r.h.s. of Eq. (40), we have

$$\begin{aligned} U(x) - U(0) &= \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^x \frac{e^z}{z^2} dz - \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^x \frac{1+z}{z^2} dz \\ &\quad - \int_0^x \sqrt{t} e^{-t} dt \int_t^x \frac{e^z}{z^2} dz. \end{aligned} \quad (B-2)$$

The first integral on the r.h.s. of Eq. (B-2) is written as

$$\int_0^{\infty} [] dt \int_0^x [] dz = \int_0^{\infty} [] dt \int_t^x [] dz + \int_0^{\infty} [] dt \int_0^t [] dz \quad (B-3)$$

and the first integral on the r.h.s. of (B-3) is combined with the last integral on the r.h.s. of (B-2) to give

$$U(x) - U(0) = \int_x^{\infty} \sqrt{t} e^{-t} dt \int_t^x \frac{e^z}{z^2} dz + \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^t \frac{e^z}{z^2} dz - \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^x \frac{1+z}{z^2} dz. \quad (B-4)$$

Now consider the last two integrals on the r.h.s. of Eq. (B-4), we have

$$\begin{aligned} &\int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^t \frac{e^z}{z^2} dz - \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^x \frac{1+z}{z^2} dz \\ &= \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^t \frac{1+z}{z^2} dz + \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^t \sum_{n=2}^{\infty} \frac{z^{n-2}}{n!} dz - \int_0^{\infty} \sqrt{t} e^{-t} dt \int_0^x \frac{1+z}{z^2} dz \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\infty} \sqrt{t} e^{-t} dt \int_x^t \frac{1+z}{z^2} dz + \sum_{n=1}^{\infty} \frac{\Gamma(n+3/2)}{n(n+1)!} \\
&= \frac{\sqrt{\pi}}{2} \left(\frac{1}{x} - \log x \right) - \sqrt{\pi} + \int_0^{\infty} \sqrt{t} e^{-t} \log t dt + \sum_{n=1}^{\infty} \frac{(2n+1)v_{n0}^{-1}}{n(n+1)} \quad (B-5)
\end{aligned}$$

where we have made use of the definition of v_{n0}^{-1} . The integral

$$\int_0^{\infty} \sqrt{t} e^{-t} \log t dt = \sqrt{\pi} (1 - \log 2 - \gamma/2)$$

is given by Bateman, see Eq. (A-4). Therefore, using the above and the expression for $U(0)$, Eq. (B-4) becomes

$$U(x) = - \int_x^{\infty} \sqrt{t} e^{-t} dt \int_x^t \frac{e^z}{z^2} dz + \frac{\sqrt{\pi}}{2} \left(\frac{1}{x} - \log 4x - \gamma \right) + \sum_{n=1}^{\infty} \frac{(3n+2)v_{n0}^{-1}}{n(1+n)}. \quad (B-6)$$

Finally, changing the order of integration in the first integral in the above gives Eq. (41).

To evaluate the infinite sum in Eq. (B-6) we consider the following sums

$$\sum_{n=1}^{\infty} \frac{v_{n0}^{-1}}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{v_{n0}^{-1}}{1+n}. \quad (B-7)$$

To evaluate the first sum, we start with the result given in Eq. (52); that is,

$$\sum_{n=1}^{\infty} \frac{L_n^{(1)}(x)}{n} = \frac{1}{x} - \log x - \gamma. \quad (B-8)$$

Multiplying the above by $xe^{-x}L_j^{(1)}(x)(1/\sqrt{x})$ and integrating from zero to infinity gives

$$\sum_{n=1}^{\infty} \frac{v_{nj}^{-1}}{n} = \int_0^{\infty} xe^{-x}L_j^{(1)}(x)x^{-3/2}dx - \int_0^{\infty} xe^{-x}L_j^{(1)}(x) \frac{\log x}{\sqrt{x}} dx - \gamma v_{j0}^{-1}. \quad (B-9)$$

Setting $j = 0$ in the above gives

$$\sum_{n=1}^{\infty} \frac{v_{n0}^{-1}}{n} = \sqrt{\pi} \log 2. \quad (B-10)$$

To evaluate the second sum, we consider the following relation

$$\frac{1}{x} - 1 = \sum_{n=1}^{\infty} \frac{L_n^{(1)}(x)}{1+n}, \quad x \neq 0 \quad (B-11)$$

which can be obtained by expanding $1/x$ in terms of the Laguerre polynomials. Again multiplying by $xe^{-x}L_j^{(1)}(x)(1/\sqrt{x})$ and integrating we obtain

$$\sum_{n=1}^{\infty} \frac{v_{nj}^{-1}}{(1+n)} = \int_0^{\infty} xe^{-x}L_j^{(1)}(x)x^{-3/2}dx - v_{j0}^{-1}. \quad (B-12)$$

Using Eq. (A-5) to evaluate the integral we have

$$\sum_{n=1}^{\infty} \frac{v_{nj}^{-1}}{(1+n)} = \frac{2\Gamma(3/2-j)}{j!} - v_{j0}^{-1} \quad (B-13)$$

and for $j = 0$

$$\sum_{n=1}^{\infty} \frac{v_{n0}^{-1}}{(1+n)} = \frac{\sqrt{\pi}}{2}. \quad (B-14)$$

Therefore using Eqs. (B-10) and (B-14) we have the result given in Eq. (42).

Appendix C

To evaluate the constant as expressed in Eq. (50), we proceed as follows: designating the l.h.s. of Eq. (51) by C, we have

$$C = \int_{\epsilon_f}^{\infty} t e^{-t} dt \int_{\epsilon_f}^{\epsilon} \frac{e^z}{z^2} dz - \int_0^{\infty} t e^{-t} dt \int_0^{\epsilon} \left(\frac{1+z}{z^2} - \frac{e^z}{z^2} \right) dz. \quad (C-1)$$

In Eq. (C-1) the order of integration is interchanged and the resulting integrations carried out to give

$$C = \lim_{x \rightarrow \infty} \left\{ \int_{\epsilon_f}^x \frac{1+z}{z^2} dz - \int_0^x \frac{1+z}{z^2} dz + \int_0^x \frac{e^{-z}(1+z)^2}{z^2} dz \right\}. \quad (C-2)$$

Combining the first and second integrals and expanding the quantity $(1+z)^2$ in the third integral gives

$$C = \lim_{x \rightarrow \infty} \left\{ - \int_0^{\epsilon_f} \frac{1+z}{z^2} dz + \int_0^x \frac{e^{-z}}{z^2} dz + 2 \int_0^x \frac{e^{-z}}{z} dz - e^{-x} + 1 \right\}. \quad (C-3)$$

Note that the expression in the brackets in Eq. (C-3) remains finite at the lower limit since the singularities cancel. Expanding the exponentials in the second and third integrals in Eq. (C-3) and combining the first integral to eliminate the singularity of the lower limit gives

$$C = \lim_{x \rightarrow \infty} \left\{ \int_{\epsilon_f}^x \frac{(1+z) dz}{z^2} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+1)!} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)!} \right\}. \quad (C-4)$$

From the definition of the exponential, e^{-x} ,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)!} = 1/x - 1 - e^{-x}/x \quad (C-5)$$

and hence it can be shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)!} - \frac{1}{x} + 1 - \log x - \int_1^x \frac{e^{-t}}{t^2} dt. \quad (C-6)$$

Substituting Eqs. (C-5) and (C-6) into (C-4) gives, after carrying out the remaining integration and limiting process,

$$C = \frac{1}{\epsilon_f} - \log \epsilon_f + \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)!} - E_2(1) \quad (C-7)$$

where $E_2(x)$ represents the usual notation for the $E_n(x)$ functions; that is,

$$E_n(x) = x^{n-1} \int_x^{\infty} t^{-n} e^{-t} dt. \quad (C-8)$$

Equation (C-7) can be further simplified to give

$$C = \frac{1}{\epsilon_f} - \log \epsilon_f - \gamma \quad (C-9)$$

where we have used the relationship for the Euler-Mascheroni constant, γ , as given by Case, de Hoffmann and Placzek (7),

$$\gamma = E_2(1) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)!} \quad (C-10)$$

Equation (C-9) is the equation given in Eq. (52).

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