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Neutron Dynamics and Control

Proceedings of the
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at The University of Arizona,
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David L. Hetrick
and
Lynn E. Weaver
Coordinators

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Preface

This volume constitutes the proceedings of the 1965 annual Symposium on Nuclear Engineering at the University of Arizona. This symposium, *Neutron Dynamics and Control*, was cosponsored by the Department of Nuclear Engineering of the University of Arizona and the United States Atomic Energy Commission. It was a sequel to the 1963 Symposium on Nuclear Engineering, *Reactor Kinetics and Control*, which was published in the AEC Symposium Series as No. 2, TID-7662.

Two years of progress in this exciting field are evident upon comparison of the two meetings. In particular, one of the primary purposes for this conference was to collect reports on the significant progress in two areas, nonlinear system stability and optimal control. The meeting brought together many of the workers in reactor dynamics with numerous contributors to recent advances in modern automatic control theory, resulting in a number of fruitful new channels of communication.

Another area of dynamics that is currently receiving much attention is that of space-time effects. This important branch of the subject, including coupled reactors, pulsed neutron research, space-dependent reactor dynamics, and space-dependent noise effects, accounts for more than a third of the papers in this volume. Finally, the continuing high interest in pulsed-reactor research is well represented in a number of papers.

Four invited lecturers served to provide the keynote for the symposium: Dr. J. E. Gibson of Purdue University (*The Present State of Nonlinear Theory in Engineering Design*), Dr. George Leitmann of the University of California (*On Some Geometrical Aspects of Optimal*

Processes), Dr. Richard Bellman of the RAND Corporation (*Dynamic Programming and Modern Control Theory*), and Dr. Elias Gyftopoulos of the Massachusetts Institute of Technology, whose lecture, with some amplification, is the only invited paper included in these proceedings. We extend particular thanks to these four distinguished gentlemen for their important contributions to the success of the symposium.

We also extend our gratitude to the four distinguished visitors who served as session chairmen: Dr. Jeffery Lewins, Mr. M. A. Schultz, Dr. Robert E. Uhrig, and Dr. E. Richard Cohen. The cooperation of the United States Atomic Energy Commission is gratefully acknowledged; we cite in particular Mr. John Cera, of the Division of Nuclear Education and Training, and Mr. James Cape, of the Office of Technical Information, for their invaluable assistance and encouragement, and Mesdames Margaret Givens and Martha Hobson of the Division of Technical Information Extension for their skillful work in preparing this volume for publication.

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October 1965

Contents

Preface	iii
Some Applications of Mathematical Methods to Nuclear Engineering at the Massachusetts Institute of Technology.....	1
<i>Elias P. Gyftopoulos</i>	
A Nonlinear Study of Reactor Systems with Linear Feedback	28
<i>A. Ziya Akcasu</i>	
Evolution of the Finite-escape-time Concept in Reactor Kinetics	38
<i>Jack Chernick</i>	
Investigations on Nonlinear Stability of Coupled Nuclear Systems	45
<i>Stephen J. Gage, F. T. Adler, and P. N. Powers</i>	
Stability of Nonlinear Coupled-core Reactors	77
<i>Hugh S. Murray and Donald G. Schultz</i>	
Centers and Limit Cycles in Reactor Kinetics	88
<i>Louis M. Shotkin</i>	
Synthesis of an Optimal Nuclear Reactor Control System	110
<i>I. Kliger</i>	
Optimal Nuclear-rocket-engine Control	137
<i>Ronald R. Mohler</i>	
Synthesis of Optimal Feedback Control for Nuclear-rocket-engine Start-up	164
<i>Phillip A. Secker and Lynn E. Weaver</i>	
Control of Absorption Cross Section for a Nuclear Rocket.....	182
<i>C. N. Shen and T. C. Liu</i>	

Time Optimal Solution to the Reactivity-- Xenon Shutdown Problem	206
<i>John J. Roberts and Harold P. Smith, Jr.</i>	
A Closed-loop, Approximately Time-optimal Control Method	207
<i>James L. Melsa</i>	
Analysis of Pulsed Neutron Experiments in Moderating Assemblies	225
<i>P. B. Daitch, R. R. Lee, and R. L. Hobson</i>	
Spatial Dependence of Pulsed Neutron Reactivity Measurements	245
<i>Martin Becker and Karl S. Quisenberry</i>	
Investigation of the Rise Portion of Pulsed Neutron Curves	277
<i>B. E. Leonard</i>	
Measurement of Local Kinetic Parameters in the Saxton Reactor	299
<i>V. Rajagopal and G. E. Swen</i>	
Space Dependence of the Reactor-noise Spectral-density Function	312
<i>James R. Sheff and Robert W. Albrecht</i>	
Kinetic Theory of Spatial and Spectral Coupling of the Reactor Neutron Field	323
<i>Robert G. Cockrell and Rafael B. Perez</i>	
Adiabatic Treatment of the Xenon Problem	342
<i>Richard Scalettar</i>	
Large Reactor Excursions Starting at High Temperature	361
<i>José Canosa</i>	
TREAT Experiments to Provide Data for Fast Reactor Safety Kinetics Calculations	371
<i>C. E. Dickerman and L. E. Robinson</i>	
Operating Characteristics of the WSMR Fast Burst Reactor	386
<i>Robert L. Long</i>	

Characteristics of Large Reactivity Insertions in a High-performance TRIGA Uranium-Zirconium Hydride Core.....	398
<i>C. O. Coffey, J. B. Dee, J. R. Shoptaugh, Jr., G. B. West, and W. L. Whittemore</i>	
Feedback Effects During Short-interval Series Pulsing	420
<i>G. H. Miley, H. A. Kurstedt, and G. P. Beck</i>	
Reactivity Effects in Uranium-Zirconium Hydride Fueled Reactor Power Excursions	437
<i>Laurence I. Moss</i>	
Study of the Dynamics of a TRIGA Type Reactor.....	457
<i>K. Hornyik and M. E. Wyman</i>	
Feasibility and Application of an Accelerator Booster	473
<i>J. M. Neill, J. R. Beyster, and J. L. Russell, Jr.</i>	
Dynamic Reactivity Effect – A Further Development	484
<i>K. M. Campe and R. E. Bailey</i>	
Ultra High Temperature Reactor Experiment On-line Process Computer	518
<i>Howard B. Demuth, Fred P. Schilling, and Larry Weintraub</i>	
Some "Theorems" on Neutron Fluctuations and Fluctuation Spectra	531
<i>R. K. Osborn and A. Z. Akcasu</i>	
Coupled-core Kinetic Behavior	544
<i>P. R. Pluta</i>	
Pressure Effects on Space-dependent Dynamics in Boiling Systems.....	566
<i>Amir N. Nahavandi and Richard F. von Hollen</i>	
List of Attendees	584
Author Index	589
Subject Index	592

Some Applications of Mathematical Methods to Nuclear Engineering at the Massachusetts Institute of Technology*

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INTRODUCTION

The purpose of this paper is to discuss some applications of mathematical methods to nuclear engineering at the Massachusetts Institute of Technology (MIT). The word *some* in the title is used advisedly because in the limited space available I will not be able to do justice to all activities of this nature in our department. I will restrict my remarks to problems that concern us and fall within the main themes of this conference.

Specifically, I would like to discuss briefly (a) a new criterion of asymptotic stability for nuclear reactors described by nonlinear equations including all delayed-neutron precursors; (b) a simple example illustrating the procedure for deriving stability criteria by investigating the nature of equilibrium states at infinity; (c) a novel approach to problems of nonlinear optimum control for systems described by means of input-output functional relations; (d) some results on questions of space-time nuclear reactor dynamics; (e) an application of dynamic programming to the problem of optimum refueling of a power nuclear reactor.

* This is an invited review paper.

A NEW CRITERION OF ASYMPTOTIC STABILITY

General Remarks

It has been shown that stability analyses of nonlinear point reactor kinetics, resulting in criteria that do not include the delayed-neutron precursor constants, are either overrestrictive or nonconservative. For example, in cases where the feedback transfer function is a lagging function of real frequencies, the analysis is overrestrictive because delayed neutrons may relax substantially the requirements for stability.¹ On the other hand, when the feedback transfer function is a leading function of real frequencies, the analysis is nonconservative because the reactor model without delayed neutrons may be absolutely stable at all operating power levels, whereas the model with delayed neutrons may be linearly stable only for a limited power range.²

In addition, it has been recognized that criteria which guarantee absolute asymptotic stability are impractical.¹ The impracticality arises from the fact that no real reactor can be operated at very high power levels.

These observations suggest that physically meaningful stability requirements should always include the delayed-neutron precursor parameters and a finite range of operating power levels. In what follows, a new criterion of stability is presented which incorporates the preceding suggestions. To the best of my knowledge, this criterion is the most general and least restrictive stability requirement that has been derived to date.

Reactor Model

If it is assumed that the reactor admits a unique equilibrium power level, P_1 , for a given reactivity input, the kinetics equations for $t > 0$ can be written in terms of normalized, dimensionless, and incremental variables as

$$\frac{dp(t)}{dt} = - \sum_i^m \frac{\beta_i}{\Lambda} [p(t) - c_i(t)] + k(t) \quad (1)$$

$$\frac{dc_i(t)}{dt} = \lambda_i [p(t) - c_i(t)] \quad (i = 1, 2, \dots, m) \quad (2)$$

$$k(t) = - \frac{P_1}{\Lambda} [1 + p(t)] \left\{ \int_{-\infty}^t f(t - \tau) p(\tau) d\tau + F_2 [p(\tau)] \right\} \quad (3)$$

$$f(t) = 0 \quad (t < 0) \quad (4)$$

The minimum physical value of the variables is $p = c_i = -1$. The meaning of Eq. 3 is that, to a first approximation, feedback effects are expressed as a linear convolution of the normalized incremental power $[\int_{-\infty}^t f(t-\tau) p(\tau) d\tau]$. Any higher order effects arising either from nonlinear feedback per se or from variations of the values of β_i , λ_i , and Λ during a transient are included in the nonlinear functional $F_2[p(\tau)]$. The properties and implications of this functional have been discussed elsewhere.¹

Before the stability analysis is discussed, it is convenient to define the following:

$$q(j\omega, t) = \int_{-\infty}^t e^{j\omega(t-\tau)} k(\tau) d\tau \quad (5)$$

$$q^*(j\omega, t) = \int_{-\infty}^t e^{-j\omega(t-\tau)} k(\tau) d\tau \quad (6)$$

$$R(s) = \left(s + \sum_i^m \frac{\beta_i}{\Lambda} \frac{s}{s + \lambda_i} \right)^{-1} = \text{zero-power-reactor transfer function} \quad (7)$$

$$r(t) = \text{inverse transform of } R(s) \quad [r(t) = 0] \quad (t < 0) \quad (8)$$

$$F(s) = \text{transform of } f(t) = \text{feedback transfer function} \quad (9)$$

With these definitions Eqs. 1 to 3 yield

$$\begin{aligned} p(t) &= \int_{-\infty}^t r(t-\tau) k(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(j\omega) q(j\omega, t) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R^*(j\omega) q^*(j\omega, t) d\omega \end{aligned} \quad (10)$$

$$\int_{-\infty}^t f(t-\tau) p(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) R(j\omega) q(j\omega, t) d\omega \quad (11)$$

$$k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(j\omega, t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} q^*(j\omega, t) d\omega \quad (12)$$

The derivation of Eqs. 10 to 12 is straightforward. It requires certain changes of order of integration which imply that

$$\int_{-\infty}^0 |p(\tau)| d\tau < \infty \quad \int_{-\infty}^0 |k(\tau)| d\tau < \infty \quad (13)$$

Conditions for Stability of the Linear Approximation

For the linear approximation of Eqs. 1 to 3 to be stable, the roots of the characteristic equation

$$1 + \frac{P_1}{\Lambda} R(s) F(s) = 0 \quad (14)$$

must lie in the left-half complex plane. It is assumed that this is true.

It is further assumed that beyond a certain critical power level, P_c , one or more of the roots of Eq. 14 move into the right-half complex plane and that the reactor becomes linearly unstable. In other words, for

$$P_c = a_c P_1 \quad (a_c > 1) \quad (15)$$

the equation

$$1 + \frac{a_c P_1}{\Lambda} R(s) F(s) = 0 \quad (16)$$

admits roots on the $j\omega$ -axis. The number a_c is a measure of the margin of linear stability with respect to the operating power level, P_1 .

Conditions for Nonlinear Asymptotic Stability

Let $F_2[p(\tau)] \equiv 0$. In other words, assume that

$$k(t) = -\frac{P_1}{\Lambda} [1 + p(t)] \int_{-\infty}^t f(t - \tau) p(\tau) d\tau \quad (17)$$

Note that this assumption does not alter any of the preceding results.

Consider the scalar function

$$V = p(t) - \ln[1 + p(t)] - \frac{1}{2d^2} p^2(t)$$

$$+ \sum_i^m \frac{\beta_i}{\lambda_i \Lambda} \left[c_i(t) - \ln[1 + c_i(t)] - \frac{1}{2d^2} c_i^2(t) \right]$$

$$+ \sum_i^m \frac{\beta_i}{\Lambda} \int_{-\infty}^t [p(\tau) - c_i(\tau)]^2 \left[\frac{1}{[1 + p(\tau)][1 + c_i(\tau)]} - \frac{1}{d^2} \right] d\tau$$

$$+ \frac{b\Lambda}{a} \int_{-\infty}^t \frac{a-1-p(\tau)}{1+p(\tau)} k^2(\tau) d\tau \quad (18)$$

where a , b , and d are positive numbers to be determined ($a, d > 1$). If $d \geq a$, the function V is positive definite in the region

$$d \geq a \quad -1 < p < a-1 \quad -1 < c_i < a-1 \quad (i = 1, 2, \dots, m) \quad (19a)$$

If $d < a$, the function V is positive definite in the region

$$d < a \quad -1 < p < d-1 \quad -1 < c_i < d-1 \quad (i = 1, 2, \dots, m) \quad (19b)$$

In addition, the function V admits continuous partial derivatives with respect to $p(t)$ and all $c_i(t)$; and V is equal to zero only for $p(t) = c_i(t) = 0$.

The time derivative of V along the trajectories of the system of Eqs. 1, 2, and 17 is

$$\begin{aligned} \frac{dV}{dt} = & -\frac{P_1}{\Lambda} p(t) \int_{-\infty}^t f(t-\tau) p(\tau) d\tau - \frac{1}{d^2} p(t) k(t) \\ & - \frac{b\Lambda}{a} k^2(t) - bP_1 k(t) \int_{-\infty}^t f(t-\tau) p(\tau) d\tau \end{aligned} \quad (20)$$

If all the terms in the right-hand side of Eq. 20 are written as a function of $q(j\omega, t)$ by means of Eqs. 10 to 12, then it is found that

$$\frac{dV}{dt} = \mp \frac{1}{4\pi^2} \left| \int_{-\infty}^{\infty} [\pm \operatorname{Re} G(j\omega)]^{1/2} q(j\omega, t) d\omega \right|^2 \quad (21)$$

where

$$G(j\omega) = \frac{P_1}{\Lambda} |R(j\omega)|^2 F(j\omega) + \frac{R^*(j\omega)}{d^2} + \frac{b\Lambda}{a} \left[1 + \frac{aP_1}{\Lambda} R(j\omega) F(j\omega) \right] \quad (22)$$

and

$$d^2 = a$$

A sufficient condition for this derivative to be negative definite is

$$\operatorname{Re} G(j\omega) > 0 \quad (23)$$

Consequently, if there exist numbers a and b such that condition 23 is satisfied, the solutions of Eqs. 1, 2, and 17 are asymptotically stable because V is a Liapunov function with a negative definite time derivative. In other words, given a reactor described by Eqs. 1, 2, and 17, the operating power level, P_1 , is asymptotically stable with respect to all initial perturbations that lie in the region (total quantities)

$$d^2 = a \quad P < aP_1 \quad C_i < aC_{1i} \quad (i = 1, 2, \dots, m) \quad (24)$$

provided that there exist positive numbers a and b which make the function $G(s)$ a positive real function without zeros on the $j\omega$ -axis.

It is evident from the discussion on linear stability that the number a must be

$$a < a_c \quad (25)$$

Its exact value as well as the exact value of b depends on the particular form of the feedback transfer function.

The sufficient condition 23 can be written in a simpler form when $d^2 = a$. Indeed, then

$$\begin{aligned} G(j\omega) &= \left[\frac{R^*(j\omega)}{a} + \frac{b\Lambda}{a} \right] \left[1 + \frac{aP_1}{\Lambda} R(j\omega) F(j\omega) \right] \\ &= \frac{1}{a} \left[1 + \frac{aP_1}{\Lambda} R(j\omega) F(j\omega) \right]^2 \frac{R^*(j\omega) + b\Lambda}{1 + (aP_1/\Lambda) R^*(j\omega) F^*(j\omega)} \end{aligned} \quad (26)$$

Therefore condition 23 is equivalent to

$$\operatorname{Re} \left[\frac{R(j\omega) + b\Lambda}{1 + (aP_1/\Lambda) R(j\omega) F(j\omega)} \right] > 0 \quad (27)$$

Another simplified sufficient criterion is derived by taking $d^2 = a$ and $b = 0$. Then condition 23 becomes

$$\operatorname{Re} \left[\frac{R(j\omega)}{1 + (aP_1/\Lambda) R(j\omega) F(j\omega)} \right] > 0 \quad (28)$$

In other words, the reactor is asymptotically stable with respect to all initial perturbations in the region 24 if the reactor transfer function at power is a positive real function without zeros on the $j\omega$ -axis.

A similar procedure can be used for the case where $F_2[p(\tau)] \neq 0$, i.e., when $k(t)$ is given by Eq. 3. For example, consider the function V in Eq. 18. Its time derivative along the trajectories of Eqs. 1 to 3 is

$$\begin{aligned} \frac{dV}{dt} = & -\frac{P_1}{\Lambda} p(t) \int_{-\infty}^t f(t-\tau) p(\tau) d\tau - \frac{1}{d^2} p(t) k(t) - \frac{b\Lambda}{a} k^2(t) \\ & - bP_1 k(t) \int_{-\infty}^t f(t-\tau) p(\tau) d\tau + F_2[p(\tau)] [p(t) + b\Lambda k(t)] \end{aligned} \quad (29)$$

If it is assumed that $F_2[p(\tau)]$ has always the opposite sign from $p(t) + b\Lambda k(t)$ and that condition 23 is satisfied, then the time derivative given by Eq. 29 is negative definite, and the reactor is asymptotically stable in the region defined by inequalities 24. Other sufficient conditions can be derived for different specifications on $F_2[p(\tau)]$.

Also, a sufficient condition for Lagrangian stability of the solutions of Eqs. 1 to 3 is given in Ref. 1. Specifically, the solutions of Eqs. 1 to 3 are Lagrange stable with respect to all initial perturbations if there exists a positive number a such that

$$\operatorname{Re} \left[\frac{R(j\omega)}{1 + (aP_1/\Lambda) R(j\omega) F(j\omega)} \right] > 0 \quad [f(t) < 0] \quad (30)$$

Comparisons with Existing Criteria for the Case $F_2[p(\tau)] \equiv 0$

Welton's sufficient criterion¹⁶ $\operatorname{Re} F(j\omega) \geq 0$ is a special case of condition 23 for $a = d = \infty$ and $b = 0$. It implies that the reactor is asymptotically stable for an infinite range of operating power levels ($a_c = \infty$), and it does not include the delayed-neutron precursor constants. It is evident that, even if the feedback transfer function, $F(s)$, does not satisfy Welton's criterion, it can satisfy inequality 23.

Popov³ derived a sufficient criterion for asymptotic stability by means of certain inequalities. In terms of the present nomenclature, this criterion is

$$\operatorname{Re} \left\{ \frac{P_1}{\Lambda} |R(j\omega)|^2 F(j\omega) + \frac{b\Lambda}{a} \left[1 + \frac{aP_1}{\Lambda} R(j\omega) F(j\omega) \right] \right\} > 0 \quad (31)$$

The range of acceptable initial perturbations is given in an implicit form. Popov's criterion is a special case of condition 23 for $d = \infty$. It is evident that the criterion derived in this paper is less restrictive than

Popov's criterion since the reactor transfer function at zero power, $R(s)$, is a positive real function [$\text{Re } R(j\omega) = \text{Re } R^*(j\omega) \geq 0$].

It is concluded that the sufficient condition for asymptotic stability stated by inequality 23 is the most general criterion that has been derived to date.

It should be noted that all the sufficient criteria for nonlinear stability discussed so far involve only parameters characteristic of the linear approximation of Eqs. 1 to 3. This is an advantage both because these parameters can be identified experimentally and because procedures for characterizing nonlinear systems are not yet fully developed.

STABILITY CRITERIA DERIVED FROM INVESTIGATIONS OF SINGULARITIES AT INFINITY

General Remarks

Next, I would like to discuss a novel technique for deriving necessary and sufficient criteria for stability of solutions of ordinary differential equations with polynomial nonlinearities. This technique requires that all singularities at infinity be locally unstable. This comes about because all trajectories of a dynamical system begin from and end at singularities. One of my students, Miguel Barandiaran, worked on this problem for his doctoral thesis.⁴

A three-dimensional example is treated in Ref. 5. To bring out the salient features of the technique without excessive mathematical complexities, I will describe another problem with two state variables.

Consider the set of differential equations:

$$\frac{d\Phi}{dt} = \frac{1}{\tau_e} \left[\delta_o - \frac{\sigma_x X}{c\sigma_f} \right] \Phi \quad (32)$$

$$\frac{dX}{dt} = y_x \sigma_f \Phi - \lambda_x X - \sigma_x \Phi X \quad (33)$$

where c , λ_x , σ_f , σ_x , τ_e , and y_x are positive constants and δ_o is a positive or a negative constant. For a physical interpretation of Eqs. 32 and 33, see Ref. 6. The region of physical interest is $\Phi, X > 0$. The purpose of this analysis is to derive necessary and sufficient conditions that must be satisfied by the constants of the system so that the solutions are asymptotically stable or bounded with respect to all physically realizable initial conditions. For completeness, singularities both in the finite phase plane and at infinity are examined.

Equilibrium States in the Finite Phase Plane

There are two equilibrium states in the finite plane:

$$\Phi_0 = 0 \quad X_0 = 0$$

and

$$\Phi_1 = \frac{c\delta_0\lambda_x}{\sigma_x(y_x - c\delta_0)} \quad X_1 = \frac{c\delta_0\sigma_f}{\sigma_x}$$

The equilibrium state (Φ_1, X_1) is physically meaningful only when

$$0 < c\delta_0 < y_x \quad (34)$$

The eigenvalues associated with the equilibrium state at the origin are

$$s_1 = -\lambda_x \quad s_2 = \frac{\delta_0}{\tau_e} \quad (35)$$

The first eigenvalue corresponds to the vertical direction $\Phi = 0$. The second eigenvalue corresponds to an eigenvector of slope $y_x\sigma_f\tau_e/(\delta_0 + \lambda_x\tau_e)$. For $\delta_0 > 0$, the origin is a saddle point, and, for $\delta_0 < 0$, it is a stable node. The origin becomes structurally unstable⁷ for $\delta_0 = 0$ and $\delta_0 = -\lambda_x\tau_e$.

The eigenvalues at the second equilibrium state are given by the characteristic equation:

$$s^2 + \frac{\lambda_x y_x}{y_x - c\delta_0} s + \frac{\delta_0 \lambda_x}{\tau_e} = 0 \quad (36)$$

For $\delta_0 < 0$, these eigenvalues are of opposite sign, and the equilibrium point is of the saddle type and nonphysical. For $\delta_0 > 0$, the eigenvalues are real or complex, and the equilibrium state becomes a focus or a node depending on whether

$$\frac{4\delta_0}{\lambda_x \tau_e} \left[1 - \frac{c\delta_0}{y_x} \right]^2 \gtrless 1 \quad (37)$$

respectively. The equilibrium point is stable for $c\delta_0 < y_x$.

Singularities at Infinity

To establish the singularities at infinity, the state variables are expressed in terms of homogeneous coordinates, and the time variable is eliminated from Eqs. 32 and 33. More specifically, if

$$\Phi = \frac{u}{z} \quad X = \frac{v}{z} \quad (38)$$

the points at infinity correspond to $z = 0$, and, without the time variable, Eqs. 32 and 33 can be written in the form

$$\begin{vmatrix} du & dv & dz \\ u & v & z \\ \frac{\delta_o}{\tau_e} uz - \frac{\sigma_x}{c\sigma_f\tau_e} uv & y_x \sigma_x uz - \lambda_x vz - \sigma_x uv & 0 \end{vmatrix} = 0 \quad (39)$$

For this equation to be satisfied for $z = 0$, regardless of the direction (du, dv, dz), either the elements of the second and third rows are proportional or the last row is zero. Therefore the following singularities exist at infinity:

1. Two points along the direction

$$\frac{v}{u} = \frac{X}{\Phi} = c\sigma_f\tau_e \quad (z = 0)$$

2. Two double points along the vertical direction

$$u = \Phi = 0 \quad (z = 0)$$

3. Two double points along the horizontal direction

$$v = X = 0 \quad (z = 0)$$

For examination of the eigenvalues at the points along the direction $v/u = c\sigma_f\tau_e$, it is expedient to transfer these points to some convenient point in finite space without altering the eigenvalues. It can be shown that this can be readily done by means of a singular projective transformation.⁴ More specifically, if the singular transformation

$$w' = \frac{u}{u} = 1 \quad v' = \frac{v}{u} \quad z' = \frac{z}{u} \quad (40)$$

is chosen, then Eq. 39 is represented in the two-dimensional Cartesian space (v', z') by the equation

$$\frac{dv'}{y_x \sigma_f z' - (\lambda_x + \frac{\delta_o}{\tau_e}) v' z' - \sigma_x v' + \frac{\sigma_x}{c \sigma_f \tau_e} v'^2} = \frac{dz'}{-\frac{\delta_o}{\tau_e} z'^2 + \frac{\sigma_x}{c \sigma_f \tau_e} v' z'} \quad (41)$$

and the singularity is at the point

$$v' = c \sigma_f \tau_e \quad z' = 0 \quad (42)$$

If the origin of the (v', z') plane is transferred to the singularity (Eq. 42), then Eq. 41 becomes

$$\frac{dv''}{\sigma_x v'' + y_x \sigma_f - (\lambda_x + \frac{\delta_o}{\tau_e}) c \sigma_f \tau_e z' + O_2(v'', z')} = \frac{dz'}{\sigma_x z' + O_2(v'', z')} \quad (43)$$

where $v'' = v' - c \sigma_f \tau_e$ and $O_2(v'', z')$ are second-order polynomials in (v'', z') . It is evident that the eigenvalues are

$$s_1 = s_2 = \sigma_x \quad (44)$$

Hence the singularity is always of the unstable star type.

The character of the double singularities along the vertical direction can be deduced by an examination of the behavior of the trajectories in the vicinity of these points. Note first that the two straight line trajectories $\phi = 0$ and $z = 0$ go through these singularities and act as separatrices. In the vicinity of the separatrices, the trajectories behave as shown in Fig. 1, as can be readily deduced from Eqs. 32 and 33 evaluated for very small Φ and very large X . Consequently one of the double singularities is an unstable saddle, whereas the other is an unstable node for all values of the system parameters.

Finally, the double singularities along the horizontal direction can be investigated in a similar fashion. Consider first the singularities for $\phi > 0$. The straight line trajectory $z = 0$ is a separatrix for this point also. The other separatrix is a curve asymptotically tangent to the line

$$X = \frac{y_x \sigma_f}{\sigma_x} \quad (45)$$

Fig. 1 - Behavior of trajectories in the vicinity of the double singularities at infinity along the direction $\Phi = 0$.

This line is also the asymptote to the 0° isocline. The nature of the singularity depends on the relative position of the 0° and the 90° isoclines. The 90° isocline is

$$X = \frac{c\delta_o \sigma_f}{\sigma_x} \quad (46)$$

The two possible positions of the isoclines and the corresponding behavior of the trajectories are shown in Fig. 2. If $c\delta_o < y_x$, the singularity is of the saddle type. If $c\delta_o > y_x$, the singularity is of the stable node type. For $c\delta_o = y_x$, the singular point (Φ_1, X_1) goes to infinity, and the singularity along the direction $X = 0$ becomes triple but maintains its nodal character because the 0° isocline remains below the 90° isocline. Thus it is concluded that the singularity along the positive Φ direction is unstable for $c\delta_o < y_x$ and stable for $c\delta_o \geq y_x$. Similar observations can be made for the singularity at $\Phi < 0$.

When the results of the investigations of all the singularities are combined, it is concluded that the necessary and sufficient condition for boundedness of all physically meaningful solutions, i.e., the necessary and sufficient condition for the physically meaningful equilibrium states at infinity to be locally unstable, is that

$$c\delta_o < y_x \quad (47)$$

Fig. 2 - Behavior of trajectories in the vicinity of the double singularity along the direction $X = 0$ ($\Phi > 0$): (a) when the 90° isocline is above the 0° isocline, i.e., when $c\delta_0 > \gamma_x$, and (b) when the 0° isocline is above the 90° isocline, i.e., $c\delta_0 < \gamma_x$.

In fact, when this condition is satisfied, the solutions are also asymptotically stable because there are no limit cycles and the equilibrium states in the finite plane are asymptotically stable. It is important to emphasize that these conclusions pertaining to a nonlinear system have been derived by considering only local (linear) properties of the singularities.

To show the topology of the trajectories, a representative phase portrait for each of the regions of the parameters in which the system is structurally stable (see Fig. 3) is given in Figs. 4 to 7. The meaning of these portraits is self-explanatory.

INPUT-OUTPUT APPROACH TO OPTIMUM CONTROL

General Remarks

Ordinarily, optimum control problems are stated in terms of differential equations. A typical statement may be made: Given a system described by the set of differential equations

$$\frac{d}{dt} \mathbf{y}(t) = \mathbf{f}[\mathbf{y}(t), \mathbf{u}(t)] \quad (48)$$

where $y(t)$ = an n -state vector

$$y(t_0) = \eta_0$$

$u(t)$ = an r -control vector

find the optimum control $u(t)$, $t \in [t_0, t_1]$, for which an extremum is attained by the cost functional

$$J[u(t)] = \int_{t_0}^t L[y(t), u(t)] dt \quad (49)$$

when $u(t)$ belongs to some constrained or unconstrained space

$$y(t_1) = \eta$$

L = a specified operator

Fig. 3 - Regions of structural stability determined in terms of the parameters $x = \delta_o / \lambda_{x_e} \tau_e$ and $y = c \delta_o / y_x$ and by means of Eq. 37 and conditions $\delta_o = 0$ and $\delta_o = -\lambda_{x_e} \tau_e$.

Fig. 4 - Typical behavior of trajectories for parameter values lying in (a) region A and (b) region B of Fig. 3.

Fig. 5 - Typical behavior of trajectories for parameter values lying in (a) region C and (b) region D of Fig. 3.

(a)

(b)

Fig. 6 - Typical behavior of trajectories for parameter values lying in (a) region E and (b) region F of Fig. 3.

(a)

(b)

Fig. 7 - Typical behavior of trajectories for parameter values lying on (a) the boundary of regions B and C and (b) the boundary of regions C and E of Fig. 3.

Another way of looking at this problem is to consider the control vector as an input and the state vector as an output. Thus the system may be represented as an input-output relation:

$$\mathbf{y}(t) = \mathbf{H}[\mathbf{u}(t)] \quad (50)$$

where $\mathbf{H}$ is, in general, a nonlinear operator. Such representations have been discussed in Refs. 8 and 9. Under these conditions the cost functional is an explicit function of the input:

$$J[\mathbf{u}(t)] = \int_{t_0}^{t_1} L\{\mathbf{H}[\mathbf{u}(t), \mathbf{u}(t)]\} dt \quad (51)$$

The explicit dependence of the cost functional on $\mathbf{u}(t)$ may facilitate the analysis of the control problem at hand, particularly when $\mathbf{H}$ is nonlinear. It is for this reason that one of my students, Sang H. Kyong, is currently investigating optimum control of nonlinear systems described by input-output relations of the form given by Eq. 50. I will not discuss all the details of this work. To illustrate its salient features, however, I will present two simple examples.

A Linear Optimum Control Problem

Consider the dynamical system

$$\mathbf{y}(t) = \int_{t_0}^t [\mathbf{h}_{ik}(t, \sigma)] \mathbf{u}(\sigma) d\sigma \quad (52)$$

where $[\mathbf{h}_{ik}(t, \sigma)]$ is a matrix conformable with $\mathbf{u}(\sigma)$ and with elements $h_{ik}(t, \sigma)$. Suppose that it is desired to bring the output vector to $\boldsymbol{\eta}$ at time t_1 , subject to the constraint that the cost

$$J_E = \int_{t_0}^{t_1} \sum_{i=1}^r u_i^2(t) dt \quad (53)$$

be minimum, where $u_i(t)$ is the i th component of $\mathbf{u}(t)$. No restrictions are imposed on the magnitude of $\mathbf{u}(t)$.

At the terminal time, Eq. 52 becomes

$$\boldsymbol{\eta} = \int_{t_0}^{t_1} [\mathbf{h}_{ik}(t_1, \sigma)] \mathbf{u}(\sigma) d\sigma \quad (54)$$

To proceed, assume that all the elements of each row of the matrix $[h_{ik}(t_1, \sigma)]$ are of the same functional character. In other words, for a given i assume that each element $h_{ik}(t_1, \sigma)$ can be expanded into a finite series in terms of the members of a characteristic, complete, and orthonormal set, $\{\phi_n^k(\sigma)\}$:

$$h_{ik}(t_1, \sigma) = \sum_{\mu=1}^{m^{ik}} A_{\mu}^{ik} \phi_{\mu}^k(\sigma) \quad (m^{ik} < \infty) \quad (55)$$

The orthonormality interval is always $[t_0, t_1]$. In addition, expand each component of $u(\sigma)$ into an infinite series:

$$u_k(\sigma) = \sum_{\mu=1}^{\infty} B_{\mu}^k \phi_{\mu}^k(\sigma) \quad (56)$$

Substitution of Eqs. 55 and 56 into Eq. 54 yields

$$\begin{aligned} \eta_1 &= \sum_{k=1}^r \sum_{\mu=1}^{m^{ik}} A_{\mu}^{ik} B_{\mu}^k \\ \eta_2 &= \sum_{k=1}^r \sum_{\mu=1}^{m^{2k}} A_{\mu}^{2k} B_{\mu}^k \end{aligned} \quad (57)$$

to

$$\eta_n = \sum_{k=1}^r \sum_{\mu=1}^{m^{nk}} A_{\mu}^{nk} B_{\mu}^k$$

Inputs of the form of Eq. 56 satisfy the system and the terminal conditions exactly if the coefficients B_{μ}^k ($k = 1, 2, \dots, r$ and $\mu = 1, 2, \dots, m_k$) satisfy Eqs. 57 and if the coefficients B_{μ}^k ($\mu > m_k$ and $k = 1, 2, \dots, r$) have arbitrary values, where m_k is the largest of the numbers m^{ik} for a given k and $i = 1, 2, \dots, n$. The number of coefficients that can be determined from Eqs. 57 is

$$m = \sum_{k=1}^r m_k \quad (58)$$

Therefore three distinct cases can arise:

1. $m < n$. Since the n -relations of Eqs. 57 are linearly independent, it is impossible to satisfy them. In other words, there is no control that accomplishes the desired task.

2. $m = n$. In this case m coefficients can be uniquely determined from Eqs. 57.

3. $m > n$. This is the most usual case encountered in practice. Here n coefficients can be determined in terms of $m-n$ others, i.e.,

$$B_r = f_r(B_{n+1}, \dots, B_m) \quad (r = 1, 2, \dots, n) \quad (59)$$

where the coefficients B_μ^k have been reordered by means of a single index r .

Next, to complete the solution, consider the cost constraint. If Eq. 56 is replaced in Eq. 57, then

$$J_E = \sum_{r=1}^m B_r^2 + \sum_{r=m+1}^{\infty} B_r^2 \quad (m = n) \quad (60a)$$

$$J_E = \sum_{r=1}^n f_r^2(B_{n+1}, \dots, B_m) + \sum_{r=n+1}^{\infty} B_r^2 \quad (m > n) \quad (60b)$$

The cost constraint (Eq. 60a for $m = n$) attains a minimum when

$$B_r = 0 \quad (r = n+1, n+2, \dots, \infty) \quad (61)$$

and the remaining n coefficients are solutions of Eqs. 57. Thus the optimum control is completely and uniquely determined.

In the case $m > n$, the cost constraint attains a minimum when

$$\sum_{r=1}^n f_r(B_{n+1}, \dots, B_m) \frac{\partial}{\partial B_p} f_r(B_{n+1}, \dots, B_m) + B_p = 0 \quad (p = n+1, \dots, m) \quad (62)$$

$$B_r = 0 \quad (r = m+1, m+2, \dots, \infty) \quad (63)$$

Equations 57, 59, 62, and 63 provide the necessary m relations for the determination of the m coefficients of the optimum input control. If these coefficients are denoted by $(B_\mu^k)^*$, then the optimum control is

$$u_k^*(t) = \sum_{\mu=1}^m (B_\mu^k)^* \phi_\mu^k(t) \quad (64)$$

Special Case As a specific case of the preceding discussion, consider the system

$$y_1(t) = \xi_1 + \xi_2 t + \int_0^t (t - \sigma) u(\sigma) d\sigma \quad (65)$$

$$y_2(t) = \xi_2 + \int_0^t u(\sigma) d\sigma \quad (66)$$

Find the input that brings the state variables to $y_1 = y_2 = 0$ at $t = T$ and minimizes the integral

$$J_E = \int_0^T u^2(\sigma) d\sigma \quad (67)$$

To this end, note that, at the terminal time,

$$-\xi_1 - T \xi_2 = \int_0^T (T - \sigma) u(\sigma) d\sigma \quad (68)$$

$$-\xi_2 = \int_0^T u(\sigma) d\sigma \quad (69)$$

The characteristic set of orthonormal functions is

$$\phi_1(t) = \left(\frac{1}{T}\right)^{1/2} \quad \phi_2(t) = \left(\frac{3}{T}\right)^{1/2} - \left(\frac{12}{T^3}\right)^{1/2} t \quad \dots \quad (70)$$

When the kernels of Eqs. 68 and 69 are expanded in terms of this set and the input is written in the form

$$u(\sigma) = \sum_{k=1}^{\infty} B_k \phi_k(\sigma) \quad (71)$$

it is found that

$$B_1 = -\frac{\xi_2}{T^{1/2}} \quad B_2 = -\left(\frac{12}{T^3}\right)^{1/2} \left(\xi_1 + \xi_2 \frac{T}{2}\right)$$

$$B_k = \text{arbitrary} \quad (k \geq 3) \quad (72)$$

Minimization of J_E (Eq. 67) results in $B_k = 0$ ($k \geq 3$). Therefore the optimum input is

$$u^*(t) = -\frac{2}{T^2} (3\xi_1 + 2\xi_2 T) + \frac{6}{T^3} (2\xi_1 + \xi_2 T)t \quad (73)$$

A Nonlinear Optimum Control Problem

The procedure for nonlinear systems is similar to that developed for linear systems (see preceding section). Admittedly, however, the resulting algebraic equations for the expansion coefficients for the optimum control can be much more involved. For brevity, only a simple example will be discussed.

Consider the nonlinear system given by

$$y_1(t) = \int_0^t (t - \sigma) u(\sigma) d\sigma + \int_0^t \int_0^t (t - \sigma_1)(t - \sigma_2) u(\sigma_1) u(\sigma_2) d\sigma_1 d\sigma_2 \quad (74)$$

$$y_2(t) = -2 \int_0^t u(\sigma) d\sigma + \int_0^t \int_0^t u(\sigma_1) u(\sigma_2) d\sigma_1 d\sigma_2 \quad (75)$$

Suppose that it is desired to find the input that brings the output to $y_1 = 1$ and $y_2 = -1$ at $t = 1$ and minimizes

$$J_E = \int_0^1 u^2(\sigma) d\sigma \quad (76)$$

To this end, note that the characteristic orthonormal set is the same as that of the special example of the preceding section for $T = 1$.

Using the expansion technique, it is found that

$$B_1 = 1 \quad B_2 = -1 \pm \sqrt{4 + \sqrt{12}} \quad (77)$$

Finally, minimization of J_E yields the optimum input as

$$u^*(t) = 1 + (-1 + \frac{1}{2}\sqrt{4 + \sqrt{12}})(\sqrt{3} - \sqrt{12}t) \quad (78)$$

In summary, it is felt that the input-output approach to nonlinear optimum control problems will yield many practical results.

SPACE-TIME DYNAMICS

General Remarks

It is well known that both the magnitude and the rapidity of flux shape changes, either in ordinary space or in velocity space, have a substantial influence on the dynamic behavior of nuclear reactors. Our efforts in this area center around the understanding of effects of such changes on a variety of measurements performed on reactors. More specifically, one of my students, Larry Foulke, has been investigating the interpretation of oscillation tests in the light of space-time effects; with S. O. Johnson, of Phillips Petroleum Co., and his group, we have also been interested in the interpretation of power excursion bursts. I will describe briefly these two efforts.

Interpretation of Oscillation Tests

Oscillation tests, or, in general, small perturbation tests, are performed to measure transfer functions either to design the reactor regulating system or to investigate stability (see also the discussion of A New Criterion of Asymptotic Stability). Mathematically the reactor transfer functions characterize the linear constant-coefficient approximation of reactor dynamics. Reactor transfer functions are by necessity related to the derived concept of reactivity, and they are functions of frequency only. Physically the reactor transfer functions describe the dynamic behavior of the undistorted fundamental flux mode. This behavior, however, may not be possible to achieve in a particular small perturbation experiment. For example, it is found experimentally that *neutron flux oscillations*, corresponding to a *localized oscillating absorber*, are functions of both the frequency of oscillation and the position of the detector.¹⁰ This is particularly true at high frequencies, namely, rapid flux changes. The position dependence results because of the excitation of higher order modes or, equivalently, because of the finite time necessary for the propagation of disturbances.

Two related questions arise at this point: (1) Are the parameters of transfer functions space dependent? (2) How does one extract the parameters of normal transfer functions from oscillation tests?

The answer to the first question is no, despite statements to the contrary which have appeared in the literature.^{10,11} The reason is that the same high modes that contribute to the dependence of the flux oscillations on position contribute also to the reactivity that corresponds to the oscillating absorber. When reactivity is computed in a consistent manner, then the ratio of flux to reac-

tivity is independent of position, and a normal, space-independent transfer function is recovered. In other words, for a given detector position, the reactivity corresponding to an oscillating absorber differs from the latter in amplitude and phase by as much as the amplitude and phase of the detected neutron flux oscillations differ from the expected space-independent amplitude and phase results. This point is illustrated in Fig. 8.

The second question can be answered, in principle, as follows: Flux oscillations must be measured simultaneously at more than one position, and the results of the measurements must be analyzed in terms of a number of appropriate eigenfunctions to establish the higher

Fig. 8 - Computed phase angles (a) between oscillating-plane absorber and ρ/Λ corresponding to different detector positions and (b) between oscillating-plane absorber and neutron oscillations at different detector positions. The reactor geometry is shown in (b). Note that the departure of the phase of ρ/Λ from 0° is identical to the corresponding departure of the phase of the neutron oscillations from that of the space-independent transfer function. Similar results have been computed for the amplitudes. --- shows the detector at $x \approx 100$ cm; ----- shows the detector at $x = 10$ cm.

mode distortion. Then reactivity can be computed, and transfer functions can be derived as described previously.

Finally, I would like to mention that the distortion at high frequencies, due to spatial modes, is relevant to the interpretation of recent techniques for the measurement of subcriticality. It seems to me that the spatial effects have not been duly considered in these subcriticality measurements.

Correlation of Power-excursion Bursts

Power-excursion bursts are often correlated by means of point reactor kinetics to establish inherent feedback coefficients and safety margins of nuclear reactors. Because of spatial dynamic effects, particular care must be exercised whenever such correlations are attempted.

To illustrate this point, some numerical results derived by means of point reactor kinetics and by means of space-time computations are presented in Refs. 5 and 12. The reactors considered in these references are excited by a step-ramp change of the fission cross sections in one region of the reactor. No feedback is included in the computations. Comparison of the results reveals that serious discrepancies can arise between the two types of calculations (see Fig. 9).

Currently S. O. Johnson and I are trying to investigate the same problem for reactors with a variety of inherent feedback mechanisms. The results will be reported¹³ at the 1965 American Nuclear Society (ANS) meeting in Gatlinburg, Tenn. For purposes of this discussion, it suffices to say that discrepancies between space-time and point kinetics computations of power-excursion bursts do exist even in the presence of feedback. These discrepancies are experienced either in the shape of the power burst or in the magnitude of the average power. Therefore special care must be exercised whenever power-excursion bursts are correlated by means of point kinetics because the burst shape is indicative of the energy dependence of the feedback mechanism and because the integral under the power burst is related to the safety margin of the reactor.

DYNAMIC PROGRAMMING

Professor Henri Fenech and Dr. Ian Wall have used dynamic programming to investigate the optimum refueling policy for a power nuclear reactor. Even though the authors have presented their findings at previous ANS meetings, I thought it might be useful if I brought their work

Fig. 9 — Curve A is the computed average power for a 240-cm slab; curve B is the computed average power for a 60-cm slab computed by means of a space-time code; curve C indicates the average power that would have been derived by means of point kinetics for both reactors (see Refs. 5 and 12).

to your attention because their papers have usually been scheduled in fuel-cycle sessions.

Ordinarily refueling policies are based either on 100% batch irradiation or on discontinuous out-in refueling procedures. For a given reactor design, the out-in policy results in unit energy cost savings of the order of 0.2 to 0.5 mill/kw-hr over that of the 100% batch policy. Typical results are summarized in Table 1.

Fenech and Wall investigated the optimum refueling policy by examining which part of the fuel should be discharged and which part should be rearranged at every refueling step during the life of the plant. By means of the method of dynamic programming, they reduced the complexity of the problem to a level amenable to com-

Table 1 - SALIENT DETAILS OF TWO NONOPTIMIZED POLICIES

Policy	Power cost, mills/kw-hr	Maximum power peaking	Maximum burnup, Mwd/ton	Average burnup, Mwd/ton	Plant life, years	Number of stages
100% batch	6.512	1.86	17,880	14,880	30.34	16
Three-zone out-in	6.140	1.74	20,620	19,750	30.31	33

Table 2 - POWER COSTS OF OPTIMIZED POLICIES AS A FUNCTION OF MAXIMUM BURNUP, POWER PEAKING, AND REFUELING TIME

Maximum permissible burnup, Mwd/ton	Maximum permissible power peaking	Refueling time, days	Power costs, mills/kw-hr	Plant life, years	Number of stages	Average burnup, Mwd/ton
20,000	2.5	14	6.152	30.93	24	18,980
20,500	2.5	14	6.147	30.27	24	19,060
21,000	2.5	14	6.089	30.86	32	20,350
21,500	2.5	14	6.062	30.62	32	20,790
21,500	2.5	28	6.224	30.60	27	20,200
21,500	2.5	42	6.365	30.29	25	20,670
22,000	2.25	14	6.009	30.61	31	21,420
22,000	2.0	14	6.073	30.40	32	20,620
22,000	1.75	14	6.115	30.43	26	19,650
22,000	1.65	14	6.303	31.40	21	17,100
22,000	1.50	14	*	*	*	*

*No admissible policies.

putation. Some typical results of the investigation are given in Table 2. It is seen from the data presented in this table that an optimized policy results in further savings over those of the out-in policy. The detailed statement of the problem and a description of the necessary digital codes are given in Ref. 14.

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A Nonlinear Study of Reactor Systems with Linear Feedback

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ABSTRACT

Criteria are presented for local, global, and Lagrange stability, as well as a criterion for the nonexistence of solutions with a finite escape time, in reactor systems with an arbitrary linear feedback.

INTRODUCTION

A reactor with a linear feedback mechanism is described by the following nonlinear integro-differential equation:

$$\frac{dP(t)}{dt} = P(t) [k_{ex}(t) + \int_0^\infty P(t-u) G(u) du] + \int_0^\infty [P(t-u) - P(t)] D(u) du \quad (1)$$

where the unit of time is chosen such that $1/\beta = 1$ and where the symbols have their conventional meanings. The linear feedback is completely described by the feedback kernel, $G(t)$, which is defined as the positive reactivity in dollars occurring at time t due to unit fission energy released at $t = 0$. By its physical nature, $G(t)$ is a causal function; namely, $G(t) = 0$ for $t < 0$ and vanishes as $t \rightarrow \infty$. The Laplace transform of $G(t)$, denoted by $H(s)$, is analytic for $\text{Re}(s) \geq 0$.

We shall assume that the reactor is stationary for $t > 0$; i.e., the externally applied reactivity, $k_{ex}(t)$, as well as all the other system parameters, remain constant for $t > 0$. The equilibrium power level, P_0 , of the stationary reactor is determined by $k_0 + H(0)P_0 = 0$, where k_0 is the constant value of the external reactivity. We shall assume that the reactor has a finite equilibrium power level; this implies that

$$H(0) < 0 \quad (2)$$

We shall take $P_0 = 1$ without loss of generality.

The stationary reactor is described for $t > 0$ by

$$\begin{aligned} \frac{dp(t)}{dt} = [1 + p(t)] \int_0^t p(t-u) G(u) du - p(t) \\ + \int_0^t p(t-u) D(u) du + F(t) \end{aligned} \quad (3)$$

where

$$F(t) = \int_0^\infty p(-u)[G(t+u) + D(t+u)] du \quad (4)$$

and

$$p(t) = P(t) - 1$$

The function $F(t)$ depends on the past history of the reactor for $t < 0$ and is assumed to be a known function of time. A unique solution of Eq. 3 is determined by specifying $F(t)$ and $p(0)$, which is the initial jump at $t = 0$. Hence $F(t)$ and $p(0)$ form the initial conditions of the problem. Since the reactor power is known to be finite before the reactor becomes stationary, i.e., $p(t)$ is bounded for $t < 0$, and, since both $D(t)$ and $G(t)$ vanish for $t \rightarrow \infty$, the function $F(t) \rightarrow 0$ for large times.

The purpose of this paper is to investigate some of the properties of the solutions of Eq. 3 for large times and discuss the question of stability, boundedness, and finite escape time in general terms without specifying the feedback kernel by particular forms.

LOCAL STABILITY

We shall first consider a reactor that is operated at the equilibrium power level prior to $t = 0$. Then $p(-u) \equiv 0$ and $F(t) \equiv 0$ as Eq. 4 indicates. The time behavior of the reactor depends on the magnitude of the initial perturbation, $p(0)$, at $t = 0$. The stationary reactor is assumed to be linearly stable; this implies that the characteristic equation

$$1 - Z(s) H(s) = 0 \quad (5)$$

has no roots in the half plane $\text{Re}(s) \geq 0$. Here $Z(s)$ is the zero-power transfer function. It is usually taken for granted that the stability of the linearized system which is obtained by neglecting $p(t)$ as compared to unity in Eq. 3 implies the stability of the original nonlinear system for small disturbances, $p(0)$. Liapunov's first method¹ shows that this is indeed true for nonlinear systems of the form

$$\frac{dX}{dt} = AX + f(t, X)$$

where A is a square matrix with constant coefficients and $f(t, X)$ is a vector function if $\|f(t, X)\| = o(\|X\|)$. Here $X(t)$ is the unknown vector whose components are the dynamical variables of the system. Although the nonlinear reactor equation can be cast into this standard form in many reactor models for which $G(t)$ is a sum of a finite number of exponential functions, one cannot replace Eq. 3 by a finite-dimensional vector equation in general. The circulating-fuel reactor whose feedback kernel is of the form²

$$\begin{aligned} G(t) &= B \left(1 - \frac{t}{\theta} \right) & (t < \theta) \\ &= 0 & (\text{otherwise}) \end{aligned} \quad (6)$$

is a typical example. Reference 3 extends Liapunov's first method to reactor systems with an arbitrary linear feedback and shows that such reactor systems are asymptotically stable if the linearized system is stable, if

$$|p(0)| < \frac{\sigma}{NK^2} \quad (7)$$

and if the power variations following such an initial perturbation, $p(0)$, satisfy

$$|p(t)| < \frac{\sigma}{NK} \quad (8)$$

Here σ and K are two positive numbers and are chosen to satisfy

$$|L(t)| \leq Ke^{-\sigma t} \quad (t \geq 0) \quad (9)$$

where $L(t)$ is the response of the linearized system to a unit reactivity impulse, i.e., the inverse Laplace transform of $Z(s)/[1 - Z(s) H(s)]$. It is

always possible to find a pair (σ, K) when the system is linearly stable. For example, σ can be chosen as any number smaller than the magnitude of the real part of the root of the characteristic equation which has the least-negative real part. Note that $K \geq 1$ because $L(0) = 1$. The number N in Eqs. 7 and 8 is given by

$$N = \int_0^{\infty} |G(t)| dt \quad (10)$$

It is concluded that a knowledge of $G(t)$ enables one to calculate the upper bound of the initial power disturbances which give rise to asymptotically stable solutions and to estimate the upper bound of the subsequent power excursions.

STABILITY IN THE LARGE

We shall now investigate the behavior of the solutions of the stationary reactor for an arbitrary power history in the past, i.e., for an arbitrary $p(t)$ in $(-\infty, 0)$. It was shown by Welton⁴ that all the solutions of the stationary reactor are asymptotically stable if $\text{Re}[H(i\omega)] \leq 0$ for all ω . In this section we shall obtain this result as a special case of a more general criterion derived by Akcasu and Dalfes⁵ by a method similar to the second method of Liapunov.¹ This criterion is applicable to any nonlinear feedback model. It is stated that, if the feedback mechanism is such that the integral

$$\int_{-\infty}^t k_f(t') p(t') dt' \quad (11)$$

has an upper bound for any continuous function $p(t) \geq -1$ defined in $(-\infty, +\infty)$, the equilibrium state, $p(t) \equiv 0$, of the stationary reactor is asymptotically stable. It is clear that the function $p(t)$ in the interval $(-\infty, 0)$ represents the power history of the reactor before it becomes stationary at $t = 0$. Therefore the class of functions in the statement of the criterion will be restricted to those functions which are bounded in $(-\infty, 0)$. In Eq. 11, $k_f(t)$ is the positive feedback reactivity, which is a functional of $p(t)$ in $(-\infty, t)$. When the feedback is linear, Eq. 11 reduces to

$$I(t) \equiv \int_{-\infty}^t du p(u) \int_{-\infty}^u dv p(v) G(u - v) \quad (12)$$

which can also be written as

$$I(t) \equiv \int_{-\infty}^t du p(u) \int_{-\infty}^t dv p(v) G(u - v)$$

where we have used the fact that $G(t)$ is a causal function. Using the Fourier representation of $G(t)$, we get

$$I(t) = \frac{1}{\pi} \int_0^{\infty} d\omega \operatorname{Re}[H(i\omega)] \left| \int_{-\infty}^t p(u) e^{i\omega u} du \right|^2 \quad (13)$$

Thus $I(t)$ is always less than zero and bounded above for any $p(t)$ if $\operatorname{Re}[H(i\omega)] \leq 0$. Since $H(s)$ has no poles on the half plane $\operatorname{Re}(s) \geq 0$, Welton's criterion requires $H(s)$ to be a negative real function. The proof⁵ of Eq. 11 indicates that the lower limit of the integral can be chosen arbitrarily inasmuch as it gives rise to a constant term in the expression of the function $U(t)$ which plays the role of a Liapunov's function with a negative definite derivative. In the present application of the criterion, it proves more convenient to choose the lower limit as $(-\infty)$ than to choose it as zero, which is more suitable to the nonlinear feedback model considered in Ref. 5.

BOUNDEDNESS

The natural step after establishing a sufficient criterion for global asymptotic stability is to investigate the time behavior of the solutions when the condition for asymptotic stability is not fulfilled. This leads to the question of boundedness. The boundedness of all solutions of the stationary-reactor equation is referred to⁶ as *stability in the sense of Lagrange*, or simply as *Lagrange stability*. Since Lagrange stability includes asymptotic stability, the criterion for boundedness includes Welton's criterion. The following criterion for boundedness was obtained by Akcasu and Noble:⁷ If $H(s)$ does not vanish on the positive real axis, then all the Laplace transformable solutions of the stationary-reactor equation are bounded. Since a negative real function has no zeros in the half plane $\operatorname{Re}(s) > 0$, and on the real axis in particular, Welton's criterion is contained in the condition for boundedness. It is also noted that $H(s)$ has no zeros on the positive real axis if $G(t)$ is always negative because $H(\lambda) = \int_0^{\infty} G(t) \exp(-\lambda t) dt$, where λ is real, is a definite function if $G(t) \leq 0$. The latter is a sufficient condition for boundedness obtained by Smets⁸ and is included in the present criterion.

FINITE ESCAPE TIME

Now the following question arises: Are there solutions of the stationary reactor which are not Laplace transformable, and, if they do exist, how do they behave in time? It is shown analytically⁷ that in some reac-

tor models there are solutions with a finite escape time which are not Laplace transformable. A solution $p(t)$ is said to have a finite escape time if there is a finite time t_e such that $p(t) \rightarrow +\infty$ as $t \rightarrow t_e$. Solutions that have been discussed until now could be extended for all $t > 0$; i.e., they have been *defined in future*.⁶ Solutions having a finite escape time are not defined beyond $t = t_e$, and therefore their Laplace transforms are not defined. It was proved generally that solutions with a finite escape time⁷ can exist only if $G(0) > 0$ and behave as

$$\frac{2}{G(0)} \frac{1}{(t_e - t)^2} \quad (14)$$

where $G(0) \neq 0$ and $t \rightarrow t_e$. If $G(t)$ and the first $(n - 1)$ derivatives vanish as $t \rightarrow 0^+$, the foregoing necessary condition becomes $G^{n-1}(0) > 0$ and expression 14 is replaced by

$$\frac{(n + 1)!}{G^{n-1}(0)} \frac{1}{(t_e - t)^{n+1}} \quad (15)$$

In most of the actual reactors, however, $G(0) \neq 0$. Physically the condition $G(0) > 0$ implies that the finite escape time can occur only in those reactor systems which have a feedback mechanism with a prompt positive reactivity coefficient. The mechanism that gives rise to a finite escape time can be explained as follows: Since $G(0) > 0$, there exists a $t_0 > 0$ such that $G(t) > 0$ in $(0, t_0)$. Although the feedback is negative for long times, i.e., $H(0) = \int_0^\infty G(t) dt < 0$, it is positive for times in $(0, t_0)$. By ignoring delayed neutrons and $F(t)$ in Eq. 3, one can show that solutions with $P(0) > 1$ always have a finite escape time if $G(t) \geq 0$ for all t . If the initial perturbation is large enough, the finite escape time will be in the interval $(0, t_0)$, and the solution will go to infinity before the delayed negative reactivity has a chance to reduce the rate of increase in power.

It is emphasized that the condition $G(0) > 0$ for finite escape time is a necessary condition. The existence of a finite escape time when this condition is fulfilled depends on the magnitude of the initial perturbations. Although we have not been able to prove it, we believe that solutions with a finite escape time will always exist if $G(0) > 0$ and the initial perturbation is sufficiently large.

It is informative to note that solutions with finite escape time cannot exist if the condition for boundedness is fulfilled. Indeed, if $H(s)$ does not vanish on the positive real axis, then $H(s) < 0$ holds for s real and positive because $H(0) < 0$. Hence $G(0) = \lim_{s \rightarrow 0} s H(s) < 0$. Conversely, if $G(0) > 0$, the condition for boundedness is not satisfied. However, $G(0) < 0$ does not necessarily imply that $H(s)$ has real and positive zeros.

The foregoing observations strongly suggest the following conjecture:* If a solution is not Laplace transformable, it must have a finite escape time. In other words, all the solutions of Eq. 3 that are defined in future are of exponential order. This conjecture implies that if $H(s)$, has no zeros on the positive real axis all the solutions of Eq. 3 are bounded and the stationary reactor is stable in the sense of Lagrange. If $H(s)$ vanishes on the positive real axis, then one can still have bounded solutions, depending on the initial conditions, in addition to the unbounded solutions with a finite escape time. Furthermore, unbounded solutions of an exponential order are also possible. Computer studies by Shotkin⁹ using the two-temperature feedback model and those in Ref. 7 using the feedback model proposed by Bethe¹⁰ to investigate the kinetics of EBR-I have verified the foregoing conclusions. It has been found numerically in Ref. 7 that a reactor which is stable for small perturbations (linear stability) can have solutions with finite escape time. The feedback kernel used in this study was

$$G(t) = X_f \omega_f e^{-\omega_f t} + X_s \frac{\omega_s \omega_f}{\omega_s - \omega_f} U(\tau - t) [e^{-\omega_f(t-\tau)} - e^{-\omega_s(t-\tau)}] \quad (16)$$

where $\omega_f = 100 \text{ sec}^{-1}$

$$\omega_s = 1 \text{ sec}^{-1}$$

$$X_f = 1 \$$$

$$X_s = -5 \$$$

$$\tau = 0.75 \text{ sec}$$

$$\beta/l = 100 \text{ sec}^{-1}$$

It was assumed that the reactor had been operating at the equilibrium power level P_0 prior to $t = 0$ at which time an initial perturbation, $P(0)$, was introduced. Figure 1 depicts the results for various ratios of $P(0)/P_0$.

CONCLUSIONS

The foregoing discussion indicates that the stability of reactor systems with a linear feedback can be investigated quite generally in terms of the feedback kernel, $G(t)$. Since the condition for various types of stability is expressed either as a sufficient or as a necessary condition, it is always possible to find stability criteria with a wider range of applicability.

*See the Appendix.

Fig. 1 - Finite escape time with delayed neutrons for a linearly stable system.

The concept of finite escape time opens up a new avenue in nonlinear reactor dynamics. The author feels that this concept is of great practical importance in reactor design and operation because it indicates that the reactor power may tend to increase very rapidly in short times in reactors with prompt positive reactivity coefficients. The rapid increase of power in the EBR-I incident may be explained⁷ by a solution with a finite escape time. Figure 2 shows the power behavior in the EBR-I incident¹¹ along with the curve representing the solution of kinetic equations following a step reactivity input of 16ϕ at the equilibrium power of 11 watts. The feedback model in Eq. 16 was used in computing the theoretical curve with the following numbers taken from Ref. 10.

$$\omega_f = 1.0 \text{ sec}^{-1}$$

$$X_s = -5.0 \times 10^{-6} \$$$

$$\omega_s = 0.001 \text{ sec}^{-1}$$

$$\beta/l \approx 100 \text{ sec}^{-1}$$

$$X_f = 2.0 \times 10^{-6} \$$$

$$\tau = 0$$

The coolant circulation time, τ , is taken to be zero because the coolant flow was stopped during the experiment. Even with this simple model of Eq. 16 and with no attempt to match the experimental curve, the asymptotic behavior of both curves is quite similar. It thus appears that although an actual physical reactor cannot have a true finite escape time

Fig. 2 – Comparison of reactor power, EBR-I, and a solution with finite escape time.

it can behave in a manner that is closely approximated, near the escape time, by a solution with a finite escape time.

APPENDIX

We have recently proved¹² the following criteria for the Laplace transformability of the solutions: When one of the following conditions is satisfied, the solutions of Eq. 3 are Laplace transformable:

$$g(0) < 0 \quad (A.1)$$

$$g(t) \leq 0 \quad \text{when } g(0) = 0 \quad (A.2)$$

where

$$g(t) = \int_0^t G(u) du$$

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Evolution of the Finite-escape-time Concept in Reactor Kinetics

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ABSTRACT

The usefulness of the finite-escape-time concept in reactor kinetics is discussed. The general problem for a reactor with several temperature coefficients is treated. For adiabatic conditions exact criteria for the existence of solutions of finite escape time are given. It is shown that, if all temperature coefficients are positive, the solutions of the reactor-excursion problem diverge in a finite time. The method of expansion in a small parameter is explored. The quenching of a reactor excursion caused by a sudden temperature pulse is discussed.

INTRODUCTION

In 1951 we studied the system of differential equations¹

$$\begin{aligned}\tau \dot{\phi} &= (\delta_0 + \sum_{i=1}^n \alpha_i T_i) \phi \\ h_i \dot{T}_i &= \phi - \lambda_i T_i\end{aligned}\tag{1}$$

as the model for a reactor controlled by a single-temperature coefficient ($n = 1$) under Newton's law of cooling. When the temperature coefficient

Note that Eq. A.1 allows a delta function $\delta(t)$ in $G(t)$, which represents a feedback proportional to the instantaneous reactor power. If, in addition to these conditions, $H(s)$ does not vanish on the positive real axis, then the Lagrange stability is guaranteed. It is to be noted that $g(0) \leq 0$ is also a sufficient condition for the nonexistence of solutions with a finite escape time when $g(0) \neq 0$. Note also that Eq. A.2 implies the nonvanishing of $H(s)$ for positive real s because the Laplace transform of $g(t)$, i.e., $H(s)/s$, is always negative if Eq. A.2 holds for real s .

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of the reactor was positive ($\alpha_1 > 0$), we showed that there exist solutions which diverge in a finite time. Even if the reactor was initially set below critical at ambient temperature ($\delta_0 < 0$), the reactor power, temperature, and multiplication factor would diverge in a finite time if the initial rate of temperature rise were sufficiently rapid.

Clearly this behavior of the reactor model is somewhat embarrassing. The average reactor will not stay put at plasma temperatures, and the number of neutrons in the universe is believed to be finite. The multiplication factor of a reactor cannot become infinite and, in fact, cannot stray very far above unity without drastic consequences which are not within the purview of the reactor model.

It is for the latter reason that Gyftopoulos² has recently suggested that the boundedness of the reactivity be introduced in some *a priori* fashion into reactor-kinetics models. In short, he would outlaw systems of differential equations, such as Eqs. 1, which yield solutions of finite escape time. The acceptance of this proposal would pose a dilemma for reactor dynamicists. If they admit solutions which diverge at infinite time, it is again the same as stating that reactor power and temperature levels can exceed any given limits in a finite time. On the other hand, if the only admissible systems were those which are *a priori* Lagrange stable, the qualitative theory of reactor kinetics would become very dull and the quantitative theory very complicated.

In a more serious vein, we know that the system of Eqs. 1 has proved itself a useful tool in studies of transients for a variety of reactor types. During rapid transients, such as reactor excursions, the reactor may indeed behave as though it had a finite escape time over the range of physical interest. The neutron generation time, τ , in modern reactors is rarely longer than 10^{-4} sec and can be orders of magnitude shorter. If the excess reactivity amounts to a large percentage ($\delta_0 \approx 0.1$), the reactor period can be of the order of milliseconds or less until the reactor is disrupted or brought under inherent control. If temperature feedback effects are linear as they frequently are in first approximation, one obtains the system of Eqs. 1 or a similar but more elaborate set if delayed neutrons are included. It may sometimes prove necessary, as Gyftopoulos suggests, to introduce nonlinear feedback terms to make the mathematical model more "true to life." On the other hand, even less complexity is desirable in many cases. For example, during rapid reactor excursions the simple, but still useful, adiabatic model of reactor kinetics is yielded if the energy-loss terms $\lambda_i T_i$ are neglected.

The connection between these reactor-kinetics models and the movable singularities of nonlinear differential equations was first pointed out by Margulies³ in 1954. A thorough investigation of the two-temperature system was recently carried out by Shotkin.⁴ Shotkin delimited the

range of parameter space for $n = 2$, which contains solutions of finite escape time. However, there are still no general results on the dependence of the existence of the solutions on initial conditions. Lellouche⁵ has also pointed out that the physical significance of the parameters for $n \geq 2$ can be altered in their transformation into the canonical form of Eqs. 1.

As is typical, the pure mathematicians were the first to raise the question of the existence of solutions of finite escape time for systems of ordinary differential equations. It is well known that such solutions do not exist for sets of linear differential equations under the usual continuity conditions. After pondering the problem for 20 years, Wintner found a similar criterion in 1945 for sets of nonlinear differential equations.⁶ The term *finite escape time*, however, is not Wintner's, but appears to be of later origin.⁷ Differential equations that possess solutions of finite escape time are easy to find. Wintner gave the simple example $x' = x^2$. Another example given by LaSalle and Lefschetz⁷ is the Van der Pol equation with the usual sign of its damping term reversed.

ADIABATIC MODEL

Some light can be cast on the finite-escape-time problem if the solutions are examined for the case $\lambda_i = 0$. The general system of Eqs. 1 is then resolvable in terms of elementary functions.

Eliminating the temperature equations, one obtains

$$\dot{\phi} = \left(\delta_0 + \sum \alpha_i T_{i0} + \sum \frac{\alpha_i}{h_i} \int_0^t \phi \, dt \right) \phi \quad (2)$$

where T_{i0} are the initial temperature conditions.

If $\sum \alpha_i/h_i < 0$, the solutions are bounded. Hence we restrict consideration to $\sum \alpha_i/\tau h_i = 2a^2 > 0$, let $(\delta_0 + \sum \alpha_i T_{i0})/\tau = \omega_0$, and write $\theta = \int \phi \, dt$. Equation 2 then integrates into

$$\phi = d\theta/dt = \phi_0 + \omega_0 \theta + a^2 \theta^2 \quad (3)$$

If the initial reactivity is positive ($\omega_0 > 0$), the solutions blow up in a finite time independently of initial conditions. For $\omega_0 < 0$, the solutions are either bounded or of finite escape time according as $\phi_0 \gtrless \omega_0^2/4a^2$.

These results yield the necessary and sufficient conditions for the existence of solutions of finite escape time for the set of Eqs. 1 with $\lambda_i = 0$. They form a straightforward extension of the results for the single-temperature problem.¹

REACTOR-EXCURSION MODEL

We show next that, if the temperature coefficients α_i are all positive, the solutions of Eqs. 1 possess finite escape times. The initial conditions assumed are those appropriate to a reactor start-up accident. We therefore seek solutions of the form

$$\phi = \sum_{m=1}^{\infty} A_m e^{m\omega_0 t} \quad \text{and} \quad T_i = \sum A_m B_{mi} e^{m\omega_0 t} \quad \text{if } \omega_0 > 0$$

so that the reactor temperatures and neutron flux will vanish at $t = -\infty$. The values of B_m are therefore given by

$$B_{mi} = \frac{1}{\lambda_i + h_i m \omega_0} \quad B_m = \sum \frac{\alpha_i}{\tau(\lambda_i + h_i m \omega_0)} \quad (4)$$

and the values of the coefficients A_m can be obtained from the recursion formula

$$(m-1)\omega_0 A_m = \sum_{r=1}^{m-1} A_r A_{m-r} B_{m-r} \quad (5)$$

Let $C = \sum [\alpha_i / (\lambda_i + h_i \omega_0)]$, whence we have

$$\begin{aligned} \frac{C}{m\tau} &= \sum \frac{\alpha_i}{\tau(\lambda_i + h_i m \omega_0)} \leq \sum \frac{\alpha_i}{\tau(\lambda_i + h_i \omega_0)} = B_m \leq \sum \frac{\alpha_i}{\tau(\lambda_i + h_i \omega_0)} \\ &= \frac{C}{\tau} \quad (6) \end{aligned}$$

Repeated use of Eqs. 5 and 6 yields the series

$$\frac{\phi C}{\omega_0 \tau} = x + a_2 x^2 + a_3 x^3 + \dots \quad (7)$$

for the flux with $x = A_1 C e^{\omega_0 t} / \omega_0 \tau$ and $n/2^{n-1} \leq a_n \leq 1$.

The power series of Eq. 7 is therefore dominated by the series

$$x + x^2 + x^3 + \dots = \frac{x}{1-x}$$

and in turn dominates the series

$$x + x^2 + \frac{3}{4}x^3 + \dots = \frac{4x}{(2-x)^2}$$

It follows that the series of Eq. 7 converges absolutely and uniformly for $x \leq 1$. Since the coefficients of the series are all positive, the neutron flux monotonely approaches infinity as x approaches the limit x_∞ of its radius of convergence with $1 < x_\infty < 2$. If the zero of the time scale is taken at $x = 1$, the corresponding escape time t_∞ is governed by the inequality $\omega_0 t_\infty < \ln 2$. These results are again natural extensions of those obtained for the one-temperature problem.¹

These brief studies show that rigorous, if incomplete, information on solutions of finite escape time can be obtained by classical methods, such as perturbation series or special analytical solutions. Quite properly recent investigators of the problem, such as Shotkin⁴ and Akcasu,⁸ have emphasized more novel, if not always rigorous, lines.

EXPANSION IN A SMALL PARAMETER

Another classical method that appears to be worth exploring is that of expansion in a small parameter since this expansion frequently leads to rigorous results. This method can be expected to yield useful approximations for solutions of finite escape time because of the basically simple character of these solutions as compared with oscillatory solutions of relaxation type. The question is what parameter to use. As the system in Eqs. 1 is written, either the h_1 or λ_1 are obvious choices. One therefore writes

$$\phi = \phi^{(0)} + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \dots$$

with $\epsilon = h$ or λ and ϵ_1/ϵ held fixed as $\epsilon \rightarrow 0$. The zero-order results yield different criteria for the existence of solutions of finite escape time when the initial reactivity is negative, the truth lying closer to those obtained for the adiabatic case.

The zero-order solution of Eq. 3 for the flux when $\lambda_1 = 0$ is given by

$$\phi^{(0)} = a^2 B^2 \sec^2 \left[a^2 B t - \left(\tan^{-1} \frac{A}{B} \right) \right] \quad (8)$$

for $\omega_0 < 0$ and $\phi_0 > \omega_0^2/4a^2$, with $A = |\omega_0|/2a^2$ and

$$B = \sqrt{\phi_0 - \omega_0^2/4a^2}/a.$$

According to Eq. 8 the flux diverges at time $t = [\pi/2 + (\tan^{-1} A/B)]/a^2 B$. Although our analytical results for the first-order approximation are incomplete, the rate of rise of flux is arrested and, in fact, appears to be overcorrected. The limiting solutions in the special case $\phi_0 = \omega_0^2/4a^2$ are also of interest since these initial conditions represent the bifurcation point for solutions of finite escape time in the adiabatic approximation.

QUENCHING OF A HOT REACTOR EXCURSION

A reactor with positive temperature coefficients can diverge from its shutdown state if the pertinent component temperatures are increased by nonnuclear means, such as electrical heating, possible chemical reactions, or a simple increase in ambient temperature.

Consider the system of Eqs. 1 in its shutdown state ($\delta_0 < 0$, $\phi_0 \approx 0$) after a sudden increase in initial temperatures, T_{i0} . If $\delta_0 + \sum \alpha_i T_{i0} > 0$, the reactor is supercritical. If ϕ_0 is sufficiently small, for some time ϕ remains small and the temperature decay is given by $T_i = T_{i0} e^{-\lambda_i/h_i t}$.

The reactor will be called *quenched* if the flux goes through a maximum and then starts to decay, i.e., if $\dot{\phi}$ becomes negative at some time $t > 0$. If the exponential law of temperature decay persists, it is clear from the first equation of Eqs. 1 that quenching occurs for $t < \infty$ if $\delta_0 < 0$. Hence quenching of a hot reactor excursion is certain if the initial flux level is sufficiently low.

More generally, let $\omega_0 = (\delta_0 + \sum \alpha_i T_{i0})/\tau$. Let $T_i < T_{i0}$. Then $\phi \leq \phi_0 e^{\omega_0 t}$ and

$$T_i \leq \frac{\phi_0 (e^{\omega_0 t} - e^{-\lambda_i/h_i t})}{\omega_0 h_i + \lambda_i} + T_{i0} e^{-\lambda_i/h_i t} < T_{i0} \quad (9)$$

and quenching occurs if

$$\delta_0 + \phi_0 \sum \frac{\alpha_i}{\omega_0 h_i + \lambda_i} (e^{\omega_0 t} - e^{-\lambda_i/h_i t}) + \sum \alpha_i T_{i0} e^{-\lambda_i/h_i t} < 0 \quad (10)$$

Thus, for $\omega_0 = 0$, sufficient conditions for quenching are obtained if ϕ_0 satisfies the inequalities

$$\phi_0 < \frac{|\delta_0|}{\sum \alpha_i / \lambda_i} \quad \text{and} \quad \phi_0 < \lambda_i T_{i0} \quad \text{for all } i$$

For any quenched system the possibility of *rekindling* should also be investigated.

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Investigations on Nonlinear Stability of Coupled Nuclear Systems

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ABSTRACT

An alternative model of the kinetics equations for a coupled-core reactor is derived from the fundamental transport equation. The applicability of this formulation to the analysis of nonlinear dynamic stability is discussed. An analysis of a two-core reactor in which the reactivity feedback mechanism is dependent on average core power is detailed. The behavior of the singular points for this simplified model is described for the cases when the delay times associated with coupling neutrons traveling from one core to another are neglected and when the time lags are approximately treated. Using the methods of Lur'e and Letov and of Aizerman, the authors have derived stability criteria for the two-core reactor with nonlinear power-dependent feedback. The relative advantages of the two approaches are compared.

INTRODUCTION

A general theory of coupled-core reactors was introduced by Avery¹ in 1958 to determine the dynamic characteristics of mixed-spectrum (or

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coupled fast-thermal) reactors. The original motivation for studying such systems was to capitalize on the increase in the effective prompt-neutron lifetime as a small thermal region was added to a fast reactor system. This safety consideration was to be balanced against the desire to retain the favorable power characteristics (high power densities and breeding) of a fast reactor. Questions were raised as to the behavior of mixed-spectrum systems under accident conditions, but a rigorous safety analysis of such reactors has never been published. Analyses of specific coupled nuclear systems have been carried out though in connection with the nuclear superheat reactors,² a fast-thermal advanced sodium reactor,³ and the Kiwi reactors of the Rover nuclear rocket program.⁴ In addition, numerous designs possessing strong coupling effects have been suggested without explicit reference to the role played by such interactions. Examples of this include the segmented-core designs⁵ and the modular core schemes.⁶ In the general area of coupling phenomena can also be included such familiar problems as xenon oscillations in large reactors, reactivity shifting, and flux tilting.

In this paper another formulation of the dynamic theory of coupled-core reactors is introduced and its applicability to the analysis of coupling phenomena in reactors is explored. This formulation may be quite valuable, it appears, in the analysis of a reactor in which discrete sections of the reactor, or the individual cores,[†] can have different nuclear fuels; different neutron spectra; different fuel-pin shapes, diameters, and cladding; different coolant flow rates; and different coolants. It is easy to imagine that such cores can have quite markedly different dynamic characteristics. An obvious example is the considerable difference in the size of $\bar{\beta}$, the effective delayed-neutron fraction, for ^{239}Pu and ^{235}U and for ^{239}Pu and ^{233}U . In a zoned ^{239}Pu and ^{233}U fast-thermal breeder or even in a zoned thermal breeder, such widely diverse dynamic properties can be only poorly represented by the point-model kinetics equations. For instance, consider the effect of a uniform step insertion of reactivity with magnitude $\bar{\beta}_{\text{Pu}} < \delta\rho < \bar{\beta}_{\text{U}}$ in such a reactor. Although the point model can predict stability for the entire reactor, in reality an individual core can be relatively unstable because of local differences in the dynamic properties.

The aim of this paper is to investigate a generalization of the point reactor model which can take into account the nonlinear feedback mechanisms associated with the individual cores in the reactor. Because this model explicitly treats the nonlinear coupling between the individual cores, it effectively extends the validity of nonlinear spatial-independent kinetics to larger deviations from the steady-state values. Thus this

[†]After Seale and Chezem,⁴ each subcritical assembly will be called a core and the array of cores will be called a reactor.

formulation should offer a better approximation than point-model kinetics without going into the complexities of space-time kinetics.

FORMULATION OF THE PROBLEM

The equations describing the time-dependent behavior of a coupled-core reactor system are derived from the multienergy time-dependent transport equation. The approach used is similar to that of Henry.⁷ In addition to the fundamental-mode adjoint function, the functions $\nu \Sigma_i(r, E, t)$ and $\Sigma_i(r, E, t)$ are used as weighting functions to facilitate comparisons with other formulations of the coupled-core kinetics equations. The use of the fission cross section as a weighting function (suggested by Gyftopoulos⁸) means that the dependent variable is a measure of the growth or decay tendencies in the average power of an individual core. Because this approach is exact in its treatment of the neutron density within the boundaries of a particular core, the only assumptions required are on the nature of the source term that provides the coupling effects from the other cores in the array. The resulting form of the coupled-core kinetics equations using the $\nu \Sigma_i(r, E, t)$ weighting function is

$$\frac{dP_j}{dt} = \frac{\rho_j - \bar{\beta}_j}{\Lambda_j} P_j + \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} P_k(t - \tau_{jk}) + \sum_{i=1}^6 \lambda_{ji} C_{ji} \quad (1)$$

$$\frac{dC_{ji}}{dt} = \frac{\bar{\beta}_{ji}}{\Lambda_j} P_j - \lambda_{ji} C_{ji} \quad (2)$$

where P_j = the time-dependent behavior of the average power in the j th core

C_{ji} = the time-dependent behavior of the average power in the j th core due to delayed neutrons in the i th group

ρ_j = the generalized reactivity expression for the j th core

$\bar{\beta}_j$ = the effective delayed-neutron fraction in the j th core

Λ_j = the prompt-neutron generation time in the j th core

λ_{ji} = the decay constant of the i th group of delayed-neutron emitters in the j th core

α_{jk} = the coupling coefficient for neutrons diffusing from the k th to the j th core

τ_{jk} = the average time delay associated with neutrons diffusing from the k th to the j th core

The precise definitions of ρ_j , $\bar{\beta}_j$, and Λ_j and the restrictions involved in this derivation are given in Appendix A.

The presence of $\nu \Sigma_f(r, E, t)$ in the integral expressions invites a comparison with the Avery formulation¹ of the coupled-core kinetics equations. The major point of agreement is that the two formulations are identical in the steady state. To describe time-dependent behavior, the Avery form uses partial power (fission source) and partial lifetime terms, whereas this formulation uses coupling terms with retarded arguments. It is worthwhile also to note that the Avery formulation requires m^2 equations to describe the power behavior in a system with m cores, whereas this model needs only m equations.

This formulation explicitly introduces the nonlinear coupling between the individual cores through the coupling terms $P_k(t - \tau_{jk})$. Because of the explicit treatment, this model may extend the validity of nonlinear spatial-independent kinetics as well as provide insight into the nature of coupled-core systems. Since the nonlinear modal couplings are neglected in point-model kinetics,⁹ coupled-core kinetics may provide better approximations over wider ranges of variations about the equilibrium values.

TREATMENT OF THE COUPLING TERMS (WITH RETARDED ARGUMENTS)

The coupled-core kinetics equations investigated are nonlinear first-order difference-differential equations. Linearization of the reactivity feedback terms leads to a form quite amenable to analysis by techniques well known in the theory of linear feedback control mechanisms. An example of this work is the study of interacting nuclear cores for rocket propulsion (Seale and Chezem⁴) by the transfer function and Nyquist methods. The presence of time lags in the linearized equations does not introduce appreciable complication because the usual transformation from the time domain to the real frequency domain converts the retarded arguments into simple exponential terms. When the reactivity is considered to be nonlinear, however, the presence of time lags causes considerably more difficulty.

One approach is to neglect the effect of the time delays entirely. Under certain conditions, e.g., very long prompt-neutron generation times and very short time delays, this is not an unreasonable assumption, but the results are largely inapplicable to many reactor types of present and future interest.

Another approach leading to more general results is to represent the coupling terms by a truncated Taylor's series expanded about the time, t . Since the time lag is inherently small, the first two terms of the expansion provide a good approximation:

$$P_k(t - \tau_{jk}) \simeq P_k(t) - \tau_{jk} \dot{P}_k(t) \quad (3)$$

Substitution of Eq. 3 into Eq. 1 leads to

$$\frac{d}{dt} \left(P_j + \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} \tau_{jk} P_k \right) = \frac{\rho_j - \bar{\beta}_j}{\Lambda_j} P_j + \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} P_k + \sum_{i=1}^6 \lambda_{ji} C_{ji} \quad (4)$$

Equation 4 can then be cast in the form of an explicit matrix equation involving only the time derivative of the $\underline{P}$ vector on the left side:

$$\dot{\underline{P}} = \underline{E}\underline{P} + \underline{H} + \underline{F} \quad (5)$$

The elements of the matrix $\underline{E}$ ($m \times m$) and of the vectors $\underline{H}$ and $\underline{F}$ contain terms in τ_{jk} as well as in the core parameters. Details of these matrices and of the operations leading to Eq. 5 are given in Appendix B.

The Taylor's series expansion of the coupling terms leads to better insight into the physical behavior of such systems. This approach is successful in a rough quantitative description of the singular points and in providing approximate stability criteria involving the τ_{jk} parameters. It is, though, an approximation that can lead to invalid conclusions unless the results are carefully interpreted.

Finally, the effect of the time lags on system behavior can be analyzed without approximation. In the analysis of the singular points, the presence of retarded arguments means that the characteristic function associated with the linearized equations is transcendental. Consequently the characteristic equation possesses an infinite number of zeros.

In the analysis of the nonlinear equations, proof of the validity of selected stability criteria must be within the framework of a generalized theory that can be based on the application of Liapunov functionals. Such complications are inherent with differential equations with advanced or retarded arguments.

KINETICS EQUATIONS WITH NONLINEAR POWER-DEPENDENT REACTIVITY FEEDBACK

To gain initial insight into the physical behavior of coupled-core reactors and to avoid complex mathematical expressions, the authors found it necessary to consider nonlinear feedback mechanisms that are functions of only the average core power. This reduces the kinetics equations to a broad class of nonlinear equations whose analysis is a part of the modern control theory. The mathematical form of this restriction is

$$\rho_j \equiv \rho_j(P_j) = \rho_j^0 + \delta\rho_j(P_j) \quad (6)$$

where ρ_j^0 corresponds to the amount of reactivity in the j th core under equilibrium conditions and $\delta\rho_j(P_j)$ is the expression for reactivity feedback in the j th core. For the purposes of the analysis in this section, it is assumed that $\delta\rho_j$ can be represented by a power series expansion of the form

$$\delta\rho_j(P_j) \equiv \gamma_{j1}(P_j - P_j^0) + \gamma_{j2}(P_j - P_j^0)^2 + \dots \quad (7)$$

where P_j^0 is the steady-state value of the average power level in the j th core.

To simplify the formalism, the authors consider that only two cores are interacting and introduce the following definitions and symbols:

$$P_j^* \equiv P_j(t) - P_j^0$$

$$C_{ji}^* \equiv C_{ji}(t) - C_{ji}^0$$

$$a_j \equiv |\rho_j^0|/\Lambda_j$$

$$\bar{a}_j \equiv a_j - (\gamma_{j1}P_j^0/\Lambda_j)$$

$$b_{jk} \equiv \alpha_{jk}/\Lambda_j$$

$$d_{ji} \equiv \beta_{ji}/\Lambda_j$$

Translation of the steady-state operating point to the origin and use of the steady-state critical relationship

$$\rho_1^0 \rho_2^0 - \alpha_{12} \alpha_{21} = 0 \quad (8)$$

leads to the following set of equations:

$$\begin{aligned} \dot{P}_1^* &= -(\bar{a}_1 + d_{11})P_1^* + b_{12}P_2^*(t - \tau_{12}) + \sum_{i=1}^6 \lambda_{1i}C_{1i}^* + g_1(P_1^*) \\ \dot{P}_2^* &= b_{21}P_1^*(t - \tau_{21}) - (\bar{a}_2 + d_{22})P_2^* + \sum_{i=1}^6 \lambda_{2i}C_{2i}^* + g_2(P_2^*) \\ \dot{C}_{1i}^* &= d_{1i}P_1^* - \lambda_{1i}C_{1i}^* \\ \dot{C}_{2i}^* &= d_{2i}P_2^* - \lambda_{2i}C_{2i}^* \end{aligned} \quad (9)$$

where $g_j(P_j^*)$ are nonlinear terms of at least second order in P_j^* . This set of equations, containing the time delays and the nonlinear reactivity feedback $g_j(P_j^*)$ explicitly, describes the nonlinear dynamics of a two-core coupled system.

Analysis of the Singular Points

Before the nonlinear analysis is discussed, the nature of the equilibrium positions, or singular points, of the dynamical system should be understood. This is important because the type of equilibrium represented by a singular point is determined by the asymptotic behavior of the trajectories in the neighborhood of the point.¹⁰ Qualitatively a singular point is asymptotically stable if all trajectories sufficiently near it tend to it asymptotically as $t \rightarrow \infty$. A singular point, conversely, is asymptotically unstable if a trajectory tends to it as $t \rightarrow -\infty$.

The nature of the singular points can be determined by analysis of the characteristic equation of the linearized coupled-core kinetics equations. In the following sections several approximate models of a two-core system are considered. In this section the effect of the delayed neutrons is neglected or is partially taken into account by adjusting the values of Λ_j . The behavior of the singular points of these models for various values of the core properties is shown on parameter space plots.

Simplified Model with Time Lags Neglected If the time lags, as well as the delayed neutrons, are neglected, the linearized form of Eqs. 9 becomes

$$\begin{aligned}\dot{x}_1 &= -\bar{a}_1 x_1 + b_{12} x_2 \\ \dot{x}_2 &= b_{21} x_1 - \bar{a}_2 x_2\end{aligned}\tag{10}$$

where $x_j = P_j^*$. The characteristic equation at the operating point is

$$s^2 + (\bar{a}_1 + \bar{a}_2)s + \bar{a}_1 \bar{a}_2 - b_{12} b_{21} = 0\tag{11}$$

which has the roots

$$s = -\frac{1}{2}(\bar{a}_1 + \bar{a}_2) \pm \frac{1}{2}\sqrt{(\bar{a}_1 - \bar{a}_2)^2 + 4b_{12}b_{21}}\tag{12}$$

Since the discriminant is greater than zero for all values of the core parameters,[†] this simplified model can never exhibit oscillatory behavior near the singular point. The singular point is therefore classified either as a node or as a saddle point. The parameter values for each of the singular point modes of behavior are given in Table 1. A parameter space representation is given in Fig. 1, which shows the regions where the various modes occur. Both curves are plotted for an arbitrary value of $\sqrt{b_{12}b_{21}} \tau$.

[†]All the core parameters are real quantities, and, in addition, the terms b_{jk} are restricted to positive values.

Table 1 - PARAMETER VALUES FOR SINGULAR POINT MODES

	Shutdown equilibrium point	Operating equilibrium point
Stable node		
1	$a_1 + a_2 > 0$	$\bar{a}_1 + \bar{a}_2 > 0$
2	$a_1 a_2 - b_{12} b_{21} > 0$	$\bar{a}_1 \bar{a}_2 - b_{12} b_{21} > 0$
Unstable node		
1	$a_1 + a_2 < 0$	$\bar{a}_1 + \bar{a}_2 < 0$
2	$a_1 a_2 - b_{12} b_{21} > 0$	$\bar{a}_1 \bar{a}_2 - b_{12} b_{21} > 0$
Saddle point (unstable)		
1	$a_1 a_2 - b_{12} b_{21} < 0$	$\bar{a}_1 \bar{a}_2 - b_{12} b_{21} < 0$

Model with Time-delay Effects Approximated If the approximate representation of the coupling terms by a truncated Taylor's series expansion is considered and delayed neutrons are neglected, the linearized form of the equations becomes

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\Delta} (\bar{a}_1 + b_{12} b_{21} \tau_{12}) x_1 + \frac{1}{\Delta} b_{12} (1 + \bar{a}_2 \tau_{12}) x_2 \\ \dot{x}_2 &= \frac{1}{\Delta} b_{21} (1 + \bar{a}_1 \tau_{21}) x_1 - \frac{1}{\Delta} (\bar{a}_2 + b_{12} b_{21} \tau_{21}) x_2\end{aligned}\quad (13)$$

where $\Delta = 1 - b_{12} b_{21} \tau_{12} \tau_{21}$. If an additional approximation, $\tau_{12} = \tau_{21} = \tau$, is used, the characteristic equation has the roots

$$\begin{aligned}s &= -\frac{1}{2\Delta} (\bar{a}_1 + \bar{a}_2 + 2b_{12} b_{21} \tau) \\ &\quad \pm \frac{1}{2\Delta} \sqrt{(\bar{a}_1 - \bar{a}_2)^2 + 4b_{12} b_{21} (1 + \bar{a}_1 \tau)(1 + \bar{a}_2 \tau)}\end{aligned}\quad (14)$$

Since the discriminant in this case can become negative for certain parameter values, the system can oscillate near the singular point. Thus, in addition to being a node or a saddle point, the singular point can behave as a focus, or a center. For $b_{12} \tau < 1$ and $b_{21} \tau < 1$, the behavior closely resembles that described in the previous model; i.e., the singular point is a node or a saddle point, as shown in Fig. 2. The Taylor's

Fig. 1 – Stability regions in parameter space if the time lags are neglected. SN is stable node; UN is unstable node; and SP is saddle point.

series approximation creates a discontinuity at the $b_{12}b_{21}\tau^2 = 1$ plane in parameter space. Although this discontinuity is nonphysical, the approximation is successful in demonstrating that oscillatory behavior is possible for at least some parameter values. The presence of regions where the system behaves as a focus near its equilibrium position is shown in Fig. 3 for $B = \sqrt{b_{12}b_{21}}\tau > 1$; i.e., above the critical plane $B = 1$.

Fig. 2 - Stability regions in parameter space if the coupling terms are approximately treated. SN is stable node; UN is unstable node; and SP is saddle point.

Exact Treatment of Time Delays If the exact form of the coupling terms is retained, the characteristic function of the linearized equations is transcendental. When the delayed neutrons are neglected, the characteristic equation becomes

$$s^2 + (\bar{a}_1 + \bar{a}_2)s + \bar{a}_1\bar{a}_2 - b_{12}b_{21}e^{-(\tau_{12} + \tau_{21})s} = 0 \quad (15)$$

Fig. 3 – Stability regions in parameter space if the coupling terms are approximately treated. SN is stable node; UN is unstable node; SP is saddle point; SF is stable focus; UF is unstable focus; and C is center.

Although this equation has an infinite number of zeros, its first two zeros are nearly the zeros of Eq. 11. To illustrate, a comparison of the zeros of Eq. 11 with the first four zeros of Eq. 15 is given in Table 2. The zeros of Eq. 15 are obtained from a Newton–Raphson numerical technique programmed for the IBM 7090. For all the examples $\alpha_{12} = \alpha_{21} = 0.5 \times 10^{-3}$ and $\tau_{12} = \tau_{21} = 1.0 \times 10^{-3}$.

These numerical results are compatible with the Seale and Chezem⁴ conclusion that the presence of time lags will affect only the high frequency behavior of the transfer function. Investigation is underway of the effect of the time lags when the reactivity feedback is strongly nonlinear.

Nonlinear Stability Criteria

The theory and application of Liapunov's direct method has been presented in many excellent reviews.¹¹⁻¹⁴ Nearly all the studies dealing with nonlinear autonomous systems have been restricted to a

Table 2 - COMPARISON OF THE ZEROS OF THE
LINEARIZED AND EXACT CHARACTERISTIC EQUATIONS

	Linearized	Exact
Case I:	$\rho_1 = -0.70 \times 10^{-3}$ $\Lambda_1 = 0.50 \times 10^{-5}$	$\rho_2 = -0.40 \times 10^{-3}$ $\Lambda_2 = 0.50 \times 10^{-3}$
1	-0.085279	-0.085158
2	-140.7147	-140.9459
3		-6619.76 $\pm i$ 3650.07
4		6891.29 $\pm i$ 7087.80
Case II:	$\rho_1 = -0.55 \times 10^{-3}$ $\Lambda_1 = 0.50 \times 10^{-4}$	$\rho_2 = -0.50 \times 10^{-3}$ $\Lambda_2 = 0.50 \times 10^{-4}$
1	-0.487508	-0.482686
2	-20.51249	-20.72164
3		-6627.30 $\pm i$ 3645.12
4		-6895.44 $\pm i$ 7082.74
Case III:	$\rho_1 = -0.35 \times 10^{-3}$ $\Lambda_1 = 0.50 \times 10^{-5}$	$\rho_2 = -0.50 \times 10^{-3}$ $\Lambda_2 = 0.50 \times 10^{-4}$
1	+3.58899	+3.50870
2	-83.5899	-85.6829
3		-5323.33 $\pm i$ 3760.14
4		-5665.35 $\pm i$ 7160.08

single nonlinear control element. This is of significance to these investigations because such a restriction means that only a small portion of the present control theory is applicable in even the simplest coupled-core system, that of two interacting cores. In multicore systems there is a distinct nonlinear reactivity feedback mechanism in each core, and the dynamics of the cores are highly coupled.

One method for analyzing coupled systems (in this case multiregion reactors) is the describing-function approach introduced by Higgins and Hill.¹⁵ The efforts of Letov and Aizerman on the problem of control systems with two or more nonlinear elements are also of interest; their

results are reviewed and applied to coupled-core reactor dynamics on the following pages.

Analysis Based on the Lur'e-Letov Formulation Under certain restrictions on the nonlinear elements, the coupled-core kinetics equations belong to a broad class of nonlinear systems considered by Lur'e¹⁶ and Letov.¹⁷ A simplified form of such equations (derived by Letov) for a two-core system where the delayed neutrons and the time lags have been neglected is given as

$$\dot{x}_k = \sum_{m=1}^2 A_{km} x_m + \sum_{n=1}^2 u_{nk} g_n(\sigma_n) \quad (k = 1, 2) \quad (16)$$

$$\sigma_n = \sum_{m=1}^2 v_{nm} x_m \quad (n = 1, 2)$$

Upon substitution of the core parameters and recognition that

$$u_{12} = u_{21} = v_{12} = v_{21} = 0 \quad (17)$$

the set becomes

$$\begin{aligned} \dot{x}_1 &= -\bar{a}_1 x_1 + b_{12} x_2 + u_{11} g_1(v_{11} x_1) \\ \dot{x}_2 &= b_{21} x_1 - \bar{a}_2 x_2 + u_{22} g_2(v_{22} x_2) \end{aligned} \quad (18)$$

The effect of the delayed neutrons can easily be factored into this formulation if the number of state variables is increased and if the A matrix is expanded to include the terms d_{ji} and λ_{ji} . The effect of the time lags can be approximately treated if the Taylor's series expansion of the coupling terms is used.

The derivation of nonlinear stability criteria in the Lur'e-Letov method is based on the assumptions that the linearized equations are unconditionally stable, i.e., that $\text{Re}(s_k) < 0$ where s_k are the zeros of the linearized system characteristic equation and that the nonlinear elements satisfy the following conditions:

$$g_j(x_j) = 0 \quad \text{when } x_j = 0 \quad (19a)$$

$$x_j g_j(x_j) > 0 \quad \text{when } x_j \neq 0 \quad (19b)$$

In extending the nonlinear stability criteria to systems with two or more control elements, Letov proposes the existence of a suitable positive definite quadratic Liapunov (V) function of the system variables. Letov

then constrains the total derivative of this function with respect to time, dV/dt , to be negative definite, i.e., that the coefficient matrix of $-dV/dt$ (also a quadratic form) must be positive definite and hence must satisfy Sylvester's theorem on positive definite forms. The positive definiteness of the $m \times m$ determinant, D_m , is sufficient to guarantee that $\text{Re}(s_k) < 0$, i.e., for the linearized equations to be unconditionally stable. The positive definiteness of the $(m+j) \times (m+j)$ determinants where $j = 1, 2, \dots, m$ provides sufficient conditions for the nonlinear system to be asymptotically stable in the large. The Liapunov functions and the stability criteria are given in Appendix C for the cases when the time delay is neglected and when the coupling terms are expanded in a Taylor's series.

Several important factors favor the application of the Lur'e-Letov formulation. The effect of delayed neutrons can easily be taken into account if the linear differential equations describing the kinetic behavior of the delayed-neutron emitters is added. The method can be generalized to more than two cores, each core containing at least one nonlinear element dependent on the average core power. Although the Sylvester inequalities rapidly become too bulky for hand calculations, numerical analysis with digital computers can provide a means of calculating them.

The major limitation of this approach lies in its restrictions on the linearized equations and on the form of the nonlinearities. Two of the stability criteria derived from the case when the delayed neutrons and the time lags are neglected are given as

$$u_{11}v_{11} < 0 \quad (20a)$$

$$u_{22}v_{22} < 0 \quad (20b)$$

For simplicity v_{11} and v_{22} can be taken to be $+1$. An obvious choice for u_{11} and u_{22} is then -1 . Thus

$$G_1(x_1) = u_{11}g_1(x_1) = -g_1(x_1) \quad (21a)$$

$$G_2(x_2) = u_{22}g_2(x_2) = -g_2(x_2) \quad (21b)$$

These results must be interpreted in light of the initial restrictions that conditions 19 imposed on the nonlinearities. The allowable range of operation for the nonlinear elements as defined by condition 19b is shown in the left half of Fig. 4. Because of the results given in Eqs. 21, the nonlinear feedback terms $u_{jj}g_j(x_j)$ are restricted to the regions shown in the right half of Fig. 4. In other words, the choice of a Liapunov function, such as the one Letov proposes, rules out the possibility of small positive nonlinear reactivity feedback in an individual core even

Fig. 4 — Initial restriction on the nonlinearities and the operating range of nonlinear terms in the Lur'e-Letov treatment.

when the total power-dependent reactivity feedback over the entire reactor is negative. This inherent restriction in the Lur'e-Letov formulation is too severe for some cases of interest.

Analysis Based on the Aizerman Problem In 1949 Aizerman¹⁸ proposed that nonlinear stability criteria for the system

$$\dot{x}_1 = \sum_{k=1}^m A_{1k} x_k + u_{11} f_1(x_1) \quad (22)$$

$$\dot{x}_j = \sum_{k=1}^m A_{jk} x_k \quad (j = 2, 3, \dots, m)$$

could be derived from the analysis of the complementary linear system

$$\dot{x}_1 = \sum_{k=1}^m A_{1k} x_k + u_{11} h_1 x_1 \quad (23)$$

$$\dot{x}_j = \sum_{k=1}^m A_{jk} x_k \quad (j = 2, 3, \dots, m)$$

where the parameter h_1 is a constant. If the linear system is asymptotically stable for

$$\phi \leq h_1 \leq \theta \quad (24)$$

where ϕ and θ are constants, Aizerman then conjectured that the null solution of the nonlinear system of Eqs. 22 is asymptotically stable in the large for

$$\phi \leq \frac{f_1(x_1)}{x_1} \leq \theta \quad (x_1 \neq 0) \quad (25)$$

where $f_1(x_1)/x_1$ is the *instantaneous slope* of the nonlinear function $f_1(x_1)$ for any value of x_1 . Another statement[†] of inequality 25 is

$$\phi x_1^2 \leq x_1 f_1(x_1) \leq \theta x_1^2 \quad (26)$$

Although Aizerman stated the problem for an automatic control system with a single nonlinear element, the problem has been generalized^{11,13,19} to systems such as

$$\dot{x}_j = \sum_{k=1}^m [A_{jk}x_k + u_{jk}f_{jk}(x_k)] \quad (j = 1, 2, \dots, m) \quad (27)$$

which contain several nonlinear elements and are therefore of interest to the present investigation. A complete solution to the problem is available only for $m = 2$. For $m \geq 3$, the Aizerman conjecture that nonlinear stability is guaranteed for

$$\phi_{jk}x_k^2 \leq x_k f_{jk}(x_k) \leq \theta_{jk}x_k^2 \quad (28)$$

is valid only when the coefficients A_{jk} are connected by certain relations.

The usual method for obtaining suitable values for the constants h_{jk} from the complementary linear equations is to use the Routh-Hurwitz inequalities. To illustrate the applicability of this method to coupled-core reactor dynamics, the simplified model for a two-core reactor (with delayed neutrons and time lags neglected) is analyzed. If Eqs. 18 with $v_{11} = v_{22} = 1$ are used and if $x_j h_j = g_j(x_j)$ is introduced, the system becomes

$$\begin{aligned} \dot{x}_1 &= -\bar{a}_1 x_1 + b_{12} x_2 + u_{11} h_1 x_1 \\ \dot{x}_2 &= b_{21} x_1 - \bar{a}_2 x_2 + u_{22} h_2 x_2 \end{aligned} \quad (29)$$

The Routh-Hurwitz conditions for asymptotic stability for this system are

[†]Inequality 26 invites comparison with the Lur'e-Letov formulation in which ϕ is set equal to zero.

$$\bar{a}_1 + \bar{a}_2 - u_{11}h_1 - u_{22}h_2 > 0 \quad (30a)$$

$$(\bar{a}_1 - u_{11}h_1)(\bar{a}_2 - u_{22}h_2) - b_{12}b_{21} > 0 \quad (30b)$$

The fundamental notion of the generalized Aizeman problem is that the nonlinear equations will be asymptotically stable in the large when

$$\bar{a}_1 + \bar{a}_2 - \frac{u_{11}x_1g_1(x_1)}{x_1^2} - \frac{u_{22}x_2g_2(x_2)}{x_2^2} > 0 \quad (31a)$$

$$\begin{aligned} \bar{a}_1\bar{a}_2 - b_{12}b_{21} - u_{11}\bar{a}_2 \frac{x_1g_1(x_1)}{x_1^2} - u_{22}\bar{a}_1 \frac{x_2g_2(x_2)}{x_2^2} \\ + u_{11}u_{22} \frac{x_1g_1(x_1)x_2g_2(x_2)}{x_1^2x_2^2} > 0 \end{aligned} \quad (31b)$$

A proof of the validity of these conditions can be carried out if Liapunov's direct method is applied. A suitable Liapunov function for this case and proof of inequalities 31 are given in Appendix D. From a careful selection of arbitrary constants used in the proof, the following limits on the instantaneous slopes of $g_1(x_1)$ and $g_2(x_2)$ can be obtained:

$$\frac{g_1(x_1)}{x_1} < \bar{a}_1 - a_1^0 \quad (32a)$$

$$\frac{g_2(x_2)}{x_2} < \bar{a}_2 - a_2^0 \quad (32b)$$

where a_1^0 and a_2^0 are the values of $\bar{a}_1$ and $\bar{a}_2$ under equilibrium conditions and $u_{11} = u_{22} = +1$.

If the linearized system is asymptotically stable, several observations can be made. First, when u_{11} and $u_{22} = -1$ and $g_1(x_1)$ and $g_2(x_2)$ are restricted by

$$x_1g_1(x_1) > 0 \quad (x_1 \neq 0) \quad (33a)$$

$$x_2g_2(x_2) > 0 \quad (x_2 \neq 0) \quad (33b)$$

then inequalities 31 are immediately satisfied. This situation corresponds to the restrictions imposed on the nonlinearities in the Lur'e-Letov formulation. Second, when $u_{11} = u_{22} = +1$, then inequalities 32 provide that the instantaneous slopes of the nonlinearities can be

positive. The Aizerman problem does allow for small positive nonlinear reactivity feedback in the individual cores. The regions to which the behavior of the nonlinear elements belong are given in Fig. 5.

Fig. 5 - Operating range of nonlinear terms in the Aizerman treatment. Slope is maximum instantaneous slope of nonlinearity.

The effect of time lags is approximately treated by representing the coupling terms by a truncated Taylor's series expansion, as previously discussed. When $b_{12}\tau_{12} \ll 1$, $b_{21}\tau_{21} \ll 1$, $\tau_{12} = \tau_{21} = \tau$, and $u_{11} = u_{22} = +1$, the Aizerman approach yields the following criteria:

$$\frac{g_1(x_1)}{x_1} < \bar{a}_1 - a_1^0 + b_{12}b_{21}\tau \left(1 - \frac{\bar{a}_2}{a_2^0}\right) \quad (34a)$$

$$\frac{g_2(x_2)}{x_2} < \bar{a}_2 - a_2^0 + b_{12}b_{21}\tau \left(1 - \frac{\bar{a}_1}{a_1^0}\right) \quad (34b)$$

In this model the restrictions on the nonlinearities are less severe than in the model just considered only when $\bar{a}_1 < a_1^0$ and $\bar{a}_2 < a_2^0$. When the linearized equations are stable, the restrictions are certainly more severe.

The major factors favoring the use of the Aizerman approach are the simpler stability criteria that can be derived, the more general nonlinear forms that are allowed, and fewer restrictions that are imposed on the linearized equations.

There are two major limitations to the method. One is the progressive difficulty involved in proving the validity of the Routh-Hurwitz-like

inequalities by Liapunov's direct method as more equations, linear or nonlinear, are added. The second is the additional restrictions required on both linear and nonlinear parts of the equation set as the problem is generalized to include delayed neutrons and other interacting cores.

Summary It is quite evident that the Lur'e-Letov and the Aizerman formulations are complementary methods in several ways. The quadratic Liapunov function used in the Lur'e-Letov approach can easily be generalized to include more effects, whereas generalization of the Aizerman form requires increasing ingenuity and effort to discover a suitable Liapunov function at each new step. The Aizerman method, though, gives general, concise stability criteria for relatively simple cases, whereas the Lur'e-Letov formulations yields overly restrictive, fairly complicated criteria for the same problems. Further application and comparison of both approaches may lead to a determination of the suitability and validity of each approach.

KINETICS EQUATIONS WITH NONLINEAR TEMPERATURE-DEPENDENT REACTIVITY FEEDBACK

As a natural generalization of the coupled-core nonlinear stability investigations, the nonlinear reactivity feedback mechanisms can be considered as dependent on several characteristic temperatures in an individual core. A familiar case of this type is the two-temperature feedback model in point reactor kinetics.^{20,21} As an illustration of the approach being taken in this study, the equations for a two-temperature feedback model for a coupled system will be presented.

In general, the reactivity feedback for the temperature-dependent feedback model depends on $T_{jj'}^*$, the variation of the temperature of the l th material or l th region in the j th core, and on $(k_f)_{jj'}$, the temperature feedback coefficient associated with the l th material or l th region in the j th core. For the two-temperature model, the mathematical form of this feedback can be written as

$$\delta\rho_j = \delta\rho_j(T_{j1}^*, T_{j2}^*) \equiv (k_f)_{j1} T_{j1}^* + (k_f)_{j2} T_{j2}^* \quad (35)$$

It is possible then by a suitable change of variables to reduce the dynamics equations to the following form:

$$\frac{dP_j}{dt} = -a_j P_j + \sum_{\substack{k=1 \\ k \neq j}}^m b_{jk} P_k + (r_{j1} T_{j1} + r_{j2} T_{j2}) P_j \quad (36a)$$

$$\frac{dT_{j1}}{dt} = \nu_{j1}(P_j - P_j^0) - \mu_{j1}T_{j1} \quad (36b)$$

$$\frac{dT_{j2}}{dt} = \nu_{j2}(P_j - P_j^0) - \mu_{j2}T_{j2} \quad (36c)$$

The terms T_{jn} , r_{jn} , ν_{jn} , and μ_{jn} can be defined by expressions analogous to those given by Gyftopoulos.²⁰

Substitution of the logarithmic expression

$$Q_j \equiv \ln \frac{P_j}{P_j^0} \quad (37)$$

into Eqs. 36 leads to

$$\begin{aligned} \dot{Q}_j &= -a_j + \sum_{\substack{k=1 \\ k \neq j}}^m b_{jk} \frac{P_k^0}{P_j^0} e^{Q_k - Q_j} + r_{j1}T_{j1} + r_{j2}T_{j2} \\ \dot{T}_{j1} &= \nu_{j1}P_j^0(e^{Q_j} - 1) - \mu_{j1}T_{j1} \end{aligned} \quad (38)$$

$$\dot{T}_{j2} = \nu_{j2}P_j^0(e^{Q_j} - 1) - \mu_{j2}T_{j2}$$

This form is quite amenable to perturbation calculations when Q_j is small. The linearized equations associated with Eqs. 38 are

$$\begin{aligned} \dot{Q}_j &= -a_j + \sum_{\substack{k=1 \\ k \neq j}}^m b_{jk} \frac{P_k^0}{P_j^0} (1 + Q_k - Q_j) + r_{j1}T_{j1} + r_{j2}T_{j2} \\ \dot{T}_{j1} &= \nu_{j1}P_j^0Q_j - \mu_{j1}T_{j1} \\ \dot{T}_{j2} &= \nu_{j2}P_j^0Q_j - \mu_{j2}T_{j2} \end{aligned} \quad (39)$$

Nonlinear effects can be considered if the third term, or higher terms, of the exponential expansion is retained. An exhaustive study of temperature-temperature feedback mechanisms may require a generalization of Eq. 35 to include terms of higher order in T_{jj}^* .

It should be noted that for even a simplified two-core model, the characteristic equation of the linearized two-temperature model is a

sixth-order polynomial, the roots of which have to be obtained by numerical methods. Stability analysis of the linearized model can be treated by the approach of Seale and Chezem.⁴ Such stability is known as *stability in the first approximation*. Techniques for handling the nonlinear stability analysis as the higher order terms in the exponential expansion are taken into account are being considered.

The effect of delayed neutrons and of delay time can also be factored into this analysis. A logarithmic substitution involving the delayed power

$$K_{ji} \equiv \ln \frac{C_{ji}}{C_{ji}^0} \quad (40)$$

can be used to introduce the effects of the delayed neutrons. When the coupling terms are represented as a truncated Taylor's series, the effect of the time lags can be approximated as previously discussed.

SUMMARY

A preliminary investigation of the nonlinear stability of coupled-core reactors is presented. An alternative formulation of the coupled-core dynamics equations is introduced. Analysis of the singular points for power-dependent reactivity feedback shows that the equilibrium point can behave as a focus for certain core parameters. This study demonstrates that nonlinear stability criteria derived from the Lur'e-Letov method are quite restrictive but that generalization of the model to include delayed neutrons and approximate effects of time lags is straightforward. Further, it shows that the Aizerman approach offers better criteria for simple models but is less amenable to generalization.

APPENDIX A

INTEGRAL DEFINITIONS OF COUPLED-CORE PARAMETERS

The fundamental assumption involved in the derivation is that the total angular flux can be separated into a *shape function* and a *decay or growth function*. If this assumption is considered as the limitation on the range of validity for the model, then the multienergy transport equation can be converted to the form shown in Eq. 1. As in the derivation by Henry,⁷ the parameters in Eq. 1 have precise integral definitions. When $\nu \Sigma_f(r, E, t)$ is used as a weighting function, the following definitions are made:

$$p_j^f = p_j^f(t) \equiv \left\{ \frac{F_j^f}{1} \int_{-\infty}^0 \int_{-\infty}^0 dr dE dE' \left[\chi(E') (1 - \beta) \right. \right.$$

$$+ \sum_{i=1}^6 \beta_i^f f_i(E') \left. \right] \nu \sum^f(r, E, t) \nu \sum^f(r, E', t) \phi^p(r, E, t)$$

$$- \int_{-\infty}^0 \int_{-\infty}^0 dr dE dE' \nu \sum^f(r, E, t) \nu \sum^f(r, E', t) \phi^p(r, E, t) \cdot \Delta \psi^p(r, E, t)$$

$$- \int_{-\infty}^0 \int_{-\infty}^0 dr dE \nu \sum^f(r, E, t) \sum_a(r, E, t) \phi^p(r, E, t) \left. \right\}$$

$$\bar{\beta}_j^f = \bar{\beta}_j^f(t)$$

$$\equiv \left[\frac{F_j^f}{1} \sum_{i=1}^6 \beta_i^f \int_{-\infty}^0 \int_{-\infty}^0 dr dE dE' f_i(E') \nu \sum^f(r, E, t) \right.$$

$$\left. \cdot \nu \sum^f(r, E', t) \phi^p(r, E, t) \right]$$

$$V_j^f = V_j^f(t)$$

$$\equiv \left[\frac{F_j^f}{1} \int_{-\infty}^0 \int_{-\infty}^0 dr dE \nu \sum^f(r, E, t) \phi^p(r, E, t) \frac{\nu(E)}{\nu \sum^f(r, E, t)} \right]$$

$$C_{ji}^f = C_{ji}^f(t)$$

$$\equiv \left[\frac{V_j^f F_j^f}{1} \int_{-\infty}^0 \int_{-\infty}^0 dr dE \nu \sum^f(r, E, t) f_i(E) C_{ji}^f(r, t) \right]$$

$$s_j^f = s_j^f(t)$$

$$\equiv \left[\frac{V_j^f F_j^f}{1} \int_{-\infty}^0 \int_{-\infty}^0 dr dE \nu \sum^f(r, E, t) \bar{s}_j^f(r, E, t) \right]$$

$$F_j = F_j(t) \\ \equiv \left\{ \int_{V_j} \int_0^\infty \int_0^\infty dr dE dE' \left[\chi(E') (1 - \beta) \right. \right. \\ \left. \left. + \sum_{i=1}^6 \beta_i f_i(E') \right] \cdot \nu \Sigma_f(r, E, t) \nu \Sigma_f(r, E', t) \phi_p(r, E, t) \right\}$$

where the limits of integration mean the following: V_j denotes integration over the volume of the j th reactor; 0 to ∞ denotes integration over all neutron energies E ; and Ω denotes integration over all solid angles.

The source distribution function $\bar{S}_j(r, E, t)$ is assumed to be composed of a series of terms representing the power coupling with the other cores in the reactor. Since there is a finite time lapse from the birth of a neutron in one core and its capture in another core causing fission energy release, the coupling power growth factors have been assigned a retarded, or lagged, argument. The assumed expression for $\bar{S}_j(r, E, t)$ is

$$\bar{S}_j(r, E, t) = \sum_{\substack{k=1 \\ k \neq j}}^m \epsilon_{jk}(r, E, t) \phi_{Pj}(r, E, t) P_k(t - \tau_{jk})$$

where $\epsilon_{jk}(r, E, t)$ is the probability that a neutron escaping the boundaries of the k th core will cause a fission at point r and at energy E in the j th core. The combination $\epsilon_{jk}(r, E, t) \phi_{Pj}(r, E, t)$ represents a "new" shape function accounting for the distorted power distribution arising from the effective surface source of coupling neutrons. One must assume, however, in introducing such an effective source term, that the contributions to the reactivity expression, ρ_j , in each core due to coupling neutrons from all other cores must be set equal to zero.

On a macroscopic basis, the source term s_j is then defined as

$$s_j \equiv \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} P_k(t - \tau_{jk})$$

with

$$\alpha_{jk} \equiv \frac{1}{F_j} \left[\int_{V_j} \int_0^\infty dr dE \nu \Sigma_f(r, E, t) \epsilon_{jk}(r, E, t) \phi_{Pj}(r, E, t) \right]$$

Implicit in the derivation is the assumption that the logarithmic term

$$\left[\frac{\partial}{\partial t} \ln \int_v \int_0^\infty dr dE \nu \Sigma_f(r, E, t) \phi_p(r, E, t) \right] P_j(t)$$

is negligible. In one sense, this assumption can be thought of as a mathematical limitation on the validity of the model. In another sense, this is a physical restriction prohibiting the power distribution function $\phi_p(r, E, t)$ from changing appreciably or from changing too rapidly by even small amounts. Since the coupled-core model separates the reactor spatial power distribution into discrete parts, the logarithmic terms may often have little effect on the dynamic behavior of the individual cores.

APPENDIX B

APPROXIMATE REPRESENTATION OF THE COUPLING TERMS

The coupled-core kinetics equations were given as

$$\frac{dP_j}{dt} = \frac{\rho_j - \bar{\beta}_j}{\Lambda_j} P_j + \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} P_k(t - \tau_{jk}) + \sum_{i=1}^6 \lambda_{ji} C_{ji} \quad (\text{B.1})$$

$$\frac{dC_{ji}}{dt} = \frac{\bar{\beta}_{ji}}{\Lambda_j} P_j - \lambda_{ji} C_{ji} \quad (\text{B.2})$$

An expansion of the coupling terms in a Taylor's series about the time, t , yields

$$\sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} P_k(t - \tau_{jk}) = \sum_{\substack{k=1 \\ k \neq j}}^m \frac{\alpha_{jk}}{\Lambda_j} \left[P_k(t) - \tau_{jk} P_k^{(1)}(t) + \tau_{jk}^2 P_k^{(2)}(t) + \dots \right] \quad (\text{B.3})$$

Since the time lags are inherently small (on the order of 10^{-3} sec), the first two terms of the expansion provide a good approximation. With the symbols introduced earlier, Eq. B.1 becomes

$$\frac{d}{dt} \left(P_j + \sum_{\substack{k=1 \\ k \neq j}}^m b_{jk} \tau_{jk} P_k \right) = -(\bar{a}_j + d_j) P_j + \sum_{\substack{k=1 \\ k \neq j}}^m b_{jk} P_k + \sum_{i=1}^6 \lambda_{ji} C_{ji} + g_j(P_j^*) \quad (\text{B.4})$$

where $g_j(P_j^*)$ represent the nonlinear terms.

Equation B.4 can be written in matrix notation

$$T\dot{\underline{P}} = A\underline{P} + \underline{C} + \underline{G} \quad (\text{B.5})$$

This formulation is applicable to any number of interacting cores. As a simple illustration the matrices for the two-core reactor are as follows:

$$T = \begin{bmatrix} 1 & b_{12}\tau_{12} \\ b_{21}\tau_{21} & 1 \end{bmatrix} \quad A = \begin{bmatrix} -(\bar{a}_1 + d_1) & b_{12} \\ b_{21} & -(\bar{a}_2 + d_2) \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \underline{C} = \begin{bmatrix} \sum \lambda_{1i} C_{1i} \\ \sum \lambda_{2i} C_{2i} \end{bmatrix} \quad \underline{G} = \begin{bmatrix} g_1(P_1^*) \\ g_2(P_2^*) \end{bmatrix}$$

If Eq. B.5 is multiplied by the inverse of T, e.g.,

$$T^{-1} = \frac{1}{\Delta} \begin{bmatrix} 1 & -b_{12}\tau_{12} \\ -b_{21}\tau_{21} & 1 \end{bmatrix} \quad (\Delta = 1 - b_{12}b_{21}\tau_{12}\tau_{21})$$

for the two-core system, the matrix equation becomes

$$\dot{\underline{P}} = T^{-1}A\underline{P} + T^{-1}\underline{C} + T^{-1}\underline{G} \quad (\text{B.6})$$

or

$$\dot{\underline{P}} = E\underline{P} + \underline{H} + \underline{F} \quad (\text{B.7})$$

E is an $(m \times m)$ matrix and $\underline{H}$ and $\underline{F}$ are $(m \times 1)$ matrices, or vectors. The expansion of the matrix equation for a two-core system is as follows:

$$\begin{aligned} \dot{P}_1 = & \frac{-1}{\Delta} (\bar{a}_1 + d_1 + b_{12}b_{21}\tau_{12})P_1 + \frac{1}{\Delta} b_{21}(1 + \bar{a}_2\tau_{12} + d_2\tau_{12})P_2 \\ & + \frac{1}{\Delta} \sum_{i=1}^6 (\lambda_{1i}C_{1i} - b_{12}\tau_{12}\lambda_{2i}C_{2i}) \\ & + \frac{1}{\Delta} g_1(P_1^*) - \frac{1}{\Delta} b_{12}\tau_{12}g_2(P_2^*) \end{aligned} \quad (\text{B.8a})$$

$$\begin{aligned}
\dot{P}_2 = & \frac{1}{\Delta} b_{21}(1 + \bar{a}_1 \tau_{21} + d_1 \tau_{21})P_1 - \frac{1}{\Delta} (\bar{a}_2 + d_2 + b_{12}b_{21}\tau_{21})P_2 \\
& + \frac{1}{\Delta} \sum_{i=1}^6 (\lambda_{2i}C_{2i} - b_{21}\tau_{21}\lambda_{1i}C_{1i}) \\
& + \frac{1}{\Delta} g_2(P_2^*) - \frac{1}{\Delta} b_{21}\tau_{21}g_1(P_1^*)
\end{aligned} \tag{B.8b}$$

APPENDIX C

LIAPUNOV FUNCTIONS AND STABILITY CRITERIA BY THE LUR'E-LETOV METHOD

If the time delay is neglected, the equations of disturbed motion can be written in matrix notation as

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{G} \tag{C.1}$$

where $\underline{G}$ includes the nonlinear terms. For this system of equations, the following positive definite form can be taken as the Liapunov function.¹⁷

$$V = W + 2 \int_0^{\sigma_1} g_1(\sigma_1) d\sigma_1 + 2 \int_0^{\sigma_2} g_2(\sigma_2) d\sigma_2 \tag{C.2}$$

where $\sigma_1 = v_{11}x_1$, $\sigma_2 = v_{22}x_2$, and W is a positive definite form, which can be defined as

$$W = \underline{x}^T \underline{B} \underline{x} \tag{C.3}$$

in matrix notation where $\underline{x}^T$ is the transpose of $\underline{x}$ or as

$$W = B_{11}x_1^2 + 2B_{12}x_1x_2 + B_{22}x_2^2 \tag{C.4}$$

The elements B_{jk} are given by

$$B_{11} = \frac{1}{2} \frac{\bar{a}_1 + \bar{a}_2}{\bar{a}_1\bar{a}_2 - b_{12}b_{21}} \tag{C.5a}$$

$$B_{12} = \frac{1}{4b_{21}} \frac{a_1(\bar{a}_1 + \bar{a}_2) - 2(\bar{a}_1\bar{a}_2 - b_{12}b_{21})}{\bar{a}_1\bar{a}_2 - b_{12}b_{21}} \tag{C.5b}$$

$$B_{22} = \frac{1}{2b_{21}^2} \frac{(\bar{a}_1\bar{a}_2 - b_{12}b_{21})(\bar{a}_2 - 3a_1) + \bar{a}_1^2(\bar{a}_1 + \bar{a}_2)}{\bar{a}_1\bar{a}_2 - b_{12}b_{21}} \quad (C.5c)$$

These elements satisfy the Sylvester inequalities for positive definite determinants, i.e., $B_{11} > 0$ and $B_{11}B_{22} - B_{12}^2 > 0$.

When the coupling terms are represented by a Taylor's series expansion, the Liapunov function Eq. C.2 can again be used provided the quadratic function W is slightly generalized. Suitable values for the elements B_{jk} are

$$B_{11} = \frac{1}{2} \frac{\bar{a}_1 + \bar{a}_2 + 2b_{12}b_{21}\tau}{\bar{a}_1\bar{a}_2 - b_{12}b_{21}}$$

$$B_{12} = \frac{(\bar{a}_1 + b_{12}b_{21}\tau)(\bar{a}_1 + \bar{a}_2 + b_{12}b_{21}\tau) - 2\Delta(\bar{a}_1\bar{a}_2 - b_{12}b_{21})}{4b_{21}(1 + \tau\bar{a}_1)(\bar{a}_1\bar{a}_2 - b_{12}b_{21})} \quad (C.6)$$

$$B_{22} = \frac{\Delta(\bar{a}_1\bar{a}_2 - b_{12}b_{21})(\bar{a}_2 - 3\bar{a}_1 - 4b_{12}b_{21}\tau) + (\bar{a}_1 + b_{12}b_{21}\tau)^2(\bar{a}_1 + \bar{a}_2 + 2b_{12}b_{21}\tau)}{2b_{21}^2(1 + \tau\bar{a}_1)^2(\bar{a}_1\bar{a}_2 - b_{12}b_{21})}$$

The preceding functions can be used to determine criteria under which the coupled-core models exhibit asymptotic stability in the large. Since the total derivative of the Liapunov functions with respect to time, dV/dt , is a quadratic in x_1 , x_2 , $g_1(x_1)$, and $g_2(x_2)$, stability criteria can be derived by restricting the expression $-dV/dt$ to be a positive definite function. First, $-dW/dt$ is restricted to be a positive definite form:

$$-\frac{dW}{dt} = U \quad (C.7)$$

where U is a positive definite quadratic of the form

$$U = \mathbf{x}^T \mathbf{C} \mathbf{x} \quad (C.8)$$

The positive definiteness of the determinant D_2 associated with the matrix $\mathbf{C}^\dagger$ is sufficient to guarantee that the linear part of the system is

[†]The elements of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are related by the following expression:

$$C_{jk} = - \sum_{i=1}^m (A_{ik}B_{ji} + A_{ij}B_{ki})$$

asymptotically stable in the large. This condition can also be expressed as

$$\operatorname{Re}(s_k) < 0 \quad (\text{C.9})$$

where s_k are the zeros of the characteristic equation. Since the determinants D_3 and D_4 include the coefficients of the $x_j g_k(x_k)$ and $g_j(x_j) g_k(x_k)$ terms, their positive definiteness

$$\begin{aligned} D_3 &> 0 \\ D_4 &> 0 \end{aligned} \quad (\text{C.10})$$

guarantees that the function $-dV/dt$ is positive definite, and therefore the nonlinear system is asymptotically stable in the large. The determinant D_4 for the simplified model in which the time delays and delayed neutrons are neglected is given as

$$D_4 = \begin{vmatrix} C_{11} & C_{12} & P_{11} & P_{12} \\ C_{21} & C_{22} & P_{21} & P_{22} \\ P_{11} & P_{21} & 2R_{11} & R_{12} + R_{21} \\ P_{12} & P_{22} & R_{12} + R_{21} & 2R_{22} \end{vmatrix} \quad (\text{C.11})$$

where the elements P_{jk} and R_{jk} are defined by the following expressions:

$$P_{jk} = - \sum_{i=1}^2 (A_{ij} v_{ki} + B_{ij} u_{ki})$$

$$R_{jk} = - \sum_{i=1}^2 v_{ki} u_{ji}$$

APPENDIX D

LIAPUNOV FUNCTIONS AND STABILITY CRITERIA BY THE AIZERMAN METHOD

To demonstrate the validity of criteria 32 and 34, a theorem (after Krasovskii¹⁹) can be stated.

The theorem is given by a system of equations[†]

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + b_{12}x_2 \\ \dot{x}_2 &= b_{21}x_1 + f_2(x_2)\end{aligned}\tag{D.1}$$

If the functions

$$\begin{aligned}h_1 &= \frac{f_1(x_1)}{x_1} \\ h_2 &= \frac{f_2(x_2)}{x_2}\end{aligned}\tag{D.2}$$

satisfy the conditions

$$\begin{aligned}h_1 + c_2 &< 0 & h_1c_2 - b_{12}b_{21} &> 0 \\ h_2 + c_1 &< 0 & c_1h_2 - b_{12}b_{21} &> 0\end{aligned}\tag{D.3}$$

where c_1 and c_2 are constants such that $c_1c_2 - b_{12}b_{21} = 0$, and, if

$$\lim_{|x_1| \rightarrow \infty} \int_0^{x_1} [f_1(x_1) - b_{12}b_{21}x_1] dx_1 = \infty\tag{D.4}$$

$$\lim_{|x_2| \rightarrow \infty} \int_0^{x_2} [f_2(x_2) - b_{12}b_{21}x_2] dx_2 = \infty$$

then the null solution of Eq. D.1 is asymptotically stable in the large.

Under the conditions of the theorem, a suitable positive definite Liapunov function is

$$\begin{aligned}V &= (c_2^2 - b_{12}b_{21})x_1^2 + b_{12}^2 \left(1 - \frac{b_{12}b_{21}}{c_1^2}\right)x_2^2 \\ &\quad - 2b_{12}c_2x_1x_2 + 2c_2 \int_0^{x_1} f_1(x_1) dx_1 + 2 \frac{b_{12}^2}{c_1} \int_0^{x_2} f_2(x_2) dx_2\end{aligned}\tag{D.5}$$

[†]For the simplified case in which the delay time and delayed neutrons are neglected, the expressions $f_1(x_1)$ and $f_2(x_2)$ are defined by

$$f_1(x_1) = -\bar{a}_1x_1 + u_{11}g_1(x_1) \qquad f_2(x_2) = -\bar{a}_2x_2 + u_{22}g_2(x_2)$$

The total derivative of V with respect to time can be written in the form

$$\begin{aligned} \frac{dV}{dt} = & (c_2 h_1 - b_{12} b_{21})(h_1 + c_2)x_1^2 \\ & + 2 \frac{b_{12}^2}{c_1^2} (c_1 h_2 - b_{12} b_{21})(h_2 + c_1)x_2^2 \end{aligned} \quad (D.6)$$

which is negative definite under the assumed restrictions.

It is easy to show that the conditions in Eqs. D.3 can be written as

$$\begin{aligned} \bar{a}_1 + c_1 & < u_{11} \frac{g_1}{x_1} < \bar{a}_1 - c_2 \\ \bar{a}_2 + c_2 & < u_{22} \frac{g_2}{x_2} < \bar{a}_2 - c_1 \end{aligned} \quad (D.7)$$

If the constants $c_1 = -a_2^0$ and $c_2 = -a_1^0$ satisfying $a_1^0 a_2^0 - b_{12} b_{21} = 0$ for the case $u_{11} = u_{22} = 1$ are chosen, criteria 32 can be quickly derived.

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Stability of Nonlinear Coupled-core Reactors

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ABSTRACT

The unique feature of the nonlinear equations describing the dynamics of coupled-core nuclear reactor systems is the presence of terms containing a retarded argument. A method based upon the ideas of the Russian authors Krasovskii⁶ and Razumikhin⁷ is developed for investigating the asymptotic stability of coupled-core reactors by means of the Second Method of Liapunov.³

Regions of stability in the state space are obtained for a simplified nonlinear model of two coupled cores. In particular, the effect of the time delay is investigated directly via the Second Method without the aid of approximations for the delayed term.

INTRODUCTION

A critical reactor that consists of two or more independently subcritical cores is a coupled-core reactor.¹ This description can be applied to many types of reactors, including reactors in which spatial subdivisions of the entire core are treated as coupled cores. Of practical importance is the case of adjacent power reactors constituting a critical system, particularly the clustering of nuclear rocket engines.²

In a coupled-core reactor, the coupling effect results from the mutual exchange of leakage neutrons between the cores. Because these

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neutrons travel between cores in a finite time, the behavior of a given core depends not only upon processes occurring at the present time but also upon the past histories of the other cores. Herein lies the uniqueness of the problem. The set of differential equations describing the kinetics of each core contains terms with the argument of the dependent variable retarded or delayed in time.

It would be useful to develop a method by which stability of coupled-core systems could be readily examined. During the last three decades, researchers have found that the most universal method of investigating stability of ordinary differential equations is the Second Method of Liapunov.³ It is natural therefore that a technique based on the Liapunov theory be derived for this problem.

The approach to the stability problem via the Second Method of Liapunov has several advantages over the more conventional methods of analysis. The kinetics equations for power reactors, such as a rocket system, are nonlinear if temperature-induced reactivity effects are included. If the cores are coupled, the kinetics equations also include a time-delay term. A conventional analysis might include a linearization of the describing equations and then perhaps a series or Padé approximant⁴ to estimate the time-delay effect.

Since the linearized equations are valid only in an arbitrarily small region about the operating point, the bound on the initial conditions in the state space cannot be determined. Also a series expansion of the time-delay term may lead to the conclusion that a system is stable when in fact many cases it is not.⁵

The Second Method, on the other hand, is directly applicable to nonlinear equations, and modification of the usual Liapunov theorems extends this applicability of the Second Method to include systems with retarded arguments. The purpose of this paper is to show how these results can be applied to problems arising in coupled-core reactors. A simplified nonlinear model of a two-core system is used for purposes of illustration.

KINETICS EQUATIONS

The model is assumed to be that of a point reactor with the various characteristic parameters representing average values with respect to space. The simplified model consists of prompt-neutron kinetics with a negative-reactivity effect proportional to the power level. The kinetics equations for the i th core of an N -core system¹ are

$$\dot{P}_i(t) = \frac{\rho_i}{l} P_i(t) + \sum_k \frac{\alpha_{ik}}{l} P_k(t - T_{ik}) \quad \text{and} \quad \rho_i = \rho_{i0} - C_P P_i(t) \quad (1)$$

where P = power level

l = effective thermal neutron lifetime

ρ = reactivity

α_{ik} = coupling coefficient

T_{ik} = delay time for effective transfer of neutrons from the k th to the i th core

C_P = a positive constant of proportionality describing the intrinsic reactivity effect

Questions of stability are examined via the Second Method by considering perturbations of the system about some equilibrium point. This point is the origin of an n -dimensional space called the state space wherein the state of the system is described at any time by a vector of the system variables. Such a state variable form is achieved by the transformation

$$X_i(t) = \frac{P_i(t) - P_{io}}{P_{io}} \quad (2)$$

where P_o is the operating power level. Equation 2, when substituted into Eq. 1, gives, for a two-core system with identical properties and power levels,

$$\begin{aligned} \dot{X}_1(t) &= -\frac{\beta}{l}(k_P + k_c) X_1(t) + \frac{\beta}{l} k_c X_1(t - T) - \frac{\beta}{l} k_P X_1^2(t) \\ \dot{X}_2(t) &= -\frac{\beta}{l}(k_P + k_c) X_2(t) + \frac{\beta}{l} k_c X_2(t - T) - \frac{\beta}{l} k_P X_2^2(t) \end{aligned} \quad (3)$$

where k_P and k_c are the negative power and coupling contributions, respectively, in fractions of β/l , or dollars, at the operating point.

NONLINEAR STABILITY WITHOUT TIME DELAY

Before the time-delay case is discussed, it is advantageous to examine first the system with zero delay. The idea of the Second, or Direct, Method of Liapunov is to determine stability for a system although the solutions of the system are not known. This is accomplished by means of the scalar function, $V(x)$, which is positive in a region H about the origin of the state space except at the origin where it is zero.

If $\dot{V}(x)$ is negative in H , then the state of the system approaches the origin asymptotically for all solutions starting in H .

The problem is then to determine a positive definite $V(x)$ with a negative definite time derivative and to establish the region H in which the system is asymptotically stable.

The method used here is to consider a general positive definite quadratic V function for the system. The time derivative of this quadratic V is then constrained to be negative definite for the linearized system. Through this process the coefficients of the quadratic V are determined. The final steps are to calculate $\dot{V}$ for the nonlinear equations and to determine the region in which $\dot{V}$ is negative. The largest V that fits in the region $\dot{V} < 0$ encloses the region of stability, H .

A general quadratic V for the linear system without delay is

$$V = X_1^2 + 2KX_1X_2 + X_2^2 \quad (4)$$

whose time derivative is

$$\frac{1}{2} \dot{V} = \dot{X}_1(X_1 + KX_2) + \dot{X}_2(X_2 + KX_1) \quad (5)$$

The function of Eq. 4 is positive definite by Sylvester's theorem³ if $|K| < 1$. Equation 5 for the linearized system with zero delay is

$$\begin{aligned} \frac{1}{2} \frac{1}{\beta} \dot{V} = & -[(k_P + k_c) - Kk_c]X_1^2 - 2X_1X_2[K(k_P + k_c) - k_c] \\ & - [(k_P + k_c) - Kk_c]X_2^2 \end{aligned}$$

which is negative definite if

$$k_P + (1 - K)k_c > 0$$

and

$$k_P(k_P + 2k_c) > 0$$

These inequalities are satisfied for all admissible values of K as long as k_P and k_c are positive. Thus the linear system is stable in the entire state space if the intrinsic reactivity effect is present.

For a clear illustration of the process of obtaining the region of stability for the nonlinear system, it is assumed that $k_P = k_c = 1$ for the

remainder of this analysis. Equation 5 for the nonlinear system without delay is

$$\frac{1}{2} \frac{I}{\beta} \dot{V} = -(2 - K)(X_1^2 + X_2^2) - 2(2K - 1)X_1X_2 - (X_1^3 + X_2^3) - KX_1X_2(X_1 + X_2) \quad (6)$$

If $K = 0$, then

$$V = X_1^2 + X_2^2$$

and the $\dot{V}$ of Eq. 6 is zero along the curve $X_2 = -X_1 - 2$. The largest V that fits in the region $\dot{V} < 0$, above the line $\dot{V} = 0$, is $V = 2$. Therefore the region of stability is a circle with a radius of $\sqrt{2}$.

The process is repeated for $K = 1/2$ and $K = 3/4$, and the regions are superimposed to obtain a larger stable region. Values of $K < 0$ add no new information. Figure 1 shows the result of this calculation and also some actual stable and unstable system trajectories. The region shown is symmetric about $X_1 = X_2$.

The line $X_2 = -1$ in Fig. 1 is the line of zero power for the system. Thus trajectories outside the region $X_2 > -1$ do not represent physically real cases. For this practical reason the region obtained for $K = 3/4$ is superfluous.

NONLINEAR STABILITY WITH TIME DELAY

The basic approach for analyzing the system with delay is the same as that for the system without delay. The linearized system is constrained to be asymptotically stable; then the region of stability is found by adding the nonlinear terms in the system equations.

Although the fundamental ideas of the Second Method are unchanged, there are important alterations in the mechanics of the solution in the time-delay problem. That such changes are necessary can be seen immediately from $\dot{V}$, calculated from Eq. 5 for the linearized Eq. 3 with time delay.

$$\begin{aligned} \frac{1}{2} \frac{I}{\beta} \dot{V} = & -2X_1^2 + X_2(t - T) X_1(t - T) X_2 + KX_2X_2(t - T) \\ & + X_1X_1(t - T) - 2X_2^2 \quad (7) \end{aligned}$$

Fig. 1 - Region of stability without delay.

The presence of cross products of the variables with and without the retarded argument raises a question as to the sign definiteness of Eq. 7.

A physical interpretation of the difficulty is seen in any first-order linear time-delay problem.⁵ For some values of the delay time in this case, the system exhibits an oscillatory response. Because $V = X^2$ for the first-order problem, V oscillates; therefore $\dot{V}$ is not always negative definite. It is known, however, that the system is asymptotically stable. The additional burden on the Second Method is to guarantee that V eventually decreases monotonically if the system is asymptotically stable. Accordingly the following conditions, derived from a theorem by Krasovskii,⁶ are imposed:

$\dot{V} \leq$ some known negative definite function (condition a)

An $f(V) > V$ exists such that $V[X(S)] < f\{V[X(t)]\}$ for all $t - T \leq S \leq t$ (condition b)

An additional condition is that the initial conditions of the system must be defined over $-T \leq t \leq 0$, and thus the region H must contain these values.

Generally, $f(V)$ can be chosen⁶ as V/q for $0 < q < 1$. The method is illustrated by using Eq. 7 for the case $K = 0$. It is clear that

$$\frac{1}{2} \frac{1}{\beta} \dot{V} \leq -2(X_1^2 + X_2^2) + |X_2(t-T)| |X_1| + |X_1(t-T)| |X_2| \quad (8)$$

and from condition b

$$X_1^2(t-T) + X_2^2(t-T) < \frac{1}{q} (X_1^2 + X_2^2)$$

This inequality is obtained in the useful form

$$|X_1(t-T)|, |X_2(t-T)| < \frac{1}{\sqrt{q}} (|X_1| + |X_2|) \quad (9)$$

Relations of type 9 are necessarily restrictive for all systems of greater than first order. Inequality 9, when substituted into inequality 8, yields

$$\frac{1}{2} \frac{l}{\beta} \dot{V} \leq - \left(2 - \frac{1}{\sqrt{q}} \right) (X_1^2 + X_2^2) + \frac{2}{\sqrt{q}} |X_1| |X_2|$$

which, by Sylvester's theorem, is indefinite; therefore stability cannot be proved for $k_p = k_c = 1$.

It is significant that the development used thus far leads to stability conclusions independent of the delay time. Razumikhin⁷ suggests a substitution based upon the mean value of X between $t-T$ and t . $X(\sigma)$ is the mean value, and

$$T X(\sigma) = \int_{t-T}^t X(S) dS$$

Thus

$$X(t-T) = X(t) - T \dot{X}(\sigma) \quad (10)$$

If the original equations are written in terms of σ , an expression for $X(t-T)$ is obtained in terms of $X(t)$, $X(\sigma)$, $X(\sigma-T)$, and T . Specifically, for the linearized Eq. 3,

$$-\frac{l}{\beta} \dot{X}_1(\sigma) = 2X_1(\sigma) - X_2(\sigma-T)$$

$$-\frac{l}{\beta} \dot{X}_2(\sigma) = 2X_2(\sigma) - X_1(\sigma-T)$$

with $X^2(\sigma)$ added for the nonlinear case. For example, in the linear case, $X_1(t-T)$ is, from Eq. 10,

$$X_1(t-T) = X_1(t) + 2 \frac{\beta}{l} T X_1(\sigma) - \frac{\beta}{l} T X_2(\sigma-T)$$

The inequalities 9 are now

$$|X_{1,2}(\sigma)| < \frac{1}{\sqrt{q_1}} (|X_1| + |X_2|)$$

$$|X_{1,2}(\sigma - T)| < \frac{1}{\sqrt{q_2}} (|X_1| + |X_2|)$$

and for $K = 0$

$$\frac{1}{2} \frac{l}{\beta} \dot{V} \leq - \left(1 - \frac{3}{2} \frac{\beta}{l} T \right) (X_1^2 + X_2^2) + \left(1 + 3 \frac{\beta}{l} T \right) |X_1| |X_2| \quad (11)$$

if q_1 and q_2 are chosen to be sufficiently close to unity.

The right side of inequality 11 is negative definite if $(\beta/l)T < 1/6$.

The entire process is repeated for the nonlinear equations and

$$\begin{aligned} \frac{1}{2} \frac{l}{\beta} \dot{V} \leq & - \left(1 - \frac{3}{2} \frac{\beta}{l} T \right) (X_1^2 + X_2^2) + \left(1 + 3 \frac{\beta}{l} T \right) |X_1| |X_2| \\ & - \frac{1}{2} \left(1 - \frac{\beta}{l} T \right) (X_1^3 + X_2^3) + \frac{1}{2} \frac{\beta}{l} T X_1 X_2 (X_1 + X_2) \quad (12) \end{aligned}$$

The radii of the largest $V = X_1^2 + X_2^2$ which fit in the regions of negative definiteness of the right side of inequality 12 as functions of $(\beta/l)T$ are

$\frac{\beta}{l}T$	$R = V_{\max}$
0.01	1.36
0.10	0.75

It is recalled that $V_{\max} = 1.41$ for $T = 0$. If $K \neq 0$, it is only slightly more difficult to find inequalities 9. For the case of $K = 1/2$, if $y = X(t - T)$, condition b yields

$$y_1^2 + y_1 y_2 + y_2^2 < \frac{1}{q} (X_1^2 + X_1 X_2 + X_2^2)$$

The ratio of the maximum to minimum radii of either elliptical function is

$$\frac{r_{\max}^2}{r_{\min}^2} = 3$$

Then, because $r^2 = X_1^2 + X_2^2$, it is certainly true that

$$X_1^2(t - T) + X_2^2(t - T) < \frac{3}{q} (X_1^2 + X_2^2)$$

For asymptotic stability of the linear system where $K = 1/2$,

$$\frac{\beta}{l} T < \frac{3}{\sqrt{18}} \quad (13)$$

Figure 2 illustrates the effect of the delay on the region of stability using the method presented in this section. The V functions for $K = 0$ and $K = 1/2$ are employed. For the case $K = 1/2$, $(\beta/l)T = 0.1$, there is no region of stability added to the circular region because linear asymptotic stability is not found in this case. This can be seen from inequality 13 since $0.1 < 3/\sqrt{18}$. The parameters in Fig. 2 are given in terms of the coupling coefficient α , which was previously assumed to be equal to β .

The actual effect of the delay on some typical trajectories is shown in Fig. 3 for $(\alpha/l)T = (\beta/l)T = 0.1$. It is seen that the upper two solutions exhibit the characteristics of a system that is becoming less stable. There is increased overshoot, and the path lengths are generally longer. The lower trajectory, however, actually becomes stable in the presence of delay.

Fig. 2 - Regions of stability with delay.

Fig. 3 - Effect of delay on trajectories.

CONCLUSIONS

The Second Method of Liapunov can be applied directly to the problem of the asymptotic stability of coupled-core reactors. In particular, the time-delay effect can be accounted for without the use of approximation.

The term *approximation* is used here to describe a process distinct from the estimation procedure used in the previous section. If the delayed term itself is approximated, it is generally not known whether the results are conservative or falsely optimistic. The approximations used in the method outlined in this paper, however, are restrictions enforced upon the system within the framework of the Second Method. As a result, the stability conclusions are always exact or conservative.

It is imperative that the results be discussed in terms of practical systems. It was mentioned in the previous section that the basic approach is obviously restrictive for second and higher order systems. Whether this leads to overly conservative results is an important point.

The assumption was that the negative intrinsic reactivity and the coupling both contribute \$1 in reactivity at the operating power level. In the case of coupled ROVER cores,² the coupling is about 0.1β at a 15-ft center-to-center core separation. If α varies approximately with the inverse separation distance squared, then a value of α equal to β is conceivable for smaller separation distances, and the assumption $k_p = 1$ is reasonable for a ROVER core at operating power.⁸

The worst case in Fig. 2 is the inner circle. If $\beta/l = 100$, then $T_{\max} = 1$ msec, and the power level can be perturbed up to 75%. This

figure is quite generous for a reactor at the high power levels achieved in rocket cores. A more reasonable number for T when $\alpha = \beta$, if T varies directly with separation, is about 0.01 msec, or one-tenth of that assumed. This case is represented by the second region in Fig. 2. The results are therefore quite good, and there appear to be no real difficulties involved either in undertaking the stability analysis or in proving asymptotic stability for systems described by the simplified model.

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Centers and Limit Cycles in Reactor Kinetics

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ABSTRACT

A practical method for computing the amplitude and frequency of centers and limit cycles is given. This is an extension of a method of Krylov and Bogoliubov. It considers stable periodic solutions that arise from linear solutions of the form $e^{(b \pm i\omega)t}$. Two asymptotic series are derived: one for $b/\omega = 0$ (μ), $\mu \ll 1$, and the other for $b/\omega = 0$ (1). Analytic results are obtained with these series for two specific problems in reactor kinetics: the two-temperature problem with no prompt power coefficient and the xenon problem with no temperature feedback.

DEFINITIONS

Stable periodic solutions of ordinary nonlinear differential equations are either centers or limit cycles. An example of each is shown in Fig. 1. A center is a nonisolated periodic motion whose amplitude and frequency depend on initial conditions. A stable limit cycle is an isolated periodic motion whose amplitude and frequency are independent of initial conditions.

INTRODUCTION

We shall consider a system of autonomous nonlinear ordinary differential equations whose variables are expressed in terms of deviations

from some particular equilibrium point. The nonlinear terms in the equations are restricted so as not to have either logarithmic or essential singularities at this equilibrium point. There may, however, be other equilibrium points, and the behavior of the terms at these points is not restricted.

Fig. 1 - Two types of stable periodic motions.

We will assume that two solutions of the linearized equations are always of the form

$$e^{(b \pm i\omega)t}$$

In this case centers can arise only for $b = 0$, and limit cycles, for $b > 0$. Since centers occur only under very specialized conditions, $b = 0$, stable periodic motions are usually limit cycles. We specifically neglect multiple limit cycles that may arise through interactions of two separate oscillations, ω_1 and ω_2 . Such behavior is possible only in fourth-order or higher systems (Ref. 1, p. 429). Our analysis also neglects the case where limit cycles arise when $b > 0$ and $\omega = 0$, as in the second-order autonomous Van der Pol equation.²

In this paper we shall use and extend a method originally derived by Krylov and Bogoliubov³ to develop a series approximation for limit cycles when $b/\omega = 0$ (μ), $\mu \ll 1$. This same series will also give the condition for centers in the limit $b \rightarrow 0$. We shall then apply this series to the two-temperature problem without a power coefficient⁴ and to the xenon problem without temperature feedback. To predict correctly certain limit cycles that arise in the xenon problem,⁵ we must derive a different asymptotic series for the cases $b/\omega = 0$ (1).

SET OF EQUATIONS CONSIDERED

Although our methods apply to very general sets of equations, we will only specifically consider here the third-order system. These equations arise in a number of problems in point reactor kinetics.^{4,6}

$$\tau \dot{\phi}(t) = [k_0 - \alpha_1 \bar{T}_1(t) - \alpha_2 \bar{T}_2(t) - \beta_0 \phi(t)] \phi(t) \quad (1a)$$

$$h_1 [\dot{\bar{T}}_1(t) + \epsilon_1 \bar{T}_1(t)] = \phi(t) \quad (1b)$$

$$h_2 [\dot{\bar{T}}_2(t) + \epsilon_2 \bar{T}_2(t)] = \phi(t) \quad (1c)$$

where ϕ = thermal flux

$\bar{T}_1, \bar{T}_2$ = two temperature-dependent or power-dependent quantities*

α_1, α_2 = two temperature or power coefficients of reactivity* ($\alpha_n > 0$ corresponding to a negative, or stabilizing, coefficient)

τ = average neutron generation time, including delayed neutrons

k_0 = constant amount of reactivity necessary to maintain operating equilibrium at a given flux level

h_1, h_2 = heat-transfer coefficients

ϵ_1, ϵ_2 = time-delay constants that are assumed positive for the physical reason that, if the flux were shut off, the temperature should decay as $e^{-\epsilon_n t}$ rather than increase

We shall examine the full set of Eqs. 1 for only two values of β_0 : $\beta_0 = 0$ and $\beta_0 = \sigma_x$, the microscopic xenon-absorption cross section. The cases $\beta_0 = 0$ and $\beta_0 = 0$ with $\epsilon_1 = 0$ have been examined in Ref. 4, and we shall reproduce these results where necessary.

The transformation of the xenon equations to Eqs. 1 is given in Appendix A.

TRANSFORMATION TO THE OPERATING EQUILIBRIUM POINT

Equations 1 possess two equilibrium points: (1) the operating equilibrium point, with coordinates

$$\phi_o = \frac{k_0}{\alpha_1/h_1\epsilon_1 + \alpha_2/h_2\epsilon_2 + \beta_0} \quad \bar{T}_{1o} = \frac{\phi_o}{\epsilon_1 h_1} \quad \bar{T}_{2o} = \frac{\phi_o}{\epsilon_2 h_2}$$

* The quantities $\bar{T}_1, \bar{T}_2, \alpha_1$, and α_2 are in most cases linear combinations of real, physical temperatures and real, physical reactivity coefficients.

and (2) the shutdown equilibrium point, with coordinates

$$\phi_s = 0 \quad \bar{T}_{1s} = 0 \quad \bar{T}_{2s} = 0$$

Since we are interested in motion about the operating equilibrium point, we introduce the notation

$$r = \ln \left(\frac{\phi}{\phi_o} \right) \quad \bar{T}_1 = \bar{T}_{1o} (1 + T_1) \quad \bar{T}_2 = \bar{T}_{2o} (1 + T_2)$$

to normalize Eqs. 1 to the operating point:

$$\dot{r} = -\frac{A_1}{\epsilon_1} T_1 - \frac{A_2}{\epsilon_2} T_2 - \beta (e^r - 1) \quad (2a)$$

$$\dot{T}_1 = \epsilon_1 (e^r - 1 - T_1) \quad (2b)$$

$$\dot{T}_2 = \epsilon_2 (e^r - 1 - T_2) \quad (2c)$$

where

$$A_1 = \frac{a_1 \phi_o}{h_1 \tau}$$

$$A_2 = \frac{a_2 \phi_o}{h_2 \tau}$$

$$\beta = \frac{\beta_o \phi_o}{\tau}$$

The coordinates of the equilibrium points are now

1. Operating: $r_o = 0$, $T_{1o} = 0$, and $T_{2o} = 0$.
2. Shutdown: $r_s = -\infty$, $T_{1s} = -1$, and $T_{2s} = -1$.

TRANSFORMATION TO CANONICAL COORDINATES

We shall now transform Eqs. 2 to canonical coordinates,⁷ i.e., coordinates corresponding to the linear eigenvalues of the system. The linear characteristic equation corresponding to Eqs. 2 is

$$\lambda^3 + \lambda^2 (\beta + \epsilon_1 + \epsilon_2) + \lambda [\beta(\epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2 + A_1 + A_2] + (\beta \epsilon_1 \epsilon_2 + A_1 \epsilon_2 + A_2 \epsilon_1) = 0 \quad (3)$$

We shall designate the three roots of Eq. 3 as λ_n , $n = 1, 2, 3$. The canonical coordinate, x_n , corresponding to the eigenvalue, λ_n , is

$$x_n = r - \frac{A_1/\epsilon_1}{\lambda_n + \epsilon_1} T_1 - \frac{A_2/\epsilon_2}{\lambda_n + \epsilon_2} T_2$$

with

$$\dot{x}_n = \lambda_n x_n + X_n(x_m)$$

where X_n contains terms of second-order and higher in the x_m , $m = 1, 2, 3$. For example, in the linearly oscillatory region to be investigated, two of the characteristic roots are always of the form $b \pm i\omega$. After separating the real and imaginary parts of the complex roots, we find the two canonical coordinates, x and y , corresponding to these roots to be

$$x = r - (b + \epsilon_1) \bar{M}_1 T_1 - (b + \epsilon_2) \bar{M}_2 T_2 \quad (4a)$$

$$y = \omega \bar{M}_1 T_1 + \omega \bar{M}_2 T_2 \quad (4b)$$

with

$$\bar{M}_j = \frac{A_j/\epsilon_j}{(b + \epsilon_j)^2 + \omega^2} \quad (j = 1, 2)$$

Using x and y , we can replace Eqs. 2 by the set

$$\dot{x} = -\omega y + bx + b(e^r - 1 - r) \quad (5a)$$

$$\dot{y} = \omega x + by + \omega(e^r - 1 - r) \quad (5b)$$

$$\dot{r} = \delta_0(x - r) + \delta_1 y - \beta_0(e^r - 1) \quad (5c)$$

with

$$\delta_0 = 2b + \epsilon_1 + \epsilon_2 \quad (6a)$$

$$\delta_1 = \frac{1}{\omega} [b(b + \epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2 - \omega^2] \quad (6b)$$

In a sufficiently small region near the origin ($x = y = r = 0$), r can be eliminated from Eqs. 5a and 5b, up to an arbitrarily high order of approxi-

*added
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mation, by a technique given and justified by Malkin (Ref. 1, par. 41). To do this, we assume a series expansion for r in terms of x and y :

$$r = C_0 x + C_1 y + C_2 x^2 + C_3 xy + C_4 y^2 + \dots$$

The coefficients of expansion are determined by requiring that

$$\dot{r} = \frac{\partial r}{\partial x} \dot{x} + \frac{\partial r}{\partial y} \dot{y}$$

where $\dot{r}$, $\dot{x}$, and $\dot{y}$ are given by Eqs. 5. This allows the system of the three Eqs. 5 to be expressed as two equations:

$$\dot{x} = -\omega y + bx + b[f_2(x, y) + f_3(x, y) + \dots] \quad (7a)$$

$$\dot{y} = \omega x + by + \omega [f_2(x, y) + f_3(x, y) + \dots] \quad (7b)$$

with

$$f_2(x, y) = \frac{1}{2} (C_0 x + C_1 y)^2 \quad (7c)$$

$$f_3(x, y) = D_1 x^3 + D_2 x^2 y + D_3 xy^2 + D_4 y^3 \quad (7d)$$

The C_n and D_n are given in Appendix B.

Terms are listed only up to the third-order approximation in x and y . It was necessary to consider terms at least up to the third order to ascertain analytically either the existence of limit cycles or the nonexistence of centers. For good agreement with machine-calculated limit cycles (see Fig. 2), it was sometimes necessary to go to fourth order, with considerable increase in algebra. Of course, for convincing proof of the existence of centers by this approximation technique, it would be necessary, somehow, to go to infinite order.

ASYMPTOTIC SERIES FOR $\frac{b}{\omega} = 0(\mu)$, $\mu \ll 1$

We shall now attempt to construct the periodic solution to Eqs. 7a and 7b by assuming an asymptotic series with $b/\omega = 0(\mu)$, $\mu \ll 1$. For a

Fig. 2 — Top: comparison of machine-calculated limit cycles with approximate theory for the two-temperature problem with $\beta_0 = 0$. Bottom: machine-calculated flux vs. time for limit cycle.

system with a third-order characteristic equation, it is possible to show* that $b/\omega \leq 1/\sqrt{4k-1}$ where $k \geq 1$.

We introduce an arbitrary small-parameter index, μ , which we will eventually set equal to unity, and replace b with μb , f_2 with μf_2 , and f_3 with $\mu^2 f_3$. With this notation Eqs. 7 become

$$\dot{x} = -\omega y + \mu b x + \mu^2 b f_2 + \mu^3 b f_3 + \dots \quad (8a)$$

$$\dot{y} = \omega x + \mu(b y + \omega f_2) + \mu^2 \omega f_3 + \dots \quad (8b)$$

We assume a solution of the form

$$x(a, \psi) = a \cos \psi + \mu U_1(a, \psi) + \mu^2 U_2(a, \psi) + \dots \quad (9a)$$

$$y(a, \psi) = a \sin \psi + \mu V_1(a, \psi) + \mu^2 V_2(a, \psi) + \dots \quad (9b)$$

*We write the characteristic equation as

$$(\lambda^2 - 2b\lambda + \bar{C})(\lambda + \alpha) = \lambda^3 + \lambda^2(\alpha - 2b) + \lambda(\bar{C} - 2b\alpha) + \bar{C}\alpha = 0$$

where α , b , and $\bar{C}$ are all > 0 and $\omega^2 = b^2 - \bar{C}$. Taking $\alpha = 2kb$ with $k \geq 1$, we get $b/\omega \leq 1/\sqrt{4k-1}$.

where the U_n and V_n are periodic functions of ψ , and require that this solution satisfy the following conditions at $\psi = 0$:

$$\begin{aligned} x(a,0) &= a & y(a,0) &= 0 \\ U_n(a,0) &= 0 & V_n(a,0) &= 0 \end{aligned} \tag{10}$$

The boundary conditions in Eqs. 10 are identical with those used by Malkin (Ref. 1, par. 38) in constructing centers. We shall use these conditions here for limit cycles because we want the $b/\omega = 0(\mu)$ asymptotic series to approach centers as $b \rightarrow 0$.

Substituting Eqs. 9 into Eqs. 8 gives, up to terms in μ^2 ,

$$\begin{aligned} \dot{x} &= -\omega a \sin \psi + \mu(-\omega V_1 + ba \cos \psi) \\ &+ \mu^2(-\omega V_2 + bU_1 + bf_2^{(0)}) + 0(\mu^3) \\ &= -\omega a \sin \psi + \mu G_0 + \mu^2 G_1 + \dots \end{aligned} \tag{11a}$$

$$\begin{aligned} \dot{y} &= \omega a \cos \psi + \mu(\omega U_1 + ba \sin \psi + \omega f_2^{(0)}) \\ &+ \mu^2[\omega U_2 + bV_1 + \omega(f_2^{(1)} + f_3^{(0)})] + 0(\mu^3) \\ &= \omega a \cos \psi + \mu G_2 + \mu^2 G_3 + \dots \end{aligned} \tag{11b}$$

and

$$f_2 = f_2^{(0)} + \mu f_2^{(1)} + \mu^2 f_2^{(2)} + \dots$$

$$f_3 = f_3^{(0)} + \mu f_3^{(1)} + \mu^2 f_3^{(2)} + \dots$$

where

$$f_2^{(0)} = M_1 + M_2 \sin 2\psi + M_3 \cos 2\psi$$

with

$$M_1 = \frac{1}{4} a^2 (C_0^2 + C_1^2)$$

$$M_2 = \frac{1}{2} a^2 C_0 C_1 \tag{12a}$$

$$M_3 = \frac{1}{4} a^2 (C_0^2 - C_1^2)$$

and

$$f_2^{(1)} + f_3^{(0)} = F_0 + \sum_{n=1}^3 (F_n \cos n\psi + \bar{F}_n \sin n\psi) \quad (12b)$$

For our present discussion we will need only F_1 and $\bar{F}_1$, which are given explicitly in Eqs. 19.

Using the method of Krylov and Bogoliubov,³ we differentiate Eqs. 9 and assume that

$$\dot{a} = \mu \bar{A}_1(a) + \mu^2 \bar{A}_2(a) + \dots \quad (13a)$$

$$\dot{\psi} = \omega + \mu \bar{B}_1(a) + \mu^2 \bar{B}_2(a) + \dots \quad (13b)$$

to get

$$\begin{aligned} \dot{x} &= -\omega a \sin \psi + \mu \left(\bar{A}_1 \cos \psi - \bar{B}_1 a \sin \psi + \omega \frac{\partial U_1}{\partial \psi} \right) \\ &+ \mu^2 \left(\bar{A}_2 \cos \psi + \bar{A}_1 \frac{\partial U_1}{\partial a} + \omega \frac{\partial U_2}{\partial \psi} + \bar{B}_1 \frac{\partial U_1}{\partial \psi} - \bar{B}_2 a \sin \psi \right) + O(\mu^3) \\ &= -\omega a \sin \psi + \mu \bar{G}_0 + \mu^2 \bar{G}_1 + \dots \end{aligned} \quad (14a)$$

$$\begin{aligned} \dot{y} &= \omega a \cos \psi + \mu \left(\bar{A}_1 \sin \psi + \bar{B}_1 a \cos \psi + \omega \frac{\partial V_1}{\partial \psi} \right) \\ &+ \mu^2 \left(\bar{A}_2 \sin \psi + \bar{A}_1 \frac{\partial V_1}{\partial a} + \omega \frac{\partial V_2}{\partial \psi} + \bar{B}_1 \frac{\partial V_1}{\partial \psi} + \bar{B}_2 a \cos \psi \right) + O(\mu^3) \\ &= \omega a \cos \psi + \mu \bar{G}_2 + \mu^2 \bar{G}_3 + \dots \end{aligned} \quad (14b)$$

If we equate terms with equal powers of μ in Eqs. 14 and 11, for terms in μ we get

$$G_0 = \bar{G}_0 \quad G_2 = \bar{G}_2 \quad (15a)$$

and for terms in μ^2 we get

$$G_1 = \bar{G}_1 \quad G_3 = \bar{G}_3 \quad (15b)$$

One can use Eqs. 15 to derive all the necessary expansion coefficients for Eqs. 9 and 13 up to the *third-order*, i.e., μ^2 approximation. To accomplish this, we assume

$$U_1(a, \psi) = \alpha_0(a) + \sum_{n=1}^2 [\tilde{\alpha}_n(a) \cos n\psi + \bar{\alpha}_n(a) \sin n\psi] \quad (16)$$

$$V_1(a, \psi) = \tilde{\beta}_0(a) + \sum_{n=1}^2 [\beta_n(a) \cos n\psi + \bar{\beta}_n(a) \sin n\psi]$$

For Eqs. 16 to satisfy Eqs. 15a, the following conditions must hold:

$$\begin{aligned} \alpha_0 &= -M_1 & \tilde{\beta}_0 &= 0 \\ \tilde{\alpha}_1 &= M_1 - \frac{1}{3} M_3 & \beta_1 &= -\bar{\alpha}_1 \\ \bar{\alpha}_1 &= -\frac{2}{3} M_2 & \bar{\beta}_1 &= \tilde{\alpha}_1 \\ \tilde{\alpha}_2 &= \frac{1}{3} M_3 & \beta_2 &= \bar{\alpha}_1 \\ \bar{\alpha}_2 &= \frac{1}{3} M_2 & \bar{\beta}_2 &= \frac{2}{3} M_3 \end{aligned}$$

and the following must be true:

$$\bar{A}_1 = b\alpha \quad (17a)$$

$$\bar{B}_1 = 0 \quad (17b)$$

Next, we assume a similar Fourier expansion for U_2 and V_2 and find that for Eqs. 15b to be satisfied by these expansions we must have

$$\bar{A}_2 = \frac{1}{2} \omega \bar{F}_1 - b\bar{\alpha}_1 \quad (18a)$$

$$\bar{B}_2 = \frac{1}{a} \left(\frac{\omega F_1}{2} - b\beta_1 \right) \quad (18b)$$

with

$$F_1 = a^3 \left(\frac{3}{4} D_1 + \frac{1}{4} D_3 \right) + C_0^2 a \left(\tilde{\alpha}_0 + \frac{\tilde{\alpha}_2}{2} \right) + C_0 C_1 a \left(\tilde{\beta}_0 + \frac{\beta_2}{2} + \frac{\bar{\alpha}_2}{2} \right) + \frac{C_1^2 a \bar{\beta}_2}{2} \quad (19a)$$

$$\bar{F}_1 = a^3 \left(\frac{D_2}{4} + \frac{3}{4} D_4 \right) + \frac{C_0^2 a \bar{\alpha}_2}{2} + C_0 C_1 a \left(\alpha_0 + \frac{\bar{\beta}_2}{2} - \frac{\tilde{\alpha}_2}{2} \right) + C_1^2 a \left(\tilde{\beta}_0 - \frac{\beta_2}{2} \right) \quad (19b)$$

The extension of this technique to get the fourth-order approximation (fifth-order and higher approximations are too laborious to be practical in most cases) is straightforward but involves more algebra.

CENTERS

It is usually much easier to prove the nonexistence of centers than to prove their existence. If one substitutes Eqs. 17a and 18a into Eq. 13a, with $b = 0$ and $\mu = 1$, then the condition that centers exist up to the third-order approximation is that $\dot{a} = 0$, or, what is equivalent, that $\bar{F}_1 = 0$. In Appendix C we show that this condition is identical with one shown by Malkin¹ for the existence of centers. The fact that $\dot{a} = 0$ up to the third-order approximation does not in any way guarantee that higher order approximations will continue to vanish. In fact, it usually turns out that $\bar{F}_1 \neq 0$, in which case (Ref. 1, p. 138), if $\bar{F}_1 < 0$, the equilibrium point is a stable spiral, and, if $\bar{F}_1 > 0$, it is an unstable spiral.

If $\beta_0 = 0$ in Eqs. 1, $\bar{F}_1$ is negative when $b = 0$ (see Appendix C). Thus the operating equilibrium point is a stable spiral in the two-temperature problem with no prompt power coefficient when the linear system is pure oscillatory.

Applying the Routh-Hurwitz condition for undamped linear oscillations to Eq. 3 gives

$$\beta^2 \epsilon + \beta \epsilon^2 + \beta (A_1 + A_2) + \epsilon \epsilon_1 \epsilon_2 + A_1 \epsilon_1 + A_2 \epsilon_2 = 0$$

$$(\epsilon \equiv \epsilon_1 + \epsilon_2) \quad (20)$$

Using the numbers for the xenon problem as given in Appendix A shows that a flux level of $\phi_0 = 1.0 \times 10^9$ satisfies Eq. 20. With this flux value a simple numerical calculation shows that $\bar{F}_1 < 0$. Thus, for the xenon problem with no temperature feedback, the operating equilibrium point is a stable spiral when the linear system is pure oscillatory.

For the two-temperature problem with $\beta_0 = 0$ and "adiabatic cooling," Eqs. 2 reduce to⁴

$$\begin{aligned}\dot{r} &= -a_1 T_1 - a_2 T_2 \\ \dot{T}_1 &= \bar{\epsilon}_1 (e^r - 1) \\ \dot{T}_2 &= \epsilon_2 (e^r - 1 - T_2)\end{aligned}\tag{21}$$

When $a_2 = 0$, the linear approximation has pure oscillatory roots. We will now show the existence of centers for this condition.

The canonical coordinates

$$x = r - \frac{a_1 b}{b^2 + \omega^2} T_1 - \frac{a_2 (b + \epsilon_2)}{(b + \epsilon_2)^2 + \omega^2} T_2$$

and

$$y = \frac{\omega a_1}{b^2 + \omega^2} T_1 + \frac{\omega a_2}{(b + \epsilon_2)^2 + \omega^2} T_2$$

transform Eqs. 21 to

$$\begin{aligned}\dot{x} &= -\omega y \\ \dot{y} &= \omega (e^x - 1)\end{aligned}\tag{22}$$

and a third equation, which we need not consider, $\dot{r} = \dot{x}$.

Applying the method of an asymptotic series for $b/\omega = 0(\mu)$, $\mu \ll 1$, to Eqs. 22 gives, up to the fourth-order approximation:

$$\begin{aligned}\dot{a} &= 0 \\ \dot{\psi} &= \omega \left(1 - \frac{a^2}{24} - \frac{a^3}{72} \right)\end{aligned}\tag{23}$$

where $a = x(0)$; $y(0) = 0$.

It is not at all easy, however, to show that higher order approximations will continue to give $\dot{a} = 0$. It turns out that one can show the existence of centers most easily by trying to construct a Liapunov function (Ref. 1, p. 125). Following the procedure in Malkin, one gets the Liapunov function

$$v = x^2 + y^2 + \sum_{n=3}^{\infty} \frac{2x^n}{n!} \quad \text{with } \dot{v} = \frac{\partial v}{\partial x} \dot{x} + \frac{\partial v}{\partial y} \dot{y} = 0$$

The fact that $\dot{v}$ vanishes for all n as n goes to infinity is sufficient to show (Ref. 1, p. 131) that one is dealing with the case of a center. The disadvantage of this Liapunov method is that it cannot give the relation between amplitude and frequency, Eqs. 23, for it does not try to construct the periodic solution itself.

One can also reduce the system of Eqs. 21 to Eqs. 22 if $\epsilon_2 = 0$ (see Ref. 4). Thus we also have a center for this case, where the linear characteristic equation has a root with a zero real part.

LIMIT CYCLES

Substituting Eqs. 18a and 17a into Eq. 13a, we set $a = 0$ and solve for the stationary amplitude, a . Substituting this value of a and the value $\mu = 1$ into Eq. 13b gives the frequency, and substituting the two values into Eq. 9 gives the shape of the limit cycle for the third-order approximation.

An example is given in Fig. 2 for the two-temperature model with $\beta_0 = 0$. Note that it was necessary to go to the fourth-order approximation for the higher value of b/ω to get good agreement with machine calculations.

Chernick⁵ has shown that limit cycles exist for the xenon problem and has been able to reproduce their frequency for values of $(k_0/\tau)/(\lambda_1 + \lambda_x) \equiv \omega_0/\lambda < 2$ (Ref. 5, p. 239). For values of $\omega_0/\lambda \geq 2$, he was less successful analytically but did show numerically four examples of "relaxation oscillations." These are shown in Fig. 3. We shall now apply the methods of an asymptotic series for $b/\omega = 0$ (μ), $\mu \ll 1$, to these four limit cycles.

The bottom two limit cycles of Fig. 3 are reproduced in the x - y plane in Fig. 4. As b/ω gets bigger for the top two limit cycles, the negative y -coordinate gets much larger and the $x_{\max} - x_0$ asymmetry increases. Machine-computed values for the four cycles are given in Table 1.

Table 1 also contains a comparison of machine values with analytic values. The amplitude and frequency computed with the $b/\omega = 0$ (μ),

Fig. 3 - Stable flux and xenon oscillations.⁵ Prompt xenon yield, $y_x = 0.002$.

REACTOR PERIOD = $\frac{T}{K_0} = 10,000$ SEC

$$\frac{b}{\omega} = 0.1044$$

REACTOR PERIOD = $\frac{T}{K_0} = 2000$ SEC

$$\frac{b}{\omega} = 0.271$$

Fig. 4 - The x - y plane representation of the 10,000- and 2000-sec limit cycles shown in Fig. 3.

$\mu \ll 1$, asymptotic series discussed earlier are shown for both the third-order and fourth-order approximations. The machine-computed value of $r_{\max}$ was in *order-of-magnitude* agreement with the third-order approximation for a in the $b/\omega = 0(\mu)$ series. This agreement is fortuitous and probably only holds for the xenon problem and parameter ranges considered.

Table 1 – COMPUTER AND ANALYTIC VALUES FOR XENON LIMIT CYCLES

Reactor period						
τ/k_0 , sec		10,000	2000	500	100	
$\frac{k_0/\tau}{\lambda_i + \lambda_x} \equiv \frac{\omega_0}{\tau}$		2	10	40	200	
Linear Theory						
$1/\omega$, hr/cycle		55.2	29.6	17.9	10.7	
b/ω		0.1044	0.271	0.348	0.426	
Computer						
x_0		2.82	8.5	18.8	58	
$x_{\max}$		2.84	12.3	44.2	1142	
Amplitude	$b/\omega = 0(\mu)$					
	third order	a	3.01	5.75	7.1	8.15
	fourth order	a	2.58	4.61	5.5	6.2
	$b/\omega = 0(1)$					
		a	4.0	9.55	12.6	19.3
Frequency, hr/cycle	Computer	$1/\omega$	89	210	362	4000
	$b/\omega = 0(\mu)$					
	third order	$1/\omega$	77.2	95.5	113	135
	fourth order	$1/\omega$	78.2	72.1	53.4	28.1
	$b/\omega = 0(1)$					
		$1/\omega$	89	224	282	152

For the higher values of b/ω , the fourth-order approximation gives worse agreement than the third order, because the approximation $b/\omega = 0(\mu)$, $\mu \ll 1$, is no longer valid for these higher values of b/ω . In an attempt to get better agreement, we rederived the series for $b/\omega = 1$. Details of the method are given in the next section. Because the boundary conditions were changed, a no longer specifically corresponds to the value of x_0 . Nevertheless, the agreement between x_0 and a with this method is reasonably good to within an order of magnitude. The agreement with the frequency is also good except for the limit cycle with a reactor period of 100 sec. Analytic values are given only for the third-order approximation. These values could be improved if one went to the fifth-order approximation (the fourth-order correction is zero), although we have not done this. For example, for the 100-sec limit cycle the third-order value of $1/\omega$ is proportional to $10^3/(16.367 - 15.221)$. Thus a

small correction (in the 15.221 number) will make a large difference in the value of $1/\omega$.

ASYMPTOTIC SERIES FOR $\frac{b}{\omega} = 0(1)$

We assume a solution to Eqs. 7 of the same form as Eqs. 9. The boundary conditions are changed so that in all order approximations the coefficients of the $\sin \psi$ and $\cos \psi$ terms vanish. This is identical with the boundary condition used by Krylov and Bogoliubov.³ With these conditions we can no longer fit the complete limit cycle, as in Fig. 2, but can only hope to get the frequency and an estimate of some amplitude associated with the cycle.

Substituting Eqs. 9 into Eqs. 7 gives, up to terms in μ^2 ,

$$\begin{aligned}\dot{x} &= -\omega a \sin \psi + ba \cos \psi + \mu(-\omega V_1 + bU_1 + bf_2^{(0)}) \\ &\quad + \mu^2[-\omega V_2 + bU_2 + b(f_2^{(1)} + f_3^{(0)})] + 0(\mu^3) \\ &= -\omega a \sin \psi + ba \cos \psi + \mu G_0 + \mu^2 G_1 + \dots\end{aligned}\quad (24a)$$

$$\begin{aligned}\dot{y} &= \omega a \cos \psi + ba \sin \psi + \mu(\omega U_1 + bV_1 + \omega f_2^{(0)}) \\ &\quad + \mu^2[\omega U_2 + bV_2 + \omega(f_2^{(1)} + f_3^{(0)})] + 0(\mu^3) \\ &= \omega a \cos \psi + ba \sin \psi + \mu G_2 + \mu^2 G_3 + \dots\end{aligned}\quad (24b)$$

where f_n , M_n , F_1 , and $\bar{F}_1$ are the same as in Eqs. 11, 12, and 19, respectively.

We now change the assumption for $\dot{a}$ in Eq. 13a to read

$$\dot{a} = ba + \mu \bar{A}_1 + \mu^2 \bar{A}_2 + \dots \quad (25)$$

The assumption for $\dot{\psi}$ is the same as in Eq. 13b. If we differentiate Eqs. 9 and use Eqs. 25 and 13b, we get for replacement of Eqs. 14

$$\begin{aligned}\dot{x} &= -\omega a \sin \psi + ba \cos \psi + \mu \left(ba \frac{\partial U_1}{\partial a} + \bar{A}_1 \cos \psi + \omega \frac{\partial U_1}{\partial \psi} - \bar{B}_1 a \sin \psi \right) \\ &\quad + \mu^2 \left(ba \frac{\partial U_2}{\partial a} + \bar{A}_1 \frac{\partial U_1}{\partial a} + \bar{A}_2 \cos \psi + \omega \frac{\partial U_2}{\partial \psi} + \bar{B}_1 \frac{\partial U_1}{\partial \psi} - \bar{B}_2 a \sin \psi \right) + 0(\mu^3) \\ &= -\omega a \sin \psi + ba \cos \psi + \mu \bar{G}_0 + \mu^2 \bar{G}_1 + \dots\end{aligned}\quad (26a)$$

$$\begin{aligned}
 \dot{y} &= \omega a \cos \psi + ba \sin \psi + \mu \left(ba \frac{\partial V_1}{\partial a} + \bar{A}_1 \sin \psi + \bar{B}_1 a \cos \psi + \omega \frac{\partial V_1}{\partial \psi} \right) \\
 &+ \mu^2 \left(ba \frac{\partial V_2}{\partial a} + \bar{A}_1 \frac{\partial V_1}{\partial a} + \bar{A}_2 \sin \psi + \omega \frac{\partial V_2}{\partial \psi} + \bar{B}_1 \frac{\partial V_1}{\partial \psi} + \bar{B}_2 a \cos \psi \right) + O(\mu^3) \\
 &= \omega a \cos \psi + ba \sin \psi + \mu \bar{G}_2 + \mu^2 \bar{G}_3 + \dots
 \end{aligned} \tag{26b}$$

We now proceed as above, equating terms with equal powers of μ (see Eqs. 15) to get

$$\alpha_0 = \frac{b^2 - \omega^2}{b^2 + \omega^2} M_1 \quad \tilde{\beta}_0 = \frac{2b\omega}{b^2 + \omega^2} M_1$$

$$\tilde{\alpha}_1 = 0 \quad \beta_1 = 0$$

$$\bar{\alpha}_1 = 0 \quad \bar{\beta}_1 = 0$$

The expressions for $\bar{\alpha}_2$, $\tilde{\alpha}_2$, β_2 , and $\bar{\beta}_2$ are algebraically complicated, and we shall not write them explicitly.

We can then get (see Eqs. 17)

$$\bar{A}_1 = 0$$

$$\bar{B}_1 = 0$$

and (see Eqs. 18)

$$\bar{A}_2 = \frac{\omega \bar{F}_1 + b F_1}{2} \quad \bar{B}_2 = \frac{\omega F_1 - b \bar{F}_1}{2}$$

The next order approximation to contribute to the series is the fifth order ($\bar{A}_4$ and $\bar{B}_4$) because $\bar{A}_3$ and $\bar{B}_3$ are identically equal to zero.

APPENDIX A

XENON EQUATIONS

The equations (in the notation of Ref. 5) for the xenon problem with no delayed neutrons and no temperature feedback are

$$\dot{\phi} = \left(\delta_0 / \tau - \frac{\sigma_x}{c\sigma_f\tau} X \right) \phi$$

$$\dot{X} = y_x \sigma_f \phi - \sigma_x X \phi + \lambda_i \mathfrak{J} - \lambda_x X$$

$$\dot{\mathfrak{J}} = y_i \sigma_f \phi - \lambda_i \mathfrak{J}$$

The transformations⁶

$$Z = \phi - X - \frac{\lambda_i}{\lambda_i - \lambda_x} I \qquad I = \frac{\mathfrak{J}}{\sigma_f c \tau}$$

lead to the equations

$$\dot{\phi} = \left(\frac{\delta_0}{\tau} + \sigma_x Z + \frac{\lambda_i \sigma_x}{\lambda_i - \lambda_x} I - \sigma_x \phi \right) \phi$$

$$\dot{Z} + \lambda_x Z = \frac{\phi}{h_1} \qquad \dot{I} + \lambda_i I = \frac{\phi}{h_2}$$

with

$$\frac{1}{h_1} = \frac{\delta_0}{\tau} - \frac{y_x}{c\tau} + \lambda_x - \frac{\lambda_i y_i}{(\lambda_i - \lambda_x) c \tau} \qquad \frac{1}{h_2} = \frac{y_i}{c\tau}$$

The notation changes

$$\tau \sigma_x = \beta_0 = -\alpha_1 \qquad \frac{\lambda_i \sigma_x \tau}{\lambda_i - \lambda_x} = -\alpha_2$$

$$Z = \bar{T}_1 \qquad I = \bar{T}_2$$

$$\delta_0 = k_0 \qquad \lambda_x = \epsilon_1 \qquad \lambda_i = \epsilon_2$$

then give Eqs. 1.

In numerical calculations we used

$$\tau = 0.1 \text{ sec}$$

$$y_i = 0.060$$

$$c = \frac{5}{3}$$

$$\sigma_x = 3.0 \times 10^{-18} \text{ cm}^2$$

$$\lambda_i = 2.9 \times 10^{-5} \text{ sec}^{-1}$$

$$y_x = 0.003$$

$$\lambda_x = 2.1 \times 10^{-5} \text{ sec}^{-1}$$

Our machine-computed values for the limit cycles in Fig. 3, which are given in Table 1, differ somewhat from those of Chernick⁵ because we used slightly different numerical values. This is especially true for the lower values of τ/k_0 .

APPENDIX B

COEFFICIENTS

The C_n and D_n are for use in Eqs. 7.

$$C_0 = \frac{\delta_0(b + \delta_0 + \beta) - \delta_1\omega}{(b + \delta_0 + \beta)^2 + \omega^2}$$

$$C_1 = \frac{\delta_0\omega + \delta_1(b + \delta_0 + \beta)}{(b + \delta_0 + \beta)^2 + \omega^2}$$

$$C_2 = \frac{-\omega C_3 - \bar{C} C_0^2}{\delta_0 + 2b + \beta}$$

$$C_3 = \frac{-2\bar{C}[C_0 C_1(\delta_0 + 2b + \beta) + \omega(C_0^2 - C_1^2)]}{(\delta_0 + 2b + \beta)^2 + 4\omega^2}$$

$$C_4 = \frac{\omega C_3 - \bar{C} C_1^2}{\delta_0 + 2b + \beta}$$

$$\bar{C} = \frac{bC_0 + \omega C_1 + \beta}{2}$$

$$D_1 = C_0 \left(C_2 + \frac{1}{6} C_0^2 \right)$$

$$D_2 = C_0 C_3 + C_1 C_2 + \frac{1}{2} C_0^2 C_1$$

$$D_3 = C_0 C_4 + C_1 C_3 + \frac{1}{2} C_0 C_1^2$$

$$D_4 = C_1 \left(C_4 + \frac{1}{6} C_1^2 \right)$$

APPENDIX C

CENTERS

Using the method of Malkin,¹ we attempted to construct a periodic solution for Eqs. 7, with $b = 0$, which depends continuously on an arbitrary small parameter, c . This parameter is taken as the initial disturbance in x , the initial value of y being zero. Therefore the periodic solution is assumed:

$$\begin{aligned}x &= cx_{(1)} + c^2x_{(2)} + \dots \\y &= cy_{(1)} + c^2y_{(2)} + \dots\end{aligned}\tag{C.1a}$$

with

$$\begin{aligned}x_{(1)}(0) &= 1 & x_{(2)}(0) &= x_{(3)}(0) = \dots = 0 \\y_{(1)}(0) &= y_{(2)}(0) = \dots = 0\end{aligned}\tag{C.1b}$$

The time, t , is also transformed to a new variable t' with the relation

$$t = \frac{t'}{\omega} (1 + h_1c + h_2c^2 + \dots)\tag{C.1c}$$

This particular form is chosen because, as c approaches 0, x and y approach 0, and the period of oscillation becomes $2\pi/\omega$, which is just the linear approximation.

If we now substitute Eqs. C.1 into Eqs. 7, with $b = 0$, and equate terms with like powers of c , we get pairs of equations of motion for the various $x_{(n)}$ and $y_{(n)}$, $n = 1, 2, 3, \dots$. Solving the $n = 1$ equations gives $x_{(1)} = \cos \tau$; $y_{(1)} = \sin \tau$; and $h_1 = 0$. Then, for $x_{(2)}$ and $y_{(2)}$ to be periodic, it is necessary and sufficient that the following two conditions be satisfied:

$$h_2 = -\frac{1}{2\pi} \int_0^{2\pi} f_{23} \cos \tau d\tau\tag{C.2a}$$

$$0 = \int_0^{2\pi} f_{23} \sin \tau d\tau\tag{C.2b}$$

with $f_{23} = f_2^{(c^3)} + f_3^{(c^3)}$ containing all terms in f_2 and f_3 that multiply c^3 .

Equations C.2 are derived by assuming a Fourier expansion for $x_{(2)}$ and $y_{(2)}$ and then requiring that this expansion be valid, i.e., that it satisfy the equations of motion.

It is always possible to choose h_2 to satisfy Eq. C.2a, but, for Eq. C.2b to be satisfied, it is necessary that the quantity¹ g , which is proportional to the integral in Eq. C.2b, be equal to zero.

For the two-temperature problem with no prompt power coefficient ($\beta_0 = 0$ in Eqs. 1), we can explicitly write g as

$$g = -\frac{\omega}{\epsilon_1 + \epsilon_2} c_1^4 + c_3 \left(c_0 + 2 \frac{\omega}{\epsilon_1 + \epsilon_2} c_1 \right)$$

It is possible to show that c_0 and c_1 are always greater than 0 and c_3 is always less than 0. Therefore g is always less than 0, and we cannot construct the center type periodic solution, Eq. C.1a, for this problem. This is sufficient to show that centers do not exist.

For the problem defined by Eqs. 1, with $\beta_0 = 0$, we can show by direct calculation that

$$\lim_{b \rightarrow 0} \bar{F}_1 = 4g \qquad \lim_{b \rightarrow 0} F_1 = -h_2$$

The fact that there is a minus sign in the second limit occurs because the definition of h_n in Eq. C.1c is just the inverse of that in Eq. 13, i.e.,

$$(1 + h_2 c^2 + \dots)^{-1} \simeq 1 - h_2 c^2 + \dots$$

Here we have used the fact that, in the limit $b = 0$, c and a are identical quantities.

Thus, up to the third-order approximation, the limit $b = 0$ in Eqs. 13 gives the same conditions for centers as Malkin¹ gives.

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Synthesis of an Optimal Nuclear Reactor Control System

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ABSTRACT

The optimal control problem of a nuclear reactor is formulated to include both a large number of desired output programs and dynamic behavior by the analysis of a generalized performance index, which is a measure of the deviation of the reactor power output from its desired response. The control input, which minimizes the performance index chosen, can be realized as a closed-loop feedback system. The optimal control input is found by the calculation of the control law in terms of the instantaneous reactor state. The obtained mathematical solution turns out to be sensitive to variations of the reactor parameters or to the adjustments of the controller parameters. So that the sensitivity can be avoided, the control scheme is reduced to one that is not sensitive to the matching between the controller and the reactor parameters. The optimal solution serves as a guide to such a reduction. Because of different kinds of disturbances that exist in the reactor, the output n may be different from the expected output in an undisturbed reactor, and an auxiliary controller is added to nullify the effects of the various disturbances.

NOMENCLATURE

n	Neutron density, neutrons/cm ³
c_i	Density of delayed-neutron precursors, atoms/cm ³
β_i	Partial delayed-neutron fraction

$\beta = \sum_{i=1}^6 \beta_i$	Delayed-neutron fraction
λ_i	Decay constant of c_i , sec^{-1}
l	Neutron lifetime, sec
δk	Excess reactivity ($k_{\text{eff}} - 1$)
ω_i	β_i / l , sec^{-1}
$\omega = \sum \omega_i$	β / l , sec^{-1}
$\rho = \delta k / \beta$	Reactivity, \$
$f(t)$	Desired output program
$\tau = n / \dot{n}$	Reactor period, sec
α	$1 / \tau$
$g(t)$	Constraint function imposed on $\dot{n}$
T	Duration of the control process
I	Unit matrix
Lower-case boldface symbols	Vectors
Capital boldface symbols	Matrices

INTRODUCTION

The synthesis of the optimal control system discussed in this paper is concerned with the problem of determining the input reactivity to a reactor in a form that can be realized as a closed-loop system. The output power has to follow, as close as possible, a given time function, $f(t)$, and its rate of change must be kept within reasonable bounds, determined by the demands for safe operation.

Two cases of interest are discussed. In the first case, which is applicable mainly during start-up, the rate of change, $\dot{n}$, is bounded through the allowed period, τ . In the second case, applicable in the power range, the temperature rate of change of the plant is bounded to prevent high thermal stresses, and this restriction reflects itself as a constraint $g(t)$ on $\dot{n}$; i.e., changes in the neutron density, n , should not be performed at a rate greater than $g(t)$. These two cases are referred to as

problems No. 1 and No. 2, respectively. Because of the similarity in the mathematical analysis, the two cases are discussed simultaneously.

Although a point reactor model having six delayed groups of neutrons is used to describe the reactor kinetics, the results of this analysis can be applied to the more accurate model, where the spatial distribution of the flux is considered.¹

A schematic representation of the controlled reactor is shown in Fig. 1. The design of the main controller is based on the zero-power-

Fig. 1 - Schematic representation of the controlled reactor.

reactor equations, which yield a nominal neutron density, $n^0(t)$. The internal-temperature feedback, which influences the input reactivity, is treated as a disturbance, and it is reduced by an auxiliary controller designed to nullify the transmittance $T_{\Delta(\delta k)}^{\Delta n}$.

STATEMENT OF THE PROBLEM

Consider the set of the seven differential equations (Eqs. 1), which describe the kinetics of a nuclear reactor having six delayed groups of neutrons:

$$\begin{aligned} \dot{n} &= -\frac{\beta}{l} n + \sum_{i=1}^6 \lambda_i c_i + \frac{n \delta k}{l} \\ \dot{c}_i &= \frac{\beta_i}{l} n - \lambda_i c_i \quad (i = 1, 2, \dots, 6) \end{aligned} \quad (1)$$

The state of the reactor at any time t is specified by the set of functions $n(t)$, $c_1(t)$, $c_2(t)$, ..., $c_6(t)$. For brevity the reactor state at time t is denoted by the vector $x(t)$, which has the seven components n , c_1 , c_2 , ..., c_6 that belong to a seven-dimensional space $\{X\}$. The set of the seven differential equations (Eqs. 1) involves one independent variable

(t), eight dependent variables (n , c_1 , c_2 , ..., c_6 , and δk), and one degree of freedom ($8 - 7 = 1$); hence some optimum requirement can be imposed on the reactivity program, $\delta k(t)$. The optimum control problem is stated as follows:* find a control law, i.e., a control function where $\delta k(t) = \delta k[t, \mathbf{x}(t)]$, in terms of the instantaneous reactor state, $\mathbf{x}(t)$, which minimizes the integral performance index, $S_{1,2}(t)|_{t=T}$, defined by the following equations.

For problem No. 1:

$$S_1(t) = \int_0^t \{ h^2 [n(\theta) - f(\theta)]^2 + [\tau \cdot \dot{n}(\theta) - n(\theta)]^2 \} d\theta \quad (2a)$$

For problem No. 2:

$$S_2(t) = \int_0^t \{ h^2 [n(\theta) - f(\theta)]^2 + [\dot{n}(\theta) - g(\theta)]^2 \} d\theta \quad (2b)$$

The integral of the first term is the mean square error of the neutron density $n(t)$ with respect to the desired program, $f(t)$. The second term is introduced because of the restriction put on $\dot{n}$, and h^2 is a piecewise-constant weighting function, which is chosen according to the relative significance in meeting the specification of $n(t)$ vs. the restriction on $\dot{n}$. In the intervals where $f(t)$ is greater than $n(t)$, h^2 is chosen to be as small as possible, and thus the second term in the performance index is weighted heavily and $n(t)$ increases at the proper rate. The input reactivity at any time t , in contrast to a conventional controller, depends not only on the relation between $n(t)$ and $f(t)$ at that time but also on the future behavior of $f(t)$ and $\dot{f}(t)$ and their relation to the period τ or to the allowed rate of change, $g(t)$.

In Fig. 2 there is a comparison between the response of a reactor controlled by a hypothetical conventional controller and that of a reactor controlled by an optimal controller; and the influence of the future values of $f(t)$ and $\dot{f}(t)$ on the input at time t' is illustrated. The problem of determining the input reactivity in terms of the instantaneous reactor state, $\mathbf{x}(t)$, such that Eqs. 2 are minimized is referred to as the synthesis problem, the solution of which yields a closed-loop controlled reactor (see Fig. 1).

*A more accurate formulation is given in Appendix B. That approach, however, does not solve the synthesis problem but can serve as a guide to determine the weighting function, h^2 , which appears in Eq. 2.

Fig. 2 — Comparison of the response of a conventional controller with an optimal one. Desired response is $f(t)$. (a) Desired response. (b) Conventional controller. (c) Optimal controller.

SOLUTION OF THE SYNTHESIS PROBLEM VIA THE MAXIMUM PRINCIPLE

The minimization of the integral of Eqs. 2 subject to the set of differential Eqs. 1 is essentially a problem in the calculus of variations. The classical approach of the calculus of variations, however, may cause some computational difficulties, especially with the synthesis problem. The maximum-principle approach² appears to be more suitable for the problem at hand.

The variables n and δk appear in the first equation of Eqs. 1 in the nonlinear form $n \cdot \delta k$. The mathematical analysis is highly simplified if a new function $u = (n \cdot \delta k)/l$ is introduced to linearize Eqs. 1 with respect to u . One does not look for $\delta k(t)$ directly; instead, the function $u(t)$ that minimizes Eqs. 2 is determined, and $\delta k(t)$ is obtained through multiplication of $u(t)$ by l/n .

The following notation is introduced for brevity:

$$\frac{1}{\tau} = \alpha \quad \omega_i = \frac{\beta_i}{l} \quad \omega = \frac{\beta}{l} \quad \rho = \frac{\delta k}{\beta} \quad u = \omega n \rho \quad (3)$$

By the use of the maximum principle, the function u that minimizes $S_{1,2}(T)$ necessarily maximizes,* at any time t , the Hamiltonian $H_{1,2}$ defined by³

*Generally, the maximization of H with respect to u is a necessary condition to the minimization of the integral of Eqs. 2; with the introduction of the function u , however, the set of differential Eqs. 1 is linear, and the maximization of H is also a sufficient condition.

$$H_{1,2} = p_1 \dot{n} - \sum_{j=1}^6 p_{j+1} \dot{c}_j - \dot{S}_{1,2}$$

$$H_{1,2} = -p_1 \omega n + p_1 \sum_{i=1}^6 \lambda_i c_i + u \cdot p_1 + \sum_{j=1}^6 p_{j+1} (\omega_j n - \lambda_j c_j) - \left\{ \begin{array}{l} h^2 (n-f)^2 + \left[\left(-\omega n + \sum_{j=1}^6 \lambda_j c_j + u \right) \tau - n \right]^2 \\ h^2 (n-f)^2 + \left[\left(-\omega n + \sum_{i=1}^6 \lambda_i c_i + u \right) - g \right]^2 \end{array} \right\} \quad (4)$$

where $p_1, p_2, \dots, p_7$ are seven time functions determined by the Hamilton equations:

$$\begin{aligned} \dot{p}_1 &= - \frac{\partial H_{1,2}}{\partial n} \\ \dot{p}_{j+1} &= - \frac{\partial H_{1,2}}{\partial c_j} \quad (j = 1, 2, \dots, 6) \end{aligned} \quad (5)$$

The vector $\mathbf{p}(t)$, which has seven components, satisfies the transversality condition at the terminal time, T ; i.e., $\mathbf{p}(T)$ is orthogonal to the manifold, M , to which $\mathbf{x}(T)$ belongs. If $n(T)$ is not specified, $\mathbf{x}(T)$ is not restricted, and M is equal to the whole space $\{X\}$; hence the transversality condition imposes the following conditions:

$$\mathbf{p}(T) = 0$$

i.e.,

$$p_i(T) = 0 \quad (i = 1, 2, \dots, 7) \quad (6)$$

Maximizing $H_{1,2}$ with respect to u gives the optimal function $u_{1,2}^0$ in terms of n, c_i , and p_1 .

For problem No. 1:

$$u_1^0(t) = \frac{1}{2} \alpha^2 p_1(t) + \omega n(t) - \sum_{i=1}^6 \lambda_i c_i(t) + \alpha n(t) = \omega n p^0 \quad (7a)$$

For problem No. 2:

$$u_2^0(t) = \frac{p_1(t)}{2} + \omega n(t) - \sum_{i=1}^6 \lambda_i c_i(t) + g(t) = \omega n \rho^0 \quad (7b)$$

The synthesis problem is thus reduced to the problem of determining $p_1(t)$ in terms of n , c_i , and the given program, $f(t)$.

Substituting the optimal u function as given by Eqs. 7 into the reactor-kinetics equations and into the set of the seven Hamilton Eqs. 5 results in the following 14 differential equations.

For problem No. 1:

$$\begin{aligned} \dot{n} &= \alpha n + \frac{1}{2} \alpha^2 p_1 \\ \dot{c}_i &= \omega_i n - \lambda_i c_i \quad (i = 1, 2, \dots, 6) \\ \dot{p}_1 &= -\frac{\partial H_1}{\partial n} = -\alpha p_1 - \sum_{j=1}^6 \omega_j p_{j+1} + 2h^2(n - f) \\ \dot{p}_{j+1} &= -\frac{\partial H_1}{\partial c_j} = \lambda_j p_{j+1} \quad (j = 1, 2, \dots, 6) \end{aligned} \quad (8)$$

For problem No. 2:

$$\begin{aligned} \dot{n} &= \frac{1}{2} p_1 + g(t) \\ \dot{c}_i &= \omega_i n - \lambda_i c_i \quad (i = 1, 2, \dots, 6) \\ \dot{p}_1 &= -\frac{\partial H_2}{\partial n} = -\sum_{j=1}^6 \omega_j p_{j+1} + 2h^2(n - f) \\ \dot{p}_{j+1} &= -\frac{\partial H_2}{\partial c_j} = \lambda_j p_{j+1} \quad (j = 1, 2, \dots, 6) \end{aligned} \quad (9)$$

By the use of a matrix notation, the preceding 14 equations can be written as one vector differential equation with 14 components (blank portions of the matrix stand for zeros).

$$\frac{d}{dt} \begin{bmatrix} n \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix} = \begin{bmatrix} \alpha & & & & & & & & & & & & & & & & \\ & \omega_1 & -\lambda_1 & & & & & & & & & & & & & & \\ & & \omega_2 & & -\lambda_2 & & & & & & & & & & & & \\ & & & \omega_3 & & -\lambda_3 & & & & & & & & & & & \\ & & & & \omega_4 & & -\lambda_4 & & & & & & & & & & \\ & & & & & \omega_5 & & -\lambda_5 & & & & & & & & & \\ & & & & & & \omega_6 & & -\lambda_6 & & & & & & & & \\ & & & & & & & -\alpha & -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 & -\omega_5 & -\omega_6 & & & \\ & & & & & & & & \lambda_1 & & & & & & & & \\ & & & & & & & & & \lambda_2 & & & & & & & \\ & & & & & & & & & & \lambda_3 & & & & & & \\ & & & & & & & & & & & \lambda_4 & & & & & \\ & & & & & & & & & & & & \lambda_5 & & & & \\ & & & & & & & & & & & & & \lambda_6 & & & \end{bmatrix} \begin{bmatrix} n \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix} - 2h^2 f \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8'')$$

For problem No. 2:

$$\frac{d}{dt} \begin{bmatrix} n \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix} = \begin{bmatrix} 0 & & & & & & & & & & & & & & \\ & \omega_1 & -\lambda_1 & & & & & & & & & & & & \\ & & \omega_2 & & -\lambda_2 & & & & & & & & & & \\ & & & \omega_3 & & -\lambda_3 & & & & & & & & & \\ & & & & \omega_4 & & -\lambda_4 & & & & & & & & \\ & & & & & \omega_5 & & -\lambda_5 & & & & & & & \\ & & & & & & \omega_6 & & -\lambda_6 & & & & & & \\ & & & & & & & 0 & -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 & -\omega_5 & -\omega_6 & \\ & & & & & & & & \lambda_1 & & & & & & \\ & & & & & & & & & \lambda_2 & & & & & \\ & & & & & & & & & & \lambda_3 & & & & \\ & & & & & & & & & & & \lambda_4 & & & \\ & & & & & & & & & & & & \lambda_5 & & \\ & & & & & & & & & & & & & \lambda_6 & \end{bmatrix} \begin{bmatrix} n \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{bmatrix} + \begin{bmatrix} g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2h^2 f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (9'')$$

When the following vectors are defined as

$$r = \begin{bmatrix} -2h^2 f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} x \\ \dots \\ p \end{bmatrix} \quad m_{1,2} = \begin{bmatrix} q_{1,2} \\ r \end{bmatrix} \quad (10)$$

and the matrix notation is used, Eqs. 8' and 9' can be rewritten as

$$\dot{\mathbf{z}} = \mathbf{A}_{1,2} \cdot \mathbf{z} + \mathbf{m}_{1,2} \quad (11)$$

where $\mathbf{A}_{1,2}$ stands for either one of the 14×14 matrices appearing in Eq. 8' or 9'. From the theory of linear differential equations, it is known that the general solution of Eq. 11 is given by³

$$\mathbf{z}(t) = \Phi_{1,2}(t, t') \cdot \mathbf{z}(t') + \int_{t'}^t \Phi_{1,2}(t, \theta) \cdot \mathbf{m}_{1,2}(\theta) \cdot d\theta \quad (12)$$

where $\Phi_{1,2}(t, t')$, which is computed in Appendix A, is the 14×14 transition matrix of the free system in Eq. 11 and $\mathbf{z}(t')$ is the value of the $\mathbf{z}$ vector at an arbitrary time t' .

If the vectors $\mathbf{z}$ and $\mathbf{m}_{1,2}$ are subdivided into $\mathbf{x}$, $\mathbf{p}$, $\mathbf{q}_{1,2}$, and $\mathbf{r}$, as defined in Eq. 10, and if the 14×14 transition matrix $\Phi_{1,2}$ is subdivided into four 7×7 submatrices, the general solution of Eq. 12 can be rewritten as

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t, t') & \Phi_{12}(t, t') \\ \Phi_{21}(t, t') & \Phi_{22}(t, t') \end{bmatrix}_{1,2} \begin{bmatrix} \mathbf{x}(t') \\ \mathbf{p}(t') \end{bmatrix} + \int_{t'}^t \begin{bmatrix} \Phi_{11}(t, \theta) & \Phi_{12}(t, \theta) \\ \Phi_{21}(t, \theta) & \Phi_{22}(t, \theta) \end{bmatrix}_{1,2} \cdot \begin{bmatrix} \mathbf{q}_{1,2}(\theta) \\ \mathbf{r}(\theta) \end{bmatrix} d\theta \quad (13)$$

where Φ_{ij} ($i, j = 1, 2$) are 7×7 submatrices of $\Phi_{14 \times 14}$. Equation 13 determines the vector $\mathbf{p}(t)$, which is

$$\begin{aligned} \mathbf{p}(t) &= [\Phi_{21}(t, t')]_{1,2} \cdot \mathbf{x}(t') + [\Phi_{22}(t, t')]_{1,2} \cdot \mathbf{p}(t') \\ &+ \int_{t'}^t \{ [\Phi_{21}(t, \theta) \cdot \mathbf{q}(\theta)]_{1,2} + [\Phi_{22}(t, \theta)]_{1,2} \cdot \mathbf{r}(\theta) \} d\theta \end{aligned} \quad (14)$$

The boundary conditions of Eqs. 6 on $\mathbf{p}(T)$ impose $\mathbf{p}(T) = 0$, from which $\mathbf{p}(t')$ for any t' is given by the following equations.

For problem No. 1 (when $\mathbf{q}_1 = 0$):

$$\begin{aligned} \mathbf{p}(t') &= -[\Phi_{22}^{-1}(T, t') \cdot \Phi_{21}(T, t')]_1 \cdot \mathbf{x}(t') \\ &- \int_{t'}^T [\Phi_{22}^{-1}(T, t') \cdot \Phi_{22}(T, \theta)]_1 \cdot \mathbf{r}(\theta) d\theta \end{aligned} \quad (15)$$

For problem No. 2:

$$\begin{aligned} p(t') = & - \left[\Phi_{22}^{-1}(T, t') \cdot \Phi_{21}(T, t') \right]_2 \cdot x(t') - \int_{t'}^T \left[\Phi_{22}^{-1}(T, t') \right]_2 \\ & \cdot \left[\Phi_{21}(T, \theta) \cdot q_2(\theta) + \Phi_{22}(T, \theta) \cdot r(\theta) \right]_2 d\theta \end{aligned} \quad (16)$$

The various matrices appearing in Eqs. 15 and 16, which have been computed in Appendix A, are functions of $T - t'$ or $T - \theta$ and are summarized here.

For problem No. 1:

$$\begin{aligned} \Phi_{21} &= \begin{bmatrix} G_1 \\ \\ \\ \\ \\ \\ \end{bmatrix}_{7 \times 7} \quad (\text{all other elements are zero}) \\ \Phi_{22} &= \begin{bmatrix} F_1 & f_1^1 & f_1^2 & f_1^3 & f_1^4 & f_1^5 & f_1^6 \\ & e^{\lambda_1 t} & & & & & \\ & & e^{\lambda_2 t} & & & & \\ & & & e^{\lambda_3 t} & & & \\ & & & & e^{\lambda_4 t} & & \\ & & & & & e^{\lambda_5 t} & \\ & & & & & & e^{\lambda_6 t} \end{bmatrix} \\ \Phi_{22}^{-1} &= \begin{bmatrix} 1/F_1 & g_1^1 & g_1^2 & g_1^3 & g_1^4 & g_1^5 & g_1^6 \\ & e^{-\lambda_1 t} & & & & & \\ & & e^{-\lambda_2 t} & & & & \\ & & & e^{-\lambda_3 t} & & & \\ & & & & e^{-\lambda_4 t} & & \\ & & & & & e^{-\lambda_5 t} & \\ & & & & & & e^{-\lambda_6 t} \end{bmatrix} \end{aligned} \quad (17)$$

where

$$G_1(t) = \frac{2h^2}{\alpha\sqrt{h^2+1}} \sinh(\sqrt{h^2+1} \alpha t)$$

$$F_1(t) = \cosh(\sqrt{h^2+1} \alpha t) - \frac{1}{\sqrt{h^2+1}} \sinh(\sqrt{h^2+1} \alpha t)$$

and

$$2h^2 F_1 = \dot{G}_1 - \alpha G_1 \quad (18)$$

For problem No. 2:

$$\Phi_{21} = \begin{bmatrix} G_2 \\ \\ \\ \\ \\ \\ \end{bmatrix}_{7 \times 7} \quad (\text{all other elements are zero})$$

$$\Phi_{22} = \begin{bmatrix} F_2 & f_2^1 & f_2^2 & f_2^3 & f_2^4 & f_2^5 & f_2^6 \\ & e^{\lambda_1 t} & & & & & \\ & & e^{\lambda_2 t} & & & & \\ & & & e^{\lambda_3 t} & & & \\ & & & & e^{\lambda_4 t} & & \\ & & & & & e^{\lambda_5 t} & \\ & & & & & & e^{\lambda_6 t} \end{bmatrix} \quad (19)$$

$$\Phi_{22}^{-1} = \begin{bmatrix} 1/F_2 & g_2^1 & g_2^2 & g_2^3 & g_2^4 & g_2^5 & g_2^6 \\ & e^{-\lambda_1 t} & & & & & \\ & & e^{-\lambda_2 t} & & & & \\ & & & e^{-\lambda_3 t} & & & \\ & & & & e^{-\lambda_4 t} & & \\ & & & & & e^{-\lambda_5 t} & \\ & & & & & & e^{-\lambda_6 t} \end{bmatrix}$$

where

$$G_2(t) = 2h \sinh(ht)$$

$$F_2(t) = \cosh(ht)$$

and

$$2h^2 F_2 = \dot{G}_2 \quad (20)$$

The functions f_j^i and g_j^i are not cited here explicitly because they do not influence the analysis to follow.

Since the optimal u function as given by Eqs. 7 depends only on p_1 , it is enough to find the first component of p as given by Eq. 15 or 16.

Substituting the matrices Φ_{21} , Φ_{22} , and Φ_{22}^{-1} , as given by Eq. 17, and the r vector, as defined in Eqs. 10, into Eq. 15 results in the first component of p for problem No. 1:

$$p_1^{(1)}(t') = -\frac{G_1(T, t')}{F_1(T, t')} n(t') + \frac{2h^2}{F_1(T, t')} \int_{t'}^T F_1(T - \theta) \cdot f(\theta) d\theta$$

Employing the relation between $2h^2 F_1$ and G_1 as given by Eq. 18 and integrating by parts yields

$$\begin{aligned} \frac{1}{2} \alpha^2 p_1^{(1)}(t') &= \frac{1}{2} \alpha^2 \frac{G_1(T - t')}{F_1(T - t')} [f(t') - n(t')] + \frac{1}{2} \alpha^2 \frac{G_1(T - t')}{F_1(T - t')} \\ &\quad \cdot \int_{t'}^T \frac{G_1(T - \theta)}{G_1(T - t')} [\dot{f}(\theta) - \alpha f(\theta)] d\theta \end{aligned} \quad (21)$$

In a similar way, substituting the various matrices of Eqs. 19 and the vectors r and q_2 , as defined in Eqs. 10, into Eq. 16 results in the first component of p for problem No. 2:

$$\begin{aligned} p_1^{(2)}(t') &= -\frac{G_2(T, t')}{F_2(T, t')} n(t') \\ &\quad + \frac{1}{F_2(T, t')} \int_{t'}^T [2h^2 F_2(T - \theta) f(\theta) - G_2(T - \theta) g(\theta)] d\theta \end{aligned}$$

Employing the relation between $2h^2 F_2$ and G_2 as given by Eq. 20 and integrating by parts yields

$$\begin{aligned} \frac{1}{2} p_1^{(2)}(t') &= \frac{1}{2} \frac{G_2(T-t')}{F_2(T-t')} [f(t') - n(t')] \\ &\quad + \frac{1}{2} \frac{G_2(T-t')}{F_2(T-t')} \int_{t'}^T \frac{G_2(T-\theta)}{G_2(T-t')} [\dot{f}(\theta) - g(\theta)] d\theta \quad (22) \end{aligned}$$

Defining the controller gain $K_{1,2}(T, t')$ as

$$K_1(T, t') = \frac{1}{2} \alpha^2 \frac{G_1(T, t')}{F_1(T, t')} = \frac{\alpha \cdot h^2}{\sqrt{h^2 + 1} \coth [\sqrt{h^2 + 1} \alpha (T - t')] - 1} \quad (23)$$

$$K_2(T, t') = \frac{1}{2} \frac{G_2(T, t')}{F_2(T, t')} = h \tanh [h(T - t')] \quad (24)$$

and substituting $p_1(t')$, as given by Eqs. 21 and 22, in the control law of Eqs. 7 gives

$$\begin{aligned} \rho_{1,2}^0(t') &= \frac{1}{\omega n(t')} [f(t') + \hat{f}_{1,2}(t') - n(t')] \cdot K_{1,2}(T - t') \\ &\quad + \frac{1}{\omega n(t')} \left[\omega n(t') - \sum_{i=1}^6 \lambda_i c_i(t') + \frac{\alpha n(t')}{g(t')} \right] \quad (25) \end{aligned}$$

where $\hat{f}_{1,2}(t')$, which depend on the future behavior of $f(t)$ and on its relation to the restrictions put on $\dot{n}$, are given by

$$\begin{aligned} \hat{f}_1(t') &= \int_{t'}^T \frac{G_1(T-\theta)}{G_1(T-t')} \left[\frac{d}{d\theta} f(\theta) - \alpha f(\theta) \right] d\theta \\ \hat{f}_2(t') &= \int_{t'}^T \frac{G_2(T-\theta)}{G_2(T-t')} \left[\frac{d}{d\theta} f(\theta) - g(\theta) \right] d\theta \quad (26) \end{aligned}$$

Since t' is arbitrary, Eq. 25 is the desired control law and can be realized as a closed-loop system (see Fig. 3). The control law of Eq. 25, which is composed of two parts, can be interpreted in the following way. The first part represents the program controller, which forces the neutron density to follow the function $f(t) + \hat{f}_{1,2}(t)$; the other part is the period $1/\alpha$, or the derivative controller, which causes $\dot{n}$ to follow $\alpha n(t)$

Fig. 3 – Schematic realization of the control law of Eq. 25.

and $g(t)$ for problems No. 1 and No. 2, respectively. The relative weight of the two controllers is determined by the value of the gain $K_{1,2}(T, t)$, which in turn depends on the weighting function h . (See Fig. 4).

An outline of the procedure of choosing the weighting function h is given in Appendix B, where the optimal control problem is reduced to the problem of Mayer in the calculus of variations. Unfortunately the approach does not solve the synthesis problem. It does, however, provide valuable information on the structure of the optimal solution from which one is able to determine the weighting function h^2 for the control period $(0, T)$.

The control law of Eq. 25 involves density of the precursors, c_i , which is not readily available and must be estimated through the measurements performed on n .

Fig. 4 – Dependence of the program controller gain on the weighting factor h and time t .

From the reactor-kinetics equations, it is easy to show that the variable $y_i = \lambda_i c_i / \omega_i$ is related to the neutron density, n , through the differential equation

$$\dot{y}_i + \lambda_i y_i = \lambda_i n \quad (27)$$

Hence, if the control process is started while the reactor is in equilibrium, $y_i(t)$ can be obtained as the output of a linear system having the transfer function $\lambda_i / (s + \lambda_i)$, and the variable $(\omega_i / \omega) (n - y_i) = (\omega_i / \omega) [n - (\lambda_i c_i / \omega_i)]$ is obtained as the output of a linear system having the transfer function

$$\frac{\omega_i}{\omega} \times \frac{s}{s + \lambda_i} \quad (28)$$

The transfer function of Eq. 27 can be realized as a simple R-C network (see Fig. 3).

The expression $1/\omega (\omega n - \sum_{i=1}^6 \lambda_i c_i)$, which appears in Eq. 25, can be rewritten as $\sum_{i=1}^6 \omega_i / \omega [n - (\lambda_i c_i / \omega_i)]$ and can be obtained by summing the outputs of the corresponding networks as given by Eq. 28 (see Fig. 3).

REDUCTIONS OF THE CONTROL SCHEME

The control law of Eq. 25 and the control scheme of Fig. 3 seem to be sensitive to a mismatch between the reactor parameters ω_i , λ_i and the corresponding parameters, ω_i' , λ_i' , as built in the inner loop of the control scheme. In this discussion the operation of the inner loop is analyzed, and another scheme, which is not sensitive to the parameter adjustments, is proposed.

The term $G(s)$ is used as the reactor transmittance from u to n . The pole-zero configuration of $G(s)$, as well as that of the inner-loop feedback network, $H(s)$, is shown in Fig. 5. If each of the λ_i' and ω_i' in the feedback network is exactly equal to the corresponding reactor parameters, the resulting positively closed system has a transmittance of $1/s$. This can be easily verified by substituting into the first of the reactor equation that part of the control law of Eq. 25 (the term in the second brackets) which is realized by the inner loop.

$$\dot{n} = -\omega n + \sum_{i=1}^6 \lambda_i c_i + u$$

Fig. 5 - Pole-zero configuration of G , H , and GH and of the positive-feedback inner loop for perfect matching of the feedback network, $H(s)$.

The part of u that is realized by the inner feedback network is

$$u' = \omega n - \sum \lambda_i c_i$$

Hence the transmittance $T_{a_n;g}^n$ of the inner positively closed loop is $1/s$. As was mentioned previously, this transmittance of $1/s$ exists only for the case of perfect matching between $H(s)$ and $G(s)$. In case of imperfect matching, the resulting system might be unstable (see root locus of $G \cdot H$ in Fig. 5) and would not yield the desired transmittance of $1/s$. The transmittance of $1/s$ can be obtained to a very good approximation if a derivative negative feedback (with a possible filtering device to reduce noise effects) is utilized and if a large gain, K_0 , is realized in the forward branch. The net result of such a system is illustrated in Figs. 6 and 7, where the pole-zero configuration and the resulting inner loop are shown. It is worthwhile to mention that in this modified scheme the achieved transmittance of $1/s$ is insensitive to the exact value of K_0 as long as $K_0 \gg 1$. Hence the exact reactivity caused by the rod is not vital to the performance of the system since it affects only the effective gain of the forward branch. The program controller gain, $K_{1,2}(T,t)$, is essentially a constant for large values of h and approaches zero as h approaches zero. In Appendix B it is shown that these are the values given to h at various time intervals during the control period; hence the time-varying gain, $K_{1,2}(T,t)$, can be replaced by a constant gain, K^0 , which is switched between zero and as large a gain as possible for the time intervals when h approaches zero and h approaches infinity, respectively. From the definitions of $\hat{f}_{1,2}(t)$ in Eqs. 26, it is easy to

Fig. 6 – Pole-zero configuration of the reduced inner loop. For $K_0 \gg 1$, the effect of the dipoles is negligible.

Fig. 7 – The modified inner loop for the control scheme as given in Fig. 3.

show that, during the time intervals when h approaches infinity, $\hat{f}_{1,2}(t)$ equals zero, and, since these are the only time intervals when the input at terminal (a) (see Fig. 3) affects the output, n , the function $f + \hat{f}_{1,2}$ can be replaced by $f(t)$.

During the time intervals when h equals zero, the output n is controlled through the input at (b) (see Fig. 3), and $\dot{n}$ follows an or g . In Appendix B it is shown that this control mode exists only when h equals zero, and the input (b) is zero when h approaches infinity. Combining all these arguments leads to the reduced control scheme of Fig. 8.

AUXILIARY CONTROLLER

The output of the controlled reactor will follow, theoretically, a known nominal pattern, $n^0(t)$; actually, because of different kinds of

Fig. 8 - Schematic realization of the modified control scheme.

disturbances, the output $n(t)$ might differ from $n^0(t)$ by $\Delta n(t)$. The purpose of the auxiliary controller is to nullify this difference. As a first step, the differential equation relating the disturbing input, $\Delta \rho$, to the perturbation, Δn , is derived. The perturbed motion, $n(t)$, is written as $n^0(t) + \Delta n(t)$. When $n^0(t) + \Delta n(t)$ and the control law of Eq. 25 are substituted into the reactor equation, the results are

$$\dot{n}^0 + \Delta \dot{n} = -\omega n + \sum_{i=1}^6 \lambda_i c_i + (f + \hat{f} - n)k_{1,2} + \omega n - \sum_{i=1}^6 \lambda_i c_i + \left\{ \begin{array}{l} \alpha(n^0 + \Delta n) \\ g \end{array} \right\} + \Delta \rho \cdot \omega n \quad (29)$$

Since $\dot{n}^0$ satisfies the unperturbed equation, Eq. 29 is reduced to

$$\Delta \dot{n} = -\Delta n \cdot K_{1,2} + \left\{ \begin{array}{l} \alpha \cdot \Delta n \\ 0 \end{array} \right\} + \Delta \rho \cdot \omega n$$

$$\Delta \dot{n} \simeq -\Delta n \cdot K_{1,2} + \left\{ \begin{array}{l} \alpha \Delta n \\ 0 \end{array} \right\} + \Delta \rho \cdot \omega n^0 \quad (30)$$

where the term $\Delta \rho \cdot \Delta n \ll \Delta \rho \cdot n^0$ is ignored.

As was mentioned in the preceding section, $K_{1,2}$ is either zero or $K_{\max}^0$, and, in the corresponding time intervals, the input at (b) (see Fig. 3) is αn (g for problem No. 2) or zero. The resulting system relating $\Delta \rho$ to Δn for the different time intervals is shown in Fig. 9.

A PID (Proportional + Integral + Derivative) feedback network can be utilized to reduce drastically the transmittance $T_{\Delta \rho}^{\Delta n}$ (see Fig. 10).

Fig. 9 - Block-diagram representation of the transmittance $T_{\Delta \rho}^{\Delta n}$.

Fig. 10 - Schematic realization of the auxiliary controller.

Equation 30, when compensated by a PID negative feedback, becomes

$$\Delta \rho - D \Delta \dot{n} - P \Delta n - I \int_0^t \Delta n \, dt = \begin{cases} \frac{\Delta \dot{n}}{\omega n^0} + \frac{K_{\max}^0}{\omega n^0} \Delta n & (h \rightarrow \infty) \\ \frac{\Delta \dot{n}}{\omega n^0} - \frac{\alpha}{\omega n^0} \Delta n & [h = 0 \text{ (Problem No. 1)}] \\ \frac{\Delta \dot{n}}{\omega n^0} & [h = 0 \text{ (Problem No. 2)}] \end{cases}$$

If $D \gg (1/\omega n^0)$ $P \gg (K_{\max}^0/\omega n^0)$, all these equations reduce to

$$D \cdot \Delta \ddot{n} + P \Delta \dot{n} + I \Delta n = \Delta \dot{\rho}$$

or in operational form these equations reduce to

$$\Delta n(s) = \frac{\Delta \rho}{D} \cdot \frac{s}{s^2 + \frac{P}{D}s + \frac{I}{D}} \quad (31)$$

Thus, even for a steady ramp disturbance of $\Delta \rho = \gamma \cdot t$, Δn approaches γ/I , and this deviation can be made negligible when a large I coefficient is chosen. For a step disturbance, Δn approaches zero.

APPENDIX A

DETERMINATION OF THE TRANSITION MATRIX $\Phi(t, t')$

If a free vector differential equation

$$\dot{\mathbf{z}}(t) = \mathbf{A} \cdot \mathbf{z}(t) \quad (\text{A. 1})$$

is given, the corresponding transition matrix, $\Phi(t', t'')$, associated with Eq. A. 1 satisfies the matrix differential equation

$$\frac{d}{dt'} \Phi(t', t'') = \mathbf{A} \cdot \Phi(t', t'') \quad (\text{A. 2})$$

and

$$\Phi(t', t'') = 1 \quad (\text{A. 3})$$

If $\mathbf{A}$ is a constant matrix, the system of Eq. A. 1 is stationary, and $\Phi(t', t'')$ is a function of t , which is the difference when t'' is subtracted from t' ; with this information $\Phi(t)$ can be determined from

$$\dot{\Phi}(t) = \mathbf{A} \cdot \Phi(t) \quad (\text{A. 4})$$

Taking the Laplace transform of Eq. A. 4 gives

$$\Phi(s) = (s\mathbf{1} - \mathbf{A})^{-1} \cdot \Phi(0)$$

Since $\Phi(0) = \mathbf{1}$, $\Phi(t)$ is given by

$$\Phi(t) = L^{-1} (s\mathbf{1} - \mathbf{A})^{-1} \quad (\text{A. 5})$$

For problem No. 1 the four submatrices of $(s\mathbf{I} - \mathbf{A}_1)^{-1}$ are

$$\Phi_{11}(s) = \begin{bmatrix} \frac{s+a}{s^2 - a^2(h^2 + 1)} & & & & & \\ \phi_1 & \frac{1}{s + \lambda_1} & & & & \\ & \phi_2 & \frac{1}{s + \lambda_2} & & & \\ & & \phi_3 & \frac{1}{s + \lambda_3} & & \\ & & & \phi_4 & \frac{1}{s + \lambda_4} & \\ & & & & \phi_5 & \frac{1}{s + \lambda_5} \\ & & & & & \phi_6 & \frac{1}{s + \lambda_6} \end{bmatrix} \quad \left\{ \phi_j = \frac{\omega_j(s+a)}{[s^2 - a^2(h^2 + 1)](s + \lambda_j)} \right\}$$

($j = 1, 2, \dots, 6$)

$$\Phi_{12}(s) = \frac{1/2 a^2}{s^2 - a^2(h^2 + 1)} \begin{bmatrix} 1 & \frac{-\omega_1}{s - \lambda_1} & \frac{-\omega_2}{s - \lambda_2} & \frac{-\omega_3}{s - \lambda_3} & \frac{-\omega_4}{s - \lambda_4} & \frac{-\omega_5}{s - \lambda_5} & \frac{-\omega_6}{s - \lambda_6} \\ \frac{\omega_1}{s + \lambda_1} & & & & & & \\ \frac{\omega_2}{s + \lambda_2} & & & & & & \\ \frac{\omega_3}{s + \lambda_3} & & \left[\frac{-\omega_i \omega_j}{(s - \lambda_i)(s + \lambda_j)} \right]_{6 \times 6} & & & & \\ \frac{\omega_4}{s + \lambda_4} & & & & & & \\ \frac{\omega_5}{s + \lambda_5} & & & & & & \\ \frac{\omega_6}{s + \lambda_6} & & & & & & \end{bmatrix}$$

$$\Phi_{21}(s) = \begin{bmatrix} \frac{2h^2}{s^2 - a^2(h^2 + 1)} \end{bmatrix}_{7 \times 7} \quad (\text{all other elements are zero})$$

$$\Phi_{22}(s) = \begin{bmatrix} \frac{s-a}{s^2 - a^2(h^2 + 1)} & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ & \frac{1}{s - \lambda_1} & & & & & \\ & & \frac{1}{s - \lambda_2} & & & & \\ & & & \frac{1}{s - \lambda_3} & & & \\ & & & & \frac{1}{s - \lambda_4} & & \\ & & & & & \frac{1}{s - \lambda_5} & \\ & & & & & & \frac{1}{s - \lambda_6} \end{bmatrix} \quad \left\{ \psi_j = \frac{-\omega_j(s-a)}{[s^2 - a^2(h^2 + 1)](s - \lambda_j)} \right\}$$

(j = 1, 2, ..., 6)

Taking the inverse Laplace transform of these matrices yields the four submatrices of the transition matrix, $\Phi(t)$, for problem No. 1.

For problem No. 2 the four submatrices of $(s\mathbf{1} - \mathbf{A}_2)^{-1}$ are

$$\Phi_{11}(s) = \begin{bmatrix} \frac{s}{s^2 - h^2} & & & & & \\ \phi_1 & \frac{1}{s + \lambda_1} & & & & \\ & \frac{1}{s + \lambda_2} & & & & \\ \phi_3 & & \frac{1}{s + \lambda_3} & & & \\ & & \frac{1}{s + \lambda_4} & & & \\ \phi_5 & & & \frac{1}{s + \lambda_5} & & \\ & & & & \frac{1}{s + \lambda_6} & \end{bmatrix} \quad \left[\phi_j = \frac{s\omega_j}{(s^2 - h^2)(s + \lambda_j)} \right]$$

(j = 1, 2, ..., 6)

$$\Phi_{12}(s) = \frac{1/2}{s^2 - h^2} \begin{bmatrix} 1 & \frac{-\omega_1}{s - \lambda_1} & \frac{-\omega_2}{s - \lambda_2} & \frac{-\omega_3}{s - \lambda_3} & \frac{-\omega_4}{s - \lambda_4} & \frac{-\omega_5}{s - \lambda_5} & \frac{-\omega_6}{s - \lambda_6} \\ \frac{\omega_1}{s + \lambda_1} & & & & & & \\ \frac{\omega_2}{s + \lambda_2} & & & & & & \\ \frac{\omega_3}{s + \lambda_3} & & \left[\frac{-\omega_i \omega_j}{(s - \lambda_i)(s + \lambda_j)} \right]_{6 \times 6} & & & & \\ \frac{\omega_4}{s + \lambda_4} & & & & & & \\ \frac{\omega_5}{s + \lambda_5} & & & & & & \\ \frac{\omega_6}{s + \lambda_6} & & & & & & \end{bmatrix}$$

$$\Phi_{21}(s) = \begin{bmatrix} \frac{2h^2}{s^2 - h^2} \end{bmatrix}_{7 \times 7} \quad (\text{all other elements are zero})$$

$$\Phi_{22}(s) = \begin{bmatrix} \frac{s}{s^2 - h^2} & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ & \frac{1}{s - \lambda_1} & & & & & \\ & & \frac{1}{s - \lambda_2} & & & & \\ & & & \frac{1}{s - \lambda_3} & & & \\ & & & & \frac{1}{s - \lambda_4} & & \\ & & & & & \frac{1}{s - \lambda_5} & \\ & & & & & & \frac{1}{s - \lambda_6} \end{bmatrix} \quad \left[\psi_j = \frac{-s\omega_j}{(s^2 - h^2)(s - \lambda_j)} \right]$$

($j = 1, 2, \dots, 6$)

Taking the inverse Laplace transform of these matrices results in the four submatrices of the transition matrix, $\Phi(t)$, for problem No. 2.

APPENDIX B

OPTIMAL CONTROL AS A PROBLEM OF MAYER IN THE CALCULUS OF VARIATIONS

The optimal control problem as formulated in the section on the statement of the problem is actually an approximation to the more accurate formulation of the control problem that follows.

When the reactor-kinetics Eqs. 1 are considered, it is necessary to determine the input reactivity such that

$$S = \int_0^T [n(t) - f(t)]^2 dt \quad (\text{B.1})$$

is minimized, subject to the condition

$$\dot{n}(t) \leq \alpha n(t) \quad (\text{B.2a})$$

or

$$\dot{n}(t) \leq g(t) \quad (\text{B.2b})$$

So that this problem can be reduced to the problem of Mayer, a new variable γ is introduced by⁴

$$\dot{n} = \alpha n - \gamma^2 \quad (\text{B.3a})$$

or

$$\dot{n} = g(t) - \gamma^2 \quad (\text{B.3b})$$

The reactor-kinetics equations, the constraint of Eqs. B.3, and the performance index of Eq. B.1 can be written as

$$\begin{aligned} \phi_1 &\equiv \dot{n} + \omega n - \sum_{i=1}^6 \lambda_i c_i - \rho \omega n + 0 \\ \phi_{i+1} &\equiv \dot{c}_i - \omega_i n + \lambda_i c_i = 0 \quad (i = 1, 2, \dots, 6) \end{aligned} \quad (\text{B.4})$$

$$\phi_8 \equiv \dot{n} - \alpha n + \gamma^2 = 0 \quad \text{or} \quad \phi_8 \equiv \dot{n} - g(t) + \gamma^2 = 0$$

$$\phi_9 \equiv \dot{s} - [n(t) - f(t)]^2 = 0$$

The optimal control problem is reduced to that of minimizing $s(t)|_{t=T}$ subject to the constraints of Eqs. B.4. A set of Lagrange multipliers

$$p_1(t), \dots, p_9(t)$$

is introduced, and the augmented function

$$F = \sum_{j=1}^9 p_j \phi_j \quad (\text{B.5})$$

is formed. From the calculus of variation, it is known⁵ that the extremal arc must satisfy the Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{y}_j} \right) = \frac{\partial F}{\partial y_j} \quad (\text{B.6})$$

where y_j stands for all the dependent variables n , c_i , ρ , γ , and S .

Substituting Eqs. B.4 into Eq. B.5 gives

$$F = p_1 \left(\dot{n} + \omega n - \sum_{i=1}^6 \lambda_i c_i - \rho \omega n \right) + \sum_{i=1}^6 p_{i+1} (\dot{c}_i - \omega_i n + \lambda_i c_i) \\ + p_8 \left(\dot{n} + \gamma^2 \begin{Bmatrix} -\alpha n \\ -g \end{Bmatrix} \right) + p_9 [\dot{s} - (n - f)^2] \quad (\text{B.7})$$

and the Euler-Lagrange equations are

$$\begin{aligned} y_j = n & \quad \dot{p}_1 + \dot{p}_8 = p_1 \omega - \rho \omega p_1 - \sum_{i=1}^6 p_{i+1} \omega_i - 2p_9(n - f) - \begin{cases} \alpha p_8 \\ 0 \end{cases} \\ y_j = c_i & \quad \dot{p}_{i+1} = -p_1 \lambda_i + p_{i+1} \lambda_i \\ y_j = \rho & \quad 0 = p_1 \omega n \quad \therefore p_1 = 0 \\ y_j = \gamma & \quad 0 = p_8 \cdot \gamma \\ y_j = s & \quad \dot{p}_9 = 0 \quad \therefore p_9 = \text{constant} \end{aligned} \quad (\text{B.8})$$

The transversality conditions are given by⁵

$$0 = \delta s + \sum \frac{\partial F}{\partial y_j} \delta y_j \bigg|_0^T = \delta s + p_8 \cdot \delta n + p_{i+1} \cdot \delta c_i + p_9 \delta s \bigg|_0^T = 0$$

hence

$$p_9 = -1 \quad p_8(T) = p_{i+1}(T) = 0$$

When discontinuities occur, the different portion of the external arc satisfies the Erdmann-Weierstrass corner conditions.⁵

$$\left(\frac{\partial F}{\partial \dot{y}_j} \right)_- = \left(\frac{\partial F}{\partial \dot{y}_j} \right)_+$$

with the result that p_8, p_{i+1} ($i = 1, 2, \dots, 6$), and p_9 are continuous.

Hence, from the second and third equations of B.8, it follows that $p_{i+1} = 0$, and the set of Eqs. B.8 is reduced to

$$\dot{p}_8 = 2(n - f) - \begin{cases} \alpha \cdot p_8 \\ 0 \end{cases} \quad (\text{B.9a})$$

$$p_8 \cdot \gamma = 0 \quad (\text{B.9b})$$

The optimal arc consists of subarcs along which $\gamma = 0$ or $p_g = 0$.

For $s(t)|_{t=T}$ to be minimum, the Legendre–Clebsch condition

$$\sum_{j,k} \frac{\partial^2 F}{\partial y_j \partial y_k} \delta y_j \delta y_k \geq 0 \quad (\text{B.10})$$

must be satisfied. For the problem at hand, Eq. B.10 yields

$$-2p_g \delta n^2 + 2p_1 \omega \delta n \delta \rho + 2p_g \cdot \delta \gamma^2 = 2\delta n^2 + 2p_g \cdot \delta \gamma^2 \geq 0$$

i.e.,

$$p_g \geq 0$$

Along the subarc where $\gamma = 0$, $\dot{n} = \begin{cases} \alpha n \\ g \end{cases}$, and the corresponding input is

$$\dot{n} = \begin{cases} \alpha n \\ g \end{cases} = -\omega n + \sum_{i=1}^6 \lambda_i c_i + \rho \omega n$$

or

$$\rho = \frac{1}{\omega n} \left(\omega n - \sum_{i=1}^6 \lambda_i c_i + \begin{cases} \alpha n \\ g \end{cases} \right) \quad (\text{B.11})$$

B.11 is identical with the second part of the control law of Eq. 25, which is realized by the inner loop in the control scheme as given in Fig. 3; i.e., along the subarcs where $\gamma = 0$ the corresponding weighting function, h , is zero, and the rate of change of the output $\dot{n}$ follows αn or g . Along the subarcs where $\gamma \neq 0$, $p_g = \dot{p}_g = 0$, and $n = f$, the output follows the program function $f(t)$ and the corresponding weighting function, h , approaches infinity. If this information is utilized, a procedure can be outlined to determine h for the control period $(0, T)$. The program function $f(t)$ is approximated by another function $f'(t)$, composed of arcs of $f(t)$ and of arcs along which $(d/dt) f'(t) = \begin{cases} \alpha f \\ g \end{cases}$ with $f'(0) = n(0)$. Along the arcs where $f' \neq f$, h is chosen to be zero, and, along the arcs where $f = f$, h approaches infinity (see Fig. B.1). Once the h function is determined and its relation to the program $f(t)$ is known, the "switching logic" part of the control scheme as given in Fig. 8 can be readily designed.

Fig. B.1 - Approximation of $f(t)$ by arcs of $f(t)$ and arcs where $\dot{f}' = \alpha f'$.

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Optimal Nuclear-rocket-engine Control

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ABSTRACT

The problem of controlling a nuclear rocket engine so as to minimize propellant consumption is analyzed in this study. It is shown that the required control consists of a maximum allowable effort process. That is, reactivity and propellant flow rate must be varied so that one or both are at a maximum or a minimum allowable value while the state of the system is inside some constraint boundary in the state space. For the trajectory to be kept on such a state-constraint boundary, however, reactivity and/or flow rate must vary appropriately. Though ostensibly a minimal-propellant-consumption process, the control is also one of minimal time. Time-optimal neutronic control is discussed in Ref. 1.

In addition to constraints in reactivity and flow rate, other necessary physical constraints (such as constraints in reactor power, rate of change of core temperature, and pump performance) are considered. These constraints considerably complicate the analysis and synthesis of the optimal control. The analysis is made with Pontryagin's maximum principle² (along with the extensions for state-constrained problems) and physical arguments.

In the analysis the system dynamics are defined by a single-lump direct-cycle heat-exchange process and a monoenergetic neutron-kinetics model. Such a system, along with the hypothetical engine design configuration (which is used in this study), is discussed in Ref. 3.

It should be recognized, however, that such a model only approximates the dynamic behavior of an actual engine. The optimal control,

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on the other hand, requires precise information in order that the state can be confined to some constraint boundary when necessary. Inaccuracies in the model and measurement (or estimation) of the state variables indicate that a suboptimal closed-loop process may very well be in the best practical control. Such a closed-loop system would also attenuate noise effects. It is shown that, with a knowledge of the optimal process, conventional closed-loop control can be synthesized so that its performance nearly approximates the optimal performance, and such control can thereby be classified as suboptimal.

INTRODUCTION

The problems of starting and controlling a simplified model of a nuclear rocket engine so as to minimize propellant consumption is analyzed in this study. Various constraints in the control and phase of the system are considered in the analysis. Utilizing the maximum principle, along with physical reasoning, shows that the optimal control is a maximum allowable effort process.

Though ostensibly a minimal-propellant process, the control derived here is also one of minimal time and requires a time-optimal neutronic control over part of the trajectory. Such optimal neutronic control is summarized in Ref. 1 and more thoroughly analyzed in Ref. 4.

Formulation of the Maximum Principle

Let us represent the system that is to be controlled by the vector differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mathbf{u}) \quad (1)$$

where the dot represents differentiation with respect to time. Here $\mathbf{x}$ is the state vector with components $x_1(t), \dots, x_n(t)$, and $\mathbf{u}$ is the control vector with components $u_1(t), \dots, u_m(t)$. In this study it is assumed that $\partial f_i / \partial x_j$ are continuous in x_q and u_k for all i, j, k , and q . Further, $\mathbf{u}(t)$ must be piecewise continuous and be in a closed region U , i.e., $\mathbf{u} \in U$.

An optimal control problem can be stated as follows: Given a process described by Eq. 1, determine an admissible control $\mathbf{u}(t)$ that transfers the process from some prescribed initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to some terminal state $\mathbf{x}(t_1) = \mathbf{x}_1$ in a manner designed to minimize an index of performance

$$J = \int_{t_0}^{t_1} C(\mathbf{x}; \mathbf{u}) dt \quad (2)$$

For a solution trajectory $\mathbf{x}(t)$ to be optimal, it is necessary that a costate vector $\mathbf{p}(t)$ be related to $\mathbf{x}(t)$ through Hamilton's equations. These equations will be discussed and will relate the state vector to a costate vector by means of a scalar function.

For convenience the system can be considered of the order $n + 1$ by letting C in Eq. 2 be f_{n+1} and adding the equation $\dot{x}_{n+1} = f_{n+1}$ to Eq. 1, with $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_1) = J$. Then Hamilton's equations, which form the maximum principle, take the following form.*

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\mathbf{x}; \mathbf{u}) = \frac{\partial R}{\partial \bar{\mathbf{p}}} \quad (3)$$

with $\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$ and $\bar{\mathbf{x}}(t_1) = \bar{\mathbf{x}}_1$; and

$$\dot{\bar{\mathbf{p}}} = - \frac{\partial R}{\partial \bar{\mathbf{x}}} \quad (4)$$

where $\bar{\mathbf{p}}$ is a costate vector (not identically zero), $\bar{\mathbf{x}}$ is a state vector, and both are of the order $n + 1$. The scalar R in Eqs. 3 and 4 is

$$R(\mathbf{x}; \bar{\mathbf{p}}; \mathbf{u}) = (\bar{\mathbf{p}}, \bar{\mathbf{f}}) = \sum_{i=1}^{n+1} p_i f_i \quad (5)$$

Then the Hamiltonian is

$$H(\mathbf{x}; \bar{\mathbf{p}}) = \sup_{\mathbf{u} \in U} R(\mathbf{x}; \bar{\mathbf{p}}; \mathbf{u}) \quad (6)$$

That is, the Hamiltonian is an absolute maximum with respect to all controls in the allowable set, and the corresponding $\mathbf{u}(t)$, which maximizes R , is the optimal control.

The presentation given here is only a summary of the excellent work given in Chap. 1 of Ref. 2. The reader is referred to this work for precise theorems and explanations. It is shown in Chap. 1 of Ref. 2 that the Hamiltonian is a constant and identically zero for the free end-time problem. Also p_{n+1} is a nonpositive constant if Eqs. 3 to 6 are satisfied. That is,

$$H[\mathbf{x}(t); \bar{\mathbf{p}}(t)] \equiv 0 \quad (7)$$

*For convenience the partial derivative of a scalar with respect to a vector is considered to be a vector rather than a row matrix in this paper.

and

$$p_{n+1} \leq 0 \quad (8)$$

From Eqs. 4 and 5, it is seen that the costate system is adjoint to the original system of Eq. 3. Hence

$$\dot{\bar{p}} = - \left[\frac{\partial \bar{f}}{\partial \bar{x}} \right]^T \bar{p} \quad (9)$$

where the superscript T refers to a matrix transpose.

The previous formulation has assumed that the terminal phase is fixed. Suppose the terminal phase is only confined to a specified smooth manifold. Then it is shown on pp. 45 to 50 of Ref. 2 that the costate vector must be orthogonal to the plane, which is tangent to the manifold at the terminal end.

State-variable Constraints

In addition to control constraints, i.e., $u \in U$, reactor processes must frequently be constrained in their state variables. Optimal control of processes with restricted phase coordinates is presented in Chap. 6 of Ref. 2. This work is related to the classical calculus variations by Berkovitz.⁵ More recent contributions in this area include publications by Chang⁶ and Bryson, Denham, and Dreyfus.⁷

The object of this formal presentation is to summarize the necessary conditions (given in Chap. 6 of Ref. 2) which are used in this study. The reader who is not interested in this complicated theory can proceed to the discussion section.

In addition to the restrictions placed on the allowable control region U , assume that $u(t)$ is composed of piecewise smooth components and, furthermore, that the admissible control set satisfies certain regularity conditions in a neighborhood of its boundary points. These regularity conditions will be defined.

Let u_1 be an arbitrary boundary point of U which belongs to U and let $q_i(u)$ ($i = 1, \dots, s$) be continuously differentiable scalar functions such that U is given by $q_i(u) \leq 0$ ($i = 1, \dots, s$) in the vicinity of u_1 and $q_i(u_1) = 0$. Furthermore, the vectors $\partial q_i / \partial u$ ($i = 1, \dots, s$) are linearly independent.

Let the region of allowed phase values be a smooth closed-region B (in the n -dimensional phase space) defined by $S(x) \leq 0$. The region $S(x)$, also, must have continuous second partial derivatives near the boundary $S(x) = 0$, and the vector $\partial S(x) / \partial x$ must not vanish on the boundary.

As was done previously, the system is made to have the order $n + 1$ by letting $\dot{x}_{n+1} = f_{n+1}(x; u) = C(x; u)$, with $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_1) = J$. The optimal trajectory must connect

$$\bar{x}(t_0) = \begin{cases} x(t_0) \\ 0 \end{cases} \quad \text{with} \quad \bar{x}(t_1) = \begin{cases} x(t_1) \\ x_{n+1}(t_1) \end{cases}$$

so that $x_{n+1}(t_1)$ takes on the minimum possible value.

Every function that depends on x can be considered a function of $\bar{x}$; e.g., $S(x) = S(\bar{x})$. Hence let the closed-region G in the $(n + 1)$ -dimensional phase space be defined by $S(\bar{x}) \leq 0$.

For $S(\bar{x})$ negative, i.e., optimal motions not on the phase-constraint boundary, the previous conditions stated by the maximum principle are valid. Necessary conditions for solutions to be optimal on the phase-constraint boundary

$$S[x(t)] \equiv S(x) \equiv S(\bar{x}) \equiv 0 \quad (10)$$

will be discussed.

Equation 10 requires that all the time derivatives $S^{(k)}$ must vanish. Assume that the required $u(t)$ can be computed from $\dot{S} = 0$, where

$$\dot{S}(\bar{x}; u) = \frac{d}{dt} S(\bar{x}) = \left[\frac{\partial S(\bar{x})}{\partial \bar{x}} \right]^T \bar{f}(\bar{x}; u) = 0 \quad (11)$$

Also, assume that the phase trajectory $\bar{x}$ is regular with respect to u_1 . That is, for each $\bar{x}$ on the trajectory, $\bar{x}$ is regular, where regular means the following conditions are satisfied:

$$\dot{S}(\bar{x}; u_1) = \dot{S}(\bar{x}; u_1) = 0 \quad (\text{Condition 1})$$

$$\frac{\partial \dot{S}(\bar{x}; u_1)}{\partial u} \neq 0 \quad (\text{Condition 2})$$

and

$$\frac{\partial \dot{S}(\bar{x}; u_1)}{\partial u}, \quad \frac{\partial q_i(u_1)}{\partial u} \quad (i = 1, \dots, s) \quad (\text{Condition 3})$$

are linearly independent. If u (as determined by Eq. 11) is an interior point, condition 3 does not appear. Let $\omega(\bar{x})$ designate the set of all admissible controls for which $\bar{x}$ is regular. If u_1 is an interior point, $\omega(\bar{x})$ is a point.

With these assumptions the maximum principle can be extended as follows (pp. 267 and 258 of Ref. 2). If $\bar{x}(t)$ is an optimal trajectory of $\dot{\bar{x}} = \bar{f}(x; u)$ for corresponding $u(t)$ and $\bar{x}$ is located entirely on the boundary of G , then a continuous $\bar{p}(t)$ and a piecewise smooth $\rho(t)$ exist such that

$$\dot{\bar{x}} = \bar{f}(x; u) = \frac{\partial R(x; \bar{p}; u)}{\partial \bar{p}} \quad (12)$$

$$\dot{\bar{p}} = - \frac{\partial R(x; \bar{p}; u)}{\partial \bar{x}} + \rho \frac{\partial \dot{S}(\bar{x}; u)}{\partial \bar{x}} \quad (13)$$

and

$$H[x(t); \bar{p}(t)] = \sup_{u \in \omega(\bar{x})} R[x(t); \bar{p}(t); u(t)] = 0 \quad (14)$$

where, as before, $R(x; \bar{p}; u) = [p, f(x; u)]$. From Eq. 14, $\rho(t)$ is determined as a Lagrange multiplier for $\partial \dot{S}(\bar{x}; u)/\partial u$ in

$$\frac{\partial R(x; \bar{p}; u)}{\partial u} = \rho \frac{\partial \dot{S}(\bar{x}; u)}{\partial u} + \sum_{j=1}^{\beta} \nu_j \frac{\partial q_j}{\partial u} \quad (15)$$

Also, $p_{n+1}(t)$ is a nonpositive constant; $\bar{p}(t_a+)$ is a nonzero vector and is not collinear with $\partial \dot{S}(\bar{x})/\partial \bar{x}$; and $\dot{\rho}(t)$ is nonpositive. The term t_a+ is the time at which the trajectory $\bar{x}(t)$ initially is on a phase-constraint boundary.

We now have necessary conditions for optimal trajectories on phase-space-constraint boundaries. For interior points the maximum principle as presented previously yields the appropriate necessary conditions. Next, consider the necessary conditions such optimal trajectories must have at their junction points, i.e., points where phase trajectories meet phase-constraint boundaries. Consider a junction time t_a such that the trajectory is on a phase-constraint boundary for $t > t_a$ and is not on a phase-constraint boundary for $t < t_a$. Then either

$$\bar{p}(t_a-) = \bar{p}(t_a+) + \mu \left. \frac{\partial \dot{S}(\bar{x})}{\partial \bar{x}} \right|_{t=t_a} \quad (16)$$

or

$$\bar{p}(t_a-) - \mu \frac{\partial \dot{S}(\bar{x})}{\partial \bar{x}} = 0 \quad (\mu \neq 0) \quad (17)$$

where μ is some real number of opposite sign to that of Ref. 2. Hence $\bar{p}(t)$ may be discontinuous at the junction time.

In Refs. 2 and 7, it is noted that an arbitrary jump, proportional to $\partial S(\bar{x})/\partial \bar{x}$, can be added to $\bar{p}$ during (t_a, t_b) . This arbitrary jump can be utilized, as is done in Ref. 7, so that $\bar{p}(t)$ can be assumed continuous at $t = t_b$ if the trajectory leaves the phase-constraint boundary at $t = t_b$. The necessary costate conditions at t_a and t_b could be interchanged, but, following the convention of Ref. 7, assume this arbitrary jump is utilized to make $\bar{p}(t)$ continuous at the exit corner, $t = t_b$. Hence

$$\bar{p}(t_b+) = \bar{p}(t_b-) \quad (18)$$

It is shown in Ref. 7, also, that $R(\mathbf{x}; \bar{p}; \mathbf{u})$ is continuous at both entrance and exit corners.

DISCUSSION

The optimal control problem analyzed in this study can be stated as follows: Given an initial engine steady state, bring the system to a desired terminal steady state so as to minimize the consumption of propellant. For a nuclear rocket engine, pounds of propellant saved will allow less bulky propellant tanks and more payload. Although nuclear rockets allow much higher specific impulse than chemical rockets, their engines are heavier and more complex than their chemical counterparts. Minimization of the nuclear-system weight becomes necessary.

A description of the dynamic behavior and operation for a nuclear rocket engine is presented in Ref. 3. In this study we will neglect dynamics in the propellant pumping system and control-rod drive mechanisms. These systems can be compensated so that their response is considerably faster than that of the neutronics heat-exchange process. The chief constraint of the pumping system seems to be the region of allowable phase of the pump. Further, a minimum amount of coolant (propellant) flow is necessary to keep certain necessary structural elements sufficiently cool.

The classical monoenergetic, spatially independent neutron-kinetic equations are used to represent the power generation in the reactor. For operation near criticality the reactor power generation is then approximated by

$$\frac{dQ}{dt} = \frac{\delta k - \beta}{l} Q + \sum_{i=1}^6 \lambda_i C_i \quad (19)$$

and

$$\frac{dC_i}{dt} = \frac{\beta_i}{l} Q - \lambda_i C_i \quad (i = 1, \dots, 6) \quad (20)$$

where Q = the rate of heat generation from fission

λ_i = the decay constant of the i th precursor group C_i (note that units of C_i must be that of power)

β_i = the portion of neutrons delayed due to the i th precursor group

δ_k = reactivity

l = the mean generation time of neutrons

A qualitative description of the direct-cycle heat-exchange dynamics can be represented by a one-stage model of the reactor (no axial or radial subdivisions).³ Such a one-stage model is represented by Eqs. 21 to 23. These equations can be formulated heuristically by an energy balance written as follows: The heat rate generated from the fission process is equal to the rate at which heat is stored in the core (which results in a core-wall temperature change) plus the rate at which heat is transferred from the core wall to the coolant. That is,

$$Q = MC \frac{dT}{dt} + hA(T - T_g) \quad (21)$$

where Q = heat rate generated from fission process, proportional to n

MC = reactor-core effective mass heat capacity

T = average core temperature

T_g = average coolant temperature

A = heat-transfer area

h = heat-transfer coefficient

t = time

The rate of heat transfer to the coolant is equal to the rate of heat storage in the coolant plus the rate of increase in enthalpy of the coolant. That is,

$$hA(T - T_g) = wc_p \frac{dT_g}{dt} + \dot{w}c_p (T_o - T_i) \quad (22)$$

where w = weight of coolant in the core

c_p = specific heat of coolant

$\dot{w}$ = coolant weight flow rate

T_o = coolant temperature at the core exit

T_i = coolant temperature at core entrance

The average coolant temperature is a weighted average of the inlet and exit temperatures,

$$T_g = \frac{T_i + \theta T_o}{1 + \theta} \quad (23)$$

where θ is a positive weighting factor dependent on the axial power distribution. For many reactors, $\theta = 2$ is found to be a good estimate.

Equations 21 and 22 are generally nonlinear since the coefficients are functions of the corresponding temperature. For gaseous coolants the mass-heat-capacity term, $w c_p \dot{T}_g$, is negligible, and at high temperature the coefficients are nearly constant with the exception of the heat-transfer coefficient. Further, for nuclear reactors the resistance to heat transfer due to conduction through the core is usually negligible compared to the resistance due to convection through the fluid film. If the coolant is a gas and the flow is turbulent, the convective heat-transfer coefficient³ is approximately proportional to $\dot{w}^{0.8}$. With these assumptions and with the coolant weight flow rate specified as a function of time, the heat exchanger is a linear nonautonomous system.

From Eqs. 22 and 23, with the coolant mass heat capacity and inlet temperature negligible, it is seen that

$$T \simeq \left(\frac{c_p \dot{w}}{hA} + \frac{\theta}{1 + \theta} \right) T_o \quad (24)$$

Substitution of Eq. 24 into Eq. 21 with T_g related to T_o by Eq. 23 shows that

$$\frac{dT}{dt} \simeq \frac{Q}{MC} - \frac{T}{\tau_h} \quad (25)$$

where (since $h \propto \dot{w}^{0.8} \simeq \dot{w}$)

$$\tau_h = MC \left[\frac{\theta}{c_p \dot{w}(\theta + 1)} + \frac{1}{hA} \right] \simeq \frac{1}{a\dot{w}} \quad (26)$$

and a is a heat-transfer constant.

Notice that Eq. 26 is approximately bilinear in $\dot{w}$ and T . This simple model is found to be a good approximation of the actual process if the effective mass heat capacity is selected according to the axial position of the temperature of interest.

The heat-exchange process is coupled to the neutron kinetics by temperature reactivity which can be approximated by

$$\delta k_t \simeq c_t T \quad (27)$$

where c_t is a temperature coefficient of reactivity (usually negative). This coupling has a stabilizing effect on the power reactor. It is important to realize that the coupled neutronics heat-exchange process is a nonlinear system even if both processes are linear.

Along with the preceding variables (which are of interest for most reactors) thrust and specific impulse are of prime significance for a nuclear rocket engine. Thrust F is approximately derived from the stagnation pressure at the nozzle entrance, P_c , and the nozzle throat area, A_t . That is,

$$F \simeq c_f A_t P_c \quad (28)$$

where c_f is a nozzle constant. Coolant or propellant temperature at the nozzle entrance (reactor exit) determines specific impulse, I_{sp} , as shown by

$$I_{sp} = \frac{F}{w} = c_s \left(\frac{T_o}{MW} \right)^{1/2} \quad (29)$$

where c_s is a constant for the nozzle and MW is propellant molecular weight.⁸ Equation 29 shows the importance of specific impulse to rocket propulsion. The promise of nuclear rockets and hydrogen propellant for space travel is shown by Eq. 29. Hydrogen is a good moderator, and hence a density reactivity further couples the heat exchanger to the neutron kinetics. Assuming the propellant is gaseous in the core, the density can be computed from the gas equation:

$$\rho = \frac{P}{RT_g} \quad (30)$$

where ρ is average propellant density in the core, R is gas constant, and P is average propellant core pressure.

The density reactivity contribution is computed by

$$\delta k_\rho = k_\rho V \rho \quad (31)$$

where k_ρ is propellant density coefficient of reactivity (positive) and V is reactor-core void volume.

The flow is assumed choked through an exit nozzle, and static conditions are approximately equal to stagnation conditions due to the low core-exit Mach number. Then pressure at the core exit is

$$P_c \sim c_n \dot{w} T_o^{1/2} \quad (32)$$

where c_n is a nozzle constant.⁸ With the approximations $P \propto P_c$ and $T \propto T_g$, the propellant density reactivity is

$$\delta k_\rho \sim \frac{c_\rho \dot{w}}{T^{1/2}} \quad (33)$$

where c_ρ is a constant. The magnitude of propellant reactivity may very well be greater than that of the classical temperature reactivity. Although the reactivity of the propellant is positive, its derivative with respect to temperature is negative. This fact further enhances reactor stability.

A hypothetical nuclear-rocket-engine configuration (at rated design conditions) is presented in Table 1.

Table 1 - HYPOTHETICAL NUCLEAR ROCKET RATED DESIGN CONDITIONS

Thrust, F	100,000 lb
Specific impulse, with losses, I_{sp}	760 sec
Reactor power, Q	2260 Mw
Propellant flow rate, $\dot{w}$	130 lb/sec
Reactor-core-exit propellant temperature, T_o	4500°R
Nozzle throat area, A_t	61 sq in.
Heat exchanger thermal time constant, τ_h	1.5 sec
Inlet propellant temperature, T_i	120°R
Neutron mean effective lifetime, l	3×10^{-5} sec
Propellant reactivity, δk_ρ	0.0065
Temperature reactivity, δk_t	-0.0065
Effective core mass heat capacity, MC	1140 Btu/°R

The computer studies that follow utilize the data given by this table unless otherwise specified.

Optimal Heat-exchange Process

Consider the optimal-start program of Eq. 25 between steady-state end points so as to minimize propellant consumption with flow rate constrained as follows:

$$u_a \leq \dot{w} \leq u_b \quad (34)$$

with $\dot{w}(t_0) = u_a$. The term u_a is a minimum flow rate necessary to cool sufficiently certain necessary structural elements. Although this constraint actually varies with temperature, it is not too well defined at this time and will be introduced as shown by Eq. 34. The term u_b is the maximum flow rate that the coolant system can develop.

For the time being, assume that the reactor power, Q , can be changed instantly with the following constraints:

$$0 \leq Q \leq Q_m$$

and

$$Q(t_1) = Q_1 \quad (35)$$

where Q_m is the rated design power of the reactor. The cost function for this problem is coolant flow rate. Then, with reference to Eq. 5, $(\bar{p}, \bar{f})$ is

$$R(T; \bar{p}; Q, \dot{w}) = \dot{T}p_1 + \dot{w}p_2 = \left(\frac{1}{MC} Q - a\dot{w}T \right) p_1 + \dot{w}p_2 \quad (36)$$

and the Hamiltonian is

$$H(T; \bar{p}) = \sup_{u \in U} R(T; \bar{p}; Q, \dot{w}) \quad (37)$$

Equation 36 is made a maximum by

$$Q^o = \begin{cases} Q_m & \text{for } p_1 > 0 \\ 0 & \text{for } p_1 < 0 \end{cases} \quad (38)$$

and

$$\dot{w}^o = \begin{cases} u_a & \text{for } p_2 - aTp_1 < 0 \\ u_b & \text{for } p_2 - aTp_1 > 0 \end{cases} \quad (39)$$

The costate is defined by the adjoint equation

$$\dot{p}_1 = a\dot{w}p_1 \quad (40)$$

and p_2 is a nonpositive constant. The solution to Eq. 40 is

$$p_1 = p_{10} \exp \left(a \int_{t_0}^t \dot{w} d\sigma \right) \quad (41)$$

where $p_{10} = p_1(t_0)$. The term p_1 cannot change sign because $a\dot{w}$ is always positive and there is no switching for $t_0 < t < t_1$. Hence, because $t_0 < t < t_1$ and p_{10} is positive, for temperature to be increased, it is required that

$$Q^o = Q_m \quad (42)$$

Since the system 25 is initially at steady state with the minimum coolant flow rate, i.e., $\dot{w}(t_0) = u_a$, the required initial power is

$$Q_o = MC_a u_a T_o \quad (43)$$

At the terminal time it is desired that $\dot{T}(t_1) = 0$, $T(t_1) = T_1$, and $Q(t_1) = Q_1$; then from system 25

$$\dot{w}(t_1) = \frac{Q_1}{MCaT_1} \quad (44)$$

Since temperature is increased most rapidly by low coolant flow rate $\dot{w}$ and high reactor power Q (see systems 25 and 26), it is obvious that the minimal coolant trajectory utilizes $\dot{w} = u_a$ between end points. Thus from Eq. 41

$$p_1 = p_{10} e^{au_a(t-t_0)} \quad (45)$$

on the interval (t_0, t_1) .

For a step change in power $\Delta Q = Q_m - Q_o$ at $t = t_0$, the solution to system 25 for $\dot{w} = u_a$ is

$$T = \left[T(t_0) - \frac{Q_m}{MCau_a} \right] e^{-au_a(t-t_0)} + \frac{Q_m}{MCau_a} \quad (46)$$

For very small flow rate, the later term in system 25 is negligible and

$$T \simeq T(t_0) + \frac{\Delta Q}{MC} (t - t_0) \quad (47)$$

Substituting Eqs. 45 and 46 into the switching function, $p_2 - aTp_1$, shows that $\dot{w} = u_a$ for $t_0 \leq t < t_1$ if p_{10} is positive (or the switching

function has no zeros). Since $\dot{H} \equiv 0$ (see Eq. 7), it is required by Eq. 36 that $p_{20} = p_{10} \{ aT(t_0) - [Q(t_1) - Q(t_0)]/MCu_a \}$. Thus the optimal process involves constant maximum power and constant minimum propellant flow rate between end points. Switching occurs at the end points to leave or reach the steady-state conditions. Such optimal-start trajectories are presented in Fig. 1.

An ideal optimal decrease in state with the preceding initial and terminal conditions reversed is caused by the previously described variation in power and flow rate with time reversed. Delayed heating may require u_a to be increased.

It is obvious from the system dynamics that the preceding minimal-propellant process is also one of minimal heat energy and minimal time. Although the power and the coolant flow rate cannot be changed instantly, their change can be very rapid compared to the temperature response. Core-temperature reactivity and propellant-density reactivity coupling, along with neutron kinetics, are shown to complicate the process.

Fig. 1 — Ideal optimal start-up of a heat-exchange process.

Optimal Neutronics Heat-exchange Process

The following analysis depends on an understanding of the system dynamics, as well as the maximum principle, to obtain an optimal trajectory. Total reactivity and coolant weight flow rate are treated as control variables with magnitude constraints. The end points of these control variables are specified by the end states in question. Further, power is constrained to have no overshoot, and propellant consumption is the performance index minimized. Let us represent the system by equations 19, 20, 25, and 26 with

$$\delta k = u_1 = u_c + \delta k_t + \delta k_\rho \quad (48)$$

and

$$u_2 = \dot{w} \quad (49)$$

where

$$\delta k_t \sim c_t T \quad (50)$$

$$\delta k_\rho \sim \frac{c_\rho u_2}{T^{1/2}} \quad (51)$$

and u_c is control reactivity.

Suppose it is desired to start the system from steady-state Q_o, T_o with $u_2 = u_a$ to some desired terminal steady-state Q_1, T_1 with power constrained such that $Q(t) \leq Q_1$. The cost function is u_2 , and $(\bar{p}, \bar{f})$ is

$$R(x; \bar{p}) = I(x; \bar{p}) + \frac{p_1 Q u_1}{I} + (p_9 - a p_8 T) u_2 \quad (52)$$

where the components of x are $Q, C_1, \dots, C_6, T, u_2$ and $I(x; \bar{p})$ is that part of R not involving the control explicitly. Again, the flow rate is constrained by expression 34. Suppose total reactivity is constrained for safety reasons such that

$$u_1 \leq \gamma_1 \beta$$

and by the finite amount available such that

$$-\gamma_2 \beta \leq u_1$$

Although γ_2 may be larger than γ_1 , we will assume symmetrical limits for convenience.

$$|u_1| \leq \gamma\beta \quad (53)$$

Then it is seen from Eq. 52 that the optimal control is

$$u_1^o = \gamma\beta \operatorname{sgn} p_1 \quad (54)$$

(where $\operatorname{sgn} p_1 = p_1/|p_1|$) and

$$u_2^o = \begin{cases} u_a & \text{for } F < 0 \\ u_b & \text{for } F > 0 \end{cases} \quad (55)$$

where

$$F = p_9 - aTp_8 \quad (56)$$

The optimal control reactivity is

$$u_c^o = \gamma\beta \operatorname{sgn} p_1 - (\delta k_t + \delta k_\rho) \quad (57)$$

Meanwhile,

$$H(x; \bar{p}) = \sup_{u \in U} R(x; \bar{p}; u) = 0 \quad (58)$$

With reactivity coupling the heat-exchange process to the neutron kinetics, Eqs. 19, 48, and 51 show that either of the control variables could be used to change power level. Since there is a constraint on total reactivity, however, no increase in flow rate u_2 would permit a faster change of state than does $u_1 = \gamma\beta$ for start-up and $u_1 = -\gamma\beta$ for shutdown. In other words, the power is changed in minimum time if $u_1 = \gamma\beta$, and any increase in flow rate u_2 would require the process to use more propellant.

At the constraint surface $Q(t) = Q_1$, it is necessary to keep steady-state power. From Eqs. 19 and 20, this requires that

$$u_1^o = l \sum_{i=1}^6 \frac{\dot{C}_i}{Q_1} \quad (59)$$

Such open-loop type control is found to be unstable in practice.⁴ Conventional type fast-acting closed-loop control can be used effectively.

A closed-loop type dither control can be synthesized, and this type control is even more accurate. An optimal engine start which utilizes a dither (multiple bang-bang) reactivity process is presented in Fig. 2. Here reactivity is rapidly varied about the theoretical value defined by Eq. 59 according to an allowed error in $Q(t)$ from Q_1^2 . At the desired terminal temperature, flow rate is increased to the required terminal value (to obtain designed thrust and hold temperature):
 $u_2(t_1) = Q_1/MCaT_1 = 130 \text{ lb/sec.}$

For the trajectories given in Fig. 2, temperature increase is almost linear for (t_0, t_1) with $u_2 = u_a = 20 \text{ lb/sec.}$ Since core thermal stresses are roughly minimized by a linear variation in average core temperature, a power magnitude constraint may also be interpreted

Fig. 2 - Minimal-propellant-consumption start-up with dither terminal control.

roughly as a $\dot{T}$ or core thermal-stress constraint for a minimal-coolant process. Such constraint might be less than the desired terminal power, Q_1 .

The preceding optimal process is hypothesized by the maximum principle along with physical reasoning. It is necessary, however, that the proper costate equations have a solution that satisfies the maximum principle as extended for phase-constrained problems. These costate equations will be analyzed.

The costate is defined by the adjoint system for an interval of no phase constraint, i.e., $t < t_a$,

$$\dot{p}_1 = \frac{\beta - u_1}{I} p_1 - \sum_{i=1}^6 \frac{\beta_i}{I} p_{i+1} - \frac{p_8}{MC} \quad (60)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) \quad (j = 2, \dots, 7) \quad (61)$$

$$\dot{p}_8 = a u_2 p_8 \quad (62)$$

and

$$\dot{p}_9 = 0 \quad (63)$$

Also, p_9 is nonpositive.

Equations 62 and 63 show that F as given by Eq. 56 cannot change sign if $p_8(t_0)$ is nonnegative since aT is positive. It is shown in Chap. II of Ref. 4 that $p_1(t)$ cannot change sign (for the proper terminal conditions) for the minimal-time neutronic process with p_8 missing in Eq. 60. The behavior of p_1 is changed only slightly by the addition of a particular solution of slow transient due to $p_8(t)$ if $p_8(t_0)$ is selected small enough. Hence one would expect that $p(t_0)$ can again be chosen so that $p_1(t)$ does not change sign.

It is shown in Eq. 76 that $p_j(t_a) = 0$ for $j = 2, \dots, 7$. This is also the terminal condition that yields no zeros for $p_1(t)$ in Chap. II of Ref. 4.

Although $Q(t) = Q_1$, it is seen from Eq. 13 that, with $t_a < t < t_1$,

$$\dot{p} = - \frac{\partial R}{\partial \bar{x}} + p \frac{\partial \dot{S}}{\partial \bar{x}} \quad (64)$$

where $S = Q - Q_1$.

Since

$$\frac{\partial \dot{S}}{\partial \bar{x}} = \begin{bmatrix} (u_1 - \beta)/I \\ \lambda_1 \\ \cdot \\ \cdot \\ \lambda_6 \\ 0 \end{bmatrix}$$

$$\dot{p}_1 = \frac{\beta - u_1}{I} p_1 - \sum_{i=1}^6 \frac{\beta_i}{I} p_{i+1} - \frac{p_8}{MC} + \frac{u_1 - \beta}{I} \rho \quad (65)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) + \lambda_{j-1} \rho \quad (j = 2, \dots, 7) \quad (66)$$

$$\dot{p}_8 = a u_2 p_8 \quad (67)$$

and

$$\dot{p}_9 = 0 \quad (68)$$

As has already been hypothesized, let $u_1 = I \sum_{i=1}^6 \dot{C}_i / Q_1$ and $u_2 = \dot{w} = u_a$ for the interval (t_a, t_1) . Then from Eq. 15

$$\frac{\partial R}{\partial \mathbf{u}} = \rho \frac{\partial \dot{S}}{\partial \mathbf{u}} + \nu \frac{\partial q}{\partial \mathbf{u}} \quad (69)$$

where

$$q = u_a - \dot{w}$$

Substituting Eq. 52 into Eq. 69, one obtains

$$\frac{p_1 Q}{I} = \frac{\rho Q}{I} \quad (70)$$

and

$$F = (p_9 - a p_8 T) = -\nu$$

Hence from Eq. 70

$$\rho = p_1 \quad (71)$$

and from Eq. 67

$$p_8 = p_{8a} \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right]$$

where $p_{8a} = p_8(t_a)$ is the value of p_8 at the point where the trajectory enters the phase-constraint boundary. Therefore

$$F = p_9 - a p_{8a} T(t) \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right] \quad (72)$$

Then since T and u_2 are positive, the switching function of Eq. 72 cannot change sign if p_{8a} is nonnegative.

The other switching function p_1 is described by

$$\dot{p}_1 = - \sum_{i=1}^6 \frac{\beta_i}{I} p_{i+1} - \frac{p_8}{MC} \quad (73)$$

where

$$\dot{p}_j = \lambda_{j-1} p_j \quad (j = 2, \dots, 7) \quad (74)$$

or

$$\dot{p}_1 = - \sum_{i=1}^6 \frac{\beta_i}{I} p_{i+1} (t_a) e^{\lambda_i(t-t_a)} - \frac{p_{8a}}{MC} \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right] \quad (75)$$

Since the terminal manifold is defined by $T(t_1) = T_1$ and $Q(t_1) = Q_1$, $p_j(t_1) = 0$ for $j = 2, \dots, 7$. Then

$$p_j(t) = 0 \quad (76)$$

and

$$\dot{p}_1 = - \frac{p_8}{MC}$$

or

$$p_1 = p_{1a} - \frac{1}{MC} \int_{t_a}^t p_8(\sigma) d\sigma \quad (77)$$

where $t_a < t \leq t_1$ and $p_{1a} = p_1(t_a+)$. When Eq. 71 is substituted into Eq. 77,

$$p_1 = p_{1a} - \frac{p_{8a}}{MC} \int_{t_a}^t \exp \left[a \int_{t_a}^{\sigma} u_2(\lambda) d\lambda \right] d\sigma \quad (78)$$

From Eq. 16

$$p_j(t_a-) = p_j(t_a+) \quad (j = 2, \dots, 9) \quad (79)$$

and

$$p_1(t_a-) = p_1(t_a+) + \mu$$

where, since there is no constraint exit corner, μ may be set to zero. Hence, if a large p_{1a} compared to p_{8a} or a large p_{10} compared to p_{80} is selected, $p_1(t)$ remains positive. Furthermore, $\dot{p}(t) = \dot{p}_1(t)$ is non-positive for $t_a < t < t_1$, and $\bar{p}(t_a)$ is not collinear with $\partial S(\bar{x})/\partial \bar{x}$ (as required).

In summary, the costate system of equations is satisfied by the optimal start-up control if (1) $p_8(t_0)$ is a small nonnegative number; (2) p_9 is a negative constant; (3) $p_1(t_0)$ is selected positive and considerably larger than $p_9(t_0)$ so that $p_1(t)$ does not change sign; (4) $p_j(t_1) = 0$, $j = 2, \dots, 7$; and (5) the actual initial values of p_1 , p_8 , and p_9 are selected so that $H = 0$.

The actual solution of the costate equation should again progress from the terminal end. When the preceding conditions 1, 3, and 5 (along with 4 at the terminal end) are selected, the costate equations can be solved, and there are no switchings in control between end points.

An example start-up trajectory with steady-state power and temperature at the end points is presented in Table 2. The switching functions $p_1(t)$ and $F(t) = p_9 - aTp_8$ are positive and negative, respectively, between end points, and hence the hypothesized constant control satisfies the maximum principle. These data were obtained from an IBM 7090 computer.

In addition to power constraint, frequently the coolant pumping system is constrained as discussed next.

Flow-system Constraints

The coolant may be supplied from a pressurized tank or may be pumped from a low-pressure supply tank. With regard to nuclear rockets, the latter is more desirable because of the saving in tank

Table 2 - OPTIMAL REACTOR START-UP TRAJECTORY WITH $Q(t) \leq Q_1$

Time, sec	Power, 10^{-4} Btu/sec	Temperature, 10^{-2}°R	$p_1(t)$, $10^6 \text{ sec}^2/\text{Btu}$	$p_8(t)$, $10^4 \text{ sec}/^{\circ}\text{R}$
0	1.3144	5.0000		
$t_0 = 0.1$	1.3144	5.0000	11.946	7.2139
0.2	13.253	5.0651	9.2838	7.2306
0.4	24.889	5.3720	5.6093	7.2640
0.6	42.642	5.9257	3.3932	7.2976
0.8	71.793	6.8790	2.0589	7.3313
1.0	119.91	8.4903	1.2663	7.3652
$t_a = 1.24$	214.20	11.911	5.0718	7.4101
1.4	214.20	14.868	4.9676	7.4375
1.6	214.20	18.549	4.8368	7.4718
1.8	214.20	22.213	4.7054	7.5064
2.0	214.20	25.860	4.5734	7.5411
2.2	214.20	29.490	4.4408	7.5759
2.4	214.20	33.104	4.3076	7.6110
2.6	214.20	36.701	4.1737	7.6461
2.8	214.20	40.281	4.0393	7.6815
$t_1 = 3.08$	214.20	45.011	3.8500	7.7751

Input Data

$$u_1 = 0, t < 0.1 \text{ sec} = t_0$$

$$u_1 = 0.9 \beta; t \geq 0.1, p_1 > 0, \text{ and } Q < Q_1 = 2.142 \times 10^6 \text{ Btu/sec}$$

$$u_1 = I \sum_{i=1}^6 \dot{C}_i / Q_1, Q = Q_1$$

$$u_2 = u_a = 2 \text{ lb/sec}, T < T_1 \text{ and } F < 0$$

$$u_2 = Q_1 / \text{MCaT}_1, T = T_1 = 4501.1^{\circ}\text{R}$$

$$H = 0, t_0 < t < t_1$$

$$a = 1.153 \times 10^{-2}, T(0) = 500^{\circ}\text{R}, \dot{Q}(0) = \dot{T}(0) = 0 = \dot{C}_i(0) \quad (i = 1, \dots, 6)$$

$$\dot{T}(t_1) = \dot{Q}(t_1) = 0; \text{ other data given by Table 1}$$

weight and because the pump is driven by a gas turbine. The output flow rate of the system is constrained by the velocity of the flow control valve (which is very fast), mechanical stresses, and certain regions of performance on the pump map. The following discussion assumes the first two constraints to be negligible.

A hypothetical pump-performance map is presented in Fig. 3. This is a plot of pressure rise across the pump (approximately proportional to nozzle-entrance pressure) vs. pump flow rate. Constant specific

Fig. 3 - Hypothetical pump-performance map. N_s = specific speed, T_1 = terminal temperature, and T_0 = initial temperature.

speed lines are parabolic on this map.⁹ Temperature lines refer to the nozzle entrance and are approximately linear because of nozzle Eq. 32. The quantity specific speed is defined according to

$$N_s = \frac{NQ^{1/2}}{H^{3/4}} \quad (80)$$

where⁹ N = pump speed (rpm), Q = flow rate (gpm), and H = pump pressure head (ft).

The importance of specific speed can best be explained by a pump map (see Fig. 3). Here the flow system is constrained by regions of stall and cavitation. Above some specific speed N_{ss} , the pump stalls, and, below some specific speed N_{sc} , cavitation exists. Operation in either of the regions is not tolerable for any finite time. A typical optimal trajectory, o-d-e-c, in Fig. 3, with a bang-bang flow variation, shows a pump constraint violation. For such a case the temperature could only be allowed to increase to T_s . At point s, mass flow rate and temperature would have to be increased so as to maintain constant specific speed N_{ss} . If flow rate were increased at some

lower temperature or power, more coolant would be consumed when the desired terminal temperature is reached. At point e, mass flow rate could be stepped to point c.

Suppose the pressure during stall specific-speed operation is approximated by

$$P = b_s \dot{w}^2 \quad (81)$$

where b_s is a constant. Substitution of nozzle Eq. 32 into Eq. 81 (with $T \propto T_0$) yields

$$\dot{w} = \frac{c_n' T^{1/2}}{b_s} \quad (82)$$

where c_n' is a constant. Then, for $T > T_s$, the optimal trajectory (as discussed previously) is altered as follows: At $T(t_s) = T_s$ (where $t_a < t_s < t_b$),

$$u_2^o = u_a = \frac{c_n' T_s^{1/2}}{b_s} \quad (83)$$

Assume $u_2(t)$ remains so small that temperature increase is approximately linear, i.e., $T < \alpha_m (t - t_s) + T_s$, where $\alpha_m = Q_1/MC$. Then it is seen from Eq. 82 that

$$u_2(t) \simeq \frac{c_n'}{b_s} [\alpha_m (t - t_s) + T_s]^{1/2} \quad (84)$$

for $t_s < t < t_1$. At the terminal end, $T = T_1$ and $Q = Q_1$; then, from Eq. 21 with $\dot{T} = 0$, it is seen that flow rate must be increased to

$$u_2(t) = \frac{Q_1}{MCaT_1} \quad (85)$$

In summary, the optimal control process as previously defined should be pieced together continuously with Eqs. 81 to 84 if the pump-stall constraint must be considered.

It is possible to bypass some of the coolant back to the tank and send only the minimum mass flow rate to the reactor, but increased energy is required to pump this bypass flow. In a bleed-cycle or topping-cycle nuclear rocket, the turbine gets this energy from the reactor.³ In practice such a process requires more coolant than the previous one because of energy losses.

The arguments presented in this section are not rigorous, but the constraint problem is so complicated (involving vector control with a constraint explicitly including both control and phase) that physical arguments are much simpler. Furthermore, the theory developed in Ref. 2 and discussed in this paper does not apply to optimization problems with constraints that involve jointly phase and control.

Again, in practice a simple closed-loop suboptimal process would be desirable for most applications. Such closed-loop systems for a nuclear rocket engine are presented in Figs. 4 and 5. There the inputs are limited to meet the necessary constraints that have been discussed. Though the latter control system is simplified by the coupling from propellant reactivity, i.e., an increase in flow rate causing an increase in power, the more complicated system allows the necessary constraints to be applied in a simpler manner. Notice that control reactivity still appears in Fig. 4 and can be used to bias appropriately the total reactivity.

As mentioned previously, specific impulse (or temperature) and thrust (or pressure) are the variables of most interest to a rocket engine.

Fig. 4 - Closed-loop nuclear-rocket-engine control system.

Fig. 5 - A simple type closed-loop nuclear-rocket-engine control system.

Hence these are the variables controlled directly in Fig. 4. Temperature and pressure determine flow rate from nozzle Eq. 32. Therefore flow rate can be held approximately constant by programming pressure demand from temperature according to the nozzle equation for constant flow rate. An alternate suboptimal scheme could use a closed $\dot{w}$ loop.

Regarding system malfunctions it might be desirable to derive one demand from another measured variable. Since the thrust loop can be considerably faster than the temperature loop, it seems better to derive thrust demand from temperature measured according to the necessary system constraint.

CONCLUSIONS

Although safety considerations of many commercial reactors require more stringent constraints than those given here, the techniques can still be applied to the optimal control design.

From the preceding study it is seen that the minimal-propellant control of a neutron-dynamics direct-cycle heat-exchange process, such as that of the nuclear rocket engine, is at least partly a bang-bang process. Though such control is one of minimal propellant, the process is also one of minimal time.

Physical constraints in maximum power level (approximate rate of change of core temperature or core thermal stress) and in the propellant flow system cause the optimal control trajectories to be a connection of those due to bang-bang and continuous types of control.

Although the control has been assumed to be inertialess in this study, the conclusions are readily extended to the actual system, and the control is a maximum allowable effort process. In other words, the process is always operating on a control-constraint boundary or a phase-constraint boundary.

In practice the state equations and the constraints are only approximations. The optimal process, on the other hand, requires precise information about the system. Hence in actual applications suboptimal closed-loop control can be utilized advantageously to approximate the optimal process in a simple manner. Such control can be used for all operations (with the necessary constraints built into the control system) or merely while the trajectories must stay on a phase constraint or hold a desired terminal phase. A closed-loop dither control also can be utilized to maintain essentially constant power while precursor levels and temperature are not at steady state.

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Synthesis of Optimal Feedback Control for Nuclear-rocket-engine Start-up

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ABSTRACT

Linear optimal feedback control theory is applied to the synthesis of an optimum nuclear-rocket-engine start-up program by use of a model-following system. This theory generates a feedback program of linear gains that alter the plant dynamics such that the rocket-engine response approaches that of the model. If small perturbations are taken about the optimum trajectories, the system differential equations become linear with time-varying coefficients. The optimum feedback control minimizes an integral quadratic performance index containing system errors and control motions. Analog-computer studies show the effect of this synthesis technique in minimizing the system error due to disturbances and the nonexact mathematical description of the system.

INTRODUCTION

In recent years a number of papers have discussed the determination of an optimum reactor control for various performance criteria.¹⁻⁴ The optimal-input-control vector and the optimal-output-state vector were computed by using Pontryagin's Maximum Principle. Since the control thus determined is open loop, the corresponding state variables will be susceptible to any internal noise or errors due to the nonexact mathematical description of the system. Slight changes in the initial and/or the final conditions of the system would require the calculation of a new control

program. This paper discusses the determination of a feedback control program to minimize the effect of these disturbances.

If small perturbations are taken about the predetermined optimal trajectories, the resulting system differential equations are linear with time-varying coefficients. Applying an integral-square-error performance index gives a linear feedback control program with time-varying gains.

Linear optimal control theory⁵ has been limited in its use as a practical synthesis technique for multivariable-control-system design because the quadratic performance criterion does not always express the true design objective. For example, the system transient response may be more meaningful. When, however, this criterion is used in a model-following control scheme, its usefulness is extended by incorporating into the performance index the error between the system and the model. This is the basis for an adaptive control system where computation of feedback gains is performed in real time and yields an optimum closed-loop control.

FORMULATION OF THE PROBLEM

Nuclear reactor systems considered in this study are described by a set of nonlinear, first-order, differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mathbf{u}, t) \quad (1)$$

where $\mathbf{x}$ is the state-variable vector, $\mathbf{u}$ is the control vector, and t is the independent variable, time. The state transition is normally uniquely determined by a set of initial conditions, $\mathbf{x}(t_0)$, and control variables, $\mathbf{u}(t)$. In this paper it will be assumed that a set of nominal control variables, $\mathbf{u}^*(t)$, which satisfies a given performance criterion, has been determined.

Small deviations from the nominal system trajectories are considered by making piecewise linear approximations about the predetermined nominal state trajectory, $\mathbf{x}^*(t)$. These deviations are represented by

$$\delta \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}^*(t) \quad (2)$$

where the quantities without an asterisk indicate the actual state of the system and $\delta \mathbf{x}(t)$ is called the *error vector*. The resulting piecewise linear differential equations describing perturbations about the nominal path in the state space are time varying and take the form

$$\begin{aligned} \delta \dot{\mathbf{x}}(t) &= \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}^*, \mathbf{u}^*} \delta \mathbf{x}(t) + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\mathbf{x}^*, \mathbf{u}^*} \delta \mathbf{u}(t) \\ &= \mathbf{F}(t) \delta \mathbf{x}(t) + \mathbf{G}(t) \delta \mathbf{u}(t) \end{aligned} \quad (3)$$

where $F(t)$ is the system matrix and $G(t)$ the input matrix.

The nominal vectors, $\mathbf{x}^*$ and $\mathbf{u}^*$, are determined from the mathematical-system model. In many cases this model is only an approximation to the actual system. In reactor kinetics, one effective group of delayed neutrons is often used to represent six or more groups. In addition, the inherent-reactivity-feedback mechanism is not clearly understood. Since it is desired to have the plant dynamics approach those of the model, the performance index is chosen to minimize the error between the model and the plant while constraining the control motion. Equation 4 gives the quadratic performance index used in this study.

$$J = \frac{1}{2} \delta \mathbf{x}^T(t_f) S(t_f) \delta \mathbf{x}(t_f) + \frac{1}{2} \int_0^{t_f} [\delta \mathbf{y}^T(t) Q_1(t) \delta \mathbf{y}(t) + \delta \mathbf{u}^T(t) Q_2(t) \delta \mathbf{u}(t)] dt \quad (4)$$

where $\delta \mathbf{y}$ = the output-error vector

$Q_1(t)$ = the error-weighting matrix

$Q_2(t)$ = the control-weighting matrix

$S(t_f)$ = the terminal-condition matrix whose elements are chosen to obtain the desired terminal accuracy

t_f = the terminal time

The output vector, $\mathbf{y}(t)$, is defined as

$$\mathbf{y}(t) = M(t) \mathbf{x}(t) \quad (5)$$

where $M(t)$ is the measurement matrix. The output vector, $\mathbf{y}$, is not identical to the state vector, $\mathbf{x}$, if all the system variables are not compared with the model variables. For example, direct measurement of the delayed-neutron state variables is not physically possible.

The feedback control that minimizes the performance index of Eq. 4 is obtained by applying Pontryagin's Maximum Principle.⁶ The pre-Hamiltonian for the system is

$$R = \frac{1}{2} \delta \mathbf{x}^T M^T Q_1 M \delta \mathbf{x} + \frac{1}{2} \delta \mathbf{u}^T Q_2 \delta \mathbf{u} + \boldsymbol{\psi}^T F(t) \delta \mathbf{x} + \boldsymbol{\psi}^T G(t) \delta \mathbf{u} \quad (6)$$

where $\boldsymbol{\psi}(t)$ is the costate vector adjoint to the error vector, $\delta \mathbf{x}$. In this equation $\delta \mathbf{y}$ is replaced by its equivalent from Eq. 5.

Hamilton's equations for the system are

$$\frac{\partial R}{\partial \dot{\mathbf{x}}} = \delta \dot{\mathbf{x}} = \mathbf{F}(t) \delta \mathbf{x} + \mathbf{G}(t) \delta \mathbf{u} \quad (7)$$

$$-\frac{\partial R}{\partial \delta \mathbf{x}} = \dot{\boldsymbol{\psi}} = -\mathbf{M}^T \mathbf{Q}_1 \mathbf{M} \delta \mathbf{x} - \mathbf{F}^T(t) \boldsymbol{\psi} \quad (8)$$

The pre-Hamiltonian R evaluated for the optimal-feedback-control vector $\delta \mathbf{u}^*$ is called the Hamiltonian. The form of $\delta \mathbf{u}^*$ is found if the partial derivative of R is taken with respect to $\delta \mathbf{u}$ and the result is equated to zero.

$$\frac{\partial R}{\partial \delta \mathbf{u}} = 0 = \mathbf{Q}_2 \delta \mathbf{u}^* + \mathbf{G}^T(t) \boldsymbol{\psi} \quad (9)$$

Thus the optimal-feedback-control vector is

$$\delta \mathbf{u}^* = -\mathbf{Q}_2^{-1} \mathbf{G}^T(t) \boldsymbol{\psi} \quad (10)$$

If it is assumed that the costate vector, $\boldsymbol{\psi}$, is of the form

$$\boldsymbol{\psi}(t) = \mathbf{P}(t) \delta \mathbf{x}(t) \quad (11)$$

then the optimum-feedback-control vector becomes

$$\delta \mathbf{u}^* = -\mathbf{Q}_2^{-1} \mathbf{G}^T(t) \mathbf{P}(t) \delta \mathbf{x} \quad (12)$$

where $-\mathbf{Q}_2^{-1} \mathbf{G}^T(t) \mathbf{P}(t)$ is the linear time-varying feedback gain which is a function of the nominal optimal state and control vectors of the model.

If Eqs. 7, 8, 11, and 12 are combined, it can be shown that $\mathbf{P}(t)$ is the solution of the following differential equation:

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t) \mathbf{F}(t) - \mathbf{F}^T(t) \mathbf{P}(t) - \mathbf{M}^T \mathbf{Q}_1 \mathbf{M} + \mathbf{P}(t) \mathbf{G}(t) \mathbf{Q}_2^{-1} \mathbf{G}^T(t) \mathbf{P}(t) \quad (13)$$

In Eq. 13, $\mathbf{P}(t)$ is a symmetric, nonlinear, time-varying matrix satisfying a differential equation of the matrix Riccati type. The boundary conditions on $\mathbf{P}(t)$ are found to be $\mathbf{P}(t_f) = \mathbf{S}(t_f)$. Thus, in order to solve for $\mathbf{P}(t)$, it is necessary to integrate Eq. 13 backwards in time.

The simplicity of this type of control system is illustrated in the block diagram of Fig. 1.

The selection of the matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is arbitrary. In practice the procedure is to vary the ratio of the weight on the error to the weight on

Fig. 1 - Solution of the optimal control process in the presence of random reactivity disturbances.

the control (Q_1/Q_2) until a satisfactory result is obtained. If too much weight is placed on the control, the response of the system will not differ significantly from that of the open-loop control owing to heavy constraints on control motion. If too much weight is placed on the error terms, the resulting feedback gains may be out of the range of physical realizability.

In general, it is desirable to have constant feedback gains. If the elements of the system matrix, $F(t)$, and the input matrix, $G(t)$, are constants, constant feedback gains could easily be achieved by solving Eq. 13 in the steady state. Constant gains for time-varying coefficients can be obtained by averaging the elements of $F(t)$ and $G(t)$ over the control interval and solving the steady-state or algebraic Riccati equation rather than a differential equation. However, the resulting feedback control system is suboptimal.

SYNTHESIS OF THE NUCLEAR-ROCKET-ENGINE START-UP PROGRAM

The dynamics of a nuclear-rocket-engine model have been formulated by several investigators.^{7,8} In general, these models consider a nuclear rocket engine with bleed turbopump or topping turbopump drive. The model considered here is that put forth by Mohler and Perry.⁷ It consists of the basic neutron-kinetics equations coupled with a heat-exchange equation via core temperature and propellant flow rate. The following nonlinear differential equations describe the system.

Neutronics:

$$\dot{Q} = \frac{\rho_T - \beta}{l} Q + \sum_{i=1}^n \lambda_i C_i \quad (14)$$

$$\dot{C}_i = \frac{\beta_i Q}{l} - \lambda_i C_i \quad (15)$$

Heat exchanger:

$$\dot{T} = \frac{Q}{M_c} - \frac{T}{\tau_h} \quad (16)$$

where the system state variables are Q, C_i , and T

Q = power level

C_i = precursor density of i th delayed-neutron group

T = average core temperature

ρ_T = total reactivity

M_c = mean effective heat capacity of reactor core

τ_h = heat-exchanger time constant

The total reactivity is composed of the control-rod reactivity, u_1 , the propellant-density reactivity, δk_p , and the temperature reactivity, δk_T .

$$\rho_T = u_1 + \delta k_T + \delta k_p \quad (17)$$

where

$$\delta k_T = C_T T \quad (18)$$

and

$$\delta k_p = \frac{C_p u_2}{\sqrt{T}} \quad (19)$$

In Eq. 19, u_2 is the coolant mass flow rate, a control variable. Usually $C_T \leq 0$ and $C_p \geq 0$. The heat-exchanger thermal time constant is

$$\tau_h = (a u_2)^{-1} \quad (20)$$

where a is a constant of proportionality at rated design flow rate.

The optimal open-loop control problem stated by Mohler is as follows: "Given an initial reactor steady state, bring the system to the desired terminal steady state so as to minimize the consumption of propellant . . .,"⁹ subject to the following constraints:

$$\begin{aligned}
 u_a &\leq u_2 \leq u_b \\
 Q &\leq Q_{\max} \\
 T &\leq T_{\max}
 \end{aligned}
 \quad
 \begin{aligned}
 -\gamma\beta &\leq \rho_T \leq \gamma\beta \\
 \dot{T}_{\max} &= \alpha
 \end{aligned}
 \tag{21}$$

where γ is a positive number greater than 1. The index of performance for the system is

$$J = \int_0^{t_f} u_2 \, dt \tag{22}$$

The optimal control program that satisfies Eqs. 21 and 22 has been formulated by Mohler. The control law is stated for five distinct intervals, or control regions, as follows:

1. Prior to start-up the reactor is at steady state with minimum propellant flow rate. The nominal control variables are

$$u_2^* = u_a \tag{23}$$

$$u_1^* = -C_T T_0 - \frac{C_P u_a}{\sqrt{T_0}} \quad \text{for } t \leq 0 \tag{24}$$

2. In the time interval $0 < t \leq t_a$, the propellant flow rate is kept at its minimum value, while the total reactivity is increased from zero to its maximum value. This allows the reactor to achieve design power level in the shortest possible time. The controls in this region are

$$u_2^* = u_a \tag{25}$$

$$u_1^* = \gamma\beta - C_T T^* - \frac{C_P u_a}{\sqrt{T^*}} \tag{26}$$

3. At the first switching time, t_a , the rate of temperature increase has reached its maximum allowable value, α . From $t_a < t \leq t_b$ the propellant flow rate is maintained at its minimum value, whereas the total reactivity is compensated to keep $\dot{T}$ equal to α . The nominal optimal controls in this region are

$$u_2^* = u_a \tag{27}$$

$$u_1^* = \frac{1}{Q^*} (a u_a M_c \alpha - \lambda C^*) + \beta - C_T T^* - \frac{C_P u_a}{\sqrt{T^*}} \tag{28}$$

4. At time t_b the total reactivity must again be set at its maximum value so that the reactor power will reach its steady-state design value when the temperature reaches its design value. The propellant flow rate must be increased so that the rate of temperature rise does not exceed α . These controls are applied to the end of the start-up control period; i.e., $t_b < t \leq t_f$. The controls in this region are

$$u_2^* = \frac{(Q^*/M_c - \alpha)}{aT^*} \quad (29)$$

$$u_1^* = \gamma\beta - C_T T^* - \frac{C_P(Q^*/M_c - \alpha)}{a\sqrt{T^{*3}}} \quad (30)$$

5. At the terminal time, t_f , when both the power and the temperature have reached their design values, the propellant is immediately increased to its design value. The control-rod-reactivity programming is such that the power maintains its design value for all time greater than t_f . The terminal controls thus become

$$u_2^* = u_2^d \quad (31)$$

$$u_1^* = \beta - \frac{I\lambda C^*}{Q_{\max}} - C_T T_{\max} - \frac{C_P u_2^d}{\sqrt{T_{\max}}} \quad (32)$$

The hypothetical nuclear-rocket-engine design parameters used in this study are as follows:

Maximum reactor power, $Q_{\max}$	2260 Mw
Minimum propellant flow rate, u_a	4 lb/sec
Design propellant flow rate, u_2^d	130 lb/sec
Design maximum average core temperature, $T_{\max}$	3400°R
Propellant inlet temperature, T_0	120°R
Mean effective neutron lifetime, l	3×10^{-5} /sec
Effective delayed-neutron fraction, β	0.0065
Design propellant reactivity, $\delta k_p(t_f)$	0.0065
Design temperature reactivity, $\delta k_T(t_f)$	-0.0065
Effective one-group delay-time constant, λ	0.10 sec ⁻¹
Effective core mass heat capacity, M_c	1140 Btu/°R
Maximum core temperature rise, α	1800°R/sec

Figures 2 and 3 show the nominal trajectories of reactor-core power and temperature for $\gamma = 1.6$. Figures 4 and 5 show the nominal control programs for control-rod reactivity and propellant flow rate for this value of γ .

Synthesis of the start-up problem was simulated on the University of Arizona Department of Nuclear Engineering DYSTAC analog computer. The reactor model used included only one group of delayed neutrons. The computing of the nominal trajectories was such that the second switching point, t_b , was eliminated. The slope of the nominal power trajectory between t_a and t_f was increased such that the desired terminal power was reached simultaneously with the desired terminal temperature. Also the propellant flow rate was maintained at its minimum value of u_a over the entire control interval.

For convenience of simulation the time-varying gains were replaced by constant gains that correspond to average values. Constant gains are easily achieved with variable potentiometers; therefore many average gain settings can be studied and the approximate values of the optimal weighting matrices, Q_1 and Q_2 , determined.

In analog-computer synthesis the model and the physical plant are simulated simultaneously. Two nearly identical sets of kinetics equa-

Fig. 2 -- Optimal power trajectory, $\gamma = 1.6$.

Fig. 3 - Optimal temperature trajectory, $\gamma = 1.6$. Fig. 4 - Optimal control-rod program, $\gamma = 1.6$.

tions are programmed, one generating the nominal control and state vectors and the other being contaminated with external noise. The contaminated system is driven by the nominal control program and, be-

Fig. 5 - Optimal propellant program, $\gamma = 1.6$.

cause of the noise, deviates from the nominal trajectories. The output-error vector is computed, multiplied by the optimal feedback gains, and added to the nominal control variables. The result is a linear optimal adaptive control.

Figures 6 and 7 give the nominal state- and control-variable programs for $\gamma = 1.6$. Figure 8 shows the open-loop controls, u_1 and u_2 , contaminated by white noise. The resulting open-loop state variables, power and temperature, are shown in Figs. 9 and 10, together with the closed-loop trajectories for several values of average gain settings.

Optimal gain settings were determined by requiring the variance of the power to be less than 0.01.

$$\overline{\sigma Q^2} = \frac{\int_0^{t_f} [Q(t) - Q^*(t)/Q^*(t)]^2 dt}{\int_0^{t_f} dt} \leq 0.01 \quad (33)$$

It was noted that the optimal gains are physically realizable. In no cases did the optimal control-rod gains exceed $1\phi/\text{Mw}$. This value is approximately twenty times the temperature coefficient of reactivity at design power. It was also observed that very large weight on the control variable u_2 does not affect the closed-loop system. In other words, optimal feedback control can be adequately approximated with the control variable u_1 .

The equation describing the perturbations about the nominal trajectories is

$$\delta \dot{\mathbf{x}} = \begin{bmatrix} f_{11}(t) & \lambda & f_{13}(t) \\ \beta/I & -\lambda & 0 \\ 1/M_c & 0 & f_{33}(t) \end{bmatrix} \delta \mathbf{x} + \begin{bmatrix} g_{11}(t) & g_{12}(t) \\ 0 & 0 \\ 0 & g_{32}(t) \end{bmatrix} \delta \mathbf{u} \quad (34)$$

where

$$\delta \mathbf{x} = \begin{bmatrix} \delta Q \\ \delta C \\ \delta T \end{bmatrix} \quad \delta \mathbf{u} = \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \quad (35)$$

and

$$f_{11}(t) = \frac{u_1^* + C_T T^* + C_P(u_2^*/\sqrt{T^*}) - \beta}{I} \quad (36)$$

$$f_{13}(t) = \frac{C_T Q^* - (C_P u_2^* Q^* / a \sqrt{T^*})}{I} \quad (37)$$

$$f_{33}(t) = -a u_2^* \quad (38)$$

$$g_{11}(t) = \frac{Q^*}{I} \quad (39)$$

$$g_{12}(t) = \frac{C_P Q^*}{I \sqrt{T^*}} \quad (40)$$

$$g_{32}(t) = -a T^* \quad (41)$$

Fig. 6 - Nominal optimal trajectories, $\gamma = 1.6$.

Fig. 7 - Nominal optimal controls, $\gamma = 1.6$.

Fig. 8 - Contaminated open-loop controls, $\gamma = 1.6$.

The output-error vector is

$$\delta \mathbf{y} = \begin{bmatrix} \delta Q \\ \delta T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta C \\ \delta T \end{bmatrix} = \mathbf{M} \delta \mathbf{x} \quad (42)$$

The forms of the error- and control-weighting matrices, $\mathbf{Q}_1$ and $\mathbf{Q}_2$, are

$$\mathbf{Q}_1 = \begin{bmatrix} {}_1q_{11} & 0 \\ 0 & {}_1q_{22} \end{bmatrix} \quad (43)$$

and

$$\mathbf{Q}_2 = \begin{bmatrix} {}_2q_{11} & 0 \\ 0 & {}_2q_{22} \end{bmatrix} \quad (44)$$

Fig. 9 - Effect of optimal feedback control.

Fig. 10 - Effect of optimal feedback control.

The Riccati type differential equations that must be solved to determine the time-varying feedback gains are given by

$$\dot{P}_1 = AP_1^2 - 2BP_1P_3 + CP_3^2 - 2f_{11}P_1 - \frac{2\beta P_2}{I} - \frac{2P_3}{M_c} - {}_1q_{11} \quad (45)$$

$$\begin{aligned} \dot{P}_2 = & AP_1P_2 - BP_1P_5 - BP_2P_3 + CP_3P_5 - \lambda P_1 \\ & + (\lambda - f_{11})P_2 - \frac{\beta P_4}{I} - \frac{P_5}{M_c} \end{aligned} \quad (46)$$

$$\begin{aligned} \dot{P}_3 = & AP_1P_3 - BP_1P_6 - BP_3^2 + CP_3P_6 - f_{13}P_1 - (f_{33} + f_{11})P_3 \\ & - \frac{\beta P_5}{I} - \frac{P_6}{M_c} \end{aligned} \quad (47)$$

$$\dot{P}_4 = AP_2^2 - 2BP_2P_5 + CP_5^2 - 2\lambda P_2 + 2\lambda P_4 \quad (48)$$

$$\begin{aligned} \dot{P}_5 = & AP_2P_3 - BP_2P_6 - BP_3P_5 + CP_5P_6 - f_{13}P_2 - \lambda P_3 \\ & - (f_{33} - \lambda)P_5 \end{aligned} \quad (49)$$

$$\dot{P}_6 = AP_3^2 - 2BP_3P_6 + CP_6^2 - 2f_{13}P_3 - 2f_{33}P_6 - {}_1q_{22} \quad (50)$$

where

$$A = \frac{g_{11}^2}{2q_{11}} + \frac{g_{12}^2}{2q_{22}} \quad (51)$$

$$B = -\frac{g_{12}g_{32}}{2q_{22}} \quad (52)$$

and

$$C = \frac{g_{32}^2}{2q_{22}} \quad (53)$$

The equations for the optimal feedback controls are

$$u_1^* = KQ1 \delta Q - KT1 \delta T = -\frac{g_{11}(t)}{2q_{11}} (P_1 \delta Q + P_3 \delta T) \quad (54)$$

$$u_2^* = -KQ2 \delta Q - KT2 \delta T = -\frac{g_{12}(t)}{2q_{22}} (P_1 \delta Q + P_3 \delta T) - \frac{g_{32}(t)}{2q_{22}} (P_3 \delta Q + P_6 \delta T) \quad (55)$$

where KQ1, KT1, KQ2, and KT2 are the time-varying gains. To date a synthesis study using time-varying gains has not been attempted. Results of a similar study,¹⁰ however, indicate that the optimal time-varying gains yield slightly better results from an error standpoint than average values of these gains over the control interval.

CONCLUSIONS

This study indicates that a linear optimal feedback control synthesized with a model-following system is useful as an adaptive closed-loop control for nuclear-rocket-engine start-up. The resulting optimal, time-varying feedback gains can be replaced by appropriate constant gains with little loss in performance.

The major disadvantage of this approach is that the optimum open-loop control program must be predetermined. In the case of the nuclear-rocket start-up, this appears to present no difficulties.

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Control of Absorption Cross Section for a Nuclear Rocket

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ABSTRACT

In this paper the two energy and one delayed-neutron-group kinetics equations with approximations are employed for the nuclear rocket. The perturbation equations for kinetics are derived with the assumption that the deviations from the reference flux are small in comparison with the reference values. The temperature effect due to the change of the Fermi age may be compensated by a feedback loop. The absorption cross section is changed accordingly so that the system will be made independent of the effect of temperature variation.

The spatial distribution of the flux and precursors is assumed to be sinusoidal in the axial direction of the nuclear rocket. For the purpose of obtaining maximum power in minimum time with the restriction that the reactivity or the absorption cross section not exceed a certain bound, the first optimization requires the reference flux and precursors to be exponential in time to the maximum power and to remain constant thereafter. A jump of flux occurs while the reactivity suddenly changes its value. A second optimization procedure is thus required to round off the jump and to minimize the error for the future, independent of the past disturbance or arbitrary starting error.

The reduced kinetics equations are obtained from the perturbed equation by introducing exponential weighting functions. The index of performance is chosen to be the integral of the square errors in perturbed neutron flux and the control effort. The maximum principle is used to derive the control laws with the use of the measured flux as feedback with a variable gain; the control laws in turn determine the absorption

cross section. The response of the flux and precursors having optimum control under different starting conditions is determined and plotted.

INTRODUCTION

Nuclear rockets^{1,2} require a fast start-up with relatively high reactivity. The power level may increase six decades in one minute. Lumped-parameter models of controlling such a system are given without linear approximations in previous papers.³⁻⁵ In this paper a distributed-parameter model of the nuclear rocket is considered. The actual control of the reactor is due to the change of the absorption cross section when the poison from the control drum in a nuclear rocket is made to vary.

KINETICS EQUATIONS FOR THE REACTOR IN DISTRIBUTED PARAMETERS

The two energy and one delayed-neutron-group reactor-kinetics equations are⁶

$$\nabla D_1 \nabla \phi_1 - \Sigma_{s1} \phi_1 + \nu(1 - \beta) \Sigma_f \phi_2 + \lambda C = \frac{1}{V_1} \frac{\partial \phi_1}{\partial t} \quad (1)$$

$$\nabla D_2 \nabla \phi_2 - \Sigma_a \phi_2 + \Sigma_{s1} \phi_1 = \frac{1}{V_2} \frac{\partial \phi_2}{\partial t} \quad (2)$$

$$\beta \nu \Sigma_f \phi_2 - \lambda C = \frac{\partial C}{\partial t} \quad (3)$$

where ϕ_1 = fast neutron flux

ϕ_2 = thermal neutron flux

D_1 = fast diffusion coefficient

D_2 = thermal diffusion coefficient

Σ_{s1} = removal cross section, spatial and time dependent

Σ_a = absorption cross section, spatial and time dependent

Σ_f = fission cross section, a constant

λ = equivalent decay constant for the one delayed-group case

C = equivalent concentration of precursor for the one delayed-group case

β = fraction of total number of fission neutrons that are delayed

ν = average number of fast neutrons released per fission

In a reactor with a relatively large fuel-loading density, the thermal leakage term, $\nabla D_2 \nabla \phi_2$, is small compared to the thermal absorption rate. Also, owing to the high velocity of fast neutrons, the term for the rate of change of fast neutrons $[(1/V_1)(\partial \phi_1 / \partial t)]$ can be neglected. With these approximations, Eqs. 1 and 2 can be written as

$$\nabla D_1 \nabla \phi_1 - \Sigma_{s1} \phi_1 + \nu(1 - \beta) \Sigma_f \phi_2 + \lambda C = 0 \quad (4)$$

$$\phi_1 = \frac{\Sigma_a}{\Sigma_{s1}} \phi_2 \left(1 + \frac{l_0}{T} \right) \quad (5a)$$

where l_0 = infinite-medium mean lifetime of neutrons

$$= 1/V_2 \Sigma_a$$

T = reactor period

$$= n/(dn/dt) = \phi_2/(d\phi_2/dt)$$

Since $l_0/T \ll 1$, Eq. 5a can be approximated as

$$\phi_1 = \frac{\Sigma_a}{\Sigma_{s1}} \phi_2 \quad (5b)$$

With the relation $\tau = D_1/\Sigma_{s1}$ and when $\frac{1}{2} \ln(\tau)$ in comparison with the term $\ln(\tau \Sigma_a \phi_2)$ is neglected, Eqs. 4 and 5b will give

$$\nabla^2(\tau \Sigma_a \phi_2) - \Sigma_a \phi_2 + \nu(1 - \beta) \Sigma_f \phi_2 + \lambda C = 0 \quad (6)$$

Rewrite Eq. 3:

$$\nu \beta \Sigma_f \phi_2 - \lambda C = \frac{\partial C}{\partial t} \quad (7)$$

(Equation 3 is repeated here as Eq. 7 for convenience to show the new set of kinetics equations, Eqs. 6 and 7.)

The Fermi age τ becomes

$$\tau = \tau_R + \tau_T \quad (\tau_R \gg \tau_T) \quad (8)$$

where τ_R is the Fermi age at reference temperature for graphite-uranium with hydrogen as coolant and τ_T is the increase of Fermi age from reference temperature to a given temperature.

PERTURBATION OF KINETICS EQUATIONS

During start-up of a nuclear reactor, some disturbances or errors may be introduced to cause the actual variable (flux) to depart from the ref-

erence (desired) variable (flux). For a nuclear rocket it can be assumed that most of the disturbances will occur along the axial direction of the reactor. The neutron flux, for a reactor with reflector, is almost constant in the transverse direction. The actual variables can be expressed by the sum of the reference variables and their deviations. If the deviations resulting from these unexpected disturbances and errors are small in comparison with the reference values, the cross-product terms of the deviations may be neglected.

Thus

$$\begin{aligned}
 \phi_2(z,t) &\longrightarrow \phi_R(z,t) + \Delta\phi(z,t) & [\Delta\phi(z,t) \ll \phi_R(z,t)] \\
 C(z,t) &\longrightarrow C_R(z,t) + \Delta C(z,t) & [\Delta C(z,t) \ll C_R(z,t)] \\
 \tau(z,t) &\longrightarrow \tau_R + \tau_T(z,t) & [\tau_T(z,t) \ll \tau_R = \text{constant}] \\
 \Sigma_a(z,t) &\longrightarrow \Sigma_R(z,t) + \Delta\Sigma_a(z,t) & [\Delta\Sigma_a(z,t) \ll \Sigma_R(z,t)]
 \end{aligned} \tag{9}$$

where the subscripts R refer to the reference variables. If the preceding quantities are substituted into Eqs. 6 and 7 and if the higher order terms are neglected, the perturbed reactor-kinetics equations are

$$\begin{aligned}
 \frac{\partial^2}{\partial z^2} \left[\tau_R \Sigma_R \phi_R \left(\frac{\tau_T}{\tau_R} + \frac{\Delta\Sigma_a}{\Sigma_R} + \frac{\Delta\phi}{\phi_R} \right) \right] - \Sigma_{aR} \phi_R \left(\frac{\Delta\Sigma_a}{\Sigma_R} + \frac{\Delta\phi}{\phi_R} \right) \\
 + \nu(1 - \beta)\Sigma_f \Delta\phi - \lambda \Delta C = 0
 \end{aligned} \tag{10}$$

$$\nu\Sigma_f \Delta\phi - \lambda \Delta C = \frac{\partial}{\partial t} (\Delta C) \tag{11}$$

FEEDBACK LOOP FOR COMPENSATING REACTIVITY FEEDBACK DUE TO TEMPERATURE EFFECT

In the kinetics equations the term most affected by temperature (including density) is the age τ_T . If a feedback loop is introduced such that the change of age due to temperature, τ_T , can be compensated by varying the absorption cross section, the perturbed kinetics equations derived from Eqs. 10 and 11 will be independent of temperature. Thus a much simpler mathematical form can be obtained. This idea can be achieved by letting

$$\Delta\Sigma_a(z,t) = \Delta\Sigma_1(z,t) + \Delta\Sigma_2(z,t) \tag{12}$$

and

$$\frac{\partial^2}{\partial z^2} \left[\tau_R \Sigma_R \phi_R \left(\frac{\tau_T}{\tau_R} + \frac{\Delta \Sigma_2}{\Sigma_R} \right) \right] - \Sigma_R \phi_R \frac{\Delta \Sigma_2}{\Sigma_R} = 0 \quad (13)$$

Then the perturbed kinetics equation (Eq. 10) becomes

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \left[\tau_R \Sigma_R \phi_R \left(\frac{\Delta \Sigma_1}{\Sigma_R} + \frac{\Delta \phi}{\phi_R} \right) \right] - \Sigma_R \phi_R \left(\frac{\Delta \Sigma_1}{\Sigma_R} + \frac{\Delta \phi}{\phi_R} \right) \\ + \nu(1 - \beta) \Sigma_f \Delta \phi + \lambda \Delta C = 0 \end{aligned} \quad (14)$$

The feedback control can be obtained by solving Eq. 13. The solution may be given in the form

$$\Delta \Sigma_2(z, t) = \frac{1}{\tau_R \phi_R(z, t)} \int_0^l G(z, \xi) \frac{\partial^2}{\partial \xi^2} [\Sigma_R \tau_T \phi_R(\xi, t)] d\xi \quad (15)$$

where $G(z, \xi)$ is the Green's function of the problem.

THE REFERENCE VARIABLES: FIRST OPTIMIZATION

Since the flux distribution in a reactor is essentially sinusoidal axially, it is reasonable to assume the reference variables as

$$\phi_R = \phi(t) \sin \frac{\pi}{l} z$$

and

$$C_R = C(t) \sin \frac{\pi}{l} z$$

Substituting the preceding equations into Eqs. 6 and 7, we obtain

$$\phi(t) = \frac{\lambda}{\Sigma_R \tau_T (\pi/l)^2 + \Sigma_R - \nu(1 - \beta) \Sigma_f} C(t)$$

and

$$\frac{d}{dt} [\ln C(t)] = \gamma(t) \quad (16)$$

where

$$\gamma(t) = \frac{\lambda \nu \beta \Sigma_f}{\Sigma_R \tau_R (\pi/l)^2 + \Sigma_R - \nu(1 - \beta) \Sigma_f} - \lambda \quad [0 \leq \gamma(t) \leq \gamma]$$

The reference absorption cross section, Σ_R , is assumed independent of z for the reference solutions in Eqs. 6 and 7. The power program, or ϕ , in a nuclear rocket requires that ϕ start at a low level, ϕ_0^a , and reach a high level in a minimum time. It is well known from the optimum theory that for a bounded control the optimum process requires the control variable $\gamma(t)$ to operate at its extreme value, γ , or zero, i.e., a bang-bang type control system. Thus the following reference variable is obtained for this purpose:

$$\begin{aligned} \phi_R(z, t) &= \phi_0^a e^{\gamma t} \sin \frac{\pi}{l} z & (0 \leq t \leq T_1) \\ C_R(z, t) &= C_0^a e^{\gamma t} \sin \frac{\pi}{l} z \\ \phi_R(z, t) &= \phi_0^b e^{\gamma T_1} \sin \frac{\pi}{l} z & (T_1 \leq t \leq T_2) \\ C_R(z, t) &= C_0^b e^{\gamma T_1} \sin \frac{\pi}{l} z \end{aligned} \quad (17)$$

where ϕ_0 , C_0 , γ , and T_1 are constants. The reference absorption cross section, Σ_R , is therefore found to be a constant at each time interval;

$$\begin{aligned} \phi_0^a &= \frac{\gamma + \lambda}{\nu \beta \Sigma_f} C_0^a & (0 \leq t \leq T_1) \\ \Sigma_R^a &= \frac{\nu \Sigma_f}{1 + \tau_R (\pi/l)^2} \left(1 - \frac{\gamma \beta}{\gamma + \lambda} \right) \\ \phi_0^b &= \frac{\lambda}{\nu \beta \Sigma_f} C_0^b & (T_1 < t \leq T_2) \\ \Sigma_R^b &= \frac{\nu \Sigma_f}{1 + \tau_R (\pi/l)^2} \end{aligned} \quad (18)$$

where $C_0^a = C_0^b$ since C_R is continuous at $t = T_1$. It is noted that a jump of reference flux occurs at $t = T_1$ since $\phi_0^a \neq \phi_0^b$.

Equation 17 gives the solution of a time-optimum problem by Eqs. 6 and 7 with the reactivity constraint ρ_c . Thus

$$k_{\text{eff}} \simeq \frac{\nu \Sigma_f}{\Sigma_a [1 + \tau_R (\pi/l)^2]} = \frac{1}{1 - [\gamma\beta/(\gamma + \lambda)]} \simeq 1 + \frac{\gamma\beta}{\gamma + \lambda} = 1 + \Delta k \quad (19)$$

or

$$\rho = \frac{\Delta k}{\beta} = \frac{\gamma}{\gamma + \lambda} \leq \rho_c \quad (\text{the constraint})$$

TRANSFORMATION OF VARIABLES FOR THE PERTURBED KINETICS EQUATIONS

Exponential weighting functions to the variables $\Delta\phi(z,t)$ and $\Delta C(z,t)$ in Eqs. 14 and 7 are introduced as

$$\phi_q^a(z,t) = \frac{\Delta\phi(z,t)}{\phi_0^a e^{\gamma t}} \quad (t \leq T_1) \quad (20)$$

$$C_q^a(z,t) = \frac{\Delta C(z,t)}{C_0^a e^{\gamma t}}$$

Thus Eqs. 14 and 7 become, respectively,

$$\begin{aligned} \tau_R \frac{\partial^2}{\partial z^2} [\Sigma_R^a \phi_q^a(z,t)] + [\nu(1 - \beta)\Sigma_f - \Sigma_R^a] \phi_q^a(z,t) \\ + \lambda \frac{C_0^a}{\phi_0^a} C_q^a(z,t) = \frac{\mu^a(z,t)}{\phi_0^a e^{\gamma t}} \end{aligned} \quad (21a)$$

$$\frac{\partial}{\partial t} C_q^a(z,t) = \nu\beta\Sigma_f \frac{\phi_0^a}{C_0^a} \phi_q^a(z,t) - (\lambda + \gamma) C_q^a(z,t) \quad (21b)$$

where

$$\mu^a(z,t) = -\tau_R \frac{\partial^2}{\partial z^2} [\phi_R^a(z,t) \Delta\Sigma_1^a(z,t)] + \phi_R^a(z,t) \Delta\Sigma_1^a(z,t) \quad (22)$$

By substituting the value of $\phi_R^a(z,t)$ in Eq. 17 into Eq. 22, we obtain

$$\mu^a(z,t) = \phi_0^a e^{\gamma t} \left[-\tau_R \frac{\partial^2}{\partial z^2} \sin \frac{\pi}{l} z \Delta\Sigma_1^a(z,t) + \sin \frac{\pi}{l} z \Delta\Sigma_1^a(z,t) \right] \quad (23)$$

Equation 21 is further transformed by assuming that the perturbed flux follows a sinusoidal distribution in the axial direction; thus

$$\begin{aligned}\phi_q^a &= X_1^a(t) \sin \frac{\pi}{l} z \\ C_q^a &= X_2^a(t) \sin \frac{\pi}{l} z\end{aligned}\quad (24)$$

The following equations are obtained by substituting Eqs. 18 and 24 into Eq. 21:

$$-A^a X_1^a(t) + A^a X_2^a(t) - u^a(t) = 0 \quad (t \leq T_1) \quad (25)$$

$$\frac{d}{dt} X_2^a = (\lambda + \gamma) X_1^a(t) - (\lambda + \gamma) X_2^a(t)$$

where

$$-A^a = -\tau_R \frac{\pi^2}{l^2} \Sigma_R^a - \Sigma_R^a + \nu(1 - \beta)\Sigma_f = -\frac{\lambda\nu\beta\Sigma_f}{\lambda + \gamma} \quad (26)$$

and

$$u^a(t) \sin \frac{\pi}{l} z = -\tau \frac{\partial^2}{\partial z^2} \left[\Delta\Sigma_1(z, t) \sin \frac{\pi}{l} z \right] + \Delta\Sigma_1(z, t) \sin \frac{\pi}{l} z \quad (27)$$

Equation 27 admits a solution of the form

$$\Delta\Sigma_1(z, t) \longrightarrow \Delta\Sigma_1(t) = \frac{1}{1 + \tau_R(\pi/l)^2} u^a(t) \quad (28)$$

THE CONTROL SYSTEM: SECOND OPTIMIZATION

If the system starts with the right initial condition C_0/ϕ_0 in Eq. 18 and the right control with a jump of Σ_R from Σ_R^a to Σ_R^b at $t = T_1$, then theoretically the reactor variables will follow Eq. 17 with no error. However, the actual initial condition is not always the value given in Eq. 18. Thus the control in Σ_a will differ from Σ_R by the amount $\Delta\Sigma_a = \Delta\Sigma_1 + \Delta\Sigma_2$. Again the quantity $\Delta\Sigma_1(z, t)$ is determined from Eq. 28 if the control $u(t)$ is known. The problem now is to find $u(t)$ for any arbitrary starting conditions $X_1(t)$ and $X_2(t)$ in Eq. 25.

The control system should be designed such that the variables would follow as closely as possible the reference variables given in Eq. 17. For an ideal system the variables X_1 , X_2 , and u in Eq. 25

should be identically zero. With this in mind we are seeking for a minimum of a functional known as the index of performance.

In the reactor start-up problem, perturbed neutron flux and control effort are of interest. An index of performance is, therefore, chosen as

$$I = \int_t^{T_1} \{ [X_1^a(\sigma)]^2 + \eta(\sigma) [u(\sigma)]^2 \} d\sigma + \int_{T_1}^{T_2} \{ [X_1^b(\sigma)]^2 + \eta(\sigma) [u(\sigma)]^2 \} d\sigma \quad (29)$$

where $\eta(\sigma)$ is a weighting function. The weighting function is assumed to be continuous except at the time $t = T_1$, where it may be discontinuous. It is noted that the lower limit is the present time t in Eq. 29. This is consistent with the principle of dynamic programming,⁷ which is to minimize the integral for the time to come no matter what the present time is. If we can measure the perturbed flux X_1 at the present time, we can find a control $u(t)$ and thus $\Delta \Sigma_1(t)$, using X_1 as feedback with a variable gain. In this manner the functional I in Eq. 29 will be kept at a minimum while subject to disturbance at any other time, not only to errors at the starting conditions. The preceding viewpoint is very important in that the source or cause of disturbance is immaterial as long as the output $X_1(t)$ can be detected and readjusted by the control element producing $\Delta \Sigma_1(t)$.

We may define a new variable, $X_3(\sigma)$, such that

$$X_3(\sigma)|_{\sigma=T_2} = I$$

Thus

$$X_3(\sigma) = \int_{\theta=t}^{\sigma} \{ [X_1^a(\theta)]^2 + \eta(\theta) [u^a(\theta)]^2 \} d\theta \quad (t \leq \sigma \leq T_1) \quad (30a)$$

and

$$X_3(\sigma) = \int_{\theta=t}^{T_1} \{ [X_1^a(\theta)]^2 + \eta(\theta) [u^a(\theta)]^2 \} d\theta + \int_{\theta=T_1}^{\sigma} \{ [X_1^b(\theta)]^2 + \eta(\theta) [u^b(\theta)]^2 \} d\theta \quad (T_1 \leq \sigma \leq T_2) \quad (30b)$$

APPLICATION OF MAXIMUM PRINCIPLE FOR A SYSTEM OF ALGEBRAIC AND DIFFERENTIAL EQUATIONS

By the rearrangement of Eq. 25 and the differentiation of Eq. 30, we obtain the following:

$$\begin{aligned}
0 &= [-A^a X_1^a(\sigma) + A^a X_2^a(\sigma) - u^a(\sigma)] = f_1 \\
\frac{d}{d\sigma} x_2^a(\sigma) &= (\lambda + \gamma) X_1^a(\sigma) - (\lambda + \sigma) X_2^a(\sigma) = f_2 \\
\frac{d}{d\sigma} x_3^a(\sigma) &= [X_1^a(\sigma)]^2 + \eta [u^a(\sigma)]^2 = f_3
\end{aligned} \tag{31}$$

where σ is the future time.

The optimum system is defined as the system for which

$$S(T_2) = \sum_{i=1}^3 C_i X_i(\sigma = T_2) \quad (C_1 = C_2 = 0) \quad (C_3 = 1) \tag{32}$$

is a minimum with respect to $u(\sigma)$. The Hamiltonian as given in Appendix A is

$$H = \sum_{i=1}^3 p_i(\sigma) f_i \tag{33}$$

where the p_i are the auxiliary variables.

Thus

$$\begin{aligned}
H &= p_1(\sigma) [-A^a X_1^a(\sigma) + A^a X_2^a(\sigma) - u^a(\sigma)] + p_2(\sigma) [(\lambda + \gamma) X_1^a(\sigma) \\
&\quad - (\lambda + \sigma) X_2^a(\sigma)] + p_3(\sigma) \{[X_1^a(\sigma)]^2 + \eta [u^a(\sigma)]^2\}
\end{aligned} \tag{34}$$

A sufficient condition for a minimum of S is that the Hamiltonian, H , be maximized with respect to the control vector for all time. For a system with unsaturated perturbed control, we have from Eq. A.15 in Appendix A

$$\frac{\partial H}{\partial u_*^a} = 0 = -p_1(\sigma) + p_3(\sigma) 2\eta u_*^a \tag{35}$$

or

$$u_*^a(\sigma) = \frac{p_1(\sigma)}{2\eta p_3(\sigma)} \tag{36}$$

where the asterisk denotes the optimum condition.

To complete the derivation of the optimum control law, we must develop the differential equations⁸ for the auxiliary variables. From Eqs. A.13 and A.14 in Appendix A, we have

$$0 = - \frac{\partial H}{\partial X_i} \quad (i = 1)$$

$$\frac{dp_i(\sigma)}{d\sigma} = - \frac{\partial H}{\partial X_i} \quad (i = 2, 3) \quad (37)$$

Applying Eq. 37 to Eq. 34, we obtain

$$0 = - [p_1(\sigma) (-A^a) + p_2(\sigma) (\lambda + \gamma) + 2p_3(\sigma) X_1^a(\sigma)] \quad (38)$$

$$\frac{dp_2}{dt} = -p_1(\sigma) A^a + p_2(\sigma) (\lambda + \gamma) \quad (39)$$

$$\frac{dp_3}{dt} = 0 \quad (t < \sigma \leq T_1) \quad (40)$$

The free terminal conditions or natural-boundary conditions are obtained from Eqs. A.9 and 32:

$$p_2(\sigma = T_2) = -C_2 = 0 \quad (41)$$

$$p_3(\sigma = T_2) = -C_3 = -1 \quad (42)$$

It is concluded from Eqs. 40 and 42 that the quantity

$$p_3(\sigma) = -1 \quad \text{for all } \sigma \quad (43)$$

Substituting Eq. 43 into Eqs. 38, 39, and 36, we obtain

$$-A^a p_1(\sigma) + (\lambda + \gamma) p_2(\sigma) - 2X_1^a(\sigma) = 0$$

$$\frac{dp_2}{d\sigma} = -A^a p_1(\sigma) + (\lambda + \sigma) p_2(\sigma)$$

$$u_*^a(\sigma) = - \frac{1}{2\eta} p_1(\sigma) \quad (44)$$

Similarly, the differential equations of the auxiliary variables for the interval $T_1 < \sigma \leq T_2$ can be obtained by changing superscript a to b and changing γ to zero.

OPTIMUM CONTROL LAW

To determine the optimum control law, we must solve the algebraic and differential equations for the auxiliary variables and the state variables. Equations similar to Eqs. 25 and 44 are given here for the interval $T_1 < t \leq T_2$:

$$-A^b X_1^b(\sigma) + A^b X_2^b(\sigma) - u_*^b(\sigma) = 0 \quad (45)$$

$$\frac{d}{d\sigma} X_2^b(\sigma) = \lambda X_1^b(\sigma) - \lambda X_2^b(\sigma) \quad (46)$$

$$-A^b p_1(\sigma) + \lambda p_2(\sigma) - 2X_1^b(\sigma) = 0 \quad (47)$$

$$\frac{d}{d\sigma} p_2(\sigma) = -A^b p_1(\sigma) + \lambda p_2(\sigma) \quad (48)$$

$$u_*^b(\sigma) = -\frac{1}{2\eta} p_1(\sigma) \quad (49)$$

If p_1 is eliminated from Eqs. 49 and 47,

$$u_*^b(\sigma) = \frac{A^b}{2\alpha^b} (-\lambda p_2 + 2X_1^b) \quad (50)$$

where

$$\alpha^b = \eta(A^b)^2 \quad (51)$$

Substituting Eq. 50 into Eq. 45 gives

$$X_2^b = \frac{1}{2\alpha^b} [-\lambda p_2 + 2(1 + \alpha^b) X_1^b] \quad (52)$$

The following can be obtained by substituting Eq. 52 into Eq. 46:

$$\left(\frac{d}{d\sigma} + \lambda - \frac{\lambda \alpha^b}{1 + \alpha^b} \right) X_1^b - \frac{\lambda}{2(1 + \alpha^b)} \left(\frac{d}{d\sigma} + \lambda \right) p_2 = 0 \quad (53)$$

By solving for p_1 from Eq. 47 and substituting into Eq. 48, we have

$$2X_1^b - \frac{d}{d\sigma} p_2 = 0 \quad (54)$$

Solutions for p_2 and X_1^b from Eqs. 53 and 54 are

$$p_2(\sigma) = F_1(t) \cosh \omega^b (\sigma - t) + F_2(t) \sinh \omega^b (\sigma - t) \quad (55)$$

and

$$X_1^b(\sigma) = \frac{\omega^b}{2} [F_1(t) \sinh \omega^b (\sigma - t) + F_2(t) \cosh \omega^b (\sigma - t)] \quad (56)$$

where

$$\omega^b = \lambda \sqrt{1 - \frac{\alpha^b}{1 + \alpha^b}} \quad (57)$$

$$\omega^b \simeq \lambda \quad \text{for small values of } \alpha^b$$

The quantity t is carried as a parameter.

From Eq. 52 we obtain

$$X_2^b = -\frac{1}{2\alpha^b} \{F_1(t) [\lambda \cosh \omega^b (\sigma - t) - \omega^b (1 + \alpha^b) \sinh \omega^b (\sigma - t)] \\ + F_2(t) [\lambda \sinh \omega^b (\sigma - t) - \omega^b (1 + \alpha^b) \cosh \omega^b (\sigma - t)]\} \quad (58)$$

With the condition that X_1^b is equal to the measured value at $\sigma = t$, Eq. 56 becomes

$$X_1^b(t) = \frac{\omega^b}{2} F_2(t) \quad (59)$$

Using boundary condition Eq. 41 for Eq. 55, we obtain

$$F_1(t) = -F_2(t) \tanh \omega^b (T_2 - t) \\ = -\frac{2}{\omega^b} \tanh \omega^b (T_2 - t) X_1^b(t) \quad (60)$$

The optimum control law is obtained from Eqs. 50, 55, and 60:

$$u_*^b(t) = u_*^b(\sigma)|_{\sigma=t} = \frac{A^b}{2\alpha^b} [-\lambda p_2|_{\sigma=t} + 2X_1^b(t)] \\ = \frac{A^b X_1^b(t)}{\alpha^b} \left[\frac{\lambda}{\omega^b} \tanh \omega^b (T_2 - t) + 1 \right] \quad (T_1 < t \leq T_2) \quad (61)$$

It should be noted that there is a jump of reference X_1 at $t = T_1$ (see Fig. 1). This discontinuity is due to the simplified mathematical expression obtained by dropping the $\partial\phi_1/\partial t$ term in kinetics Eq. 1. How-

Fig. 1 - Response of flux.

ever, the jump does not physically exist in the reactor if the term $\partial\phi_1/\partial t$ is not dropped from Eq. 1. For this situation to be modified, the actual flux must be continuous everywhere. This requirement is equivalent to letting the precursor and its derivatives be continuous at $t = T_1$. From this point of view, the conditions that will be used for derivation of the optimum control law in the interval $t \leq T_1$ are imposed as follows:

1. The natural-boundary condition is given in Eq. 41.
2. The initial condition is expressed as

$$X_1^a|_{\sigma=t} = X_1^a(t) \quad (\text{measured value}) \quad (62)$$

3. For the precursors to remain continuous at $t = T_1$, it is required that

$$X_2^a(\sigma)|_{\sigma=T_1} = X_2^b(\sigma)|_{\substack{\sigma=t \\ t=T_1}} \quad (63)$$

4. The requirement for the continuity of the derivative of the precursors at $t = T_1$ is mathematically expressed as (see Appendix B)

$$\frac{\partial}{\partial \sigma} X_2^a(\sigma)|_{\sigma=T_1} = \frac{\partial}{\partial \sigma} X_2^b(\sigma)|_{\substack{\sigma=t \\ t=T_1}} - \gamma - \gamma X_2^a(\sigma)|_{\sigma=T_1} \quad (64)$$

Thus the optimum control law derived in Appendix C for $t \leq T_1$ is

$$u_*^a(t) = \frac{A^a}{\alpha^a} \left\{ (\lambda + \gamma) \frac{(1/\omega^a) [(\gamma + \omega^b \Omega)G + \omega^a H] X_1^a(t) - \alpha^a \gamma}{\omega^a G + (\gamma + \omega^b \Omega)H} + X_1^a(t) \right\} \quad (65)$$

where $\alpha^a = \eta(A^a)^2$

$$\omega^a = (\lambda + \gamma) \sqrt{1 - (\alpha^a / 1 + \alpha^a)}$$

$$\simeq \lambda + \gamma \text{ for small values of } \alpha^a$$

$$\Omega = \frac{\lambda + \omega^b (1 + \alpha^b) \tanh \omega^b (T_2 - T_1)}{\omega^b (1 + \alpha^b) + \lambda \tanh \omega^b (T_2 - T_1)}$$

$$G = (\lambda + \gamma) \sinh \omega^a (T_1 - t) - \omega^a (1 + \alpha^a) \cosh \omega^a (T_1 - t)$$

$$H = (\lambda + \gamma) \cosh \omega^a (T_1 - t) - \omega^a (1 + \alpha^a) \sinh \omega^a (T_1 - t)$$

For the homogeneous C - ^{235}U (hydrogen as coolant) reactor with the atom ratio about 500, if $\lambda \simeq 0.1 \text{ sec}^{-1}$, $\beta \simeq 7.5 \times 10^{-3}$, $\nu \simeq 2.5$, $\Sigma_f \simeq 0.09 \text{ cm}^{-1}$, $\tau_R \simeq 325 \text{ cm}^2$, $T_2 - T_1 \geq 60 \text{ sec}$, $\gamma \simeq 0.23 \text{ sec}^{-1}$ (equivalent to 69¢ reactivity), and $\eta \simeq 2.5 \times 10^4 \text{ cm}^2$, we can approximate the following quantities as

$$A^a = \frac{\lambda \nu \beta \Sigma_f}{\lambda + \gamma} \simeq 5.7 \times 10^{-4} \text{ cm}^{-1}$$

$$A^b = \nu \beta \Sigma_f \simeq 1.875 \times 10^{-3} \text{ cm}^{-1}$$

$$\alpha^a \simeq 0.01 \quad \alpha^b \simeq 0.1$$

$$\omega^b \simeq \lambda \quad \omega^a \simeq \lambda + \gamma \simeq \gamma + \omega^b$$

$$\tanh \omega^b (T_2 - T_1) \simeq 1$$

Substituting some of these approximate quantities into Eq. 63, we have

$$\Omega \simeq \frac{\lambda + \omega^b (1 + \alpha^b)}{\omega^b (1 + \alpha^b) + \lambda} = 1$$

$$\omega^a G + (\gamma + \omega^b \Omega)H \simeq \omega^a (G + H) \simeq -(\lambda + \gamma) \alpha^a e^{(\lambda + \gamma)(T_1 - t)}$$

$$(\gamma + \omega^b \Omega)H + \omega^a G \simeq \omega^a (G + H) \simeq -(\lambda + \gamma) \alpha^a e^{(\lambda + \gamma)(T_1 - t)}$$

and

$$u_*^a(t) \simeq \frac{2A^a}{\alpha^a} X_1^a(t) + \frac{A^a \gamma}{\alpha^a(\lambda + \gamma)} e^{-(\lambda + \gamma)(T_1 - t)} \quad (66)$$

RESPONSE OF THE FLUX AND PRECURSOR HAVING OPTIMUM CONTROL

The optimum flux, ϕ_* , subjected to initial errors can be obtained by solving the following differential equations, which are derived from Eqs. 66, 61, 25, and 45:

$$\frac{d}{dt} X_{1*}^a + \frac{2(\lambda + \gamma)}{\alpha^a + 2} X_{1*}^a = -\frac{2\gamma}{\alpha^a + 2} e^{-(\lambda + \gamma)(T_1 - t)} \quad (t \leq T_1) \quad (67)$$

$$\frac{d}{dt} X_{1*}^b + \frac{2\lambda}{\alpha^b + 2} X_{1*}^b \simeq 0 \quad (T_1 \leq t < T_2) \quad (68)$$

For $\alpha^b \simeq 0.1$ and $\alpha^a \simeq 0.01$, i.e., $\alpha^a + 2 \simeq 2$ and $\alpha^b + 2 \simeq 2$, the solutions of Eqs. 67 and 68 are

$$X_{1*}^a = X_{10}^a e^{-(\lambda + \gamma)t} - \frac{\gamma}{\lambda + \gamma} e^{-(\lambda + \gamma)T_1} \sinh(\lambda + \gamma)t \quad (t \leq T_1) \quad (69)$$

$$X_{1*}^b \simeq X_{10}^b e^{-\lambda(t - T_1)} \quad (T_1 \leq t < T_2) \quad (70)$$

where X_{10}^a is the value of X_1^a at $t = 0$ and X_{10}^b is the value of X_1^b at $t = T_1$.

By using the definition of X_1^a and X_1^b in Eqs. 24 and 20, we obtain

$$\Delta\phi^a(z, t) = \left[\Delta\phi_0^a e^{-\lambda t} - \frac{\gamma}{\lambda + \gamma} \phi_0^a e^{-(\lambda + \gamma)T_1 + \gamma t} \sinh(\lambda + \gamma)t \right] \sin \frac{\pi}{l} z \quad (t \leq T_1) \quad (71)$$

$$\Delta\phi^b(z, t) \simeq [\Delta\phi_0^b e^{-\lambda(t - T_1)}] e^{\gamma T_1} \sin \frac{\pi}{l} z \quad (T_1 \leq t < T_2) \quad (72)$$

where

$$\frac{\gamma}{\lambda + \gamma} \phi_0^a = \phi_0^a - \phi_0^b \quad \text{from Eq. 18}$$

$$\begin{aligned}\Delta\phi_0^b e^{\gamma T_1} &= (\phi_0^a - \phi_0^b) e^{\gamma T_1} - \Delta\phi^a|_{t=T_1} \\ &\simeq \frac{\phi_0^a - \phi_0^b}{2} e^{\gamma T_1} \quad (\text{see Fig. 1})\end{aligned}\quad (73)$$

Then

$$\begin{aligned}\phi_* &= \phi_R + \Delta\phi \\ \phi_*^a &= [\phi_0^a e^{\gamma t} + \phi_0^a e^{-\lambda t} - (\phi_0^a - \phi_0^b) e^{-(\lambda+\gamma)T_1 + \gamma t} \\ &\quad \times \sinh(\lambda + \gamma)t] \sin \frac{\pi}{l} z \quad (t \leq T_1)\end{aligned}\quad (74)$$

$$\begin{aligned}\phi_*^b &\simeq \left[\frac{\phi_0^a - \phi_0^b}{2} e^{\gamma T_1} e^{-\lambda(t-T_1)} + \phi_0^b e^{\gamma T_1} \right] \sin \frac{\pi}{l} z \\ &\quad (T_1 \leq t < T_2)\end{aligned}\quad (75)$$

When $t = T_1$, it is known that

$$\phi_0^a e^{-\lambda T_1} \simeq 0 \quad \text{and} \quad \phi_*^a|_{t=T_1} \simeq \phi_*^b|_{t=T_1}$$

and the derivative of ϕ_* at $t = T_1$ is discontinuous.

If Eqs. 74 and 75 are substituted into Eq. 7, the optimum precursors are obtained:

$$\begin{aligned}C_*^a &= \left\{ C_0^a e^{\gamma t} + [(\lambda + \gamma)t + 1] e^{-\lambda t} \Delta C_0^a + \frac{\gamma}{2} e^{-(\lambda+\gamma)T_1} \left[t e^{-\lambda t} \right. \right. \\ &\quad \left. \left. - \frac{e^{\gamma t}}{\lambda + \gamma} \sinh(\lambda + \gamma)t \right] \right\} \sin \frac{\pi}{l} z \quad (t \leq T_1)\end{aligned}\quad (76)$$

$$\begin{aligned}C_*^b &\simeq C_*^a|_{t=T_1} e^{-\lambda(t-T_1)} + C_0^a e^{\gamma T_1} + \frac{C_0^a \gamma}{2} e^{\gamma T_1} t e^{-\lambda(t-T_1)} \\ &\quad \times \sin \frac{\pi}{l} z \quad (T_1 \leq t < T_2)\end{aligned}\quad (77)$$

It can be proved that at $t = T_1$ both C_* and $(\partial/\partial t) C_*$ are continuous.

The responses ϕ_* and C_* are calculated for the typical nuclear-powered rocket start-up, i.e., $\gamma \simeq 0.23 \text{ sec}^{-1}$, $\lambda \simeq 0.1 \text{ sec}^{-1}$, by using Eqs. 72 to 75 with initial errors of 100% and -50%. The reference ini-

tial power is 10 kw, and the final power is 10^6 kw. The rate of rise of power is 1 decade per 10 sec, corresponding to 69¢ of reactivity. Two curves for ϕ_*/ϕ_0^a and C_*/C_0^a vs. time are plotted in Figs. 1 and 2, re-

Fig. 2 - Response of precursors.

spectively. The actual response shown in Fig. 1 is very close to the desired response, even when subjected to an initial error as high as 100%.

CONCLUSION

Attempts are made to control the absorption cross section of the nuclear rocket with distributed-parameter kinetics. Two optimization procedures are taken, one with bang-bang control for the reference flux and precursors and the other to eliminate the jump by the maximum principle. Closed-form solutions for the control laws are obtained.

APPENDIX A

MAXIMUM PRINCIPLE FOR A SYSTEM OF ALGEBRAIC AND DIFFERENTIAL EQUATIONS

The system is described by the following set of algebraic and differential equations:

$$0 = f_i(X_1, \dots, X_m, X_{m+1}, \dots, X_n, X_{n+1}, u_1, \dots, u_k, \dots, u_r) \quad (i = 1, \dots, m) \quad (A.1)$$

$$\frac{dx_i(\sigma)}{d\sigma} = f_i(X_1, \dots, X_m, X_{m+1}, \dots, X_n, X_{n+1}, u_1, \dots, u_k, \dots, u_r) \quad (i = m+1, \dots, n+1) \quad (A.2)$$

$$X_i(\sigma = t) = X_i(t) \quad (i = 1, \dots, m, m+1, \dots, n+1) \quad (A.3)$$

where $x_i(\sigma)$ are state variables and u_k are control variables. The problem is to minimize

$$\begin{aligned} S(T_2) &= \sum_{i=1}^{n+1} C_i X_i(\sigma = T_2) \\ &= \int_t^{T_2} \sum_{i=1}^{n+1} C_i \dot{X}_i d\sigma + \sum_{i=1}^{n+1} C_i X_i(\sigma = t) \end{aligned} \quad (A.4)$$

The method of calculus of variations may be applied to derive the maximum principle for the case of unbounded control variables for fixed time. The quantity $S(T_2)$ in Eq. A.4 remains the same by introducing Lagrange multipliers $p_i(\sigma)$ to adjoin the constraints imposed by Eqs. A.1 and A.2:

$$\begin{aligned} S(T_2) &= \int_t^{T_2} \sum_{i=1}^{n+1} C_i \dot{X}_i d\sigma + \sum_{i=1}^{n+1} C_i X_i(\sigma = t) \\ &\quad + \int_t^{T_2} \left[\sum_{i=1}^m p_i(0 - f_i) + \sum_{i=m+1}^{n+1} p_i(\dot{X}_i - f_i) \right] d\sigma \end{aligned} \quad (A.5)$$

First-order variations are taken for the state and control variables about a stationary path as

$$\begin{aligned} X_i &= \bar{X}_i + \epsilon \xi_i(t) \\ \dot{X}_i &= \dot{\bar{X}}_i + \epsilon \dot{\xi}_i(t) \\ u_k &= \bar{u}_k + \epsilon \zeta_k(t) \end{aligned} \quad (A.6)$$

Stationary values of $S(T_2)$ in Eq. A.5 are determined by taking the partial derivative of $S(T_2)$ with respect to ϵ and setting the result equal to zero:

$$\begin{aligned} \frac{\partial S}{\partial \epsilon} = & \int_t^{T_2} \left(\sum_{i=1}^{n+1} C_i \dot{\xi}_i + \sum_{i=m+1}^{n+1} p_i \dot{\xi}_i - \sum_{i=1}^{n+1} p_i \frac{\partial \bar{f}_i}{\partial X_j} \xi_j \right) d\sigma \\ & - \int_t^{T_2} \left(\sum_{i=1}^{n+1} \sum_{k=1}^r p_i \frac{\partial \bar{f}_i}{\partial u_k} \zeta_k \right) d\sigma = 0 \end{aligned} \quad (A.7)$$

The first two terms of the right-hand side of Eq. A.7 may be integrated by parts to obtain the following:

$$\begin{aligned} \sum_{i=1}^m C_i \xi_i \Big|_t^{T_2} + \sum_{i=m+1}^{n+1} (C_i + p_i) \xi_i \Big|_t^{T_2} = & \int_t^{T_2} \left(\sum_{i=m+1}^{n+1} \dot{p}_i \right. \\ & \left. + \sum_{j=1}^{n+1} p_j \frac{\partial \bar{f}_j}{\partial X_i} \right) \xi_i d\sigma + \int_t^{T_2} \sum_{i=1}^{n+1} \sum_{k=1}^r p_i \frac{\partial \bar{f}_i}{\partial u_k} \zeta_k d\sigma \end{aligned} \quad (A.8)$$

Since Eq. A.8 must be satisfied for arbitrary ξ_i and ζ_k , the coefficients of these terms must be zero; i.e.,

$$C_i = 0 \quad (i = 1, \dots, m) \quad (A.9)$$

$$C_i = -p_i(T_2) \quad (i = m+1, \dots, n+1)$$

$$0 = - \sum_{j=1}^{n+1} p_j \frac{\partial \bar{f}_j}{\partial X_i} \quad (i = 1, \dots, m) \quad (A.10)$$

$$p_i = - \sum_{j=1}^{n+1} p_j \frac{\partial \bar{f}_j}{\partial X_i} \quad (i = m+1, \dots, n+1)$$

$$\sum_{i=1}^{n+1} p_i \frac{\partial \bar{f}_i}{\partial u_k} = 0 \quad (k = 1, \dots, r) \quad (A.11)$$

If the Hamiltonian is defined as

$$H = \sum_{i=1}^{n+1} p_i \bar{f}_i \quad (A.12)$$

then

$$0 = - \frac{\partial H}{\partial X_i} \quad (i = 1, \dots, m) \quad (A.13)$$

$$\frac{dp_i}{d\sigma} = - \frac{\partial H}{\partial X_i} \quad (i = m + 1, \dots, n + 1) \quad (\text{A.14})$$

$$\frac{\partial H}{\partial u_k} = 0 \quad (k = 1, \dots, r) \quad (\text{A.15})$$

APPENDIX B

CONDITION FOR THE DERIVATIVE OF A PRECURSOR TO BE CONTINUOUS AT $t = T_1$

By definition from Eqs. 20, 24, and 17,

$$\begin{aligned} X_2^a(\sigma) &= \frac{C_q^a(\sigma, z)}{\sin(\pi/l) z} = \frac{\Delta C^a(\sigma, z)}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} = \frac{C^a - C_R^a}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} \\ &= \frac{C^a}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} - 1 \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} X_2^b(\sigma) &= \frac{C_q^b(\sigma, z)}{\sin(\pi/l) z} = \frac{\Delta C^b(\sigma, z)}{C_o^a e^{\gamma T_1} \sin(\pi/l) z} = \frac{C^b - C_R^b}{C_o^a e^{\gamma T_1} \sin(\pi/l) z} \\ &= \frac{C^b}{C_o^a e^{\gamma T_1} \sin(\pi/l) z} - 1 \end{aligned} \quad (\text{B.2})$$

Thus

$$\begin{aligned} \frac{\partial}{\partial \sigma} X_2^a(\sigma) \Big|_{\sigma=T_1} - \frac{\partial}{\partial \sigma} X_2^b(\sigma) \Big|_{\substack{\sigma=t \\ t=T_1}} &= \frac{\partial}{\partial \sigma} \frac{C^a}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} \Big|_{\sigma=T_1} \\ &- \frac{\partial}{\partial \sigma} \frac{C^b}{C_o^a e^{\gamma T_1} \sin(\pi/l) z} \Big|_{\substack{\sigma=t \\ t=T_1}} = \frac{\partial C^a / \partial \sigma}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} \Big|_{\sigma=T_1} \\ &- \frac{\gamma C^a}{C_o^a e^{\gamma\sigma} \sin(\pi/l) z} \Big|_{\sigma=T_1} - \frac{\partial C^b / \partial \sigma}{C_o^a e^{\gamma T_1} \sin(\pi/l) z} \Big|_{\substack{\sigma=t \\ t=T_1}} \end{aligned} \quad (\text{B.3})$$

For continuous slope of precursors at $\sigma = T_1$, i.e.,

$$\frac{\partial C^a(\sigma)}{\partial \sigma} \Big|_{\sigma=T_1} = \frac{\partial C^b(\sigma)}{\partial \sigma} \Big|_{\substack{\sigma=t \\ t=T_1}}$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial \sigma} X_2^a(\sigma)|_{\sigma=T_1} - \frac{\partial}{\partial \sigma} X_2^b(\sigma)|_{\substack{\sigma=t \\ t=T_1}} &= - \frac{\gamma C^a(\sigma, z)}{C_0^a e^{\gamma \sigma} \sin(\pi/l) z} \Big|_{\sigma=T_1} \\ &- \gamma \frac{C_R^a + \Delta C^a}{C_0^a e^{\gamma \sigma} \sin(\pi/l) z} \Big|_{\sigma=T_1} = -\gamma - \gamma X_2^a(\sigma)|_{\sigma=T_1} \end{aligned} \quad (B.4)$$

APPENDIX C

OPTIMUM CONTROL LAW

The optimum control law for the time interval $t \leq T_1$ can be derived by solving Eqs. 25 and 44. The solutions are similar to the solution for the interval $T_1 < t \leq T_2$ except that superscript b is changed to a and the arbitrary constants F are changed to E:

$$u_*^a(\sigma) = \frac{A^a}{2\alpha^a} [-(\lambda + \gamma)p_2 + 2X_1^a] \quad (C.1)$$

$$p_2(\sigma) = E_1(t) \cosh \omega^a(\sigma - t) + E_2(t) \sinh \omega^a(\sigma - t) \quad (C.2)$$

$$X_1^a = \frac{\omega^a}{2} [E_1(t) \sinh \omega^a(\sigma - t) + E_2(t) \cosh \omega^a(\sigma - t)] \quad (C.3)$$

$$\begin{aligned} X_2^a &= -\frac{1}{2\alpha^a} \{E_1(t) [(\lambda + \gamma) \cosh \omega^a(\sigma - t) - \omega^a(1 + \alpha^a) \sinh \omega^a(\sigma - t)] \\ &+ E_2(t) [(\lambda + \gamma) \sinh \omega^a(\sigma - t) - \omega^a(1 + \alpha^a) \cosh \omega^a(\sigma - t)]\} \end{aligned} \quad (C.4)$$

where $\omega^a = (\lambda + \gamma) \sqrt{1 - \alpha^a/(1 + \alpha^a)} \sim \lambda + \gamma$ for small values of α^a and the quantity t is carried as a parameter.

Arbitrary constants can be determined by using the imposed conditions in Eqs. 63 and 64.

Substituting Eq. 58 into Eqs. 63 and 64, we obtain the following equations, respectively:

$$X_2^a(\sigma)|_{\sigma=T_1} = \frac{-1}{2\alpha^b} [\lambda F_1(T_1) - \omega^b(1 + \alpha^b) F_2(T_1)] \quad (C.5)$$

$$\begin{aligned} \frac{\partial}{\partial \sigma} X_2^a(\sigma)|_{\sigma=T_1} + \gamma X_2^a(\sigma)|_{\sigma=T_1} + \gamma &= \frac{-\omega^b}{2\alpha^b} [-\omega^b(1 + \alpha^b) F_1(T_1) \\ &+ \lambda F_2(T_1)] \end{aligned} \quad (C.6)$$

Let $t = T_1$ in Eq. 60; we have

$$F_1(T_1) = -F_2(T_1) \tanh \omega^b (T_2 - T_1) \quad (C.7)$$

Eliminating $F_1(T_1)$ and $F_2(T_1)$ among the preceding three equations leads to

$$\frac{\partial}{\partial \sigma} X_2^a(\sigma) \Big|_{\sigma=T_1} + (\gamma + \omega^b \Omega) X_2^a(\sigma) \Big|_{\sigma=T_1} + \gamma = 0 \quad (C.8)$$

where

$$\Omega = \frac{\lambda + \omega^b(1 + \alpha^b) \tanh \omega^b(T_2 - T_1)}{\omega^b(1 + \alpha^b) + \lambda \tanh \omega^b(T_2 - T_1)} \quad (C.9)$$

Substituting Eq. C.4 into Eq. C.8, we have

$$E_1(t) [\omega^a G + (\gamma + \omega^b \Omega) H] + E_2(t) [(\gamma + \omega^b \Omega) G + \omega^a H] - 2\alpha^a \gamma = 0 \quad (C.10)$$

where

$$G = (\lambda + \gamma) \sinh \omega^a(T_1 - t) - \omega^a(1 + \alpha^a) \cosh \omega^a(T_1 - t)$$

$$H = (\lambda + \gamma) \cosh \omega^a(T_1 - t) - \omega^a(1 + \alpha^a) \sinh \omega^a(T_1 - t)$$

Using Eq. C.3, we obtain the following equation:

$$X_1^a(t) = \frac{\omega^a}{2} E_2(t) \quad (C.11)$$

Combining Eq. C.11 with Eq. C.10 gives

$$E_1(t) = - \frac{(2/\omega^a) [(\gamma + \omega^b \Omega) G + \omega^a H] X_1^a(t) - 2\alpha^a \gamma}{\omega^a G + (\gamma + \omega^b \Omega) H} \quad (C.12)$$

The optimum control law is obtained from Eq. C.1:

$$u_*^a(t) = u_*^a(\sigma) \Big|_{\sigma=t} = \frac{A^a}{2\alpha^a} [-(\lambda + \gamma) E_1(t) + 2X_1^a(t)] = \frac{A^a}{\alpha^a} \left\{ (\lambda + \gamma) \right. \\ \left. \times \frac{(1/\omega^a) [(\gamma + \omega^b \Omega) G + \omega^a H] X_1^a(t) - \alpha^a \gamma}{\omega^a G + (\gamma + \omega^b \Omega) H} + X_1^a(t) \right\} \\ (t \leq T_1) \quad (C.13)$$

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Time Optimal Solution to the Reactivity–Xenon Shutdown Problem*

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ABSTRACT

A time optimal criterion for the reactivity–xenon shutdown problem has been formulated and solved by application of the theorems of optimal processes as developed by L. S. Pontryagin et al. [*The Mathematical Theory of Optimal Processes* (L. W. Neustadt, Ed.), Interscience Publishers, a division of John Wiley & Sons, Inc., New York, 1963]. We desire the minimum time trajectory between any point of operation in the xenon–iodine phase space to a zero-power shutdown curve subject to the constraint that neither the shutdown curve nor the trajectory to the shutdown curve allow the xenon concentration to exceed an arbitrarily specified maximum. The optimal solution is shown to usually require sequential operation at zero power, variable (decreasing) power, and finally full power. The exact power program is calculated for one example.

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A Closed-loop, Approximately Time-optimal Control Method

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ABSTRACT

In this paper a special class of solutions of the Hamilton–Jacobi equation, called eigenvector scalar products, is used to develop a method of closed-loop, approximately time-optimal control. The method is developed first for linear systems in which the control matrix is nonsingular and then for systems in which the control matrix is singular. Finally, the extension of the method to nonlinear systems is considered.

The procedure to be followed in each case is systematic and is illustrated by three examples. When a closed-loop system is designed with this method, the controller–computer must solve only algebraic equations, and hence the control can be computed continuously, in contrast to many of the present methods, which require on-line solution of two-point boundary value problems and hence discrete control.

INTRODUCTION

Within the last few years, several, rather elegant, general methods of solving the optimal control problem have been presented.^{1,2} In general, these methods involve unwieldy computations for all but trivial problems. Also in many cases the control, once obtained, is of an open-loop nature; i.e., it is valid for only one initial condition and no disturbances.

The difficulties associated with these methods have led to a growing gap between theoretical and practical control work. To fill this gap, there has been an ever-increasing development of techniques of sub-optimal control, control that is acceptably close to the true optimal, but practicable.³⁻⁵

In this paper a special class of solutions of the Hamilton-Jacobi equation, called eigenvector scalar products, is used to develop a method of closed-loop, approximately time-optimal control. The system is described in state variable form by a set of n first-order differential equations

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_r) \quad (i = 1, 2, \dots, n) \quad (1)$$

which may be written in vector notation as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (2)$$

where $\mathbf{x}$ is the n -dimensional state vector and $\mathbf{u}$ is the r -dimensional control vector. The system is assumed to be completely controllable and observable.

It is desired to find a closed-loop control, $\mathbf{u}(\mathbf{x})$, that transfers the system from an arbitrary initial state, $\mathbf{x}^0$, to the terminal state, $\mathbf{x}^1$, in minimum time. The control region, U , is the set of all control vectors, $\mathbf{u}$, such that $\|\mathbf{u}\| \leq \rho$, i.e., a total effort constraint.

HAMILTON-JACOBI EQUATION

In this and the following two sections, two additional assumptions are made concerning the optimization problem of the previous section. First, the system² is assumed to be linear and is written in vector-matrix notation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (3)$$

where $\mathbf{A}$ is a constant n by n matrix with real, distinct, and nonpositive eigenvalues and $\mathbf{B}$ is a constant n by r matrix. Second, the terminal state $\mathbf{x}^1$ is assumed to be the origin.

Now let $V(\mathbf{x}^0)$ be defined as the minimum transition time from $\mathbf{x}^0$ to the origin. Then, if it is assumed that V is continuously differentiable with respect to $\mathbf{x}$, V satisfies the Hamilton-Jacobi equation⁶

$$H^0[\mathbf{x}, \nabla V(\mathbf{x})] + \frac{\partial V}{\partial t} = 0 \quad (4)$$

where

$$H^0[\mathbf{x}, \nabla V(\mathbf{x})] = \min_{\mathbf{u} \in U} H[\mathbf{x}, \nabla V(\mathbf{x}), \mathbf{u}] \quad (5)$$

For the preceding problem the Hamilton–Jacobi equation becomes⁷

$$\nabla V(\mathbf{x})' \mathbf{A} \mathbf{x} - \rho \|\mathbf{B}' \nabla V(\mathbf{x})\| + 1 = 0 \quad (6)$$

and the optimal control is given by

$$\mathbf{u}^0(\mathbf{x}) = \frac{-\rho \mathbf{B}' \nabla V(\mathbf{x})}{\|\mathbf{B}' \nabla V(\mathbf{x})\|} \quad (7)$$

Unfortunately no general method of solving Eq. 6 is known, and therefore it is necessary to obtain an approximate solution. From the special class of solutions of the Hamilton–Jacobi equation described in the next section, a method for designing effective suboptimal systems is developed.

EIGENVECTOR SCALAR PRODUCTS

In this section a particular class of solutions of the Hamilton–Jacobi equation (Eq. 6) is shown to exist. Because of the manner in which these solutions are formed, they are called eigenvector scalar products.

Theorem: If $\mathbf{q}$ is an eigenvector of $\mathbf{A}'$ and λ is the associated eigenvalue, then for $\lambda \neq 0$

$$V(\mathbf{x}) = \frac{1}{-\lambda} \ln \left[\frac{-\lambda |\mathbf{q}' \mathbf{x}|}{\rho \|\mathbf{B}' \mathbf{q}\|} + 1 \right] \quad (8a)$$

or for $\lambda = 0$

$$V(\mathbf{x}) = \frac{|\mathbf{q}' \mathbf{x}|}{\rho \|\mathbf{B}' \mathbf{q}\|} \quad (8b)$$

is a solution to the Hamilton–Jacobi equation, and the corresponding optimal control is given by

$$\mathbf{u}^0(\mathbf{x}) = \frac{-\rho \mathbf{B}' \mathbf{q} \mathbf{q}' \mathbf{x}}{\|\mathbf{B}' \mathbf{q} \mathbf{q}' \mathbf{x}\|} \quad (9)$$

The proof of this theorem involves a rather routine substitution of $V(\mathbf{x})$ into Eq. 6 and therefore is not presented here. These solutions, however, cannot be used directly, since $V(\mathbf{x})$ is zero if $\mathbf{q}^i \mathbf{x}$ is zero. Therefore the value of $V(\mathbf{x})$ does not correspond to the minimum transition time from $\mathbf{x}^0$ to the origin but rather from $\mathbf{x}^0$ to the hyperplane $\mathbf{q}^i \mathbf{x} = 0$. Since the numerical value of $V(\mathbf{x})$ is dependent on both $\mathbf{x}$ and ρ , $V(\mathbf{x})$ is written as $V(\mathbf{x}, \rho)$ to indicate this relation. Similarly $\mathbf{u}$ is written as $\mathbf{u}(\mathbf{x}, \rho)$.

SUBOPTIMAL CONTROL FOR NONSINGULAR CONTROL MATRIX

In this section a method of suboptimal control of linear systems is developed on the basis of the eigenvector scalar product solutions of the previous section. The control matrix, $\mathbf{B}$, is assumed to be nonsingular in this section.

Consider a general n th-order system with n real, nonpositive, and distinct eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_n$. With each eigenvalue there is associated an eigenvector of $\mathbf{A}^*$, $\mathbf{q}^1, \mathbf{q}^2, \dots, \mathbf{q}^n$, and an eigenvector scalar product solution, $V_1(\mathbf{x}, \rho), V_2(\mathbf{x}, \rho), \dots, V_n(\mathbf{x}, \rho)$. For each eigenvector scalar product solution, there is a corresponding optimal control, $\mathbf{u}^1(\mathbf{x}, \rho), \mathbf{u}^2(\mathbf{x}, \rho), \dots, \mathbf{u}^n(\mathbf{x}, \rho)$.

It may seem strange to speak of n optimal controls. The reader, however, is reminded of the significance of these controls. The control $\mathbf{u}^1(\mathbf{x}, \rho)$ transfers the system from any initial point $\mathbf{x}^0$ to some point $\mathbf{x}^1$ on the surface $V_1(\mathbf{x}, \rho) = 0$ in minimum time. (Typical points and the corresponding optimal trajectory for a second-order system are shown in Fig. 1. In the case of second-order systems, surfaces of $V(\mathbf{x})$ equals a constant become lines.) On the other hand, $\mathbf{u}^2(\mathbf{x}, \rho)$ transfers the system to the surface $V_2 = 0$ etc. Hence it is possible to have n optimal controls since the problem is different in each case.

It is easy to show⁷ that each optimal control, $\mathbf{u}^i(\mathbf{x}, \rho)$, is constant during the transition from $\mathbf{x}^0$ to $V_i(\mathbf{x}, \rho) = 0$ and that for any other constant control $\mathbf{u}$ only the portion of $\mathbf{u}$ that is parallel to $\mathbf{u}^i(\mathbf{x}, \rho)$ determines the transition time from $\mathbf{x}^0$ to $V_i(\mathbf{x}, \rho) = 0$.

For some initial state $\mathbf{x}^0$, let the magnitude of the optimum control vectors, $\rho_1, \rho_2, \dots, \rho_n$, be chosen such that $V_1(\mathbf{x}^0, \rho_1) = V_2(\mathbf{x}^0, \rho_2) = \dots = V_n(\mathbf{x}^0, \rho_n) = T$. Therefore the control vector $\mathbf{u}^i(\mathbf{x}^0, \rho_i)$ transfers the system from $\mathbf{x}^0$ to the hyperplane $V_i = 0$ in T seconds. Now consider a vector $\mathbf{u}$ such that the portion of $\mathbf{u}$ that is parallel to $\mathbf{u}^i$ is equal to $\mathbf{u}^i(\mathbf{x}^0, \rho_i)$. See Fig. 2 for a graphical representation of this situation for a second-order case. The magnitude, ρ_i , of the portion of $\mathbf{u}$ that is

Fig. 1 — Surfaces of constant V for eigenvector scalar product solutions.

Fig. 2 — Representation of a vector in terms of the optimal control vectors.

parallel to $\mathbf{u}^i$ is obtained from the scalar product of $\mathbf{u}$ and a unit vector parallel to $\mathbf{u}^i$. Hence

$$\rho_i = \mathbf{u}' \mathbf{u}^i(\mathbf{x}^0, 1) \quad (10)$$

One might be tempted to say that

$$\mathbf{u} = \sum_{i=1}^n \rho_i \mathbf{u}^i(\mathbf{x}^0, 1)$$

This is not true, as can be readily shown by example, since the optimal control vectors are not orthogonal.

Since only the portion of $\mathbf{u}$ that is parallel to $\mathbf{u}^i$ has any effect on the time necessary to transfer the system from $\mathbf{x}^0$ to $V_i = 0$, $\mathbf{u}$ must transfer the system from $\mathbf{x}^0$ to $V_1 = V_2 = \dots = V_n = 0$ in the same time. But $V_1 = V_2 = \dots = V_n = 0$ can occur only at the origin; hence $\mathbf{u}$ must transfer the system from $\mathbf{x}^0$ to the origin in $V_1(\mathbf{x}^0, \rho_1) = T$ seconds.

Since setting $V_1(\mathbf{x}^0, \rho_1) = V_2(\mathbf{x}^0, \rho_2) = \dots = V_n(\mathbf{x}^0, \rho_n)$ yields only $n - 1$ equations in the n unknowns, $\rho_1, \rho_2, \dots, \rho_n$, there is an infinite set of control vectors that satisfy these equations. From this set there is one vector whose norm is equal to ρ . This vector is then an admissible control that transfers the system from any point $\mathbf{x}^0$ to the origin in a finite time T .

Two significant features of this suboptimal control should be noted. First, in the absence of disturbances, the control is constant for the entire transition time. Since the control vector does not require rapid variations, only a minimum of recalculation of the control is necessary during the transition time. This should simplify the instrumentation of the controller.

Second, although it is not trivial, the simultaneous solution of the equations $V_1(\mathbf{x}^0, \rho_1) = \dots = V_n(\mathbf{x}^0, \rho_n)$ and $\|\mathbf{u}\| = \rho$ is an algebraic manipulation, in contrast to many of the presently advocated methods for which it is necessary to solve simultaneously the usual nonlinear differential equations of the two-point boundary value nature. The computational advantage is obvious. Since these computations are algebraic, it is possible to carry them out continuously on an analog computer to create *continuous* closed-loop control.

Before a numerical example to illustrate the method is considered, it is perhaps of value to outline the complete method for reference.

1. Obtain the eigenvalues and eigenvectors of $\mathbf{A}'$.
2. Obtain the eigenvector scalar product solutions as given by Eq. 8, $V_1(\mathbf{x}, \rho)$, $V_2(\mathbf{x}, \rho)$, $\dots$, $V_n(\mathbf{x}, \rho)$ and their corresponding optimal controls, $\mathbf{u}^1(\mathbf{x}, \rho)$, $\mathbf{u}^2(\mathbf{x}, \rho)$, $\dots$, $\mathbf{u}^n(\mathbf{x}, \rho)$.

3. For a given point, $\mathbf{x}^0$, solve the relations $\rho_i = \mathbf{u}' \mathbf{u}^i(\mathbf{x}^0, 1)$ to obtain $\mathbf{u}_i$'s in terms of the ρ_i 's.

4. Solve $V_1(\mathbf{x}^0, \rho_1) = V_2(\mathbf{x}^0, \rho_2) = \dots = V_n(\mathbf{x}^0, \rho_n)$ and $\|\mathbf{u}\| = \rho$ simultaneously to obtain $\rho_1, \rho_2, \dots, \rho_n$.

5. By the use of the relations obtained in step 3, find $\mathbf{u}$, the desired suboptimal control.

The last three steps in this procedure can be mechanized by the use of a digital or analog computer to create a closed-loop system.

Example 1: Consider the following system in order to illustrate the application of the preceding method of suboptimal control.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (11)$$

It is desired to transfer the system from the point $\mathbf{x}^0 = (2, 1)$ to the origin with $\|\mathbf{u}\| \leq 1$.

By standard methods the eigenvalues are found to be -1 and -2 with the corresponding eigenvectors of $\mathbf{A}'$ being $(2, 1)$ and $(1, 1)$. The two eigenvector scalar product solutions as given by Eq. 8 are

$$V_1(\mathbf{x}, \rho) = \ln \left[\frac{|2x_1 + x_2|}{2\rho} + 1 \right] \quad (12)$$

$$V_2(\mathbf{x}, \rho) = \frac{1}{2} \ln \left[\frac{4|x_1 + x_2|}{\sqrt{5}\rho} + 1 \right] \quad (13)$$

The corresponding optimal controls as given by Eq. 9 are

$$\begin{bmatrix} u_1^1(\mathbf{x}, \rho) \\ u_2^1(\mathbf{x}, \rho) \end{bmatrix} = -\rho \frac{(2x_1 + x_2)}{|2x_1 + x_2|} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} u_1^2(\mathbf{x}, \rho) \\ u_2^2(\mathbf{x}, \rho) \end{bmatrix} = -\rho \frac{(x_1 + x_2)}{|x_1 + x_2|} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad (15)$$

This completes steps 1 and 2 in the procedure outlined previously. Using the relation $\rho_i = \mathbf{u}' \mathbf{u}^i(\mathbf{x}^0, 1)$ and solving for u_1 and u_2 in terms of ρ_1 and ρ_2 and setting $\|\mathbf{u}\| = 1$, one obtains

$$(-\rho_1)^2 + (+2\rho_1 - \sqrt{5}\rho_2)^2 = 1 \quad (16)$$

Setting $V_1(\mathbf{x}^0, \rho_1) = V_2(\mathbf{x}^0, \rho_2)$, one obtains

$$\ln \left(\frac{5}{2\rho_1} + 1 \right) = \frac{1}{2} \ln \left(\frac{12}{\sqrt{5}\rho_2} + 1 \right) \quad (17)$$

By solving Eqs. 16 and 17 simultaneously, one obtains

$$\rho_1 = 0.645$$

$$\rho_2 = 0.236$$

Therefore a control vector $\mathbf{u} = (-0.645, +0.763)$ transfers the system from the point $\mathbf{x}^0 = (2, 1)$ to the origin in $V_1(\mathbf{x}^0, \rho_1) = 1.58$ sec. It should be noted that, although this example is presented for a specific initial state, the policy was implemented by use of an analog computer to provide continuous closed-loop operation.⁷

In the previous paragraphs a method of obtaining suboptimal control of systems in which the control matrix is nonsingular was developed. Every suboptimal control method should satisfy two requirements. First, the method should facilitate the design and implementation of the suboptimal control system. Second, the performance of the suboptimal control system should be acceptably close to the optimum.

The aspects of the method that assist the design and implementation of the suboptimal system have been mentioned previously. Unfortunately, showing that the performance is acceptable is a more difficult task. It is possible to show⁷ that there are $2n$ curves in the state space for which the optimal control and suboptimal control are identical, and, owing to the continuity of the control variables, they are also close in some region about each of these curves. This is, in no sense, a proof that the quality of the suboptimal performance is acceptably close to the optimal, although one may conclude that they are close in a reasonably large portion of the state space. The problem of judging suboptimal systems without obtaining the optimal solution is of great practical significance and is a topic of current research at The University of Arizona.

Since the number of practical systems for which $\mathbf{B}$ is nonsingular is limited, it is necessary to extend the method to systems for which $\mathbf{B}$ is singular. This is done in the next section.

SUBOPTIMAL CONTROL WITH SINGULAR CONTROL MATRIX

In the previous section a method for the suboptimal control of systems with nonsingular control matrices was developed. This restriction on the control matrix is removed in this section. In addition to the fact that the control matrix is singular, it is assumed that the dimension of the control space, r , is equal to the rank of the matrix $\mathbf{B}$. If the control space, r , is unequal to the rank of the matrix $\mathbf{B}$, then one may always reduce the number of control variables by a simple linear transformation.⁷

Since the eigenvector scalar product solutions do not require a nonsingular control matrix, the method developed in this section is, once again, based on the eigenvector scalar products. Although this method cannot treat all problems where $\mathbf{B}$ is singular, there is a large class of problems for which it is applicable. Consider first the case where $r = n - 1$.

The suboptimal control method for systems with a nonsingular control matrix requires the simultaneous solution of the n algebraic equations. These n equations can be solved simultaneously because the control space is n -dimensional and hence there are n control variables.

It is not possible to apply such an approach directly if the number of control variables is only $n - 1$. However, there is an unbounded $(n - 1)$ -dimensional subset, R , of the state space in which all n equations can be satisfied by an $(n - 1)$ -dimensional control. From any point in R , it is possible to determine a constant control vector, $\mathbf{u}$, with magnitude ρ , which transfers the system to the origin in finite time by the method of the previous section.

However, since the subset R is only $(n - 1)$ -dimensional, an arbitrary initial state is not in R , and it is necessary to devise a method for transferring the system to R . Before discussion of this method, it is convenient to introduce some additional notation. Let $\mathbf{u}^{2,n}(\mathbf{x}^0, \rho)$ be the vector of magnitude ρ such that $V_2(\mathbf{x}^0, \rho_2) = V_3(\mathbf{x}^0, \rho_3) = \dots = V_n(\mathbf{x}^0, \rho_n)$, where as usual $\rho_i = \mathbf{u}^{2,n}(\mathbf{x}^0, \rho)^T \mathbf{u}^i(\mathbf{x}^0, 1)$. Then let the common value of these r eigenvector scalar product solutions be designated by $V_{2,n}(\mathbf{x}^0, \rho) = V_2(\mathbf{x}^0, \rho_2) = \dots = V_n(\mathbf{x}^0, \rho_n)$. Let the eigenvector scalar product solutions be ordered such that $|\lambda_i| < |\lambda_j|$ if $i < j$. Now divide the state space into three regions.

Region I: $\mathbf{u}^1(\mathbf{x}, \rho)^T \mathbf{u}^{2,n}(\mathbf{x}, \rho) > 0$ and $V_1(\mathbf{x}, \rho_1) > V_{2,n}(\mathbf{x}, \rho)$

Region II: $\mathbf{u}^1(\mathbf{x}, \rho)^T \mathbf{u}^{2,n}(\mathbf{x}, \rho) < 0$

Region III: $\mathbf{u}^1(\mathbf{x}, \rho)^T \mathbf{u}^{2,n}(\mathbf{x}, \rho) > 0$ and $V_1(\mathbf{x}, \rho_1) < V_{2,n}(\mathbf{x}, \rho)$

Regions I and III are further subdivided into positive and negative regions, depending on whether $\mathbf{q}^1 \mathbf{x}$ is positive or negative. The control policy is then stated in the following form.

1. If $\mathbf{u}^1 \mathbf{u}^{2,n} < 0$, use $\mathbf{u} = \mathbf{u}^1$.
2. If $\mathbf{u}^1 \mathbf{u}^{2,n} > 0$ and $V_1 > V_{2,n}$, use $\mathbf{u} = \mathbf{u}^1$.
3. If $\mathbf{u}^1 \mathbf{u}^{2,n} > 0$ and $V_1 < V_{2,n}$, use $\mathbf{u} = -\mathbf{u}^1$.
4. If $\mathbf{u}^1 \mathbf{u}^{2,n} > 0$ and $V_1 = V_{2,n}$, use $\mathbf{u} = \mathbf{u}^{2,n}$.

This policy is shown graphically in Fig. 3. This policy can be implemented on an analog computer to yield continuous closed-loop, suboptimal control. The suboptimal policy is based on two factors. First, the technique reduces to true time-optimal control for second-order systems.⁷ Second, the method works; i.e., it transfers the system from an initial state to the R subset and thence to the origin in finite time. This second factor can be demonstrated by the following argument.

Consider an initial state $\mathbf{x}^0$ arbitrarily located in region I⁻. Then by the use of the preceding technique, the control $\mathbf{u}$ is set equal to $\mathbf{u}^1$. Since $\mathbf{u}^1 \mathbf{u}^{2,n} > 0$, the value of both V_1 and $V_{2,n}$ must be decreasing as time increases. Hence there are two possibilities to be considered.

First, the value of V_1 may reach zero before the value of $V_{2,n}$ does. But, if this were to happen, then there is some intermediate state at which $V_1 = V_{2,n}$, since initially V_1 was greater than $V_{2,n}$. But that state is in R, and hence the system has been transferred to R.

Second, the direction of $\mathbf{u}^{2,n}$ may change such that $\mathbf{u}^1 \mathbf{u}^{2,n} < 0$ and the system has crossed into region II. The control $\mathbf{u}$, however, continues to be equal to $\mathbf{u}^1$; since $\mathbf{u}^1$ has not changed direction, the control is constant.

Fig. 3 - The suboptimal control policy.

Since in region II $\mathbf{u}^1 \dot{\mathbf{u}}^{2,n} < 0$, the value of V_1 continues to decrease with time, while $V_{2,n}$ is increasing. After a finite time the value of V_1 becomes zero. As the system crosses the $V_1 = 0$ line, $\mathbf{u}^1$ changes direction and $\mathbf{u}^1 \dot{\mathbf{u}}^{2,n}$ again becomes positive.

The system is now in region III⁺, and $V_1 < V_{2,n}$ and $\mathbf{u}^1 \dot{\mathbf{u}}^{2,n} > 0$. Since $\mathbf{u} = -\mathbf{u}^1$, the value of both V_1 and $V_{2,n}$ is increasing with time. Therefore it is necessary to show that V_1 will become greater than $V_{2,n}$ again and hence that the system will reach R.

Although it could not be shown that this is true in general, the condition is satisfied⁷ for all second-order systems and for all systems where $\lambda_1 = 0$. Therefore it seems reasonable to assume that the general case is also true. However, one should check this assumption for any particular system of interest.

Let us review the process. From an arbitrary initial state in region I⁻, the system is transferred to region II, then to III⁺, and finally it reaches R, which is the boundary separating III⁺ and I⁺. Since the initial point in region I⁻ is arbitrary, the argument for any initial state in regions II and III⁺ is identical from that point on. Similarly for initial states in regions I⁺ and III⁻, the arguments are essentially identical to the preceding ones. Hence for any initial state the preceding technique transfers the system to the subset R. Once in R, the control $\mathbf{u} = \mathbf{u}^{2,n}$ transfers the system directly to the origin.

As an illustration of this technique, consider the following third-order system with two control variables.

Example 2: The laws of motion for the system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

It is desired to drive the system from the initial state (1.0, -1.0, -2.0) to the origin in approximately minimum time with $\|\mathbf{u}\| \leq 1$. It is easily verified that from region III the system is driven to region I, and hence the procedure described can be applied.

Figure 4 shows the time response of this system when subjected to the preceding suboptimal procedure. The system begins in region I⁻, and the control is therefore $\mathbf{u} = \mathbf{u}^1(\mathbf{x}^0, 1) = (0, 1)$. At approximately $t = 0.9$, the system crosses into region II, and the control remains equal to (0, 1). The system enters region III⁺ at approximately $t = 1.1$, while the control remains constant. The R subset is reached at $t = 1.8$, and the control is switched to $\mathbf{u}^{2,n}(\mathbf{x}, 1) = (-0.285, -0.959)$. The system reaches the origin at $t = 2.23$.

Hence by the preceding suboptimal procedure the system has been transferred from an initial state to the origin. Although this example is

Fig. 4 - Time response of a third-order system with two control variables.

presented from a specific initial state, the policy was implemented on an analog computer to provide continuous closed-loop operation.⁷

If $r < n - 1$, then the method presented here cannot be applied directly. In this case it is necessary for the system to pass through a sequence of subsets with ever-decreasing dimensionality until the r -dimensional subset, R , is reached. Each member of this sequence of subsets must satisfy two requirements: (1) it must contain all the following subsets, and (2) it must be possible for the system to remain in the subset under a constant control.

One possible sequence is $V_{n-r} = V_{1+n-r,n}$, $V_{n-r-1} = V_{n-r,n}$, ..., $V_1 = V_{2,n}$. Although this sequence does satisfy the first requirement, it

is easy to show, by example, that it does not satisfy the second. The search is being continued for such a sequence in terms of the eigenvector scalar products.

There is another possible approach that yields acceptable performance for a large class of systems. This approach is based on the concept of using the preceding technique to drive the system to a state where the first $r + 1$ eigenvector scalar product solutions are zero. At that point the control is turned off, and the system is allowed to coast uncontrolled toward the origin. Since, at the time that the control is turned off, the first $r + 1$ eigenvector scalar product solutions are zero, they remain zero, and the eigenvalues associated with them are, in effect, removed from the system. Hence the system approaches the origin after the control is terminated exponentially with a time constant of $1/|\lambda_{r+2}|$.

If the magnitude of $r + 1$ or less of the eigenvalues is small relative to the remaining eigenvalues, then during the controlled interval the small eigenvalues are effectively eliminated from the system. The system then approaches the origin with only the relatively fast time constants present.

Although this method yields good performance for a large number of systems, it is still far from a complete solution. Further research is needed in order to provide an acceptable answer for all systems.

AN APPLICATION TO NEUTRONIC CONTROL

In this section a procedure is outlined for applying the closed-loop, approximately time-optimal control methods of the two preceding sections to some practical control problems.

One encounters three obstacles in attempting to apply the previously developed methods to practical control problems:

1. The desired terminal state is not the origin.
2. The differential equations describing the system are generally nonlinear.

3. The control vector is not constrained in the norm.

Since each of these three factors violates the basic assumptions of the previous sections, it is necessary that the problems be modified before these methods can be applied. Appropriate modifications in order to overcome each of these three factors are presented in the following development.

The first two difficulties may be removed by making the transformation of coordinates

$$y_i = x_i - x_i^1 \quad (i = 1, 2, \dots, n) \quad (18)$$

in order to move the terminal state $\mathbf{x}^1$ to the origin and then by linearizing about the origin. For the suboptimal methods to be applied to this linearized version of the system, the eigenvalues must be real, distinct, and nonpositive so that the class of systems that can be treated are limited to some extent. The effectiveness of the suboptimal control methods depends, to a large measure, on the validity of this linearization procedure. If, for example, most initial states lie within the region for which the linearization is accurate, then good performance can be expected. If, on the other hand, many initial states are outside the region of validity, then the quality of the performance may be in question. However, owing to the closed-loop nature of the system, as long as the linearization is even "reasonably" accurate, the system should be rapidly driven toward the region of validity, and hence the performance may still be good. Once again closed-loop control is shown to be highly desirable.

The fact that in many cases the control vector is not constrained in the norm is perhaps the most difficult obstacle to overcome in the application of the suboptimal methods to practical control problems. The method of attack is to find an approximation to the given control region in the form of a constraint on the norm of the control vector. In other words, even though the control vector is not initially constrained in the norm, a norm constraint is generated and employed. In order to make this approximation most conveniently, consider the generalized norm $\|\mathbf{D}\mathbf{u}\| \leq 1$. The problem is then to pick the elements of $\mathbf{D}$ which generate the best approximation to the given control region. After $\mathbf{D}$ is chosen, if a new set of control variables is defined by $\mathbf{w} = \mathbf{D}\mathbf{u}$, then the control region becomes $\|\mathbf{w}\| \leq 1$, and the problem has been put into proper form for the application of the suboptimal control methods.

As an illustration of this procedure, consider the second-order control region defined by

$$-5 \leq u_1 \leq 4 \qquad -1 \leq u_2 \leq 1$$

(See Fig. 5.) In this case $\mathbf{D}$ might be selected as

$$\mathbf{D} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$$

in which case the approximate control region is $\|\mathbf{D}\mathbf{u}\| \leq 1$, shown as the cross-hatched area in Fig. 5.

In summary, the procedure for preparing a practical control problem for the application of the previously developed suboptimal control methods is as follows:

1. Transform the desired terminal state to the origin.
2. Linearize the system about the origin.
3. Approximate the control region with a norm constraint.

Fig. 5 - Approximation of a typical control region.

As an illustration of this procedure, consider the time-optimal control of single-precursor group neutron kinetics. The system is defined by the differential equations⁸

$$\begin{aligned}\dot{n} &= -\frac{\beta}{l} n + \lambda c + \frac{\beta}{l} nu \\ \dot{c} &= \frac{\beta}{l} n - \lambda c\end{aligned}\tag{19}$$

where n is the neutron density, c is the precursor concentration, and u is the reactivity control. The problem is to develop a closed-loop controller that transfers the system from any initial state to a desired terminal state $n(T)$ and $c(T)$ in approximately minimal time. The control region is given by $-0.5 \leq u \leq +0.5$.

The problem was treated numerically by use of the following parameters:

$$\begin{aligned}\lambda &= 0.1 \text{ sec}^{-1} \\ \beta &= 6.0 \times 10^{-3} \\ l &= 3 \times 10^{-5} \text{ sec}\end{aligned}$$

In this case the system equations (Eq. 19) become

$$\begin{aligned}\dot{n} &= -200n + 0.1c + 200nu \\ \dot{c} &= 200n - 0.1c\end{aligned}\quad (20)$$

Using the terminal state

$$\begin{aligned}n(T) &= 1.5 \\ c(T) &= 3 \times 10^3\end{aligned}$$

and linearizing the system, one obtains

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -200 & 0.1 \\ 200 & -0.1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 300 \\ 0 \end{bmatrix} u \quad (21)$$

where

$$\begin{aligned}y_1 &= n - 1.5 \\ y_2 &= c - 3 \times 10^3\end{aligned}$$

The eigenvalues of the linearized system are 0, -200.1 , and therefore the requirement for real, distinct, and nonpositive eigenvalues is satisfied.

The control may be put into the correct form by making the substitution $w = 2u$ and using the control region

$$|w| = \|w\| \leq 1 \quad (22)$$

and the system equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -200 & 0.1 \\ 200 & -0.1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 150 \\ 0 \end{bmatrix} w \quad (23)$$

The method of the previous section was used in the implementation of the closed-loop control system on a digital computer. For an initial state

$$\begin{aligned}n &= 1.0 \\ c &= 2.0 \times 10^3\end{aligned}$$

the optimal transition time is 4.084 sec, and the suboptimal is 4.086, for a difference of less than 0.1%. It cannot be overemphasized that the suboptimal system is closed loop and correspondingly yields good results for any initial state. Although this problem is perhaps the exception

rather than the rule as concerns the closeness of optimal and suboptimal performance, the approach produces a relatively simple, closed-loop system in a systematic manner.

The success of the procedure of this section for any particular problem depends on several factors and hence is difficult to predict. In particular, the degree of validity of the linearization and the closeness to which the admissible control region can be approximated in step 3 are important factors in determining the success or failure of the method. The degree to which the suboptimal performance approximates the true optimal is also unknown, and hence one must simply decide if the suboptimal system is adequate for any given problem.

CONCLUSIONS

In this paper a special class of solutions of the Hamilton–Jacobi equation, called eigenvector scalar products, is used to develop several methods of closed-loop, suboptimal control. The procedure to be followed in each case is systematic and is illustrated by example.

The systematic procedures of this work should be contrasted with the arbitrary selection of V -functions required by previously suggested methods.^{4,5} In addition, these previous methods use quadratic forms for V -functions, and hence the “best” V -function depends on the initial state of the system. In the case of the Kalman–Bertram method,⁵ chattering near the origin degrades the performance considerably. Nahi’s method,⁴ on the other hand, requires that $\mathbf{B}$ be nonsingular.

It is the author’s opinion that the concept of closed-loop, suboptimal control provides the greatest potential for the application of modern control theory to practical problems. This paper is only a small, but hopefully significant, step toward more practical utilization of optimal control theory.

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11: 29/78

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Analysis of Pulsed Neutron Experiments in Moderating Assemblies

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ABSTRACT

In this paper some features of the present state of the analysis of pulsed neutron phenomena in moderating assemblies are discussed. Much of the analysis is based on the use of a discrete mesh or groups to represent the energy variable and the use of spherical harmonic approximations to represent the angles. Although this type of approach does not yield a continuum of eigenvalues (which is present in other analyses), it is justified on physical grounds and is sufficient to represent the situations studied. Illustrative examples of eigenvalues and eigenfunctions are presented for mass-1 free gas and beryllium over a large range of bucklings; both discrete and continuum eigenfunctions are illustrated. The major features of the real-time eigenvalues as they vary with sinusoidal and hyperbolic bucklings are shown in a schematic map of the real λ vs. B^2 plane. The analysis is then used in order to discuss some of the major areas of time-dependent thermalization investigation.

INTRODUCTION

A great deal of information has been gained over the past decade by exposing a block of neutron-moderating material to a short intense burst of neutrons with an energy of a few million electron volts and then measuring and analyzing the subsequent behavior of the neutrons.

*Based in part on Ph. D. Theses of R. R. Lee and R. L. Hobson.

The time scale involved in the most interesting of these thermalization problems runs from a few microseconds to several milliseconds. In general, although not exclusively, timing over known distance is used to determine neutron energies, and the experiments frequently make use of flight paths, choppers, and electronic timing equipment.

In this paper we would like to discuss the analysis of such experiments from a particular, rather general point of view. Although many experiments in this field are performed to furnish fiducial data against which more or less elaborate theories may be tested, our emphasis here will be on the systematics and meaning of the calculations and analysis. However, our methods will also be inherently adapted to realistic models.

The method of this study is that of an eigenvalue-eigenfunction analysis based on the use of a discrete mesh to represent the energy variable. The eigenvalue-eigenfunction approach seems most natural for pulsed neutron experiments in moderators since measurements are made on the systems when there are no sources present; that is, we are dealing with an essentially homogeneous system. A discrete mesh as the representation of the energy variable is very useful in thermalization studies at this point of development because it is adapted to representing the rapidly varying functions that abound in this area, such as the cross sections reflecting crystalline and quantum level effects. A discrete mesh is also very useful in treating the practically singular eigenfunctions.

ANGLE AND ENERGY REPRESENTATIONS

In the rigorous treatment of the variations with angle of the neutron transport problem, one obtains a discrete set and also a singular set of eigenfunctions or distributions and a continuum of eigenvalues.¹ In the monoenergetic case the use of the spherical harmonics to represent the angles yields regular eigenfunctions and a discrete set of diffusion lengths² without the continuum of the rigorous treatment. In a similar fashion the energy variable in thermalization problems may be treated rigorously as a continuous variable or may be represented by sets of groups or discrete points. The rigorous treatment yields a continuum³ of singular eigenfunctions⁴ with or without⁵ a discrete set depending on the parameters of the problem. Here also the other representations such as polynomials and discrete meshes usually yield only discrete sets.

Eventually our theories must compare with experiment. It is obviously inherent that just about all thermalization experiments involve only a finite number of measurements. Hence, for example,

they can yield a comparison with only a finite number of eigenvalues. If one proceeds from a rigorous theory, one sums with the resolution of the measuring device over the eigenfunctions including the continuum of singular distributions to obtain the results to compare with experiment. On the other hand, the use of theories involving only a discrete set of eigenvalues may be directly adapted to the finite set of measurements available. For example, measurements of energies in a set of energy "channels" may be compared with a theory that will use these channels as groups. The important point is that the rigorous theory is the limit of the approximate theories, in this case those theories yielding only a discrete set of eigenvalues. If no physical information is lost by not going to the limit, then one may commute, say, the processes of averaging over channels and predicting eigenvalues. Indeed, one may question the limiting process when forming the original differential-integral equation from a form involving differences and finite sums. The central point appears to be that the model adequately represent the physical situation.

CALCULATION OF EIGENVALUES

Many measurements are made which give the flux of thermal neutrons as a function of time after the introduction of a pulse. The fitting of this time behavior is usually attempted as a series of exponentials, the fitted relaxation times being interpreted as eigenvalues. The integral equation for the flux in the infinite-medium case is

$$\frac{1}{v} \frac{\partial F(E, t)}{\partial t} = -\Sigma_T(E) F(E, t) + \int \Sigma_s(E' \rightarrow E) F(E', t) dE' \quad (1)$$

where Σ_T is the total cross section and $\Sigma_s(E' \rightarrow E)$ is the P_0 component of the scattering cross section from E' to E . If we look for solutions having a separated time dependence of the form

$$F(E, t) = F(E) e^{-\lambda t} \quad (2)$$

then Eq. 1 becomes

$$[\Sigma_T(E) - \lambda/v] F(E) = \int \Sigma_s(E' \rightarrow E) F(E') dE' \quad (3)$$

By the use of detailed balance, the kernel in the integral equation is made symmetric, and, by the introduction of a discrete mesh for the

energy variable, the problem is reduced to the problem of finding the eigenvalues of a finite real symmetric matrix.⁶

In Table 1 are listed some meshes that have been used, covering the range 10^{-6} to 1.0 ev, with 37 mesh points. The mesh intervals are equal intervals in each variable considered. It is immediately noteworthy that equally spaced energy intervals are not adapted to the thermal problem, $kT = \frac{1}{40}$ ev, because of the lack of mesh points to represent the curvature of the flux and upscattering phenomena at energies of the order of kT and below.

With the mass-1 free gas kernel as an example and the variables listed in Table 1, the eigenvalues were found by using the Jacobi method and are listed in Table 2. The eigenvalues computed by using the energy variable lead to computing-machine overflows and are not listed because of the resultant inaccuracies. The first few eigenvalues listed in the table appear to be related; but, as one proceeds down the list, the eigenvalues for the various meshes become quite unrelated to each other. All of these three sets of eigenvalues and eigenfunctions are as adequate to represent the time derivative of the neutron flux as they are to represent the scattering in the integral of Eq. 1 or Eq. 3. Besides the cross section this integrand contains only the flux, which is a more or less smooth function as observed. The first few eigenvalues, those less than the minimum of $v\Sigma_T$, are discrete eigenvalues and represent measurable quantities. The higher eigenvalues represent selections from the continuum;⁶ the selections are just those necessary to give the correct value of the time derivative of the flux at the chosen energy points.

If the time dependence were found by measuring instruments such as a series of crystal spectrometers or time-of-flight channels that selected the energies of a particular mesh, then one might expect to find the particular eigenvalues selected by that mesh. As was stated before, the integration scheme should be adequate to represent the physical circumstances.

Many polynomial expansions have been used to find the eigenvalues of Eq. 3. To shed some more light on the use of a discrete mesh, we note here some results we obtained (following closely the work of Bailly du Bois, Horowitz, and Maurette⁷) by using the method of directly expanding the flux and cross-sectional quantities of Eq. 3 in odd-order Hermite polynomials. It is particularly instructive to apply this method to the mass-1 free gas case because the necessary integrals are then available analytically. By finding the integrals necessary to determine the eigenvalues in this method both numerically and analytically, we ascertained the effect of mesh spacing on the eigenvalues. These results are listed in Table 3. The upper limit

Table 1 - ENERGIES AT THE MESH POINTS COVERING
THE INTERVAL 10^{-6} TO 1.0 EV IN 37 EQUAL INTERVALS
IN VARIABLES X , Z_1 , Z_2 , AND E/kT

Mesh points	Energy, ev			E/kT
	X $(\sqrt{E/kT})$	Z_1 $[X^2/(1 + X^2)]$	Z_2 $[X/(1 + X)]$	
1	0.000001	0.000001	0.000001	0.000001
2	0.000826	0.000705	0.0000242	0.0277
3	0.00319	0.00145	0.0000819	0.0555
4	0.00709	0.00223	0.000179	0.0833
5	0.0125	0.00307	0.000322	0.111
6	0.0195	0.00396	0.000518	0.138
7	0.0280	0.00491	0.000775	0.166
8	0.0381	0.00592	0.00110	0.194
9	0.0497	0.00700	0.00151	0.222
10	0.0628	0.00815	0.00202	0.250
11	0.0775	0.00940	0.00264	0.277
12	0.0937	0.0107	0.00339	0.305
13	0.111	0.0121	0.00429	0.333
14	0.130	0.0137	0.00537	0.361
15	0.151	0.0154	0.00667	0.388
16	0.174	0.0173	0.00822	0.416
17	0.198	0.0193	0.0100	0.444
18	0.223	0.0216	0.0122	0.472
19	0.250	0.0240	0.0149	0.500
20	0.279	0.0268	0.0181	0.527
21	0.309	0.0299	0.0129	0.555
22	0.340	0.0334	0.0265	0.583
23	0.374	0.0373	0.0320	0.611
24	0.408	0.0418	0.0388	0.638
25	0.444	0.0470	0.0471	0.666
26	0.482	0.0531	0.0574	0.694
27	0.522	0.0603	0.0701	0.722
28	0.563	0.0689	0.0862	0.750
29	0.605	0.0795	0.106	0.777
30	0.649	0.0927	0.132	0.805
31	0.694	0.109	0.167	0.833
32	0.742	0.132	0.213	0.861
33	0.790	0.165	0.276	0.888
34	0.840	0.213	0.364	0.916
35	0.892	0.295	0.492	0.944
36	0.945	0.463	0.686	0.972
37	1.000	1.000	1.000	1.000

Table 2 – EIGENVALUES IN UNITS OF $v_{kT} \sum_{epi}$ OBTAINED
WITH THE DISCRETE MESHES IN TABLE 1 FOR
THE MASS-1 FREE GAS*

Mesh points	$\frac{X}{(\sqrt{E/kT})}$	Z_1 $[X^2/(1 + X^2)]$	Z_2 $[X/(1 + X)]$
1	0.56×10^{-6}	0.59×10^{-6}	0.48×10^{-6}
2	0.9238	0.9242	0.9241
3	1.100	1.106	1.105
4	1.127	1.119	1.126
5	1.154	1.138	1.1282
6	1.212	1.150	1.1286
7	1.294	1.166	1.129
8	1.405	1.177	1.132
9	1.502	1.194	1.135
10	1.650	1.206	1.139
11	1.758	1.225	1.145
12	1.930	1.239	1.151
13	2.044	1.259	1.162
14	2.231	1.276	1.170
15	2.349	1.298	1.185
16	2.546	1.318	1.198
17	2.667	1.342	1.220
18	2.869	1.366	1.238
19	2.993	1.394	1.267
20	3.198	1.423	1.296
21	3.325	1.454	1.334
22	3.531	1.490	1.377
23	3.662	1.527	1.426
24	3.867	1.571	1.489
25	4.002	1.615	1.553
26	4.205	1.672	1.646
27	4.345	1.725	1.727
28	4.544	1.800	1.864
29	4.690	1.867	1.969
30	4.884	1.971	2.170
31	5.036	2.057	2.310
32	5.226	2.211	2.588
33	5.384	2.324	2.869
34	5.569	2.572	3.132
35	5.740	2.881	3.840
36	5.916	2.993	4.037
37	6.112	4.156	5.369

* The top entry in each column is an indication of the numerical accuracy since this value would be zero in an analytic treatment.

Table 3 - EIGENVALUE SENSITIVITY, IN THE MASS-1 FREE GAS CASE, TO NUMERICAL MESH WHEN 16 HERMITE POLYNOMIALS ARE USED IN THE METHOD OF BAILLY *du BOIS*, HOROWITZ, AND MAURETTE⁷

Hermite polynomials	Numerical integrations			Analytical integration
	21 mesh points	41 mesh points	61 mesh points	
1	0	0	7	0
2	0.47032-2	0.88474	0.92327	0.92431
3	0.21885	1.0691	1.1423	1.1457
4	0.95143	1.3365	1.3542	1.3555
5	1.3921	1.6259	1.6356	1.6365
6	1.8741	1.9562	1.9635	1.9644
7	2.3960	2.3168	2.3233	2.3243
8	2.9396	2.7004	2.7068	2.7079
9	3.0681	3.1035	3.1102	3.1115
10	3.5172	3.5254	3.5325	3.5341
11	4.0905	3.9672	3.9550	3.9769
12	4.6701	4.4323	4.4409	4.4431
13	5.2624	4.9266	4.9362	4.9387
14	5.8704	5.4611	5.4719	5.4749
15	6.0082	6.0577	6.0699	6.0734
16	6.6351	6.7747	6.7887	6.7931

of the numerical integration in these cases was 2 ev. We note that a rather fine mesh is needed if one uses a method that requires numerical quadrature of Hermite functions times cross sections; this requirement is a result of the closely spaced zeros of the Hermite functions. Also, the distribution of eigenvalues and the representation of the integral are not as transparently flexible as they are in the direct use of a discrete mesh.

Another form for the calculation of the eigenvalues is based on rewriting Eq. 3 as

$$F(E) = P \frac{1}{\Sigma_T(E) - \lambda/v} \int \Sigma_s(E' \rightarrow E) F(E') dE' + h(\lambda) \delta[\lambda - v \Sigma_T(E)] \quad (4)$$

where P stands for principal value and δ is the Dirac delta function. This method, where the factor $[\Sigma_T(E) - \lambda/v]^{-1}$ is not further expanded but all other quantities in the equation are expanded in polynomials, was introduced by Corngold, Michael, and Wollman⁴ and by Koppel.⁸ In some examples we have rather closely followed Koppel's method including the expansion of the kernel and cross sections in Hermite polynomials. Table 4 indicates the rate of convergence of the eigenvalues when Eq. 3 is solved by expanding $F(E)$ in various numbers of Hermite polynomials, and a comparison is made when the expression $[\Sigma_T(E) - \lambda/v]^{-1}$ is factored out. Without this factoring the Hermite polynomial expansion is seen to converge slowly. The results of the factored method may also be seen to agree with the eigenvalues in Table 2 when discrete meshes are used.

CALCULATION OF EIGENFUNCTIONS

Experimental measurements have been made of asymptotic spectra, and these are just the lowest eigenfunction of the system. Time-dependent spectra and the time integral of the spectrum after a pulse have also been frequently measured, and they may also be related to eigenfunctions.

Table 4 - EIGENVALUE SENSITIVITY TO NUMBER OF HERMITE POLYNOMIALS USED IN THE METHOD OF BAILLY du BOIS, HOROWITZ, AND MAURETTE⁷ TO CALCULATE THE INFINITE-MEDIUM MASS-1 FREE GAS CASE

Hermite analytical			Factored*
8 polynomials	16 polynomials	31 polynomials	
0.0	0.0	0.0	0.0
0.92732	0.92431	0.92417	0.9242
1.2429	1.1457	1.11505	1.1055
1.6473	1.3555	1.22322	
2.1562	1.6365	1.37954	
2.7441	0.19644	1.57189	
3.4170	2.3243	1.79029	
4.2312	2.7079	2.02750	

*The results in this column were obtained by the method of Corngold, Michael, and Wollman⁴ and Koppel⁸ and may be compared with the results shown in Table 2.

In the mass-1 free gas case for an infinite medium with any non-zero amount of $1/v$ absorption, the asymptotic eigenfunction is a Maxwellian. This eigenfunction is produced as an exact result in the polynomial expansion methods and is also given to a very high degree of accuracy when numerical methods and a discrete mesh are used. The third eigenfunction for this case is illustrated in Fig. 1. The agree-

Fig. 1 - The third eigenfunction in the infinite-medium mass-1 free gas case calculated by using a discrete mesh,⁶ Hermite polynomials,⁷ and the factored method.^{4,8} ($1/v$ absorption.)

ment between the results found by the discrete mesh calculation and those found by the factored method is quite good; however, there is considerable disagreement between these results and the direct expansion of the eigenfunction in Hermite polynomials. In Fig. 2 a continuum eigenfunction for the same problem computed by the same three methods is illustrated. In this eigenfunction the quantity $\Sigma_T(E) - \lambda/v$ has a zero, and the factored method yields a function that has meaning under an integral sign. The discrete mesh eigenfunction and the other functions do not happen to have the same eigenvalue picked from the continuum but are nonetheless easily comparable. The discrete mesh calculation and the expansion with the singularity factored out agree quite well. Again the direct expansion of the eigenfunction in Hermite polynomials does not appear to agree with the others; there is an apparent difficulty here in that it is rather difficult for the ex-

Fig. 2 - A continuum eigenfunction in the infinite-medium mass-1 free gas case calculated by using a discrete mesh,⁶ Hermite polynomials,⁷ and the factored method.^{4,8} ($1/\nu$ absorption.)

pansion in polynomials to match a violent fluctuation in the neighborhood of the singularity. This difficulty may not only lead to an error in this eigenfunction but may also be responsible for the discrepancy noted with respect to Fig. 1; a change in one eigenfunction may be reflected in others because they span a certain function space in an orthogonal fashion.

Equation 1 must be modified to account for leakage losses in the case of noninfinite systems. The use of time-dependent diffusion theory with a P_1 scattering kernel that is diagonal in E and E' and the consideration of a flux that varies as $\cos(Bx)$ leads to the following modification of Eq. 3:

$$[D(E) B^2 + \Sigma_T(E) - \lambda/\nu] F(E) = \int \Sigma_s(E' \rightarrow E) F(E') dE' \quad (5)$$

where D is an energy-dependent diffusion constant.

Figure 3 shows the energy dependence of the loss rate as a function of energy for the terms on the left side of Eq. 5 evaluated for beryllium. Since the loss rate is not monotonic in this example, a value of λ greater than the minimum loss rate may lead to more than one singularity in the eigenfunction. Some of the continuum eigenfunctions for this case, calculated by using a discrete mesh, are shown in Fig. 4.

Further calculations in this same series in beryllium are shown in Fig. 5; in particular, the lowest eigenfunction is shown for a series

Fig. 3 - The loss term in the energy-dependent diffusion equation for beryllium.

Fig. 4 - Some typical continuum eigenfunctions calculated for beryllium with $B^2 = 0.026 \text{ cm}^{-2}$ by using energy-dependent diffusion theory and a discrete mesh.

Fig. 5 - The lowest eigenfunction calculated for beryllium at the indicated bucklings using energy-dependent diffusion theory and a discrete mesh.

of buckling values. At the higher values of buckling, the lowest eigenfunction behaves like a continuum eigenfunction and concentrates the flux at the lowest energy point in the mesh used (0.00005 eV). The energy-exchange kernel used had inaccuracies of about 0.1%, which may be important because of the large variation of the amplitude of the fluxes in Fig. 5. The merging of the asymptotic eigenvalue into the continuum of eigenvalues has been found previously in numerical calculations for crystalline materials by Corngold and Shapiro.⁵

Time-dependent spectra for the infinite-medium mass-1 free gas kernel with an absorption lifetime of 200 μ sec are shown in Fig. 6, with various meshes used. The calculations use the eigenfunctions to match an initial-value problem with a source given as a pulse in time of a slowing-down distribution of a mass-1 free gas. The only discrepancies in the use of the three different meshes are at the earliest

Fig. 6 - Infinite-medium spectra for the mass-1 free gas kernel with $\nu \Sigma_a = (200 \times 10^{-6} \text{ sec})^{-1}$ at the indicated times, with equal-spaced meshes used in the indicated variables [$X = \sqrt{E/kT}$, $Z_1 = X^2/(1 + X^2)$, and $Z_2 = X/(1 + X)$].

times where the highest energies are important and it is clear that the mesh spacing at these energies is somewhat coarse.

THE REAL λ VS. B^2 PLANE

In this section we limit the discussion to plane-geometrical configurations where parameters may vary with the x but not the y and z spatial coordinates and further where the energy-exchange kernel has an arbitrary isotropic component but all higher angular components are diagonal in E and E' . The angular part of the transport problem is then treated by means of the N th-order spherical harmonics approximation. The form of the solution for the scalar flux is written as

$$F(E) e^{-\lambda t} \cos Bx \quad (6)$$

(a $\sin Bx$ term may also be considered for problems involving some spatial asymmetry), where $F(E)$ is a solution of

$$L_N[B, \Sigma_a(E) - \lambda/v, \Sigma] F(E) = \int \Sigma_s(E' \rightarrow E) F(E') dE' \quad (7)$$

where $L_N[B, \Sigma_a(E) - \lambda/v, \Sigma]$ is an operator depending on various powers of B and $\Sigma_a(E) - \lambda/v$ as well as on the other energy-dependent cross sections; the operator also changes with the order N . This operator may be derived as the extension to the time-dependent case of Davison's general formulation of the spherical harmonics method.² Equation 7 reduces to Eq. 3 in the infinite-medium case for all N . In the time-dependent P_1 approximations, Eq. 7 is given explicitly by

$$\left\{ \frac{B^2}{3[\Sigma_s(E) - \Sigma_{s1}(E) + \Sigma_a(E) - \lambda/v]} + \Sigma_s(E) + \Sigma_a(E) - \lambda/v \right\} \times F(E) = \int \Sigma_s(E' \rightarrow E) F(E') dE' \quad (8)$$

This equation represents an energy-dependent telegrapher's equation. The usual time-dependent diffusion theory, Eq. 5, results from the neglect of the λ/v term in the denominator of the B^2 term of Eq. 8.

The permissible values of λ and B^2 are connected by the transport equation used. We are particularly motivated to study all real values of λ because of several important special cases and because it furnishes a start in understanding the details of more complicated cases involving complex values of λ in the particular form we have chosen for the scalar flux. An attempt has been made to illustrate some of the known features of the problems involving real values of λ by means of the schematic plot of real λ vs. B^2 in Fig. 7. Some of the salient features of this plot are as follows:

1. Positive values of λ correspond to decaying modes and belong to the area of interest in pulsed experiments.

2. Negative or zero values of λ , which are solutions of the same equation, as are the positive values,⁹ correspond to various amounts of $1/v$ absorption in exponential measurements of relaxation lengths and similar static problems. This is evident from the form of Eq. 7.

3. $B^2 > 0$ corresponds to sinusoidal type of spatial dependence.

4. $B^2 = 0$ corresponds to an infinite medium.

5. $B^2 < 0$ corresponds to hyperbolic sine or exponential type of spatial dependence.

6. There is a continuum of eigenvalues^{3,4} in the infinite-medium time-dependent case indicated by the heavy line on the axis. This has been shown to be a complete set in the continuous⁸ and discrete mesh⁶ cases.

7. The continuum of item 6 starts at a value $\lambda^* = (v\Sigma_{\min})_{\text{inelastic}}$ (Refs. 4, 6, and 10) above which λ forms a continuum.

8. There is a value of buckling, B^{*2} , beyond which no discrete eigenvalues exist.¹¹ This circumstance is an energy effect and not a transport effect since it exists in diffusion theory. See, for example, the continuum lowest eigenfunction at large bucklings in Fig. 5.

9. There is a value of buckling, $B^{\dagger 2}$, beyond which no real-time eigenvalues of the form of Eq. 6 exist.¹² This is a transport effect since it exists even in one-group time-dependent P_N theory but not in time-dependent diffusion theory.

10. There is a curve, labeled 1 in Fig. 7, which is the asymptotic time eigenvalue as a function of buckling in the first quadrant and is the asymptotic diffusion length as a function of absorption in the third quadrant.

11. There may be other discrete eigenvalues such as curve 2.

12. There is usually (constant cross sections being the notable exception) a limiting value of λ , λ^{**} below which no discrete diffusion length occurs.¹¹

13. The second quadrant contains spatial transients in the time-dependent problem. This is illustrated subsequently.

14. The third quadrant contains the continuum necessary in the static case.

15. The fourth quadrant is empty because no sinusoidal solutions are expected in the homogeneous source-free static case.

Fig. 7 - Schematic map of the real λ vs. B^2 plane.

In Fig. 8 are shown the real eigenvalues for a three-group calculation in both the P_1 and diffusion approximations that roughly represent beryllium. In N -group diffusion theory at any value of B^2 , there are enough real eigenvalues to assure matching any N -group scalar flux. If a zero-flux boundary condition is used, then the possibility of utilizing all values of B^2 assures completeness in this approximation. Furthermore, because of this completeness with the form given by Eq. 6, it is clear that diffusion theory yields no traveling waves.

In P_1 theory an upper limit on the values of B^2 that yield real values of λ is evident. The restricted range of B^2 covered by the real values of λ is clearly not sufficient to match arbitrary initial conditions in the space variable for the time-dependent problems.¹³⁻¹⁵

The P_1 curves in the vicinity of the origin are shown on an expanded scale in Fig. 9. In the commonly used one-buckling approximation, the fundamental spatial buckling or one of its harmonics is matched in energy by a linear combination of the eigenfunctions as-

Fig. 8 - A plot in the real λ vs. B^2 plane of a theory using three groups (with cross sections representing beryllium) in two different time-dependent transport approximations. Note that all the curves extend indefinitely in roughly the directions indicated.

Fig. 9 — A plot in the vicinity of the origin of the real λ vs. B^2 plane of a theory that uses three groups (with cross sections representing beryllium) in the time-dependent P_1 approximation.

sociated with the eigenvalues at that particular buckling. This is possible with real relaxation times only for very small bucklings. As the buckling is increased, complex eigenvalues must be brought in to satisfy the initial conditions. At large enough bucklings one would have to use only complex values of λ to match initial conditions, but here the single-buckling approximation has broken down.

The single-buckling approximation or zero-flux boundary condition may be replaced by a Marshak or similar type of boundary condition. In this case all the values of B^2 having the same λ are combined linearly to satisfy the boundary condition. In the time-dependent case, the minimum value of λ is $v\Sigma_a$, which is for the case of $B^2 = 0$. For slightly larger values of λ , one value of B^2 is positive and is accompanied by two negative values of B^2 ; these negative bucklings yield exponential transients near the boundary to satisfy the Marshak type conditions. For somewhat larger values of B^2 there may be two or

three positive values of B^2 for the same value of λ ; in these cases various groups would behave all through the slab as differing sinusoids with different extrapolation distances. The set of real values of λ is obviously not complete in the P_1 theory, since this set does not yield the traveling wave fronts known to be included in the time-dependent P_1 approximation.

CONCLUSIONS

One of the major motivations for experimental and analytical work in pulsed neutron phenomena in moderating assemblies is to shed light on the thermal region of reactor problems. The analysis presented in this paper is a framework with which one may try to understand and to see some of the limitations of certain types of measurement. Also, the theory as presented can be used directly to furnish experimental predictions in many cases. Further extension and understanding, especially into the role of complex values^{13,14} of λ in the form given by Eq. 2, should open up to more penetrating analysis the study of the complete formation in time of thermal spectra. The time development of the spectrum may be thought of as following essentially in real time what a computer does in Monte Carlo techniques for the steady-state spectrum.

By means of the pulsed method in large assemblies, one may study the asymptotic eigenvalue in terms of a power series analysis,

$$\lambda = \overline{v\Sigma_a} + \overline{vD} B^2 - CB^4 + \dots \quad (9)$$

The first two terms in this series have been determined remarkably easily and accurately for a number of materials. The third term, involving C , the diffusion cooling constant, is evasive in this particular development. In large assemblies when B^2 is small and accurate measurement is possible, the entire third term does not contribute appreciably. As one goes to smaller assemblies in order to obtain larger values of B^2 , one gets into the region where a sharp bend¹² may be encountered in the curve of the lowest eigenvalue, as well as the merging of the asymptotic eigenvalue into the continuum,^{3,4} which vitiates the use of the series of Eq. 9 to determine C . Fullwood¹⁶ has shown how a measurement of the asymptotic spectrum may be used to yield a value for the diffusion cooling constant, and this appears to yield much better agreement between theory and experiment than the use of the measurement of the asymptotic eigenvalue.

At this point we may also note that the asymptotic eigenvalue has been used to determine the thermal average of non- $1/v$ absorbers rather successfully.¹⁷

The direct measurement of higher eigenvalues has not been pursued to any great extent. In cases such as water, where more than one discrete eigenvalue exists, the numerical extraction of the eigenvalues requires extraordinary experimental accuracy.¹⁸ In the case of crystalline materials such as beryllium and carbon, there may be a separated discrete eigenvalue or no discrete eigenvalues at all. Ghatak and Honeck¹⁰ have suggested the measurement of speed or energy moments as a function of time in these cases. Another alternative, which carries a great deal of information, is the measurement of the time-dependent spectrum. We note that the type of analysis presented here leads directly to a detailed comparison of such experiments and theory. Study of this comparison can easily show in a semiquantitative manner, in cases where there is disagreement between experiment and theory, the areas of energy transfer that are being improperly treated. In principle the time-dependent spectrum in large assemblies contains enough information to determine the energy-exchange scattering kernel.¹⁹

The understanding of the eigenvalue spectrum should be of considerable help in the understanding and treating of the rethermalization problem, i.e., the case of neutron transport involving regions at different temperatures. The temperature discontinuity induces energy transients in the spectrum, and the type of analysis used, unless it is a complete analytic treatment, will select from the continuum a discrete set of transient relaxation or rethermalization lengths.¹⁵ This understanding may be used to select a set adequate to represent the physical situation.

The large amount of work in neutron thermalization is sharpening our intuition in the area of thermal-neutron work. It is leading to a meaningful predictive capability in the treatment of temperature coefficients,^{20,21} since, besides density effects, it is only the thermal scattering kernel which carries the temperature into the neutron transport problem. It is also leading to a raising of the thermal cutoff energy and a realistic treatment of absorption rates in non- $1/v$ absorbers such as ²³⁹Pu in the range of a few tenths of an electron volt, where chemical binding makes scattering kernels most complicated.

In the comparison of theory and experiment in this field, both the agreement and disagreement have led to progress. The agreement, of course, has led to progress because it has verified the use of certain cross-section data and theoretical models for scattering kernels. The disagreements are important because they have greatly stimulated work in two major areas. First, they have stimulated work on the application of solid-state physics in the construction of kernels important to reactor work. Second, they have stimulated the reactor analyst to develop better tools for using in an economic and penetrating way the better cross sections thus provided.

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Spatial Dependence of Pulsed Neutron Reactivity Measurements

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ABSTRACT

A series of pulsed neutron reactivity measurements is reported in which strong spatial dependence of the results is observed. The difference between prompt and delayed fundamental-mode shapes is found to account for this dependence. The major pulsed neutron techniques are rederived by the use of a kinetics analysis in which the prompt and delayed fluxes are each assumed to be separable in space and time but in which the total flux is not. It is concluded that standard experimental methods yield a meaningful reactivity if an easily obtained spatial correction factor is applied.

INTRODUCTION

In the past few years, much attention has been focused on the development of pulsed neutron techniques for the measurement of the reactivity of a subcritical assembly. Of the principal methods proposed, only the method of Simmons and King,¹ which depends only on the asymptotic prompt decay constants of the subcritical assembly and of a reference critical configuration, yields a result that is independent of the position of the detector. None of the others rigorously eliminates spatial dependence. When this method is applied, however, the generation times for the critical and subcritical configurations are usually assumed to be equal.

The earliest technique of direct reactivity measurement was proposed by Sjostrand.² His *area-ratio* method assumes that the fundamental mode dominates the response; thus this procedure is sensitive to the presence

of higher harmonics. Gozani³ attempted to alleviate this sensitivity with an *extrapolated-area-ratio* method, which eliminated the prompt but not the delayed higher harmonics. Garelis and Russell⁴ tried to modify the Simmons–King method (by introducing a parameter to replace the critical decay constant of the Simmons–King method) so as to make it independent of both the constant-generation-time assumption and higher harmonics. The conclusions of Garelis and Russell, however, depend for validity upon the assumption of a bare reactor containing monoenergetic neutrons.

In addition to the presence of higher harmonics, a factor that can cause spatial dependence of reactivity measurements is that the prompt and the delayed eigenfunctions differ. Very little attention of a quantitative nature has been paid to this *modal-shape difference*, even though it is known^{5,6} and has been cited as a possible source of difficulty in pulsed neutron experiments.⁶

This paper advances the thesis that modal-shape difference is a major cause of the spatial dependence of pulsed neutron reactivity measurements. We report a series of experiments performed on the Flexible Plastic Reactor⁷ (FPR) at the Knolls Atomic Power Laboratory (KAPL), the results of which were strongly dependent on detector position. These results demonstrate that this position dependence is accounted for by the difference in spatial shape between the prompt and the delayed fundamental modes.

When the prompt and the delayed fundamental modes differ significantly, the reactor kinetics analyses of Refs. 2 to 4 are no longer valid. It is therefore necessary to show that the quantity determined in a pulsed neutron experiment can be interpreted as the product of a reactivity and of a spatial correction. Such an analysis is presented in this paper.

In order that the paper be self-contained, it begins with a critical review of the various pulsed neutron techniques. Next, prompt and delayed eigenfunctions and why they differ are discussed. The various pulsed neutron procedures are then derived by the use of a reactor kinetics formulation that accounts for modal-shape difference. The experimental procedure is described, and the experimental results are presented and shown to bear out the contention of the importance of modal-shape difference.

A SURVEY OF PULSING TECHNIQUES

This section briefly describes the methods developed in Refs. 1 to 4 and points out some difficulties associated with these methods.

Simmons and King¹ assume that the prompt decay constant is given by

$$\alpha = \frac{\rho + \beta_e}{\Lambda} \quad (1)$$

where β_e , Λ , and ρ are effective delayed-neutron fraction, generation time, and subcritical reactivity. When the reactor is critical, the reactivity is zero and the decay constant becomes

$$\alpha_c = \left(\frac{\beta_e}{\Lambda} \right)_c \quad (2)$$

If the delayed-neutron fraction and generation time do not change when the system is made critical, then it follows that the reactivity is given by

$$\frac{\rho}{\beta_e} = \frac{\alpha}{\alpha_c} - 1 \quad (3)$$

The assumption of a subcriticality-independent generation time is an important one, as Simmons himself points out.¹ It has often been necessary to apply a generation-time correction to the Simmons-King formula, a correction that can be difficult to make reliably.⁸

A more fundamental objection, which includes the generation-time assumption as one of its facets, is that the method yields a result dependent not only on the subcritical configuration under study but also on an additional critical configuration. It is therefore necessary to bring the reactor to criticality in order to make a measurement of the subcriticality. The procedure of bringing the reactor to criticality is often complicated and time-consuming experimentally, and there are cases where this is not even possible. In addition, one must face the conceptual problem that there is not a unique critical configuration (nor, consequently, a unique critical decay constant) relative to the subcritical situation being investigated.

Garelis and Russell⁴ attempted to eliminate the dependence on a critical configuration and on a critical decay constant by determining reactivity from the formula

$$\frac{\rho}{\beta_e} = \frac{\alpha}{\alpha_G} - 1 \quad (4)$$

They show that in a repetitively pulsed, bare, monoenergetic reactor the parameter α_G , interpreted as β_e/Λ analogous to Eq. 2, is given by

$$\int_0^T dt \left(e^{\alpha_G t} - 1 \right) \phi_P = \int_0^T dt \phi_D \quad (5)$$

where $\phi_p(t)$ is the prompt flux, $\phi_D(t)$ is the delayed flux, and T is the repetition period. Thus the *effective critical decay constant*, α_G , is determined from the delayed-neutron behavior of the subcritical system (rather than from the prompt-neutron behavior of a critical configuration). Within the context of their model, the results Garelis and Russell obtain are independent of the presence of higher harmonics. Henry⁵ and Corngold,⁹ however, have pointed out that several important assumptions made by Garelis and Russell break down when more sophisticated reactor models are used. If higher harmonics are not important and if the prompt and delayed fundamental modes have the same shape, the Garelis–Russell formulation is still valid.

An interesting property of the Garelis–Russell formulation is the tendency toward experimental elimination of prompt higher harmonics. The $e^{\alpha_G t} - 1$ weighting given to the prompt flux tends to minimize the consideration given to the flux until after the higher harmonics have decayed. As will be shown, our experiments demonstrate that the Garelis–Russell method actually yields results very close to those obtained from the extrapolated-area-ratio method (which rigorously eliminates prompt higher harmonics as will be discussed in the next paragraph).

The area-ratio method of Sjostrand² obtains the reactivity from

$$\frac{\rho}{\beta_e} = \frac{\int_0^T \phi_P dt}{\int_0^T \phi_D dt} \quad (6)$$

Sjostrand's formulation neglects modal-shape difference and higher harmonics. In the extrapolated-area-ratio method, Gozani eliminates the prompt higher harmonics by waiting until they have decayed before counting prompt neutrons and then by extrapolating the prompt area back to zero time. The reactivity is then given by

$$\frac{\rho}{\beta_e} = e^{\alpha t_w} \frac{\int_{t_w}^T \phi_P dt}{\int_0^T \phi_D dt} \quad (7)$$

where t_w is the waiting time. Both Sjostrand² and Gozani³ neglected modal-shape differences in these formulations. The extrapolated-area-ratio method still depends on delayed-neutron harmonics.

The de-emphasis or elimination of prompt harmonics while delayed harmonics are retained has been mentioned in connection with two methods. The question is: If harmonic effects are important, why would not the presence of delayed harmonics be sufficient to invalidate the results? Delayed harmonics constitute a less severe contamination

than do prompt harmonics. The reason is that not all prompt effects are "remembered" well by the delayed flux. The delayed flux tends to remember the fission distribution. Thus certain reflector effects (such as those to be discussed in the next section) tend to be "forgotten." In addition, energy transients, such as the slowing-down transient that occurs when a fast pulse thermalizes into a fundamental-mode spectrum,¹⁰ tend to be forgotten. Following Fultz,¹⁰ we write the thermal flux as

$$\phi(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha_s t} + A_3 e^{-\lambda t} \quad (8)$$

where α is the prompt fundamental decay constant, α_s is a slowing-down time constant, and λ is a precursor decay constant. The second term corresponds to a slowing-down source (from the pulsed source), which does not have corresponding delayed neutrons.

PROMPT AND DELAYED EIGENFUNCTIONS

The assumption is often made (as in Refs. 2 and 4 and for the fundamental mode in Ref. 3) that the time-dependent flux for a subcritical system can be written in the form (with the use of the one-group theory for convenience)

$$\phi(r, t) = \sum_{n=0}^{\infty} \phi_n(r) \sum_{j=0}^M a_{nj} e^{-\alpha_{nj} t} \quad (9)$$

where M is the number of delayed-neutron groups. Thus the assumption is made that the prompt modes have the same spatial distributions as do the corresponding delayed modes. This section illustrates the significance of this assumption.

For simplicity, consider a reflected slab reactor in a one-group diffusion theory representation in which one delayed-neutron group is used. The core flux, precursor density, and reflector flux are given by

$$\frac{1}{v} \frac{\partial \phi_c}{\partial t} = D_c \nabla^2 \phi_c + [k_c(1 - \beta) - 1] \sum_{ac} \phi_c + \lambda C \quad (10)$$

$$\frac{\partial C}{\partial t} = \beta k_c \sum_{ac} \phi_c - \lambda C \quad (11)$$

$$\frac{1}{v} \frac{\partial \phi_r}{\partial t} = D_r \nabla^2 \phi_r - \sum_{ar} \phi_r \quad (12)$$

where the subscripts c and r denote core and reflector, k_c is the infinite-medium multiplication factor of the core material, and the remaining notation is standard. If we assume eigenfunction solutions of the decaying exponential type

$$\phi_c(r,t) \sim \phi_{cnj}(r) e^{-\alpha_{nj}t} \quad (13)$$

$$C(r,t) \simeq C_{nj}(r) e^{-\alpha_{nj}t} \quad (14)$$

$$\phi_r(r,t) \simeq \phi_{rnj}(r) e^{-\alpha_{nj}t} \quad (15)$$

and note that

$$C_{nj}(r) = \frac{\beta k_c \sum_{ac}}{\lambda - \alpha_{nj}} \phi_{cnj}(r) \quad (16)$$

we obtain the eigenvalue equations

$$\nabla^2 \phi_{cnj} + \frac{\sum_{ac}}{D_c} \left(k_c - 1 + \frac{\alpha_{nj}}{v \sum_{ac}} + \frac{\beta k_c \alpha_{nj}}{\lambda - \alpha_{nj}} \right) \phi_{cnj} = 0 \quad (17a)$$

$$\nabla^2 \phi_{rnj} + \frac{\sum_{ar}}{D_r} \left(\frac{\alpha_{nj}}{v \sum_{ar}} - 1 \right) \phi_{rnj} = 0 \quad (17b)$$

The eigenvalues α_{nj} will occur in pairs, α_{nP} and α_{nD} , corresponding to the prompt and delayed *submodes*.⁵ The prompt eigenvalues constitute a monotonically increasing set ranging from the fundamental value α_{0P} to infinity. The delayed eigenvalues, however, are all in the vicinity of the precursor decay constant, λ .

Consider the spatial shapes of the prompt and delayed eigenfunctions in the reflector. Because

$$\frac{\lambda}{v \sum_{ar}} \ll 1 \quad (18)$$

in realistic systems, all the delayed modes (see Eq. 17b) have shapes characterized by the static diffusion length

$$L_r = \sqrt{\frac{D_r}{\sum_{ar}}} \quad (19)$$

These shapes are exponential in nature. The prompt modes, however, are more complicated because $\alpha_{n_P}/v\Sigma_{ar}$ is generally not negligible compared to unity even for the fundamental mode. Typical thermal parameters and fundamental prompt decay constant for the polyethylene-reflected FPR are

$$\begin{aligned}\alpha_{0P} &\simeq 10^3 \text{ sec}^{-1} \\ v &\simeq 2 \times 10^5 \text{ cm/sec} \\ \Sigma_{ar} &\simeq 0.02 \text{ cm}^{-1} \\ \alpha_{0P}/v\Sigma_{ar} &\simeq 0.25\end{aligned}\tag{20}$$

Thus the reflector diffusion length characterizing the prompt fundamental mode will be greater than the static diffusion length.

In more sophisticated multigroup models, similar effects will be observed. The effective reduction in the absorption cross section will intensify the flux rise in the reflector. (Figures 4 to 8 will illustrate the difference between flux shapes for the prompt and the delayed thermal fluxes for the FPR experimental configurations considered in this study.) Prompt flux is calculated by removing from core and reflector an amount of boron equivalent to the decay constant. Delayed flux is obtained from a conventional diffusion calculation.

At high neutron energies the term $\alpha_{n_P}/v\Sigma_{ar}$ becomes small because the speed v becomes large. Reflector materials, however, generally have absorption cross sections that vary as $1/v$. Modal-shape difference should therefore be present at all energies. In Fig. 1 the prompt and delayed fast (energy above 0.8 Mev) fluxes are displayed for one case to illustrate the difference between fast-flux shapes. In the fast group, of course, reflector peaking is not present, and the effects are more closely related to the simple one-group model used in the preceding equations than are the thermal-group effects.

The deviation of the prompt mode from the corresponding delayed mode increases for higher eigenfunctions. In addition, a difference in kind (as well as in degree) is observed since there exists a harmonic for which

$$\frac{\alpha_{n_P}}{v\Sigma_{ar}} > 1\tag{21}$$

This prompt harmonic and all succeeding higher harmonics have trigonometric (not exponential) shapes in the reflector.

Fig. 1 - Core 5 fast fluxes.

The fact that prompt and delayed modes can differ substantially indicates that some of the analyses of Refs. 2 and 4 cannot be extended to reflected systems unless these analyses are modified so as to account for modal-shape difference.

ANALYSIS OF A PULSED NEUTRON EXPERIMENT

This section presents a reactor kinetics analysis that accounts for modal-shape difference. In particular, we will show how the area-ratio, the extrapolated-area-ratio, and the Garelis-Russell methods can be modified to account for modal-shape difference. The relative merits of these modified techniques will be studied in subsequent sections.

The prompt and delayed fluxes and the precursor densities are given by

$$\frac{1}{v} \frac{\partial \phi_P}{\partial t} = [-L + (1 - \beta) \chi_P F - A] \phi_P + S(r, E, \Omega, t) \quad (22)$$

$$\frac{1}{v} \frac{\partial \phi_D}{\partial t} = [-L + (1 - \beta) \chi_P F - A] \phi_D + \sum_{i=1}^M \lambda_i \chi_i C_i \quad (23)$$

$$\frac{\partial C_i}{\partial t} = \beta_i F \phi_D - \lambda_i C_i + \beta_i F \phi_P \quad (24)$$

where L = a leakage and energy transfer operator

F = the fission operator

χ_P and χ_i = prompt and delayed fission spectra, respectively

A = an absorption operator

S = an extraneous source

Let these equations be integrated over time from $t = 0$ (just before the source is turned on) to $t = \infty$. Since the fluxes and precursor densities are zero at $t = 0$ and $t = \infty$, the left sides of the equations vanish yielding

$$[-L + (1 - \beta)\chi_P F - A] \int_0^\infty \phi_P dt + \int_0^\infty S dt = 0 \quad (25)$$

$$[-L + (1 - \beta)\chi_P F - A] \int_0^\infty \phi_D dt + \sum_{i=1}^M \lambda_i \chi_i \int_0^\infty C_i dt = 0 \quad (26)$$

$$\beta_i F \int_0^\infty \phi_D dt - \lambda_i \int_0^\infty C_i dt + \beta_i F \int_0^\infty \phi_P dt = 0 \quad (27)$$

Eliminating the precursor densities from Eqs. 26 and 27 yields

$$(-L + \chi_t F - A) \int_0^\infty \phi_D dt + \sum_{i=1}^M \beta_i \chi_i F \int_0^\infty \phi_P dt = 0 \quad (28)$$

where χ_t is the total fission spectrum

$$\chi_t = (1 - \beta)\chi_P + \sum_{i=1}^M \beta_i \chi_i \quad (29)$$

Assume that the prompt and delayed fluxes are each separable into a function of time and a function of space, energy, and direction:

$$\phi_P(r, E, \Omega, t) = \psi_P(r, E, \Omega) \theta_P(t) \quad (30)$$

$$\phi_D(r, E, \Omega, t) = \psi_D(r, E, \Omega) \theta_D(t) \quad (31)$$

We then obtain*

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = - \frac{(-L + \chi_t F - A)\psi_D}{\sum_{i=1}^M \beta_i \chi_i F \psi_P} \quad (32)$$

We have pointed out that the delayed flux is essentially a *static* distribution, i.e., that this flux is characterized by a static reflector diffusion length. It thus seems reasonable to choose ψ_D so that it satisfies the eigenvalue equation

$$\left(-L + \frac{1}{k} \chi_t F - A\right)\psi_D = 0 \quad (33)$$

With the static subcritical reactivity defined by

$$\rho = \frac{1 - k}{k} \quad (34)$$

we obtain

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = \rho \frac{\chi_t F \psi_D}{\sum_{i=1}^M \beta_i \chi_i F \psi_P} \quad (35)$$

Integrating the numerator and denominator of the right side over energy (since the left side is a constant, we may cross-multiply, integrate, and divide back to the original form) leads to

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = \rho \frac{\int_0^\infty dE W_1(E) \chi_t F \psi_D}{\int_0^\infty dE W_1(E) \sum_{i=1}^M \beta_i \chi_i F \psi_P} \quad (36)$$

*Actually we should deal with prompt and delayed detector responses, since these are what the experiment yields. If D_P and D_D are the prompt and delayed detector responses, then a similar analysis yields, instead of Eq. 40,

$$\frac{\int_0^\infty D_P dt}{\int_0^\infty D_D dt} = \frac{\rho \int_E dE \int_\Omega d\Omega \sigma(E, \Omega) \psi_P(r, E, \Omega)}{\beta_e \int_E dE \int_\Omega d\Omega \sigma(E, \Omega) \psi_D(r, E, \Omega)}$$

where $\sigma(E, \Omega)$ is the detector sensitivity.

where $W_1(E)$ is an arbitrary energy weighting function. If the effective delayed-neutron fraction β_e is defined by

$$\beta_e = \frac{\int_0^\infty dE W_1(E) \sum_{i=1}^M \beta_i \chi_i F \psi_P}{\int_0^\infty dE W_1(E) \chi_t F \psi_P} \quad (37)$$

then

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = \frac{\rho}{\beta_e} \frac{\int_0^\infty dE W_1(E) \chi_t F \psi_D}{\int_0^\infty dE W_1(E) \chi_t F \psi_P} \quad (38)$$

The expression on the left side is a constant. For the equation to be valid, the expression on the right side must also be a constant. This means that the prompt- and delayed-fission distributions in space must be the same. This is not a bad assumption because, as can be seen from Figs. 4 to 8, the prompt and delayed thermal fluxes are almost identical over most of the core (where the fissions take place). It then follows that

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = \frac{\rho}{\beta_e} \quad (39)$$

An area-ratio pulsed neutron reactivity measurement thus yields

$$\frac{\int_0^\infty \phi_P dt}{\int_0^\infty \phi_D dt} = \frac{\rho}{\beta_e} \frac{\psi_P}{\psi_D} \quad (40)$$

The area-ratio method should therefore have a spatial dependence on detector location given by the ratio of the prompt and delayed fundamental modes.

If the prompt and delayed fundamental modes are not identical in the core, part of the preceding analysis becomes invalid. We then return to Eq. 32. Let us integrate the numerator and denominator of the right

side over space, energy, and direction, with the bra-ket notation denoting the multivariable integration.

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = - \frac{\langle W_2, (-L + \chi_t F - A)\psi_D \rangle}{\langle W_2, \sum_{i=1}^M \beta_i \chi_i F \psi_P \rangle} \quad (41)$$

where W_2 is an arbitrary function of r, E, Ω . If we define reactivity as

$$\rho = - \frac{\langle W_2, (-L + \chi_t F - A)\psi_D \rangle}{\langle W_2, \chi_t F \psi_D \rangle} \quad (42)$$

and effective delayed-neutron fraction as

$$\beta_e = \frac{\langle W_2, \sum_{i=1}^M \beta_i \chi_i F \psi_P \rangle}{\langle W_2, \chi_t F \psi_P \rangle} \quad (43)$$

then we obtain

$$\frac{\int_0^\infty \theta_P dt}{\int_0^\infty \theta_D dt} = \frac{\rho}{\beta_e} \frac{\langle W_2, \chi_t F \psi_D \rangle}{\langle W_2, \chi_t F \psi_P \rangle} \quad (44)$$

If we had chosen to normalize the shapes ψ_P and ψ_D such that

$$\langle W_2, \chi_t F \psi_P \rangle = \langle W_2, \chi_t F \psi_D \rangle \quad (45)$$

(instead of to unity in the center), then we would have obtained Eqs. 39 and 40 again.

The assumption of Eqs. 30 and 31 is equivalent to the assumption of fundamental-mode dominance. Equation 40 is therefore a general formula for the area-ratio method, and it reduces to Sjostrand's expression (Eq. 6) when the prompt and delayed eigenfunctions are essentially the same, as in a bare reactor. The formula for the extrapolated-area-ratio method is obtained from Eq. 40 in the same way that Eq. 7 is obtained from Eq. 6:

$$e^{\alpha t_w} \frac{\int_{t_w}^\infty \phi_P dt}{\int_0^\infty \phi_D dt} = \frac{\rho}{\beta_e} \frac{\psi_P}{\psi_D} \quad (46)$$

To obtain the generalized Garelis–Russell formula, let us define α_G according to

$$\int_{0+}^{\infty} \theta_P (e^{\alpha_G t} - 1) dt = \int_0^{\infty} \theta_D dt \quad (47)$$

With θ_P varying as $e^{-\alpha t}$, this becomes

$$\left(\frac{1}{\alpha - \alpha_G} - \frac{1}{\alpha} \right) \theta_P(0+) = \int_0^{\infty} \theta_D dt \quad (48)$$

The $0+$ here refers to the time immediately following the pulse. (Earlier $t = 0$ was used to denote a time just before the pulse so that the initial flux values were zero.) With

$$\frac{1}{\alpha} \theta_P(0+) = \int_0^{\infty} \theta_P dt \quad (49)$$

Eq. 48 becomes

$$\left(\frac{\alpha}{\alpha - \alpha_G} - 1 \right) \int_0^{\infty} \theta_P dt = \int_0^{\infty} \theta_D dt \quad (50)$$

Invoking Eq. 39 yields

$$\frac{\rho}{\beta_e} = \frac{\alpha}{\alpha_G} - 1 \quad (51)$$

Thus the Garelis–Russell formula is obtained if Eq. 5 is replaced by

$$\frac{\int_0^{\infty} dt (e^{\alpha_G t} - 1) \phi_P}{\int_0^{\infty} dt \phi_D} = \frac{\psi_P}{\psi_D} \quad (52)$$

Basically our analysis thus far has been for a single pulse followed by an infinite integration time. Adequate delayed-neutron statistics, however, are generally obtained by the use of rapid repetitive pulsing. All the formulas derived here hold for repetitive pulsing at “equilibrium” if we replace the upper limit by the repetition period T . Equations 25 to 27 (with upper limit T) are again obtained because at equilibrium the left sides vanish owing to periodicity; e.g., $\phi_D(0) = \phi_D(T)$.

A problem of no little concern in repetitive pulsing experiments is fluctuation of pulse amplitude.¹¹ Most of the pulsed neutron sources in current use exhibit large burst-to-burst intensity variations, and such variations tend to invalidate the argument of periodicity. This problem can be eliminated by the alternative of counting not only when the system is in equilibrium and supposedly periodic but also when the equilibrium is being obtained, and after the last pulse, until the delayed-neutron contribution is zero (i.e., from times zero to infinity). One is then performing a superposition of many single-pulse experiments for which the analysis is valid. In fact, since the source in Eq. 22 was not specified, the analysis applies for general sources. Experimentally, one must be able to separate the prompt and the delayed fluxes, a matter that will be discussed later in this paper.

EXPERIMENTAL ARRANGEMENT

For the experiments described here, which were performed on the FPR at this laboratory,⁷ a core 18 by 18 by 36 in. was constructed in which enriched ²³⁵U metal fuel, aluminum structural material, and polyethylene moderator were used. The core was enclosed in a 6 in.-thick polyethylene reflector. Mylar tape coated with ¹⁰B enriched boron was used as poison. The poison density and the moderator density of the core were adjusted as necessary to provide the desired subcriticality. Table 1 is a summary of the compositions of the various cores used. Except for the case at delayed critical, all measurements were made with control rods removed, and therefore lumped poison corrections were not required in the subcritical calculations. Reactivity measurements of the subcriticality of each case were made by the inverse multiplication technique,¹² and these results are shown in Table 1. It should be noted that inverse multiplication results are, in general, increasingly unreliable as the subcritical reactivity increases and that the error for core 5 may thus be as large as $\pm 20\%$.

The calculated reactivity for each case was obtained with a four-group two-dimensional diffusion calculation by the use of the KARE system¹³ on the KAPL Philco-2000 digital computer. Group constants were obtained with the latest available cross-section libraries and the FOMENT¹⁴ and SWAK¹⁵ codes. The core materials used in the FPR are physically arranged so that it is a good calculational approximation to assume that the total amount of a given material is distributed uniformly throughout the core. Number densities of all important materials used are listed in Table 1 together with the calculational results. Reactivities are extracted from eigenvalues by the use of $\beta_e = 0.0072$. The calculated reactivity values of Table 1 are assigned an overall error of about $\pm 10\%$.

Table 1 - CORE-PARAMETER SUMMARY

Reactivity		Core composition, atoms/cm ³ × 10 ⁻²⁴				
Calculated*	Measured†	²³⁵ U	Aluminum	Hydrogen	Carbon	Boron
Core 1 (Critical Case)						
0.00‡	0.00	0.0005757	0.025862	0.029125	0.015319	0.0000033568
Core 2 (Poison Added)						
2.02	2.30	0.0005757	0.025862	0.029180	0.015388	0.000035874
Core 3 (Polyethylene Removed)						
3.6	4.25	0.0005757	0.025862	0.026143	0.013831	0.000033686
Core 4 (Poison Added)						
5.9	7.0	0.0005757	0.025862	0.026195	0.013896	0.000035874
Core 5 (Polyethylene Removed)						
10.0	12.0	0.0005757	0.025862	0.023054	0.012385	0.000035874
Reflector Material						
			0.0059429	0.06168	0.03084	

*Measured by means of the inverse multiplication method.

†Calculated by a four-group two-dimension diffusion code.

‡Assumed to eliminate the calculational bias.

The assumed subcritical reactivity listed in Table 2 is a composite, or best value, obtained by averaging the calculated and measured values of Table 1. They are listed for comparison purposes in the judgment that follows of the correctness of the pulsed neutron results, and they are used as nominal values throughout.

Pulsed neutron reactivity measurements were made as a function of detector position for each of the listed subcriticalities. Because the primary purpose of this investigation was to study detector-position effects, the detector pattern shown in Fig. 2 was employed. Reactivity changes caused by different positions of the active detectors were eliminated by a dummy detector set that was always in place in the core during pulse measurements. Six dummy detectors and two active detectors

Fig. 2 - Neutron-detector positions.

filled all eight available positions in one plane (horizontal) or the other (vertical) but not both simultaneously. Thus the reactivity of the core being pulsed was constant with respect to the active detector position, except for the minor effect of the connecting cable. The reactivity effects of the horizontal and the vertical arrays were the same.

The detectors used were Westinghouse WX4247 BF_3 -filled proportional counters. The active detectors were placed in symmetric positions so that, as far as possible, contributions from higher harmonics would be balanced. Like-numbered or -lettered positions (Fig. 2) were used together as the active positions, while the remaining six numbered or lettered positions were filled, respectively, with dummy detectors. When numbered positions were used, the lettered positions contained fuel, and when lettered positions were used, the numbered positions contained fuel.

Each detector was connected to its own preamplifier, amplifier, and discriminator, but a common high voltage power supply was used. The discriminator output signals were mixed and analyzed with a 256-channel pulsed neutron logic time analyzer.* The measured resolution time of the overall counting system was about $\tau = 0.33 \mu\text{sec}$.

A Kaman A-800 pulsed neutron source[†] was used for all of these measurements. A pulse-repetition rate of 10 pulses/sec usually was used, with a slower rate introduced occasionally. For the cases described here, the burst intensity was more than adequate, and the source generally was operated at greatly reduced power so that system counting losses would be kept below 10% in all channels. For all measurements the source tube was placed outside the reactor about 6 in. from the center of one vertical side of the reflector, as shown in Fig. 2. This position puts the source tube in the horizontal detector plane and on a line perpendicular to the center of the vertical detector plane.

To utilize the Garelis–Russell method with as little uncertainty as possible, one should accumulate data describing the entire prompt shape following a source burst. Standard pulsed neutron logic analyzers, however, introduce a time delay between the firing of the source and the activation of the first data channel; therefore extrapolation is required in order to estimate the contribution to the prompt shape during the analyzer delay time. The extrapolation uncertainty was eliminated by a digital delay generator[‡] placed in series with the neutron-source trigger so that

*Model CN-110, Technical Measurement Corporation, North Haven, Conn.

[†]Kaman Nuclear, Division of Kaman Aircraft Corporation, Garden of the Gods Road, Colorado Springs, Colorado.

[‡]Model 218A, Hewlett-Packard Corporation, Palo Alto, Calif.

the firing of the source was precisely delayed. The neutron pulse was made to occur between two of the early data channels, usually channels 2 and 3. In this manner the entire prompt shape is recorded in the analyzer.

To obtain the average delayed-neutron count rate with good statistical accuracy and to reduce extrapolation errors, one should acquire data during as much of the time between source bursts as is possible. With the time analyzer used, the single-cycle time ranges from 0.01 to 0.09 sec, depending upon the particular channel width and other adjustable analyzer parameters chosen for the particular measurement. Because the pulse-repetition rate of the neutron source is restricted to a maximum of about 10 pulses/sec, a time of nearly 0.10 sec is available for delayed-neutron data accumulation between source bursts. Following a procedure based upon that of Garelis,¹¹ the analyzer was swept for as many complete cycles as possible in the time between source bursts to take advantage of all possible delayed-neutron counting time. A digital dividing circuit provided variable control over the number of analyzer sweeps per source burst. This factor varied from 1:1 to 8:1 in this series of experiments.

DATA ACQUISITION AND REDUCTION

The actual experimental data-taking procedure was arranged to provide with one measurement raw data that could be analyzed by three methods: the area-ratio method, the extrapolated-area-ratio method, and the Garelis-Russell method. The analyzer was swept with the neutron source turned off to measure the steady-state neutron background. Accuracy in this measurement becomes particularly important at large subcriticalities because the delayed to steady-state background count ratio is low. The background measurement was made immediately prior to an actual pulsing run. Data were then accumulated with the source operating for a time long enough to provide good statistics in both the prompt and the delayed portions. After the source was turned off, analyzer cycles continued until the delayed-neutron contribution became insignificant. An example of the resulting raw data is shown in Fig. 3.

The raw data must be separated into prompt and delayed components in order for the reactivity calculations to be made. This separation may be easily made if it is assumed that the information concerning the delayed-neutron intensity obtained immediately before and just after the prompt burst is sufficient to predict the contribution of the delayed neutrons to the total neutron count in the prompt portion. A single data channel 256 times as long as the regular data channels is activated at the beginning of each analyzer cycle prior to the source burst. The time

Fig. 3 - Raw pulsed neutron data.

width of the regular data channels is adjusted so that the prompt-neutron contribution has decayed completely before channels 100 to 150. This leaves 155 to 105 data channels after the burst to sample delayed neutrons, so that the delayed-neutron portion of the neutron population is sampled for the equivalent of 350 to 400 channels each analyzer cycle.

Consider only the portion of the analyzer cycle where delayed neutrons are present. Beginning at $t = 0$, the analyzer, using the available channels, samples the delayed-neutron intensity at a constant repetition rate. This sampling continues until the delayed contribution again returns to zero, some time after the last source burst. "Return to zero" means that the count rate reaches background level. If the total delayed-

neutron contribution varies slowly with respect to the sampling rate (a reasonable assumption), then knowledge of the total elapsed time for the experiment, the number of analyzer cycles, and the analyzer parameters is sufficient to allow calculation of the delayed-neutron flux integral from $t = 0$ to $t = \infty$.

The delayed contribution may then be used for the separation of the prompt portion of the data from the total. To the prompt data the delayed portion represents a constant background that is simply subtracted from each data point. The delayed-neutron data appearing immediately after the prompt-neutron data in the analyzer are an excellent measure of this background. Before background subtraction the prompt-neutron data points are corrected for counting losses, as required, by the use of the known system resolution time. The linear approximation is used to make this correction, and the intensity is adjusted to restrict count losses to a maximum of 10% in all cases. Actually, in much of the data that follow, count losses were less than 5%.

The process of data reduction and subsequent analysis was performed with the use of a FORTRAN program written for the IBM 1620-II computer. Given the raw data for both the steady-state background and the pulsed-source cases, along with the various parameters such as experiment elapsed time, number of source bursts, number of analyzer sweeps, and analyzer parameters, the computer performed background subtraction, count-loss correction, and separation of the prompt and the delayed portions. This code also obtained the fundamental-mode decay constant α from the prompt distribution. Following basically the procedure of Behrens,¹⁶ we used a weighted least-squares fit to a single exponential. For each experimental case a set of α values was produced. We calculated each successive value in this set by neglecting an increasing portion of the early-time prompt data. This information allowed a test of the overall validity of the results, and this test helped us decide whether or not the value obtained did indeed represent the fundamental prompt mode.

Appended to the data-reduction code are three reactivity-calculation subprograms that calculate the reactivity by using the prompt and the delayed data previously separated. One subprogram calculates reactivity by the extrapolated area-ratio method, giving a result for each value of α obtained in the α calculations. A second subprogram analyzes the data according to the Garelis-Russell method. The prompt shape information, corrected for delayed background and count losses, is retained from the reduction routine. The details of this part of the code were taken from the code PUNS.¹⁷ Finally the data are analyzed according to the area-ratio method.

DATA INTERPRETATION

To interpret the reactivity results obtained in the pulsed-neutron measurements, we must obtain the spatial correction factors previously proposed. Figures 4 to 8 show the results obtained when the experimentally determined fundamental-mode decay constant is used to calculate the shape of the prompt thermal flux. Figure 8 is included to show that the effect exists to a small degree even at delayed critical. In Fig. 9 are shown the ratios of the prompt-to-delayed fluxes obtained from Figs. 4 to 8. The spatial correction factors for the detector locations used (Fig. 2) are obtained from these flux ratios by the application of the fission normalization condition (Eq. 45). These correction factors are given in Table 2. It should be mentioned here that the spatial correction factors are generally insensitive to the exact value of α and to the specific details of the calculational model used to obtain the flux shapes. We assume that the thermal-flux spatial correction factor is the correct one to use here because the detector contains boron (a $1/v$ absorber) as the neutron-sensitive element.

Summarized in Tables 3 to 6 are the position-dependent reactivity results obtained by pulse measurements made in each of the seven detector positions previously described (Fig. 2) and by an analysis of the

Fig. 4 - Core 2 calculated thermal fluxes.

Fig. 5 - Core 3 calculated thermal fluxes.

Fig. 6 - Core 4 calculated thermal fluxes.

Fig. 7 - Core 5 calculated thermal fluxes.

Fig. 8 - Core 1 calculated thermal fluxes.

Fig. 9 - Calculated thermal-flux ratios.

Table 2 - SPATIAL-CORRECTION-FACTOR SUMMARY

Core No.	Subcritical assumed reactivity ρ/β	Detector positions*			
		Center fuel 1 or A	Fuel 2 or B	Reflector peak 3 or C	Outer reflector 4 or D
2	2.15	0.967	0.978	1.20	1.30
3	4.0	0.951	0.965	1.28	1.42
4	6.5	0.921	0.944	1.46	1.74
5	11.0	0.874	0.910	1.70	2.18

* See Fig. 2.

results obtained with the three different methods under investigation. Each table pertains to one of the reactivities listed in Tables 1 and 2 and contains spatially corrected results.

It is interesting to note that, except for detector positions far out into the reflector in the far subcritical cases, results obtained with the extrapolated-area-ratio and Garelis-Russell analysis techniques agree. The area-ratio results, however, not only do not agree with the other two results but also do not give a result independent of detector position

Table 3 - PULSED NEUTRON RESULTS SUMMARY, CORE 2*

Detector position (Fig. 2)	Reactivity in ρ/β						Measured α , sec^{-1}	$\frac{\alpha}{\alpha_c} - 1$ ($\alpha_c = 396$)
	Extrapolated-area-ratio method		Garelis--Russell method		Area-ratio method			
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected		
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected		
1	2.26	2.34	2.27	2.34	2.29	2.37	1300	2.28
A								
2	2.32	2.37	2.32	2.37	2.31	2.36	1304	2.29
B	2.35	2.40	2.35	2.40	2.40	2.45	1296	2.27
3	2.80	2.34	2.79	2.33	2.58	2.15	1298	2.28
C	2.81	2.35	2.75	2.30	2.39	1.99	1320	2.33
4	2.95	2.27	2.89	2.23	2.55	1.97	1288	2.25
D	3.01	2.32	2.93	2.26	2.50	1.93	1299	2.28
Average value		2.34		2.32			1301	2.28

* $\rho/\beta = 2.15$ assumed.

Table 4 - PULSED NEUTRON RESULT SUMMARY, CORE 3*

Detector position (Fig. 2)	Reactivity in ρ/β						Measured α , sec^{-1}	$\frac{\alpha}{\alpha_c} - 1$ ($\alpha_c = 396$)
	Extrapolated-area-ratio method		Garelis—Russell method		Area-ratio method			
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected		
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected		
1	4.07	4.28	4.09	4.30	4.16	4.37	1717	3.34
A								
2	4.16	4.31	4.16	4.31	4.11	4.26	1734	3.38
B	4.14	4.29	4.14	4.29	4.10	4.25	1735	3.39
3	5.62	4.39	5.47	4.28	4.74	3.70	1690	3.28
C	5.35	4.17	5.16	4.03	4.43	3.46	1695	3.29
4	6.16	4.34	5.90	4.16	5.06	3.56	1723	3.35
D	6.05	4.25	5.71	4.03	4.68	3.30	1710	3.32
Average value		4.29		4.20			1715	3.33

* $\rho/\beta = 4.0$ assumed.

Table 5 – PULSED NEUTRON RESULT SUMMARY, CORE 4*

Detector position (Fig. 2)	Reactivity in ρ/β						Measured α , sec ⁻¹	$\frac{\alpha}{\alpha_c} - 1$ ($\alpha_c = 396$)
	Extrapolated-area-ratio method		Garelis—Russell method		Area-ratio method			
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected		
1								
A	6.35	6.89	6.46	7.01	7.43	8.07	2450	5.20
2	6.04	6.40	6.20	6.57	6.93	7.35	2407	5.08
B	6.25	6.62	6.35	6.72	7.12	7.55	2475	5.25
3	9.59	6.54	8.95	6.13	7.32	5.00	2448	5.19
C	9.62	6.58	8.85	6.07	7.11	4.86	2462	5.22
4	11.17	6.42	10.14	5.85	8.03	4.61	2461	5.25
D	11.24	6.45	10.00	5.77	7.54	4.33	2450	5.19
Average value		6.56		6.30			2451	5.20

* $\rho/\beta = 6.5$ assumed.

Table 6 - PULSED NEUTRON RESULT SUMMARY, CORE 5*

Detector position (Fig. 2)	Reactivity in ρ/β							α , sec^{-1}	$\frac{\alpha}{\alpha_c} - 1$	
	Extrapolated-area-ratio method		Garelis—Russell method		Area-ratio method		α_c		$(\alpha_c = 396)$	$(\alpha_c = 333)$
	Uncorrected	Corrected	Uncorrected	Corrected	Uncorrected	Corrected				
1	9.7	11.1	10.4	11.9	11.6	13.2	3224	7.14	8.66	
A	10.1	11.6	10.3	11.8	10.9	12.4	3224	7.15	8.67	
2	10.1	11.1	10.6	11.6	11.6	12.7	3255	7.22	8.77	
B	10.4	11.5	11.0	12.1	12.4	13.7	3245	7.20	8.74	
3	18.9	11.1	17.6	10.4	13.8	8.1	3166	7.00	8.50	
C	18.6	10.9	17.1	10.1	13.2	7.7	3170	7.01	8.51	
4	24.0	11.0	21.6	9.9	16.6	7.6	3191	7.06	8.57	
D	23.4	10.8	20.4	9.4	14.9	6.8	3182	7.04	8.55	
Average value		11.1		10.9			3207	7.10	8.62	

* $\rho/\beta = 11.0$ assumed.

either before or after spatial correction. It thus appears that prompt non-fundamental-mode effects are present. The extrapolated-area-ratio and Garelis–Russell methods, which effectively eliminate these effects, are therefore preferable to the area-ratio method.

In Tables 3 to 6 are also listed the reactivity values obtained with the method of Simmons and King.¹ The value $\alpha_c = 396 \text{ sec}^{-1}$ was obtained experimentally before poison-insertion or moderator-removal core changes were made. As the summary of Table 7 shows, the underestimate of the subcriticality tends to get worse with increasing subcriticality. For the case of core 5 (Table 6), a second set of values is listed, based upon $\alpha_c = 333 \text{ sec}^{-1}$. This value of α_c was obtained experimentally by uniform removal of boron from the core (but not from the reflector) to achieve a new delayed-critical configuration. This second reference critical should be a better reference point because it is a closer approximation to the uniform removal of $1/v$ absorber from core and reflector recommended by Simmons.¹ Indeed, the prompt- α reactivity value obtained in this manner is much closer to the presumed correct result. Of course, the procedures of removing poison uniformly from the core or from both the core and the reflector to obtain the reference critical are in most real situations not possible. The two sets of prompt- α results in Table 6 also illustrate the problem of the nonuniqueness of the reference critical state.

Experimental errors are somewhat difficult to assess properly, partly because they are not independent of reactivity and partly because statistical considerations apparently do not reflect all experimental uncertainty. The values shown in Tables 3 to 6 are usually the result of a single measurement, but in a few cases they are the average of two results. These repeat measurements suggest that experimental reproducibility is about ± 1 to 2% at $\rho/\beta = 2$, increasing to ± 4 to 5% at $\rho/\beta = 10$. In the more subcritical cases, accuracy in the delayed-neutron portion is obtained only by careful steady-state background measurement and by long data-accumulation times.

Figure 10 shows graphically the effect of the correction factor on the extrapolated-area results. Note that in all the cases studied the raw reactivity result for each detector position is corrected to a common result. This common result is assumed to be correct within experimental and calculational errors. The results shown in Fig. 10 are for the extrapolated-area method, but reference to Tables 3 to 6 shows that the Garelis–Russell results also correct fairly well but not quite as well as the extrapolated-area method. Note in the cases of cores 4 and 5 the trend toward greater spread in the corrected Garelis–Russell results. The fact that the difference between prompt and delayed fundamental-mode shapes accounts for the total spatial dependence of the reactivity measurements (after prompt higher harmonics have been eliminated) in-

Fig. 10 - Position-dependent reactivity.

indicates that delayed higher harmonics do not make an important contribution in these experiments (see discussion in the second section).

An overall summary of the extrapolated-area-ratio and the Garelis-Russell results is given in Table 7. The pulsed-neutron reactivity results are simple averages of corrected values taken from Tables 3 to 6.

In connection with measurement accuracy, it is of interest to note that the extrapolated area method is perhaps preferable for better and more rapid data acquisition because it does not depend upon the early time data in any way. If the early-time portion of the prompt shape is not required, the neutron-source strength can be increased considerably with a corresponding increase in delayed neutrons per pulse and hence in a reduction of the steady-state background interference. A second advantage of the extrapolated-area-ratio method is that it eliminates prompt higher harmonics completely whereas the Garelis-Russell method eliminates them approximately (see the second section of this paper).

SUMMARY

These results clearly establish modal-shape difference as a major contributor to the spatial dependence of pulsed-neutron reactivity measure-

Table 7 - PULSED NEUTRON REACTIVITY SUMMARY

Core No.	Assumed reactivity (Table 2)	ρ/β Values		
		Pulsed neutron reactivity *		
		Extrapolated- area-ratio method	Garelis- Russell method	$\frac{\alpha}{\alpha_c} - 1$
2	2.15	2.34	2.32	2.28
3	4.0	4.29	4.20	3.33
4	6.5	6.56	6.30	5.20
5	11.0	11.1	10.9	7.10

* See Tables 3 to 6.

ments on assemblies containing regions in which α/v is comparable to the total cross section. Prompt nonfundamental-mode effects are also important, as can be seen from the discrepancies between the area-ratio and the extrapolated-area-ratio results. Delayed nonfundamental-mode effects do not appear to be important in these experiments.

Our results have a direct practical implication. If it is feasible to place one's detectors at positions where the modal-shape difference correction is slowly varying, then the extrapolated-area-ratio method or the Garelis-Russell method can be used to obtain a meaningful reactivity measurement that does not require an auxiliary criticality measurement.

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Investigation of the Rise Portion of Pulsed Neutron Curves

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ABSTRACT

For many years the "die-away" portion of a pulsed neutron curve, determined by a thermal-neutron detector, has been used to measure thermal-neutron constants for both multiplying and nonmultiplying media. Ramanna et al.¹ reported use of the die-away portion to measure the Fermi age of a medium. Intuitively one would suspect that the "rise" portion of the curve should contain slowing-down information since during this time the fast neutrons from the pulse are slowing down and diffusing. Two-group theory has been applied to the problem and shows that the rise portion of the curve directly yields the Fermi age after correction for higher modes. By the Fermi age slowing-down theory, the rise portion again provides the Fermi age and other slowing-down constants, but a time correction is necessary to account for the slowing and diffusion from pulsed-source neutron energy to fission neutron energy. Corrections are discussed. Pulsed neutron methods thus hold promise of providing a measurement of Fermi age superior to the presently used indium resonance method.

Also, an integrated flux method by foil activation over the core during pulsing is shown to provide a measure of the product of the fast k_{eff} factors, ϵ and p , and the fast-leakage factor.

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INTRODUCTION

In very recent years, emphasis has been placed on the consideration of spatial effects in problems related to reactor kinetics. Highly sensitive nanosecond detectors and electronic-analysis systems have brought about improved precision in measurements of transient behavior of nuclear reactors. With this improved sensitivity, more sophisticated space-time kinetics theories have become necessary to adequately analyze experimental data. Some of the areas where the point reactor model has not been sufficient are

1. Xenon oscillations.
2. Boiling-water-reactor excursions.
3. Perturbation in one region of large reactor cores.
4. Fuel burnup.
5. Coupled-core reactors.
6. Transfer-function measurements.
7. Neutron-wave studies.

Just as in these other areas, pulsed neutron theory for subcritical systems has undergone a process of evolution: Initially, a one-velocity point reactor model was applied, and from this a decreasing exponential behavior was predicted. When the α -decay constant for finite systems was computed, it was found that only in the asymptotic region of the decay curve did the neutron population vary in accordance with a single decaying exponential.

This nonexponential initial behavior was accounted for by the application of a space-dependent and time-dependent theory. Higher spatial harmonics were predicted such that the neutron decay was shown to behave as a sum of exponentials where these exponentials were time-dependent coefficients for an infinite series of spatial eigenfunctions forming a complete orthogonal set. Any continuous function can be described by such an infinite set with the actual function being determined by the coefficients. Thus, with the space-dependent and time-dependent pulsed neutron model in a finite medium, the neutron distribution undergoes a continuous shape shift from an initial *delta function* for a point pulsed neutron source (see Fig. 1, curve 1) at the time of the pulse to a quasi-equilibrium zero harmonic shape (Fig. 1, curve 5) which proceeds to die out exponentially with the fundamental-mode α -decay constant as predicted for the point reactor model.

To obtain more accurate relations for the die-away portion of pulsed neutron curves, investigators have examined the effect of energy

Fig. 1 - Spatial diffusion of neutron pulse.

dependence.^{2,3} In particular the "diffusion cooling" by fast neutron leakage has been examined carefully. Ramanna et al.¹ used the die-away portion of a pulsed neutron curve to measure the Fermi age of nonmultiplying media. This was a departure from usual application of pulsed neutron experiments in that most are for the purpose of measuring thermal-neutron-group properties whereas the Fermi age is a slowing-down constant.

The objective of this paper is to propose the possibility of the use of pulsed neutron curves to determine fast-group and slowing-down constants. In particular the rise portion of pulsed neutron curves is examined.

MOTIVATION FOR INVESTIGATION AND METHOD OF APPROACH

Pulsed neutron generators generally produce fast neutrons; the most common inexpensive pulsed neutron source comes from the deuterium-tritium reaction yielding 14.4-Mev neutrons. In a multiplying medium the fast neutrons of the initial burst slow down to thermal energy, some are absorbed in the fuel, and thus fission is produced. Prompt and delayed neutrons from fission appear of 2.0 and 0.5 Mev average energy, respectively. This regeneration process thus continues.

A block diagram of this process is shown in Fig. 2. Four energy boundaries are selected by virtue of their natural convenience: 14.4 Mev, 2.0 Mev, 0.5 Mev, and thermal energy.

Qualitatively the neutron-population variation as a function of time as measured by a thermal detector at some point $\bar{r}$ in a medium from a prompt-neutron burst at another point $\bar{r}_0$ should be expected to be of the form shown in Fig. 3.

A pulsed neutron curve can be naturally divided into four time zones. In the first two time zones, slowing down and spatial diffusion occur simultaneously, but thermal diffusion continues for some time after thermalization of the burst. In the third zone the quasi-equilibrium, spatial, fundamental-mode distribution is established and decays exponentially. Finally, for a multiplying medium, after the prompt burst has died away, delayed neutrons continue to be present because of the decay of the fission products produced during the regeneration process. These decay with relatively long decay constants. Intuitively, from Figs. 2 and 3, one would suspect that the rise portion of the curve should contain slowing-down and fast-group information. Here we apply the two-group and the Fermi age slowing-down theories with the

Fig. 2 - Neutron regeneration from fast-neutron pulse.

specific objective of studying the rise portion of pulsed neutron curves.

Experimental investigation of this portion of the curve has recently become considerably more feasible. Inexpensive high-output neutron generators with outputs of 10^{10} neutrons/sec, time analyzers with

Fig. 3 - Pulsed neutron curve.

resolving times on the order of nanoseconds and channel times of less than $0.1 \mu\text{sec}$, and detector systems with similar performance are now commercially available.

The method of approach and presentation is to develop two-group and Fermi age theory expressions for pulsed neutron phenomenon in parallel to offer a basis of comparison. The two theories are formulated only in outline form since they are presented elsewhere.

We first look at the slowing down of the initial 14.4-Mev neutron burst into the thermal-neutron energy band. The regeneration process is then considered and the rise portion of pulsed neutron curves examined.

FORMULATION OF THE FERMİ AGE AND THE TWO-GROUP THEORIES AS METHODS OF SOLUTION

Fermi Age Theory

The Fermi age equation can be developed⁴ from generalized transport theory if the following assumptions are specified:

1. The mass of the moderator material is sensibly larger than that of the neutron.
2. Neutron scattering is isotropic in the center-of-mass system.
3. No large sources or sinks exist in the medium in addition to the pulsed neutron source.
4. The medium does not contain large voids.
5. The medium is isotropic and homogeneous.
6. The dimensions of the medium are much greater than the mean free path of neutrons for the energy ranges considered.
7. The macroscopic absorption cross section of the medium is much less than the macroscopic scattering cross section over the neutron-energy ranges investigated.

The transport equation as a function of lethargy, space, and time can be reduced to

$$D(u) \nabla^2 \phi(u, r, t) - \Sigma_a(u) \phi(u, r, t) - \frac{\partial}{\partial u} \phi(u, r, t) + S(u, r, t) = \frac{1}{v(u)} \frac{\partial}{\partial t} \phi(u, r, t) \quad (1)$$

where

$D(u)$ = diffusion coefficient

$\Sigma_a(u)$ = macroscopic absorption cross section

$v(u)$ = neutron velocity

$q(u, r, t)$ = neutron slowing-down density

$\phi(u, r, t)$ = neutron flux

$S(u, r, t)$ = fission and pulsed source production rate

If we assume the slowing-down distribution is essentially the same as for an infinite medium, we can write

$$q(u, r, t) = \xi \Sigma_t(u) \phi(u, r, t) \quad (2)$$

where ξ is the average logarithmic neutron-energy change per collision and $\Sigma_t(u)$ is the total macroscopic cross section.

We now follow conventional procedure by transforming from lethargy to age space and separating the resonance absorption. We let

$$d\tau = \frac{D(u)}{\xi \Sigma_t(u)} du \quad (3)$$

and

$$q(u, r, t) = P(0, u) Q(u, r, t) \quad (4)$$

where

$$P(0, u) = \exp \left[- \int_0^u \frac{\Sigma_a(u^1)}{\xi \Sigma_t(u^1)} du^1 \right] \quad (5)$$

or

$$P(0, \tau) = \exp \left[- \int_0^\tau \frac{\Sigma_a(u^1)}{D(u^1)} d\tau^1 \right] \quad (6)$$

A time-dependent Fermi age equation is thus obtained:

$$\begin{aligned} \nabla^2 Q(\tau, r, t) + \frac{\Sigma_a(\tau)}{D(\tau)} Q(\tau, r, t) - \frac{\partial}{\partial t} Q(\tau, r, t) \\ + \bar{S}(\tau, r, t) = \frac{1}{v(\tau) D(\tau)} \frac{\partial}{\partial t} Q(\tau, r, t) \end{aligned} \quad (7)$$

where

$$\bar{S}(\tau, r, t) = \frac{\xi \Sigma_t(\tau)}{D(\tau)} S(\tau, r, t) \quad (8)$$

In the solution of Eq. 7 for the specific cases to be investigated, we wish to pursue an approach compatible with methods that have been used to solve the space-dependent and time-dependent model. Here then the modified slowing-down density is expanded into an infinite series of spatial eigenfunctions forming a complete orthogonal set.

We let

$$Q(\tau, r, t) = \sum_{n=0}^{\infty} Q_n(\tau, t) R_n(r) \quad (9)$$

The choice of eigenfunctions is arbitrary, and by this expansion we do not place any restrictions on the solution as far as uniqueness is concerned.⁵ We choose solutions of the Helmholtz equation,

$$\nabla^2 R_n(r) + B_n^2 R_n(r) = 0 \quad (10)$$

It should be pointed out that, although the solution of the Helmholtz equation involves application of boundary conditions, Eq. 9 does not demand that the neutrons have a knowledge of the boundaries of the medium at all times.

The source terms given by Eq. 8 contain a term for the pulsed neutron source $\bar{S}_p(\tau, r, t)$. We also expand this quantity by letting

$$\bar{S}_p(\tau, r, t) = \sum_{n=0}^{\infty} S_0 A_n R_n(r) \delta(t) \delta(\tau) \quad (11)$$

$$= S_0 \delta(r - r_0) \delta(t) \delta(\tau) \quad (12)$$

Two-group Theory

Physically the two-group model assumes that the neutron transport can be represented by a single fast-neutron group with a characteristic single velocity and suitable fast-neutron-group constants. The thermal group is represented by a second group in the usual way. We shall not here develop the theory but simply state the equations as follows:

$$D_f \nabla^2 \phi_f(r, t) - \Sigma_1 \phi_f(r, t) + S(r, t) = \frac{1}{v_f} \frac{\partial}{\partial t} \phi_f(r, t) \quad (13)$$

$$D_t \nabla^2 \phi_t(r, t) - \Sigma_a \phi_t(r, t) + p \Sigma_1 \phi_f(r, t) = \frac{1}{v_t} \frac{\partial}{\partial t} \phi_t(r, t) \quad (14)$$

where f and t = fast and thermal group properties, respectively

D = diffusion coefficient

v = neutron velocity

Σ_1 = fast group macroscopic removal cross section

p = resonance escape probability

S(r, t) = fission and pulsed neutron sources

Again, the series-expansion method of solution is applied:

$$\phi_f(r,t) = \sum_{n=0}^{\infty} T_n^f(t) R_n(r) \quad (15)$$

$$\phi_t(r,t) = \sum_{n=0}^{\infty} T_n^t(t) R_n(r) \quad (16)$$

where

$$\nabla^2 R_n(r) + B_n^2 R_n(r) = 0 \quad (17)$$

SLOWING DOWN OF THE INITIAL FAST BURST TO THERMAL ENERGY

Here the initial pulsed neutron burst is considered as it slows down to thermal energy. Fast fission is neglected in that no new neutrons are assumed to be generated until the neutrons arrive in the thermal group. Both the Fermi age and the two-group theories are investigated.

Fermi Age Theory

The range of energies is 14.4 Mev to thermal. We let

$$\tau = 0 \quad \text{for } E = 14.4 \text{ Mev}$$

The source term in Eq. 7 is given by Eq. 12. Applying the relations in Eqs. 9, 10, and 11 and collecting the terms of the n th harmonic, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} Q_n(\tau,t) = & - \left[B_n^2 + \frac{\Sigma_a(\tau)}{D(\tau)} + \frac{1}{v(\tau) D(\tau)} \frac{\partial}{\partial t} \right] Q_n(\tau,t) \\ & + \frac{S_0 A_n}{p(0,\tau)} \delta(t) \delta(\tau) \end{aligned} \quad (18)$$

Taking the Laplace transform of the time variable, we have

$$\begin{aligned} \frac{\partial}{\partial \tau} Q_n(\tau,s) = & - \left[B_n^2 + \frac{\Sigma_a(\tau)}{D(\tau)} + \frac{s}{v(\tau) D(\tau)} \right] Q_n(\tau,s) \\ & + \frac{S_0 A_n \delta(\tau)}{p(0,\tau)} \end{aligned} \quad (19)$$

Integration yields

$$Q_n(\tau, s) = \frac{S_0 A_n}{p(0,0)} \exp \left\{ - \int_0^\tau \left[B_n^2 + \frac{\Sigma_a(\tau^1)}{D(\tau^1)} + \frac{s}{v(\tau^1) D(\tau^1)} \right] d\tau^1 \right\} \times H(\tau) \quad (20)$$

where $H(\tau)$ is the Heaviside function

$$H(\tau) = 0 \quad (\tau < 0)$$

$$H(\tau) = 1 \quad (\tau \geq 0)$$

The inverse transform of Eq. 20 is

$$Q_n(\tau, t) = S_0 A_n \exp \left\{ - \left[B_n^2 \tau + \int_0^\tau \frac{\Sigma_a(\tau^1)}{D(\tau^1)} d\tau^1 \right] \right\} \times H(\tau) \delta(t - T) \quad (21)$$

where T is the slowing-down time to age τ :

$$T = \tau \int_0^\tau \frac{d\tau^1}{v(\tau^1) D(\tau^1)} \quad (22)$$

The slowing-down density to thermal energy ($\tau = \tau_t$) is

$$q(\tau_t, r, t) = \sum_{n=0}^{\infty} S_0 A_n \exp \left[-B_n^2 \tau_t \right] R_n(r) \times \exp \left[- \int_0^{\tau_t} \frac{\Sigma_a(\tau^1)}{D(\tau^1)} d\tau^1 \right] H(\tau) \delta(t - T_t) \quad (23)$$

where T_t is the slowing-down time to thermal age.

Two-group Theory

In the two-group treatment, the pulsed source for Eq. 13 is

$$\begin{aligned} S(r, t) &= S_0 \delta(r - r_0) \delta(t) \\ &= \sum S_0 A_n R_n(r) \delta(t) \end{aligned} \quad (24)$$

By the series expansion defined by Eq. 15, we can integrate with respect to time to obtain the solution

$$T_n^f(t) = v_f S_0 A_n e^{-a_n t} H(t) \quad (25)$$

where

$$a_n = v_f \Sigma_1 (1 + B_n^2 \tau_t) \quad (26)$$

and

$$\tau_t = \frac{D_f}{\Sigma_1} = \text{two-group Fermi age} \quad (27)$$

The thermal-neutron source from the fast group resulting from the pulse is

$$p \Sigma_1 \phi_f(r, t) = p(0, \tau_t) \Sigma_1 v_f \sum S_0 A_n e^{-a_n t} R_n(r) \delta(t) \quad (28)$$

Comparison of Results of Two Theories

If we assume that, in the Fermi age solution, the neutrons slow to a one-velocity thermal group, the thermal source is

$$q(\tau_t, v, t) \quad (29)$$

From Eqs. 2, 8, 24, and 30, it can be shown that the ratio of the modal coefficients of the Fermi age to the two-group solutions are

$$\frac{\exp(-B_n^2 \tau_t) \times \exp \left[- \int_0^{\tau_t} (\Sigma_a/D) d\tau \right] H(\tau) \delta(t - T_t)}{\exp(-a_n t) H(t)} \quad (30)$$

With the presence of the delta function, $\delta(t - T_t)$, the slowing-down source to thermal is zero except for time ($t = T_t$). This implies that the neutrons from the fast pulse are diminished by leakage and absorption during slowing but that they appear at the thermal boundary in a bunch at time ($t = T_t$).

The two-group solution indicates a continuous flow of the fast neutrons into the thermal group. Further discussion of these results is given in later sections.

THERMAL-NEUTRON DISTRIBUTION FROM A FAST BURST AND REGENERATION

We shall here develop general solutions for the thermal-neutron distribution for a multiplying medium as a function of time after the initial fast burst. We again apply the two-group and the Fermi age theories and consider the production of prompt and delayed neutrons from fission, as depicted in Fig. 2.

Two-group Treatment

The treatment here is that of Jenkins.⁶ We assume the average energy of the delayed neutrons is the same as for prompt neutrons. The source term for Eq. 13 is

$$S(r,t) = \frac{k(1-\beta)}{p} \Sigma_a \phi_t(r,t) + \Sigma \lambda_i C_i(r,t) + S_0 \delta(r-r_0) \delta(t) \quad (31)$$

where β = delayed-neutron fraction

p = resonance-escape probability

λ_i = decay constant for i th delayed-neutron group

$C_i(r,t)$ = modified delayed-neutron-emitter concentration for i th group

To account for the delayed-neutron-emitter production, we need an equation for $C_i(r,t)$:

$$\frac{\partial}{\partial t} C_i(r,t) = -\lambda_i C_i(r,t) + \beta_i k \Sigma_a \phi_t(r,t) \quad (32)$$

We expand $C_i(r,t)$ such that

$$C_i(r,t) = \sum_{n=0}^{\infty} T_n^i(t) R_n(r) \quad (33)$$

Substituting the relations given by Eqs. 33, 15, 16, and 17 and taking the Laplace transform of the time variable, we obtain

$$T_n^f(s) = \frac{I_n^f S_0 A_n (I_n^s s + 1)}{I_n^f I_n^s s^2 + k_{en} \left(\Lambda_n s - \rho_n + \sum \frac{\beta_{ie} s}{s + \lambda_i} \right)} \quad (34)$$

$$T_{n(s)}^i = \frac{k_{en}\beta_{ie}S_0A_n}{(s + \lambda_i) \left[I_n^f I_n^s s^2 + k_{en} \left(\Lambda_n s - \rho_n + \sum \frac{\beta_{ie}s}{s + \lambda_i} \right) \right]} \quad (35)$$

$$T_{n(s)}^s = \frac{\frac{I_n^f I_n^s}{\tau_f} \rho S_0 A_n}{I_n^f I_n^s s^2 + k_{en} \left(\Lambda_n s - \rho_n + \sum \frac{\beta_{ie}s}{s + \lambda_i} \right)} \quad (36)$$

where

$$\Lambda_n = \frac{I_n^f + I_n^s}{k_{en}} = \text{neutron generation time}$$

$$\rho_n = \frac{k_{en} - 1}{k_{en}} = \text{reactivity}$$

$$k_{en} = \frac{k}{(1 + B_n^2 L_f^2)(1 + B_n^2 L_s^2)} = \text{finite-medium multiplication factor of the } n\text{th harmonic}$$

$$L_f^2 = \frac{D_f}{\Sigma_1} = \text{two-group age}$$

$$L_s^2 = \frac{D_s}{\Sigma_a} = \text{square of thermal diffusion length}$$

$$\tau_f = \frac{1}{\Sigma_1 v_f} = \text{infinite-medium slowing-down time}$$

$$\tau_s = \frac{1}{\Sigma_a v_s} = \text{infinite-medium thermal mean lifetime}$$

$$I_n^f = \frac{\tau_f}{1 + B_n^2 L_f^2} = \text{finite-medium slowing-down time of } n\text{th harmonic}$$

$$I_n^s = \frac{\tau_s}{1 + B_n^2 L_s^2} = \text{finite-medium thermal mean lifetime of } n\text{th harmonic}$$

To find the inverse transform of Eqs. 34, 35, and 36, we must find the roots of

$$I_n^f I_n^s s^2 + k_{en} \left(\Lambda_n s - \rho_n + \sum \frac{\beta_{ie} s}{s + \lambda_i} \right) = 0 \quad (37)$$

By assuming that

$$\frac{\beta_e - \rho_n}{\Lambda_n} \ll \lambda_i + \frac{k_{en} \Lambda_n}{I_n^f I_n^s} \quad (38)$$

and

$$\frac{\rho_n \lambda_i}{\Lambda_n} \ll \frac{k_{en} \Lambda_n}{I_n^f I_n^s} \left(\lambda_i + \frac{\beta_e - \rho_n}{\Lambda_n} \right) \quad (39)$$

we can rearrange Eq. 37 for a model of a one delayed-neutron group such that

$$\frac{I_n^f I_n^s}{s + \lambda} (s + a_n)(s + b_n)(s + c_n) = 0 \quad (40)$$

where

$$a_n = \frac{k_{en} \Lambda_n}{I_n^f I_n^s} \quad (41)$$

$$b_n = \frac{\beta_e - \rho_n}{\Lambda_n} \quad (42)$$

$$c_n = \frac{\rho_n \lambda}{\beta_e - \rho_n} \quad (43)$$

The inverse Laplace transforms (with the subscript n dropped) for one delayed-neutron group are

$$T^f = \frac{1}{I_s} S \left[\frac{(1 - aI^s)(\lambda - a)}{(b - a)(c - a)} e^{-at} + \frac{(1 - bI^s)(\lambda - b)}{(a - b)(c - b)} e^{-bt} + \frac{(1 - cI^s)(\lambda - c)}{(a - c)(b - c)} e^{-ct} \right] \quad (44)$$

$$T^s = \frac{p}{\tau_f} S \left[\frac{(\lambda - a)}{(b - a)(c - a)} e^{-at} + \frac{(\lambda - b)}{(a - b)(c - b)} e^{-bt} + \frac{(\lambda - c)}{(a - c)(b - c)} e^{-ct} \right] \quad (45)$$

$$T^i = \frac{k_e \beta_{ie}}{f I^s} S \left[\frac{1}{(b - a)(c - a)} e^{-at} + \frac{1}{(a - b)(c - b)} e^{-bt} + \frac{1}{(a - c)(b - c)} e^{-ct} \right] \quad (46)$$

For m delayed-neutron groups, we have

$$T_n^s(t) = \frac{p}{\tau_f} \Lambda_n S_0 A_n \left[A_{0n} e^{-a_n t} + e^{-b_n t} + \sum_{j=2}^{\infty} A_{jn} e^{\omega_{jn} t} \right] \quad (47)$$

where the ω_j are roots of the inhour relation

$$\Lambda_n s - \rho_n + \sum_i^m \frac{s \beta_{ie}}{s + \lambda_i} = 0 \quad (48)$$

Fermi Age Treatment

The general method here is the same as in the discussion of Fermi age theory in the preceding section except that the source term in Eq. 7 does not contain prompt-neutron and delayed-neutron production. Here we deviate from the two-group form in that we consider the difference in average energies at which prompt and delayed neutrons are born (2.0 vs. 0.5 Mev). Then for Eq. 7 we let

$$\begin{aligned} \bar{S}(\tau, r, t) = S_0 \delta(r - r_0) \delta(t) \delta(\tau) + \frac{k(1 - \beta) \Sigma_a}{p(\tau_f, \tau)} \\ \times \phi_t(r, t) \delta(\tau - \tau_f) + \sum \lambda_i C_i(r, t) \delta(\tau_1 - \tau_a) \end{aligned} \quad (49)$$

where

τ_f = age of fission neutrons (14.4 - 2.0 Mev)

τ_d = age of delayed neutrons (14.4 - 0.5 Mev)

and where Eqs. 13 and 32 are used to specify ϕ_t and the C_i terms. The source term for Eq. 13 is given by Eq. 29.

We again expand the functions, as specified by Eqs. 33, 9, 10, and 11, and take the Laplace transform such that

$$\begin{aligned} \frac{\partial}{\partial \tau} Q_n(\tau, r, s) = & - \left[B_n^2 + \frac{\Sigma_a(\tau)}{D(\tau)} + \frac{s}{v(\tau) D(\tau)} \right] Q_n(\tau, r, s) \\ & + \frac{S_0 A_n}{p(0, \tau)} \delta(\tau) + \frac{k(1 - \beta) v_t \Sigma_a}{p(\tau_f, \tau)} T_n(s) \delta(\tau - \tau_f) \\ & + \sum \frac{\lambda_i \beta_{ie} k v_t \Sigma_a}{p(\tau_d, \tau) (s + \lambda_i)} T_n(s) \delta(\tau - \tau_d) \end{aligned} \quad (50)$$

Integrating with respect to τ , we obtain

$$\begin{aligned} Q_n(\tau, r, s) = & S_0 A_n p(0, \tau) \exp(-B_n^2 \tau) \exp[-s T(0)] \\ & + k(1 - \beta) v_t \Sigma_a T_n^s(s) \exp[-B_n^2(\tau - \tau_f)] \exp(-s T_f) \\ & + \sum \frac{\lambda_i \beta_{ie} k v_t \Sigma_a}{s + \lambda_i} T_n^s(s) \exp[-B_n^2(\tau - \tau_d)] \exp(-s T_d) \end{aligned} \quad (51)$$

where

$$p(0, \tau) = \int_0^\tau \frac{\Sigma_a(\tau^1)}{D(\tau^1)} d\tau^1 \quad (52)$$

and

$$T_i = \int_{\tau_i}^\tau \frac{d\tau^1}{v(\tau^1) D(\tau^1)} = \text{slowing-down time to age } \tau \text{ from age } \tau_i \quad (53)$$

If we again take the Laplace transform of Eq. 13, substitute Eq. 51 evaluated at the thermal age τ_t , and solve for $T_n^s(s)$, we obtain

$$T_n^s(s) = \frac{S_{\text{effn}} f_n^s e^{-sT_t}}{s f_n^s + 1 - k_{\text{en}}^p (1 - \beta) e^{-sT_t} - \sum \frac{\lambda_i \beta_i k_{\text{en}}^d}{s + \lambda_i} e^{-sT_d}} \quad (54)$$

where

$S_{\text{effn}} = S_0 A_n p(0, \tau_t) \exp(-B_n^2 \tau_t) = \text{effective thermal-source strength}$

$$k_{\text{en}}^p = \frac{k}{1 + B_n^2 L_s^2} \exp[-B_n^2(\tau_t - \tau_f)]$$

$$k_{\text{en}}^d = \frac{k}{1 + B_n^2 L_s^2} \exp[-B_n^2(\tau_t - \tau_d)]$$

If we analyze Eq. 54, we see intuitively that the results are plausible. If $f(t)$ is the inverse Laplace transform of $F(s)$, then $f(t + T)$ is the inverse transform of $F(s)e^{-sT}$. The thermal-neutron population responds to the 14.4-Mev burst at time T_t after the burst, where T_t is the slowing-down time from 14.4 Mev to thermal. In the denominator the thermal-neutron contribution from fission neutrons experiences a time lag T_t , the time required for the fission neutrons to slow down and a similar lag for the delayed neutrons.

The usual procedure for finding the inverse transform is to solve for the poles of the transform, i.e., the roots of the denominator. The denominator of Eq. 54 is transcendental in the function s . Various approximations of the equation

$$s f_n^s + 1 - k_{\text{en}}^p (1 - \beta) e^{-sT_t} - \sum \frac{\lambda_i \beta_i}{s + \lambda_i} k_{\text{en}}^d e^{-sT_d} = 0 \quad (55)$$

can, however, be made. It can be observed from Eq. 54 that the Fermi age theory does not predict a rise portion to a pulsed neutron curve and that maximum thermal-neutron population occurs when the delta-function prompt burst arrives at thermal energy.

If we consider the case for $T \ll l_n^s$, i.e., slowing-down time much less than thermal lifetime, and if

$$e^{-sT} \simeq \frac{1}{1 + sT} \quad (56)$$

we obtain a form similar to the two-group results. In addition, by assuming the same energy for prompt and delayed neutrons, we can manipulate Eq. 55 into a form similar to Eq. 37.

APPLICATION OF TWO-GROUP THEORY TO POTENTIAL EXPERIMENTS

The intent of this paper is not to present definite results regarding the investigation of the rise portion of pulsed neutron curves, but simply to suggest the possibility of obtaining slowing-down data from this portion of the curve.

The theories developed here are by no means adequate to apply to the analysis of the rise portion. If, however, the two-group results here were valid, we could show that the same information regarding fast-group constants may be obtained from the rise portion as has for many years been obtained for the thermal group constants from the die-away portion.

In Eq. 45 we show that the thermal-neutron population for each harmonic is described by the sum of three descending exponentials. By examination of Eqs. 41, 42, and 43 and also the coefficients, it can be shown that

$$T_n^s = -|A_n| e^{-a_n t} + |B_n| e^{-b_n t} + |C_n| e^{-c_n t} \quad (57)$$

where the coefficient of the first term is negative and represents the slowing of the fast neutrons into the thermal group. The second term is the prompt neutron die-away, and the third is the slowly decaying delayed-neutron tail as predicted in Fig. 3.

Now

$$|A_n| \simeq |B_n| \gg |C_n|$$

and thus during the rise portion

$$T_n^s(t) \simeq |B_n| (e^{-b_n t} - e^{-a_n t}) \quad (58)$$

If the source detector system is positioned such that the higher harmonics are minimized,

$$\phi_t(t) \simeq v_t R_0(r) |B_0| (e^{-b_0 t} - e^{-a_0 t}) \quad (59)$$

Since

$$a_0 \gg b_0$$

in physical reactor systems, b_0 can, as usual, be obtained from the asymptotic part of the curve. Subtracting the total flux from the prompt die-away yields

$$v_t R_0 | B_0 | e^{-a_0 t} = v_t R_0 | B_0 | e^{-b_0 t} - \phi_t$$

and a_0 can thus be obtained. We have

$$a_0 = v_f \Sigma_1 (1 + B_0^2 L_f^2) \quad (60)$$

By varying B_0^2 , we can extrapolate to $B_0^2 = 0$ (infinite medium) and obtain $v_f \Sigma_1$. The slope of the a_0 -vs. $-B_0^2$ curve yields $v_f D_f$. The ratio of the slope to the value of a_0 at $B_0^2 = 0$ provides the two-group form of the Fermi age to thermal energy:

$$\left. \frac{1}{a_0} \frac{da_0}{dB_0^2} \right|_{B_0^2 = 0} = \frac{D_f}{\Sigma_1} = \tau_t \quad (61)$$

If we examine also the coefficients, we can show that

$$a_0 \simeq \frac{\epsilon p S_0}{\tau_f a_0} = \frac{\epsilon p S_0}{\tau_f} l_n^f \simeq \frac{\epsilon p S_0}{1 + B_0^2 L_f^2}$$

Thus

$$\frac{1}{a_0} = \frac{1 + B_0^2 L_f^2}{\epsilon p S_0} \quad (62)$$

which is linear in B_0^2 . If we extrapolate again for $B_0^2 = 0$, we obtain $\epsilon p S_0$. If, also, we calculate the slope and form the ratio of slope to value at $B_0^2 = 0$, we have

$$a_0 \left. \frac{d}{dB_0^2} \frac{1}{a_0} \right|_{B_0^2 = 0} = L_f^2 = \tau_t = \text{Fermi age to thermal energy}$$

Since S_0 can be calculated, we can thus measure the following fast-group constants

$$\frac{\epsilon p}{1 + B_0^2 L_f^2} \quad (\epsilon p, L_f^2 = \tau_t, D_f, \Sigma_1) \quad (63)$$

As a final experiment we consider the positioning of activation foils in a grid arrangement similar to that discussed by Uhrig.⁷ If the system is pulsed with a pulsed neutron source, the activity of the foil at the i th position is

$$A_{ci} = K \int_0^\infty \phi_t(r_i, t) dt \quad (64)$$

where K is a constant related to the macroscopic cross section of the foil material, the number of pulses, etc. If the activity given by Eq. 64 for each grid position is weighed by the value of the spatial zero-order harmonic eigenfunction at that point and then summed, we have⁷

$$\begin{aligned} \sum_i A_{ci}(r_i) R_0(r_i) \Delta V_i &\simeq \int_0^\infty \int_v K \sum T_n^s(t) R_n(r_i) R_0(r_i) dV \\ &= K \int_0^\infty T_0^s(t) dt \end{aligned} \quad (65)$$

as a result of the orthogonality of the R_n .

From Eqs. 57 and 45,

$$\int_0^\infty T_0^s(t) dt = -\frac{|A_0|}{a_0} + \frac{|B_0|}{b_0} + \frac{|C_0|}{c_0} = \frac{\epsilon_p S_i \lambda}{\tau_f a_0 b_0 c_0} \quad (66)$$

Referring again to Eqs. 41, 42, 43, and 45, we can show that

$$\int_0^\infty T_0^s(t) dt \simeq \frac{f \eta S_0 \tau_s}{\rho_0} \simeq \frac{\sum_f \eta S_0 v_t}{\rho_0} \quad (67)$$

SUMMARY AND CONCLUSIONS

Purohit³ has provided perhaps the most rigorous development of energy dependence of pulsed neutron experiments. The prime purpose, however, was to obtain a higher ordered approximation for the relation between the thermalization time constant and the diffusion cooling coefficient (which is generally obtained from die-away portions of pulsed neutron curves) by the Laguerre polynomial expansion into energy eigenfunctions. Barnard et al.² have measured time-dependent neutron spectra in graphite and have obtained experimental data on the average neutron energy as a function of time after a pulse in the time range of 300 to 1000 μ sec. At 400 μ sec, however, the spectra had achieved a nearly Maxwellian energy distribution with a nearly thermal average energy of approximately 0.04 ev.

Here, by Fermi age and two-group theories, we have found that the neutrons arrive at thermal energies: in the former case, in a delta-function bunch and, in the latter, by approximately a continuous exponential buildup of the form

$$A_0(1 - e^{-\alpha_0 t})$$

From a knowledge of slowing-down theory, it is obvious that a delta-function pulse of fast neutrons will not retain the delta-function distribution because with each collision a single neutron has a possibility of a range of energies depending on the angle of scatter. It can be shown that, for neutrons of initial energy E_0 moderated in a medium of atomic number A , after n collisions there is a finite probability of a neutron containing an energy

$$E = \alpha^n E_0$$

where n is the number of collisions and

$$\alpha = \left(\frac{A - 1}{A + 1} \right)^2$$

For a thermal-neutron detector, a pulsed neutron curve then contains a rise portion similar to that shown in Fig. 3. We have shown from two-group theory that information regarding the slowing-down properties of the medium should be contained within this rise portion. Measurements of the rise portion would appear to be more fundamental than calculations of the slowing-down properties from the die-away portion by diffusion cooling data.

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Measurement of Local Kinetic Parameters in the Saxton Reactor

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ABSTRACT

Experimental measurements of local transfer functions were performed in the Saxton experimental power reactor, with the use of a movable in-core neutron detector, with a view to determine local kinetic parameters such as prompt power coefficient of reactivity and fuel time constant. The measurements consisted in perturbing the core reactivity through a reactivity modulator in a pseudorandom manner and measuring the response signal from an in-core detector at various locations. Input and response signals were recorded on magnetic tape and processed in a cross-power spectrum analyzer to determine the transfer functions. The experimental data were processed in a digital computer and fitted to a mathematical model in a least-squares sense to yield local kinetic parameters. Applications and the usefulness of such techniques in obtaining information on local kinetic behavior of a power reactor are discussed.

INTRODUCTION

Saxton Reactor

The Saxton reactor is a small, closed-cycle, pressurized light-water reactor developed by the Westinghouse Electric Corp. and the Saxton Nuclear Experimental Corp. (SNEC), a nonprofit subsidiary of the General Public Utilities Corporation. The reactor is being utilized for a joint five-year experimental program by Westinghouse and SNEC

with the primary objective of improving pressurized water reactor technology.

The reactor has a nominal thermal output of 20 Mw. Heat generated in the reactor is removed by a vertical U-tube steam generator that operates in the primary loop. Under normal operating conditions a canned-motor pump circulates loop coolant at a rate of 7600 gal/min. The coolant enters the reactor at 520°F and leaves at 540°F.

The Saxton initial core is composed of 21 fuel assemblies of substantially square geometry. Each fuel assembly contains approximately 72 individual fuel rods filled with slightly enriched uranium oxide. The assembled core has an effective diameter of 28 in. and an effective height of 36 in. Provision has been made for five removable subassemblies that can be inserted and withdrawn through special ports in the reactor-vessel head without removing the head.

Frequency-response Measurements

The usefulness of frequency-response or transfer-function measurements of nuclear reactors for the purpose of determining kinetic parameters and stability of power reactors has become increasingly evident as a result of the developments in the experimental and analysis techniques. Applications of the frequency-response techniques for determining gross reactor parameters and for studying the effects of chemical shim (boric acid in the coolant water) control on dynamic behavior of the Saxton reactor have been reported.¹

When the power of a power reactor is modulated, the resulting temperature changes in the fuel and moderator vary in a manner that is dependent upon their location in the core. The reactivity effect of a given localized physical change in a reactor-core parameter will be proportional to the square of the flux (one-group theory) or to the product of the flux times the adjoint flux (two-group theory). In order to obtain a detailed understanding of the core kinetic behavior, it is necessary to determine the parameters in local regions. A series of tests was conducted at Saxton for the primary purpose of demonstrating the feasibility of local frequency-response measurements with in-core detectors and for the determination of local kinetic parameters.

General Theory

The transient behavior in any region of a reactor core is basically dependent upon the feedback effects due to temperature and power. There is also a slow response dependence on xenon, but such effect is extremely slow and need not be considered in comparison with the temperature- and power-dependent feedback effects. The general theory and equations covering the effects of feedback in nuclear reactors have been

thoroughly covered in the literature. However, a simple block-diagram illustration of these effects is shown in Figs. 1 and 2.

Figure 1 is the basic diagram and illustrates the manner in which the zero-power transfer function, $G(s)$, of a reactor is transformed by the power-dependent feedback term, $H(s)$. The zero-power transfer function is dependent only upon the delayed-neutron fractions, β_i and λ_i , and upon the ratio of prompt-neutron lifetime, l , of the reactor to the effective fraction of delayed neutrons, β . However, $H(s)$ is dependent upon phenomena in local regions as well as upon those in the reactor system.

Fig. 1 - Block diagram of basic reactor transfer function.

Fig. 2 - $P(s)$, the reactor transfer function at power.

Figure 2 is a breakdown of some of these effects. The fuel effects occurring within a migration length (about 8 cm) around the detector and moderator effects, which are governed by the system characteristics, weighted for the detector location affect the feedback reactivity in the local region.

If the at-power transfer function is denoted as $P(s)$, the feedback transfer function, $H(s)$, is then

$$H(s) = \frac{1}{P(s)} - \frac{1}{G(s)} \quad (1)$$

When the reactivity input is sinusoidal, the Laplace operator, s , is set equal to $j\omega$ in the transfer function. Under this operation the transfer function becomes the frequency response. In Eq. 1, $H(j\omega)$ is the frequency response of the local feedback process.

From Fig. 2,

$$H(j\omega) = \alpha_f \left(\frac{\delta \bar{T}_f}{\delta n} \right) (j\omega) + \alpha_m \left(\frac{\delta \bar{T}_m}{\delta n} \right) (j\omega) \quad (2)$$

The α 's are referred to as the isothermal coefficients of reactivity. The physical process that affects temperatures of the fuel and moderator in a local region is complex but may be approximated by a mathematical model characterized by two terms: (1) a prompt term associated with Doppler reactivity effects and pellet to coolant conductance properties and approximated by a gain constant and time constant and (2) a relatively delayed term associated with thermal effects in the coolant and structure and approximated by a gain constant and a time constant. The approximate feedback model is written as

$$H(j\omega) = \frac{\alpha_1}{1 + \tau_f(j\omega)} + \frac{\alpha_2}{1 + \tau_m(j\omega)} \quad (3)$$

The experimental frequency-response data can be fitted to Eq. 3 to yield the four unknowns. Separation of terms is possible since the experimental data show that the break frequencies of the moderator and fuel are separated by well over a decade.

In experimental measurements Eq. 1 is normalized to

$$\frac{n_0}{\beta} H(j\omega) = \frac{1}{(\beta/n_0) G(j\omega)} - \frac{1}{(\beta/n_0) P(j\omega)} \quad (4)$$

The normalized zero-power transfer function is known from theoretical calculations and experimental measurement. One way to obtain normalized feedback frequency response is by the use of Eq. 1. Since the normalized feedback term approaches zero at sufficiently high frequencies, the amplitudes of the at-power and the zero-power frequency responses are the same at high frequencies.

EXPERIMENTAL EQUIPMENT

Reactor Instrumentation

The primary side of the Saxton reactor is provided with extensive instrumentation with respect to that normally provided for an operating power reactor. In particular, the core is extensively instrumented to permit operational evaluation of the core performance. Flux measurements are obtainable by means of flux wires inserted in the core, irradiated for a predetermined period, and then withdrawn from the core and automatically scanned. Miniature detectors, such as ion or fission chambers (0.25-in. outside diameter and 1.0-in. sensitive length), can be inserted in special thimbles. The axial position of the detector is read on a digital position transmitter. The locations of the flux-wire thimbles and the in-core detector thimble (x1) are shown in a cross-sectional view of the core in Fig. 3.

Thermocouples in the inlet and outlet plenums measure water-temperature distribution at the reactor inlet and outlet. Flow distribution at the reactor-core inlet and outlet is measured by way of pitot tubes positioned at the inlet and outlet nozzles of selected fuel assemblies. Additional plant instrumentation includes compensated ion chambers, resistance bulb thermometers, and flow meters in the main coolant loop.

Reactivity-modulator System

The system for modulating reactivity consisted of oscillator tubes located in a subassembly at B3 (Fig. 3) and a drive system with associated electronics. The oscillator tubes consist of a stationary tube, $\frac{1}{2}$ in. in outside diameter, with ten 0.020-in.-thick by 1.6-in.-long hafnium sleeves spaced on 3.2-in. centers and, inside the stationary tube, an oscillating tube with similar hafnium sleeves. Reactivity changes occur when the hafnium absorber sections in the oscillating tube move in and out of the shadow of stationary hafnium sleeves. A low-pressure helium atmosphere was provided for these tubes to reduce the absorber temperature to about 1200°F from a possible 2000°F and to prevent oxidation of the oscillator parts.

The drive for the oscillator consisted of a modified commercial electrodynamic system (Unholtz-Dickie Corporation model 28) with a position transducer (Schaevitz Engineering type 500 XS-ZRT). This

Fig. 3 - Saxton reactor-core cross section.

apparatus has operated satisfactorily at ambient temperatures of 170°F, humidity of 100%, and high radiation levels.

The peak to peak amplitude of reactivity obtained was 0.5×10^{-4} under chemical shim operation and 1×10^{-4} under rodged shim operation. The input signal consisted of a maximum-length binary sequence read by a paper-tape reader. A block diagram of the setup for measurements is shown in Fig. 4.

Instrumentation and Analysis System

The response signal from the in-core detector for reactivity modulations was small compared to the steady level. Operational amplifiers, potentiometers, and reference supply in an analog computer (Electronic

Fig. 4 – Block diagram of the instrumentation and analysis system.

Associates, Inc., model PC-12) were used to condition the input and response signals to approximately 1-volt level.

The conditioned signals were recorded on a multichannel (seven) multispeed (six from $1\frac{7}{8}$ to 60 in./sec) tape recorder (Precision Instrument Co. model PS 207). Simultaneous visual recording was also made on a Visicorder.

The power and cross-power spectral analyses were performed in a commercial spectrum analyzer (Weston-Boonshaft and Fuchs model 711 C). The analyzer computes input, output, and cross-power spectral densities, the amplitude ratio in decibels and phase. The results are computed frequency by frequency. An input band-pass filter, if used, tracks the analysis frequency and attenuates the frequency components (10 times, twice, or once) that are above or below the analysis frequency.

EXPERIMENTAL RESULTS

Reactor Conditions for Measurements

At-power measurements for the local transfer functions were conducted twice with the in-core detector located at three axial positions in thimble x1. The center of the detector was located at distances of 1, 10, and 20 in. from the top of the core in thimble x1. Control-rod number 5, adjacent to the flux thimble, was positioned 11.5 in. from the top of the core. All other control rods were in the out position. All tests were conducted at equilibrium xenon, and the reactor power and coolant temperature were kept at a quasi-equilibrium state whereby, for the duration of the time required for data acquisition for each run, the reactor power level and average coolant temperature would not change significantly with control rods kept in a fixed position.

Oscillation Tests

In these oscillation tests a binary pseudorandom signal was used to drive the reactivity modulator. The signal consisted of five cycles of a 128-bit series with minimum event intervals of approximately 6 sec. The duration of the test was about 1 hr. Prior experiments and analyses established that this input signal was most satisfactory for covering the range of frequencies of interest (0.003 to 2 cycles/sec) in a short time.

The reactivity input and detector response signals were recorded on magnetic tape at a speed of $1\frac{7}{8}$ in./sec. During recording, the response signal from the detector and the reactor power and coolant temperature signals were monitored in a multichannel Visicorder and a dual-beam oscilloscope. Whenever appreciable drift in power or temperature of the reactor occurred, the recordings of such runs were discarded, and the experiment was rerun.

Analysis of the recorded data was performed in the cross-power spectrum analyzer at a playback speed of 60 in./sec. With this speedup of data (a factor of 32), analysis for a frequency could be carried out in about 2 min.

Local Power to Reactivity Transfer Functions

The local power to reactivity transfer functions measured with the detector located at 1, 10, and 20 in. from the top of the core, along with the zero-power transfer function, are shown in Fig. 5. The amplitude curve was normalized to a 0-db gain at high frequencies (0.5 to 2 cycles/sec) to match into the zero-power curve. No change was made in the phase curves.

Based on comparison with sinusoidal measurements and rough error analysis for duration of data and analysis methods, errors of amplitude

Fig. 5 - (a) Local transfer functions (gain). (b) Local transfer functions (phase).

measurements by the binary input method were found to be within 0.5 db and phase measurements within 3° . In the two runs made with the detector at the same location, the results for 0- and 20-in. positions agreed within the quoted limits, whereas those for the 10-in. position were slightly above the limits. At the 10-in. position, the detector was close to the tip of a control rod, and any discrepancy in the detector position would alter the response characteristics significantly. The expected discrepancy in the detector position is within 0.5 in.

Several qualitative observations can be made from Fig. 5. The combined fuel-temperature and moderator-temperature feedback effect is

negative at different regions of the core, and the negative feedback has a stabilizing effect on the reactor. If any unstable tendencies exist in any region of the core, the phase will approach -180° . Local feedback effects are widely different at different regions.

Analysis for Local Kinetic Parameters

For a complete understanding of the mechanism by which different regions of the reactor core respond to the stimulus of a change in reactivity, not only the local frequency response but also the individual components as set forth in Eq. 2 must be known.

Derivation of local transfer functions for the fuel and coolant, by use of appropriate equations for nuclear, thermal, and hydraulic phenomena, has been attempted. The resultant equations were quite involved. However, for simple analysis of the experimental data, the equations were reduced to a simple time-constant system with appropriate gain, as represented in Eq. 3.

Determination of the normalized local feedback frequency response

$\left[\frac{n_0}{\beta} H(j\omega) \right]$ was accomplished from $\left[\frac{n_0}{\beta} P(j\omega) \right]$ and calculated

$\left[\frac{n_0}{\beta} G(j\omega) \right]$ as indicated in Eq. 4. This computation was performed by a digital code in an IBM 7094 computer.

The local frequency response according to the simplified model is given by

$$H(j\omega) = \frac{(\alpha_1 + \alpha_2) \left[1 + \frac{\alpha_1 \tau_m + \alpha_2 \tau_f}{(\alpha_1 + \alpha_2)} j\omega \right]}{(1 + j\omega \tau_f)(1 + j\omega \tau_m)} \quad (5)$$

The data for $H(j\omega)$ obtained from measurements were fitted into the mathematical model (Eq. 5) by a digital-computer program in a least-squares sense to yield time constants and gain constants for both the fuel and the moderator.

The values for $H(j\omega)$ obtained from experimental data and the "best" computer fit for the data are shown in Fig. 6. As can be seen from the figure, the fit for the 10- and 20-in. positions is quite good over the complete range, but, for the 1-in. position, the agreement is poor at low frequencies. The poor agreement may be attributed to the effects of coolant mixing at the outlet plenum.

Fig. 6 - (a) Feedback frequency response (gain). (b) Feedback frequency response (phase).

The power distribution in the two fuel assemblies adjacent to the detector thimble (assemblies D3 and D4) was obtained from in-core flux-wire measurements.² The average heat fluxes for two adjacent assemblies at the points of measurement were obtained from these data.

Values of Local Kinetic Parameters

The local-kinetic-parameter values and average heat fluxes corresponding to the various locations of measurement are given in Table 1.

**Table 1 – Values of Local Kinetic Parameters and Average Heat Fluxes
at Various Locations in the Core**

Parameter	Distance from top of core		
	20 in.	10 in.	1 in.
Local heat flux, Btu/(hr) (sq ft)	274,000	175,000	73,000
Doppler gain constant	1.15	0.95	0.69
Local fuel time constant, sec	3.23	3.34	3.80
Local prompt power coefficient $\frac{\partial P}{\partial Q} \left(\frac{10^{-3}}{6850 \text{ Btu/(hr) (sq ft)}} \right)$	0.19	0.24	0.42
Moderator gain constant	3.80	1.53	

These data clearly show that the feedback effect attributable to fuel temperature and Doppler effects decreases toward the top of the core. The feedback effect attributable to moderator temperature decreases more rapidly toward the top. The assumed mathematical model could not be fitted well in the low-frequency region for 1 in. from top of core data to obtain the value of moderator gain constant. Indications are that it is small and may even be negative.

Comparison with Calculations

Local prompt power coefficient values can be compared with calculated values at different heat fluxes. The calculations are based on a correlation of the effective fuel-pellet temperature for the Doppler broadening of ^{238}U resonance at various heat fluxes.³ The correlation presently available is valid up to a heat flux of only 200,000 Btu/(hr) (sq ft). Measured and calculated values are given in Table 2. The calculated and the measured values are in agreement.

**Table 2 – Measured and Calculated Prompt Power Coefficients
at Different Heat Fluxes**

Prompt power coefficient $\left(\frac{10^{-3}}{6850 \text{ Btu/(hr) (sq ft)}} \right)$	Average heat flux		
	274,000	175,000	73,000
Measured	0.19	0.24	0.42
Calculated		0.23	0.41

Space Dependence of the Reactor-noise Spectral-density Function

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ABSTRACT

The authors' recent space-dependent theory for the reactor-noise auto-spectral- and cross-spectral-density function is applied to a series of bare cubical reactors. In its present form the method is based upon one-energy-group diffusion theory.

The spectral-density function was calculated for both critical and subcritical bare cubes of three sizes. The results are compared to the space-independent-theory prediction of the spectral density.

It was found that, for detector location away from the axes of symmetry and/or for subcritical systems, the comparison between the two theories is poor. For the system sizes considered, the space-independent representation is adequate for the critical case with detector located at the cube center, but it becomes less so for large systems.

Although inclusion of the energy and transport effects will reduce the magnitude of the space-dependent effects, it is felt that for certain situations (i.e., larger subcritical systems with off-axes detector location) space-dependent effects should not be ignored.

INTRODUCTION

At this time several theories have been developed for analysis of space-dependent reactor fluctuations. The systematic application of

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CONCLUSIONS

1. Oscillation techniques in conjunction with cross-power spectral methods of analysis and in-core detectors afford an excellent method of determining local transfer functions.

2. Specific local kinetic parameters for a simple model are separable, and the agreement between the simple model and experimental data is quite good.

3. Simplification of experimental and analytical techniques are possible to determine specific parameters such as local power coefficient.

In general, the results prove that significant information and insight on the spatial kinetic behavior of a power reactor can be obtained by employing oscillation techniques.

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these theories to specific problems, however, has been limited. In this paper the space-dependent autospectral-density function, including delayed-neutron effects, has been calculated for a series of three different bare homogeneous uranium-beryllium reactors. These calculations are an extension of those previously published by Sheff.¹ The calculations are done in one-speed diffusion theory, which, it is expected, will accentuate spatial effects more than a corresponding multi-energy calculation would.

The solution for the autospectral-density function in a simple homogeneous cubical reactor reveals space dependencies that enter as high-frequency "breaks." These effects are measurable in subcritical systems, and their magnitudes and frequencies are determined by the higher space modes present. Previous papers^{1,2} show how these effects determine the infinite-medium result, and recent experiments^{3,4} tend to corroborate the theory given here. Note that, whereas prior theories and subsequent comparison to experiments involving "multibreak frequencies" have been confined to effects produced by large lumped regions, i.e., reflector, coupled cores, etc., the effects reported herein occur in even the simple homogeneous, bare, cubical system.

THEORY

The autospectral-density function for the counting rate of a small detector of volume ΔV in a homogeneous, bare, cubical, one-speed reactor can be expressed as follows:¹

$$\begin{aligned} \psi_{z_1 z_1}(\bar{r}_1, \omega) = c \left(\frac{2}{\pi S} \right)^3 \Delta V \left\{ \sum_{\bar{g}} G_{\bar{g}} R_{\bar{g}}^2(\bar{r}_1) \frac{(\delta k_{\bar{1}} + k_{\bar{1}}\beta)/l_{\bar{1}}}{(\delta k_{\bar{g}} + k_{\bar{g}}\beta)/l_{\bar{g}}} \right. \\ + 2 \sum_{\bar{g}\bar{h}} G_{\bar{g}\bar{h}} R_{\bar{g}}(\bar{r}_1) R_{\bar{h}}(\bar{r}_1) \frac{(\delta k_{\bar{1}} + k_{\bar{1}}\beta)/l_{\bar{1}}}{(\delta k_{\bar{g}\bar{h}} + k_{\bar{g}\bar{h}}\beta)/l_{\bar{g}\bar{h}}} \\ + \epsilon \left(\frac{k_{\bar{1}}}{l_{\bar{1}}} \right)^2 \left| 1 - i\omega \sum_{j=1}^N \frac{\beta_j}{i\omega + \lambda_j} \right|^2 \left(\frac{\bar{\nu}^2 - \bar{\nu}}{\bar{\nu}^2} \right) \left[\sum_{\bar{g}} \frac{G_{\bar{g}} R_{\bar{g}}^2(\bar{r}_1)}{\alpha_{\bar{g}}^2 + (\gamma_{\bar{g}}\omega)^2} \right. \\ \left. + \sum_{\bar{g}\bar{h}} \frac{G_{\bar{g}\bar{h}} R_{\bar{g}}(\bar{r}_1) R_{\bar{h}}(\bar{r}_1)}{\alpha_{\bar{g}\bar{h}}^2} \left(\frac{\alpha_{\bar{g}}}{\alpha_{\bar{g}}^2 + (\gamma_{\bar{g}}\omega)^2} + \frac{\alpha_{\bar{h}}}{\alpha_{\bar{h}}^2 + (\gamma_{\bar{h}}\omega)^2} \right) \right] \left. \right\} \quad (1) \end{aligned}$$

The detailed definitions of the quantities in Eq. 1 are given in the appendix. Briefly, the G 's are constants, $R_{\bar{g}}(\bar{r}_1)$ is a three-dimensional space eigenfunction, $\alpha_{\bar{g}}$ is approximately $(\delta k_{\bar{g}} + k_{\bar{g}}\beta)/l_{\bar{g}}$ for large ω , $\gamma_{\bar{g}}$ is approximately 1 for large ω , $\bar{g}$ and $\bar{h}$ are triple-summation indexes ($\bar{h} < \bar{g}$), and $l_{\bar{g}}$ is the mean neutron lifetime for the $\bar{g}$ space mode. The terms $\alpha_{\bar{g}}(\omega)$ and $\gamma_{\bar{g}}(\omega)$, aside from their dependence upon higher order bucklings, are the usual quantities found in the spectral density, including delayed-neutron effects, and are the real and imaginary parts of the inhour equation where the roots have been taken as $i\omega$.

The first two lines of the spectral-density equation (Eq. 1) are independent of frequency and correspond to the "white" term in the space-independent spectral-density function. The fundamental mode, i.e., the term for $\bar{g}(g, m, p) = 1, 1, 1$ is

$$\psi_{z_1 z_1}^{(\bar{1})}(\bar{r}_1, \omega) = c \Delta V \left(\frac{2}{\pi S} \right)^3 G_{\bar{1}} R_{\bar{1}}^2(\bar{r}_1) \times \left[1 + \epsilon \left(\frac{k_{\bar{1}}}{l_{\bar{1}}} \right)^2 \left| 1 - i\omega \sum_{j=1}^N \frac{\beta_j}{i\omega + \lambda_j} \right|^2 \left(\frac{\bar{\nu}^2 - \bar{\nu}}{\bar{\nu}^2} \right) \frac{1}{\alpha_{\bar{1}}^2 + (\gamma_{\bar{1}}\omega)^2} \right]$$

This bracketed quantity is identically the same as the space-independent result with the possible exception of the definition of ϵ . Here

$$\epsilon = \frac{\sum_d}{\sum_f} = \frac{v \sum_d n}{v \sum_f n}$$

and is the ratio of detector counts per unit volume to fissions per unit volume. The definition most frequently used in the space-independent theory is the ratio of total counts in the detector to total fissions in the reactor.

One most interesting fact shown by this result is that, when the space-independent form, or the leading term of the series, dominates, it is not so much due to rapid convergence as to a high degree of cancellation among the higher order terms. As a result, in a bare cube the individual breaks for higher order terms will not be observed as such, the deviation from the fundamental term being a complex sum of many terms.

BARE URANIUM-BERYLLIUM CORE DATA

For a calculational investigation of Eq. 1, the bare homogeneous uranium-beryllium critical-assembly data of Krasin⁵ was picked. The critical-assembly data were selected for the calculation so that (1) there would be a consistent set of systems covering a reasonable range of sizes, (2) estimation of the reflector savings would be unnecessary, and (3) problems of inhomogeneity would be eliminated.

The quantities required by Eq. 1 are B^2 , L , l_∞ , and the delayed-neutron-group parameters. The experimental values of B^2 obtained by Krasin were used to obtain the cube edge, s , which is the input required by the computer. The L and l_∞ values were calculated with the use of the core compositions given by Krasin. The six-group delayed-neutron data of Keepin⁶ for ^{235}U were used, and the delayed-photon neutron effect characteristic of beryllium was neglected.

Figure 1 shows the variation of L^2 , B^2 , and L^2B^2 for the range of reactor-core sizes for which data were available. Note that, although

Fig. 1 - Variation of the diffusion length squared, L^2 , the buckling squared, B^2 , and the product L^2B^2 with core size.

B^2 decreases as the core size increases, the product L^2B^2 , which is the dimensionless buckling squared in terms of the diffusion length, increases. The quantity L^2B^2 determines the extent to which space-dependent effects will be found in the spectral-density function for a point detector. Thus, for the case considered here, the space-dependent effects are most pronounced in the geometrically smaller systems rather than in the larger ones. Actually, the effect of finite detector size on the measurement tends to remove some of the space-dependent effects. In this case the smaller reactor would probably benefit from the relatively larger detector.

The principal cause of the increasing L^2B^2 with increasing reactor size is the decreasing absorption cross section produced by the smaller fuel concentrations necessary in the larger systems. Clearly, as the infinite-medium fuel concentration is approached, the diffusion length L will become approximately constant, and L^2B^2 will become proportional to B^2 .

The three systems considered had the following infinite-medium neutron lifetimes:

<u>s, cm</u>	<u>$l_{\infty} = 1/v \sum_a, \mu\text{sec}$</u>
88.3	752
67.9	190
59.4	24.5

RESULTS

Numerical calculations of the autospectral density (Eq. 1) were made both including and excluding the white, or frequency-independent, part, i.e., the first two lines of Eq. 1. The white terms were excluded so that the space-dependent effects could be more easily seen.

Figure 2 shows the autospectral density, excluding the white term, for a detector located at the geometric center of the cube; i.e., $x = y = z = 0$ where the cube faces are at $\pm s/2$. All three reactors are critical, and the asymptotic space-independent behavior is shown by the dashed lines about $\omega = \beta/l$, where l is the finite-medium fundamental-mode prompt-neutron lifetime. Ten terms in each dimension of the six-dimensional series were used for these calculations. For large ω , i.e., $\omega > 10^3$, more terms are required. The dashed lines in the range $\omega = 10^3$ to 10^4 represent 20 terms. In most of the figures that follow, only the 10-term result will be given because the machine time required to calculate 20 terms is prohibitive. Also, it has been found that the error is usually about the same in all cases tested.

From Figure 2 one sees that for the largest L^2B^2 , 0.3031, or the smallest reactor in terms of diffusion lengths (but largest in physical size), the space-dependent effects are almost absent. Those effects

Fig. 2 - Counting rate autospectral density at the center of three critical, homogeneous, bare, cubical reactors.

which do occur do so at such high frequencies and low amplitudes as to be immeasurable. The high-frequency space-independent constant over ω^2 form is valid for frequencies up to about 300 radians/sec and for amplitudes 2.5 decades (50 db) below the plateau. As one proceeds to the smaller values of L^2B^2 , however, the space-dependent effects become more pronounced. For the smallest L^2B^2 , 0.0197, the spectral-density function is poorly approximated by the space-independent form for all frequencies greater than about the break frequency, 267 radians/sec.

Figure 3 shows the autospectral density of the same critical reactors for a detector located at one-half the distance from the cube center to a corner; i.e., $x = y = z = s/4$. Here one notes that for the largest L^2B^2 , 0.3031, the space-independent form is still somewhat dominant, the approximation being valid up to about 80 radians/sec and 1.5 decades (30 db) below the plateau. For the small L^2B^2 case, the space-independent behavior is completely lacking for ω on the order of β/l and greater. Indeed, the asymptotic behavior above ω equals β/l is more like the infinite-medium point-detector result,^{1,2} which asymptotically behaves as a constant over $\omega^{1/2}$.

Fig. 3 - Counting rate autospectral density at $x = y = z = s/4$ in three critical, bare, cubical reactors.

An additional feature not readily apparent is the similarity of Figs. 2 and 3 at high frequencies, here greater than 10^4 . This effect occurs for all detector locations. Thus at high frequencies the spectral densities for all detector locations asymptotically approach each other. For frequencies less than those in the asymptotic region, the effect of moving nearer the boundaries is a "whitening" of the spectrum and a decrease of amplitude. In the limit of the reactor face, the spectrum becomes flat, or white, at amplitude zero.

Figure 4 shows the effect of \$3 and \$6 subcriticality upon the spectral density of a detector located at the center of the cube, $x = y = z = 0$. Again, where present, the dashed lines at high frequency indicate the more accurate 20-term results. The small L^2B^2 core is seen to be poorly approximated by the space-independent form.

Figure 5 shows the subcritical effects for the $x = y = z = s/4$ detector location. Here all three cores are poorly approximated by the space-independent form.

The effect of including the white, or frequency-independent, term in the autospectral density is shown in Fig. 6. The detector efficiency, ϵ , was arbitrarily picked to give the behavior seen for $x = y = z = 0$. The frequency-dependent part is seen to vanish as compared with the white terms as the outer edge of the cube is approached. This reflects not only the relative magnitudes of the two parts of the spectrum but also the whitening of the frequency-dependent terms.

Fig. 4 - Counting rate autospectral density at the center of \$3 and \$6 subcritical reactors.

Fig. 5 - Counting rate autospectral density at $x = y = z = s/4$ in \$3 and \$6 subcritical reactors.

Fig. 6 - Effect of including the white term in the autospectral-density function.

CONCLUSIONS

Within the limitations of the model used, i.e., the one-group diffusion theory, the space-dependent autospectral-density function in a bare cube shows departure from the space-independent form. For detector locations away from the axes of symmetry and/or in a subcritical core, the deviations appear to be sufficiently large to enable measurements.

The restrictions imposed by the one-group diffusion theory are such that it may be considered as the limiting case insofar as space-dependent effects are concerned. The inclusion of energy and/or angle dependence would tend to reduce the space-dependent effects.

APPENDIX

The quantities in Eq. 1 are defined as follows:

$$R_{\vec{g}}(\vec{r}) = \sin \frac{g\pi}{s} \left(x + \frac{s}{2} \right) \sin \frac{m\pi}{s} \left(y + \frac{s}{2} \right) \sin \frac{p\pi}{s} \left(z + \frac{s}{2} \right)$$

$$\alpha_{\bar{g}}(\omega) = \frac{\delta k_{\bar{g}}}{l_{\bar{g}}} + \frac{k_{\bar{g}} \omega^2}{l_{\bar{g}}} \sum_{j=1}^N \frac{\beta_j}{\omega^2 + \lambda_j^2}$$

$$\gamma_{\bar{g}}(\omega) = 1 + \frac{k_{\bar{g}}}{l_{\bar{g}}} \sum_{j=1}^N \frac{\lambda_j \beta_j}{\omega^2 + \lambda_j^2}$$

$$\delta k_{\bar{g}} = 1 - k_{\bar{g}} = 1 - \frac{k}{1 + L^2 B_{\bar{g}}^2}$$

$$l_{\bar{g}} = \frac{l_{\infty}}{1 + L^2 B_{\bar{g}}^2} = \frac{1/v \Sigma_a}{1 + L^2 B_{\bar{g}}^2}$$

$$B_{\bar{g}}^2 = \left(\frac{g\pi}{s} \right)^2 + \left(\frac{m\pi}{s} \right)^2 + \left(\frac{p\pi}{s} \right)^2$$

$$B_{\bar{g}\bar{h}}^2 = \frac{1}{2} (B_{\bar{g}}^2 + B_{\bar{h}}^2)$$

where $\delta k_{\bar{g}\bar{h}}$ and $l_{\bar{g}\bar{h}}$ are defined as $\delta k_{\bar{g}}$ and $l_{\bar{g}}$ but include $B_{\bar{g}\bar{h}}^2$ rather than $B_{\bar{g}}^2$.

$$G_{\bar{g}\bar{h}} = \prod_{\bar{g}} \frac{1}{\bar{g}} [(-1)^{\bar{g}+\bar{h}+1} - 1] \left[\frac{1}{1 - \left(\frac{\bar{h}+1}{\bar{g}} \right)^2} - \frac{1}{1 - \left(\frac{\bar{h}-1}{\bar{g}} \right)^2} \right]$$

where the product is taken over the pairs gh , mn , and pq .

$$G_{\bar{g}} = G_{\bar{g}\bar{g}}$$

The detector volume is ΔV . The subscripts $\bar{g}$ and $\bar{h}$ stand for the triple subscripts g, m, p and h, n, q . The subscripts $\bar{g}(g, m, p)$ each take on the integers 1 to infinity. The triple subscripts $\bar{h}(h, n, q)$ also take on integer values but are restricted to values less than $\bar{g}$; i.e., $h < g$, $n < m$, and $q < p$.

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Kinetic Theory of Spatial and Spectral Coupling of the Reactor Neutron Field

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ABSTRACT

From the space, angle, energy, and time-dependent integro-differential form of the Boltzmann neutron-transport equation, a set of coupled, ordinary first-order differential equations is derived which describe the kinetics of the spatial and spectral coupling of the reactor neutron field. The derivation is very general and is not restricted to any particular reactor configuration or type. The coefficients in the derived expression are defined in terms of diffusion theory as well as transport theory. Existing computer codes for performing the necessary computations are discussed. The results of calculations made on the University of Florida Training Reactor (UFTR), an Argonaut type reactor, are presented.

INTRODUCTION

Considerable interest in the kinetics of the spatial and spectral coupling of the neutron field within a reactor has arisen in several areas of reactor design. Chief among these are developments in the areas of coupled fast-thermal assemblies,¹ Argonaut type reactors,² coupled nuclear rocket engines,³ and modular cores for large power reactors.⁴ The usual approach has been the formulation of the problem on the basis of a

[†]This work was done by Mr. Cockrell in partial fulfillment of the requirements for the Ph.D. degree at the University of Florida.

coupled lumped-parameter model with *ad hoc* definitions of coupling coefficients and region lifetimes;^{1,5} such a formulation, however, was essentially unable to provide the structure of these coupling coefficients and lifetimes.

This paper presents a set of kinetics equations that have been rigorously derived from the space, angle, energy, and time-dependent integro-differential Boltzmann neutron-transport equation.⁶ These equations describe the coupling of the reactor neutron field in terms of the adjoint weighted-neutron population with respect to the place of origin of the neutrons within the reactor. The equations are generally applicable and are not restricted by any assumptions in the mathematical formulation to any particular reactor configuration with the one minor exception that the reactor must have fixed fuel.

GENERAL FORMULATION OF THE COUPLED KINETICS EQUATIONS

The space, angle, energy, and time-dependent neutron-transport equation describing the reactor neutron field can be written in the form

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \psi(r, E, \Omega, t) = & \left[\chi(E) - \sum_{i=1}^D \beta_i f_i(E) \right] P\psi(r, E', \Omega', t) \\ & + \sum_{i=1}^D \lambda_i f_i(E) c_i(r, t) + s(r, E, \Omega, t) + T\psi(r, E', \Omega', t) \\ & - \Omega \cdot \nabla \psi(r, E, \Omega, t) - \Sigma_t(r, E) \psi(r, E, \Omega, t) \end{aligned} \quad (1)$$

The kinetics of the precursor concentration is given by

$$\begin{aligned} \frac{\partial}{\partial t} f_i(E) c_i(r, t) = & \beta_i f_i(E) P\psi(r, E', \Omega', t) \\ & - \lambda_i f_i(E) c_i(r, t) \quad (i = 1, 2, \dots, D) \end{aligned} \quad (2)$$

where Eq. 2 has been multiplied through by $f_i(E)$ for convenience and the symbols are defined as follows:

$\psi(r, E, \Omega, t)$ = angular flux⁶ (the number of neutrons per square centimeter per second per unit solid angle per unit energy in the volume element dr at r traveling in the solid angle $d\Omega$ about Ω and having an energy in the range dE containing E) at time t

$$P\psi(r, E', \Omega', t) = \int_{\Omega'} d\Omega' \int_{E'} dE' \nu(E') \Sigma_f(r, E') \psi(r, E', \Omega', t)$$

$$T\psi(r, E', \Omega', t) = \int_{\Omega'} d\Omega' \int_{E'} dE' \Sigma_S(r, E' \rightarrow E, \Omega' \rightarrow \Omega) \psi(r, E', \Omega', t)$$

$\chi(E)$ = fission spectrum for all neutrons resulting from a fission

$f_i(E)$ = delayed-neutron spectrum for neutrons emitted by the i th group of precursors

$c_i(r, t)$ = concentration of the i th precursor group at point r and time t

$s(r, E, \Omega, t)$ = number of neutrons (per unit time per unit volume per unit energy per unit solid angle in volume element dr about r having an energy in the range dE containing E and traveling in the solid angle $d\Omega$ about Ω) at time t from an external source

The remaining symbols have the usual definitions and therefore will not be defined here.

Implicit in the formulation of Eqs. 1 and 2 is the assumption that the precursors do not diffuse; in effect, this limits the treatment to fixed-fuel reactors. It would appear that Eqs. 1 and 2 are also limited to reactors with only one fissile isotope; reactors with many fissile isotopes are included, however, if

$$\chi(E) P\psi(r, E', \Omega', t) = \sum_j \chi^j(E) \int_{\Omega'} d\Omega' \int_{E'} dE' \nu^j(E') \Sigma_f^j(r, E') \psi(r, E', \Omega', t)$$

and

$$\beta_i f_i(E) P\psi(r, E', \Omega', t) = \sum_j \beta_i^j f_i \int_{\Omega'} d\Omega' \int_{E'} dE' \nu^j(E') \Sigma_f^j(r, E') \psi(r, E', \Omega', t)$$

where j refers to some j th fissile isotope and the i th precursor group is assumed to be the same for all species.

We wish to write Eqs. 1 and 2 in terms of the quantities $\psi_{j/a}(r, E, \Omega, t)$ where

$$\psi_{j/a}(r, E, \Omega, t) \equiv \text{the angular flux due to neutrons that originated in region } R_j \text{ and energy group } E_a \quad (3)$$

where R_j is a region of the reactor configuration and E_a is a range of energies within the spectrum of energies that describe the neutron field.

If the phase space of the neutron field is divided into M configuration regions ($R_1, R_2, \dots, R_j, \dots, R_M$) and N energy groups ($E_1, E_2, \dots, E_a, \dots, E_N$), then, on the basis of Eqs. 1 and 3, the neutron field resulting from neutrons originating in region R_j and group E_a is described by

$$\begin{aligned}
\frac{1}{v} \frac{\partial}{\partial t} \psi_j(r, E, \Omega, t) = & U(r; R_j) U(E; E_\alpha) \left\{ \left[\chi(E) - \sum_{i=1}^D \beta_i f_i(E) \right] \right. \\
& \times P\psi(r, E', \Omega', t) + \sum_{i=1}^D \lambda_i f_i(E) c_i(r, t) + s(r, E, \Omega, t) \left. \right\} \\
& + T\psi_j(r, E', \Omega', t) - \Omega \cdot \nabla \psi_j(r, E, \Omega, t) \\
& - \Sigma_t(r, E) \psi_j(r, E, \Omega, t) \quad \begin{matrix} (j = 1, 2, \dots, M) \\ (\alpha = 1, 2, \dots, N) \end{matrix} \quad (4)
\end{aligned}$$

where $U(r; R_j) \equiv u(r - R_j^{\text{inner}}) - u(r - R_j^{\text{outer}})$

$U(E; E_\alpha) = u(E - E_\alpha^{\text{lower}}) - u(E - E_\alpha^{\text{upper}})$

$u(x - y) = 0 \quad \text{if } x < y$

$u(x - y) = 1 \quad \text{if } x \geq y$

$R_j^{\text{inner}}, R_j^{\text{outer}} = \text{boundaries of region } R_j$

$E_\alpha^{\text{upper}}, E_\alpha^{\text{lower}} = \text{boundaries of group } E_\alpha$

The adjoint weighted population of the j - α neutrons (the neutrons that originated in region R_j and group E_α) in region R_k and group E_β is found by multiplying each term of Eq. 4 by the adjoint angular flux $\psi^*(r, E, \Omega)$ and integrating over region R_k , group E_β , and all directions. The resulting expression is

$$\begin{aligned}
\frac{d}{dt} (\psi^* v^{-1} \psi_j)_{\alpha \beta} = & \delta_{jk} \delta_{\alpha \beta} \left[(\psi^* \chi, P\psi)_{\beta} - \sum_{i=1}^D (\psi^* \beta_i f_i, P\psi)_{\beta} \right. \\
& + \sum_{i=1}^D \lambda_i (\psi^* f_i c_i)_{\beta} + (\psi^* s)_{\beta} \left. \right] + (\psi^*, T\psi_j)_{\alpha \beta} \\
& - (\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha \beta} - (\psi^* \Sigma_t \psi_j)_{\alpha \beta} \quad \begin{matrix} (j \text{ and } k = 1, 2, \dots, M) \\ (\alpha \text{ and } \beta = 1, 2, \dots, N) \end{matrix} \quad (5)
\end{aligned}$$

where the parenthetic notation

$$(\psi^* v^{-1} \psi_j)_{\alpha \beta} = \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \psi^*(r, E, \Omega) v(E)^{-1} \psi(r, E, \Omega, t)$$

has been used and where the symbols δ_{jk} and $\delta_{\alpha\beta}$ are Kronecker deltas.⁷ In a similar manner an expression is found for the kinetics of the i th precursor group:

$$\frac{d}{dt} (\psi^* f_i c_i)_{\beta} = \beta_i (\psi^* \chi, P \psi)_{\beta} - \lambda_i (\psi^* f_i c_i)_{\beta} \quad \begin{matrix} (k = 1, 2, \dots, M) \\ (\beta = 1, 2, \dots, N) \\ (i = 1, 2, \dots, D) \end{matrix} \quad (6)$$

An equation⁸ for the adjoint angular flux ψ^* can be obtained from the sourceless, steady-state form of Eq. 1 by interchanging the primed and unprimed independent variables in the χ^P and T operators and replacing $-\Omega \cdot \nabla$ by $\Omega \cdot \nabla$. The result is

$$\begin{aligned} -\Omega \cdot \nabla \psi^*(r, E, \Omega) + \Sigma_t(r, E) \psi^*(r, E, \Omega) \\ = \nu(E) \Sigma_f(r, E) \int_{E'} dE' \chi(E') \psi^*(r, E', \Omega) \\ + \int_{\Omega} d\Omega \int_{E'} dE' \Sigma_s(r, E \rightarrow E', \Omega \rightarrow \Omega') \psi^*(r, E', \Omega') \end{aligned} \quad (7)$$

It is required that ψ^* satisfy the adjoint boundary conditions.

By a direct application of the following definitions of the multiplication constant and the neutron lifetime, Eqs. 5 and 6 can be put in a form that is similar to the equation for the lumped-parameter, or point, model:⁹

k = effective multiplication constant

$$\equiv \frac{\text{importance weighted-neutron production rate}}{\text{importance weighted-neutron loss rate}}$$

l = prompt-neutron lifetime

$$\equiv \frac{\text{importance weighted-neutron population}}{\text{importance weighted-neutron loss rate}}$$

The resulting set of coupled-reactor-kinetics equations is expressed by

$$\begin{aligned}
\frac{d}{dt} N_{jk}^{\alpha\beta}(t) = & \delta_{jk} \delta_{\alpha\beta} \left[\sum_{m=1}^M \sum_{\gamma=1}^N (1 - \bar{\beta}_{mk}^{\gamma\beta}) \frac{k_{mk}^{\gamma\beta}}{l_{mk}^{\gamma\beta}} N_{mk}^{\gamma\beta}(t) \right. \\
& + \sum_{i=1}^D \lambda_i C_{ik}^{\alpha\beta}(t) + \bar{\psi}_{S_k}^* S_k^{\alpha\beta}(t) \left. \right] - \frac{N_{jk}^{\alpha\beta}(t)}{\theta_{jk}^{\alpha\beta}} \\
& + \sum_{\gamma=\alpha}^{\beta} \frac{N_{jk}^{\alpha\gamma}(t)}{\tau_{jk}^{\alpha\gamma}} + \sum_{\substack{m=1 \\ m \neq k}}^M \frac{N_{jm}^{\alpha\beta}(t)}{\sigma_{jm \rightarrow k}^{\alpha\beta}} \quad (j \text{ and } k = 1, 2, \dots, M) \\
& (\alpha \text{ and } \beta = 1, 2, \dots, N) \quad (8)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} C_{ik}^{\alpha\beta}(t) = & \sum_{j=1}^M \sum_{\alpha=1}^N \bar{\beta}_{ijk}^{\alpha\beta} \frac{k_{jk}^{\alpha\beta}}{l_{jk}^{\alpha\beta}} N_{jk}^{\alpha\beta}(t) \\
& - \lambda_i C_{ik}^{\alpha\beta}(t) \quad \begin{array}{l} (i = 1, 2, \dots, D) \\ (k = 1, 2, \dots, M) \\ (\beta = 1, 2, \dots, N) \end{array} \quad (9)
\end{aligned}$$

where

$N_{jk}^{\alpha\beta}(t) \equiv (\psi^* v^{-1} \psi_j)_k$ = the number of neutrons in region R_k and group E_β at time t which originated in region R_j and group E_α

$C_{ik}^{\alpha\beta}(t) \equiv (\psi^* f_i C_i)_k$ = the number of precursors of the i th precursor group which emit neutrons into region R_k and group E_β

$$k_{jk}^{\alpha\beta} \equiv \frac{(\psi^* \chi, P \psi_j)_{\alpha\beta}}{(\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha\beta} + (\psi^* \Sigma_t \psi_j)_{\alpha\beta} - (\psi^*, T \psi_j)_{\alpha\beta}}$$

= the effective multiplication constant for j - α neutrons in region R_k and group E_β

$$l_{jk}^{\alpha\beta} \equiv \frac{(\psi^* v^{-1} \psi_j)_{\alpha\beta}}{(\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha\beta} + (\psi^* \Sigma_t \psi_j)_{\alpha\beta} - (\psi^*, T \psi_j)_{\alpha\beta}}$$

= the average lifetime of a j - α neutron in region R_k and group E_β

$$\bar{\beta}_{ijk}^{\alpha\beta} \equiv \frac{(\psi^* \beta_i f_i, P \psi_j)_{\alpha\beta}}{(\psi^* \chi, P \psi_j)_{\alpha\beta}}$$

= the effective delayed-neutron fraction averaged over the j - α neutron spectrum in region R_k and group E_β

$$\bar{\beta}_{jk}^{\alpha\beta} \equiv \sum_{i=1}^D \bar{\beta}_{ijk}^{\alpha\beta}$$

$$S_k(t) \equiv \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr s(r, E, \Omega, t)$$

= the rate at which external-source neutrons are introduced into region R_k with energies in group E_β at time t

$$\psi_{S_k}^* \equiv \frac{\int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \psi^*(r, E, \Omega) s(r, E, \Omega, t)}{\int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr s(r, E, \Omega, t)}$$

$\psi_{S_k}^*$ = the average importance of the external-source neutrons in region R_k and group E_β

The three remaining coefficients $\theta_{jk/\alpha\beta}$, $\tau_{jk/\alpha\gamma \rightarrow \beta}$, and $\sigma_{jm \rightarrow k/\alpha\beta}$ do not have counterparts in the point-model kinetics equation, and their appearance in the coupled kinetics equations requires an explanation. The rate of removal of the j - α neutrons from region R_k and group E_β is expressed by the last three terms in Eq. 5, where

$$(\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha\beta} = \text{leakage rate}$$

and

$$(\psi^* \Sigma_t \psi_j)_{\alpha\beta} - (\psi^*, T \psi_j)_{\alpha\beta} = \text{net rate of removal by collision}$$

It is through the leakage term that the spatial boundary conditions enter the coupled kinetics equations. The leakage rate is given by

$$(\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha \beta} = \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \psi^*(r, E, \Omega) \times \left[\Omega \cdot \nabla \psi_j(r, E, \Omega, t) \right]_{\alpha} \quad (10)$$

Since $\psi^*(r, E, \Omega)$ is a scalar, Eq. 10 can be written as

$$(\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha \beta} = \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \times \{ \Omega \cdot [\psi^*(r, E, \Omega) \nabla \psi_j(r, E, \Omega, t)] \}_{\alpha} \quad (11)$$

where it is understood that ∇ operates only to the right. Applying the divergence theorem¹⁰ to the integral over R_k yields

$$\begin{aligned} (\psi^*, \Omega \cdot \nabla \psi_j)_{\alpha \beta} &= \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \\ &\times \Omega \cdot \int_{S_k} \mathbf{n} \psi^*(r, E, \Omega) \psi_j(r, E, \Omega, t) ds \\ &- \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \psi_j(r, E, \Omega, t) \\ &\times [\Omega \cdot \nabla \psi^*(r, E, \Omega)] \end{aligned} \quad (12)$$

where S_k is the surface of region R_k and $\mathbf{n}$ is the positive (outward-drawn) unit normal to S_k . Since region R_k has a surface in common with all regions surrounding it, we can write the first term on the right-hand side of Eq. 12 as

$$\begin{aligned} &\int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \Omega \cdot \int_{S_k} \mathbf{n} \psi^* \psi_j ds \\ &= \sum_{\substack{m=1 \\ m \neq k}}^M \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \Omega \cdot \int_{S_{k \rightarrow m}} \mathbf{n} \psi^* \psi_j ds \end{aligned} \quad (13)$$

where $S_{k \rightarrow m}$ is the surface of the interface between regions R_k and R_m . The arrow drawn from k to m indicates that $\mathbf{n}$ is taken as positive in the direction from region R_k to region R_m . Requiring that the flux vanish at

the outer boundary causes all the terms that contain an outer-boundary interface to vanish, and they are therefore not included in the summation over m . If the outward normal from region R_k to region R_m is reversed, it becomes the outward normal from region R_m to R_k . With this point of view in mind, we can express Eq. 13 as

$$\begin{aligned} \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \Omega \cdot \int_{S_k} \mathbf{n} \psi^* \psi_j \frac{ds}{\alpha} \\ = - \sum_{\substack{m=1 \\ m \neq k}}^M \int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \Omega \\ \cdot \int_{S_{m \rightarrow k}} \mathbf{n} \psi^* \psi_j \frac{ds}{\alpha} \end{aligned} \quad (14)$$

We can now define $\sigma_{jm \rightarrow k/\alpha\beta}$:

$$\begin{aligned} \sigma_{jm \rightarrow k/\alpha\beta} \equiv \frac{(\psi^* v^{-1} \psi_j)_{\alpha\beta}^m}{\int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \Omega \cdot \int_{S_{m \rightarrow k}} \mathbf{n} \psi^* \psi_j \frac{ds}{\alpha}} \\ = \text{the inverse of the average rate at which } j\text{--}\alpha \text{ neutrons in group } \\ E_\beta \text{ traverse from region } R_m \text{ into region } R_k \end{aligned}$$

The second term on the right-hand side of Eq. 12 is combined with the terms giving the net removal rate to define $\theta_{jk/\alpha\beta}$ and $\tau_{jk/\alpha\gamma \rightarrow \beta}$:

$$\begin{aligned} \theta_{jk/\alpha\beta} \equiv \frac{(\psi^* v^{-1} \psi_j)_{\alpha\beta}^k}{(\psi^* \Sigma_t \psi_j)_{\alpha\beta}^k - (\psi_j, \Omega \cdot \nabla \psi^*)_{\alpha\beta}^k} \\ = \text{the inverse of the average rate at which } j\text{--}\alpha \text{ neutrons are removed} \\ \text{from group } E_\beta \text{ in region } R_k \end{aligned}$$

$$\tau_{jk/\alpha\gamma \rightarrow \beta} \equiv \frac{(\psi^* v^{-1} \psi_j)_{\alpha\gamma}^k}{\int_{E_\beta} dE \int_{\text{all } \Omega} d\Omega \int_{R_k} dr \psi^*(r, E, \Omega) \int_{\text{all } \Omega'} d\Omega' \int_{E_\gamma} dE' \Sigma_S(r, E' \rightarrow E, \Omega' \cdot \Omega) \psi_j(r, E', \Omega', t)}$$

The system-averaged parameters N , k , $\bar{\beta}_i$, and l are found as shown in Eqs. 15 to 18. The total weighted-neutron population in the reactor at time t is given by

$$N(t) = \sum_{j=1}^M \sum_{k=1}^M \sum_{\alpha=1}^N \sum_{\beta=1}^N N_{jk}(t)_{\alpha\beta} \quad (15)$$

The average prompt-neutron lifetime for the reactor at time t is given by

$$l(t)^{-1} = \frac{\sum_{j=1}^M \sum_{k=1}^M \sum_{\alpha=1}^N \sum_{\beta=1}^N l_{jk}^{-1} N_{jk}(t)}{N(t)} \quad (16)$$

The multiplication constant is given by

$$k(t) = \frac{\sum_{j=1}^M \sum_{k=1}^M \sum_{\alpha=1}^N \sum_{\beta=1}^N k_{jk} l_{jk}^{-1} N_{jk}(t)}{N(t)/l(t)} \quad (17)$$

The effective delayed-neutron fraction for the i th precursor group is given by

$$\bar{\beta}_i(t) = \frac{\sum_{j=1}^M \sum_{k=1}^M \sum_{\alpha=1}^N \sum_{\beta=1}^N \bar{\beta}_{ijk} k_{jk} l_{jk}^{-1} N_{jk}(t)}{k(t)N(t)/l(t)} \quad (18)$$

Basically, three criteria guided the formulation of Eqs. 8 and 9 from Eqs. 5 and 6:

1. The dependent variables should be the weighted-neutron population and the precursor concentrations in each region.
2. The coefficients of the dependent variables should be as weakly time dependent as this formulation would permit.
3. Insofar as possible, the coefficients should resemble the familiar point-model kinetics equation.

The coefficients in the kinetics equation have been defined in terms of a time-dependent flux; if, however, the regions and groups are chosen such that the space dependence and energy dependence of the j - α neutrons within some region R_k and group E_β are constant with respect to time, then it is reasonable to assume separability of the time dependence. If the time dependence of the flux is separable, then the time-varying factor in the numerator cancels that in the denominator, and the coefficients are constant with respect to time.

It was necessary to introduce a new parameter, $\sigma_{jm \rightarrow k/\alpha\beta}$, which does not have an analogue in the usual point-model kinetics equation. The inverse of $\sigma_{jm \rightarrow k/\alpha\beta}$ gives the average rate at which j - α neutrons in group E_β are transferred from region R_m to region R_k . It is this parameter that gives the coupling among the regions.

Another parameter appearing in Eq. 8 is $\tau_{jk/\alpha\gamma \rightarrow \beta}$. It gives the average rate of transfer from group E_γ to E_β of j - α neutrons in region R_k .

Finally, if Eqs. 8 and 9 are summed over all regions R_j and R_k and all groups E_α and E_β , the resulting expression is the set of point-model kinetics equations:¹¹

$$\frac{d}{dt} N(t) = (1 - \bar{\beta}) \frac{k}{l} N(t) - \frac{N(t)}{l} + \sum_{i=1}^D \lambda_i C_i(t) \quad (19)$$

$$\frac{d}{dt} C_i(t) = \bar{\beta}_i \frac{k}{l} N(t) - \lambda_i C_i(t) \quad (20)$$

where the external-source term has been omitted. A simpler way to arrive at Eqs. 19 and 20 is to write them down directly from Eqs. 8 and 9 by setting $M = N = 1$, in which case the summation over $\sigma_{jm \rightarrow k/\alpha\beta}$ does not appear.

In summary, we have derived a set of spatially and spectrally coupled kinetics equations directly from the Boltzmann equation without relying on physical intuition or *ad hoc* definitions. As a result this set of equations should prove highly useful in the study of space- and energy-dependent kinetics for a variety of problems and reactor types. Since in practice one is usually forced to use diffusion theory instead of transport theory, the next section points out the changes that should be made in the definitions of the kinetics coefficients to compute them on the basis of diffusion theory.

DIFFUSION THEORY FORM OF THE KINETICS COEFFICIENTS

Except for $\theta_{jk/\alpha\beta}$ and $\sigma_{jm \rightarrow k/\alpha\beta}$, the mathematical definitions of the kinetics parameters are correct if we replace the leakage operator $\Omega \cdot \nabla$ by $-\nabla \cdot D \nabla$, where $D(r, E)$ is the diffusion coefficient. It is this change in leakage operator that brings about the changes in $\theta_{jk/\alpha\beta}$ and $\sigma_{jm \rightarrow k/\alpha\beta}$. The diffusion theory expressions for these coefficients are

$$\theta_{jk/\alpha\beta} = \frac{(\phi^* v^{-1} \phi_j)_{\alpha\beta k}}{(\nabla \phi^*, D \nabla \phi_j)_{\alpha\beta k} + (\phi^* \Sigma_t \phi_j)_{\alpha\beta k}}$$

and

$$\sigma_{jm \rightarrow k}^{a\beta} = \frac{(\phi^* v^{-1} \phi_j)_{\alpha\beta}}{\int_{E\beta} dE \int_{S_{m \rightarrow k}} ds D(r,E) \phi^*(r,E) \mathbf{n} \cdot \nabla \phi_j(r,E,t)}$$

where

$$\phi(r,E,t) = \int_{\text{all } \Omega} d\Omega \psi(r,E,\Omega,t)$$

and $D(r,E)$ is assumed to be independent of r at all points within a region but can vary from region to region. For diffusion theory the parenthetic notations do not include integration over Ω .

COMPUTER CODES FOR STUDYING COUPLED KINETICS PROBLEMS

Two computer codes have been written and a third is being developed to perform the computations for a kinetics study of the spatial and spectral coupling of the reactor neutron field. The first is called SAIM and is a modification of the FAIM code,^{1,2} a one-dimensional multi-group diffusion code. This code computes the partial fluxes $\phi_{j/a}(r,E)$ and the adjoint function $\phi^*(r,E)$. A second code, COUPKIN, uses punched-card output from the SAIM code to compute the coefficients in the kinetics equation and the corresponding initial values of the $N_{jk/a\beta}$. A third code is being developed which will integrate the set of equations given by Eqs. 8 and 9.

The SAIM and COUPKIN codes are written in FORTRAN II for use on an IBM 709 computer. They are limited to one-dimensional geometries: slab, sphere, or cylinder.

The basic operation of the SAIM code is as follows: The code first solves the eigenvalue problem expressed by

$$-D^i \nabla^2 \phi^i(r) + \sum_t^i \phi^i(r) - \sum_{k=q}^{i-1} \sum_{S_{k \rightarrow i}} \phi^k(r) = \chi^i(r) S(r)$$

for $1 \leq i \leq \text{NOG} \leq 18$ (21)

where NOG = number of groups

q = the highest group scattering into group i

D^i , $\sum_t^i$, and $\sum_{S_{k \rightarrow i}}$ = group constants for diffusion, removal, and transfer, respectively

and the source term is given by

$$S(r) = \frac{1}{\lambda} \sum_{i=1}^{NOG} (\nu \Sigma_f)^i \phi^i(r) \quad (22)$$

where λ is the eigenvalue. The term $\chi^i(r)$ gives the fraction of neutrons that are born into the i th group and is region dependent.

After the group fluxes, $\phi^i(r)$, have been determined, the code then solves the nonhomogeneous set of equations expressed by Eq. 23 for the $\phi_{j/a}^i(r)$:

$$\begin{aligned} -D^i \nabla^2 \phi_{j/a}^i(r) + \Sigma_t^i \phi_{j/a}^i(r) - \sum_{k=q}^{i-1} \Sigma_{S_{k \rightarrow i}} \phi_{j/a}^k(r) \\ = \chi_{j/a}^i(r) S(r) \quad (\alpha = 1, 2, \dots, N) \\ (j = 1, 2, \dots, M) \end{aligned} \quad (23)$$

where $\chi_{j/a}^i(r) = 1$ if $i \in E_\alpha$ and $r \in R_j$
 $= 0$ otherwise

$\phi_j^i(r)$ = the flux in group i at point r which originated within group E_α in region R_j

$S(r)$ = definition given by Eq. 22

The output from SAIM is read into the COUPKIN code, which uses a modified trapezoidal-rule numerical integration formula¹⁴ to compute the coupled kinetics parameters. This integration formula has the same order of accuracy as Simpson's rule.

COMPUTATION OF KINETICS PARAMETERS FOR THE UFTR

The SAIM and COUPKIN codes have been used to determine the partial fluxes $\phi_{j/a}(r, E)$ and the coupled kinetics coefficients. Figure 1 is a graph of the flux and the partial fluxes in the UFTR. The flux configuration is nonsymmetric because the UFTR has a control blade partially inserted in one fuel slab when the reactor is critical.

Cross-section calculations¹⁵ for the UFTR were based on a loading of 20% enriched-uranium fuel with a critical mass of 3500 g of ^{235}U . The thermal group constants were computed by the TEMPEST code¹⁶ for the energy group ranging from 0 to 0.683 ev. The GAM-1 code¹⁷ was used to compute the group constants for 17 groups between 0.683 ev and 10 Mev.

Fig. 1 - Flux in the UFTR as computed by the SAIM code; ϕ_i is the flux due to neutrons born in the i th region.

These group constants were for the core with all control blades completely withdrawn.

The TEMPEST and GAM-1 group constants were read into the SAIM code, which then searched for the value of the transverse buckling that would give a value¹⁵ for k_{eff} of 1.0057. The value of the transverse buckling as determined by the SAIM code was 0.00254 cm^{-2} . To arrive at a value of unity for k_{eff} , the SAIM code searched to determine the amount of poison necessary in one fuel slab to make the system critical. The poison required for criticality was about 1.6% as large as the thermal macroscopic cross section for the poisoned slab.

The partial fluxes were computed according to the method discussed earlier in this section. The 18-group fluxes from the SAIM code were summed over all energy groups, and the resulting total flux and partial fluxes are shown in Fig. 1.

The COUPKIN code used the adjoint fluxes and partial fluxes determined by the SAIM code to compute the kinetics coefficients according

to the formulas given in the section General Formulation of the Coupled Kinetics Equations. The results of these computations are given in Tables 1 and 2. Table 1 gives the values for the coefficients when the system is divided into two regions as shown in Fig. 1. Table 2 gives the values for the system divided into five regions according to the material in the region, i.e., graphite or fuel.

In the past the usual approach to a study of a two-slab Argonaut type reactor has been to consider it as two weakly coupled point reactors. Then, on the basis of this assumption, the kinetics of the system was described by a point-model kinetics equation for each region with an additional term to account for the interchange of neutrons between the two slabs.¹⁸ The formulation of this extra term was based on physical intuition and included a factor that represented the additional reactivity introduced by the coupling among the regions and a factor that represented the transit time for neutrons to travel from one region to the next. Because of the *ad hoc* means by which these extra quantities were introduced into the kinetics equation, a method for evaluating them was not always clear.^{5,19}

In the consistent method presented in this paper, the same problem can be determined in terms of any number of desired regions and groups. The limit to the number of regions and groups depends on our ability to integrate the resulting set of coupled first-order differential equations

Table 1 - COUPLED KINETICS PARAMETERS FOR THE UETR
FOR TWO REGIONS AND ONE GROUP

j	k	m	$\frac{l_{jk}}{k_{jk}}$ 11	$\left(\frac{1}{\theta_{jk}} - \frac{1}{\tau_{jk}} \right)^{-1}$ 11	$\sigma_{jk \rightarrow m}$ [†] 11	Γ_{jk} [‡] 11	$N_{jk}(0)$ 11
			μsec	μsec	μsec		
1	1	2	129.3	128.8	163.6	1.086	0.3692
1	2	1	347.5	368.3	-366.9	1.088	0.0831
2	1	2	368.0	348.1	-346.8	1.084	0.0875
2	2	1	128.1	128.6	183.2	1.090	0.4602

[†]A negative sign indicates that the net neutron current is from m to k instead of k to m.

[‡] $\Gamma_{jk} \equiv \bar{\beta}_{jk}/\beta_0$

Table 2 - COUPLED KINETICS PARAMETERS FOR THE UFTR FOR FIVE REGIONS AND ONE GROUP

j	k	m	$\frac{l_{jk}}{k_{jk}}$ 11 μsec	$\left(\frac{1}{\theta_{jk}} - \frac{1}{\tau_{jk}}\right)^{-1}$ 11 μsec	$\sigma_{jk \rightarrow m}^{\dagger}$ 11 μsec	$\Gamma_{jk}^{\ddagger}$ 11	$N_{jk}(0)$ 11
2	1	2	∞	315.4	-315.0		0.1191
2	2	1	45.7	57.6	345.2	1.086	0.1306
2	2	3			291.5		
2	3	2	∞	768.2	-421.9		0.1896
2	3	4			938.6		
2	4	3	46.5	56.4	-55.3	1.088	0.0111
2	4	5			2566.7		
2	5	4	∞	451.8	-452.0		0.0020
4	1	2	∞	445.4	-445.6		0.0019
4	2	1	46.0	54.7	2558.9	1.084	0.0109
4	2	3			-53.7		
4	3	2	∞	634.5	1057.0		0.2165
4	3	4			-396.1		
4	4	3	46.3	59.2	305.2	1.090	0.1663
4	4	5			351.1		
4	5	4	∞	321.3	-320.9		0.1520

$\dagger$ A negative sign indicates that the net neutron current is from m to k instead of from k to m.

$\ddagger \Gamma_{jk} \equiv \bar{\beta}_{jk}/\beta$.

expressed by Eqs. 8 and 9 rather than on our intuitive reasoning concerning the coupling between the regions and the groups. A simple example will serve to illustrate the point. Suppose the UFTR is represented by two coupled point reactors; then, according to the steady-state values of the $N_{jk/11}$ given in Table 1, 19% of the weighted neutron population in region R_1 originated in region R_2 and 15% of the weighted neutron population in region R_2 originated in region R_1 . On the basis of these figures, the assumption of weak coupling does not appear to be valid. As Fig. 1 illustrates, however, most of the neutrons in region R_1 that were born in region R_2 are in the graphite, and the same is true for the neutrons in R_2 .

Let us now consider Table 2, where the figures represent the five regions in Fig. 1 (divided according to the material used, i.e., graphite or fuel) as if the regions were numbered from left to right as R_1 to R_5 . Of the neutrons in fuel region R_2 , only 7.7% were born in the other fuel region, R_4 ; and, of the neutrons in R_4 , only 6.3% were born in R_2 . In this case the validity of the weakly coupled approximation is easily justified. Furthermore, the graphite region R_3 between regions R_2 and R_4 tends to "decouple" the fuel slab in such a way that there is a lag time between the origination of a perturbation in one fuel slab and its appearance in the other fuel slab. This lag time, as well as the coupling between the regions, is implicit in the formulation of the spatially and spectrally coupled kinetics equations.

One further point that is well worth our consideration is the comparison of coupled generation times given in Tables 1 and 2. The term *coupled generation time* refers to the ratio l/k and is given by

$$\frac{l_{jk}}{k_{jk}} = \frac{(\phi^* v^{-1} \phi_j)_{\alpha\beta}}{(\phi^* \chi P \phi_j)_{\alpha\beta}} \quad (24)$$

Equation 24 follows directly from the definitions of $l_{jk/\alpha\beta}$ and $k_{jk/\alpha\beta}$. Observe that in the two-region case given by Table 1 the generation times are much longer than in the five-region case given by Table 2. This longer generation time is a result of averaging the weighted-neutron production rate over a large region, although the production occurs only in a small part of that region. In the five-region case production occurs in all parts of a region over which the production rate is averaged, and the coupled generation time is almost independent of where the neutron was born or where it causes fission. Since the two fuel slabs of the UFTR are almost identical, this is as it should be and tends to indicate that five regions are sufficient to describe the kinetics of the system.

CONCLUSIONS

The theory of coupled reactors has been established on a firm basis by deriving the pertinent set of equations from the general Boltzmann equation as opposed to the heuristic methods used in the past. This allows the computation of coupling parameters from first principles in a consistent way. The set of kinetics equations presented here is in a very suitable form for studying the dynamic behavior of composite systems.

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AUTHORS' NOTE

Since the presentation of this paper, the authors have reformulated the coupled-reactor-kinetics equations on the basis of a semidirect variational method. This new formulation is discussed in detail in Ref. 13.

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Adiabatic Treatment of the Xenon Problem

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ABSTRACT

The coupled spatio-temporal variation of reactor neutron flux, induced by the decay and production of xenon, is investigated in the adiabatic approximation. It is shown that the slow variation of the Hamiltonian permits the approximate representation of the flux as the product of a well-determined function of the time alone and a combined function of position and time. A systematic method for the computation of higher corrections is described, and a W.K.B. description of the spatial variation permits a more detailed determination of the analytical form for the solution.

INTRODUCTION

In recent years widely varying attempts have been made to describe the combined spatial and temporal variation of reactors; an excellent summary of progress through 1964 has been given in the Geneva Conference paper of Kaplan et al.¹ In particular many investigations have been carried out on the effect of the nonlinear coupling of xenon production with the spatio-temporal variations of reactor flux. We shall refer to the above-mentioned survey article of Kaplan et al. for a detailed evaluation of research to date, but we should like briefly to review the alternative approaches before describing our own.

As was emphasized long ago by Ward,² small variations in flux level may, because of the delay introduced by the iodine decay, cause increases or decreases in reactivity associated with the new xenon con-

centration and resultant perpetuations of the original deviation. Similarly shifts in control-rod position, for example, may, even when the mean power is kept constant, introduce local flux changes that result in a new local xenon concentration; the new local xenon concentration then enhances the initial flux change, with associated combined spatio-temporal shifts and instabilities. The existence of these spatial oscillations has been demonstrated experimentally, and many people have observed that an attempt to control the oscillations may, in fact, amplify them.¹ A very considerable effort to attain a quantitative description of these and associated phenomena has thus resulted.

The most numerous investigations have centered around linearized theory; i.e., products of deviations from some mean or initial values are neglected. In this class fall the aforementioned work of Ward,² the investigations of Chernick,³ and the work of many others. Although useful information concerning small changes can thus be obtained, no reliable knowledge concerning the behavior of the system, when the approximation of small deviations fails, can be found. This is, of course, precisely a domain of potential interest. Moreover, linearized theory frequently involves a point model, and a point model cannot provide information concerning spatial changes. Extensions to include the effects of spatial changes have been made by Randall and St. John.⁴

Direct brute-force digital studies have thus far been carried out only to a limited extent. A one-group problem has been studied by Barss,⁵ and the question of the error induced by finite time steps in a preliminary linearized theory has been investigated by Davison.⁶ Generally, however, the attacks have thus far involved lengthy computations, and methods for improvement are clearly desirable.

In this paper we shall present the fundamental theory of an *adiabatic* attack, a theory that appears promising, both in cutting down computational time and in giving additional physical insight into the combined spatio-temporal variations. An adiabatic motivation is implicit in the early work of Henry⁷ on reactor kinetics in which the reactor flux is separated into a product of a time-dependent function alone and into a function of all the other relevant variables, including the time; if the latter factor is slowly varying in time, the approach is adiabatic. The usual treatments, however, do not provide a systematic way of estimating the slowness of this variation or of computing corrections to approximate assumed forms for this function. It appears to have been Hapke⁸ who first recognized that such a decomposition may often be justified as part of a systematic-expansion procedure analogous to that used in the quantum-mechanical adiabatic approximation.⁹ He was primarily concerned with short-duration problems, however, and provided only a cursory survey of long-term changes.

We shall see in the section on the adiabatic method that, under quite general conditions, if the *Hamiltonian* is slowly varying with the time (which is the essence of an adiabatic approximation!), the previously mentioned decomposition can be justified, the form for the pair of functions is determined, and error estimates and corrections can be systematically computed. Moreover, we shall see that in the particular case when the time variation is induced by the slow xenon change with time, the smallness parameter which measures the applicability of the adiabatic method (as the term is used herein) is indeed small. Thus a quite accurate picture of the combined spatio-temporal variation is expected to be provided by our solution.

In the section on the zeroth approximation, we investigate this approximation more closely, solving the iodine-xenon decay equations in terms of the flux and sketching an iterative procedure;⁵ this investigation should be satisfactory for a description of the reactor behavior over quite long time intervals. We further note that, in those cases where gross reactor inhomogeneity due to the xenon and other causes is slowly varying *spatially*, a W.K.B. evaluation⁹ of the eigenfunctions arising in the solution is feasible. This is illustrated in the one-group case for a bare reactor, but a multigroup generalization of this procedure has been constructed by the author. For illustrative purposes we have confined our attention to a relatively simple example; the inclusion, however, of several more realistic effects is quite feasible. A rather clear picture of the coupled modifications of time and spatial behavior emerges in this illustration, and we reemphasize that a major portion of effects of this type are included in the general solution outlined in the next section. A numerical study of the multigroup form for the solution is under way and will be reported elsewhere. In the Appendix we outline very briefly the scheme for computing higher order corrections to the solution thus obtained.

ADIABATIC METHOD

In this section we shall derive in a rather general form the results of an adiabatic treatment of time-dependent reactor problems, emphasizing the formal differences that arise when the *Hamiltonian* is a functional of functions (e.g., xenon concentration) of position and time. It will be seen that the method is of wide applicability so long as the Hamiltonian varies slowly with time; this statement will be made more precise in our subsequent discussion.

We are dealing with a time-dependent equation of the form

$$H[u(\mathbf{r}', t)] | \psi \rangle = \frac{\partial | \psi \rangle}{\partial t} \quad (1)$$

Here $u(\mathbf{r}', t)$ is any function (say, the xenon concentration at position $\mathbf{r}'$ at time t) on which H may depend. The bracket notation, $H[u]$, is meant to imply that H is, in general, a functional of values of u at all positions $\mathbf{r}'$. The term $|\psi\rangle$ is the neutron-flux vector, and we use the Dirac notation,¹⁰ $|\rangle$, to emphasize the distinction between a function and its value at a particular point and also to help from a manipulative viewpoint; the reader who prefers can replace $|\psi\rangle$ by ψ , *mutatis mutandis*, in what follows.

We now consider a set of eigenvectors of H

$$H[u] |\phi_n[u]\rangle = \lambda_n[u] |\phi_n[u]\rangle \quad (2)$$

where our notation emphasizes the fact that the eigenvalues $\lambda_n[u]$ and the eigenvectors $|\phi_n[u]\rangle$ are functionals of the function u and therefore functions of the time. To emphasize the latter point, we shall sometimes write $|\phi_n[u]\rangle = |\phi_n[u(t)]\rangle \equiv |\phi_n(t)\rangle$ and $\lambda_n[u] \equiv \lambda_n(t)$ for brevity so that, more concisely, Eq. 2 becomes

$$H(t) |\phi_n(t)\rangle = \lambda_n(t) |\phi_n(t)\rangle \quad (3)$$

In this form we need only keep in mind the physical origin of the time dependence, e.g., the variation with time of, say, the xenon concentration at all points in the reactor. To avoid confusion, we remark that it is a fact that, in general, H , ϕ_n , and λ_n depend on the values of u at *all positions* $\mathbf{r}'$; this makes H , ϕ_n , and λ_n , by definition, functionally dependent on u . Formal clarification of the concept of a functional can be found in standard works.¹¹

We now assume the possibility of an expansion of $|\psi\rangle$ in terms of the time-dependent eigenfunctions $|\phi_n(t)\rangle$.

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n(t) \exp \left[\int_0^t \lambda_n(t') dt' \right] |\phi_n(t)\rangle \quad (4)$$

The first few steps of our discussion are formally identical with conventional procedure⁹ and are included for completeness. The exponential in Eq. 4 simplifies slightly the subsequent treatment. Questions of convergence, completeness, the possible inclusion of a continuum, and improper eigenfunctions all need to be examined for particular choices of H . For present purposes we assume that Eq. 4 is valid.

The determination of the amplitudes $c_n(t)$ follows from the substitution of Eq. 4 into Eq. 1 and the use of Eq. 3.

$$\sum_{m=1}^{\infty} \left[\frac{dc_m}{dt} |\phi_m(t)\rangle + c_m \frac{d|\phi_m\rangle}{dt} \right] \exp \left[\int_0^t \lambda_m(t') dt' \right] = 0 \quad (5)$$

To simplify Eq. 5, we introduce the eigenvectors $|\phi_n^+\rangle$ of the Hermitian conjugate operator, H^+ :

$$H^+ |\phi_n^+\rangle = \lambda_n^* |\phi_n^+\rangle \quad (6)$$

where it is known¹² that the eigenvalues λ_n^* of Eq. 6 are the complex conjugates of the eigenvalues of Eq. 3. If H is Hermitian, $H = H^+$, the λ_n are real. The Hermitian conjugate of Eq. 6 is

$$\langle \phi_n^+ | H = \lambda_n \langle \phi_n^+ | \quad (7)$$

and we can choose these so that the orthonormality conditions

$$\langle \phi_n^+ | \phi_m \rangle = \delta_{nm} \quad (8)$$

are satisfied. Then, multiplying Eq. 5 by $\langle \phi_n^+ |$, we obtain

$$\frac{dc_n}{dt} = - \sum_{m=1}^{\infty} c_m \langle \phi_n^+ | \frac{d\phi_m}{dt} \rangle \exp \left\{ \int_0^t \left[\lambda_m(t') - \lambda_n(t') \right] dt' \right\} \quad (9)$$

The time dependence of ϕ_m is clearly induced by the time dependence of H , and a more easily interpreted expression for the scalar product on the right-hand side of Eq. 9 is obtained by differentiating Eq. 3:

$$\frac{dH(t)}{dt} |\phi_m\rangle + H \left| \frac{d\phi_m}{dt} \right\rangle = \frac{d\lambda_m}{dt} |\phi_m\rangle + \lambda_m \left| \frac{d\phi_m}{dt} \right\rangle \quad (10)$$

Multiplying scalarly on the left by $\langle \phi_n^+ |$ and using Eq. 7, we obtain

$$(\lambda_m - \lambda_n) \langle \phi_n^+ | \frac{d\phi_m}{dt} \rangle + \frac{d\lambda_m}{dt} \delta_{nm} = \langle \phi_n^+ | \frac{dH}{dt} | \phi_m \rangle \quad (11)$$

For $m = n$ we get

$$\frac{d\lambda_n}{dt} = \langle \phi_n^+ | \frac{dH}{dt} | \phi_n \rangle \quad (12)$$

Thus, if H varies slowly with time, so does λ_n as shown by Eq. 12. For $m \neq n$, if no degeneracy obtains, Eq. 11 yields

$$\left\langle \phi_n^+ \left| \frac{d\phi_m}{dt} \right. \right\rangle = \frac{\left\langle \phi_n^+ \left| \frac{dH}{dt} \right| \phi_m \right\rangle}{\lambda_m - \lambda_n} \quad (m \neq n; \lambda_m \neq \lambda_n) \quad (13)$$

We now have a convenient expression for the scalar product occurring in Eq. 9 for $m \neq n$. For the case $m = n$, we shall now show that ϕ_n can be so chosen that

$$\left\langle \phi_n^+ \left| \frac{d\phi_n}{dt} \right. \right\rangle = 0 \quad (14)$$

First, $|\phi_n\rangle$ is undetermined to within a numerical factor Z_n so that we write

$$|\phi_n'\rangle = Z_n |\phi_n\rangle \quad (15)$$

We also write

$$|\phi_n^{+'}\rangle = Y_n |\phi_n^+\rangle \quad (16)$$

Normalization of both the old and the new vectors is required:

$$\langle \phi_n^{+'} | \phi_n' \rangle = Y_n^* Z_n = 1 \quad (17)$$

$$Y_n = \frac{1}{Z_n^*} \quad (18)$$

Now

$$\frac{d|\phi_n'\rangle}{dt} = \frac{dZ_n}{dt} |\phi_n\rangle + Z_n \left| \frac{d\phi_n}{dt} \right\rangle \quad (19)$$

If we multiply Eq. 19 by Eq. 16,

$$\left\langle \phi_n^{+'} \left| \frac{d\phi_n'}{dt} \right. \right\rangle = Y_n^* \frac{dZ_n}{dt} + Y_n^* Z_n \left\langle \phi_n^+ \left| \frac{d\phi_n}{dt} \right. \right\rangle \quad (20)$$

and if we demand that

$$\left\langle \phi_n^{+'} \left| \frac{d\phi_n'}{dt} \right. \right\rangle = 0 \quad (21)$$

then

$$\frac{d \ln Z_n}{dt} = - \left\langle \phi_n^+ \left| \frac{d\phi_n}{dt} \right. \right\rangle \quad (22)$$

Thus, if we choose

$$Z_n(t) = Z_n(0) \exp \left[- \int_0^t \left\langle \phi_n^+ \left| \frac{d\phi_n}{dt'} \right. \right\rangle (t') dt' \right] \quad (23)$$

Eq. 21 is automatically satisfied.

Hence we may assume that Eq. 14 has been satisfied. Thus Eq. 9 becomes

$$\frac{dc_n}{dt} = + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{\left\langle \phi_n^+ \left| \frac{dH}{dt} \right| \phi_m \right\rangle}{\lambda_n - \lambda_m} \exp \left[- \int_0^t (\lambda_n - \lambda_m) (t') dt' \right] c_m(t) \quad (24)$$

We are now in a position to state more precisely the meaning of the slow variation of H . For simplicity we consider the case of usual interest when the system is initially in some single mode:[†]

$$c_m(0) = c_k(0) \delta_{mk} \quad (25)$$

and we estimate the right-hand side of Eq. 24 assuming that the λ_n are approximately constant and that the c_m may be replaced by their initial values. Then

$$\frac{dc_k}{dt} \simeq 0 \quad c_k(t) \simeq c_k(0) \quad (26)$$

and

$$\frac{dc_n}{dt} \simeq c_k(0) \frac{\left\langle \phi_n^+ \left| \frac{dH}{dt} \right| \phi_k \right\rangle}{(\lambda_n - \lambda_k)} \exp [- (\lambda_n - \lambda_k)t] \quad (27)$$

[†]In the interest of clarity, we emphasize that for the xenon problem this initial mode can be one which is spatially asymmetric, as induced, for example, by control-rod shifts, but which leaves the reactor otherwise critical (initially).

On carrying out one more integration that assumes approximate constancy, we obtain

$$c_n(t) \simeq c_n(0) \frac{\langle \phi_n^+ | \frac{dH}{dt} | \phi_n \rangle}{(\lambda_n - \lambda_k)^2} \{ 1 - \exp [- (\lambda_n - \lambda_k)t] \} \quad (28)$$

The approximate ratio of the total amplitude of the n th to the k th mode as a function of time is

$$\frac{c_n(t) \exp (\lambda_n t)}{c_k(t) \exp (\lambda_k t)} \simeq \frac{\langle \phi_n^+ | \frac{dH}{dt} | \phi_k \rangle}{(\lambda_n - \lambda_k)^2} \{ \exp [(\lambda_n - \lambda_k)t] - 1 \} \quad (29)$$

If $\text{Re}(\lambda_n - \lambda_k) < 0$, then the time-dependent factor in braces is bounded by unity, and the relative magnitude of the excitation of the n th mode is determined by its coefficient in Eq. 28. We shall see that the preceding condition on the eigenvalues is satisfied in an example later. If

$$\frac{|\langle \phi_n^+ | \frac{dH}{dt} | \phi_k \rangle|}{(\lambda_n - \lambda_k)^2} \ll 1 \quad (30)$$

then this is the condition that will make a good first approximation to the solution of Eq. 24 be

$$c_k(t) = c_k(0) \quad (31a)$$

$$c_n(t) = 0 \quad \text{for } n \neq k \quad (31b)$$

$$|\psi \rangle = c_k(0) \exp \left[\int_0^t \lambda_k(t') dt' \right] |\phi_k(t) \rangle \quad (32)$$

We shall examine more closely the validity of Eq. 30 in a definite example later. Let us now go on to the specific case where the temporal behavior of H is induced by its functional dependence on some quantity $u(\mathbf{r}', t)$.

In this case

$$\frac{dH}{dt} = \int dr' \frac{\delta H}{\delta u(r', t)} \frac{\partial u(r', t)}{\partial t} \quad (33)$$

where $\delta/\delta u(r', t)$ denotes the functional derivative. Then Eq. 24 becomes

$$\begin{aligned} \frac{dc_n}{dt} = & \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \int dr' \frac{\partial u(r', t)}{\partial t} \frac{\langle \phi_n^+ | \frac{\delta H}{\delta u(r', t)} | \phi_m \rangle}{\lambda_n - \lambda_m} c_m(t) \\ & \times \exp \left[- \int_0^t (\lambda_n - \lambda_m)(t') dt' \right] \end{aligned} \quad (34)$$

If, for example, $u = N_{xe}(r', t)$, the concentration of xenon atoms, then Eq. 34 enables one to compute, from the equation for $\partial N_{xe}/\partial t$, corrections to our first approximation, Eq. 32. In the Appendix we shall elaborate on this point, but it is important to reemphasize that Eq. 32 should already by itself give a reasonably good representation of the flux in position and time.

In closing this general discussion, we record two results which are the analogs of Eqs. 12 and 13 and which are derived by functionally differentiating Eq. 2 with respect to $u(r')$:

$$\frac{\delta \lambda_m}{\delta u(r')} = \langle \phi_m^+ | \frac{\delta H}{\delta u(r')} | \phi_m \rangle \quad (35)$$

$$\langle \phi_n^+ | \frac{\delta \phi_m}{\delta u(r')} \rangle = \frac{1}{\lambda_m - \lambda_n} \langle \phi_n^+ | \frac{\delta H}{\delta u(r')} | \phi_m \rangle \quad (m \neq n) \quad (36)$$

These give the first-order changes in eigenvalues and eigenvectors as a result of local changes in $u(r')$.

ZEROth APPROXIMATION

As we have already indicated, the neutron flux in the case of a slowly varying Hamiltonian should be well represented by

$$\psi(r, t) = c_1(0) \exp \left[\int_0^t \lambda_1(t') dt' \right] \phi_1(r, t) \quad (37)$$

We have here introduced the following notation: we have set $k = 1$ in anticipation of the fact that a usual initial condition refers to the reactor in its fundamental mode.[†] We have explicitly introduced the position-independent variable, r , and shall not in this discussion attempt to describe the formal modifications required if a more elaborate description (angular variations) of the flux is included; these are probably not of importance for the xenon problem. The vectorial symbols ψ and ϕ_1 now refer to the energy variation that we assume to have been "discretized" in multigroup form so that $\psi(r, t)$, for example, has components $\psi_g(r, t)$, where g is the energy group index.[‡] The factor $c_1(0)$ is a constant determined by the initial power normalization. We shall subsequently consider corrections to the use of Eq. 37.

The factors λ_1 and ϕ_1 are functionals of the xenon concentration, and thus it is necessary to study the variation of the latter with time. We have for the equations of motion

$$\frac{\partial N_I}{\partial t} = \sum_g y_g \sigma_{fg} \psi_g(r, t) - \frac{N_I}{\tau_I} \quad (38)$$

$$\frac{\partial N_{Xe}}{\partial t} = \frac{N_I}{\tau_I} - \frac{N_{Xe}}{\tau_{Xe}} - \sum_g \sigma_{Xe,g} \psi_g N_{Xe} \quad (39)$$

where

σ_{fg} = the macroscopic fission cross section for the g th energy group

$\sigma_{Xe,g}$ = the microscopic xenon cross section for the g th energy group[§]

y_g = the iodine fission yield for the g th group

$N_I(r, t)$ = the concentration of iodine atoms at position r and time t

$N_{Xe}(r, t)$ = the concentration of xenon atoms at position r and time t

τ_I, τ_{Xe} = the associated mean lifetimes

[†] See footnote on page 348.

[‡] The relation between $|\psi\rangle$ and ψ is that the components of $|\psi\rangle$ are labeled by r and g , and those of ψ by g alone.

[§] Throughout the paper, σ denotes macroscopic cross section except for $\sigma_{Xe,g}$, which is the only case for microscopic cross section.

The solution of Eq. 38 is

$$N_I(r, t) = N_I(r, 0) \exp\left(-\frac{t}{\tau_I}\right) + \sum_g y_g \sigma_{fg} \int_0^t dt' \psi_g(r, t') \exp\left(-\frac{t-t'}{\tau_I}\right) \quad (40)$$

The resultant solution of Eq. 39 is

$$\begin{aligned} N_{Xe}(r, t) = & N_{Xe}(r, 0) \exp\left[-\frac{t}{\tau_{Xe}} - \sum_g \sigma_{Xe, g} \int_0^t dt' \psi_g(r, t')\right] \\ & + \int_0^t dt' \exp\left[-\frac{t-t'}{\tau_{Xe}} - \sum_g \sigma_{Xe, g} \int_{t'}^t dt'' \psi_g(r, t'')\right] \\ & \times \left[\frac{N_I(r, 0)}{\tau_I} \exp\left(-\frac{t'}{\tau_I}\right) \right. \\ & \left. + \sum_g \frac{\sigma_{fg} y_g}{\tau_I} \int_0^{t'} dt'' \psi_g(r, t'') \exp\left(-\frac{t'-t''}{\tau_I}\right) \right] \quad (41) \end{aligned}$$

The $\psi_g(r, t')$ terms that occur in this expression are given by the components of Eq. 37:

$$\psi_g(r, t) = c_1(0) \exp\left[\int_0^t \lambda_1(t'') dt''\right] \phi_{1g}(r, t) \quad (42)$$

Equations 41 and 42 must be solved together, and it is clear that, in general, this can be accomplished only numerically. Such a study is under way, but one can get some insight into the general nature of the result as follows: We assume that at $t = 0$ the reactor is on a reciprocal rise time,[†] $\lambda_1^{(0)}$, with a flux distribution determined by $\phi_{1g}^{(0)}(r, 0)$. For some considerable time thereafter, say for $0 < t < t_1$, it should be a reasonable approximation⁵ to determine the xenon destruction in Eq. 41 by the use of

$$c_1(0) \exp(\lambda_1^{(0)} t) \phi_{1g}(r, 0)$$

[†]In concord with our previous remarks (see footnote, page 348), we can have $\lambda_1^{(0)} = 0$ and an asymmetric initial mode.

Thus

$$\begin{aligned}
 N_{Xe}(r, t) \simeq N_{Xe}(r, 0) \exp \left[-\frac{t}{\tau_{Xe}} - \sum_g \sigma_{Xe, g} c_1(0) \phi_{1g}(r, 0) \frac{\exp(\lambda_1^{(0)} t) - 1}{\lambda_1^{(0)}} \right] \\
 + \int_0^t dt' \exp \left[-\frac{t-t'}{\tau_{Xe}} \right. \\
 \left. - \sum_g c_1(0) \sigma_{Xe, g} \phi_{1g}(r, 0) \frac{\exp(\lambda_1^{(0)} t) - \exp(\lambda_1^{(0)} t')}{\lambda_1^{(0)}} \right] \\
 \times \left[\frac{N_I(r, 0) \exp(-t'/\tau_I)}{\tau_I} \right. \\
 \left. + \sum_g \frac{y_g \sigma_{fg}}{\tau_I} c_1(0) \phi_{1g}(r, 0) \frac{\exp(\lambda_1^{(0)} t') - \exp(-t'/\tau_I)}{\lambda_1^{(0)} + 1/\tau_I} \right] \\
 \text{for } 0 < t < t_1 \quad (43)
 \end{aligned}$$

For this time interval $N_{Xe}(r, t)$ is thus a *known* function of position and time. Equation 2 is to be solved with this dependence of the xenon concentration on position and time

$$H[N_{Xe}(r', t)] |\phi_1[N_{Xe}(r', t)]\rangle = \lambda_1[N_{Xe}(r', t)] |\phi_1\rangle \quad (44)$$

We shall say something about the expected form for this solution shortly, but the point to be emphasized now is that the result for Eq. 44 gives the flux, Eq. 37 or Eq. 42:

$$\psi_g(r, t) = c_1(0) \exp \left[\int_0^t \lambda_1(t') dt' \right] \phi_{1g}(r, t) \quad (45)$$

as a function of position and time for the time interval

$$0 < t < t_1 \quad (46)$$

It is clear how this procedure is to be extended to longer time intervals: Let $N_{Xe}^{i+1}(r, t)$ denote the result of the calculation of Eq. 41 with ψ_g replaced by $\psi_g^{(i)}$. This value of N_{Xe}^{i+1} is used in the eigenvalue equation, Eq. 44, to generate $\phi_1^{(i+1)}$ and λ_1^{i+1} and in Eq. 45 to generate

the associated flux $\psi^{i+1}(r,t)$ valid in the interval $t_i < t < t_{i+1}$. Here $t_0 = 0$, $\psi_g^{(0)}$ = whatever is appropriate from the initial conditions (see preceding example), and $i = 0, 1, 2, \dots$. The intervals will, in general, need to be determined by numerical experimentation, but preliminary results⁵ and the preceding arguments indicate that the description given by Eq. 45 is valid for quite long time intervals.

In concluding this section, we wish to point out that, at least in one-dimensional geometries, it is possible to get a fair idea of the shape of the eigenfunctions and the values of the eigenvalues if the reactor can be regarded as being approximately homogeneous independent of the xenon. It is known that the spatial variation of the xenon does not induce a radical spatial variation of the absorption cross section, and this is its main effect; e.g., the effect in the diffusion constant is small. Hence a W.K.B. method⁹ is applicable.

The author has developed a generalization of the W.K.B. method which is applicable to multigroup diffusion theory. For the sake of brevity, however, we shall illustrate the form for the eigenfunctions and eigenvalues in the case of a one-group diffusion theory. We have for a slab reactor

$$\frac{d^2 \phi_n}{dx^2} + \kappa_n^2(x) \phi_n(x) = 0 \quad (47)$$

where

$$\kappa_n^2(x) \equiv \frac{\nu\sigma_f - \sigma_a - \lambda_n}{v} \frac{1}{D} \quad (48)$$

and the notation is standard except that the σ 's denote macroscopic cross sections; the slight effect of the variation of xenon concentration on D has been neglected, although this is taken into account in the generalized W.K.B. method mentioned previously. The reactor is imagined to extend over the interval $-a/2 < x < a/2$; the eigenfunctions defined by

$$H\phi_n = \lambda_n \phi_n \quad H = Dv \frac{d^2}{dx^2} + v(\nu\sigma_f - \sigma_a) \quad (49)$$

are then

$$\phi_n(x) = \frac{A_n}{\sqrt{\kappa_n(x)}} \sin \left[\int_{-a/2}^x \kappa_n(x') dx' \right] \quad (50)$$

where A_n is a normalization constant and the eigenvalues are determined from

$$\int_{-a/2}^{+a/2} \kappa_n(x') dx' = n\pi \quad (n = 1, 2, \dots) \quad (51)$$

If the fractional effect of the xenon absorption is not too large, it can be shown that

$$\kappa_n(x, t) = \frac{n\pi}{a} \left\{ 1 + \frac{\sigma_{Xe} a^2}{2Dn^2\pi^2} \left[\frac{1}{a} \int_{-a/2}^{+a/2} N_{Xe}(x', t) dx' - N_{Xe}(x, t) \right] \right\} \quad (52)$$

so that the correction is associated with the local deviation of the xenon concentration from its average value. The eigenvalues are given by

$$\frac{\lambda_v}{v} = \frac{\lambda_n^{(0)}}{v} - \frac{\sigma_{Xe}}{a} \int_{-a/2}^{+a/2} N_{Xe}(x', t) dx' \quad (53)$$

$$\frac{\lambda_n^{(0)}}{v} = \nu\sigma_f - \sigma_a^{(0)} - \frac{Dn^2\pi^2}{a^2}$$

which exhibit explicitly the functional dependence of λ_n on $N_{Xe}(x', t)$ in this simple case. The eigenfunctions are given by

$$\phi_n(x, t) = \frac{A_n}{\sqrt{\kappa_n(x)}} \sin \frac{n\pi}{a} (x + a/2) \left\{ 1 + \frac{\sigma_{Xe} a^2}{2Dn^2\pi^2} \right. \\ \left. \times \left[\frac{1}{a} \int_{-a/2}^{+a/2} N_{Xe}(x', t) dx' - \frac{1}{x + a/2} \int_{-a/2}^x N_{Xe}(x', t) dx' \right] \right\} \quad (54)$$

which also indicates explicitly the functional dependence of ϕ_n on $N_{Xe}(x', t)$.

The temporal and spatial behavior in the fundamental mode is given by

$$\begin{aligned} \psi_1(x, t) = & \frac{c_1(0) A_1}{\sqrt{\kappa_1(x, t)}} \exp \left\{ \lambda_1^{(0)} t - \frac{v \sigma_{Xe}}{a} \int_0^t dt' \int_{-a/2}^{+a/2} N_{Xe}(x', t') dx' \right\} \\ & \times \sin \frac{\pi}{a} (x + a/2) \left\{ 1 + \frac{\sigma_{Xe} a^2}{2D\pi^2} \left[\frac{1}{a} \int_{-a/2}^{+a/2} N_{Xe}(x', t) dx' \right. \right. \\ & \left. \left. - \frac{1}{x + a/2} \int_{-a/2}^x N_{Xe}(x', t) dx' \right] \right\} \end{aligned} \quad (55)$$

The correction to the initial reciprocal relaxation time is thus

$$\delta \lambda_1 = \lambda_1^{eff} - \lambda_1^{(0)} = - \frac{1}{at} \int_0^t dt' \int_{-a/2}^{+a/2} dx' v \sigma_{Xe} N_{Xe}(x', t') \quad (56)$$

i.e., essentially the space-time average of the xenon concentration. Here the effective reciprocal relaxation time, $\lambda_1^{eff}(t)$, is defined by Eq. 56, and the reactor moves as

$$\exp \left[\lambda_1^{eff}(t) t \right] \quad (57)$$

Similarly the correction to the wave number is

$$\frac{\delta k_1}{\kappa_1^{(0)}} = \frac{\sigma_{Xe} a^2}{2D\pi^2} \left[\frac{1}{a} \int_{-a/2}^{+a/2} N_{Xe}(x', t) dx' - \frac{1}{x + a/2} \int_{-a/2}^x N_{Xe}(x', t) dx' \right]$$

which has a local spatial and temporal dependence. Thus a fairly clear picture of the nonlinear effect on the reactor period and the temporal changes in shape has been obtained. It is, furthermore, easy to convince oneself that the condition for the validity of this procedure, Eq. 30, is satisfied for a wide range of values of reactor nuclear parameters and sizes.

It is, in fact, the case, from Eqs. 58 and 61, that

$$\lambda_n - \lambda_1 = - \frac{Dv\pi^2}{a^2} (n^2 - 1) \quad (58)$$

We note first that this justifies our remark subsequent to Eq. 29 that the time-dependent term therein does not grow with time. We further remark that the smallness parameter defined by Eq. 30 is

$$\epsilon_n \equiv \frac{|\langle \phi_n^+ | \frac{dH}{dt} | \phi_n \rangle|}{(\lambda_n - \lambda_1)^2} \simeq \frac{1}{(n^2 - 1)^2} \left(\frac{a}{\pi D} \right)^2 - \left(\frac{a}{\pi v \bar{\tau}_{xe}} \right) \left(\frac{a \sigma_{xe} N_{xe}}{\pi} \right) \quad (59)$$

where $\langle \phi_n^+ | \frac{dH}{dt} | \phi_n \rangle = \frac{v \sigma_{xe}}{\tau_{xe}} \bar{N}_{xe}$ with $\bar{N}_{xe}$ the mean xenon concentration weighted with ϕ_n^2 and $\bar{\tau}_{xe}$ is an effective xenon relaxation time. To make a numerical estimate, we write $\sigma_{xe} \bar{N}_{xe} \simeq y \sigma_f$, where $y = 0.059$ is the xenon fission yield and the equilibrium value of the mean xenon concentration at high flux is used as an approximation for $\bar{N}_{xe}$.

$$\epsilon_n \simeq y \left(\frac{a}{n\pi} \right)^4 \frac{\sigma_f}{v \bar{\tau}_{xe} D^2} = y \frac{k}{\nu} \left(\frac{a}{n\pi L} \right)^2 \left(\frac{a}{n\pi D} \right) \left(\frac{a}{n\pi \bar{\tau}_{xe}} \right) \quad (60)$$

where $L^2 = D/\sigma_a$

$$k = v \sigma_f / \sigma_a$$

With $D \simeq 0.25$ cm

$$\sigma_f \simeq 0.05 \text{ cm}^{-1}$$

$$k/\nu \simeq 1/3$$

$$L^2 \simeq 3.0 \text{ cm}$$

$$v \simeq 3 \times 10^5 \text{ cm/sec}$$

$$\bar{\tau}_{xe} \simeq 4 \times 10^4 \text{ sec}$$

$$\epsilon_2 \simeq 10^{-15} a^4 \simeq 10^{-5} \quad \text{for } a = 300 \text{ cm} \quad (61)$$

The first smallness parameter, ϵ_2 , is negligible for $a \simeq 10$ ft; variations by a factor of 2 or even more in the nuclear parameters do not affect the validity of $\epsilon_n \ll 1$, and the replacement of L^2 by M^2 helps. An increase of a by a factor of 10 may begin to introduce the necessity for corrections discussed in the Appendix. We finally remark that a little thought concerning Eq. 55 will show the reader that the flux tilting and its temporal variation predicted thereby is qualitatively in the direction to

be expected on physical grounds. A complete discussion of the solution will be presented elsewhere.

APPENDIX

HIGHER APPROXIMATIONS

We shall sketch briefly the computation of higher corrections to the zeroth approximation discussed in the previous section. We write Eq. 24 in the form

$$\frac{dc_n}{dt} = \sum_{m=1}^{\infty} V_{nm}(t) c_m(t) \quad (\text{A.1})$$

with

$$V_{nm} = \int dr' \frac{\partial N_{xe}(r', t)}{\partial t} \frac{\langle \phi_n^+ | \frac{\delta H}{\delta N_{xe}(r, t)} | \phi_m \rangle}{\lambda_n - \lambda_m} \times \exp \left[- \int_0^t (\lambda_n - \lambda_m) dt' \right] \quad \text{for } n \neq m \quad (\text{A.2a})$$

$$V_{nm} = 0 \quad \text{for } n = m \quad (\text{A.2b})$$

In vectorial form this becomes

$$\frac{d\mathbf{c}}{dt} = \mathbf{V}(t) \mathbf{c} \quad (\text{A.3})$$

where $\mathbf{c}$ is a column vector with c_n as components and $\mathbf{V}$ is the matrix with V_{nm} as elements. The solution to Eq. A.3 can be written compactly as

$$\mathbf{c}(t) = \mathbf{P} \exp \left\{ \int_0^t dt' [\mathbf{V}(t')]_t \right\} \mathbf{c}(0) \quad (\text{A.4})$$

where the ordered exponential¹³ on the right is defined by

$$P \exp \left\{ \int_0^t dt' [V(t')]_t \right\} = \sum_{j=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \times \int_0^{t_{j-1}} dt_j V(t_1) V(t_2) \dots V(t_j) \quad (\text{A.5})$$

Returning to component notation, we have for Eq. A.4

$$c_n(t) = \sum_{j=0}^{\infty} \sum_{n_1=1}^{\infty} \dots \sum_{n_j=1}^{\infty} \int_0^t dt_1 V_{nn_1}(t_1) \int_0^{t_1} dt_2 V_{n_1 n_2}(t_2) \dots \times \int_0^{t_{j-1}} dt_j V_{n_{j-1} n_j}(t_j) c_{n_j}(0) \quad (\text{A.6})$$

If we assume only one mode initially present, then

$$c_n(0) = c_1(0) \delta_{n1} \quad (\text{A.7})$$

and

$$c_1(t) = c_1(0) + \sum_{\substack{n_1=1 \\ n_1 \neq 1}}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 V_{1n_1}(t_1) V_{n_1}(t_2) c_1(0) + \dots \quad (\text{A.8})$$

$$c_n(t) = \int_0^t dt_1 V_1(t_1) c_1(0) + \sum_{\substack{n_1=1 \\ n_1 \neq 1, n}}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 V_{nn_1}(t_1) V_{n_1}(t_2) c_1(0) + \dots \quad (\text{A.9})$$

Equations A.8 and A.9 can be used to compute higher corrections if necessary. But we note that Eq. A.8 reveals that corrections to the nonconstancy of c_1 are, in fact, proportional to the square of the smallness parameter, Eq. 30.

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Large Reactor Excursions Starting at High Temperature

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ABSTRACT

The maximum temperature reached by a reactor core in several step excursions with different Doppler reactivity feedbacks is calculated analytically with the delayed neutrons taken into account. The analytical values are compared with those obtained from a numerical solution of the point kinetics equations, and it is found that the agreement is extremely good for the more severe excursions; in these cases the difference between the respective results is less than 1%.

INTRODUCTION

This paper is an extension of previous work¹ on large-step excursions in reactors and includes excursions starting at operating, steady-state conditions or at high initial temperature and very low power. The previous work was limited to excursions starting at zero power and room temperature.

Taking into account the delayed neutrons, we calculate analytically *the maximum temperature reached by a reactor core* in several step excursions with different Doppler reactivity feedbacks. The analytical results are compared with those obtained with the TER 9 code,² a program used at the General Electric Company Atomic Power Equipment Department (APED) for certain types of reactor-excursion calculations. We find that the agreement is extremely good for the more severe excursions, the difference between the respective results being less than 1%.

FORMULATION OF THE PROBLEM

The reactor excursions are described by the following nonlinear kinetic equations:¹

$$\begin{aligned}\frac{dn}{dt} &= \frac{\rho_0 - \beta + f(T)}{l} n + \lambda C \\ \frac{dC}{dt} &= \frac{\beta}{l} n - \lambda C \\ \frac{dT}{dt} &= \gamma n\end{aligned}\tag{1}$$

Here $f(T)$ is the temperature-dependent Doppler reactivity feedback. When the reactor is critical and a step insertion of reactivity is introduced, at $t = 0$ the total reactivity equals the step insertion ρ_0 , and $(n, C, T) = (n_0, C_0, T_0)$ are the values of the power, the power due to delayed neutrons, and the temperature at the critical initial state. The first integral of Eq. 1 in Ref. 1 reads

$$n + C - \frac{\rho_0}{\gamma l} T - \frac{1}{\gamma l} \int f(T) dT = A\tag{2}$$

Here A is an arbitrary constant giving the initial condition. In our case this is $(n, C, T) = (n_0, C_0, T_0)$, and thus

$$A = \left(1 + \frac{\beta}{\lambda l}\right) n_0 - \frac{1}{\gamma l} \left[\int f(T) dT \right]_{T=T_0}\tag{3}$$

with $C_0 = (\beta/\lambda l) n_0$. From here, using Eq. 3, we can follow exactly the treatment of Ref. 1 to obtain *all* the results corresponding to the general ones presented there for excursions starting at shutdown, where the initial condition was $(n, C, T) \simeq (0, 0, 0)$ and

$$A = \left(-\frac{1}{\gamma l}\right) \left[\int f(T) dT \right]_{T=0}$$

APPLICATIONS

As an application, we have calculated the maximum core temperature resulting from a step insertion of reactivity for the following Doppler

reactivity feedbacks: case 1, $\rho = \rho_0 - a_1(T - T_0)$; case 2, $\rho = \rho_0 - a_1(\sqrt{T} - \sqrt{T_0})$; case 3, $\rho = \rho_0 - a_1 \left(e^{a_2 \sqrt{T}} - e^{a_2 \sqrt{T_0}} \right)$. Here ρ is the total reactivity, ρ_0 is the step reactivity, T is the core temperature, and T_0 is the initial core temperature, which in all cases is 547°F.* The initial power is very small so that the term with n_0 in Eq. 3 is neglected.

Case 1: $\rho = \rho_0 - a_1(T - T_0)$

In this linear case the analytical formula giving the maximum core temperature is

$$T_{\max} = T_0 + \frac{2\rho_0}{a_1} \quad (4)$$

The delayed neutrons produced in the excursion begin to be released at a certain time after peak power, creating the so-called delayed-neutron tail. If the excursion is terminated before any appreciable delayed-neutron release, the final temperature is given by Eq. 4 with the use of $\rho_0 - \beta$ (prompt reactivity) instead of ρ_0 (total reactivity). This conclusion and the following discussion are valid for any Doppler reactivity feedback (see Ref. 1).

Table 1 gives the maximum core temperatures ($T_{\max}$) calculated from Eq. 4 for both the prompt ($\rho_0 - \beta$) and the total (ρ_0) reactivities along with those obtained from the TER 9 code. The $T_{\max}$ values as calculated by TER are the core temperatures at times between 10^{-1} and 1 sec after peak power. The agreement between the values obtained by the TER code and those obtained from Eq. 4 with $\rho_0 - \beta$ instead of ρ_0 shows that, at these times after peak power, the delayed-neutron release is quite small and entirely negligible for the excursions with the bigger step insertions: for the \$4 excursions ($\rho_0 = 0.03$, $\beta = 0.0075$), the difference between the TER and the analytical values is less than 0.1%. For the \$2 excursions the results from TER are about 10% higher than those obtained from Eq. 4 with $\rho_0 - \beta$, i.e., if we neglect entirely the delayed-neutron heating. For the excursions that are due to smaller step insertions, for which the burst is wider in time, the delayed-neutron heating begins to be felt sooner after the power peak. It must be emphasized that the analytical solution takes rigorously into account one average group of delayed neutrons, and a temperature vs. time plot can be obtained from it. This requires some numerical computation (see

* This is approximately the saturated steam temperature at 1000 psi, which is the operating steam pressure of a large boiling-water reactor like Dresden.

Appendix A). For the excursions due to the larger step insertions, however, there is no need for any further computation since the values from Eq. 4 and from TER are identical. A potentially important safety point, which is not revealed in the TER calculations, is shown in Table 1.

Table 1 - MAXIMUM CORE TEMPERATURE

$$[\rho = \rho_0 - a_1(T - T_0), T_0 = 547^\circ\text{F}, \beta = 0.0075]$$

Excursion	$a_1, ^\circ\text{F}^{-1}$	ρ_0	$T_{\max}, ^\circ\text{F}^*$			Difference between ana-	
						lytic [†] solu-	
			<u>Analytic[†]</u>			tion and TER	
			TER 9	$\rho_0 - \beta$	ρ_0	9 solution, %	
				$\rho_0 - \beta$	ρ_0		
1	1.7502×10^{-5}	0.015	1567	1404	2252	14.5	36
2	1.7502×10^{-5}	0.030	3115	3118	3977	0.1	24
3	1.93284×10^{-5}	0.015	1472	1323	2197	10.6	40
4	1.93284×10^{-5}	0.030	2872	2875	3647	0.1	24

* Note that these are final core temperatures. The temperature increase is obtained by subtracting 547°F from these values.

[†] Prompt heating is indicated by $\rho_0 - \beta$; total heating is indicated by ρ_0 .

This is the considerable energy stored by the delayed neutrons. More specifically, the delayed neutrons produced during the excursion are multiplied by the non-negligible multiplication factor that still exists at the end of the prompt burst. Theoretically, this energy would be released in an infinite time and would make the core temperatures 20 to 40% higher than the values at the time when the delayed-neutron heating is negligible. Some experimental evidence of this has been reported for a \$2 excursion in TRIGA,³ where the total energy release is 46% higher than the prompt energy release. This increase compares with the 36 and 40% increases for the \$2 excursions shown in Table 1. The prompt reactivity feedback of TRIGA is a linear function of the fuel temperature increase just as is the case shown in Table 1. Of course, these figures cannot be taken literally because, when a sufficiently long time has elapsed after peak power, heat transfer to the moderator takes place and moderator reactivity feedback effects become important; these are not taken into account in the theory.

Case 2: $\rho = \rho_0 - a_1(T - \sqrt{T_0})$

In this case the analytical formula for $T_{\max}$ (in dimensionless variables) is*

$$T_{\max} = \frac{9}{16} \left[\rho_0 + \frac{\sqrt{T_0}}{3} + \sqrt{(\rho_0 + \sqrt{T_0})^2 + \frac{4}{3} \rho_0 \sqrt{T_0}} \right]^2 \quad (5a)$$

The conversion from dimensionless temperature T to physical temperature $\{T\}$ is given by

$$\{T\} = \frac{T}{a_1^2} \quad (5b)$$

The values of $T_{\max}$ obtained from TER and from Eq. 5a for both ρ_0 and $\rho_0 - \beta$ are given in Table 2. All the discussion of case 1 applies equally

Table 2 - MAXIMUM CORE TEMPERATURE

$[\rho = \rho_0 - a_1(\sqrt{T} - \sqrt{T_0}), T_0 = 547^\circ\text{F}, \beta = 0.0075]$

Excursion	$a_1, ^\circ\text{K}^{-1/2}$	ρ_0	$T_{\max}, ^\circ\text{F}$			Difference between ana-	
			TER 9	Analytic*		lytic* solution and TER 9 solution, %	
				$\rho_0 - \beta$	ρ_0		
						$\rho_0 - \beta$	ρ_0
1	0.001782	0.015	1536	1346	2312	13	40
2	0.001782	0.030	3451	3422	4922	0.8	35
3	0.001968	0.015	1430	1263	1808	12	23
4	0.001968	0.030	3115	3088	3968	0.9	24

* Prompt heating is indicated by $\rho_0 - \beta$; total heating is indicated by ρ_0 .

for any Doppler reactivity feedback, the most important point being that, if $\rho_0 - \beta$ is used instead of ρ_0 to compute $T_{\max}$, then the temperature obtained is the temperature of the core before the (very slow) delayed-neutron heating produces an appreciable effect. It must be emphasized again that, once this delayed heating becomes appreciable, so does the moderator feedback and that neither these TER calculations nor the

* For the derivation of Eq. 5a, see Appendix B.

theory takes this into account. Therefore the fact that the TER values are about 10% higher than the analytical ones for the excursions due to the smaller step insertions does not make the TER values more reliable; they are an overestimate if the moderator temperature coefficient is negative, as is the case in SPERT I (Special Power Excursion Reactor Test No. 1).⁴

$$\text{Case 3: } \rho = \rho_0 - a_1 \left(e^{a_2 \sqrt{T}} - e^{a_2 \sqrt{T_0}} \right)$$

Here the value of $T_{\max}$ cannot be obtained explicitly; it is given* by the highest of the two roots of the transcendental equation (in dimensionless variables):

$$\begin{aligned} \frac{\rho_0 + a_1 e^{\sqrt{T_0}}}{2a_1} T - e^{\sqrt{T}}(\sqrt{T} - 1) - \frac{\rho_0 + a_1 e^{\sqrt{T_0}}}{2a_1} T_0 \\ + e^{\sqrt{T_0}}(\sqrt{T_0} - 1) = 0 \end{aligned} \quad (6a)$$

The conversion from dimensionless temperature T to physical temperature $\{T\}$ is given by

$$\{T\} = \frac{T}{a_2^2} \quad (6b)$$

The values of $T_{\max}$ obtained from TER and from Eq. 6a for both ρ_0 and $\rho_0 - \beta$ are given in Table 3. The discussion of the two previous cases applies here also.

CONCLUSION

It has been shown that in step excursions two time scales are relevant in considering the heating of the core: that associated with the prompt-neutron heating and that associated with the delayed-neutron heating. If the time constant for heat transfer to the moderator were long enough (e.g., above 30 sec), then the analytical method presented here would yield results as reliable as those obtained with any computer program. The tables show that the delayed-neutron heating would be considerable if there were no moderator reactivity effects; however, if these effects are appreciable and are neglected in the calculations, the retention of the delayed-neutron heating does not make these calcula-

* For the derivation of Eq. 6a, see Appendix C.

Table 3 - MAXIMUM CORE TEMPERATURE

$$\left[\rho = \rho_0 - a_1 \left(e^{a_2 \sqrt{T}} - e^{a_2 \sqrt{T_0}} \right), T_0 = 547^\circ\text{F}, \beta = 0.0075 \right]$$

Excursion	a_1	$a_2, ^\circ\text{K}^{-1/2}$	ρ_0	$T_{\text{max}}, ^\circ\text{F}$			Difference between ana-	
				Analytic*			lytic* solu-	
							tion and TER	
				TER 9	$\rho_0 - \beta$	ρ_0	9 solution, %	
				$\rho_0 - \beta$	ρ_0			
1	0.2002	0.006	0.015	1688	1612	2809	4.6	50
2	0.231	0.006	0.015	1525	1461	2469	4.3	47
3	0.2002	0.006	0.025	3310	3288	4727	0.66	34

* Prompt heating is indicated by $\rho_0 - \beta$; total heating is indicated by ρ_0 .

tions any more reliable. In the case where heat transfer to the moderator is important, more elaborate computer programs than TER 9 are used at APED, where all pertinent effects such as moderator feedback and delayed neutrons are taken into account.

It must be emphasized that the maximum temperature reached in a very large excursion is *independent* of the prompt-neutron lifetime, as is shown in Eqs. 4 to 6. Therefore, in the context of the maximum energy release, accidents due to step excursions in fast reactors are not more severe than in thermal reactors.

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APPENDIX A

DETERMINATION OF THE TIME DEPENDENCE OF THE POWER AND TEMPERATURE

The procedure to determine the power and temperature as functions of time [$n(t)$ and $T(t)$] during the excursion is only mentioned briefly because it is a matter of numerical computation beyond the scope of this paper. In Ref. 1, after having eliminated the delayed neutrons by using the first integral (Eq. 2), we have solved a system of the form

$$\begin{aligned}\frac{dn}{dt} &= F_1(n, T) \\ \frac{dT}{dt} &= F_2(n, T)\end{aligned}\tag{A.1}$$

Here $F_1(n, T)$ is a nonlinear function of its arguments. The solution was obtained by the method of the phase-plane (power-temperature). Namely, the trajectories of Eqs. A.1 are given by the solution of the first-order nonlinear DE

$$\frac{dn}{dT} = \frac{F_1(n, T)}{F_2(n, T)}\tag{A.2}$$

and the particular solution corresponding to the initial condition of interest was obtained. This solution was given as two asymptotic series of the form

$$n(T, \epsilon) = \sum_{l=0}^{\infty} f_l(T, \epsilon)\tag{A.3}$$

whose respective ranges of validity were determined.

The time dependence of the temperature T can be obtained with Eq. A.1 (second expression) and Eq. A.3:

$$t = \int_{T_0}^T \frac{dT}{F_2\left[\sum_{l=0}^{\infty} f_l(T, \epsilon), T\right]}\tag{A.4}$$

Here t is the time after the beginning of the excursion and T_0 is the initial temperature. In Eq. A.4 we need only to use the first term of the series since this converges asymptotically. Therefore we determine $T(t)$ by solving numerically the definite integral of Eq. A.4. The time-dependent power can be determined directly from the temperature because the second expression in Eq. A.1 is in all cases $dT/dt = \gamma n$, and thus

$$n = \frac{1}{\gamma} \frac{dT}{dt}$$

It has been shown in the main text that, to compute the maximum temperature reached in the excursion, we do not need to obtain $n(t)$ and

$T(t)$. Therefore the procedure of this appendix is unnecessary in this respect.

APPENDIX B

CASE 2 $[\rho = \rho_0 - a_1(\sqrt{T} - \sqrt{T_0})]$: DERIVATION OF EQ. 5a

Substituting $f(T) = -a_1(\sqrt{T} - \sqrt{T_0})$ into Eq. 3,* we obtain A, and therefore the first integral (Eq. 2) becomes

$$n + C - \frac{\rho_0}{\gamma l} (T - T_0) + \frac{a_1 T}{\gamma l} \left(\frac{2}{3} \sqrt{T} - \sqrt{T_0} \right) + \frac{a_1 T_0 \sqrt{T_0}}{3\gamma l} = 0 \quad (\text{B.1a})$$

Equation B.1a is used to eliminate the variable C from Eqs. 1 of the main text, and the system in n and T equivalent to Eq. 3 of Ref. 1 is obtained. After defining the dimensionless power and temperature,

$$\{n\} = \frac{\lambda}{\gamma a_1^2} n \quad (\text{B.1b})$$

$$\{T\} = \frac{T}{a_1^2}$$

and dividing the first equation by the second equation of the system equivalent to Eq. 3 of Ref. 1, we obtain the differential equation corresponding to Eq. 12 of Ref. 1:

$$\frac{dn}{dT} = \frac{(\rho_0 - \beta)n - (\sqrt{T} - \sqrt{T_0})n + \rho_0(T - T_0) - T(2\sqrt{T}/3 - \sqrt{T_0}) - T_0\sqrt{T_0}/3}{\epsilon n} \quad (\text{B.2})$$

We recall from Ref. 1 that the maximum temperature reached in the excursion is the highest root of the equation obtained when the numerator of Eq. B.2 is set equal to zero with $n = 0$. Therefore

$$T_{\max} = \frac{9}{16} \left[\rho_0 + \frac{\sqrt{T_0}}{3} + \sqrt{(\rho_0 + \sqrt{T_0})^2 + \frac{4}{3} \rho_0 \sqrt{T_0}} \right]^2 \quad (\text{B.3})$$

*The initial power n_0 is in all cases 10^{-6} of the operating power and is neglected in Eq. 3.

APPENDIX C

CASE 3 $\left[\rho = \rho_0 - a_1 \left(e^{a_2 \sqrt{T}} - e^{a_2 \sqrt{T_0}} \right) \right]$: DERIVATION OF EQ. 6a

The procedure is formally the same as that in Appendix B. The equation corresponding to Eq. B.2 is in this case

$$\frac{dn}{dT} = \left\{ (\rho_0 - \beta)n - a_1 \left(e^{\sqrt{T}} - e^{\sqrt{T_0}} \right) n + \rho_0(T - T_0) - 2a_1 \right. \\ \left. \times \left[e^{\sqrt{T}}(\sqrt{T} - 1) - e^{\sqrt{T_0}}(\sqrt{T_0} - 1) \right] + a_1 e^{\sqrt{T_0}}(T - T_0) \right\} / \epsilon n \quad (C.1)$$

Setting $n = 0$ in the numerator and making this equal to zero, we obtain the transcendental Eq. 6a:

$$\frac{\rho_0 + a_1 e^{\sqrt{T_0}}}{2a_1} T - e^{\sqrt{T}} (\sqrt{T} - 1) - \frac{\rho_0 + a_1 e^{\sqrt{T_0}}}{2a_1} T_0 \\ + e^{\sqrt{T_0}} (\sqrt{T_0} - 1) = 0 \quad (C.2)$$

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TREAT Experiments to Provide Data for Fast Reactor Safety Kinetics Calculations

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ABSTRACT

For kinetics calculations on the behavior of fast reactors during hypothetical meltdown accidents to be performed, data must be obtained on such phenomena as fuel-failure thresholds, sequences of fuel motion, pressure transients, coolant vaporization, and coolant movements. The Transient Reactor Test facility (TREAT) was designed and built specifically for experimental investigation of these phenomena under transient conditions. For the time scales of meltdown accidents, for which shock effects are not important, typical time spans for clipped-transient meltdown experiments in TREAT are comparable to the critical time constants of physical phenomena in fast reactor fuel-element failure. These phenomena include cladding-penetration rates, fuel-expulsion trajectories, and inertial interactions of adjacent elements.

Data developed from TREAT experiments were used to make calculations of hypothetical accidents occurring in a core similar to that of the Experimental Breeder Reactor No. II (EBR-II); these calculations take into account variations, over the reactor core, of power and coolant flow. These variations can account for a reduction of an order of magnitude or more in the reactivity-insertion rates estimated from the failure of a single fuel pin.

From photographic experiments significantly different modes of failure and failure thresholds were found for UO_2 specimens of fast reactor fuel size. Similar samples of US, when exposed to comparable transient conditions, demonstrated resistance to mechanical damage

superior to that of the uranium oxide. These differences in behavior can be explained on the basis of physical properties.

Transient expansion and modes of failure of irradiated metallic fuel are more complex than those of unirradiated specimens. However, EBR-II type irradiated samples exhibit some failure kinetics behavior in common with unirradiated pins. An unrestrained specimen displayed transient expansion under heating during a TREAT burst which was the same, within experimental uncertainties, as that reported for unirradiated material up to about 650°C. Thus, although irradiated metal fuel shows sufficient swelling under meltdown conditions to make substantial changes in the course of meltdown incidents, rates of swelling must be properly taken into account in accident conditions.

INTRODUCTION

Data on such phenomena as fuel-failure thresholds, sequences of fuel motion, pressure transients, coolant vaporization, and coolant movements must be obtained to perform kinetics calculations on the behavior of fast reactors during hypothetical meltdown accidents, in which motion of core materials can contribute significantly to the reactivity. Because of the complexity and the interrelations of the possible mechanisms involved in meltdown, these mechanisms are being studied in a program centering on in-reactor experiments being performed in TREAT.¹

TREAT was designed and built specifically for experimental investigation of these phenomena under representative transient conditions, rather than as a safety evaluation of a reactor concept.² The kinetic analyses of transient behavior of TREAT have been reported.³ Although the shortest asymptotic reactor period attainable with the current core loading under present operating restrictions is 40 msec, the machine is appropriate for the study of accidents that may have typical time scales of the order of fractions of a second to tens of seconds. Furthermore, time spans for TREAT transients can be made short enough to be comparable to critical time constants of physical phenomena in fast reactor fuel-element failure. These phenomena include thermal time constants for ceramic fuel elements, rates of cladding penetrations by molten metals, fuel-movement trajectories, and inertial interactions of adjacent elements.

The basic philosophy of this experimental program is to separate out the basic physical processes of possible interest and study them, rather than to concentrate on mocking up specific accident conditions. Proof tests, e.g., measurement of the time at which a fuel element will fail, are made to check reliability and applicability of data. Experiments

were conducted initially on single metallic-uranium fuel samples¹ that were exposed to reactor transients in an inert gas in opaque capsules. The program has since been expanded to include the survey of the effects of clustering; the behavior of the ceramic fuels uranium dioxide,⁴ uranium monosulfide,⁵ and uranium monocarbide; the changes in metallic-fuel meltdown brought about by prior irradiation to appreciable fission-gas inventories;^{6,7} and the behavior of a thorium-based uranium fuel.⁸ Fuel, both unirradiated and irradiated, has been exposed in transparent containers that permit observation by high-speed color motion-picture photography;⁹ in addition, unirradiated samples have been run in stagnant sodium and in flowing sodium.¹⁰ A key phase of the program consists in the theoretical study of related phenomena, analysis of experimental results, and calculations relating results to reactor-safety analyses.

COHERENCE OF FAILURE CALCULATIONS

Although a fast reactor meltdown could lead to a reduction in reactivity, under certain circumstances fuel failure could add reactivity. The simultaneous or near-simultaneous failure of large numbers of fuel pins could produce excessive rates of reactivity addition. A study has been made of the coherence of failure for three hypothetical meltdown accidents in a reactor of the EBR-II type.¹¹ Calculations were made from a point reactor model, with temperature-dependent feedback data for an intact core. Provision was made for fuel-plate change, flowing coolant, coolant-flow decays, and coolant-flow blockage. Power calculated from this model was then used to calculate temperatures as a function of time for various channels throughout the core.¹² In the individual channel calculations, effects taken into account included radial and axial power variations, coolant-flow changes from subassembly to subassembly, and variations in effective coolant cross-section area for different pins inside a single subassembly.¹²

On the basis of experimental results,^{1,13,14} the threshold for short-term failure of an EBR-II pin was taken to be a temperature of 1015°C at the fuel-cladding interface. Failure was attributed to the dissolution of the cladding and the subsequent expulsion of molten fuel alloy by the vapor pressure of the sodium that was present inside the pin to provide a thermal bond between the fuel and the cladding and by the argon that was present inside the expansion space in the top of the element. In-reactor measurement of the time of penetration of the cladding was about 1 sec,¹⁴ which is in good agreement with isothermal-furnace results.¹⁵ At 1200°C, failure is from bursting of the cladding, and the delay for penetration does not occur.

Figure 1 shows maximum fuel-cladding interface temperatures calculated for complete pumping-power failure at constant reactor power. Temperatures are given for internal and peripheral pins in the first-row (central) subassembly and for the internal pin, the peripheral pin nearest the core center, and the peripheral pin farthest from the core center in

Fig. 1 - Maximum fuel-cladding interface temperatures for pumping-power failure at constant reactor power.

the fourth-row subassembly. The difference in the time it takes the two internal pins to attain the 1015°C threshold is about 1 sec. Spreads within a subassembly are even larger, about 5 to 6 sec. (Because of possible propagation of failure from pin to pin, a maximum time of about 2 sec probably should be used within a single subassembly.) Slower rates of temperature rise occur if feedback is included.

Maximum fuel-cladding interface temperatures calculated for the internal and peripheral pins of the central (row 1) subassembly are shown in Fig. 2 for two different inlet-orifice blockages for constant power. For an instantaneous block there is a time spread of about 0.15 sec before 1015°C is reached. A linear-flow decay from full flow to no flow occurring over 1 sec displaces the curves but changes the time spread only 0.03 sec. For this accident the degree of coherence in failure is dominated by the power variations between subassemblies and the experimentally observed variations in penetration times.^{14,15} Figure 3 shows the time spread over the reactor, for peripheral pins in four different

Fig. 2 - Maximum fuel-cladding interface temperatures for two assumptions on flow blockage at constant reactor power.

rows, for full feedback. Figure 4 is a qualitative graph of relative reactivity change as a function of time produced by failure of central subassembly pins and fourth-row subassembly pins for blocked flow.

Although this study does not include the possible feedback effects from material motions, it does indicate that significant time variations in a failure can be calculated if realistic spatial variations in reactor parameters and time-dependent meltdown effects obtained experimentally are taken into account.

Fig. 3 - Maximum fuel-cladding interface temperatures for peripheral pins in four subassembly positions for full feedback when instantaneous flow blockage occurs.

Fig. 4 - Qualitative effect of variations in time at which pins reach failure threshold when the orifice is blocked and the reactor power is constant.

CERAMIC-FUEL EXPERIMENTS

Inert-gas-bonded UO_2 specimens of EBR-II size fuel were run in an inert-gas atmosphere in TREAT.⁴ The uranium oxide samples were approximately 0.381 cm in diameter, stacked in lengths of about 36 cm and clad with EBR-II size tubing 0.442 cm in outside diameter and about 0.025 cm in wall thickness. In general, failure was nonviolent, and failure of steel-clad samples occurred near the cladding melting point in reasonable agreement with predictions of failure from internal pressure of the gas bond. Figure 5 is a sequence of still photographs taken from the motion-picture record of the failure of a stainless-steel-clad UO_2 pin. The initial failure (part b of Fig. 5) is a release of hot gas through a small cladding break. Subsequent photographs show the collapse, under influence of gravity upon melting of the cladding. High-temperature failures of tantalum-clad specimens resulted in vigorous expulsion of molten and fragmented fuel by internal gas pressure. Maximum central fuel temperatures above approximately 2200°C produced a central zone of granular material; higher temperature exposures tended to cause the oxide cylinders to crack into shorter axial lengths.

Similar specimens of US fuel showed essentially no mechanical damage but did display appreciable plastic deformation. The general features of the US behavior and the differences between the US and the UO_2 samples can be explained on the basis of physical properties.⁵ Figure 6 shows US pellets that reached an estimated maximum temperature of 2440°C (melting point¹⁶ of US is 2462°C). Selective melting of the

Fig. 5 - Still photographs showing the failure of a steel-clad EBR-II size UO_2 sample (sample on left; mirror image on right). (a) Pretransient condition. (b) Failure. (c), (d), and (e) Collapse of element.

fuel structure, flow, and solidification against the tantalum cladding are clearly apparent. Mechanical damage and central voids are absent.

RECENT EXPERIMENTAL RESULTS

Investigation of the behavior of ceramic fast reactor fuels has continued, with transient studies on UC clad with niobium-1 wt. % zirconium. Failure phenomena studied include mechanical stresses, melting, and pressure. Figure 7 shows the posttransient condition of a UC specimen that failed as a result of radial pressure exerted on the

Fig. 6 — Appearance of US pellets after TREAT exposure (maximum estimated fuel temperature of 2440°C).

cladding by pieces of ceramic that had fractured because of thermal expansion during a transient axial loading.

Experimentation on previously irradiated metallic fuel pins showed density decreases upon meltdown sufficiently large to produce significant changes in reactivity.^{6,7} The competition between failure mechanisms, and the transient expansion of irradiated uranium-5 wt. % fission alloy, the EBR-II Mark I fuel, have been studied in recent experiments.¹⁷

Two unclad samples were run for expansion studies, one with a heating rate about $80^{\circ}\text{C}/\text{sec}$ and one with a rate about $1000^{\circ}\text{C}/\text{sec}$ over the temperature range of interest. Up to approximately 700°C the transient expansion is the same (within experimental uncertainties of about $\pm 15\%$) as that predicted on the basis of data for unirradiated material.¹⁸ Expansion at higher temperatures was appreciably greater, and the final residue of fuel that melted had densities only

Fig. 7 - Appearance of niobium-alloy-clad UC element after failure produced by transient mechanical loading of ceramic fuel.

a fraction of the original value. Motion-picture photographs taken during the experiments on preirradiated metallic fuel indicate that neither the threshold of gross failure due to dissolution of the cladding nor the time required for penetration is changed significantly. However, at a maximum cladding temperature of approximately 1000°C , small localized failures may occur, limited amounts of vapor being emitted. When gross failure does occur, it produces a vigorous ejection of fuel under the influence of internal pressure, just as for unirradiated specimens.¹

A study has begun of the phenomena of meltdown in flowing sodium. On the basis of available data, the threshold for EBR-II Mark I specimens occurs at a fuel-cladding interface temperature (about 50°C) close to that for specimens run in an inert-gas atmosphere. The details of failure are different from those for the inert-gas environment because of different temperature distributions and time scales; it appears that the key require-

(a)

(b)

(c)

(d)

Fig. 8 - EBR-II Mark I specimen carried almost to the failure threshold in a flowing-sodium loop. (a) Close-up of surface of fuel cylinder showing solidified fuel-cladding eutectic alloy that flowed inside the cladding without penetrating it. (b), (c), and (d) Close-ups of the fuel cylinder showing surface cracks containing solidified fuel-cladding eutectic.

ment is that the interface temperature become high enough that the region of the fuel near the surface is not strong enough to contain the molten central fuel, thus permitting the inner fuel to escape and attack the cladding locally. Figure 8 shows four close-up photographs of the uranium fuel cylinder of an EBR-II pin carried almost to the failure threshold in a flowing-sodium loop experiment. Part a of Fig. 8 shows solidified fuel-cladding eutectic alloy that flowed downward inside the annulus between fuel and cladding and froze against the cladding without penetration. Parts b, c, and d of Fig. 8 show surface cracks filled with fuel-cladding eutectic formed by reaction of high-temperature fuel (maximum recorded central fuel temperature was 1285°C) with the cladding.

Half-length EBR-II pins¹⁸ with thorium-20 wt. % uranium metallic fuel clad with stainless steel have been run in inert-gas atmosphere recently.⁸ The threshold for cladding failure was found to occur at a cladding-surface temperature of approximately 1350°C , in reasonable agreement with a bursting-pressure estimate of 1312°C made for an isothermal sample model. A sample with maximum cladding temperatures of 1315°C and 1350°C recorded from two thermocouples did not fail but showed evidence of selective melting and flow of a molten phase that solidified against the inside surface of the cladding without penetrating it. A sample elevated just above the failure threshold showed greater fuel-cladding alloy formation and minor fuel flow, apparently caused by gravity alone, from the breached cladding. Two specimens tested to extensive failure also gave evidence of fuel motion predominantly caused by gravity.

The high-speed motion pictures taken of the tests showed high-velocity release of sodium at the time of cladding failure. However, because the fuel was still relatively intact, it was not accelerated by the sodium. Thus one mechanism for producing relatively rapid movement of fuel upon meltdown apparently is not operative for the steel-clad sodium-bonded thorium-uranium metallic-fuel type. This behavior might be modified significantly under different exposure conditions.

The thorium-uranium fuel behavior is illustrated in Figs. 9 to 11. Figure 9 shows the appearance of sample 3 with dissolution of cladding along the fuel, a localized rupture at the bottom (apparently due to internal pressure), and drops of material that had run down the inside of the cladding and solidified at the region of rupture. Sample 4, displayed in Fig. 10, has suffered somewhat greater cladding dissolution. Figure 11 is a photograph of two samples that were run to extensive failure. Although sodium was expelled vigorously from the cladding at the time of failure, the predominant mode of fuel motion was downward, under gravity.

Fig. 9 - Posttransient appearance of thorium-uranium sample 3. Thermocouple wires are still attached to the cladding.

Fig. 10 - Posttransient appearance of thorium-uranium sample 4. Thermocouple wires are still attached to the cladding.

Fig. 11 - Posttransient appearance of two thorium-uranium samples run to extensive failure.

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Operating Characteristics of the WSMR Fast Burst Reactor

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ABSTRACT

The White Sands Missile Range Fast Burst Reactor is an enriched uranium-molybdenum alloy assembly, unmoderated and unreflected, which provides a routine output of 5×10^{16} fissions per burst with a pulse width at half maximum power of 46 μ sec. A description of the reactor stand and core designs and the results from the first two months of operation are given. Effects on the core and reactor stand of the thermo-mechanical shocks generated during burst operations are described. Characteristics of burst yield vs. reciprocal period, burst width vs. reciprocal period, and reciprocal period vs. reactivity are given.

INTRODUCTION

The design of the Fast Burst Reactor (FBR) core at the White Sands Missile Range (WSMR) was based on the Oak Ridge National Laboratory Health Physics Research Reactor (HPRR).¹ Criticality tests and approximately fifty burst operations at Los Alamos Scientific Laboratory (LASL) indicated the adequacy of the core design. The LASL tests also gave the first indications that the maximum burst yield would be determined by the ability of the core and the reactor stand to withstand the massive shocks produced during a burst. On the basis of HPRR data,

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it had been thought that temperature limitations based on metallurgical phase changes and on the melting temperature of the fuel would determine the maximum burst yield.

The FBR has been operated at WSMR since September 1964. Over 300 bursts* have been accomplished during the conducting of a great variety of experiments. The reactor is used primarily to test missile system and component behavior in short-duration high-intensity radiation fields. Since most of the experiments are located within a few centimeters of the reactor fuel, the effects on reactivity and burst characteristics of a variety of reflecting materials have been carefully studied. The results of these studies are presently being prepared for publication.† This paper is limited to reporting the basic operational characteristics determined during the first two months of operation and to describing qualitatively the shock effects observed at LASL and at WSMR.

DESCRIPTION OF REACTOR CORE

The FBR is an unreflected, unmoderated, enriched (93.2% ^{235}U) uranium assembly. The fuel material is uranium alloyed with 10 wt. % molybdenum. The main fuel assembly (Fig. 1) consists of a stack of six stepped annular ring sections that form a right circular cylinder 20.6 cm in diameter and 19.3 cm high. The fuel sections are fastened together and to a 1.27-cm-thick stainless-steel support plate with three 1.9-cm-diameter Inconel X-750 bolts. A 10.16-cm-diameter safety-block cavity is concentric with the main fuel-assembly axis. The three cavities for the two control rods and the burst rod are 2.65 cm in diameter and are displaced at 120° on a 7.95-cm radius, as measured from the axis of the assembly.

The safety block is a right circular cylinder of U-10 wt. % Mo alloy that fits into the center cavity of the core. It is 9.75 cm in diameter and 14.6 cm long. A 3.17-cm-diameter stainless-steel support shaft extends through the center of the safety block.

The two control rods and the burst rod are also made from uranium-molybdenum alloy. The control rods are 2.54 cm in diameter and 16.5 cm long. The burst rod is 2.54 cm in diameter and 13.0 cm long. The uranium-molybdenum center insert in the top fuel sections contains a stainless-steel mass adjustment ring that may be removed to compensate for large additions of reactivity caused by the location of experimental

* At the time of publication, over 600 bursts had been accomplished.

† The results of these studies have been published; see R. L. Long, Effects of Reflectors on the Burst Characteristics of the WSMR Fast Burst Reactor, *Trans. Am. Nucl. Soc.*, 8: 2 (1965).

Fig. 1 – Isometric Section of FBR core.

setups external to the core. Table 1 gives the masses of the various fuel components. All exposed surfaces of the main fuel sections and the control components are plated with 0.005 cm of nickel.

DESCRIPTION OF REACTOR STAND

Figure 2 is a photograph of the reactor stand. The lower part of the stand is 61.0 cm square and 81.5 cm high and consists of a forklift-truck pallet, four 10.2-cm-diameter aluminum-tubing support columns, and two deck plates. Contained in the lower part of the stand are the safety-block pneumatic and electromechanical drives, the safety-block scram-magnet housing, the neutron-source drive and neutron-source shield, the burst-shape photodiode, the pneumatic pressure regulators, cooling-air manifolds, and solenoid valves for operating the pneumatic systems.

The upper part of the reactor stand is 28.3 cm in diameter and 79.5 cm high and consists of three sets of three equally spaced stainless-steel rods that separate a spider plate, a middle accessory plate, and the fuel support plate. Contained in the upper part of the stand are the control-rod

Table 1 - MASSES OF FBR FUEL COMPONENTS

Component	Uranium-molybdenum and plating mass, g
Bottom fuel section	11962
Center section, bottom	12038
Center section, middle	12050
Center section, top	12040
Intermediate section	12813
Top section	12350
Safety block	16530
Center insert	3511
Burst rod	1082
Control rod	1384
Control rod	1382
Total	97142

Fig. 2 - FBR stand and fuel assembly.

electromechanical drives and the burst-rod pneumatic drive cylinder. The guide bushings for the control elements are mounted in the middle accessory plate, and when the elements are withdrawn from the core they rest in the space between the middle accessory plate and the fuel support plate.

During normal operations both the lower and the upper parts of the reactor stand are enclosed with aluminum dust covers. The fuel is enclosed in a cadmium-plated safety shield that acts as a manifold for forced air cooling of the reactor core.

REACTIVITY-WORTH MEASUREMENTS

After assembly of the reactor core at WSMR, the integral and differential rod worths were determined from measurements of the positive stable reactor period for incremental changes in rod position. The integral and the maximum differential worths of a control rod were determined to be \$1.60 and \$0.365/in., respectively. The integral and the maximum differential worths of the burst rod were determined to be \$1.11 and \$0.36/in., respectively.

After the calibration of the control rods, reactivity worths of the various reactor-core components were determined, and these results are summarized in Table 2. All measurements made at WSMR were in good agreement with those made during the LASL criticality tests. As indicated in Table 2, the first 2.54 cm of withdrawal of the safety block

Table 2 - REACTIVITY WORTHS OF FBR COMPONENTS

Component	Reactivity worth, \$
Control rod	1.60
Burst rod	1.11
Uranium-molybdenum mass adjustment ring	2.91
Steel mass adjustment ring	1.26
0.635-cm steel retaining plate	0.41
Hex nut and washer	0.024
Uranium-molybdenum bolt, two washers, and nut	1.20
Inconel bolt, two washers, and nut	0.54
First 2.54 cm of safety-block withdrawal	3.79
Safety shield	0.65

was measured to have a reactivity worth of \$3.79. The total shutdown worth of the safety block was calculated by T. Wimett of LASL to be \$30. The calculation was made with a DDK computer-code method with a normal mode source assumed.

BURST CHARACTERISTICS

Figures 3 and 4 are best-line fits to the experimental data obtained from the first 55 bursts of the FBR at WSMR. Total fission yield was determined from radiochemical analysis of oralloy foil placed on the safety block, from heat calculations based on temperature rise, and from sulfur-threshold flux measurements made at 1 m from the core center line. Determination of the yield from fission foil and heat calculations was based on fission density distributions calculated with a LASL two-dimensional DDK computer code. Yield determinations by the three methods agreed within $\pm 15\%$. A value of 2.45×10^{14} fissions in the

Fig. 3 - Burst yield vs. reciprocal period.

Fig. 4 - Burst width at half power vs. reciprocal period.

entire core per degree centigrade maximum temperature rise was used to obtain the temperature scale in Fig. 3.

Data points from which the best-line fit of Fig. 3 was drawn indicated a scatter of $\pm 15\%$ from the line. This scatter was probably due to two factors. The first was the uncertainty in the measurement of the initial reactor period as obtained from oscilloscope photographs of the burst-shape photodiode output. The second was the increase in the temperature rise for bursts that had a 9- to 10-msec plateau. Examination of the burst-shape photographs indicated that, for most of the bursts, the safety-block scram-magnet armature plate was separated from the magnet as a consequence of shocks generated during the burst. These same shocks were large enough to change the position of the safety-block-seated switch actuators so that for some bursts the safety-block mandrel and fuel support plate were not making physical contact. On bursts where contact was not made, the scram-magnet armature plate was not released until the fast-level scrams were triggered. This scram time is of the order of 10 msec. The increase in total number of fissions contributed by the plateau gave a temperature rise that was 10 to 15°C higher for a given reciprocal period.

The largest burst yield of 8×10^{16} fissions obtained at WSMR corresponded to a maximum temperature rise of 325°C and had a burst width at half maximum power of $37 \mu\text{sec}$.^{*} The largest burst yield of 1.2×10^{17} fissions obtained at LASL corresponded to a maximum temperature rise of 450°C and had a burst width of $35 \mu\text{sec}$. For burst yields greater than 7×10^{16} to 8×10^{16} fissions, various shock effects were observed. These effects are described in the next section.

From graphs of reciprocal period as a function of reactivity for both indoor and outdoor operation of the FBR, the measured change in reciprocal period per dollar reactivity above prompt critical was determined to be $(0.77 \pm 0.03) \times 10^6 \text{ sec}^{-1}$ where the reactivity calibration was based on positive-period measurements made below prompt critical. This measurement was in good agreement with LASL measurements of the Rossi alpha made at delayed critical with the equipment described by J. D. Orndoff.² Although the slopes of the indoor and outdoor graphs were the same, the burst-rod worth was increased by \$0.01 for outdoor operations. This shift was the result of a change in the effective delayed-neutron fraction between the indoor and outdoor sites. At the indoor site, the effective delayed-neutron fraction is higher because of an apparent delayed-neutron group created by neutrons that are scattered back to the reactor from the concrete walls, floor, and ceiling of the reactor cell.

Peak fission rate as a function of reciprocal period is plotted in Fig. 5. These data were obtained from a planimeter measurement of the burst-shape area as seen on enlarged photographs of the photodiode output. The area was equated to the total burst yield to permit calculation of the peak fission rate for each burst. For the burst of 8×10^{16} fissions, J. Snyder of Sandia Corporation measured a peak fission rate of 2.3×10^{21} fissions/sec with a Sandia Corporation calibrated fission chamber. This measurement was in good agreement with the fission rates determined from the burst-shape photographs. The break from a square law to a fourth-power dependence on reciprocal period is apparent.³

Figure 6 is a plot of FBR fission yield as a function of reactivity above prompt critical. The data from yields of 5×10^{15} to 8×10^{16} fissions were fitted to the first-order equation suggested in Ref. 3. This equation was then used to extrapolate the data to yields of 10^{19} total fissions. The characteristic time for pressure waves to travel through the fuel can be determined from the first-order equation. This time was found to be $12 \mu\text{sec}$, as compared with the value of $14 \mu\text{sec}$ determined in the LASL tests and $9 \mu\text{sec}$ for the Godiva I reactor. With the experimental value of 5.1×10^{14} fissions per average degree centigrade temperature rise, the dynamic temperature coefficient was calculated to be $\$0.0025/^{\circ}\text{C}$.

^{*} Burst yields have now been obtained at WSMR at yields up to 1.1×10^{17} fissions and burst widths of $31 \mu\text{sec}$.

Fig. 5 — Peak fission rate vs. reciprocal period.

From changes in delayed-critical rod positions as a function of fuel temperature, the static temperature coefficient was determined to be $0.0030/^{\circ}\text{C}$.

THERMAL-SHOCK EFFECTS

As indicated in the previous section, burst operations at both LASL and WSMR revealed evidence of massive shocks produced during burst

Fig. 6 - Fission yield vs. initial reactivity.

operations. For routine operations the burst yield of the FBR had to be limited to yields less than 7×10^{16} fissions because of the probability of producing permanent damage to the reactor fuel and stand at higher yield levels.

At LASL 10 bursts were obtained at yields greater than 7×10^{16} fissions. A number of damage effects were observed. The original 0.318-cm-thick stainless-steel retaining plate used to hold the center insert and mass adjustment ring in place was warped upward approximately 0.32 cm so that the plate looked as if it had been hit with a sledge hammer. The three 10-32 bolts holding the retaining plate were bent and stretched and had to be replaced several times. The 0.318-cm retaining plate was replaced with a plate 0.635 cm thick and, for bursts greater than 1×10^{17}

fissions, the thicker plate was also warped upward as much as 0.085 cm. On the maximum burst of 1.2×10^{17} fissions, one of the 10-32 screws sheared completely.

During the first two months of operation at WSMR, 3 bursts greater than 6×10^{16} fissions, 20 bursts between 4 and 6×10^{16} fissions, and 35 bursts less than 4×10^{16} fissions were obtained. Stretching of the 10-32 retaining-plate bolts was observed during all the higher yield bursts. Two successive bursts of 7×10^{16} and 8×10^{16} fissions stretched the 1.27-cm-long bolts to a length of 1.43 cm. Prior to the opening of the FBR facility for use by experimenters, a number of modifications were made on the FBR stand and fuel assembly. One of these was the enlargement of the tapped holes in the top fuel section for use of $\frac{5}{16}$ -in. 18-thread high-tensile-strength retaining-plate bolts.

The core assembly bolts had to be tightened after each of the larger bursts at LASL. To give some idea of the force on the bolts, a 3.81-cm-diameter metal washer was placed on the head of one of the bolts during several burst operations. During a burst of 2×10^{16} fissions, the washer was observed with closed-circuit television to jump about 2 cm into the air. For double this burst size, the washer jumped about 30 cm into the air. For a burst of 10^{17} fissions, the washer jumped an estimated 2 to 3 m into the air and probably hit the roof of the reactor kiva. The maximum LASL burst caused a net elongation of approximately 0.048 cm in each of the three core assembly bolts.

After the maximum LASL burst, the core was disassembled and carefully examined. This examination revealed that the 1.27-cm-thick stainless-steel fuel support plate had been permanently bowed downward approximately 0.13 cm. The bowing was symmetrical with respect to the three assembly bolts. A press force of $7\frac{1}{2}$ tons was required to flatten the plate. As a result of this distortion, the diameter of the fuel support plate increased 0.015 to 0.020 cm in diameter.

In the tests at LASL, there was a gap of approximately 0.075 cm between the bottom of the fuel support plate and the top of the safety-block support mandrel flange. The flange showed definite marks of "pounding" from the deflection of the fuel support plate during a burst. At WSMR this same effect was observed, as evidenced by the release of the scram-magnet armature plate on all bursts where contact was maintained between the fuel support plate and the safety-block mandrel. It was estimated that this release occurred about 250 μ sec after the peak of the burst.

On the FBR stand used at WSMR, the seated position for the safety block was determined by limit switches that were actuated by a small metal yoke clamped to the safety-block drive shaft. This yoke was slipping upward on the drive shaft as a consequence of its own inertia and the downward acceleration of the safety block following a burst.

Thus, on an operation following a large burst, the seated switches would stop the safety block with a gap of 0.075 to 0.125 cm between the mandrel and the fuel support plate. As mentioned previously, when this gap existed the scram-magnet armature was not released until the fast-level scrams triggered. A 10-msec plateau then followed the burst peak and produced an extra 10 to 15°C temperature rise. The yoke was modified to prevent movement.

Other effects observed which give an indication of the magnitude of the shock effects were (1) the distortion of the rod-hole liners from radial shock movements, (2) the Brinelling of the control-rod-drive lead screws from forces transmitted to the ball nut during a burst, and (3) the peculiar triangular oxidation patterns on the fuel section interfaces.

Since the evidence obtained on the FBR indicated that shock effects rather than temperature limits determined the maximum-yield output on cores similar to the FBR, studies are in progress at WSMR and Sandia Corporation to obtain quantitative measurements of the magnitude of these shock forces. In the engineering design of future fast-burst reactors, such things as supporting of the fuel assembly at its center of mass and shock mounting of the assembly bolts should be considered.

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Characteristics of Large Reactivity Insertions in a High-performance TRIGA Uranium-Zirconium Hydride Core

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ABSTRACT

Reactivity insertions up to \$4.60 (3.22% $\delta k/k$) have been performed safely in the Advanced TRIGA Prototype Reactor (ATPR), located at General Atomic. This work was done as a continuing program to determine the operational capabilities of a TRIGA uranium-zirconium hydride core.

The large reactivity insertions, with the associated fuel temperatures and pressures, were made with TRIGA fuel elements. This fuel uses a uranium-zirconium hydride alloy with a hydrogen/zirconium atomic ratio of 1.7 and is clad with 20-mil type 304 stainless steel. The U-ZrH_{1.7} alloy has the property of being single phase well above 1000°C.

The characteristic reactor parameters for pulsed reactors such as peak reactor power, reactor period, pulse width, energy release, fuel temperatures, and pressure within the fuel can were measured as a function of reactivity insertion. These measurements compare favorably with calculated values as predicted by the Fuchs-Nordheim model with variable heat capacity as modified by R. Scalettar. Also of interest is the fact that the experimental pulses imply an average negative temperature coefficient for the uranium-zirconium hydride reactor system in agreement with the calculated value.

For a measured fuel temperature of 700°C in the hottest fuel element, the following reactor performance was obtained: reactivity insertion, \$4.60; peak power, 6480 Mw; energy release, 44.5 Mw-sec; re-

actor period, 1.68 msec; pulse width at one-half peak power, 6.2 msec; measured fuel temperature, 700°C; mass of fuel-moderator alloy, 200 kg.

Experimental evidence indicates that, for the preceding reactivity insertions, film boiling has occurred in isolated spots on some of the hotter fuel elements. Computer calculations confirm the possibility that film boiling may occur under some circumstances.

It has been shown by this program that it is safe and feasible to operate the reactor with reactivity insertions of the magnitude reported here.

5.

INTRODUCTION

In a continuing program to determine the operational capabilities of a TRIGA uranium-zirconium hydride core, the ATPR¹ has been safely subjected to reactivity insertions up to \$4.60 (3.22% $\delta k/k$). These large reactivity insertions were made with an improved-design TRIGA fuel element clad in stainless steel. The same core has been repetitively pulsed to peak powers of 6500 Mw as well as operated at 1.5 Mw steady state. The purpose of this paper is to report and discuss the results obtained during the recent pulsing program in which more than 400 pulses have been made for reactivity insertions above \$4.00 and up to \$4.60.

DESCRIPTION OF CORE

The TRIGA fuel uses a zirconium hydride moderator having a hydrogen/zirconium atomic ratio of about 1.7 homogeneously combined with uranium. The uranium fuel loading is 8 to 8.5 wt. % of the total element weight enriched to 20% in ²³⁵U. The fuel-moderator material is sealed in type 304 stainless-steel cladding having a wall thickness of 0.020 in. Spaces between the internal fuel-element components and the cladding are filled with air at ambient temperature and pressure. Table 1 gives some pertinent specifications for the fuel-moderator element. The advantage of the U-ZrH_{1.7} alloy lies in the fact that it is single phase well above 1000°C. With zirconium hydride having hydrogen content less than 57 at. %, a phase change and an associated volume change occur at approximately 530°C. Figure 1 shows these relationships.² A potential limitation of the increased hydrogen/zirconium ratio is the increased equilibrium hydrogen pressure that results when the alloy is subjected to temperatures greater than 900°C. However, it should be noted that even for a temperature of 800°C the predicted equilibrium hydrogen pressure is only 2.5 atm and, further, when the alloy is cooled, rehydriding will occur so that no residual hydrogen is left in the fuel-

Table 1 - SPECIFICATIONS FOR TRIGA FUEL-MODERATOR ELEMENT

Overall length	28.37 in.
Outside diameter	1.47 in.
Fuel outside diameter	1.43 in.
Fuel length	15.0 in.
Weight of ^{235}U	36.5 g (average)
Uranium content	8 wt. %
Uranium enrichment	20%
Hydrogen/zirconium atomic ratio	1.68 (average)
Cladding material	304 stainless steel
Cladding thickness	20 mils

cladding gap. Figure 2 gives values for the dissociation-pressure isochores of zirconium hydride.³

The reactor core is composed of 87 to 123 (depending on the excess reactivity desired) of these fuel elements, positioned in concentric rings and supported by upper and lower grid plates. This configuration and the placement of control rods is shown in Fig. 3. Also noted in this figure is the placement of fuel elements instrumented to measure fuel temperature and equilibrium hydrogen pressure in the fuel can. The core interstices between fuel elements are water filled to give a metal to water ratio of 2.

CHARACTERISTIC PULSED REACTOR PARAMETERS

Some of the characteristic parameters of interest for a pulsed reactor system are (1) peak reactor power, (2) energy release during transient, (3) prompt reactor period, (4) pulse width, and (5) fuel temperature. These parameters are a function of reactivity insertion.

For transient operation the reactor is first brought to criticality at approximately 50 watts with one or two transient rods in the core, depending on the reactivity insertion needed. The reactor transient is produced by the rapid withdrawal of the transient rods. When the rods are removed, the reactor power increases to a value that produces a fuel temperature that compensates for the excess reactivity inserted; this is the peak reactor power. The fuel temperature continues to rise with an increasing loss of reactivity, and the reactor power decreases to a

Fig. 1 - Zirconium-hydrogen phase diagram. [From R. L. Beck, *Trans. Am. Soc. Metals*, 55: 542-555 (1962).]

comparatively low steady-state level. This final steady-state power level depends on the reactivity insertion and the heat-transfer characteristics of the fuel element. The steady-state power level after the transient is referred to as the after-pulse "tail." Approximately 2 sec after initiation of the transient, the reactor is scrammed for operational convenience. A typical pulse is shown in Fig. 4.

The peak reactor power during the transient can be described with the use of a modification by Scalettar⁴ of the Fuchs-Nordheim model of

Fig. 2 – Dissociation-pressure isochores of zirconium hydride.³

reactor response to step reactivity insertions. A linearly varying heat capacity for the reactor system is considered in the following equation:

$$P_{\max} = \left[P_0 + \frac{(\delta k_p)^2 (C_0 + \gamma T_0)}{2\alpha l} \right] + \frac{1}{6} \frac{(\delta k_p)^3 \gamma}{\alpha^2 l}$$

where δk_p = prompt reactivity insertion

$P_{\max}$ = peak reactor power, watts

T_0 = average initial temperature of core above 25°C

C_0 = initial heat capacity of the fuel-moderator (1088 watt-sec/°C/element)

γ = rate of change in heat capacity of the fuel-moderator [1.43×10^2 watt-sec/(°C)²]

α = prompt negative temperature coefficient (-1.25×10^{-4} $\delta k/k/^\circ\text{C}$) average value for average core temperature up to 450°C

l = prompt-neutron lifetime (39 μsec)

P_0 = initial reactor power, watts

Fig. 3 - Reactor-core loading.

Calculations indicate that the prompt negative temperature coefficient varies with fuel temperature as shown in Fig. 5. Figure 6 gives the relation of reactivity insertion to measured peak reactor power. The experimental data show that the peak power varies linearly with the square of the prompt reactivity insertion over the range of reactivity insertions performed, as is predicted by the Fuchs-Nordheim model. The calculated values shown were made with both the variable heat capacity and an average value for the prompt negative temperature coefficient of the fuel-moderator material taken into consideration. The agreement with the calculational model indicates that, over the temperature range investi-

Fig. 4 - Typical reactor power pulse.

gated, the average prompt negative temperature coefficient is not greatly different from that calculated. The differences in the values of the experimentally determined reactor peak powers and those obtained from calculation are attributed to instrumental error as well as uncertainty in exact assignment of the values for the reactivity insertion. The uncertainty in the values of reactivity insertion stems from inherent differ-

Fig. 5 - Prompt negative temperature coefficient vs. average fuel temperature.

ences between reactivity for the control-rod configuration during calibration of the transient rods and reactivity for the configuration during a transient. The experimental data indicate that there is no large deviation from the predicted heat capacity of the fuel and its variation with increasing temperature.

The prompt energy release during the transient may also be described by the Fuchs-Nordheim model in the form

$$E = C_0(T_\infty - T_0) + \frac{\gamma}{2} (T_\infty^2 - T_0^2)$$

where E is the prompt energy release and T_∞ is the average temperature rise at the conclusion of the transient. An approximation of T_∞ is

$$T_\infty = -\frac{3}{8}(\sigma - 1) \left\{ 1 \pm \left[1 + \frac{16}{3} \frac{\sigma}{(\sigma - 1)^2} \right]^{1/2} \right\} [2\Delta T(t_{\max})]$$

where

$$\sigma = \frac{\alpha(C_0 + \gamma T_0)}{\gamma \delta k_p}$$

and

$$\Delta T(t_{\max}) = \frac{\delta k_p}{\alpha}$$

= average temperature rise when the power maximum occurs

Fig. 6 - Peak reactor power per fuel element vs. square of prompt reactivity insertion.

Figure 7 relates the prompt energy release during a transient and the prompt reactivity insertion. The experimental data were obtained by integrating the reactor power as a function of time. This integration gives

Fig. 7 - Prompt energy release per fuel element vs. prompt reactivity insertion.

only the prompt energy release since the instrumentation used to obtain reactor power information is placed on a range that makes it insensitive to reactor power levels less than 10 Mw. The after-pulse tail is not included in the integration. As with peak reactor power, the experimental values of energy release compare quite favorably with the calculated values, again indicating the validity of the calculated values of negative temperature coefficient and heat capacity of the fuel-moderator elements.

The prompt reactor period is inversely related to the prompt reactivity insertion and directly related to the prompt-neutron lifetime. As can be seen in the equation for peak reactor power, the peak power is related to the period through the prompt-neutron lifetime, which for the ATPR is taken as 39 μsec . Figure 8 relates the inverse period to the prompt re-

Fig. 8 - Inverse prompt reactor period vs. prompt reactivity insertion.

activity insertion and shows the decreasing value of the reactor period as the reactivity insertions increase. The reactor periods were determined from reactor power as a function of time in the time region after the transient rods have been removed and before fuel-temperature effects cause a loss in reactivity.

For convenience the width of the transient is defined as the time interval between the half-peak-power points of the power pulse. Experimental data are given in Fig. 9, where the reciprocal width is related to prompt reactivity insertion.

Fig. 9 - Inverse pulse width vs. prompt reactivity insertion.

As has been mentioned, fuel temperature is an important consideration in the higher hydrided uranium-zirconium hydride fuel since the equilibrium hydrogen pressure within the fuel can is a function of temperature. Figure 10 shows the increase in measured fuel temperature with increasing prompt reactivity insertion. The location of the thermocouples in the fuel is shown in Fig. 11. It should be noted that the location of the thermocouples, midway between the fuel axial center line and the fuel edge, is not the hottest position of the fuel element for the duration of the pulse. As will be shown later, the temperature distribution during adiabatic heating of the fuel element will have the same general U shape as the power distribution within the element, and thus the peak temperature

Fig. 10 - Measured fuel temperature vs. prompt reactivity insertion (95 fuel elements in core).

Fig. 11 – Thermocouple locations in fuel-moderator section.

will occur at the fuel-element edge for short time periods before heat transfer becomes significant. The measured fuel temperature is approximately 13% lower than the calculated peak fuel temperature and approximately 3% higher than the calculated average fuel temperature.

An instrumented fuel element was used to monitor pressures during transient operation. At the largest reactivity insertion (\$4.60), the maximum observed pressure in the fuel can was less than 10 psi above the initial pressure level in the can, which is much lower than predicted. Figure 12 shows the increase in pressure in the fuel can with prompt reactivity insertion. The pressure increase noted is linear with the prompt reactivity insertion in the same manner as the fuel-temperature increase. If the pressure increase were caused by a release of hydrogen from the zirconium hydride, the increase would be exponential with increasing reactivity insertion as is indicated in Fig. 2. Therefore the increase in pressure is attributed to the expansion of the gas used to pressurize the instrumented fuel-element can and expansion of the fuel meat.

INSTRUMENTATION

Instrumentation used in obtaining data for this pulsing program may be grouped into two categories: (1) instrumentation for the determination of the power generation in the reactor core and (2) instrumentation devoted to measuring other effects and parameters related to the power generation.

For measuring transient reactor power performance, three types of instrumentation channels have been used: (1) neutron-sensitive uncompensated ion chambers, (2) a gamma-sensitive ion chamber, and (3) a high-current-capability vacuum photodiode arranged to monitor Čerenkov

Fig. 12 - Internal fuel-can pressure vs. prompt reactivity insertion.

light from the entire upper face of the core. Outputs from these detectors were fed into multirange linear micromicroammeters whose current drive output is fed, with appropriate shunting, to fast-response galvanometers in a high-speed recorder, as shown in part a of Fig. 13. The power-level calibration of each channel is obtained by relating the channel output to the reactor power level as determined by thermal measurements. The galvanometer deflection in terms of current into the micromicroammeters is subsequently found by electrical means.

The resulting calibrations allow measurement of the reactor power as a function of time by means of the trace deflections and timing markers on the recorder strip. Chart speeds of approximately 300 cm/sec and maximum useful deflections of approximately 30 cm allow timing and power-level determinations to be made with good accuracy. Overall accuracy, including all calibration, is estimated to be better than 90%.

The second category of instrumentation required is that related to determination of fuel temperatures and any resulting pressures within the

Fig. 13. Recording channels. (a) Detector recording channel.
(b) Fuel-can pressure-transducer channel.

cladding. Fuel temperatures are measured by thermocouples and are recorded in as direct a manner as possible to avoid the difficulties that arise as a result of electronic amplifiers. The output of all thermocouples is fed directly to the galvanometers in the recorder, as shown in part b of Fig. 13. In this mode of operation, the thermocouples are used as a current source, and calibrations must be made between the infinite-impedance emf output as measured by a potentiometer and the deflection obtained on the galvanometer for a wide range of junction temperatures. By elimination of electronic amplifiers in the system, the usual zero drift and gain calibration problems are eliminated. Sheathed thermocouples are used and, in the interest of reliability and ruggedness, a reasonably heavy-gauge wire is used in the couple. The combination of heat capacity and the thermal bond between the thermocouple and fuel-moderator material produce a thermocouple having a time response of 50 to 100 msec. This time is long relative to the temperature-rise rates encountered in the fuel, and therefore

the measurements obtained are indicative of time-averaged temperatures rather than instantaneous values. Under such measuring conditions it is necessary to rely upon calculated values to estimate the instantaneous peak temperatures, although, as will be shown later, the peak temperature is maintained for a large fraction of a second.

Pressures are measured by a diaphragm type strain-gauge pressure transducer having a frequency response of several kilocycles per second. The device is essentially a resistance bridge circuit whose degree of unbalance is a measure of differential pressure on the sensing diaphragm. In most of our applications for which extreme sensitivity is not demanded (i.e., pressures greater than 10 psig), the output terminals of the strain-gauge bridge are fed directly into the recording galvanometers as indicated in Fig. 13. In this use the transducers must be calibrated by subjecting them to known pressures. In applications for which higher sensitivity is demanded, electronic amplifiers are used to allow increased gain to the system. By use of these methods, a sensitivity at the recorder of approximately 1 mm deflection for 1 psi (in the unamplified case) to several centimeters deflection for 1 psi (with electronic gain) can be obtained from transducers having an upper limit of 750 psi.

THERMAL CHARACTERISTICS OF TRIGA FUEL

Detailed calculations have been made to obtain a greater understanding of the temperature distribution within a TRIGA fuel element during and after a pulse. Not only the temperature distribution in the fuel element and its cladding but also the temporal changes in temperature are important during a pulse. The temperature behavior is of interest because calculations indicate that the higher pulses may cause film boiling, temporarily surrounding the fuel element with an insulating sheath of steam. The calculations were made to estimate to what extent film boiling may be expected since experimental evidence indicated that film boiling did occur on some fuel elements.

At this point it may be useful to comment on the temperature dependence of the strength of the type 304 stainless-steel cladding material. Table 2 shows that, for a temperature of approximately 400°C, the tensile strength of the cladding is not significantly reduced.⁵ Even for cladding temperatures as high as 800°C, the strength of the cladding is still one-fourth that at room temperature. Standard information⁵ on the yield strength indicates essentially no decrease in strength for temperatures up to 600°C. Calculations have shown that the pressure within the fuel-element cladding during a pulse having a fuel temperature of 800°C is approximately 200 psi. This pressure is due both to the predicted small quantities of hydrogen and to fission-product gases produced by infinitely long steady-state operation at 1.5 Mw. The room-temperature bursting pressure for the steel

Table 2 – TENSILE STRENGTH AT RUPTURE POINT *

Temperature		Tensile strength, psi
°F	°C	
Room temperature		87,000
400	205	74,000
800	427	69,000
1,200	650	44,500
1,500	815	22,000

*From *Metals Handbook*, American Society for Metals, Metals Park, Ohio, 1948.

cladding is 1800 psi. If the cladding temperature were in equilibrium with the fuel meat at 800°C, the rupture pressure for the cladding would be 450 psi. Of course, it is unreasonable to expect that the cladding will be as hot as the fuel meat even under film-boiling conditions because vapor cooling and radiation will cause a substantial cooling. In fact, calculations for the case of film boiling have indicated that the maximum cladding temperature is about two-thirds the maximum fuel temperature. Thus there is no serious problem due to loss of cladding strength.

It is of interest at this point to discuss the time and temperature variation in a fuel element and its cladding. In the following considerations it should be recognized that the coolant water at a depth of 22 ft boils at temperatures considerably above the normal boiling point at atmospheric pressure. The temperature–time distribution throughout the body of the fuel was derived from computer calculations for the case of a maximum fuel temperature of 750°C and is shown in Fig. 14. A somewhat artificial boundary condition was imposed, namely, that only nucleate boiling will occur at the water–cladding interface regardless of the heat flow, and it was further assumed that there was no thermal resistance at the fuel–cladding interface. These assumptions are conservative in the sense that they lead to calculation of a maximum value of heat flux and a maximum cooling rate. This does not change the peak fuel temperature. It is interesting to note that the temperature peaking that occurs near the edge of the element during a pulse has not changed significantly in 0.071 sec and the measured peak temperatures are only beginning to show significant changes in 1 to 3 sec after a pulse. The practical consequence is that the thermocouples placed at a radius of 0.3 in. have at least 3 sec to indicate the temperature.

Based on the preceding calculations, Fig. 15 shows a composite of the time change in the heat flux, cladding temperature, and water-coolant

Fig. 14 - Temperature distribution in fuel meat following pulse.
(Calculations from RAT computer code; $T_{\max} = 750^{\circ}\text{C}$; $\bar{T} = 650^{\circ}\text{C}$.)

temperature following pulse. It can be seen that the maximum heat flux from the cladding occurs at about 0.071 sec after the pulse when the water-coolant temperature is still relatively unchanged. For a water temperature of 100°F , the heat flux required for film boiling is about $2.2 \times 10^6 \text{ Btu}/(\text{hr})(\text{sq ft})$ under these transient conditions. The calculated heat flux of $1.85 \times 10^6 \text{ Btu}/(\text{hr})(\text{sq ft})$ indicates that film boiling probably does not occur at this time. However, as the water temperature rises, the heat flux required for film boiling drops rapidly, with the result that at a time of about 0.3 sec the calculated heat flux exceeds that required for initiating film boiling. This condition occurs a short time after the cladding temperature reaches a maximum. However, because of the nature of the calculation, one must be careful not to attach undue significance to the specific quantitative details of this calculation. In particular, the prediction of the time at which film boiling may occur can be in substantial error. It should also be noted that there is a random variation in the mechanism for initiation of film boiling which leads to an inherent uncertainty in the critical heat flux of the order of 20 to 30%.

The calculations given here indicate that film boiling does not necessarily occur generally but may occur locally depending upon local conditions and upon the nature of the thermal resistance at the fuel-body and cladding interface. If film boiling does not occur, the cladding and water temperatures should be about the same as those shown in Fig. 15.

Fig. 15 - Heat flux, cladding temperature, and water temperature (zero flow) following pulse. (Calculations from RAT computer code; $T_{\text{max}} = 750^\circ\text{C}$; $\bar{T} = 650^\circ\text{C}$.)

If film boiling does occur, however, the temperature of the cladding at the cladding-water interface will rise considerably higher until a new balance is reached. As pointed out previously, the temperature of the cladding could approach two-thirds that of the fuel body for a time of the order of 1 sec or more. Calculations indicate that in the absence of forced-water cooling the wall temperature will probably approach 400 to 500°C when film boiling occurs, depending on the exact circumstances. Stainless steel heated to 400 to 500°C in the presence of water vapor will oxidize and form a blue coating. Still higher temperatures of longer time duration will produce a much darker, bluish-gray, flaky coating of oxidation. Visual inspection of the TRIGA fuel elements revealed occasional spots of shiny blue oxidation on elements located at random near the midplane of the core. Hotter elements in the inner core positions had larger areas of oxidation. In no case was the blue oxidized area larger than a few square centimeters. Several other elements were only slightly discolored (straw colored) and thus had not been as hot as 400 to 500°C long enough for blue oxidation to form. A laboratory experiment was conducted to show that type 304 stainless steel at 400°C for 10 min in water vapor would produce the same shade of blue as that observed on the fuel elements near the center of the core. It is clear that film boiling occurs in isolated spots for the larger reactivity insertions. This film boiling, however, in

Fig. 16 - Fuel temperature required for a given steady-state reactor power as a function of transient size performed on fuel.

no way impairs the fuel containment and presents no problems in reactor control.

Another aspect of the thermal considerations is related to the expansion of the fuel-cladding material beyond its elastic limit. This consideration becomes important for steady-state operation because an increase in the fuel-cladding gap reduces heat transfer and results in a higher fuel temperature. Thus, because of the large prompt negative temperature coefficient ascribed to this core, a larger quantity of reactivity is required to overcome the temperature increase. The stretching of the cladding occurs during a large pulse, when the fuel expands and pushes the cladding, causing it to stretch. The effect of pulsing on raising the fuel temperature for steady-state operation (and thus increasing the reactivity loss) was investigated: A virgin fuel element with temperature instrumentation was operated at various steady-state power levels before pulsing and after pulsing at different reactivity insertions, and the results of this study are shown in Fig. 16. From this and other experimental data, we have determined that the reactivity loss and operating fuel temperature for a given high steady-state power are relatively unchanged by repeated additional pulses unless the fuel is pulsed to higher levels than it has previously experienced.

CONCLUSION

Information obtained in making reactivity insertions up to \$4.60 in the ATPR indicates that the parameters associated with large reactivity

insertions agree with those predicted by the variable-heat-capacity Fuchs–Nordheim model as modified by Scalettar.⁴ It is also important to note that the average prompt negative coefficient and the temperature-dependent heat capacity as used in the theoretical treatment lead to satisfactory predictions of the pulsed reactor parameters. Because the gas pressures developed within the fuel can are small and lower than predicted for the temperatures involved and because pressure increases linearly with reactivity insertion, we believe that the pressures observed have not arisen primarily as a result of hydrogen release from the uranium–zirconium hydride.

Film boiling occurred on some of the larger reactivity insertions; however, calculations show that the cladding strength, even in the worst case (when the cladding reached the temperature of the fuel meat for the largest pulse), is great enough to prevent rupture of the cladding.

Experience with the TRIGA uranium–zirconium hydride fuel has shown that this fuel can be operated safely with large reactivity insertions of at least \$4.60. General Atomic is continuing its test program on TRIGA fuel elements to substantially higher reactivity insertions and peak reactor powers.

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Feedback Effects During Short-interval Series Pulsing

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ABSTRACT

Experimental results are reported for short-interval series pulsing with a TRIGA Mark II reactor using repetition rates from $\frac{1}{2}$ to 3 pulses/min. The influence of repetition rate, pulse-rod drop time, and initial pulse reactivity on both the equilibrium pulse amplitude and the time to reach equilibrium is discussed. Results indicate that the reactivity coefficient associated with a water temperature increase has the opposite sign from the fuel coefficient. Some methods for optimization of series pulsing which do not involve major design modifications are noted.

Also, a calculational model that is designed for rapid calculation of series pulsing is discussed. The model utilizes an integral formulation to solve directly for the time-dependent temperature.

INTRODUCTION

Continued investigations of short-interval series pulsing with a TRIGA reactor are discussed in this paper. The first experiments of this type were reported by the authors¹ in December 1964.

Repetition rates varying from $\frac{1}{2}$ to 3 pulses/min have been studied. Higher rates are not practical with the TRIGA reactor without design modifications. For completeness, however, it is of interest to review briefly the development of reactors especially designed for very high rates.

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Zubarev² discussed such a reactor in 1960. He suggested a design based on revolving reflector sections with repetition rates of the order of 1 pulse/sec. Also, he suggested using a cluster of these reactors with their pulses slightly displaced in time to achieve an average thermal flux approaching the peak pulse value. This approach bypasses the usual flux limitation imposed by heat-transfer considerations on standard high-flux reactor designs.

A reactor now in operation at the Joint Nuclear Research Institute in Dubna, Russia, is reported^{3,4} to yield 23-Mw pulses at repetition rates of 3.3 pulses/sec. This is equivalent to an average power of 3 kw. Instead of Zubarev's suggested revolving reflector, a section of the core rotates. Present research using this reactor includes inelastic neutron scattering and spectrometry studies.

Although the repetition rates of the present study are comparatively modest, such operation can still result in significant improvement over standard pulsing for experiments such as neutron spectrometry, time of flight, and pulse propagation. In addition, such operation makes possible a unique study of transient feedback effects and also the measurement of certain reactor parameters.

EXPERIMENTAL PROCEDURE

The use of the TRIGA Mark II reactor and its standard pulsing procedures are well documented.^{5,6} In the present work the series-pulsing operation was initiated from a 10-watt steady-state condition. For a pulse to be initiated, the pulse rod was rapidly raised from its fully inserted position in the core up to a position predetermined by the desired reactivity. Normally, after each pulse, approximately 1.5 sec elapsed before the pulse rod was dropped back into the core; this interval helped to reduce the temperature after the pulse and to "ready" the pulse rod for the next pulse. This time is significant and is discussed later. For a given series of pulses, the same drop time and up-position of the pulse rod were used for each pulse.

Both power and temperature traces were recorded by the use of suitable electronics and a visicorder. Typical results are shown in Figs. 1 to 3.

Ideally temperature coefficients etc. would be based on an average core temperature. To do this experimentally, however, requires a prohibitive number of thermocouples. The present results are based on a single thermocouple at an arbitrary but fixed fuel-element location in the core. The use of this technique introduces questions about the effect of variations in both the gross core and internal fuel-element temperature profiles during a transient. The method will be acceptable

Fig. 1 - Power-temperature traces for a \$3, 60 sec, and 1.5 sec series.

Fig. 2 - A \$2, 120 sec, and 1.5 sec series.

Fig. 3 - A 2, 50 sec, and 1.5 sec series.

as long as all parameters, such as temperature coefficients and heat-transfer coefficients, are determined by the use of the same thermocouple and a unique single-valued function of temperature is found for each parameter. It is conceivable that, for times of the order of the pulse width (milliseconds), the uniqueness requirement may not be met since fuel-element temperature-shape changes are particularly drastic.⁷ For the time scale of importance to the present work (seconds), however, this requirement appears to be satisfied, since, as shown in later sections, unique parameters have been found. Also, the reverse implication, namely, that the profile is slowly varying during this time, has been investigated experimentally as illustrated in Fig. 4. Figure 4 compares the calculated reactivity loss vs. temperature (measured with

Fig. 4 - Comparison of reactivity vs. fuel temperature rise from steady-state and transient measurements.

the single couple) by the use of data from both steady-state power-coefficient measurements and transient measurements by the use of two pulses in series. The curves coincide within experimental accuracy, and this lends support to the argument. Transient reactivities were calculated from the measured temperature entering the second pulse and the peak power of that pulse by the use of the relation

$$\frac{\Delta k}{k} (\Delta T) \equiv \sum_{i=w,f} \alpha_i \langle e^{|\mathbf{T}_i|/2} \rangle = \rho_2 - \rho_R \quad (1)$$

where ρ_R is known and ρ_2 was taken from a separate experimental correlation of peak pulse power vs. reactivity. The nomenclature is defined following Eq. 2.

EXPERIMENTAL RESULTS

The effect of three parameters — time interval between pulses, rod drop time, and initial pulse amplitude — has been investigated and each will be discussed.

Results derived from time intervals ranging from 50 to 120 sec with a fixed initial pulse reactivity and rod drop time of \$2 and 1.5 sec, respectively, are summarized in Figs. 5 and 6. Power and temperature traces from two series in this set (\$2, 120 sec, and 1.5 sec for one series; \$2, 50 sec, and 1.5 sec for the other series[†]) are shown in Figs. 2 and 3, respectively. From these figures and Fig. 5, it is seen that the pulse amplitudes vary considerably during the start of a series. Note that the second pulse is always the smallest. As shown in Fig. 5, however, this oscillation damps out quite rapidly for the larger time intervals. Physically these oscillations are the result of a very delicate balance between the heat produced during any pulse and the heat transferred away from the core between pulses. This determines the temperature entering the next pulse, which, of course, determines the amplitude of that pulse.

In this paper pulses after the initial oscillations have died away (Fig. 5) are designated as *equilibrium pulses* or pulses with *equilibrium amplitudes*. The peak power for equilibrium pulses is plotted as a function of time interval between pulses in Fig. 6. For times less than 100 sec, the peak power decreases with time interval in an almost linear fashion. For the method of operation now used, the maximum

[†]The convention used throughout is to describe each series by its three characteristic parameters — initial pulse reactivity, time interval, and pulse-rod drop time — in that order.

Fig. 5 - Initial transients for various repetition rates.

Fig. 6 - Variation of equilibrium pulse amplitude with repetition rate for a \$2 initial pulse and 1.5-sec rod drop.

temperature occurs during the initial pulse rather than during the equilibrium pulses (Figs. 2 and 3).

Series with time intervals shorter than 50 sec have been run also. These data cannot be compared directly with the above, however, because the pulse reactivities became quite small. As a result, the pulse rod must be held up longer than 1.5 sec to prevent "clipping off" the peak of these pulses. In fact, when the pulsing interval is shortened to 30 sec, the second pulse must be clipped off to "ready" the pulse rod for the next pulse. This is discussed further under the heading Optimization. A series has been run, nevertheless, with a clipped second pulse using a 20-sec interval. The oscillations were fairly large and lasted for over 10 pulses, after which an equilibrium pulse height of about 2 Mw was achieved.

The effect of pulse-rod drop time is illustrated in Fig. 7. Peak powers for the second pulse in a series are shown for drop times from 1.9 to 10.1 sec for an initial pulse of \$2 (250 Mw) and various repetition rates. As might be expected, the peak power increases as the drop time is shortened. The percentage increase is most dramatic for high repetition rates. For example, at 1.2 pulses/min, a 12-Mw second pulse is obtained for a 1.9-sec drop time vs. a 4-Mw pulse for a 10.1-sec drop time — an increase of a factor of 3. (Although short drop times

Fig. 7—The effect of pulse-rod drop time.

are desirable, a lower limit is reached when the rod starts clipping the peak of the second pulse.) Some points shown in Fig. 7, where all three rods were dropped, show that this technique is most effective. Unlike the pulse rod, however, the control-rod and shim-rod drives are not pneumatic, and it is difficult to raise them quickly enough to prepare for later pulses.

Finally, some results have been obtained by the use of different initial pulse amplitudes; e.g., a \$3, 60 sec, and 1.5 sec series is illustrated in Fig. 1. This series leads to a 250-Mw equilibrium-pulse amplitude, representing about one-fourth of the amplitude of the initial pulse (1000 Mw). For comparison, the series of \$2, 60 sec, and 1.5 sec already discussed had an equilibrium amplitude equal to one-eighth of the initial height. Similar results were noted from a \$1.50, 120 sec, and 1.5 sec series vs. a \$2, 120 sec, and 1.5 sec series. Here the fractions were one-twenty-fifth and two-fifths, respectively. Thus, for the range of parameters considered, this fraction increases with larger reactivity initial pulses.

DELAYED TEMPERATURE COEFFICIENT

Temperature coefficient calculations reported for the TRIGA reactor imply that the prompt and delayed coefficients are negative and almost equal.⁸ "Bath-coefficient" experiments, however, indicate that increases in water and structure temperature may contribute a positive coefficient of magnitude comparable with the fuel temperature coefficient.^{6,9} This would imply a much smaller delayed coefficient than calculated. Differences in rod configuration between the calculation and experiment, and also problems associated with rod movements during the bath experiment, are thought to be possible causes for this discrepancy.¹⁰

Results from the present work can be used as an independent check on the experiment. This requires the development of a simple theory. Assume a two-coefficient model: i.e., define α_f and α_w as temperature coefficients associated with fuel temperature and water temperature changes, respectively. For the purposes of the present model, these coefficients are taken to be independent of temperature (as seen later, this is justified, at least for α_f).

Then, since a null excess prompt reactivity exists at the peak of any pulse,

$$\rho_R - 1 + \sum_{i=w,f} \left(\int \frac{\langle e | T_i | n \rangle}{\langle e | T_i | 1 \rangle} \alpha_i dT + \int \frac{\langle p | T_i | n \rangle}{\langle e | T_i | n \rangle} \alpha_i dT \right) = 0 \quad (2)$$

where the symbol $\langle r|T_i|n \rangle$ represents the temperature of the i th medium (w = water; f = fuel) at time r (e = entering a pulse; p = peak power of the pulse; and m = maximum temperature) for the n th pulse ($n = 1, 2, 3, \dots$); ρ_R is the excess reactivity due to the pulse-rod motion.

Since the characteristic time constant for heat transfer is large in comparison to the pulse width, to a good accuracy

$$\langle e|T_w|n \rangle = \langle p|T_w|n \rangle \quad (3)$$

The reactivity for the n th pulse, ρ_n , is defined as the excess reactivity due to the rod motion less losses due to the elevated water and fuel temperature entering the pulse;[†] i.e.,

$$\begin{aligned} \rho_n - 1 &= \rho_R - 1 + \sum_{i=f,w} \alpha_i (\langle e|T_i|n \rangle - \langle e|T_i|1 \rangle) \\ &= -\xi(n) \alpha_f (\langle m|T_f|n \rangle - \langle e|T_f|n \rangle) \end{aligned} \quad (4)$$

where

$$\xi(n) = \frac{\langle p|T_f|n \rangle - \langle e|T_f|n \rangle}{\langle m|T_f|n \rangle - \langle e|T_f|n \rangle} \quad (5)$$

The quantity $\xi(n)$ is introduced here because the thermocouple time constant was too large to permit an accurate determination of $\langle p|T_f|n \rangle$. (See Ref. 7.) As discussed later, results indicate that $\langle m|T_f|n \rangle$ and hence the fuel temperature differences in Eqs. 4 and 4' can probably be measured fairly accurately.

When Eq. 4 is rearranged, it is found that

$$\alpha_w = - \frac{(\langle e|T_f|n \rangle - \langle e|T_f|1 \rangle)}{(\langle e|T_w|n \rangle - \langle e|T_w|1 \rangle)} (W_n + \alpha_f) \quad (6)$$

where

$$W_n \equiv \frac{\rho_R - \rho_n}{(\langle e|T_f|n \rangle - \langle e|T_f|1 \rangle)} \quad (7)$$

The two temperature differences of Eq. 6 and the reactivity difference $(\rho_R - \rho_n)$ of Eq. 7 are always positive; but the temperature

[†]Note that for the first pulse, $n = 1$ and $\rho_n = \rho_R$; thus Eq. 4 reduces to $\rho_R - 1 = -\xi(1) \alpha_f (\langle m|T_f|1 \rangle - \langle e|T_f|1 \rangle)$ (4')

coefficient of reactivity of the fuel, α_f , is negative. Thus the sign of the temperature coefficient of reactivity of the water, α_w , can be determined from the relative sizes of $|\alpha_f|$ and W_n . In fact, α_w is positive if

$$|\alpha_f| \geq W_n \quad (8)$$

Experimental data listed in Table 1 show that, for a number of pulses checked, $0.49 < W_n < 0.64$. Also, from Table 2 it is noted that

$$|\alpha_f| \sim \frac{0.64}{\xi(1)}$$

Table 1 - TEMPERATURE MEASUREMENTS:[†] TYPICAL VALUES FOR W_n

Pulse No., n	ρ_n , \$	Time interval, sec	$\langle e T_f n \rangle$, [‡] °C	W_n , [§] °C
2	1.55	100	118	0.54
2	1.40	80	145	0.54
2	1.13	60	175	0.61
2	0.95	50	197	0.64
3	1.62	100	112	0.49
3	1.50	80	131	0.51
3	1.40	60	144	0.54
3	1.39	50	144	0.54

[†] $\rho_R = \$2$ and $\langle e|T_f|1 \rangle = 33^\circ\text{C}$ for all pulses.

[‡]All nomenclature defined in text.

[§]Calculated by use of Eq. 7.

Table 2 - TEMPERATURE MEASUREMENTS: TYPICAL VALUES FOR $\xi(1) \alpha_f$

ρ_R , \$	$\langle e T_f 1 \rangle$, °C	$\langle m T_f 1 \rangle$, °C	$\xi(1) \alpha_f$, [†] °C
3.00	26	498	-0.64
1.95	23	340	-0.64
1.00	18	182	-0.62

[†]Calculated by use of Eq. 4'.

By the use of these data, Eq. 8 is satisfied unless $\xi(1)$ is larger than 1.0 which is a physical impossibility in the present case. This argument indicates that α_w is positive. [The expression $\xi(1) = \frac{1}{2}$ holds for the adiabatic pulse model without delayed neutrons. Actually $\xi(1)$ will differ somewhat from $\frac{1}{2}$ because of energy from delayed neutrons and from heat transfer.] An additional argument⁸ is that for the TRIGA $|\alpha_f|_{av}$ is approximately $1\text{¢}/^\circ\text{C}$, considerably larger than the largest observed W_n of 0.64. The conclusion that α_w is positive agrees with the bath-coefficient experiment already noted but disagrees with the coefficient calculations of Ref. 8.

Finally, it should be noted that, as seen in Table 2, values of $\xi(1)\alpha_f$ calculated from three pulses of widely different reactivity turn out to be quite constant. This seems to justify the assumption that $\alpha_f \neq f(T)$ and also supports the assertion that the thermocouple time constant is satisfactory for the measurement of $\langle m|T_f|n \rangle$ (as opposed to $\langle p|T_f|n \rangle$).

CALCULATIONS

Although repetitive pulsing could in theory be analyzed by the use of a standard kinetics code, the computation time would be prohibitive. For this reason a specialized technique that allows rapid survey calculations has been devised.

The key item of interest for repetitive pulsing predictions is the fuel-element temperature entering each pulse. This in turn requires an accurate temperature calculation for the "tail" of each pulse. For pulses over \$1, heat transfer out of the fuel rods can be neglected for the TRIGA during the pulse itself. Assume that heat transfer becomes important at a time t_a . (In calculations to date, t_a has been chosen as that time at which the temperature reaches 60% of its maximum value.) Define the ratios of the volume-averaged temperature, $\bar{T}(t)$, and the surface temperature, $T_s(t)$, to the temperature $T(t)$ recorded by the thermocouple used in the study as

$$\bar{T}(t)/T(t) = r_1(t) \quad (9)$$

$$T_s(t)/T(t) = r_2(t) \quad (10)$$

Then, if the temperature of the coolant water relative to the surface temperature is neglected, the heat balance becomes

$$\dot{T}(t) - \beta(T) P(t) = \begin{cases} 0 & (0 < t < t_a) \\ -\gamma(T) T(t) & (t > t_a) \end{cases} \quad (11)$$

where

$$\beta(T) = \frac{a}{C_p \rho V r_1} \quad \gamma(T) = \left[\frac{h A_s}{C_p \rho V} \frac{r_2}{r_1} - \frac{\dot{r}_1}{r_1} \right] \quad (12)$$

when a = a unit conversion factor

C_p = the heat capacity

ρ = the density

V = the volume

A_s = the surface area

h = the surface coefficient of heat transfer for the fuel element

$P(t)$ = the reactor power level

In the present work $\beta(T)$ and $\gamma(T)$ were obtained by solving Eq. 11 for $T(t)$ as will be described and fitting the result to data from single pulses. Following some exploratory calculations, the following forms were selected:

$$\beta(T) = \beta_1 \quad (13)$$

$$\gamma(T) = \gamma_1 + \gamma_2 T \quad (14)$$

where β_1 , γ_1 , and γ_2 are constants. Strictly speaking, $\beta(T)$ should be a function of temperature through C_p and r_1 . The use of a constant is common in the adiabatic pulse model,⁶ however, and any error for $t > t_a$ will be absorbed when $\gamma(T)$ is fitted. The values used in calculations to date are $\beta_1 = 0.023^\circ\text{C/kw-sec}$, $\gamma_1 = 0.05 \text{ sec}^{-1}$, and $\gamma_2 = 1 \times 10^{-4} \text{ sec}^{-1} \text{ }^\circ\text{C}^{-1}$.

Equation 11 is solved by the use of the integral formulation of the reactor-kinetics equations discussed by Keepin and Cox.¹¹ Neutron flux is not solved for; instead, temperatures are found directly. This technique has the advantage that, since time constants for the temperature variations are much larger than for the flux transients, larger time steps can be used in the numerical evaluation. This feature combined with the proven speed of the integral formulation¹¹ yields a very rapid technique for survey work.

By the use of Eqs. 11 and 2.14 of Ref. 12, the following relations are found for $T(t)$:

$$\dot{T}(t) = \begin{cases} \beta_1 P(0) + \int_0^t G(t-\tau) \rho(\tau) \dot{T}(\tau) d\tau & (0 < t < t_a) \\ \dot{T}(t_a) - \gamma(T) T(t) + \int_{t_a}^t G(t-\tau) \rho(\tau) [\dot{T}(\tau) + \gamma(\tau) T(\tau)] d\tau & (t > t_a) \end{cases} \quad (15)$$

where

$$G(t) = \sum_{r=1}^7 A_r e^{(-P_r t)} \quad (16)$$

and

$$\rho(t) = \begin{cases} \rho_R - \alpha_f T(t) & (0 < t < t_R) \\ -\alpha_f T(t) & (t > t_R) \end{cases} \quad (17)$$

The coefficients A_r and P_r for the kernel $G(t)$ are tabulated in Ref. 11; α_f is the fuel-element temperature coefficient normalized to the temperature $T(t)$, and ρ_R is the excess reactivity introduced by the pulse rod which is held up until the time t_R . (Since pulsing begins from a critical condition with the pulse rod fully inserted, the excess reactivity *due to rods* is zero after the scram, i.e., after t_R .)

Some operations indicated in Eq. 15 can be carried out analytically, and it is found that the solution for the temperature involves only a single integration. For example, for $t_a < t < t_R$

$$\begin{aligned} T(t) = f(t) + \sum_{n=1}^3 C_n \int_{t_a}^t T^n(\tau) d\tau - \sum_{n=1}^3 B_{n1} \int_{t_a}^t \tau T^n(\tau) d\tau \\ - \sum_{n=1}^3 \sum_{m=2}^7 D_{nm} \int_{t_a}^t e^{-P_m(t-\tau)} T^n(\tau) d\tau \end{aligned} \quad (18)$$

where

$$f(t) = G_1 + G_2 t + \sum_{j=2}^7 E_j e^{-P_j t} \quad (19)$$

$$G_1 = T(t_a) - G_2 t_a - \sum_{r=2}^7 \frac{A_r}{P_r} T(t_a) \left[\frac{\alpha_f}{2} T(t_a) + \rho_R \right] \quad (20)$$

$$G_2 = \dot{T}(t_a) + T(t_a) \left[\frac{\alpha_f}{2} A_0 T(t_a) - \rho_R A_0 \right] \quad (21)$$

$$E_j = \frac{A_j}{P_j} T(t_a) e^{-P_j t_a} \left[\rho_R - \frac{\alpha_f}{2} T(t_a) \right] \quad (22)$$

$$C_n = A_n^* + \sum_{m=2}^7 \frac{B_{nm}}{P_m} + t B_{n1} \quad (23)$$

$$[A^*] = \begin{bmatrix} \rho_R \sum_{r=2}^7 A_r - \gamma_1 + \rho_R A_1 \\ -\frac{\alpha_f}{2} \sum_{r=2}^7 A_r - \gamma_2 - \frac{\alpha_f}{2} A_1 \\ 0 \end{bmatrix} \quad (24)$$

$$[B]_1 = \begin{bmatrix} \rho_R A_1 \gamma_1 \\ -\alpha_f A_1 \left(\gamma_1 - \frac{\rho_R \gamma_2}{\alpha_f} \right) \\ -\alpha_f \gamma_2 A_1 \end{bmatrix} \quad (25)$$

$$[B]_m = \begin{bmatrix} \rho_R A_m (\gamma_1 - P_m) \\ -\frac{\alpha_f}{2} A_m \left(2\gamma_1 - \frac{2\rho_R \gamma_2}{\alpha_f} - P_m \right) \\ -\alpha_f \gamma_2 A_m \end{bmatrix} \quad (26)$$

Note that the indices of P_r and A_r normally listed from 0 to 6 have been redefined to run for 1 to 7 in these equations.

Equation 18 is solved by an evaluation of the integrals by the trapezoidal rule. Temperature $T(t)$ is then found at every time step by solving the quadratic equation:

$$Q_1(t_k) T^2(t_k) + Q_2(t_k) T(t_k) + Q_3(t_k) = 0 \quad (27)$$

where

$$t_k = kh + t_a \quad (k = 1, 2, 3, \dots) \quad (28)$$

The term h is a constant selected to control step size. The Q 's are given by

$$Q_n(t_k) = \frac{h}{2} \left[C_n - \sum_{m=2}^7 \frac{B_{nm}}{P_m} - t_k B_{n1} \right] - \delta_{2n} \quad (n = 1, 2) \quad (29)$$

$$Q_3(t_k) = \begin{cases} \eta(t_k) & (k = 1) \\ \eta(t_k) + h \sum_{n=1}^3 \sum_{i=1}^{k-1} C_n T^n(t_i) - h \sum_{n=1}^3 \sum_{m=2}^7 \sum_{i=1}^{k-1} \frac{B_{nm}}{P_m} \\ \quad \times e^{-P_m(t_k - t_i)} T^n(t_i) - h \sum_{n=1}^3 \sum_{i=1}^{k-1} B_{n1} t_i T^n(t_i) & (k \geq 2) \end{cases} \quad (30)$$

where

$$\eta(t_k) = f(t_k) + \frac{h}{2} \sum_{n=1}^3 C_n T^n(t_a) - \frac{h}{2} \sum_{n=1}^3 \sum_{m=2}^7 \frac{B_{nm}}{P_m} e^{-P_m(t_k - t_a)} \\ \times T^n(t_k) - \frac{h}{2} \sum_{n=1}^3 B_{n1} t_a T^n(t_a)$$

These equations are solved by a computer code¹³ KESIT-I, written for the IBM 7094. Equivalent equations are solved at all times. An exception is that, if $\rho(t)$ evaluated at the inception of a pulse is over \$1, the well-known analytic solution for $T(t)$ for an adiabatic pulse¹⁴ is used for $t < t_a$.

Although the reactor power is not found directly from this code, it can be found at any time that $T(t)$ is known by the use of Eq. 11. Reactor power is not normally found, however, since the main item of interest is the peak power of each pulse. Peak power is found easily, once the entering temperature is known, by the use of Eq. 17 to calculate the pulse reactivity. In the present study an experimental correlation of pulse amplitude and reactivity is then applied, but the Fuchs model could just as easily have been used for the larger pulses.

This technique has been employed to calculate temperatures for the pulses shown in Figs. 1 to 3. (Results are not shown so as not to confuse the drawings.) Depending upon the particular pulse, the calculated temperatures are within 10% of the corresponding experimental values. For subsequent pulses larger than \$1 (which is normally the case), the percentage error in the prediction of pulse reactivity is less than the percentage error in temperature.

Work is continuing to optimize the code both from the point of view of speed and accuracy. It is also being tested against other and more varied experimental data, and these results will be published later.

OPTIMIZATION

The criteria for optimization of repetitive pulsing will vary depending upon the intended use. For example, in experiments investigating dose-rate effects, one might wish to maximize the ratio of the neutron yield recorded for fluxes above 10^{13} to the total neutron yield. For other experiments, such as pulse propagation, it might be desirable simply to maximize the pulse repetition rate for pulses over \$1 etc. There are some techniques and minor modifications which might enable the operator to tailor repetitive pulsing to the users' needs. (Major modifications or special reactor designs will not be considered here.)

If a fixed repetition rate is assumed, the most obvious adjustment of subsequent pulse amplitudes comes through varying the pulse-rod drop time. This has already been discussed under experimental results. As noted, pulses are especially sensitive to the rod drop time for high repetition rates. Care must be taken, however, to ensure that the rod is not dropped so quickly that it interrupts the peak of the pulse. The shortest pulse-rod drop in Fig. 7 is 1.9 sec. The pulse rod may be dropped at shorter times down to 0.5 sec without cutting off the peak of pulses over \$1, or approximately 2 Mw. If the pulses are above 30 Mw, a drop time of 0.3 sec will not cut off the peaks.

An additional pulse rod, such as that on the TRIGA Mark F, would add flexibility and allow insertion of larger negative reactivity between pulses. Dropping the standard control rods as discussed in connection with Fig. 7 is not satisfactory because of the long time required to raise them again.

Aside from rod programming, several other techniques should be mentioned. Pulsing described here always used the same pulse-rod travel and was initiated from a cold critical condition. Thus the first pulse was always the largest. The maximum pulse amplitude allowed under our hazards analysis is set by a maximum temperature limitation. With the first pulse being the largest, it (rather than the following "equilibrium" pulses) represents the limiting temperature case. This opens several possibilities. The pulse-rod travel might be continually increased until later pulses in the series reached the temperature limit. Alternately, one might operate the reactor at a high steady-state power to heat the fuel to a predetermined level and then actually begin pulsing from this power level. The initial fuel temperature achieved in this way might be thought of as simulating that entering any pulse in a series; i.e., proper selection of this temperature would eliminate the initial oscillation in a series.

A third alternative, which might be used to accomplish equivalent results, would be to vary the time interval between pulses.

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Reactivity Effects in Uranium–Zirconium Hydride Fueled Reactor Power Excursions

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ABSTRACT

Studies are in progress to determine the reactivity effects in uranium–zirconium hydride fueled reactor power excursions. These studies have as an objective the evaluation of the consequences of SNAP reactor excursions that might occur under postulated accident conditions. In this paper some theoretical considerations involved in establishing the prompt fuel-temperature coefficient are discussed. The development of a simple approximation to describe reactivity-feedback effects during reactor expansion and disassembly is presented. Calculations by these approximations are compared with experimental results. The suitability, or the lack of suitability, of the point kinetics approximation for these reactor systems is discussed. Finally, ways in which the rather unique properties of uranium–zirconium hydride influence the behavior of the reactor with respect to energy and fission-product release during destructive excursions are noted.

THEORETICAL CONSIDERATIONS OF REACTIVITY FEEDBACK

Spectral Effects

As the fuel temperature increases during a reactor transient, there is reactivity feedback that arises from what are usually referred to as “spectral effects.” These consist in (1) the effect of a hardening of the neutron spectrum on the ratio of fissions to absorptions in ^{235}U and (2) the effect of this hardening on increasing the neutron leakage

in the finite reactor assembly. The magnitude of the first effect has been calculated from the experimental data of Beyster and coworkers.¹ With the use of Beyster's neutron spectra, it is estimated that a reactivity coefficient of $-0.025\text{¢}/^{\circ}\text{F}$ should exist in an infinite-medium system of the SNAP reactor type. The magnitude of the leakage effect² is also about $-0.025\text{¢}/^{\circ}\text{F}$.

Thermal Expansion

Also, as the fuel increases in temperature, thermal expansion will result in greater leakage from the reactor. This effect has been calculated, and the results have been confirmed independently in critical experiments. It is $-0.042\text{¢}/^{\circ}\text{F}$ in magnitude.

When these contributions are summed, the result is a total fuel-temperature coefficient of $-0.092\text{¢}/^{\circ}\text{F}$. If the effect of importance weighting is included in order to put the coefficient on the basis of degrees Fahrenheit of fuel-averaged temperature rise, then the total fuel coefficient becomes $-0.11\text{¢}/^{\circ}\text{F}$.

Mechanical Disassembly and Expansion

There are rather unique properties of uranium-zirconium hydride which relate to the description of extreme power excursions.³⁻⁵ The equilibrium pressure of hydrogen as a function of temperature for a typical uranium-zirconium hydride fuel is shown in Fig. 1. Experimental data have been taken for only the solid-line portion of the curve. These data are well correlated by the usual activation-energy model in which the log of the pressure is proportional to the reciprocal of the absolute temperature. If it is assumed that this model is valid over two more decades of pressure rise, then at a temperature of 1300°K the equilibrium pressure of hydrogen will be approximately 1000 atm. At about this internal pressure, if the external pressure is low, fracturing of the fuel might be expected to occur.

The rate at which the equilibrium pressure is approached is indicated by the magnitude of the diffusion coefficient. In Fig. 2 some recent data for this coefficient are shown as a function of temperature. Some points not shown in Fig. 2, obtained from capsule experiments in the TREAT reactor, indicate a value for the diffusion coefficient of about $10^{-2}\text{ cm}^2/\text{sec}$ at a temperature of 1300°K . This value is, of course, considerably higher than that obtained by an exponential extrapolation of the Atomics International data in Fig. 2 to the same temperature (about $10^{-4}\text{ cm}^2/\text{sec}$). The discrepancy is presently being investigated.

A diffusion coefficient in the range of 10^{-4} to $10^{-2}\text{ cm}^2/\text{sec}$ is sufficient to establish essentially instantaneous equilibrium between

Fig. 1 - Equilibrium pressure of hydrogen as a function of temperature for a typical uranium-zirconium hydride fuel.

fuel and hydrogen in the microvoids of the fuel structure. It is not sufficiently high to establish equilibrium with respect to hydrogen outside the massive fuel rod. Consequently the internal gas pressure in the fuel matrix can far exceed the gas pressure outside the fuel. This becomes significant if the pressure difference is sufficient to cause extensive fracturing and disintegration of the fuel matrix.

The situation is represented schematically in Fig. 3 in which temperature-composition trajectories during rapid heating are shown for three heating rates. At a heating rate of 5×10^3 $^{\circ}\text{C}/\text{sec}$ (case C),

Fig. 2 - Diffusion coefficient of hydrogen in zirconium hydride as a function of reciprocal temperature. Atomics International data: O, beta phase; Δ , delta phase. [From S. D. Harkness and W. A. Young, The Diffusion Coefficient of Hydrogen in Delta-phase Zirconium Hydride, USAEC Report NAA-10516, p. 11, Atomics International, 1965. (Classified)]

the loss of hydrogen from the fuel by simple diffusion is great enough to prevent the attainment of a temperature at which the internal pressure would be sufficient to cause disintegration. On the other hand, at the heating rate of 5×10^5 °C/sec (case A), essentially no hydrogen is lost by diffusion before the characteristic temperature of disintegration is achieved. The heating rate of 5×10^4 °C/sec (case B) represents an intermediate situation; for this rate some hydrogen is lost by diffusion before fuel-matrix disintegration is achieved. The heating rates of case A are typical for high-reactivity-input destructive transients with the SNAPTRAN reactors and for capsule experiments in conjunction with the KIWI-TNT destructive experiment. The heating rates of case B have been measured in rapid-electrical-heating experiments. For case

Fig. 3 - Temperature-composition trajectories for three heating rates with zirconium hydride.

C both electrical heating and TREAT capsule experiments have been employed.

A longitudinal section of a uranium-zirconium hydride cylinder that was exposed to the flux of the KIWI-TNT destructive transient is shown in Fig. 4. This cylinder was rigidly constrained on the top, on the bottom, and on the right-hand side. On the left-hand side a small void volume was available for fuel motion. Fuel that was near the left-hand boundary was displaced and partially filled the void volume. An enlarged (100 $\times$) section of the fuel near this boundary is shown in Fig. 5. Very little hydrogen was lost from this fuel, as demonstrated by the presence of the characteristic banding of the epsilon-phase structure. There are both microcracks and macrocracks dispersed throughout the structure in a random manner, such as those which might occur from high internal pressures.

When fuel disintegration occurs during an extreme reactor transient, there will be a very rapid release of hydrogen gas to the surroundings. This will first occur in the highest power region of the core (normally in the center). The pressure of hydrogen gas in this region will act to produce mechanical expansion of the entire core. The time scale and the reactivity effect of this expansion are such that in many cases there is a significant effect in the shape and magnitude of the power pulse and the energy release.

An analytical model has been formulated to give an approximation to the reactivity effects from mechanical expansion and disassembly.

Fig. 4 - Fuel recovered from a KIWI-TNT capsule.

Fig. 5 - An enlarged (100X) section of fuel specimen shown in Fig. 4.

The formulation of this approximation includes the following equations.

1. The usual space-independent neutron-kinetics equation (with delayed neutrons neglected), with prompt temperature and reactor-expansion reactivity effects:

$$\frac{d\bar{P}(t)}{dt} = \frac{\beta}{l^*} \left\{ \rho_p + \frac{\alpha_T}{\beta} \bar{T}(t) + \frac{\alpha_e}{\beta} [R(t) - R_0] \right\} \bar{P}(t) \quad (1)$$

where $\bar{P}$ = volume-averaged power

β = effective precursor delayed-neutron fraction

l^* = effective prompt-neutron lifetime

ρ_p = superprompt-critical reactivity input, in units of β

α_T/β = prompt (fuel) temperature coefficient of reactivity

$\bar{T}$ = volume-averaged temperature

α_e/β = expansion coefficient of reactivity

R = radius of the outer boundary of the core

R_0 = initial value of R

2. The space-dependent energy-balance equations, in which it is assumed that below a temperature, T_D , which characterizes fuel disintegration, all energy goes to sensible heat and that at this temperature all energy goes to dissociate hydrogen gas from the hydride:

$$\int_0^t P(r,t) dt = \int_{T_0}^{T_D} C_p(T) dT(r,t) + \Delta H_D \int_0^{N_D} dN_D(r,t) \quad (2)$$

with

$$N_D = 0 \quad \text{when } T(r,t) < T_D$$

and

$$\frac{dT}{dt} = 0 \quad \text{when } T(r,t) = T_D$$

where P = local power

C_p = heat capacity of the fuel

T = local temperature

ΔH_D = heat of dissociation of the hydride

N_D = quantity of hydrogen dissociated from fuel after disintegration

3. The equations (as written below, for a spherical geometry) for the pressure in the disintegrated fuel region, in which it is assumed that, at the instant the fuel temperature reaches T_D , an initial quantity of hydrogen gas becomes unrestrained by the fuel structure and is available to act on the surroundings and, furthermore, that the volume available to the gas consists of the sum of (1) the volume of microscopic voids that previously existed in the fuel that has disintegrated and (2) the incremental volume arising from the movement of the outer fuel boundary of the reactor:

$$p(t) = \frac{R_G T_D \bar{N}(t)}{4/3 \pi [x r_D^3(t) + R^3(t) - R_0^3]} \quad (3)$$

with

$$\bar{N}(t) = 0 \quad \text{when } T(r,t) < T_D$$

and

$$\bar{N}(t) = \frac{4/3 \pi p_D x r_D^3(t)}{R_G T_D} + \bar{N}_D(t) \quad \text{when } T(r,t) = T_D \quad (4)$$

where p = pressure in the disintegrated-fuel region

R_G = gas constant

$\bar{N}$ = total quantity of hydrogen released from the fuel

x = volume fraction of microscopic voids in the fuel structure

r_D = radius of the disintegrated-fuel region

p_D = pressure of hydrogen in the microscopic voids which will cause fuel disintegration

4. An equation of motion (also, as written, for a spherical geometry) for the portion of the core in which fuel disintegration has not yet occurred:

$$4\pi r_D^2 p(t) = \frac{d}{dt} \left[M(t) \frac{dR(t)}{dt} \right] \quad (5)$$

with

$$M(t) = M_0 \left[1 - \frac{r_D^3(t)}{R_0^3} \right] \quad (6)$$

where M is the mass of all materials that will be accelerated (fuel, cladding, coolant, etc.) and M_0 is the initial value of M .

The preceding equations have been solved numerically in an equivalent spherical geometry, as shown in Fig. 6. The solution of these equations results in a prediction of the power, fuel temperature, radius of disintegrated-fuel region, pressure in the disintegrated-fuel region, core-boundary displacement, and reactivity vs. time traces as functions of the initial reactivity insertion, reactivity coefficients, and properties of the fuel-moderator, including the fuel-moderator disintegration temperature and pressure.

REACTOR EXPERIMENTS

SNAPTRAN-1

The SNAPTRAN-1 experiments are presently being conducted at the National Reactor Testing Station (NRTS) in Idaho by the Phillips Petroleum Company. In these experiments a fuel loading similar to that of the SNAP-10A/2 reactor is being used. This loading consists of 37 fuel rods arranged in a hexagonal array as shown in Fig. 7. Each rod consists of a cylinder of uranium-zirconium hydride, 1.21 in. in diameter and 12.22 in. long, canned in Hastelloy N. Six beryllium inserts round out the cross section from the hexagonal array to the cylindrical inner surface of the stainless-steel core vessel. The core vessel is 9.0 in. in diameter.

Fig. 6 - Schematic diagram of geometry used for mechanical-expansion calculation.

An overall view of the SNAPTRAN-1 machine is shown in Fig. 8. The beryllium reflectors, which surround the core vessel, are located in the central portion of the cubical structure shown in the figure. The mechanisms that produce extremely rapid motion of the beryllium control drums, thereby providing for the high rates of addition and removal of reactivity required for certain of the experiments, account for a large part of the bulk of the machine.

For each of the transients conducted with the SNAPTRAN-1 machine, a power history was obtained. The power trace was analyzed with the use of the point kinetics model to give a reactivity history. With the measured temperature as a function of time, which was obtained with the use of intrinsic-junction thermocouples referred to as energy probes,^{5,6} the compensated reactivity as a function of increase in temperature was plotted for each transient. The results for 23 step-reactivity-input transients, with initial asymptotic periods from 1.8 msec to tens of seconds, are shown in Fig. 9. Over the range of tem-

Fig. 7 - View of SNAPTRAN fuel elements loaded into the core vessel.

Fig. 8 - Overall view of the SNAPTRAN-1 machine.

Fig. 9 - Compensated reactivity as a function of temperature rise for 23 SNAPTRAN-1 transients.

peratures thus far studied, the experimental points can be bracketed by temperature coefficients between $-0.10\text{¢}/^{\circ}\text{F}$ and $-0.15\text{¢}/^{\circ}\text{F}$. This is considered to be reasonable agreement with the coefficient of $-0.11\text{¢}/^{\circ}\text{F}$ derived from theoretical considerations.

The SNAPTRAN-1 experiments are being continued to give reactivity-feedback information for transients with shorter reactor periods (to about 0.25 msec), as well as those resulting in higher temperatures.

SNAPTRAN-3

In the SNAPTRAN-3 destructive experiment (which is essentially the only experiment with the SNAPTRAN-3 reactor), a fuel-loading and core vessel, in all significant respects identical to that described for SNAPTRAN-1, was submerged in water and held subcritical by means of a Binal annulus surrounding the vessel. The transient was initiated by removing the poison sleeve at high velocity. The sleeve was completely removed prior to reactivity feedback from temperature rise. About 3.60¢ of reactivity was inserted by this means.^{5,7}

Figure 10 shows that the compensated reactivity vs. temperature curve for the SNAPTRAN-3 case is significantly different in slope from

Fig. 10 - Compensated reactivity as a function of temperature for the SNAPTRAN-3 destructive experiment. Above approximately 1200°F the calculated peak fuel-moderator temperature has exceeded the disintegration temperature.

that of the SNAPTRAN-1 transients. The apparent value for the fuel-temperature coefficient of SNAPTRAN-3 is $-0.22\phi/^{\circ}\text{F}$.

The expansion history of the core vessel was recorded photographically by Edgerton, Germeshausen & Grier, Inc. A sequence of typical photographs is shown in Fig. 11. The first picture was taken 0.75 msec following peak power. (The asymptotic period was 0.65 msec.) From these photographs equal-time contours were obtained of the core-vessel

Fig. 11 - Sequence of photographs of core-vessel expansion during the SNAPTRAN-3 destructive experiment.

silhouette, as shown in Fig. 12. In Fig. 13 the points of maximum radial expansion, as determined from the contour maps, are shown for both left- and right-hand sides. An expansion coefficient of reactivity was then deduced by plotting the compensated reactivity during the time of expansion, less the reactivity compensated by energy (assuming the energy coefficient of reactivity was constant during that time), as a function of change in radius. This is shown in Fig. 14. After the initial change in radius of about 0.3 cm, the reactivity coefficient of

Fig. 12 - Equal-time contours of the core-vessel silhouette during the SNAPTRAN-3 destructive experiment. (Contours 226.25 μ sec apart; film No. 07; frame Nos. 22, 26, 30, 34, 38, 42, 46, 50, 54, and 58.)

Fig. 13 - Maximum radial expansion of the core vessel as a function of time during the SNAPTRAN-3 destructive experiment. O, left. Δ , right.

expansion is estimated to be $-\$11/\text{in.}$ This value is in approximate agreement with theoretical estimates and with critical experiments performed at Atomics International.

With the use of the deduced feedback coefficients of temperature and of expansion in the analytical approximations derived in Eqs. 1 to 6, the power, reactivity, temperature, and expansion histories of the SNAPTRAN-3 experiment were calculated. The comparison of the results with the experiment is shown in Fig. 15. The significance of Fig. 15 does not lie in the agreement between the calculated and the experimental power, reactivity, and temperature traces; it would indeed be surprising if there were not satisfactory agreement considering that

Fig. 14 – Expansion-compensated reactivity as a function of change in core-vessel radius during the SNAPTRAN-3 destructive experiment.

the feedback coefficients were derived from the same point kinetics equations used in the formulation of Eqs. 1 to 6. Rather, the significance lies in the good agreement of the expansion histories since these are determined simply by the driving force of the power history and fuel disintegration and by the mechanical-expansion parameters used in the analysis. When the extent of the simplifying approximations made in the analysis is considered, this agreement is satisfactory.

It is interesting to observe the condition of fuel recovered from the SNAPTRAN-3 experiment. In Fig. 16 a single fuel rod is shown. This rod is the only one in which the top and the bottom were still connected. It was located in the next-to-the-last ring of the reactor. Needless to say, it shows evidence of having been subjected to a very substantial pressure from within the reactor, probably originating from the center. The fuel at the central portion of this rod was cracked and porous and depleted in hydrogen. The fuel toward the ends of this same rod was undamaged because it was located in a lower power region of the reactor. There was no depletion of hydrogen from the fuel at the ends.

Fig. 15 - Comparison of theory and experiment for the SNAPTRAN-3 destructive test. —, experiment. — — —, theory.

Fig. 16 - View of middle portion of a fuel element recovered after the SNAPTRAN-3 destructive experiment.

In Fig. 17 a photomicrograph of the fuel structure shows extensive microcracking throughout the region. Many of these cracks originate at sites within the matrix where material foreign to the zirconium hydride lattice is located. The fine specks consist of uranium precipitate of about 0.5 to $1.0\ \mu$ in diameter. Many of the cracks originate from these tiny inclusions. Since these inclusions would be appropriate sites for collection of hydrogen when high temperatures were reached during the transient, these observations are consistent with the fuel-disintegration hypothesis earlier proposed.

Fig. 17 - Photomicrograph of fuel recovered from the SNAPTRAN-3 destructive experiment.

SUMMARY

The prompt fuel-temperature coefficient deduced from the first series of SNAPTRAN-1 experiments appears to be in agreement with

the theoretical predictions. This is not the case for the coefficient deduced from the SNAPTRAN-3 experiment. It is speculated that in the SNAPTRAN-3 case the presence of a water reflector, with the large thermalization delay times that would be associated with a small fraction of the reflector-returned neutrons, may be introducing a spatially dependent neutron-kinetics effect that is being interpreted as a contribution to the prompt temperature coefficient when the point kinetics equations are used to deduce a coefficient. This possibility is currently under study. The SNAPTRAN-3 case will be analyzed with the use of a two-node kinetics approach, the synthesis-method approach, and an exact solution of the spatially dependent kinetics equations. The comparison of the results of these techniques should prove interesting.

A phenomenon herein referred to as fuel disintegration is a significant feedback mechanism in extreme transients with reactors fueled with uranium-zirconium hydride. A rather simple model will give a good approximation to the expansion-feedback effects for these reactors. The model may very well have application to reactors fueled with other materials in which vaporization of the fuel can result in substantial gas pressures within the core. The comparison of theory and experiment for the case of the SNAPTRAN-3 destructive transient with this approximation was satisfactory.

As a result of fuel disintegration, for all practical systems fueled with uranium-zirconium hydride, there will be no melting of fuel, even in extreme transients. As a consequence the release of fission products from the fuel is orders of magnitude less than would otherwise occur. In the SNAPTRAN-3 experiment, a person standing immediately outside the water tank in the downwind direction would have been exposed to a total dose from the fission-product cloud of only a few millirems.

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Study of the Dynamics of a TRIGA Type Reactor

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ABSTRACT

A number of measurements of coolant flow and coolant temperatures were performed on a TRIGA Mark II pulsing reactor under both steady-state and transient conditions. Based on these measurements, as well as on other pertinent information, a simple analytical model was developed and found to be in good agreement over the whole range of operations including the transients following fast insertion of reactivity in excess of \$1. A non-uniform heat-source distribution over the radius of the fuel rod causing a temporary peaking of the fuel temperature near the surface of the rod during a pulse transient,¹ as well as the natural-circulation cooling conditions, requires an adequate treatment of the heat-transfer model for the temperature feedback phenomena to be described with sufficient accuracy.

NOMENCLATURE

<i>Symbol</i>	<i>Units</i>	<i>Definition</i>
$a_{rs}^c, a_{rs}^d, a_{rs}^w$ α	kcal/kw-sec $^{\circ}\text{C}^{-1}$	Power-distribution function Volumetric-expansion coefficient
β		Total fraction of power due to delayed neutrons
β_i		Partial fraction of power due to delayed neutrons, group i

<i>Symbol</i>	<i>Units</i>	<i>Definition</i>
c		Superscript: center region of fuel
c^f, c^w	kcal/°C-cm ³	Specific heat of fuel and water, respectively
C_i	kw	Power due to delayed-neutron fissions, group i
d		Superscript: outer fuel region
D	sec/cm	Flow drag coefficient
f		Volume ratio: water to center fuel region
g	cm/sec ²	Acceleration due to earth gravity
h	cm	Height of slab
H	cm	Height of active core
h	kcal/sec-°C-cm ²	Film coefficient
i		Subscript: delayed-neutron group
k	kcal/sec-°C-cm	Thermal conductivity
k, k_0		Reactivity
Δk_c	\$	Controlled reactivity change
l	sec	Prompt-neutron lifetime
λ_i	sec ⁻¹	Decay constant of delayed neutron, group i
n	kw	Total reactor power
N_P		Number of delayed-neutron groups
N_R		Number of rings
N_S		Number of slabs
q		Volume ratio: outside region to center region
Q	gpm	Flow rate of coolant in one channel
r		Subscript: ring r
R	cm	Radius of fuel element
R^c	cm	Radius of center region of fuel
R^f	\$/°C	Temperature coefficient of reactivity in fuel
R^w	\$/°C	Temperature coefficient of reactivity in water

<i>Symbol</i>	<i>Units</i>	<i>Definition</i>
<i>s</i>		Subscript: slab <i>s</i>
<i>t</i>	sec	Time
T^f	°C	Fuel temperature, relative to tank water
$T_{rs}^c, T_{rs}^d, T_{rs}^w$	°C	Temperature of center fuel region, outside fuel region, and water, respectively, in cell <i>rs</i>
ΔT_{rs}^w	°C	Increase in water temperature through cell <i>rs</i>
T^{in}	°C	Inlet temperature of water into core
$\Delta T, T^w$	°C	Water temperature at outlet, relative to inlet
v_r	cm/sec	Water flow velocity in ring <i>r</i>
$W_{rs}^c, W_{rs}^d, W_{rs}^w$		Weighting functions for reactivity

EXPERIMENTAL WORK

All experimental work was performed on the University of Illinois TRIGA Mark II reactor, which is licensed to operate at 250 kw maximum power and to be pulsed to 1000 Mw (Fig. 1). The core consists of concentric arrays of cylindrical fuel elements (1.47 by 23 in.) suspended by two grid plates and surrounded by a graphite reflector. The core is situated in a water tank and cooled by natural convection of the water that enters the core through holes in the bottom grid plate and passes through openings between the top ends of the fuel elements and the top grid plate.

The fuel-moderator elements consist of a zirconium-uranium-hydrogen alloy that has a high negative prompt temperature coefficient of reactivity.² This feature permits fast insertion of reactivity in excess of \$1. The resulting excursion terminates itself when the Zr-U-H temperature becomes sufficiently high to compensate for the reactivity insert. A typical transient, such as one resulting from insertions of 1.46% reactivity (\$2), has a period of about 8.5 msec, a peak power of 245 Mw, a width of 35 msec, and a total-energy release under the peak of about 10 Mw-sec (Fig. 2).

Fuel Temperature Transient Measurements

A previous analysis¹ has indicated that the observed early peaking of the fuel temperature near the surface of the fuel rods (Fig. 3) after a

Fig. 1 - Schematic of Illinois TRIGA core.

Fig. 2 - Temperature-power comparison. Thermocouple 1 (TC-1) is located at the center of the fuel, and TC-2 is displaced somewhat radially. Thermocouple 4 is located at a greater radial position than TC-2. All thermocouples were slow in response, having time constants from 0.3 to 0.5 sec. This accounts for the lateness of temperature curves relative to the integrated power.¹

Fig. 3 - Temperature vs. radial position in a fuel element. Each curve of the family of curves is for a different time following a 245-Mw pulse. Six different radial positions were used to determine the data.¹

pulse is a result of the relatively higher power density in this fuel region (Fig. 4). This phenomenon is important in both the heat-transfer process and the feedback of reactivity and was considered to be so significant that it was incorporated in an approximate way into the model that is described later.

Water Flow Rate and Water Temperature Increase Measurements

For these measurements it was necessary to construct a rotor type flow-rate meter that could measure accurately the flow rate of water

Fig. 4 - Source term vs. radial position in a fuel element. The data are for various times after a $\$2$ transient. The source term is given in units of calories per cubic centimeter per second. Equilibrium 100-kw power results are shown for comparison.¹

emerging from the top grid plate around any particular fuel element. A pipe (1.5 in. in inside diameter by 4 in. in length) is centered around the top of a fuel element and rests flush with its lower edge against the top grid plate (Fig. 5). A low-inertia impeller supported by a steel pin and teflon bearings is rotated by the water flow. This rotation produces a pulsating signal in a magnet-coil arrangement. This signal was recorded

Fig. 5 - Sketch of flow meter.

on an optical galvanometer system (Visicorder). Calibration was carried out with a mock-up of the fuel end fixture and the top grid plate by the use of the signal frequency as an indicator of the predetermined flow of water through the device (Fig. 6). The calibration and the time response of the device were found more than adequate for measuring transients of flow rate that might result from the TRIGA core. In addition, two thermocouples were placed into the flow path to facilitate measurement of the energy released through the stream of water around each fuel element.

Steady-state measurements were taken when the reactor was operating at power levels up to 250 kw. The water flow rate, as well as the temperature increase, at all possible fuel positions was measured. Even though control rods, neutron start-up source, pneumatic tubes, and irregularities in the fuel rods caused some local disturbances, these measurements yielded a number of significant results:

1. The rate at which energy was conducted from the core in the form of heat in the stream of water was in good agreement with the power measurement obtained from the regular instrumentation. At a nominal power level of 250 kw, the measured energy flow rate totaled 232 kw.

2. A linear relation between the flow rate and the temperature rise of the water at each fuel position for steady-state operation was obtained and seemed to hold for most measurements, as is shown in Fig. 7. Some of the spread that can be observed may be due to the local disturbances mentioned previously.

3. When the flow rate of energy within each ring of fuel elements at steady-state operation of 100 and 250 kw was averaged, a power distribution was obtained which is shown graphically in Fig. 8. (The rings are lettered alphabetically from the inside out, starting with B at the innermost ring of fuel elements. The values for the B ring had to be extrapolated in parts since measurement of the flow was technically not possible there.)

The data indicate that the power-density-distribution function across the core is very nearly the same at 100- and 250-kw steady-state operation. This result is seen to differ slightly from results obtained from a multigroup-multiregion diffusion calculation for a TRIGA Mark II core.³ One possible explanation for the discrepancy might be that the energy stream has a tendency to flow toward the center of the core because of the pressure differences set up by the higher flow velocities near the center. Radial flow is inhibited by the rather narrow passages between the fuel elements, but this is thought to be a rather minor effect compared to the vertical flow. The transient flow measurements are discussed together with calculated results in a later section of this paper.

Measurement of the Fuel-cladding Temperature and the Water Temperature Inside the Core

One fuel element in the D-ring was instrumented with a number of thermocouples. As shown in Fig. 9, one of these was located at the cen-

Fig. 6 - Results of the flow-meter calibration.

Fig. 7 - Q vs. ΔT at steady-state operation. Each point represents a flow rate in gallons per minute and is plotted at the measured temperature ($^{\circ}\text{C}$) increase for that channel.

Fig. 8 - Average power at various ring positions. The power is expressed as gallon-degrees centigrade per minute. The letters refer to the fuel-element rings, B being closest to the core center. Normalized curves: —, experimental line. - - -, theoretical line (theoretical data from General Atomic).

ter of the fuel (No. 1), and three were placed on the surface of the cladding (C3, C4, and C5, respectively). In addition, four other thermocouples, supported by a wire strung parallel to the fuel element axis about

Fig. 9 — Measured steady-state temperature of fuel, cladding, and water. All indicated temperatures are relative to the inlet water temperature ($W1 = 25^{\circ}\text{C}$).

3/16 in. away from the surface, recorded the temperature of the water in these locations (W1, W3, W4, and W5). The corresponding hole in the top grid plate had to be widened so that this fuel element could be installed in its proper place.

Some of the temperatures recorded at steady-state operation are also given in Fig. 9. The rather large variations in the recorded temperatures even at 50 kw steady state seem to be of a random nature and indicate the presence of disturbances in the flow, the cause of which has not yet been determined with certainty.

The fact that at high power levels the temperature in W5 is lower than in W4 seems to support the hypothesis that a flow toward the center of the core exists which transfers cooler water from the outer rings into the inner ones. The transient measurements are discussed in a subsequent section.

ANALYTIC MODEL

The analytic model consists of three parts:

1. The neutronic model.
2. The heat-transfer model.
3. The coolant-flow model.

The three are coupled by the heat generation due to thermal-neutron fission, by the feedback of reactivity, and by the temperature-determined buoyancy forces in the coolant which generate the natural-circulation flow.

For the neutronic model the well-known point reactor model was adopted.

Neutron balance:

$$\frac{dn}{dt} = \frac{n}{l} [k(1 - \beta) - 1] + \sum_i \lambda_i C_i$$

Delayed-neutron emitter balance:

$$\frac{dC_i}{dt} = \frac{n}{l} \beta_i k - \lambda_i C_i \quad (i = 1, 2, \dots, N_p)$$

Reactivity:

$$k = k_0 + \beta \Delta k_c + \beta \sum_{rs} (T_{rs}^c W_{rs}^c + T_{rs}^d W_{rs}^d) R^f \\ + T_{rs}^w W_{rs}^w R^w \quad \begin{matrix} (r = 1, 2, \dots, N_R) \\ (s = 1, 2, \dots, N_S) \end{matrix}$$

We realize that this simple model is justified only partially by the convenience it provides. For a small core, however, the underlying assumptions are not too unrealistic. The coupling between the space-independent neutronic model and the space-dependent heat-transfer model is treated by the introduction of weighting functions W_{rs} for the feedback of reactivity⁴ and density functions a_{rs} for the heat generation from fission. Work is being undertaken to test the validity of this approach.

For the heat-transfer model, a geometry is chosen as shown in Fig. 10. Fictitious interfaces are introduced to obtain a cell structure of the whole core:

1. The individual fuel elements are subdivided into a center region (radius R^c) and an outer region (outside radius R) to account for the characteristic transient behavior pointed out previously.¹

2. A portion of the water surrounding each fuel element is associated with the fuel element to form a "channel," and motion of water is confined to vertical flow in this channel only.

Fig. 10 - Geometry of the TRIGA analytic model.

3. A vertical subdivision into slabs (height h) is introduced to account for the vertical variations of the temperatures; it is also assumed that no heat flows vertically within the fuel elements.

Under these assumptions the following equations describe the heat-transfer model.

Heat balance in center fuel region:

$$c^f \frac{dT_{rs}^c}{dt} = a_{rs}^c n - \frac{2k}{R \cdot R^c} (T_{rs}^c - T_{rs}^d)$$

Heat balance in outer fuel region:

$$c^f \frac{dT_{rs}^d}{dt} = a_{rs}^d n - \frac{2Rh}{q(R^c)^2} (T_{rs}^d - T_{rs}^w) + \frac{2k}{qR \cdot R^c} (T_{rs}^c - T_{rs}^d)$$

Heat balance in water of fuel channel:

$$c^w \frac{dT_{rs}^w}{dt} = a_{rs}^w n - c^w \frac{v_r}{h} \Delta T_{rs}^w + \frac{2Rh}{f(R^c)^2} (T_{rs}^d - T_{rs}^w) \quad (r = 1, 2, \dots, N_R) \\ (s = 1, 2, \dots, N_S)$$

If we assume "slug" flow in each channel, we can write the equation of motion as follows:

Water motion in fuel channel:

$$\frac{1}{g} \frac{dv_r}{dt} = \alpha \left(\frac{1}{N_S} \sum_s T_{rs}^w - T^{in} \right) - D \cdot v_r \quad (r = 1, 2, \dots, N_R)$$

In addition, the temperature dependence is accounted for in the following properties:

$c^f = c^f(T^c) = c^f(T^d) \dots$ = specific heat of fuel

$h = h(T^c, T^d, T^w) \dots$ = film coefficient

$k = k(T^c) = k(T^d) \dots$ = thermal condition of fuel

$R^f = R^f(T^c) = R^f(T^d) \dots$ = temperature coefficient of reactivity (fuel)

A great number of the parameters in this model were obtained directly from steady-state measurements on the reactor, in which case the steady-state equilibrium solution of the system of differential equations has to apply. Other sources for numerical values of the parameters are reported in the literature.^{1,5-8}

The preceding equations constitute the model that was used to describe the dynamics of the reactor system. A computer code was written for the IBM 7094 to obtain a numerical solution. The differential equations were integrated by the use of the Nordsieck scheme, which provided automatic interval control for optimum speed and accuracy.⁹ The calculation of a pulse transient as shown in the next section takes about 10 min of computer time (12 cells).

COMPARISON OF MEASURED AND CALCULATED TRANSIENTS

The first part of a power pulse from a TRIGA reactor can be described adequately by an adiabatic model. Thus it is fairly easy to simulate this portion of the transient (exponential power increase, peak power, and pulse width) because period, peak power, and pulse width are determined by simple formulae.¹⁰ A proper choice of a few parameters will guarantee agreement in this part of the transient.

Figure 11 shows the part of the transient following the power spike, for which the adiabatic model is no longer valid. The transients of fuel temperatures, water temperatures, and flow rates were measured simultaneously by the use of a multichannel optical recorder. Thermocouple 1 is used as a monitor of the center region; TC-4, for the outer region. In this experiment the pulse rod was held in its *up* position; this causes the power to level off at a rather high value after passing through a minimum about 20 sec after the pulse. The fuel temperatures show the characteristic early peak near the surface. Both flow rate and water-exit temperature go through peak values a few seconds after the peaking of power.

Figures 12 to 14 show these experimental transients in comparison with transients calculated from the model described in the preceding sections. In all cases good qualitative agreement can be observed. Calculated and measured maxima and minima coincide in most cases, as do the

Fig. 11 — Measured transients, \$2.50 pulse. The times were measured relative to the occurrence of the pulse peak. The peak of the power pulse was 540 Mw.

Fig. 12 — Power and fuel temperature transients, \$2.50 pulse.

Fig. 13 - Transients of water flow and temperature increase, \$2.50 pulse. The temperature increase of the channel coolant is measured and calculated relative to channel inlet temperature.

Fig. 14 - Water temperature transients in a fuel channel, \$2.50 pulse. The experimental data have been smoothed after 2 sec so that the trends can be more easily seen. The data flags represent the magnitude of the temperature variations.

extreme values themselves. Further improvement is expected from a more critical inspection of the experimental data as well as a determination of a better set of parameters, for which only an initial choice was made at this stage.

It is obvious from these graphs that the behavior of the coolant is particularly difficult to interpret and describe. The core design does not lend itself readily to a simple and accurate analysis. Hence the analysis will remain somewhat approximate. Detailed description of this particular effect, however, is not considered to be of major importance for the overall validity of the model.

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Feasibility and Application of an Accelerator Booster

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ABSTRACT

The concept is explored of pulsing a fast reactor by mechanical means and simultaneously pulsing it by photoneutrons from a linear electron accelerator. The resulting burst of neutrons can be thermalized to give a pulsed source with an instantaneous flux greater than 10^{16} neutrons/cm²/sec. The application of this pulsed thermal source to time-of-flight studies is considered. A comparison of this pulsed source with other sources is made.

INTRODUCTION

The concept of pulsing a supercritical fast reactor is discussed in this paper. The purpose of the pulsing is to obtain short-burst high-peak powers that will be of use in research programs, particularly those in which neutron energies are measured by time-of-flight techniques.

Before we discuss the concept, let us examine briefly the counting intensity (counts measured per analyzer time channel) in a pulsed-source experiment. If a moderator is subjected to a square neutron pulse of strength S_0 and width t_1 , the counts N received in an analyzer channel of width W are given by the following expression:

$$N = C S(E) E^{3/2} \eta(E) \frac{W f S_0 t_1}{L^3} \quad (1)$$

where C = a normalization constant

L = the flight-path length

f = the pulsing rate

$\eta(E)$ = the moderator efficiency at energy E

$S(E)$ = the detector sensitivity

The total timing uncertainty, ΔT , is related to the energy resolution, R , by

$$R = \frac{\Delta E}{E} = \frac{2\Delta T}{T} \propto \frac{\Delta T E^{1/2}}{L} \quad (2)$$

The total timing uncertainty is a function of the burst width, t_1 , the slowing-down time in the moderator, t_2 , the resolving time of the detectors and electronics, t_3 , and the analyzer channel width, W . This dependence can be expressed approximately as

$$\Delta T = \sqrt{t_1^2 + t_2^2 + t_3^2 + W^2} \quad (3)$$

If we choose W to be proportional to ΔT , as is common practice, then W is also proportional to L for a constant R . In an experiment it may be desirable to increase the source intensity, S_0 , but in practice this is often accompanied by an increase in the burst width, t_1 . The flight-path length, L , may be increased to maintain the same energy resolution, R , but increasing L will decrease the neutron intensity at the detector. What relation then must there be between S_0 and t_1 if the same R and N are to be maintained? It is easy to show that S_0 and t_1 may be increased until the burst width, t_1 , becomes a large contribution to the total timing uncertainty, ΔT . When t_1 becomes the principal contributor to ΔT , a common situation, then the relation necessary for constant R and N is

$$S_0 \propto t_1/f \quad (4)$$

In many cases, particularly at low energy, $f \propto 1/L$ and hence $f \propto 1/t_1$ if pulse overlap is to be avoided. On this basis it is found that the source intensity, S_0 , must be increased without a proportionate increase in the burst width t_1 or t_1^2 if f is restrictive.

This simplified analysis is intended to point out the premium that can be obtained in time-of-flight studies by having as short a flight path as is possible to maintain a desired energy resolution and statistical accuracy. The analysis includes varying the size of the detector to maintain a constant solid angle since the cost of the detector is roughly proportional to the square of its size. This fact and the cost factor involved in increasing the flight-path length further substantiate

the requirement of having as short a flight path as possible. In some cases a neutron chopper is employed, instead of the natural-source pulse width, to define the neutron burst. In such cases the chopper utilizes the pulsed source more effectively when the source burst width is as short as possible. If the integrated source is limited by average power considerations, a further incentive is provided to shorten the pulse width and increase the peak pulse intensity. Alternatively, the source burst width alone may be sufficiently short to eliminate entirely the use of a chopper; such elimination is highly desirable because of the relatively small throat size of a fast chopper.

PULSED NEUTRON SOURCES

Let us now examine some of the important types of high-intensity neutron sources presently used or proposed for time-of-flight measurements.

1. Steady-state reactor with chopper, e.g., Brookhaven High Flux Beam Research Reactor.
2. Pulsed thermal reactor, e.g., the TRIGA type.
3. Mechanically pulsed reactor, e.g., the Euratom SORA and the Russian IBR type.
4. Linear electron accelerator, e.g., the General Atomic (GA) and the Rensselaer Polytechnic Institute Linacs.
5. Linear electron accelerator with fixed booster, e.g., the Harwell type.
6. Linear electron accelerator with mechanically pulsed booster.

Device 6 is the primary concept discussed in this paper; a review of the first five devices, however, will give some perspective on its application.

Device 1 allows peak thermal fluxes as high as 10^{15} neutrons/cm²/sec to be obtained. Use of a steady-state reactor entirely for pulsed neutron research is wasteful of neutrons because of the very low duty cycle. In addition, the interpulse background is the highest of all the devices listed.

Device 2 is an improvement over device 1 since higher peak power and lower interpulse background are possible. The maximum pulsing rate of this device at peak powers of 2000 Mw is very low (~ 5 to 10 pulses/hr). Pulsing at 20 pulses/sec is possible,¹ however; this pulsing rate would give peak leakage fluxes of 10^{15} neutrons/cm²/sec.

Device 3 will be discussed later in a comparison with device 6.

Device 4, the straight linear electron accelerator, such as the GA Linac, gives pulse widths as short as 10 nsec or as long as $4.8 \mu\text{sec}$, variable frequency up to 720 pulses/sec, and peak neutron yields of

higher than 10^{17} neutrons/sec. The linac is very flexible and is presently used with success for neutron spectroscopy from energies of 0.001 ev to 10 Mev. The straight linac system, however, could use to good advantage a higher intensity for measurements in the thermal and low epithermal energy range in cases where the experimental assembly has a long "die-away" and requires use of a long flight path to obtain the desired energy resolution.

Device 5 has been proposed as a means to increase the available neutron intensity in the intermediate energy region. The linac is coupled to a subcritical multiplying assembly with a multiplication of about 10 and still has sufficiently short die-away to permit its use directly without a chopper. An example of this can be seen in the Harwell booster,² which has the following design characteristics:

Burst width	0.25 μ sec
Peak source intensity	10^{18} neutrons/sec
Frequency	550 to 750 pulses/sec

PULSED FAST REACTORS

In discussing device 3, we briefly describe the proposed SORA reactor³ since it is a good basic example of this class. It is to be a fast reactor with a reflected, spherical enriched-uranium core and a sodium-potassium coolant. It is pulsed by moving a nickel or beryllium neutron reflector across a window in the core vessel at a speed of about 300 m/sec. The reflector is worth about 5β and, for peak power to be achieved, the reactor must be held above prompt critical for about 80 μ sec, during which interval it reaches a peak reactivity of 0.12β above prompt critical. Ports are provided for fast and thermal fluxes, and peak thermal fluxes of 10^{16} neutrons/cm²/sec are anticipated.

The linac booster comprises a reactor whose reactivity is varied periodically by mechanical means. At the peak reactivity the reactor is subjected to a burst of neutrons from a linear electron accelerator obtained by means of (γ, n) or (γ, f) reactions. A maximum reactivity of 0.13β below prompt critical is envisaged here, which corresponds to a multiplication of about 1000.

The equations governing these two devices are obtained by the following point kinetic theory

$$\frac{dn}{dt} = \frac{[k(1 - \beta) - 1]n}{l} + \lambda C + S_0 \quad (5)$$

$$\frac{dC}{dt} = -\lambda C + \frac{k\beta n}{l} \quad (6)$$

where the nomenclature is standard.

The delayed neutrons can be treated as a constant background when the pulsing interval τ is much less than the mean half-life of the delayed-neutron precursors. This allows the response of the mechanically pulsed reactor to be obtained simply as

$$n(t) = n(t_0) \exp \left\{ \int_{t_0}^t [k(1 - \beta) - 1] dt/l \right\} \quad (7)$$

For the linac booster the equations are further simplified by the use of a constant multiplication factor k over the pulse although this detracts from the booster performance. In addition, the multiplication of the background level $n(t_0)$ during the power pulse is ignored since it is small compared to the linac source term S_0 . The response is then

Linac on:

$$n(t) = \frac{S_0}{1 - k(1 - \beta)} \left(1 - \exp \{ [k(1 - \beta) - 1] t/l \} \right) \quad (0 \leq t \leq t_1) \quad (8)$$

Linac off:

$$n(t) = n(t_1) \exp \{ [k(1 - \beta) - 1] (t - t_1)/l \} \quad (t \geq t_1) \quad (9)$$

The interpulse background n_b for both systems at high pulsing rates is given approximately by

$$n_b = \frac{\int_0^\tau k\beta n(t) dt}{\tau[1 - k_i(1 - \beta)]} \quad (10)$$

where k_i is the interpulse multiplication factor (presumed constant). For practical purposes the upper bound τ of the integral in Eq. 10 can be replaced by t_1 since most of the production of neutron precursors occurs during the pulse.

From Eq. 10 it can be seen that n_b will be lowered if the interpulse multiplication k_i is as small as possible. The interpulse background also becomes smaller as τ increases; i.e., the pulsing rate is reduced. Finally, n_b can be reduced by minimizing β , and this suggests the use of ^{239}Pu as a fuel with a β of 0.0021 instead of ^{235}U with a β of 0.0065.

Uranium-233 with β equal to 0.0027 is another possibility. With the requirement that k be above prompt critical during the pulse and as small as possible between the pulses, the mechanically pulsed reactor must move reactivities as large as possible in and out of the reactor as quickly as possible. Not so the linac booster.

The performance of the two devices is compared in Fig. 1. The reactivity variation of the mechanically pulsed reactor is similar to that of the SORA reactor, whereas the linac booster response has been simplified by the use of a constant reactivity. Both systems are compared on the basis of a mean lifetime of 10 nsec. The linac source pulse duration is 5 μ sec, and both peaks have been normalized to the same integrated neutron yield. It can be seen that the linac booster gives a shorter, more intense pulse. The neutron flux in the linac booster die-away with a prompt period of 10 μ sec, whereas the die-away of the mechanically pulsed reactor is initially larger, being determined more by the speed with which reactivity can be removed. After 98 μ sec

Fig. 1 - Comparison of mechanically pulsed reactor and linac booster.

(see Fig. 1) the reactivity of the mechanically pulsed device is down to that assumed constant value for the linac booster. Thereafter on this simplified basis the die-away of the linac booster is the slower.

The neutron burst from the booster is terminated by shutting off the electron beam, whereas the neutron burst from the mechanically pulsed reactor is terminated by the reactivity removal. Consequently it is not necessary in the linac booster that the reactivity-bearing component travel at such high speeds. Furthermore, since the linac booster has a higher peak power for the same mean power, the amount of reactivity to be removed can be reduced for the same peak-to-background-intensity ratio. Both these features lead to a system that is inherently safer than the mechanically pulsed reactor.

Two more disadvantages of the mechanically pulsed device should be mentioned. First is the problem of moving safely such large reactivities in and out of the reactor. Slight fluctuations in speed or wobble of the flywheel bearing the reflector can cause excessive power peaks. Second is the fact that the burst width increases as the flywheel speed is reduced to obtain lower repetition rates. The linac booster, on the other hand, has its burst width and intensity controlled by means of the linac. The linac booster is therefore essentially similar to the subcritical operation of the Russian IBR reactor⁴ when it is pulsed by an electron linear accelerator, except that, by the use of the higher linac-beam conditions and by operation above delayed critical, one can easily surpass the performance of the IBR reactor operating in the mechanically pulsed mode.

The question naturally arises as to whether the linac intensity is sufficient to drive the booster reactor to powers equivalent to those of the mechanically pulsed system. Evidence on this is available from recent pulsing studies⁵ made at GA on a ^{235}U metal ball whose leakage and power were compared to that of Godiva. These studies are illustrated in Fig. 2. The metal-ball assembly had a prompt multiplication of about 250 and a neutron lifetime of about 6 nsec. These values can be used to show that the characteristics expected from point kinetic theory are exhibited clearly. Further information on the available source intensities can be obtained by scaling the parameters of the IBR reactor. The scaled-up device is compared to SORA and to the IBR subcritical performance in Table 1. The peak power of the linac booster can clearly exceed that of the SORA reactor. Peak thermal fluxes can therefore be scaled accordingly. If the heat-removal rate is the limiting factor in both reactors, the linac booster device can yield peak powers considerably larger than those of the mechanically pulsed reactor, peak powers depending, of course, on linac specifications and reactor properties. Further, because the linac booster permits the use of a variable repetition rate, advantage can be taken at times with lower pulse rates to yield even higher

Fig. 2 - Linac pulse and response from a ^{235}U metal ball. (a) Linac electron pulse. (b) Burst shape of fission power pulse.

Table 1 - COMPARATIVE CHARACTERISTICS OF PULSED FAST ASSEMBLIES

Characteristics	Mechanically pulsed SORA reactor	Subcritical IBR reactor	Linac booster scaled-up from the subcritical IBR
Pulse rate	Fixed, 100 pulses/sec	Variable, approximately 100 pulses/sec	Variable, approximately 100 pulses/sec
Reactivity	Variable, 1.12β max.	Fixed, -35β	Variable, 0.87β max.
Burst width	50 μsec	1 μsec (linac)	5 μsec (linac) 10 μsec (overall)
Driving source, S_0		5×10^{15} neutrons/sec	5×10^{17} neutrons/sec
Peak power	40 Mw	7.2 Mw	24,000 Mw
Mean power	30 kw	720 watts	24 Mw

peak powers in the pulse, although the same mean power is maintained. In practice limitations of the device caused by thermal and mechanical shock will be approached and will require detailed study.

APPLICATION OF THE ACCELERATOR BOOSTER

Since we have claimed that the linac booster is superior in most respects to the mechanically pulsed reactor, it would seem legitimate

to make similar claims for its use as are made for SORA. This is the case for short-lived-isotope production and radiation-effects studies where the higher peak powers and improved flexibility of the linac booster provide a superior operating performance; this is not the case, however, for time-of-flight measurements. The point is that with a mechanical booster one would use it whenever possible. With a linac booster one has both a linear accelerator and a fast reactor. It can be shown that in many cases the performance of the accelerator by itself is better than that of the coupled device. Consequently, if one has both a linac and a booster, he can afford to be particular about when they should be coupled.

If time-of-flight measurements were to be made at fast-neutron energies (say above 3 kev) with the linac booster, a neutron chopper would be required. The chopper, however, reduces the effective source intensity by a factor of up to 100 because the viewing area is much smaller with a chopper than it is without a chopper. In addition, timing resolution of the chopper, 0.5 μ sec at best, is much larger than the useful minimum burst width of 20 nsec of the straight linac. The net effect of the timing and viewing-area restrictions is that required intensities from the booster are unrealistic, and thus the booster is not as efficient as the straight linac in the energy range above 3 kev. The 3-kev limit is obtained when the slowing-down time is the major contribution to the timing uncertainty. The slowing-down time of a neutron in a hydrogenous material to 3 kev is roughly equal to the shortest useful burst width of the linac (20 nsec).

Even with a chopper, the booster will be competitive with the straight linac below 10 ev. When the booster is used, however, the linac burst width and reactivity level of the booster can be varied so as to optimize the source intensity and overall burst width. It can be shown in principle that a higher effective source intensity can be obtained by the adjustment of the variables of the booster device to produce a short natural burst width rather than by the use of a chopper. In practice a design for the lower reactivity level would probably be different. This is true at all energies below 3 kev, but it is of particular usefulness between 1 ev and 3 kev, since below 1 ev a chopper would not normally be employed anyway. Thus we are led to the conclusion that, if the booster is to be used, it should not be used in conjunction with a neutron chopper. Second, we have concluded that the linac booster is not competitive with the straight linac for time-of-flight studies above 3 kev. These contentions are supported when the cost of the booster device, the neutron chopper, and the longer flight paths are taken into account.

The most suitable areas then for the application of the linac booster will be in the low epithermal and thermal energies where a

neutron chopper is not necessary. There the moderation time is of the order of the source burst width so that advantage can be taken of higher neutron yield despite its being coupled to a larger burst width. It should be noted that even though the source burst width is of comparable size to the thermalization time, the source width should not be allowed to increase more than is absolutely necessary. This viewpoint differs from that of Raievski,³ but it can be illustrated clearly by Eq. 3. If one takes typical values for the slowing-down and diffusion time, for example, $t_2 = 30 \mu\text{sec}$, for the electronics uncertainty, for example, $t_3 = 2 \mu\text{sec}$, and for the channel width, for example, $W = 10 \mu\text{sec}$, it obviously makes a difference to the total timing uncertainty, ΔT , whether the burst width, t_1 , is $50 \mu\text{sec}$ as for the SORA reactor or $10 \mu\text{sec}$ for the linac booster.

Important applications of the linac booster will be in the measurement of double differential scattering cross sections, particularly for the common moderators. The booster will allow production of an intense cold pulsed neutron source for inelastic scattering measurements and diffraction studies. The device will also permit total cross sections and neutron spectra in the thermal energy range to be measured with higher precision than is now possible. The device will allow measurements that are better than those with a straight linac in the case of neutron spectra that are strongly positionally dependent and require small collimators for a precise definition of the measurement point (cell spectra, for instance). In addition, the greater intensity of the booster would improve spectral measurements in multiplying assemblies where a longer flight path is necessary because of the long mean emission time from the assembly.

It should also be mentioned that, by adjustment of the reactivity level, the output of the device for a few pulses may be increased to make it appropriate for weapon-effect studies and other transient radiation damage phenomena. In situations such as these, the capability of the device to produce short-lived isotopes is superior to that of the mechanically pulsed reactor since much greater control over pulse duration is obtainable.

SUMMARY

The main potentialities and advantages of the linac booster appear to be as follows:

1. Peak thermal flux levels of 10^{16} neutrons/cm²/sec can be attained at a fraction of the cost of typical advanced high flux reactors.
2. Signal-to-background ratios are greatly improved over those in steady-state high flux reactors and are improved in most energy regions over those for mechanically pulsed reactors.

3. Neutron burst widths significantly shorter than those for mechanically pulsed reactors can be obtained. In addition, pulse widths can be varied easily and independently of the pulse repetition rate.

4. Moving components possessing substantial reactivity worth can operate at slower speeds than those for a mechanically pulsed reactor, since the reactivity variation does not establish the pulse width. Reactor safety is relatively good since there is no operation above prompt critical.

5. Pulse repetition rate and power level can be easily changed without varying the reactivity.

6. Controlled production of very short-lived isotopes is possible, and radiation-damage effects can be easily studied by means of the linac booster.

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Dynamic Reactivity Effect— A Further Development

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ABSTRACT

In a recent publication¹ a prediction has been made, based on some results obtained from simple diffusion theory, that there possibly may exist a dynamic reactivity effect in a compressible fissile medium whose geometric size is forced to change periodically. Within the assumptions inherent in the mathematical model representing the physical behavior of the system, it was found, according to diffusion theory, that, if one were able to force the rate of change of the size of the system faster than the neutron population could adjust itself to the said change, a net reactivity gain could be obtained.

Supporting studies have been made by the use of the telegrapher's equation for the slab geometry and the diffusion equation for spherical and cylindrical geometries. The conditions for the dynamic reactivity effect (DRE) are presented for each geometry. Also, some additional reasoning pertaining to the physical causes of the DRE is presented.

INTRODUCTION

The present paper contains some further developments of the DRE. First of all, it was felt that a neutron-balance equation with higher order time derivatives was needed to put the diffusion-theory results on a firmer foundation. Thus the telegrapher's equation of neutron balance was applied to a compressible fissile system of slab geometry. The increased mathematical complexity prohibited a complete analytical solution, and at this time only asymptotic results have been obtained for

small frequencies. These results, however, do support the existence of the DRE, and, in addition, they do not exhibit some of the anomalous properties associated with the simple diffusion-theory results.

Second, it was noted that the slab geometry contained some unique traits from the point of view of static geometry criticality criteria; e.g., a critical slab at a given width remains critical at any other width so long as the expansion or compression to the new width is performed slowly. This peculiarity, coupled with the fact that any practically realizable system would necessarily be of finite size, prompted the study of the DRE in finite geometries, specifically in spherical and cylindrical cases. In both geometries, results similar to that of the slab geometry were obtained.

MATHEMATICAL MODEL OF THE COMPRESSIBLE SYSTEM

The basic assumptions inherent in the mathematical model representing the physical system are essentially the same as in Ref. 1. They are listed briefly.

1. Neutrons are all at a single energy.
2. Compression and expansion are isothermal.
3. All neutrons from fission are prompt.
4. The density of the compressible medium changes uniformly and instantaneously throughout the system at any given time.
5. A perfect absorber, e.g., vacuum, exists beyond the boundaries of the system.

The last assumption has been made for the sake of simplicity, although reflected systems can also be treated.

Although the motion of the boundaries of the system is considered the same for all geometries, the resulting density and cross-section variation differs from one geometry to another. Relevant formulae describing the periodic compression of the various systems are listed.

1. Slab geometry (see Fig. 1)

$$x_B(t) = \frac{x_0}{2} (1 + h \cos \omega t) \quad (1a)$$

$$\rho(t) = \frac{\rho_0}{1 + h \cos \omega t} \quad (1b)$$

$$\Sigma(t) = \frac{\Sigma_0}{1 + h \cos \omega t} \quad (1c)$$

where $x_B(t)$ = time-dependent position of the boundary

$\rho(t)$ = time-dependent density

x_0 = mean slab width, including the linear extrapolation distance

$\Sigma(t)$ = time-dependent macroscopic cross section

ρ_0, Σ_0 = density and macroscopic cross section corresponding to the slab at its mean width

h = amplitude of boundary oscillation ($0 \leq h < 1$)

ω = frequency of boundary oscillation

Fig. 1 - Semi-infinite slab configuration.

2. Spherical geometry (see Fig. 2)

$$r_B(t) = r_0(1 + h \cos \omega t) \quad (2a)$$

$$\rho(t) = \frac{\rho_0}{(1 + h \cos \omega t)^3} \quad (2b)$$

$$\Sigma(t) = \frac{\Sigma_0}{(1 + h \cos \omega t)^3} \quad (2c)$$

where h , ω , r_B , r_0 , ρ , and Σ have interpretations analogous to the previous case.

3. Cylindrical geometry (see Fig. 3)

$$z_B(t) = \frac{z_0}{2} (1 + h \cos \omega t) \quad (3a)$$

$$\rho(t) = \frac{\rho_0}{1 + h \cos \omega t} \quad (3b)$$

$$\Sigma(t) = \frac{\Sigma_0}{1 + h \cos \omega t} \quad (3c)$$

where z_0 is the mean cylinder height, including linear extrapolation distance.

APPLICATION OF THE TELEGRAPHER'S EQUATION TO THE BARE, SOURCE-FREE, COMPRESSIBLE SLAB REACTOR

For a bare slab reactor without external sources, the telegrapher's equation describing the neutron balance is one of the form

$$\phi_{tt} + \frac{\alpha_1}{1 + h \cos \omega t} \phi_t + \frac{\alpha_2(1 + \beta h \sin \omega t)}{(1 + h \cos \omega t)^2} \phi = c^2 \phi_{xx} \quad (4)$$

where ϕ = neutron flux

$$\alpha_1 = v\Sigma_{t0}[1 + (\Sigma_{a0}/\Sigma_{t0})(1 - k_\infty)] \equiv v\Sigma_{t0} \delta$$

$$\alpha_2 = v^2\Sigma_{t0}\Sigma_{a0}(1 - k_\infty)$$

$$\beta = \omega/\Sigma_{t0}v$$

Fig. 2 - Spherical configuration.

Fig. 3 - Cylindrical configuration (axial compression mode).

$$c^2 = v^2/3$$

v = neutron velocity

Σ_{t0} = macroscopic transport cross section at mean boundary position

Σ_{a0} = macroscopic absorption cross section at mean boundary position

k_{∞} = infinite multiplication factor = ηf

η = average number of neutrons liberated per neutron absorbed in the fuel

f = thermal utilization

The boundary conditions are

$$\phi(x, 0) = F(x) \quad (|x| \leq \infty)$$

$$\phi \left[\pm \frac{x_0}{2} (1 + h \cos \omega t), t \right] = 0$$

$$\phi(x, t) = 0 \quad \left[|x| \geq \frac{x_0}{2} (1 + h \cos \omega t) \right]$$

When the condition

$$\frac{h\omega x_0}{v} \ll 1$$

is satisfied, Eq. 4 can be simplified to a point where a separation of variables in a suitably transformed plane can be made. Thus one gets solutions of the form (see Appendix A)

$$\phi_n(x, t) = A_n \cos \frac{n\pi x}{x_0(1 + h \cos \omega t)} \cdot T(t) \quad (n = 1, 3, 5, \dots) \quad (5)$$

where

$$T(t) = S(z)$$

is the solution of the second-order differential equation

$$\epsilon^2 \frac{d^2 S}{dz^2} + \frac{4\epsilon}{\sqrt{2}} g(z) \frac{dS}{dz} + [2v(z) + m^2] S = 0 \quad (6)$$

with

$$\epsilon = \frac{3\sqrt{2}}{4} \frac{\omega \Sigma_{t0} x_0^2}{v} \delta \sqrt{1-h}^2$$

$$g(z) = \frac{3}{2} \Sigma_{t0}^2 x_0^2 \delta^2 (1-h) \frac{\sum_{i=1}^5 \sigma_{1i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{2i} \tan^{2i-2} z}$$

$$v(z) = \frac{9}{2} \Sigma_{t0}^3 \Sigma_{a0} x_0^4 \delta^2 (1-k_\infty) \frac{\sum_{i=1}^5 \sigma_{3i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{2i} \tan^{2i-2} z}$$

$$m^2 = 6 \left(\frac{n\pi}{2} x_0 \Sigma_{t0} \delta \right)^2 \quad (n = 1, 3, 5, \dots)$$

$$A_n = \frac{4}{x_0(1+h)} \int_0^{1/2 x_0(1+h)} F(x) \cos \frac{n\pi x}{x_0(1+h)} dx \quad (n = 1, 3, 5, \dots)$$

Because of the obvious complexity of the coefficients in Eq. 6, the first attempt was to seek an asymptotic solution as ϵ^2 became very small, or, in effect, as $\omega \rightarrow 0$. The solution obtained is of the form

$$T(t) = \exp \left[-\frac{\sqrt{2}}{4\epsilon} \int_0^{\tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right)} \frac{2v(z) + m^2}{g(z)} dz \right] \quad (7)$$

The boundary between diverging and decaying solutions is the criticality condition

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{1}{2} \frac{(1-h) \int_0^\pi \frac{\sum_{i=1}^5 \sigma_{2i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{1i} \tan^{2i-2} z} dz}{\int_0^\pi \frac{\sum_{i=1}^5 \sigma_{3i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{1i} \tan^{2i-2} z} dz} \quad (8)$$

where

$$\bar{B}_m^2 = \text{material buckling of the slab reactor at its mean width} \\ = 3 \sum_{a0} \sum_{t0} (k_{\infty} - 1)$$

$$\bar{B}_g^2 = \text{geometric buckling of the slab reactor at its mean width} = (\pi/x_0)^2$$

Figure 4 shows a graphical comparison of the lines of criticality as predicted by diffusion theory and by the asymptotic solution of the telegrapher's equation for small frequencies. Higher frequencies are presently being studied by the use of a numerical procedure. Since Floquet theory predicts solutions of Eq. 6 in terms of an exponential part times a periodic function, i.e.,

$$S(z) = e^{\mu z} B(z)$$

where

$$B(z + 2\pi) = B(z)$$

then criticality conditions would be obtained from the nature of the characteristic exponent μ . For critical operation, i.e., excluding diverging and decaying solutions, we can set up the criterion that μ be a pure imaginary number. Thus the numerical approach is to search for the particular sets of system constants (ω , h , Σ_{t0} , $\bar{B}_m^2$, and $\bar{B}_g^2$) which yield a pure imaginary value for μ . A straightforward numerical scheme for the computation of μ can be found in Ref. 2.

It should be mentioned that some of the peculiarities that were found in the time behavior of the solutions of the diffusion equation are notably absent in $T(t)$ as given by Eq. 7. Specifically, for the critical and supercritical cases, it was noted that the oscillatory flux at any given point in the slab had constant and increasing peaks, respectively; this is to be expected. The minima, however, were situated on an exponentially decreasing locus in both the critical and the supercritical cases. Figure 5 shows a qualitative sketch of the relative flux at a point x_1 in a critical slab reactor. In the same figure the corresponding case as predicted by Eq. 7 is also shown.

As noted in Ref. 1, it is quite disturbing that the flux minima at any point in the critical oscillating slab continually approached zero, while the maxima remained constant. Since the solution describing this flux behavior is compatible with the usual solution of the diffusion equation for a stationary slab as ω or h approaches zero, it was difficult to find a reasonable explanation for this anomalous effect.

As seen in Fig. 5, however, the telegrapher's equation predicts a reasonable type of flux behavior for the critical slab, namely, a sustained

Fig. 4 - Comparison of the criticality ratio $\bar{B}_m^2/\bar{B}_g^2$ as predicted by diffusion theory (high frequencies) and the telegrapher's equation (low frequencies).

oscillation about a mean value. It should be pointed out that, although the curve given by Eq. 7 in Fig. 5 corresponds to the limiting case of $\omega \rightarrow 0$, a similar type of behavior should be expected from the non-asymptotic solution of Eq. 7. This is based on the fact that sustained oscillations are predicted by Floquet theory for the critical slab, as mentioned previously.

Briefly, then, one can see that the use of the telegrapher's equation for a neutronics problem of this nature may have more validity than diffusion theory, but associated with it is the increased complexity of the mathematics, wherein only asymptotic cases can be handled readily. It should be stressed, however, that at present the main interest is in validating predictions of the DRE for various geometries. This usually involves a criticality criterion of some form, but it does not necessitate the accurate prediction of the space-time flux distribution. To put it

Fig. 5 - A qualitative comparison of the relative neutron flux at a point x_1 in the slab reactor as a function of ωt , as predicted by the diffusion and the telegrapher's equations.

more clearly, the approximation which allowed a separation of variables and thus the derivation of the solution given by Eq. 5 may cause an inaccurate flux shape prediction; i.e., the shape function

$$\cos \frac{n\pi x}{x_0(1 + h \cos \omega t)}$$

may not be entirely correct. From the point of view of criticality, however, this is not important, since criticality criteria are determined solely by the function $T(t)$; and $T(t)$ is not affected by the previously mentioned approximation. For this reason we find diffusion theory adequate as a first approximation.

APPLICATION OF THE DIFFUSION EQUATION TO BARE, SOURCE-FREE COMPRESSIBLE REACTORS OF FINITE GEOMETRY

Spherical Geometry

Keeping in mind Eqs. 2 a and 2 c, we find the diffusion equation for the compressible bare sphere without sources to be of the form

$$\frac{\delta}{v} \phi_t + \left[1 + \frac{3h\omega}{\Sigma_{t0} v} \sin \omega t (1 + h \cos \omega t)^2 \right] \frac{\Sigma_{a0} (1 - k_{\infty})}{(1 + h \cos \omega t)^3} \phi$$

$$= \frac{(1 + h \cos \omega t)^3}{3\Sigma_{t0}} \nabla^2 \phi \quad (9)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}$$

and the boundary conditions and initial conditions are

$$\phi(r, 0) = F(r) \quad (|r| < \infty)$$

$$\phi[r_0(1 + h \cos \omega t), t] = 0$$

$$\phi(r, t) = 0 \quad [r \geq r_0(1 + h \cos \omega t)]$$

The solution of Eq. 9 is derived in Appendix B and is of the form

$$\phi(r, t) = \left(\frac{1 + h \cos \omega t}{1 + h} \right)^{3\tau} \sum_{i=1,3,5,\dots}^{\infty} \left[4 \int_0^1 F(r) \sin \frac{l\pi r}{r_0(1+h)} \frac{dr}{r_0(1+h)} \right]$$

$$\times \exp \left\{ \frac{4h}{\sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right) - \frac{3\Sigma_{t0} v}{\omega} \tau \right.$$

$$\times \left(\omega t - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+3) h^{n+1} \int \cos^{h+1} \omega t \, d\omega t \right)$$

$$+ \frac{l^2 \Sigma_{t0} v}{\omega} \frac{\bar{B}_g^2}{\bar{B}_m^2} 3\tau \left[\left(1 + \frac{3}{2} h^2 \right) \omega t + \left(1 + \frac{1}{2} h \cos \omega t + \frac{1}{3} h^2 \right) 3h \right.$$

$$\left. - \frac{1}{3} h^3 \sin^3 \omega t \right] \frac{1}{(1 + h \cos \omega t)^2} \Bigg\}$$

$$\times \frac{r_0(1 + h \cos \omega t)}{r} \sin \frac{l\pi r}{r_0(1 + h \cos \omega t)} \quad (10)$$

where

$$\tau \equiv \frac{(\Sigma_{a0}/\Sigma_{t0})(k_{\infty} - 1)}{1 + (\Sigma_{a0}/\Sigma_{t0})(k_{\infty} - 1)}$$

and $\bar{B}_m^2$ and $\bar{B}_g^3$ are interpreted as the material and geometric buckling, respectively, of the sphere at its mean radius, r_0 .

The criticality conditions are of the form

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{-\left(1 + \frac{3}{2}h^2\right) \bigg/ \left(1 + h\right)^2}{\left(1 + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (n+2)h^{2n+1}\right) - \frac{2h\omega}{3\tau\Sigma_{t0}v\sqrt{1-h^2}}} \quad (11)$$

which does in fact reduce to the familiar static criticality condition when $h \rightarrow 0$. The similar process of letting $\omega \rightarrow 0$ cannot be done in Eq. 11, since it was derived on the basis that ω does not approach zero. It is shown, however, in Appendix B that the criticality conditions for $\omega \rightarrow 0$ correspond to those obtained in the static case. For very high frequencies, on the other hand, one obtains the approximation

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{3\tau\Sigma_{t0}v}{\omega} \frac{\left(1 + \frac{3}{2}h^2\right) \sqrt{1-h^2}}{2h(1+h)^2} \quad (12)$$

This indicates the sensitivity of the DRE to the inverse of the frequency.

Cylindrical Geometry

In the case of the cylinder, we have the choice of two modes of compression, namely, lateral and axial. For the sake of simplicity, we choose the axial-compression case, with the associated motion described by Eqs. 3a and 3c. The diffusion equation becomes

$$\frac{\delta}{v} \phi_t + \frac{1 + (\beta/v) \sin \omega t}{(1 + h \cos \omega t)} \Sigma_{a0}(1 - k_{\infty})\phi = \frac{(1 + h \cos \omega t)}{3\Sigma_{t0}} \nabla^2 \phi \quad (13)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and the boundary conditions and the initial conditions are

$$\phi(r_B, z, t) = 0$$

$$\phi \left[r, \frac{z_0}{2} (1 + h \cos \omega t), t \right] = 0$$

$$\phi(r, z, 0) = F(r, z)$$

$$\phi(r, z, t) = 0 \quad (r \geq r_B) \quad \left[|z| \geq \frac{z_0}{2} (1 + h \cos \omega t) \right]$$

where r_B is the radial boundary including the linear extrapolation distance.

The solution of Eq. 13, derived in Appendix C, is of the form

$$\begin{aligned} \phi(r, z, t) = & \left(\frac{(1 + h \cos \omega t)}{1 + h} \right)^{\tau-1} \\ & \times \sum_{l=1,3,5 \dots}^{\infty} \left[4 \int_0^{1/2} F(r, z) \cos \frac{l\pi z}{z_0(1+h)} \frac{dz}{z_0(1+h)} \right] \\ & \times \exp \left\{ - \left[\frac{l^2 \pi^2 (\omega t + h \sin \omega t)}{(1 + h \cos \omega t)^2} + \pi^2 \frac{\bar{B}_{gr}^2}{\bar{B}_{ga}^2} \omega t + 2\pi^2 \frac{\bar{B}_m^2}{\bar{B}_{ga}^2} \right. \right. \\ & \times \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right) \left. \left. \left(\frac{v}{3 \Sigma_{t0} z_0^2 \omega \delta} \right) \right\} \right. \\ & \times \cos \frac{l\pi z}{z_0(1+h \cos \omega t)} \Bigg) R(r) \end{aligned} \quad (14)$$

If we write the geometric buckling of the cylinder at its mean height, it will be of the form

$$\bar{B}_g^2 = \left(\frac{2.405}{r_B} \right)^2 + \left(\frac{\pi}{z_0} \right)^2 \equiv \bar{B}_{gr}^2 + \bar{B}_{ga}^2 \quad (15)$$

Then we obtain the criticality condition that

$$\frac{\bar{B}_m^2}{\bar{B}_{ga}^2} = \sqrt{1-h^2} \frac{\bar{B}_{gr}^2}{\bar{B}_{ga}^2} + \frac{\sqrt{1-h^2}}{(1+h)^2} \quad (16)$$

where $\bar{B}_m^2$ is the material buckling of the cylinder at its mean height. For $r_B \rightarrow \infty$, Eq. 16 reduces to the results obtained for the slab case, namely,

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{\sqrt{1-h^2}}{(1+h)^2} \quad (17)$$

SOME PHYSICAL INTERPRETATIONS OF THE DRE

Although some insight in the physical explanation of the DRE has been presented in Ref. 1, the full understanding of the behavior of a physical compressible system is by no means available. In fact, a limit is reached to the amount of information that can be extracted from any further interpretation of the dynamics of the mathematical model used to describe the actual system behavior. This limit is determined in part by the inherent discrepancies that will always exist between the physical system and the mathematical model.

Some further discussion of the causes responsible for the DRE is still desirable at this point, however, especially if an account of the phenomena can be made by the use of the physical arguments that are more or less independent of the mathematical model used.

Figures 6 to 8 show qualitatively the behavior of material and geometric bucklings of static systems as a function of the geometric size of the system. In the case of the slab, because of the fact that two of the three spatial dimensions of the slab are infinite, the material and geometric bucklings belong to the same functional family of curves, namely, hyperbolae. Because of this unique situation, we can immediately infer that for criticality, when the fuel-moderator loading is adjusted properly in relation to the geometric size of the system, the two curves in Fig. 6 would coincide and criticality would be maintained independently of slab width, provided, of course, that the slab width is changed in a quasi-steady fashion.

From experience we know that for finite-size reactors, spherical or cylindrical in shape, a concept of critical geometric size is meaningful. This is borne out in Figs. 7 and 8, where the respective curves for geometric and material buckling intersect, the intersection occurring at a

Fig. 6 - A qualitative plot of the material and the geometric bucklings of a static slab reactor of constant unit mass per unit height as a function of slab width.

Fig. 7 - A qualitative plot of the material and the geometric bucklings of a static spherical reactor of constant total mass as a function of the sphere radius.

Fig. 8 - A qualitative plot of the material and the geometric bucklings of a static cylindrical reactor of constant total mass as a function of the cylinder height.

critical radius in the case of the sphere and at a critical height in the case of the cylinder.

The preceding concepts are true only for static systems. What about systems where the size and density of the system change as a function of time in a prescribed manner? The simple substitution of time-dependent system size into the expression for the geometric buckling or the substitution of time-dependent macroscopic cross sections into the expression for the material buckling is meaningless. The parameter called "buckling" is derived from a wave equation applicable only to a static system. The only possibility of obtaining a meaning for the buckling in a dynamic system is to make an association of the buckling with some physical concept. Thus, in a very general sense, one may postulate that in a static system the geometric buckling signifies a nonleakage probability, whereas the material buckling is associated with a noninteraction probability. At this point, however, it becomes more meaningful to transcend to the dynamic system and try to reason out how the nonleakage and noninteraction probabilities behave as a function of space and time. In this perspective one can carry out some physical

"thought experiments." When a boundary of a reactor is moving away from a free neutron, the situation can be depicted qualitatively as shown in Fig. 9. Linear expansion is considered here for the sake of simplicity.

Fig. 9 - A configuration of a static and a linearly expanding slab reactor in the (x,t) plane. For simplicity the wall velocity has been chosen such that the neutron flight path in the compressible medium is of the same length for both the static and the expanding slab reactors.

In this specific example it is seen that the neutron flight path is of the same length whether the boundary is stationary or moving. The average density of the medium as experienced by the neutron, however, is higher in the case of the moving boundary. The net effect is increased non-leakage probability.

At first glance this maneuver does not seem to gain very much, since one can argue in the opposite sense; i.e., the walls have to be brought back together again for cyclical operation, and the increased leakage during this part of the cycle would result in an equivalent loss. If we keep in mind, however, that we are dealing with a multiplying system, a new gain of neutrons is not necessarily forbidden. To illustrate this, we consider a very general form of the neutron balance in a sub-critical system:

$$\left. \begin{aligned} & \left(\text{production} \right) - \left(\text{absorption} \right) - \left(\text{leakage} \right) < 0 \\ & \text{rate} \quad \text{rate} \quad \text{rate} \end{aligned} \right\} \text{static system}$$

During a linear expansion of the slab boundaries, the production and the absorption terms are characterized by the time-dependent fission and the absorption cross sections, respectively. This is qualitatively shown in Fig. 10. The important thing to note is that the resulting curve is a function of position only. The leakage, on the other hand, is definitely a function of velocity, as shown in the figure. If the slab is sufficiently close to criticality to begin with, then a high enough wall velocity might reduce leakage to the point where we would have the following:

$$\left(\begin{array}{c} \text{production} \\ \text{rate} \end{array} \right) - \left(\begin{array}{c} \text{absorption} \\ \text{rate} \end{array} \right) - \left(\begin{array}{c} \text{leakage} \\ \text{rate} \end{array} \right) \geq 0 \quad \left. \vphantom{\begin{array}{c} \text{production} \\ \text{rate} \end{array}} \right\} \begin{array}{l} \text{static} \\ \text{system} \end{array}$$

During this part of the cycle, we would have no limit to the net gain of neutrons. It is conceivable that one could generate enough "extra" neutrons to overcome the high losses during the next part of the cycle (compression) and have enough to start the next cycle with.

Although this has been a highly qualitative discussion of the physical phenomena involved, nevertheless, it is the authors' opinion that some physical explanation of the DRE has been achieved.

Fig. 10 - A qualitative sketch showing the functional dependence of the neutron leakage probability on the wall velocity of an expanding slab. V = wall velocity; $V_2 > V_1$.

CONCLUSIONS

In this paper three new results have been presented with respect to the DRE:

1. The telegrapher's equation predicts the DRE for low frequencies of oscillation.
2. Criticality relations for spherical and cylindrical geometries are obtained.
3. Some thought experiments are described, wherein a physical interpretation, slightly different from that of Ref. 1, is made.

It is hoped that future work on the telegrapher's equation for higher frequencies will amplify the understanding of the DRE. Also of considerable interest is the study of the effect of delayed neutrons on the DRE, where phase relations between oscillation frequencies and delayed-neutron time constants should show attenuation or resonance phenomena.

APPENDIX A

SOLUTION OF THE TELEGRAPHER'S EQUATION FOR THE BARE SOURCE-FREE, COMPRESSIBLE SLAB REACTOR

For a bare, compressible slab reactor without external sources, the telegrapher's equation, represented by Eq. 4, can be transformed so that the moving boundaries become fixed. The transformation is of the form

$$\eta = \frac{x}{x_0(1 + h \cos \omega t)}$$

$$\xi = \frac{v}{3 \sum_{t_0} X_0^2 \delta} t = \gamma_1 t$$

$$\phi(x, t) = \psi(\eta, \xi)$$

$$\omega' = \frac{\omega}{\gamma_1}$$

where δ is defined in Eq. 4. The transformed equation is

$$\begin{aligned}
& \left[\frac{(\eta h \omega)^2 \sin^2 \omega' \xi - \frac{1}{3} \left(\frac{v}{x_0} \right)^2}{1 + h \cos \omega' \xi} \right] \psi_{\eta\eta} + \eta h \omega \left[\omega \cos \omega' \xi \right. \\
& + \left. \frac{v \sum_{t_0} \delta \sin \omega' \xi + h \omega \sin^2 \omega' \xi}{1 + h \cos \omega' \xi} \right] \psi_{\eta} + \left[\frac{v^2 \sum_{t_0} \sum_{a_0} (1 - k_{\infty}) \left(1 + \frac{h \omega}{\sum_{t_0} v} \sin \omega' \xi \right)}{1 + h \cos \omega' \xi} \right] \psi \\
& + \left[\frac{v^2 (1 + h \cos \omega' \xi)}{9 \sum_{t_0}^2 x_0^4 \delta^2} \right] \psi_{\xi\xi} \\
& + \left[\frac{1}{3} \left(\frac{v}{x_0} \right)^2 \right] \psi_{\xi} + \left[\frac{2 \eta h \omega v \sin \omega' \xi}{3 \sum_{t_0} x_0^2 \delta} \right] \psi_{\eta\xi} = 0 \quad (\text{A.1})
\end{aligned}$$

with the new boundary conditions

$$\begin{aligned}
\psi(\eta, 0) &= P(\eta) \\
\psi(\pm \frac{1}{2}, \xi) &= 0 \\
\psi(\eta, \xi) &= 0 \quad (|\eta| \geq \frac{1}{2}) \\
P(\eta) &= F(x)
\end{aligned}$$

Neglecting terms of order $h\omega/\sum_{t_0} v$ in the coefficients of Eq. A.1, we can write the equation in the canonical form

$$A\psi_{\eta\eta} + 2B\psi_{\eta\xi} + C\psi_{\xi\xi} = D \quad (\text{A.2})$$

where

$$\begin{aligned}
A &= \frac{-\frac{1}{3} \left(\frac{v}{x_0} \right)^2}{1 + h \cos \omega' \xi} & C &= \frac{v^2 (1 + h \cos \omega' \xi)}{9 \sum_{t_0}^2 x_0^4 \delta^2} \\
2B &= \eta \frac{2h\omega v}{3 \sum_{t_0} x_0^2 \delta} \sin \omega' \xi & D &= D(\eta, \xi, \psi_{\eta}, \psi_{\xi})
\end{aligned}$$

The discriminant of Eq. A.2 is found to be positive, i.e.,

$$B^2 - AC > 0$$

and so Eq. A.2 is a second-order partial differential equation of the hyperbolic type. Using the notation of Shapiro,³ we can write the equation of characteristics in the form

$$\left(\frac{d\xi}{d\eta}\right)_{\text{characteristic}} = \frac{B \pm \sqrt{B^2 - AC}}{A} \quad (\text{A.3})$$

For

$$\frac{h\omega x_0}{v} \ll 1$$

the equation of characteristics simplifies to

$$\left(\frac{d\xi}{d\eta}\right)_{\text{characteristic}} \approx \frac{1 + h \cos \omega' \xi}{\sqrt{3} \Sigma_{t0} x_0 \delta} \quad (\text{A.4})$$

We see that the characteristics, as given by Eq. A.4, for the case where $h \rightarrow 0$, form a mesh of intersecting straight lines in the (η, ξ) plane (Fig. 11). For nonvanishing values of h , on the other hand, the character-

Fig. 11 - An illustration of the characteristics in the (η, ξ) plane, as generated by Eqs. A.5 and A.6.

istics oscillate about the straight-line approximations. Integration of Eq. A.4 gives us the two families of characteristics, namely,

$$\frac{2}{\omega' \sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega' \xi}{2} \right) = \frac{\eta}{\sqrt{3} \Sigma_{t0} x_0 \delta} + \text{constant} \quad (\text{A.5})$$

$$\frac{2}{\omega' \sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega' \xi}{2} \right) = \frac{-\eta}{\sqrt{3} \Sigma_{t0} x_0 \delta} + \text{constant} \quad (\text{A.6})$$

To transform Eq. A.2 to the normal form of the hyperbolic equation, we define new independent variables⁴

$$\theta = \frac{2}{\omega' \sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega' \xi}{2} \right) - \frac{\eta}{\sqrt{3} \Sigma_{t0} x_0 \delta} \quad (\text{A.7})$$

$$\mu = \frac{2}{\omega' \sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega' \xi}{2} \right) + \frac{\eta}{\sqrt{3} \Sigma_{t0} x_0 \delta} \quad (\text{A.8})$$

The effect of the transformation is to rotate the plane through 45° and also to straighten out the wavy characteristics into a linear orthogonal grid in the (θ, μ) plane. The new independent variables θ and μ transform Eq. A.2 into the normal hyperbolic form

$$L(\psi) \equiv \psi_{\theta\mu} + a\psi_{\theta} + b\psi_{\mu} + c\psi = 0 \quad (\text{A.9})$$

where*

$$\begin{aligned} \left(\begin{array}{c} a \\ b \end{array} \right) = & - \frac{\frac{1}{3} \left(\frac{v}{x_0} \right)^2}{(1+h) \cos^2 \omega' \xi + (1-h) \sin^2 \omega' \xi} \left(\begin{array}{c} + \\ - \end{array} \right) \frac{\eta}{\sqrt{3} \Sigma_{t0} x_0 \delta} \left(\frac{h \omega^2 \cos \omega' \xi + \frac{v \Sigma_{t0} h \omega \delta \cos \omega' \xi + h^2 \omega^2 \sin^2 \omega' \xi}{1+h \cos \omega' \xi}}{2 \left(\frac{v^2}{9 \Sigma_{t0}^2 x_0^4 \delta^2} \right) \left(\frac{1}{1+h \cos \omega' \xi} + \frac{1+h \cos \omega' \xi}{[(1+h) \cos^2 \omega' \xi + (1-h) \sin^2 \omega' \xi]^2} \right)} \right) \end{aligned} \quad (\text{A.10})$$

$$c = \frac{\frac{v^2 \Sigma_{t0} \Sigma_{a0} (1-k_{\infty})}{1+h \cos \omega' \xi}}{2 \left(\frac{v^2}{9 \Sigma_{t0}^2 x_0^4 \delta^2} \right) \left(\frac{1}{1+h \cos \omega' \xi} + \frac{1+h \cos \omega' \xi}{[(1+h) \cos^2 \omega' \xi + (1-h) \sin^2 \omega' \xi]^2} \right)} \quad (\text{A.11})$$

At this point a standard numerical procedure could be employed in solving for a detailed map of $\psi(\theta, \mu)$ in the (θ, μ) plane. We are interested

*In $\left(\begin{array}{c} a \\ b \end{array} \right)$ in Eq. 10, the a goes with the plus and the b goes with the minus in $\left(\begin{array}{c} + \\ - \end{array} \right)$.

in getting criticality criteria, however, and, to this end, would like to work with an ordinary differential equation instead.

If we consider another transformation of the independent variable, defining

$$s = \frac{1}{\sqrt{2}} (\mu + \theta)$$

$$r = \frac{1}{\sqrt{2}} (\mu - \theta) \quad (\text{A.12})$$

then Eq. A.9 transforms into

$$\psi_{ss} - \psi_{rr} + \frac{4}{\sqrt{2}} g(s) \psi_s + \frac{4}{\sqrt{2}} r q(s) \psi_r + 2v(s) \psi = 0 \quad (\text{A.13})$$

This equation as it stands does not readily lend itself to the method of separation of variables because of the "mixed" coefficient of ψ_r . For purposes of simplification and want of an immediate result, we consider the following comparison of the order of magnitude of the coefficients of Eq. A.13.

$$\psi_{ss} \sim 1$$

$$\psi_{rr} \sim 1$$

$$\psi_s \sim \frac{4}{\sqrt{2}} \mathcal{O} \left(\frac{3}{4} \Sigma_{t0}^2 x_0^2 \delta^2 \right)$$

$$\psi_r \sim \frac{4}{\sqrt{2}} \mathcal{O} \left(\frac{9}{4\sqrt{3}} \frac{h\omega \Sigma_{t0}^2 x_0^3 \delta^2}{v} \right)$$

$$\psi \sim 2 \mathcal{O} \left[\frac{9}{4} \Sigma_{t0}^3 \Sigma_{a0} x_0^4 \delta^2 (1 - k_\infty) \right]$$

Thus it is reasonable to neglect the term involving ψ_r because of the smallness of its coefficient. Therefore we have, approximately, from Eq. A.13

$$\psi_{ss} - \psi_{rr} + \frac{4}{\sqrt{2}} g(s) \psi_s + 2v(s) \psi = 0 \quad (\text{A.14})$$

It is evident from Eq. A.14 that the ψ_{rr} term will yield the familiar cosine modes of flux shape in r . The deleted term involving ψ_r may be interpreted as a "damping term" in the sense that the expected cosine

modes would be distorted somewhat due to its presence. It is felt that this approximation takes away only detailed information about the spatial flux distribution but does not disturb the conditions of criticality associated with the time behavior of the flux.

Considering a separation of variables, we can write

$$\psi(r, s) = R(r) S(s)$$

and so obtain from Eq. A.14

$$\frac{S''}{S} = \frac{4}{\sqrt{2}} g(s) \frac{S'}{S} + 2v(s) = \frac{R''}{R} = -m^2 \quad (\text{A.15})$$

where m^2 is the separation constant. Equation A.15 yields

$$R'' + m^2 R = 0 \quad (\text{A.16})$$

and

$$S'' + \frac{4}{\sqrt{2}} g(s) S' + [2v(s) + m^2] S = 0 \quad (\text{A.17})$$

Applying boundary conditions to the solution of Eq. A.16, we get for the value of m^2

$$m^2 = 6 \left(\frac{n\pi}{2} \sum_{t_0} x_0 \delta \right)^2 \quad (n = 1, 3, 5, \dots) \quad (\text{A.18})$$

Equation A.17, under the transformation

$$\epsilon S = z \quad \left(\epsilon = \frac{\omega'}{4} \sqrt{2} \sqrt{1 - h^2} \right) \quad (\text{A.19})$$

becomes

$$\epsilon^2 S'' + \frac{4}{\sqrt{2}} g(z) \epsilon S' + [2v(z) + m^2] S = 0 \quad (\text{A.20})$$

which, in the limit as $\epsilon \rightarrow 0$, i.e., $\omega \rightarrow 0$, reduces to

$$S' + \frac{\sqrt{2}}{4\epsilon} \left(\frac{2v(z) + m^2}{g(z)} \right) S = 0 \quad (\text{A.21})$$

whose solution is⁵

$$S(z) = \exp \left[-\frac{\sqrt{2}}{4\epsilon} \int_0^z \frac{2v(z) + m^2}{g(z)} dz \right] \quad (\text{A.22})$$

In terms of time t , Eq. A.22 can be written as

$$S(z) = T(t) = \exp \left[-\frac{\sqrt{2}}{4\epsilon} \int_0^{\tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right)} \frac{2v(z) + m^2}{g(z)} dz \right] \quad (\text{A.23})$$

The integral in the exponent of Eq. A.23 describes two functions of time which oscillate about some straight-line loci. When the slopes of these loci are identical, the loci coincide and criticality is achieved. If the loci do not coincide, the difference between these two functions will keep growing with time, and, depending upon the sign of the difference, the solution will either diverge or decay with time. The slope of each locus is given by the average value of the corresponding integrand over one cycle. Thus the values of the two slopes are given by

$$\frac{1}{\pi} \int_0^\pi \frac{2v(z)}{g(z)} dz \quad \text{and} \quad \frac{1}{\pi} \int_0^\pi \frac{m^2}{g(z)} dz \quad (\text{A.24})$$

If the integrands are written in terms of the original parameters and in terms of the geometric and material bucklings defined by

$$\bar{B}_g^2 = \left(\frac{\pi}{x_0} \right)^2 \quad (\text{A.25})$$

$$\bar{B}_m^2 = 3 \sum_{a0} \sum_{t0} (k_\infty - 1)$$

then equating the two slopes gives the equation

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{1}{2} \frac{(1-h) \int_0^\pi \frac{\sum_{i=1}^5 \sigma_{2i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{1i} \tan^{2i-2} z} dz}{\int_0^\pi \frac{\sum_{i=1}^5 \sigma_{3i} \tan^{2i-2} z}{\sum_{i=1}^5 \sigma_{1i} \tan^{2i-2} z} dz} \quad (\text{A.26})$$

where

$$\begin{array}{lll}
 \sigma_{11} = (1-h)^3 & \sigma_{21} = 2(1-h)^4 & \sigma_{31} = (1-h)^4 \\
 \sigma_{12} = (1-h)^2(4-5h) & \sigma_{22} = 4(1-h)^3(2-3h) & \sigma_{32} = 4(1-h)^3(1-3h) \\
 \sigma_{13} = 2(1-h)(3-5h) & \sigma_{23} = 4(1-h)^2(3-5h+9h^2) & \sigma_{33} = 2(1-h)^2(3-10h+19h^2) \\
 \sigma_{14} = 2(1+h)(2-3h+3h^2) & \sigma_{24} = 4(1-h^2)(2-3h-3h^2) & \sigma_{34} = 4(1-h)(1+h)^2(1-3h) \\
 \sigma_{15} = (1+h)^3 & \sigma_{25} = 2(1+h)^2(1+h^2) & \sigma_{35} = (1+h)^4
 \end{array}$$

APPENDIX B

SOLUTION OF THE DIFFUSION EQUATION FOR THE BARE, SOURCE-FREE, COMPRESSIBLE SPHERICAL REACTOR

As in the case of the slab reactor, the diffusion equation represented by Eq. 9 can be transformed through a similar transformation of the independent variables so that the boundaries become fixed. Thus we define

$$\eta = \frac{r}{r_0(1+h \cos \omega t)}$$

$$\xi = \frac{v}{3 \sum_{t0} r_0^2 \delta} t \equiv \gamma_1 t$$

$$\omega' = \frac{\omega}{\gamma_1}$$

$$\phi(r, t) = \psi(\eta, \xi)$$

and accordingly obtain the transformed equation

$$\begin{aligned}
 \frac{a_0}{v(1+h \cos \omega' \xi)} \psi_{\xi} + \left[\frac{a_1 \sin \omega' \xi}{v(1+h \cos \omega' \xi)^2} \eta - \frac{2}{\eta} \right] \psi_{\eta} \\
 + a_2 \left[\frac{1 + (\beta/v) \sin \omega' \xi (1+h \cos \omega' \xi)^2}{(1+h \cos \omega' \xi)^4} \right] \psi = \psi_{\eta\eta} \quad (B.1)
 \end{aligned}$$

where

$$\begin{aligned}
 a_0 &= 3 \sum_{t0} r_0^2 \gamma_1 \delta \\
 a_1 &= 3 \sum_{t0} r_0^2 h \omega \delta \\
 a_2 &= 3 \sum_{t0} \sum_{a0} r_0^2 (1 - k_{\infty}) \\
 \beta &= 3h\omega / \sum_{t0}
 \end{aligned} \quad (B.2)$$

If we now consider the transformation of the dependent variable $\psi(\eta, \xi)$ such that

$$u(\eta, \xi) = \eta \psi(\eta, \xi)$$

then Eq. B.1 transforms into

$$p(\xi) u_{\xi} + q(\xi) \eta u_{\eta} + [s(\xi) - q(\xi)] u = u_{\eta\eta} \quad (\text{B.3})$$

where

$$\begin{aligned} p(\xi) &= \frac{a_0}{v(1 + h \cos \omega' \xi)} \\ q(\xi) &= \frac{a_1 \sin \omega' \xi}{v(1 + h \cos \omega' \xi)^2} \\ s(\xi) &= \frac{a_2 [1 + (\beta/v) \sin \omega' \xi (1 + h \cos \omega' \xi)^2]}{(1 + h \cos \omega' \xi)^4} \end{aligned} \quad (\text{B.4})$$

Equation B.3 is of the same type as the equation obtained in Ref. 1 for the slab with fixed boundaries; thus the same type technique of finding the solution is possible. Briefly the procedure is to apply a Fourier transform of the type

$$\bar{u}(\alpha, \xi) \equiv 2 \int_0^{\infty} u(\eta, \xi) e^{i\alpha\eta} d\eta$$

to Eq. B.3. The transformed equation in $\bar{u}(\alpha, \xi)$ becomes a first-order partial differential equation of the Lagrange type. Its solution is found through direct integration. Transforming back to the (r, t) plane, we get the solution of the form

$$\begin{aligned} \phi(r, t) &= \left(\frac{1 + h \cos \omega t}{1 + h} \right)^{3\tau} \sum_{l=1,3,5,\dots}^{\infty} \left[4 \int_0^1 F(r) \sin \frac{l\pi r}{r_0(1+h)} \frac{dr}{r_0(1+h)} \right] \\ &\times \exp \left\{ \frac{-4h}{\sqrt{1-h^2}} \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right) + \frac{3\Sigma_{t0} v \tau}{\omega} \right. \\ &\times \left[\omega t - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+3)h^{n+1} \int \cos^{n+1} \omega t d\omega t \right] - \frac{l^2 \Sigma_{t0} v}{\omega} \frac{\bar{B}^2}{\bar{B}_m^2} \end{aligned}$$

$$\times 3\tau \left[\frac{(1 + 3/2h^2)\omega t + (1 + 1/2h \cos \omega t + 1/3h^2)3h - 1/3h^3 \sin^3 \omega t}{(1 + h \cos \omega t)^2} \right] \times \frac{r_0(1 + h \cos \omega t)}{r} \sin \frac{l\pi}{r_0(1 + h \cos \omega t)} \quad (\text{B.5})$$

where

$$\tau \equiv \frac{(\Sigma_{a0}/\Sigma_{t0})(1 - k_{\infty})}{1 + (\Sigma_{a0}/\Sigma_{t0})(1 - k_{\infty})}$$

For criticality considerations it is convenient to write the exponent of Eq. B.5 in its equivalent integral form, namely,

$$\frac{2a_1}{\omega'a_0} \int \frac{d\omega t}{1 + h \cos \omega t} - \frac{a_2 v}{\omega'a_0} \int \frac{d\omega t}{(1 + h \cos \omega t)^3} + \frac{l^2 v \pi^2}{\omega'a_0} \times \frac{\int (1 + h \cos \omega t) d\omega t}{(1 + h \cos \omega t)^2} \quad (\text{B.6})$$

In a manner analogous to Ref. 1, we require that the exponentially oscillatory part of the solution not become unbounded. Thus we consider the expression B.6 for $l = 1$. The corresponding terms for $l > 1$ constitute fast-decaying transient contributions to the l -sum in Eq. B.5 and so are not considered in the stability analysis. The first two terms of this expression oscillate about straight-line loci. The third term is also oscillatory, but with a linearly increasing amplitude. Here we consider the locus of points generated by the local minima of the third term. For large values of ωt , i.e., for

$$\omega t = 2N\pi \quad (N \gg 1)$$

the local minima occur at

$$\omega t = 2N\pi + \Delta_N$$

where $\Delta_N \ll 2\pi$ and the slope of the straight-line locus of these points is

$$\frac{1 + 3/2h^2}{(1 + h)^2}$$

We then impose the requirement that the sum of the slopes of the loci about which the various terms oscillate be zero. Thus we have the condition

$$\frac{2h}{\sqrt{1-h^2}} - \frac{3\tau \Sigma_{t0} v}{\omega} \left[1 + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (n+2)h^{2(n+1)} \right] - \frac{3\tau \Sigma_{t0} v}{\omega} \frac{\bar{B}_g^2}{\bar{B}_m^2} \frac{(1+3/2 h^2)}{(1+h)^2} = 0 \quad (B.7)$$

which can be reformulated as

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = \frac{-(1+3/2 h^2)/(1+h)^2}{\left[1 + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (n+2)h^{2(n+1)} \right] - \frac{2h\omega}{\sqrt{1-h^2} v \Sigma_{t0} 3\tau}} \quad (B.8)$$

For the limiting case when $\omega \rightarrow 0$, the value of the same slope becomes simplified to

$$1+h$$

so that the criticality condition is modified to the form

$$\lim_{\omega \rightarrow 0} \left[\frac{2h\omega}{\sqrt{1-h^2}} - \frac{3\tau \Sigma_{t0} v}{(1+h)^3} - 3\tau \Sigma_{t0} v \frac{\bar{B}_g^2}{\bar{B}_m^2} (1+h) \right] = 0$$

and so

$$\frac{\bar{B}_m^2}{\bar{B}_g^2} = (1+h)^4$$

It is important to note that, unlike the slab reactor, the spherical system does not yield a unity ratio of the mean material and geometric bucklings in the limit as $\omega \rightarrow 0$. For a critical sphere this may appear to be disturbing at first. But one should realize that the sphere boundary radius for $\omega = 0$ is

$$r_B = r_0(1+h)$$

On the other hand, the bars in $\bar{B}_m^2$ and $\bar{B}_g^2$ imply static conditions for

$$r_B = r_0$$

To make this more tangible, we can consider the ratio of the material and geometric bucklings of the system at $\omega = 0$:

$$r_B = r_0(1 + h)$$

namely,

$$B_m^2 = \frac{3 \sum_{t0} \sum_{a0} (k_{\infty} - 1)}{(1 + h)^6} = \frac{\bar{B}_m^2}{(1 + h)^6}$$

$$B_g^2 = \frac{\pi^2}{r_0^2(1 + h)^2} = \frac{\bar{B}_g^2}{(1 + h)^2}$$

Clearly then

$$\frac{B_m^2}{B_g^2} = \frac{\bar{B}_m^2}{\bar{B}_g^2} \frac{1}{(1 + h)^4} = 1$$

APPENDIX C

SOLUTION OF THE DIFFUSION EQUATION FOR THE BARE, SOURCE-FREE, COMPRESSIBLE CYLINDRICAL REACTOR

The transformation used to fix the boundaries for Eq. 13, the diffusion equation for a cylindrical reactor compressed periodically in the axial direction, is

$$\eta = \frac{z}{z_0(1 + h \cos \omega t)}$$

$$\xi = \frac{v}{3 \sum_{t0} z_0^2 \delta} t = \gamma_1 t$$

$$\omega' = \frac{\omega}{\gamma_1}$$

$$\phi(z, r, t) = \psi(\eta, r, \xi)$$

Thus the transformed equation becomes

$$\begin{aligned} \frac{\delta}{v} \left(\eta \frac{h\omega \sin \omega' \xi}{1 + h \cos \omega' \xi} \psi_{\eta} + \gamma_1 \psi_{\xi} \right) + \frac{1 + (\omega h / v \Sigma_{t0}) \sin \omega' \xi}{1 + h \cos \omega' \xi} \Sigma_{a0} (1 - k_{\infty}) \psi \\ = \frac{1 + h \cos \omega' \xi}{3 \Sigma_{t0}} \left[\psi_{rr} + \frac{1}{r} \psi_r + \frac{1}{z_0^2 (1 + h \cos \omega' \xi)^2} \psi_{\eta\eta} \right] \quad (C.1) \end{aligned}$$

with the new boundary conditions

$$\begin{aligned} \psi(\eta, r_B, \xi) &= 0 \\ \psi\left(\frac{1}{2}, r, \xi\right) &= 0 \\ \psi(\eta, r, 0) &= P(\eta, r) \\ \psi(\eta, r, \xi) &= 0 \quad (r \geq r_B) \quad \left(\left| \eta \right| \geq \frac{1}{2} \right) \end{aligned}$$

The radial dependence easily separates out so that we can write

$$\psi(\eta, r, \xi) = \chi(\eta, \xi) R(r)$$

and thus obtain

$$\begin{aligned} \frac{3 \Sigma_{t0} \delta}{v(1 + h \cos \omega' \xi)} \left[\eta \frac{h\omega \sin \omega' \xi}{1 + h \cos \omega' \xi} \frac{\chi_{\eta}}{\chi} + \gamma_1 \frac{\chi_{\xi}}{\chi} \right] \\ + \frac{1 + (\omega h / \Sigma_{t0} v) \sin \omega' \xi}{(1 + h \cos \omega' \xi)^2} 3 \Sigma_{t0} \Sigma_{a0} (1 - k_{\infty}) \\ - \frac{1}{z_0^2 (1 + h \cos \omega' \xi)^2} \frac{\chi_{\eta\eta}}{\chi} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -m^2 \quad (C.2) \end{aligned}$$

where m^2 is the separation constant. The resulting equation involving only η and ξ is of the form

$$a(\xi) \eta \chi_{\eta} + b(\xi) \psi_{\xi} + [c(\xi) + m^2 z_0^2 (1 + h \cos \omega' \xi)] \chi = \chi_{\eta\eta} \quad (C.3)$$

where

$$a(\xi) = \frac{3\Sigma_{t0} z_0^2 \delta}{v} h \omega \sin \omega' \xi$$

$$b(\xi) = \frac{3\Sigma_{t0} z_0^2 \gamma_1 \delta}{v} (1 + h \cos \omega' \xi) \quad (C.4)$$

$$c(\xi) = 3\Sigma_{t0} \Sigma_{a0} z_0^2 [1 + (h\omega/\Sigma_{t0} v) \sin \omega' \xi]$$

Again, we can apply a Fourier transform, i.e.,

$$\bar{\chi}(\alpha, \xi) = 2 \int_0^\infty \chi(\eta, \xi) e^{i\alpha\eta} d\eta \quad (C.5)$$

and, applying direct integration to the resulting Lagrange type first-order partial differential equation, we can get, after suitable inverse transformations, the solution

$$\phi(r, z, t) = \left(\left(\frac{1 + h \cos \omega t}{1 + h} \right)^{\tau-1} \sum_{l=1,3,5,\dots}^\infty \left[4 \int_0^{1/2} F(r, z) \cos \frac{l\pi z}{z_0(1+h)} \frac{dz}{z_0(1+h)} \right] \right. \\ \times \exp \left\{ - \left[\frac{l^2 \pi^2 (\omega t + h \sin \omega t)}{(1 + h \cos \omega t)^2} + \pi^2 \frac{\bar{B}_{gr}^2}{\bar{B}_{ga}^2} \omega t - 2\pi^2 \frac{\bar{B}_m^2}{\bar{B}_{ga}^2} \right. \right. \quad (C.6) \\ \left. \left. \times \tan^{-1} \left(\sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \right) \right] \left[\frac{v}{3\Sigma_{t0} z_0^2 \omega \delta} \right] \right\} \cos \frac{l\pi z}{z_0(1+h \cos \omega t)} \right) R(r)$$

As in the previous cases, the exponential part of the solution for $l = 1$ is of interest with respect to criticality. Thus we require that the expression

$$\frac{\pi^2 (\omega t + h \sin \omega t)}{(1 + h \cos \omega t)^2} + \pi^2 \frac{\bar{B}_{gr}^2}{\bar{B}_{ga}^2} \omega t - \frac{2\pi^2 \bar{B}_m^2}{\bar{B}_{ga}^2} \frac{1}{\sqrt{1-h^2}} \tan^{-1} \\ \times \sqrt{\frac{1-h}{1+h}} \tan \frac{\omega t}{2} \quad (C.7)$$

be nondivergent. The slope of the locus about which the last term oscillates is

$$\pi^2 \frac{\bar{B}_m^2}{\bar{B}_{ga}^2} \frac{1}{\sqrt{1-h^2}}$$

and the slope of the locus generated by the middle term is

$$\pi^2 \frac{\bar{B}^2}{\bar{B}_{ga}^2}$$

The first term, as in the slab and the spherical cases, oscillates with a linearly increasing amplitude, and we can consider only the slope of the locus line passing through the local minima of the term itself. Again, we require that the sum of the three slopes be zero and so obtain the condition

$$\frac{\pi^2}{(1+h)^2} + \pi^2 \frac{\bar{B}^2}{\bar{B}_{ga}^2} - \pi^2 \frac{\bar{B}_m^2}{\bar{B}_{ga}^2} \sqrt{1-h^2} = 0 \quad (C.8)$$

or

$$\frac{\bar{B}_m^2}{\bar{B}_{ga}^2} = \sqrt{1-h^2} \frac{\bar{B}^2}{\bar{B}_{ga}^2} + \frac{\sqrt{1-h^2}}{(1+h)^2} \quad (C.9)$$

From Eq. C.2 we also have

$$\ddot{R}'' + \frac{1}{r} R' + m^2 R = 0 \quad (C.10)$$

the solution of which is of the form

$$R = c_1 J_0(mr) + c_2 Y_0(mr)$$

Application of boundary conditions yields

$$R(r) = c_1 J_0(mr) \quad (C.11)$$

where J_0 is the Bessel function of the first kind.

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Ultra High Temperature Reactor Experiment On-line Process Computer

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ABSTRACT

An on-line process computer has been purchased for the Ultra High Temperature Reactor Experiment (UHTREX). It will be used to aid in the operation of the UHTREX reactor, the reactor coolant loops, and the coolant purification system.

The computer system includes a central processor with 4000 words of magnetic-core memory, 32,000 words of drum memory, an input-output typewriter, and a paper-tape punch and reader. There are four logging typewriters and one visual-display unit. The system can sample some 600 analog and 300 digital inputs, and it can set 444 digital and 24 analog outputs. Twenty interrupt channels will be used to allow "time sharing" of input-output equipment and to initiate special programs dependent on conditions in the UHTREX system.

The computer will sample analog channels at a rate of 200 channels per second and monitor them for out-of-limit conditions. It will also prepare logs, perform on-line calculations, keep fuel-element and other system histories, and monitor reactor-operator activities. Capability is being provided for sequencing valves in operations such as fuel loading and for controlling coolant flow rates, thermal-power output of the cooling system, etc. In addition, system diagnostic programs and emergency shutdown programs will be available to help in system operation.

INTRODUCTION

The UHTREX involves the use of simple, unclad, refractory fuel elements in a helium-cooled graphite-moderated reactor. The reactor will be operated at a thermal power of 3 Mw and a maximum coolant temperature of 2400°F. Unique features of UHTREX are as follows:

1. Use of a very simple fuel element.
2. Capability to load and unload fuel during operation at full power.
3. Design of the reactor and coolant system for the high temperature of the coolant outlet.
4. Design of the coolant system with the expectation of complete release of fission products from the fuel elements.

These features make UHTREX especially well suited for testing a variety of refractory fuel-element designs that have application in the gas-cooled advanced-converter program.

The reactor (Fig. 1) has a vertical cylindrical graphite core surrounded by carbon reflector and insulation. It is encased in a 13-ft-diameter spherical pressure vessel. Hollow 5-in.-long cylindrical fuel elements, with a 1.0-in. outside diameter and a 0.5-in.-diameter central hole, lie freely in 1.1-in.-diameter radial holes in the core. There are 312 such channels with 4 elements per channel for a total of 1248 elements.

Since the reactor may be loaded at power, it is planned to adjust temperature by the use of fuel load and to depend on the negative coefficient of reactivity for control. Poison rods will be used for reactivity control during start-up, shutdown, and restart operations following a shutdown. For normal operation the rods are withdrawn. The relative ease of fuel loading also simplifies many problems of fuel-element testing.

Fuel is loaded by means of conveyors and gas lock valves, which feed the elements into the fuel elevator. In the elevator each element is positioned vertically and is pushed into a core fuel channel by a fuel-loader ram, which then forces the innermost element to drop down the fuel-discharge slot. All channels in the core may be fueled by rotation of the core until any of the 24 radial columns of channels is aligned with the fuel-loading ram system.

The associated coolant system is shown schematically in Fig. 2. The small block labeled "gas cleanup" represents a considerable number of components required for purification, helium storage, and pressure regulation. The purification system has a copper oxide bed, molecular sieve beds, several activated-carbon beds, and the associated complex set of valves, compressors, heat exchangers, coolers, etc.

A mathematical model of the reactor and its coolant loops has been derived for UHTREX. A digital simulation program has been written and used to study the transient response of the system. Simulation results

Fig. 1 - Reactor vertical section.

indicate that the reactor is stable and that the overall system is well behaved. Typical time constants are in the range of 100 to 1000 sec; and, since performance requirements are not severe, it is anticipated that relatively simple, low-band-width control systems will be adequate.

Fig. 2 - UHTREX coolant-flow diagram.

PROCESS COMPUTER

The control and data system for UHTREX will include an on-line computer that will aid in the operation of the reactor system. It is anticipated that the reactor will be run for extended periods of time at steady conditions. The computer will relieve the tedium of record keeping while greatly improving the quality of the records, and it will provide supervision and control to increase the reliability of the operation.

The central processor of the system is a small, solid-state general-purpose computer which has some additional features that make it suitable for on-line process use. The equipment that is included with the central processor and is characteristic of a general-purpose computer is listed in Part A of Table 1. Part B of this same table lists equipment ordinarily associated with process computers. The large number of analog and digital inputs and outputs in Part B provide considerable capability for monitoring and controlling. The special equipment in Part C of Table 1 is of particular significance in the application of a process computer and is discussed later in more detail.

There are 20 system interrupts, ordered as follows:

1. Power on
 2. Power outage
 3. W-buffer, end of word
 4. W-buffer, end of block
 5. Watchdog timer
 6. Real time clock
 7. Analog to digital converter complete
- } Drum-core transfers

8. Elapsed-time counter complete
 9. UHTREX system
 10. UHTREX system
 11. UHTREX system
 12. UHTREX system
 13. Relay output (matrix) complete (digital output matrix)
 14. Analog output complete
 15. Punch complete
 16. Alarm typewriter complete
 17. Log typewriter complete
 18. Computer console
 19. Main system operations
 20. Auxiliary system operations
- } From UHTREX-system operation
panels

These interrupts are used to facilitate time sharing of input-output equipment and to provide immediate recognition of special external or

Table 1 - PROCESS COMPUTER COMPONENTS

Part A - General-purpose Computer Equipment	
Arithmetic unit	24-bit word plus parity bit 16 μ sec to add 32 μ sec to multiply
4096-word magnetic-core storage	
31,680-word drum storage	
Input-Output typewriter	10 characters/sec
Paper-tape punch	60 characters/sec
Paper-tape reader	300 characters/sec
Part B - Process Computer Equipment	
Visual display	
4 logging and alarm typewriters	10 characters/sec
608 analog inputs	Scanned at 200/sec (max.)
300 digital inputs	Scanned at 190,000/sec
24 analog outputs	Updated twice/sec
444 digital outputs	20 to 4000 outputs/sec
Part C - Special Equipment	
20 interrupts	
Automatic safe shutdown on power failure	
Stall warning	
Watchdog timer	
Real time clock	

internal conditions. The computer automatically scans the interrupts after executing each instruction (with three exceptions) and will either transfer control to the program associated with the highest level interrupt that is present or continue the current sequence of instructions if no new interrupt has occurred. When an interrupt program is completed, control is returned to the sequence of instructions from which control was transferred. However, after this return the interrupts are again scanned, so that the current highest level interrupt will be effective. Thus the program associated with the highest level interrupt is always the one being executed. Fortunately the computer will neither forget an interrupt nor lose track of it if multiple interrupts occur.

Most of the time, however, there will be no active interrupts, and the computer will be in the "executive" program or another program controlled from the executive. In the executive the computer is constantly looping through a list of about 30 programs to see if it is time to run one of them. These programs include the input scans (which are done once per second), two logs (which are done once per hour), operator-console programs (which are done only on request of the operator), etc. There will be a sporadic sequence of interrupts to the executive from input-output equipment, indicating that information is ready to be transferred into memory (every 5 msec during an analog scan, for example) or that an output has been completed and another output may be made (every 100 msec in the case of a log).

The executive program and the ordering of the interrupts assure that computer programs are handled in a sensible sequence, with high priority given to the most important activity. As an illustration of this point, consider the case of automatic shutdown when an input power failure occurs. As soon as low voltage is detected, well before the computer d-c power is low enough to interfere with normal operation, the power-failure interrupt (No. 2) is activated, and a transfer is made to the automatic-shutdown program. This program will store the contents of all registers and stop the computer on a *Halt* instruction. There is time to finish this program before the power can fall low enough to hinder operation, even if the computer power is removed "instantly." When power is resumed, interrupt 1 takes the computer to a restart program that could simply restore the registers and resume operation. It is more probable that the restart program will go through an initialization and start the computer out anew. Thus the automatic-shutdown equipment, through its use of a high-priority interrupt, assures a safe computer shutdown and restart with no loss of information. This feature is particularly appealing in a reactor system.

Interrupts 9 through 12 are reserved for special UHTREX signals, at least some of which will be associated with emergency conditions. Thus certain neutronic signals, indicating such conditions as loss of pressure in the primary or secondary loops, will activate interrupts. One might

think that these "emergency" interrupts should be higher on the list. Clearly, however, they should not be above interrupts 1 and 2 because, if the computer power is off, nothing can be done anyway; and they should not be above 3 and 4 because chances are that one would like to transfer a diagnostic or an emergency program from the drum to the core immediately. Similar arguments lead to the placement of the UHTREX-system interrupts in positions 9 through 12. Admittedly, the particular ordering is somewhat, but not completely, arbitrary.

The stall-warning equipment, listed in Part C of Table 1, is provided to take the computer off-line automatically upon receipt of certain signals. In the event of a memory parity error, the manual operation of the *Inhibit Computer* pushbutton, the failure of certain power supplies, or a watchdog-timer interrupt (described in the next paragraph), the computer will be put in the so-called *stall* condition. Except for parity error and some power-supply failures, the computer operation is not stopped during stall. However, the computer is prevented from making changes in any of the analog or digital outputs that are connected to process equipment. The computer has no control capability in the stall condition, but, depending on the source of the stall signal, it will continue to monitor inputs, run typewriters, operate lights and recorders, etc. Thus the stall-warning equipment serves to transfer the whole system from computer operation if a computer malfunction is detected.

The watchdog timer is another safety device, and it has a fairly high position on the interrupt list. It is basically a 5-sec alarm clock that is ordinarily reset to zero by the executive routine many times per second. However, if the computer stops, if an endless loop is encountered, or if for any reason the computer does not return to the executive program and reset the watchdog timer at least once every 5 sec, the timer "times out," generates interrupt number 5, and puts the computer in stall. When a watchdog-timer interrupt is received, a computer diagnostic program will be run (if the computer is running at all).

The real time clock generates an interrupt (No. 6) every second, causing transfer to a program that increments the real world "time" count stored in the computer, does a couple of other short jobs, and returns to the interrupted program. Time is used by the executive program to initialize programs that are to be run periodically.

DATA ACQUISITION AND MONITORING APPLICATIONS

It is believed that the computer system, in comparison to conventional instrumentation and data-handling techniques, will make UHTREX more reliable and the quality of data obtained from the

experiment higher. Some of the computer jobs, which are all passive in the sense that they do not involve any control functions, are described in the following sections.

Monitor and Alarm

About 50 vital analog signals will be sampled, linearized, and compared with high and low limits each second. If any signal is found to be out of the limits, an annunciator will be operated, and a message will be typed on the alarm typewriter. A second set of 50 analog signals will be sampled every 3 sec and treated in the same way. The remaining 508 signals, which include calibration signals, will be sampled once every 15 sec.

All digital inputs will be scanned once per second, and comparisons will be made with acceptable configurations. An alarm will be initiated if an unacceptable or inconsistent configuration is found.

Calculations

A small number of process variables such as flow rates, blower coefficients, and reactivity will be calculated each second and compared with limits in the same manner as the analog signals. Calculations such as heat balances and a reactivity balance will be made once a minute, and any unusual results will be printed as an alarm message. A larger number of calculations of heat balances, blower efficiencies, heat-transfer coefficients, helium inventories, molecular-sieve-bed saturations, etc., will be calculated when needed for logs or histories or when requested by the operator.

It is planned to have a calculation of dynamic reactivity as a function of time, based on reactor power, available as a demand program for use in operation.

Hourly and Demand Logs

The computer will prepare and type two hourly logs of some 40 entries each. These logs will be on paper with preprinted headings and will be in engineering units. Two of the four typewriters will be used for these logs. In addition, there will be as many as 20 demand logs that can be requested by the operator and will be typed on the third typewriter. The fourth typewriter is reserved for alarm messages. Data that can be requested on demand logs include the current values of all analog points, a summary of points currently on alarm, a summary of limits, a summary of points that are not being scanned for one reason or another, a summary of all surface temperatures or all internal temperatures, a summary of strain-gauge readings, results of special calculations, etc. The fuel-history log,

which is discussed later in this paper, will be the longest demand log.

Operation Monitoring

The control switches on the UHTREX-system operator's panels are connected to a computer digital-input matrix and to interrupt 19 or 20. Thus, every time the operator actuates one of these momentary control switches to open a valve, the computer receives an interrupt, and, by scanning the digital inputs related to that interrupt, the computer is able to identify the particular control switch that has been actuated. As a minimum, it is planned to have the computer type a message on the alarm typewriter to record the action and the time. In some cases the computer may check this action to see if it is acceptable on the basis of the current status of other valves, rod positions, analog signals, etc. The computer may take some action if it determines that the operator made a mistake. The computer could simply initiate an alarm, or it could prevent the operator action through an interlock. In any event, the computer does have capability to provide a good operation log.

System Histories

The computer will keep a modest number of histories such as the number of megawatt-days of operation, number of days of operation, amount of contaminants removed in the gas purification system, number of revolutions of the core, etc., as seems appropriate and useful. The one large job of this type will be to keep a history of the 1248 fuel elements in the core. A fuel-history program will include a record of the serial number, the loading, and the position in the reactor as a function of time for each fuel element. These data, when combined with a computer record of reactor power, operating time, and temperature, will allow the computer to type or punch the history of each fuel element in considerable detail. This history will be available to the operator on demand and will be logged each time an element is discharged from the computer.

The computer-maintained histories of fuel elements, power, temperature, flow, etc., will be logged on the paper-tape punch at some interval such as once per day or week in a format that is compatible with the Laboratory electronic data processing equipment. Thus historical data from the UHTREX system will be available in a convenient form for later analysis.

COMPUTER CONTROL

As shown in Table 1, the computer has 24 analog outputs and 444 digital outputs. Twelve of the analog outputs are 0 to +20 volts and

twelve are 0 to 20 ma. The digital outputs include 288 magnetically latching relays, 102 48-volt single shots of various durations, and 54 flip-flop type outputs producing 48-volt signals. Some of these outputs will be used to run recorders and panel lights. Many outputs, however, will be available to operate valves and motors in the reactor, the coolant system, and the gas purification system, and they will be used whenever it is sensible to do so.

Control Modes

Each control switch for a valve or motor in the UHTREX system will have an associated mode selector switch. The mode selector switch has three positions: *Manual*, *Computer*, and *Both*. When the mode selector switch is in the Manual position and the operator actuates a control switch (or adjusts a potentiometer), the signal is transmitted directly to the controlled device as is normal for manual control. The computer can monitor this action but has no active control. If the selector switch is in the Computer position, the situation is just reversed, and the computer output is transmitted to the device.* If the selector switch is in the Both position, the circuit requires simultaneous signals from both the operator and the computer to transmit a signal to the system.

The mode selection feature provides considerable flexibility in dividing authority between the computer and the operator. Also, it can be used in checking out computer programs, and it provides the mechanism for manual control when it is desired.

Sequencing and Interlocking

The computer has the capability to operate any of the valves or motors in the system and can be programmed to do so in the proper sequence. Loading fuel, for instance, is a natural sequencing job for the computer. After receiving the fuel-element identification number, the fuel load of the element, and the channel to which the element is to be loaded, the computer may schedule the operation of the core-indexing motor, the two secondary gas-lock valves, conveyor No. 1, three more gas-lock valves, three pressure-equalization valves, the elevator, and a loading ram. A similar sequence of operations is required to remove the discharged fuel element. Checks on valve positions, loader positions, system pressures, etc., will be made throughout the loading and unloading sequence.

*As noted previously, a variety of computer malfunctions will automatically put the computer into stall; and while in stall the computer cannot change any output that is connected to the system. The operator can put the computer into stall by pushing the *Inhibit Computer* switch.

Fuel loading is a job that must be done with utmost care to avoid damage to the loading mechanisms or fuel elements and to prevent any leakage of contaminated gas to the containment structure. It is a tedious job that must be repeated many times. Thus it is suited for computer operation. An additional advantage in having the computer do this particular job is that the computer will be programmed to refuse to load a fuel element until the operator has entered the fuel-element data required for the fuel history. This will greatly reduce the possibility of errors in the fuel histories.

Until confidence in the computer is established and the programs are thoroughly tested, many of the mode-selector switches in a sequence of operations will be in either the Both or the Manual position. Then the sequence of operation by the computer always will stop at these points until the operator actuates the correct switch. If most switches are put in the Manual position but a few are left in the Both position, the computer can be used to check a sequence of operations done by the operator. Thus the computer can provide interlocks on the operator actions, just as it is possible for the operator to check the computer operation.

Reactor start-up, shutdown, helium venting to reduce inventory, and bed regeneration are other jobs that are being considered for computer sequencing.

Emergency Shutdown

There is a nuclear-safety system completely separate from the computer which will shut down the reactor in the event of too high a power, too short a period, etc. There are other system failures, however, such as failure of the heat-dump fan, which call for rapid system shutdown, and the computer can handle these. If the analog or digital scan programs discover an alarm or if an emergency interrupt signal is received, the computer will transfer to a diagnostic program. This program will determine if a true emergency exists or if perhaps a single transducer has failed. If it is an emergency, the computer will determine the type of emergency and initiate the shutdown of the system. A good diagnostic program will help to avoid unnecessary shutdowns. Typically a diagnostic program can be executed within 1 sec of receipt of the alarm.

On-line Control

Provision will be made in the design of the control system for computer control of primary- and secondary-coolant flow rates and pressures, reactor-outlet-gas temperature, rod position, reactor power, and recovery from a scram. Control systems that require sampling as

frequently as once a second will be handled by programs associated with the real time clock interrupt. Systems that require sampling less frequently will have timed or periodic compensation programs initiated by the executive program.

The compensation program for any particular control system can be quite complex if necessary. Switching from one compensation program to another dependent, say, on system operating conditions, is a simple matter in the computer.

A process computer with time and storage available suggests many interesting analysis and control activities. Correlation programs to determine system characteristics come to mind immediately. It appears now that the computer will have some free time, perhaps half of every second, but the storage question is another matter.

DISCUSSION

Status

The computer is due for delivery in mid-June 1965. When installed at Los Alamos, the factory demonstration tests will be repeated, and a six-month 24-hour-per-day reliability-demonstration run will be made.* The specifications for the system require that, during this six-month demonstration test, the computer, excluding typewriters and the paper-tape punch and reader, have a 99% availability and no more than four failures (about 1000 hr mean time between failure).

The executive, analog scan, linearization, log, and 1-sec-calculation programs are being done now, and it is hoped that they will be complete at the start of the demonstration run, so that the computer can be doing useful work during this period. All programs, of necessity, will be checked prior to bringing the reactor to design conditions. The hardware and any programs that are prepared for control of the reactor and its associated equipment will be given especially close attention. Dry runs with simulated neutronics and reactor heat exchanger will be made to check connections and interaction of the computer, rod system, and reactor.

*The computer arrived in September. The first reliability test was terminated 9 days after it began because an analog-output card having no replacement failed. Subsequently design improvements were made in the drum circuits. In almost 3 weeks of operation during the second reliability test, one analog-output failure has occurred. The reliability-demonstration program, which includes many permanent programs, has been invaluable for computer debugging and programming experience. Major final-system programs have been, or are being, written. The magnetic-core storage appears to be just adequate, and the drum storage is more than adequate for programs now planned.

Computer Ability

The question of whether the computer that we are getting is powerful enough to do all its current jobs arises constantly. At this point it is clear that we do not really know the answer. It is suspected that the 4096 words of core storage and 31,680 words of drum storage may not be sufficient. If either storage system should prove to be insufficient, another core module of 4096 words can be added and the drum can be expanded to 84,840 words by adding read/write heads. It now appears that the computer is sufficiently fast to do all the anticipated jobs.

Safety

It has been noted by the authors that other people who are not especially familiar with computers feel a certain apprehension, bordering on suspicion, about letting a computer run any part of the reactor system. An analogy between "a computer run amok" and "a monkey loose in the control room" has been made. There is no basis for such an analogy, for a computer run amok either stops very quickly or gets itself into a tight loop and stays there. As those who have done some programming know, it is, in fact, difficult to get a computer to do anything at all. Correct instructions must be written and placed in the right sequence to produce an output. For instance, the computer that we will use requires from two to six 24-bit instructions of exactly the right form to change a digital or an analog output that might affect the UHTREX system, and this output will have the same safety interlock as the corresponding manual control switch.

It is recognized, however, that computer failures will occur, and a number of operational and programming features have been included to assure safe operation. These include the following:

1. The reliability demonstration
2. Hardware and program check-out with simulated inputs
3. Stall-warning device
4. Watchdog timer
5. Automatic safe shutdown on power failure
6. Memory parity error checking
7. UHTREX system interrupts
8. On-line computer diagnostic programs

We think and certainly hope that the computer system will prove to be reliable and that its use will improve the safety and operation of UHTREX.

Some "Theorems" on Neutron Fluctuations and Fluctuation Spectra

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ABSTRACT

The notion of fluctuation in critical systems is explored. The equivalence of the singlet (Langevin) and kinetic theories of fluctuation is noted. Solutions for cross-power spectral densities are presented.

INTRODUCTION

Neutron distributions are usually considered to be described adequately by a singlet density,¹ since most experiments measure quantities that are in turn interpretable in terms of such a density. In principle, however, no distribution can be so simply characterized, and, in fact, the equation for the neutron singlet density obtained with the assumption of binary collisions only (if neutron-neutron collisions, as well as delayed neutrons, are ignored completely) reads

$$\begin{aligned} \frac{\partial f_1^N}{\partial t} + v_j \frac{\partial f_1^N}{\partial x_j} = S - \int d^3V |\mathbf{v} - \mathbf{V}| \sigma_a(|\mathbf{v} - \mathbf{V}|) f_2^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}, \mathbf{V}, t) \\ + \int d^3\mathbf{v}' d^3\mathbf{V} |\mathbf{v}' - \mathbf{V}| \sigma_f(|\mathbf{v}' - \mathbf{V}|) f_2^{NA}(\mathbf{x}, \mathbf{v}', \mathbf{x}, \mathbf{V}, t) \sum_{j\alpha} \alpha B_{\alpha}^j(\mathbf{v}', \mathbf{v}) \\ + \int d\Omega d^3\mathbf{V} |\mathbf{v} - \mathbf{V}| \sigma_s(|\mathbf{v} - \mathbf{V}|, \oplus) [f_2^{NA}(\mathbf{x}, \mathbf{v}', \mathbf{x}, \mathbf{V}', t) - f_2^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}, \mathbf{V}, t)] \quad (1) \end{aligned}$$

In this equation f_1^N is the usual neutron singlet density (or first moment of the basic probability distribution that satisfies the Liouville equation, henceforth to be referred to as b.p.d.), f_2^{NA} is a neutron-atom cross density (a second moment of the b.p.d.), S is any source of neutrons that is independent of the neutron density, the σ 's are the conventional mi-

cross sections, and $B_{\alpha}^j(\mathbf{v}', \mathbf{v}) d^3v$ is the probability that a fission induced by a neutron of velocity $\mathbf{v}'$ produces exactly j prompt neutrons (delayed neutrons are neglected in Eq. 1), α of which have velocities in d^3v about $\mathbf{v}$. Evidently the neutron singlet density is not, in general, independent of the higher order densities required for the complete description of reactor systems. Equation 1 was introduced explicitly for the purpose of emphasizing this point, although it is standard to assume that

$$f_2^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}, \mathbf{V}, t) \simeq f_1^N(\mathbf{x}, \mathbf{v}, t) f_1^A(\mathbf{x}, \mathbf{V}, t)$$

in which case Eq. 1 immediately reduces to the usual neutron-transport equation. This approximation, and others in complete analogy to it, will be implicit in the whole of the subsequent discussion. However, there are, no doubt, circumstances in which the validity of these approximations is to be questioned, and quite probably a case in point arises in connection with the discussion of fluctuations in critical systems.

Derivations of the equations (including Eq. 1) required for a minimal discussion of neutron fluctuations, and the interpretation of measurements of neutron fluctuations, have been described in varying amounts of detail in various places^{2,3} and hence will be given no attention here. Instead, we will concern ourselves with three theorems based upon these equations. Since the equations are approximate, the theorems will have obverse implications both for the validity of the equations themselves and for the physical situations purportedly described by the equations.

The first theorem relates to the meaning of the notion of fluctuations in critical systems as viewed from the present perspective. It is in connection with this matter that we pointed up earlier the emphasis surrounding Eq. 1. As is fairly well known,⁴ the so-called point-reactor kinetic theory of fluctuations contains the implication that the neutron fluctuations increase linearly with time in a fictitious system operated steadily in the critical state. It is one of our purposes here to show that this implication is not peculiar to point-reactor kinetics but inheres equally in phase-space kinetic theory³ provided neutron-atom correlations are ignored. It will be noted that this implication, in the context of both point-reactor kinetics and phase-space kinetics, is unaffected by the presence or absence of delayed neutrons.

The second theorem has to do with the establishment of a rather striking connection between the singlet and the doublet theories of fluctuations.⁵ Here again the distinction between Eq. 1 and the usual neutron-transport equation must be borne in mind because it vitally affects the validity of this connection. The singlet theory contends that an equation that describes the usual neutron singlet density can be used

to deduce a useful and interpretable equation for the doublet density. The doublet theory proceeds to the derivation of an equation for the doublet density directly from first principles. As will be seen, an inhomogeneous equation for the doublet density is deduced by both arguments. The corresponding homogeneous equations are identical; but the inhomogeneous parts appear ill-defined in the singlet theory, whereas they are explicit and unambiguous functions in the doublet theory.

The third theorem establishes in quite general, though formal, terms the similarities and dissimilarities between the singlet and the doublet interpretations of a fluctuation-spectrum measurement.

The equations that we accept for present purposes as descriptive of singlet, doublet, and cross-doublet densities³ of neutrons, one group of delayed-neutron precursors, and detected particles are

$$\frac{\partial f^N}{\partial t} + Bf^N - \Lambda f^A = S \quad (2a)$$

$$\frac{\partial f^A}{\partial t} + Df^A - \Gamma f^N = 0 \quad (2b)$$

$$\frac{\partial f^D}{\partial t} + Jf^D - \Lambda f^N = 0 \quad (2c)$$

$$\begin{aligned} \frac{\partial F^N}{\partial t} + (B + B')F^N - \Lambda'F^{NA} - \Lambda F^{AN} \\ = S'f^N + Sf^{N'} + \delta(x - x') \delta(v - v') S + H^N \end{aligned} \quad (2d)$$

$$\frac{\partial F^A}{\partial t} + (D + D')F^A - \Gamma F^{NA} - \Gamma'F^{AN} = H^A \quad (2e)$$

$$\frac{\partial F^D}{\partial t} + (J + J')F^D - \Lambda F^{ND} - \Lambda'F^{DN} = H^D \quad (2f)$$

$$\frac{\partial F^{NA}}{\partial t} + (B + D')F^{NA} - \Gamma'F^N - \Lambda F^A = Sf^{A'} + H^{NA} \quad (2g)$$

$$\frac{\partial F^{AN}}{\partial t} + (B' + D)F^{AN} - \Gamma F^N - \Lambda'F^A = S'f^A + H^{AN} \quad (2h)$$

$$\frac{\partial F^{ND}}{\partial t} + (B + J')F^{ND} - \Lambda'F^N - \Lambda F^{AD} = Sf^{D'} + H^{ND} \quad (2i)$$

$$\frac{\partial F^{DN}}{\partial t} + (B' + J)F^{DN} - \Lambda F^N - \Lambda'F^{DA} = S'f^D + H^{DN} \quad (2j)$$

$$\frac{\partial F^{AD}}{\partial t} + (D + J')F^{AD} - \Lambda'F^{AN} - \Gamma F^{ND} = 0 \quad (2k)$$

and

$$\frac{\partial F^{DA}}{\partial t} + (D' + J)F^{DA} - \Delta F^{NA} - \Gamma' F^{DN} = 0 \quad (21)$$

In these equations:

$$f^N(\mathbf{x}, \mathbf{v}, t), f^A(\mathbf{x}, \mathbf{v}, t), \text{ and } f^D(\mathbf{x}, \mathbf{v}, t)$$

are the neutron, delayed-neutron-precursor, and detected-particle (alpha particles produced in an ionization chamber, for example) singlet densities;

$$F^N(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), F^A(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), \text{ and } F^D(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

are the neutron, delayed-neutron-precursor, and detected-particle doublet densities (sometimes referred to as binary densities⁶ or as two-phase-point moments of the b.p.d.); and

$$F^{AN}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), F^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), F^{AD}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), F^{DA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), \\ F^{ND}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t), \text{ and } F^{DN}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

are the several doublet cross-densities relevant to our system. Note, for example, that

$$F^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = F^{AN}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t)$$

The operators B , Λ , D , Γ , J , and Δ operate on primed or unprimed phase points depending upon whether they themselves are primed or unprimed, and they are defined by

$$Bf^N \equiv \mathbf{v} \cdot \nabla f^N + \mathbf{v} \Sigma_t f^N - \int d^3 \mathbf{v}' \mathbf{v}' \Sigma_s(\mathbf{v}' \rightarrow \mathbf{v}) f^N(\mathbf{v}') \\ - \int d^3 \mathbf{v}' \mathbf{v}' \Sigma_f(\mathbf{v}') \sum_{j\alpha} \alpha B_{\alpha}^j(\mathbf{v}', \mathbf{v}) f^N(\mathbf{v}') \quad (3a)$$

$$\Lambda f^A \equiv \lambda \beta(\mathbf{v}) \int d^3 \mathbf{v}' f^A(\mathbf{v}') \quad (3b)$$

$$Df^A \equiv \mathbf{v} \cdot \nabla f^A + \lambda f^A \quad (3c)$$

$$\Gamma f^N \equiv \mu(\mathbf{v}) \int d^3 \mathbf{v}' \mathbf{v}' \Sigma_f(\mathbf{v}') f^N(\mathbf{v}') \quad (3d)$$

$$Jf^D \equiv \mathbf{v} \cdot \nabla f^D \quad (3e)$$

and

$$\Delta f^N \equiv \int d^3v' v' \Sigma_D(v' \rightarrow v) f^N(v') \quad (3f)$$

Most of the quantities appearing in definitions 3a through 3f are conventional, except for the following:

$$B_\alpha^j(v', v) d^3v \equiv \text{the probability that a fission induced by a neutron with velocity } v' \text{ will produce exactly } j \text{ new neutrons, } \alpha \text{ of which have velocities in } d^3v \quad (4a)$$

$$\beta(v) d^3v \equiv \text{the probability that a delayed neutron will be produced with a velocity in } d^3v \quad (4b)$$

$$\mu(v) d^3v \equiv \text{the probability that, in a given fission, a delayed-neutron precursor will be produced with a velocity in } d^3v \quad (4c)$$

$$\Sigma_D(v' \rightarrow v) d^3v \equiv \text{the macroscopic cross section for a neutron of velocity } v' \text{ to be captured and produce a detection particle with a velocity in } d^3v \quad (4d)$$

The symbol $S(x, v, t)$ represents any neutron source that is independent of the neutron density. The quantities H^N , H^A , H^D , H^{NA} , H^{AN} , H^{ND} , and H^{DN} are complicated but explicit linear functionals of the neutron and delayed-neutron-precursor singlet densities. They are independent of the sources, S , and of the detected-particle densities. Since we will have only limited need for the details of these quantities and since they have been explicitly displayed elsewhere,³ we will not discuss them further at this point.

FLUCTUATIONS IN CRITICAL SYSTEMS

For the brief discussion in this section, we do not need the full set of preceding equations since we will not be concerned with the measurement of, but only the notion of, fluctuations. Thus detected particles are of no concern, and we consider only Eqs. 2a, 2b, 2d, 2e, 2g, and 2h. We can neatly summarize these, in the absence of sources (S), by defining the column matrices

$$\Psi = \begin{bmatrix} f^N \\ f^A \end{bmatrix} \quad (5a)$$

$$\Psi = \begin{bmatrix} F^N \\ F^{AN} \\ F^{NA} \\ F^A \end{bmatrix} \quad (5b)$$

and

$$H = \begin{bmatrix} H^N \\ H^{AN} \\ H^{NA} \\ H^A \end{bmatrix} \quad (5c)$$

and the operator matrices

$$M = \begin{bmatrix} B & -\Lambda \\ -\Gamma & D \end{bmatrix} \quad (6a)$$

and

$$R = \begin{bmatrix} M & O \\ O & M \end{bmatrix} \quad (6b)$$

Then the neutron and delayed-neutron-precursor equations are comprised in

$$\frac{\partial \Psi}{\partial t} + M\Psi = 0 \quad (7a)$$

and

$$\frac{\partial \Psi}{\partial t} + (R + QR'Q)\Psi = H \quad (7b)$$

The matrix Q is its own inverse, i.e., $Q^{-1} = Q$, and is explicitly given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Of course, the analytical problem posed by Eqs. 7a and 7b (plus appropriate boundary conditions) is ridiculously difficult, and therefore any discussion of solutions must be conducted only in the most formal terms. Nevertheless, the conclusions indicated by such arguments seem suggestive and plausible and can be demonstrated explicitly in certain simpler, special cases.

We assume the existence of a complete set of eigenfunctions of the non-self-adjoint operator, M ; i.e.,

$$M\phi_j = \lambda_j \phi_j \quad (9)$$

The eigenvalues λ_j may be discretely and/or continuously distributed. Let the adjoint operator be $M^\dagger$ and the adjoint eigenfunction set be defined by

$$M^\dagger \phi_j^\dagger = \lambda_j \phi_j^\dagger \quad (10)$$

where it is assumed that the eigenfunctions and the adjoint eigenfunctions have the same eigenvalue spectrum. None of these assumptions have been proved justified in the present general context. Still, we persist with the arguments based upon them.

Expanding

$$\psi = \sum_j a_j(t) \phi_j \quad (11)$$

where the symbol $\sum_j$ means summation over the discrete points and integration over the continuum, we find from Eq. 7a that

$$\frac{da_j}{dt} + \lambda_j a_j = 0 \quad (12)$$

Consequently Eq. 11 becomes

$$\psi = \sum_j a_j(0) e^{-\lambda_j t} \phi_j \quad (13)$$

For the critical system we surmise that the smallest eigenvalue is zero and that furthermore this smallest eigenvalue is discrete. Then we may display

$$\psi = a_0(0) \phi_0 + \sum_{j>0} a_j(0) e^{-\lambda_j t} \phi_j \quad (14)$$

where we have labeled the zero eigenvalue by $j = 0$. Evidently, for large times

$$\Psi \rightarrow a_0(0) \Phi_0 \quad (15)$$

so that Φ_0 represents the critical state of the system.

To solve Eq. 7b, recall that

$$\Phi_j = \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \end{bmatrix} \quad (16)$$

and construct four-rowed column matrices according to

$$\Phi_{jk} = \begin{bmatrix} \phi_{1j} & \phi'_{1k} \\ \phi_{2j} & \phi'_{1k} \\ \phi_{1j} & \phi'_{2k} \\ \phi_{2j} & \phi'_{2k} \end{bmatrix} \quad (17)$$

This function set is biorthogonal and complete over the extended domain, (x, v, x', v') . Expanding Ψ as

$$\Psi = \sum_{jk} A_{jk}(t) \Phi_{jk} \quad (18a)$$

and H as

$$H = \sum_{jk} C_{jk} \Phi_{jk} \quad (18b)$$

we find from Eq. 7b that

$$\frac{dA_{jk}}{dt} + (\lambda_j + \lambda_k)A_{jk} = C_{jk} \quad (19)$$

The solutions to Eq. 19 are

$$A_{jk}(t) = A_{jk}(0) e^{-(\lambda_j + \lambda_k)t} + \int_0^t dt' C_{jk}(t') e^{-(\lambda_j + \lambda_k)(t-t')} \quad (20)$$

Recall that H is a linear functional of the neutron and delayed-neutron singlet densities only. Thus, according to Eq. 14, there is at least one C_{jk} , namely C_{00} , which is independent of the time. Consequently

$$A_{00}(t) = A_{00}(0) + tC_{00} \quad (21)$$

and therefore Ψ , which describes the neutron and delayed-neutron fluctuations, grows linearly with time according to this description of the critical system. This result is implicit in some earlier studies of fluctuations according to point reactor kinetics⁴ and explicit in an investigation of the space-dependent neutron doublet density in which a one-speed, consistent $-P_1$ calculational model³ is used. Here it is suggested (though hardly proved) to be a general consequence of the assumption that the solution Eq. 14 may be employed to describe singlet densities in kinetic studies of critical systems.

This result is neither paradoxical nor nonsensical. As will be pointed out in the next section, the doublet densities and variances satisfy identical equations (according to Eqs. 2a through 21) in critical systems and hence also are predicted to grow with time. But these equations (Eqs. 2a through 21) are obtained from equations like Eq. 1 by the neglect of variances when the variances are compared to singlets squared. Consequently the prediction of large variances merely implies that the equations upon which the predictions were based are themselves invalid. The main point to the present discussion of this matter is to emphasize that the differences between equations like Eqs. 1 and 2a should not be uncritically ignored.

THE SINGLET VS. THE DOUBLET THEORY OF FLUCTUATIONS

A first-order estimate of the fluctuations of a given distribution is usually taken to be a correlation function, which, for the neutron distribution, may be defined as

$$G^N(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = F^N(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) - f^N(\mathbf{x}, \mathbf{v}, t) f^N(\mathbf{x}', \mathbf{v}', t) \quad (22)$$

The function $G^N(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$ is the phase-space variance of the neutrons. If Eq. 22 vanishes identically, then the neutron distribution (at least to second order) may be regarded as nonfluctuating, its stochastic properties completely inferrable from a knowledge of its first moment (the singlet density) only. Conversely, if Eq. 22 does not vanish everywhere, the neutron distribution does exhibit fluctuations whose character cannot be completely understood from a knowledge of the singlet density, i.e., from the solutions of Eqs. 2a and 2b only. The content of this remark is practically self-evident, for if delayed neutrons are neglected, it is seen that G^N satisfies the equation

$$\frac{\partial G^N}{\partial t} + (B + B')G^N = \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{v} - \mathbf{v}') S + H^N \quad (23)$$

and hence that the spectrum of G^N is determined by the same operator, B , as is the spectrum of the singlet density, f^N . (Analogous results obtain and conclusions are inferred when delayed neutrons are included; their neglect is for reasons of illustrative simplicity only.) Thus the function $f^N f^N$, known from Eq. 2a, satisfies the same homogeneous equation as F^N . Consequently all the information pertaining uniquely to fluctuations is carried exclusively by the inhomogeneous terms in Eq. 23, particularly by H^N , which we will here refer to as the noise source terms. (Again, it must be borne in mind that these conclusions are not general but are a direct consequence of diverse approximations analogous to those involved in replacing Eq. 1 by Eq. 2a.) It is evident therefore that the principal, immediately practical contribution of the present theory of fluctuations is the deduction of an explicit noise source without recourse to independent, semi-intuitive argument. However, it has been shown by Akcasu⁵ that a generalization of Cohn's argument⁷ appropriate to space- and velocity-dependent distributions yields the same noise source as is here displayed in Eq. 23. Thus the general, formal equivalence of Cohn's treatment of pile noise and the present theory of fluctuations is thereby established.

THE SPECTRUM OF NEUTRON FLUCTUATIONS

The problem of measuring and interpreting fluctuation spectra in neutron distributions has been dwelt upon at length in many places⁸ and hence will be briefly dealt with here. Our main purpose in this section is to develop a general formula for the interpretation of such measurements which, in the present context, is formally rigorous and provides a full accounting of the space and velocity dependence of the neutron distribution. In accordance with the preceding section, we find that the break frequencies of the fluctuation spectrum are completely determined by the eigenvalues of the operator that characterizes the singlet densities and that the uniquely noise information is carried entirely by the coefficients that measure the relative importance of the breaks.

The present theory relates the two-phase-point power spectral density to the doublet density of the detected particles according to⁹

$$\Phi(\mathbf{Q}, \mathbf{Q}', \omega) = \int_0^\infty dt \frac{1}{2} \frac{\partial^2 F^D(\mathbf{Q}, \mathbf{Q}', t)}{\partial t^2} \cos \omega t \quad (24)$$

In this expression we have introduced the symbol $\mathbf{Q}$ to represent a phase point $(\mathbf{x}, \mathbf{v})$. Now define G^A , G^D , G^{NA} , G^{AN} , G^{ND} , G^{DN} , G^{AD} , and G^{DA} in

strict analogy to the definition (Eq. 22) for the neutron correlation function. From Eq. 2c (neglecting the operator J here and elsewhere throughout the remainder of the present discussion, since the detected particles are not presumed to "stream" significantly), we see that

$$f^D = t \Delta f^N \quad (25)$$

Here, and from here on, we assume that all neutron and delayed-neutron densities and correlation functions are constant in time and that all detected-particle densities and correlation functions are zero at the origin of time. Thus we find that

$$\frac{\partial^2 F^D}{\partial t^2} = \frac{\partial^2 G^D}{\partial t^2} + 2 (\Delta f^N) (\Delta' f^{N'}) \quad (26)$$

If we restrict our attention to Φ for $\omega > 0$, we then have

$$\Phi(Q, Q', \omega) = \frac{1}{2} \int_0^\infty dt \frac{\partial^2 G^D(Q, Q', t)}{\partial t^2} \cos \omega t \quad (27)$$

Since

$$\frac{\partial G^D}{\partial t} = \Delta G^{ND} + \Delta' G^{DN} + H^D \quad (28)$$

and H^D is a linear function of the neutron singlet density and is thus constant in time, we then have

$$\frac{\partial^2 G^D}{\partial t^2} = \Delta \frac{\partial G^{ND}}{\partial t} + \Delta' \frac{\partial G^{DN}}{\partial t} \quad (29)$$

and therefore

$$\Phi = \frac{1}{2} \Delta \int_0^\infty dt \frac{\partial G^{ND}}{\partial t} \cos \omega t + \frac{1}{2} \Delta' \int_0^\infty dt \frac{\partial G^{DN}}{\partial t} \cos \omega t \quad (30)$$

The correlation functions G^{ND} and G^{DN} are coupled to the functions G^{AD} and G^{DA} , and the relevant systems of equations may be summarized as

$$\frac{\partial X^{ND}}{\partial t} + M X^{ND} = Y^{ND} \quad (31a)$$

$$\frac{\partial \mathbf{X}^{\text{DN}}}{\partial t} + \mathbf{M}' \mathbf{X}^{\text{DN}} = \mathbf{Y}^{\text{DN}} \quad (31b)$$

where

$$\mathbf{X}^{\text{ND}} = \begin{bmatrix} \mathbf{G}^{\text{ND}} \\ \mathbf{G}^{\text{AD}} \end{bmatrix} \quad (32a)$$

$$\mathbf{Y}^{\text{ND}} = \begin{bmatrix} \mathbf{H}^{\text{ND}} + \Delta' \mathbf{G}^{\text{N}} \\ \Delta' \mathbf{G}^{\text{AN}} \end{bmatrix} \quad (32b)$$

and similarly for $\mathbf{X}^{\text{DN}}$ and $\mathbf{Y}^{\text{DN}}$. The operator matrix $\mathbf{M}$ was defined in Eq. 6a. According to previous remarks, the source functions, $\mathbf{Y}$, are constant in time. Equations 31a and 31b are trivially solved to obtain

$$\mathbf{X}^{\text{ND}} = \mathbf{M}^{-1}(\mathbf{I} - e^{-t\mathbf{M}}) \mathbf{Y}^{\text{ND}} \quad (33a)$$

and

$$\mathbf{X}^{\text{DN}} = \mathbf{M}'^{-1}(\mathbf{I} - e^{-t\mathbf{M}'}) \mathbf{Y}^{\text{DN}} \quad (33b)$$

where $\mathbf{I}$ is the two-dimensional unit matrix. Now define the unit vector:

$$\mathbf{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (34)$$

so that, for example,

$$\mathbf{G}^{\text{ND}} = (\mathbf{E}, \mathbf{X}^{\text{ND}}) \quad (35)$$

Inserting Eq. 35, along with Eqs. 33a and 33b, into Eq. 30, we find

$$\begin{aligned} \Phi(\mathbf{Q}, \mathbf{Q}', \omega) = & \left[\mathbf{E}, \Delta \frac{\mathbf{M}}{\mathbf{M}^2 + \omega^2} \mathbf{Y}^{\text{ND}}(\mathbf{Q}, \mathbf{Q}') \right] \\ & + \left[\mathbf{E}, \Delta' \frac{\mathbf{M}'}{\mathbf{M}'^2 + \omega^2} \mathbf{Y}^{\text{DN}}(\mathbf{Q}, \mathbf{Q}') \right] \end{aligned} \quad (36)$$

for the fluctuation spectrum.* Recalling Eq. 7a, we see, as indicated previously, that the break frequencies are completely determined by the

*Equation 36 is in error because of the omission of terms arising from the statistics of the detection process itself. For a discussion of this point, consult Ref. 5.

eigenvalues of the same operator M that determines the system singlet eigenfunctions and eigenvalues. Fluctuation information is contained in the noise source functions, Y^{ND} and Y^{DN} .

According to the preceding section, the singlet theory of fluctuations, which derives the homogeneous equations for the doublets from the singlet equations and infers the inhomogeneous terms representing noise sources by a generalization of Cohn's deduction of equivalent noise sources, leads to the same formula for the fluctuation spectrum as Eq. 36.

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Coupled-core Kinetic Behavior

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ABSTRACT

The scope of a system model designed to predict the transient behavior of a coupled boiling water—centrally located superheat reactor power plant is discussed. Quantitative results describing the time-dependent power changes in two coupled cores resulting from ramp reactivity insertions into each core separately are given in terms of the degree of nuclear coupling between the cores. Several of the more interesting non-nuclear features of the plant simulations are introduced.

INTRODUCTION

Boiling-water reactors with integral superheat are a natural extension of boiling-water-reactor technology. General Electric is presently evaluating two integral superheat concepts. The Integral Thermal Superheat Reactor* (SBWR) features a centrally located thermal-spectrum superheat region. The Mixed Spectrum Superheat Reactor† (MSSR) features a centrally located fast-spectrum superheat region.

In the SBWR the moderator is separate from the coolant in the superheat zone. The design study uses jet pumps, internal steam separators, and an outlet steam headering system. In many respects the SBWR has mechanical features strikingly similar to those of a boiling-water reactor.

*Designed under contract to the Empire State Atomic Development Associates, Inc.

†Designed under the Nuclear Superheat Performance and Evaluation Program being conducted in cooperation with the German Federal Republic and participating German industrial and utility firms.

The MSSR core cross section is represented schematically in Fig. 1. A single cylindrical shroud separates the boiler-region water from the superheater-region steam. The superheater region incorporates hexagonal-shaped fuel bundles of closely packed fuel rods in a triangular pitch. Two

Fig. 1 - MSSR core cross section.

buffer regions assure a good radial power distribution and provide a breeding region for the fast leakage flux.

For detailed information concerning the plant data sheets and power costs, see the discussion by Imhoff.¹

SCOPE OF SIMULATION OF PLANT TRANSIENT BEHAVIOR

As part of the normal design evaluation, a model simulating the transient response of these plants is in the process of being coded for a large digital computer. The model will be used to optimize the response of the final designs with respect to the nonstatic events that may occur, such as control-rod withdrawal, pump-power changes, and coolant-density changes. Figures 2 and 3 schematically show the coolant-moderator flow paths and the regions that are used in the model to represent the SBWR and the MSSR systems, respectively. In the SBWR the moderator flow path for the thermal superheat region is down, in the same direction as the superheated steam, on the outside of the insulated superheater process tubes. The return liquid flow is also down in the bypass region outside the boiler bundles. The live water flow inside the boiler fuel bundles is

Fig. 2 - Flow schematic for SBWR.

Fig. 3 - Flow schematic for MSSR.

up, against the force of gravity. In the MSSR the flow in the boiler section is up in the bypass and fuel regions, similar to the flow in a conventional boiling-water reactor. There is no moderator liquid in the superheat region of the MSSR during normal power operation. In each plant two separate banks of drive pumps are allowed for the optional determination of conservative recirculation rates with respect to the jet-pump feed sparger. In both plants the saturated steam flow is down through the superheater region, after internal steam separation and drying.

Because elements of the system are represented as regions, such as the boiler and the superheater regions, detailed axial and radial behavior (beyond details obtained by the use of peaking and other factors) is not obtained from the present model. More special-purpose tools, such as the digital code,² which gives the coupled nuclear-thermal-hydraulic behavior in a boiling-water-reactor bundle, are required for detailed axial integrations.

Within the limitation of the regional description given in Figs. 2 and 3, the model provides nuclear coupling between the boiler and the superheater regions, mass storage in all regions, pressure-dependent thermodynamic and material properties of water, and full feedback control in turbine and bypass-condenser steam flow rates and in feedwater flow rate. The model is designed for transients in which no extensive damage to the fuel or the vessel is postulated.

Complete system results will be described in detail for transients of interest at a later time. In this paper special emphasis is given to some aspects of the coupled-core kinetic behavior without reactivity feedback and to several of the more interesting non-nuclear features of the plant simulation.

SOME KINETIC BEHAVIOR

Because of limitations on digital-computer storage and machine-running time and because of the excellent results that have been obtained with the one-reactor space-independent or point kinetics equations in predictions of the gross behavior of boiling-water-reactor plants, an equivalent two-region point kinetics representation was chosen. Avery described this next logical step in complexity at the second Geneva conference.³ Previous coupled thermal superheat studies have used the single-region point kinetics equations to describe the core.^{4,5} Other coupled-core theoretical analyses^{6,7} have been similar in principle to Avery's, and the results have been evaluated in the frequency domain. An interesting integral solution⁸ of Baldwin's formulation appeared recently in the time domain for a step reactivity change in two coupled cores.

In this paper we will use Avery's formulation to describe the time-dependent power changes in the two reactor regions, with a minor change to allow different delayed-neutron precursor values in each region. The equations will be solved numerically since, in general, the time-dependent multiplication value will not permit analytical solutions. The prompt jump approximate solution technique will be used when appropriate to economize on the machine-running-time requirements for the plant simulation. The point kinetics equations for two cores are

$$\frac{dN_{11}}{dt} = k_{11}(1 - \beta_1)S_1 - S_{11} + k_{11} \sum_{i=1}^J \lambda_{i1} C_{i1}$$

$$\frac{dN_{21}}{dt} = k_{21}(1 - \beta_1)S_1 - S_{21} + k_{21} \sum_{i=1}^J \lambda_{i1} C_{i1}$$

$$\frac{dN_{12}}{dt} = k_{12}(1 - \beta_2)S_2 - S_{12} + k_{12} \sum_{i=1}^J \lambda_{i2} C_{i2}$$

$$\frac{dN_{22}}{dt} = k_{22}(1 - \beta_2)S_2 - S_{22} + k_{22} \sum_{i=1}^J \lambda_{i2} C_{i2}$$

$$\frac{dC_{i1}}{dt} = \beta_{i1} S_1 - \lambda_{i1} C_{i1} \quad (i = i, J)$$

$$\frac{dC_{i2}}{dt} = \beta_{i2} S_2 - \lambda_{i2} C_{i2} \quad (i = i, J)$$

where

$$S_{jk} = N_{jk}/l_{jk} \quad (j = 1, 2) \quad (k = 1, 2)$$

and

$$S_k = S_{k1} + S_{k2} \quad (k = 1, 2)$$

Physically,

k_{jk} = the expectation value that a neutron resulting from a fission event in region k will cause a next-generation fission neutron in region j

l_{jk} = the prompt-neutron generation time for the process characterized by k_{jk}

S_{jk} = the total fission source rate in region j due to neutrons from region k

S_k = the total source rate in region k

β_k = the average delayed-neutron fraction in region k

The initial steady-state multiplication values, k_{jk} , can be determined from full-core eigenvalue solutions in which the number of prompt neutrons per fission in first one region and then the other are perturbed. The detailed algebraic manipulations required to determine the k_{jk} from the overall-system eigenvalues are straightforward. By the same procedure the values of k_{jj} can be developed as coefficient maps as a function of macroscopic changes in the composition of region j [for example, $(\Delta k_{11}/k_{11}^0)/(\Delta \rho/\rho)$ as a function of ρ , the superheated steam density with some appropriate parameter]. In this way the multiplication values for each region will be changed independently on the basis of system-calculated values for fuel temperature, moderator density, etc., in each region. The two cores are dynamically coupled by the 16 differential equations previously described, in the case of six delayed-neutron groups. The cross-coupling multiplication values, k_{jk} , are changed during a transient by the relative change in k_{jj}

$$k_{jk} = k_{jk}^0 \frac{k_{jj}}{k_{jj}^0}$$

Since the l_{jk} are on the order of 10^{-6} and 50×10^{-6} sec for the fast superheater region and the boiler and thermal superheater regions, respectively, the prompt jump approximation will be realistic for most transients of interest below prompt critical. Figure 4 illustrates this point by comparing the prompt-jump solution to the exact solutions for a ramp reactivity insertion rate of $10\text{c}/\text{sec}$ for six delayed-neutron groups based on thermal fission ^{235}U parameters in a one-region reactor.

To show parametrically the quantitative time-dependent behavior of coupled cores to changes in the multiplication values of each region, we will give the results of several calculations. The prompt jump approximation technique will be used for all coupled-core transient results discussed. The delayed-neutron and precursor-atom characteristics are assumed to be the same in both regions. Six delayed-neutron groups are assumed in both regions. The driving function is chosen to be a ramp,

$$k_{11}(t) = k_{11}(t=0) + 0.1\beta t$$

The results are shown as the relative power change in the superheater, P_s , which corresponds to region 1, and the relative power change in the

Fig. 4 - Approximate and exact solutions to the point kinetics equations for a 10¢/sec reactivity ramp in a one-region reactor.

boiler, P_B , which corresponds to region 2. The original cross-coupling k_{jk} values are determined from the original values of P_s/P_B and k_{jj} and the criticality condition $k_{jk} = S_{jk}/S_k$. As the degree of coupling between the cores decreases, both k_{jk} and S_{jk} decrease. It should be clearly understood that the following results are not representative of the expected transient response of either the SBWR or the MSSR plant. The prediction of the transient responses of these plants will require starting with the design values for the initial k_{jk} and including feedback

reactivity. Neither of these conditions is met for the following results, which are meant only to illustrate the dependence of the power on the degree of coupling.

Figure 5 shows the dependence of the relative power changes in both regions to a ramp multiplication change in region 1 when the cores are loosely and symmetrically coupled. The initial condition corresponding to two completely uncoupled cores, $k_{11} = k_{22} = 1.0$, gives the expected result: The power change in region 1 is the same as that for the one-region-reactor case shown in Fig. 4, and the region 2 power remains

Fig. 5 - Power transients for symmetrically coupled cores.

unchanged. The driven-region response decreases slightly as k_{11} decreases, and the passive-region response increases as the coupling increases (k_{11} decreases).

Figure 6 continues the trend of increasing coupling from $k_{11} = k_{22} = 0.999$ to the completely and evenly coupled condition $k_{11} = k_{22} = k_{12} = k_{21} = 0.50$. As the initial k_{11} decreases, the relative power change in the two regions becomes less dissimilar owing to increased coupling. For practical purposes the power changes are about the same in both regions for initial values of $k_{11} = k_{22}$ below 0.95. In all cases the

Fig. 6 – Power transients for symmetrically coupled cores.

hypothetical two-region system represented by a symmetric pair of k_{jk} values is initially critical without external sources. The system response to an absolute magnitude change in k_{11} as a function of time, $0.1\beta t$, is a decreasing function at the end of the ramp insertion as the initial value of k_{11} decreases from 1.0 to about 0.94 but an increasing function for smaller k_{11} values, as shown in Figs. 6 and 7. This reversal should not be surprising because the fractional change in k_{11} is increased for $k_{11} \lesssim 0.99$ during the transient. For example, the $k_{11} = k_{22} = 0.50$ case is being driven twice as hard as the $k_{11} = k_{22} = 1.0$ case to give essentially the same power history in both regions 1 and 2 as was found

Fig. 7 - Power transients for symmetrically coupled cores.

Fig. 8 - Dependence of final relative power on the degree of initial symmetric coupling.

for the driven, uncoupled region 1 in Fig. 5. For values of $k_{11} = k_{22} < 0.50$, the predominate driving function becomes the cross-coupling multiplication values because of the relation between k_{jk} and k_{jj} previously stated. For example, a change in k_{11} of $0.1\beta t$ for $k_{11} = 10^{-4}$ would correspond to a \$1000/sec change in k_{12} , which is equivalent to a \$1000/sec change in $k_{11} = 0.9999$ for the loosely coupled cores considered in Fig. 5. Accordingly, the almost totally coupled case considered in Fig. 7, $k_{11} = k_{22} = 10^{-4}$, was driven by $10^{-5}\beta t$ to make the change in the predominant multiplication value, k_{12} , equivalent to the change introduced in Fig. 5.

Figure 8 compares the relative power changes in the two regions at the end of 10.05 sec of ramp reactivity addition to k_{11} for two conditions, $k_{11}(t) = k_{11}^0 + 0.1\beta t$ and $k_{11}(t) = k_{11}^0(1.0 + 0.1\beta t)$. As expected, the results are either the same for $k_{11} = k_{22} \gtrsim 0.95$ or the difference between an increasing and a monotonically decreasing function for $k_{11} = k_{22} \lesssim 0.95$. The relative trade-off between power coupling and uncoupling is shown very clearly by the use of the inverted axis, $1 - k_{11}^0$. The overall-system

response is decreased when the power coupling becomes significant, and this decrease is equivalent to moving away from the prompt-critical point when only one region is being driven. In all instances the original power split has no appreciable effect on the relative power changes in each region, although it has a significant effect on the absolute values of S_{jk} and k_{jk} .

Figure 9 shows the relative power changes for an absolute change in k_{11} of $0.1\beta t$ when the initial k_{jj} are not symmetric. For example, the two cases where $k_{11} \approx 1.0$ and $k_{22} \ll 1.0$, which are analogous to a critical

Fig. 9 - Power transients for unsymmetrically coupled cores.

reflected system with the reflector or blanket thought of as a separate region, show that the relative power change is essentially the same as for one critical reactor, with the passive region following the driven region. When the coupling between region 1 and region 2 is increased and the dependence of region 2 on region 1 is decreased, we expect to find the diminished relative change in P_s and the dissimilarity between the relative changes in the two regions as shown in Fig. 9.

Figure 10 compares the effect of driving the less critical (the more dependent) region, $k_{22}(t) = k_{22} + 0.1\beta t$, and two symmetric, more highly

Fig. 10 - Power transients for unsymmetrically coupled cores, when the passive core (region 2) is being driven.

coupled cores. The more highly coupled cores are both more responsive than their counterparts. For the unsymmetric core the response in region 1, the more self-sufficient region, is severely limited. In the limit, e.g., the two totally dependent cases of Fig. 9, the response in both the dependent and the less dependent regions would be negligible.

Figure 11 shows the relative power changes where there is appreciable coupling between the cores and the more critical region 1 is being driven by a $0.1\beta t$ addition. As the coupling increases, the relative responses become identical for both regions, and the absolute response decreases as we move away from the prompt-critical point for the system.

Fig. 11 - Power transients for unsymmetrically coupled cores.

Figure 12, a composite of several of the earlier figures, illustrates the relative power vs. coupling trade-off. The most highly coupled system ($k_{11} = 0.999$, $k_{22} = 0.83$) has the greatest relative response, with both regions changing at the same rate when region 1 is driven. The less coupled system ($k_{11} = 0.999$, $k_{22} = 0.99$) has less response, but region 2 follows the driven region closely. Driving the passive region ($k_{11} = 0.999$, $k_{22} = 0.9999$) gives the same relative change in the passive region as that for the less coupled system and a severely limited response in the less dependent region 2, as expected.

Fig. 12 - Power transients for unsymmetrically coupled cores.

The effects of different delayed-neutron and precursor-atom characteristics in both regions, of different prompt-neutron lifetimes, of two different, simultaneous driving functions in each region, and of reactivity feedback on the time-dependent behavior of coupled-core kinetics will be reported in detail in a separate paper.

FREQUENCY DOMAIN

Our primary interest is in time-domain solutions to the overall system equations with dynamic feedback. Stability analyses will be performed by exciting the nonlinear system simulation with sinusoidal disturbances in control-rod position, drive-pump power, turbine-admission-valve position, etc., for a number of discrete frequencies to obtain the conventional gain and phase margins and by interpreting the system response to step changes in plant variables such as control-rod position. Although the digital system model will not be linearized and set up for matrix reduction in the frequency domain, the results of some special-purpose frequency-domain analyses may be worth passing mention. Using one group of delayed neutrons and the neutronic characteristics of the mixed-spectrum critical assembly at Vallecitos Atomic Laboratory⁹ ($k_{11} = 0.94$, $k_{22} = 0.83$), we compare the normalized response to that of critical and subcritical ($k = 0.94$ and $k = 0.83$) single-region reactors. The terms "fast" and "thermal" for these systems refer to the prompt-neutron lifetime characteristics of regions 1 and 2, respectively. The frequency range is limited to 10 cycles/sec, since there are no significant physical-system gains leading to important feedback in a power plant above 10 to 20 cycles/sec and the β/l roll-off occurs on the order of 15 cycles/sec in the SBWR and on the order of 300 cycles/sec in the MSSR. Figures 13 and 14 show the single-region critical and subcritical transfer functions. The fast system shows no gain or phase change at the higher frequencies because the fast region lifetime was taken to be 156 times smaller than in the thermal region. As expected, the subcritical systems are very unresponsive, with the less critical thermal-parameter system being the less responsive. The subcritical phase angles are essentially frequency independent at slightly less than 0° . Figure 15 shows the normalized response in regions 1 and 2 for the coupled system to an excitation in region 1 (the fast core), and Fig. 16 illustrates the response in regions 1 and 2 to an excitation in the more passive region 2. The phase angles are very nearly identical in both cases and are controlled at the higher frequencies by the longer thermal lifetime. The gain responses are similar in shape to those of the single-region critical case, but they are attenuated 2.5 db when driven in the fast region ($k_{11} = 0.94$) as opposed to 12 db when driven in the thermal region ($k_{22} = 0.83$), as is qualitatively expected.

Fig. 13 - Single-region critical and subcritical fast reactor transfer functions.

Fig. 14 - Single-region critical and subcritical thermal reactor transfer functions.

Fig. 15 - Coupled-core transfer functions when fast core is oscillated.

Fig. 16 - Coupled-core transfer functions when thermal core is oscillated.

REMAINDER OF SYSTEM SIMULATION

It would not be appropriate to discuss all the remaining aspects of the system simulation in the required detail at this time. However, some of the more interesting features will be briefly considered because of their novelty or continuing research interest. The dynamics of steam formation and transport in the boiler bundle is in the latter category. Our heat-transfer model allows for different radial power distributions in the subcooled and bulk boiling zones of the bundle; different power densities in the two zones; temperature-dependent thermal conductivity, fuel density, and heat capacity; coolant-dependent film coefficients; and variable coolant temperature. The dynamic bulk boiling length is calculated. Mass storage and pressure-dependent coolant properties are allowed in the bulk zone. All attempts to describe the dynamic relation between heat flux into the coolant and the average steam fraction response require a number of assumptions as to the form of the dynamic shear forces between steam and water or the mechanics of transverse mixing and usually a large number of simplifications to allow partial or complete analytical solutions or to ease the numerical problems of solving the nonlinear set of partial differential equations that describe conservation of mass, energy, and momentum in the bundle. Attempts by various authors have differed widely in their degree of sophistication and validity. Some of the more interesting approaches have been made by Jones,¹⁰ Kirchenmayer,¹¹ Solberg,¹² and Hudson et al.¹³ One area that has not received its proportionate attention is a comparison of the results of various models with power-to-void data. The axial, high-pressure power-to-void transfer-function data of Christensen¹⁴ has been vector-weighted over the bulk boiling zone, and in Fig. 17 the phase-angle results are compared to the 90° [or $1/(1+s)$, $s \rightarrow j\omega$] sweep model that is commonly used to represent this effect and to a smoothed 180° model (which has received very little attention). When plotted against this dimensionless parameter, $\omega L/u = \omega\tau$, where ω is the frequency of the power oscillation in radians per second and τ is the effective steam transit time across the two-phase length, Zivi's atmospheric data¹⁵ are found to be not unlike Christensen's. In the present study a modified 180° sweep operator is used which features a time-dependent two-phase transit time and fitting coefficients that are determined from power-to-void data taken from an out-of-pile hydraulic loop that approximates the power density and hydraulic conditions in the boiler bundles under study. The loss of phase lag that results from vector-weighting a number of parallel boiler bundles is incorporated in the sweep-operator coefficients.

The internal jet pumps used in both plants represent an advance in the simplicity of the recirculation system. Jet pumps, in which a high-velocity motive fluid aspirates a greater total flow at a lower speed, have

Fig. 17 - Channel average power-to-void phase angle vs. $\omega\tau$.

been known for a long period.¹⁶ With the advent of large boiling-water reactors, General Electric Company has improved the efficiency of jet pumps by design modifications to the point where they are normal hardware components that may be advantageous in certain plant systems. Jet pumps are characterized by a flow ratio

$$M = \frac{\text{suction flow rate}}{\text{drive flow rate}}$$

and a head ratio

$$N = \frac{\text{total head difference across the pump}}{\text{total head difference between the drive nozzle and pump outlet}}$$

It is well known that the steady-state relation between M and N is of the form $M = aN + b$ over the operating range of interest. Table 1 shows some representative atmospheric and high-pressure jet-pump transients in terms of the time-dependent M and N values. (It should be noted that these data are not from jet-pump designs that would be used in an actual plant, although the data trends are representative.) Table 1 also shows the values of M and N that would have been consistent with the measured value of N and M , respectively, had the steady-state correlations been

Table 1 - TYPICAL REDUCED TRANSIENT JET-PUMP DATA

Time, sec	Drive flow rate, lb/hr	Observed M	Observed N	M consistent with observed N	N consistent with observed M
Moderate Decrease in Drive Flow Rate; Atmospheric Pressure; Seven-nozzle Unit					
0.0	231,000	2.06 ± 0.03	0.133 ± 0.002	2.06	0.133
1.0	214,000	2.16 ± 0.03	0.141 ± 0.002	1.93	0.128
2.0	198,000	2.27 ± 0.03	0.144 ± 0.002	1.88	0.122
3.0	174,000	2.44 ± 0.03	0.138 ± 0.002	1.98	0.113
5.0	139,000	2.62 ± 0.03	0.139 ± 0.002	1.96	0.103
10.0	128,000	2.22 ± 0.03	0.132 ± 0.002	2.09	0.125
Extreme Decrease in Drive Flow Rate; Nominally 1000 psig; One-nozzle Unit					
0.0	119,000	0.98 ± 0.05	0.37 ± 0.05	0.98	0.37
0.25	97,000	1.1 ± 0.05	0.44 ± 0.05	0.83	0.31
0.50	63,000	1.2 ± 0.05	0.33 ± 0.05	1.1	0.27
1.0	50,000	1.1 ± 0.05	0.25 ± 0.05	1.2	0.31
2.0	24,000	0.95 ± 0.05	0.16 ± 0.05	1.4	0.38
2.4	10,000	1.0 ± 0.05	0.08 ± 0.05	1.6	0.36

valid during all portions of the transient. The discrepancy, which should be expected, is due to the temporal acceleration of the hydraulic loop in which the jet pump is operating and the dynamics of the drive line-suction flow momentum interchange in the jet-pump inlet and throat. Data taken on the jet-pump design which will be incorporated in a plant, over the range of operating conditions that can be expected, are used to represent the jet-pump behavior in the system model.

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Pressure Effects on Space-dependent Dynamics in Boiling Systems

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ABSTRACT

The effects of system pressure variation on the space-dependent dynamics of two-phase systems such as boiling-water reactors and conventional steam generators are investigated. The mathematical formulation developed for this study consists of three main parts: (1) a space-dependent recirculation-loop dynamic model simulating the behavior of the heating zone, riser, steam separator, and downcomer; (2) a steam-drum lumped-parameter dynamic model; and (3) a steam-main dynamic model simulating the steam flow from the drum to the turbine steam-admission valve. A constant feedwater flow rate and the occurrence of choking flow at the turbine admission valve are among the simplifying assumptions employed. The governing space-dependent dynamic equations are solved numerically on an IBM 7094 digital computer. The mathematical analysis is applied to the dynamic analysis of a typical direct-cycle forced-circulation boiling-water reactor with the pressure regulator operative. The results of this study indicate that the effect of system pressure variation combined with the nuclear-thermal-hydraulic interaction renders the system transient behavior more susceptible to oscillations. Improperly selected pressure-controller parameters could adversely affect the operation of an inherently stable system by introducing undesirable oscillations in the system-dependent variables.

NOMENCLATURE

A	Coolant flow cross-sectional area, sq ft
B_1, B_2	Functions defined by Eqs. 18 and 19
b	Pressure-regulator gain setting, sq ft/sec
C_1, C_2	Functions defined by Eqs. 20 and 21
g	Acceleration of gravity, ft/sec ²
h	Coolant enthalpy, Btu/lb
J	A conversion factor = $J'/144$, where $J' = 778$
M	Coolant mass, lb
M_h, M_v	Functions defined by Eqs. A.4 and A.2
m	Coolant mass flow rate, lb/sec
p	System pressure, psi
s	Entropy, Btu/(lb)(°F)
t	Time, sec
u	Coolant velocity, ft/sec
V	Volume, cu ft
v	Specific volume, cu ft/lb
v'	Energy averaged specific volume, cu ft/lb
v^*	Momentum averaged specific volume, cu ft/lb
X	Steam quality
y	Water level, ft
α	Void fraction
γ	Slip ratio
δ	A function defined by Eq. 27
ρ	Coolant density, lb/cu ft
τ	Measurement time lag, sec

Subscripts

d	Downcomer inlet
f	Saturated liquid
fg	Saturated liquid to saturated vapor
g	Saturated vapor
r	Riser outlet
t	Steam main to turbine
v	Turbine steam-admission valve
o	Initial value

INTRODUCTION

In a previous article the authors have presented a mathematical model for the space-dependent dynamic analysis of boiling-water reactor systems.¹ This model consisted of space-dependent neutron-kinetics

equations for the reactor core and flow-conservation equations for the reactor coolant system developed in terms of length along the flow path and time. The resulting set of nonlinear partial differential equations was solved numerically to obtain the space- and time-dependent system variables. One of the simplifying assumptions employed in the above-mentioned study is the neglect of system pressure variation. The reactor was assumed to operate under the "turbine slaved to reactor" scheme. In this scheme reactor pressure is positively controlled by a pressure regulator adjusting the turbine steam-admission valve in such a way as to pass only sufficient steam to maintain the primary system pressure as close to a reference pressure as possible.

The assumption of constant system pressure may not always be justified. In most practical applications, under the normal mode of operation, the pressure regulator compares the pressure upstream of the turbine steam-admission valve with a reference pressure. Any error signal thus generated actuates the turbine valve servomotor, with a subsequent change in valve flow area, to maintain the upstream pressure within a specified range.² In the event of a large change in utility-load demand, the turbine speed governor overrides the pressure-regulation mode and varies the admission-valve flow area to meet power demand. Under this condition gross changes in system pressure may be encountered.

The main objectives of the present study are to include the effect of pressure variation in the space-dependent dynamic model for the analysis of two-phase steam-water systems such as boiling-water reactors and conventional steam generators and to determine the system transient response with either the pressure control system operative or under a simulated large-load-change condition when the pressure-regulator operation is overridden by the turbine speed governor. The mathematical formulation, developed for this study, consists of three main parts:

1. A recirculation-loop dynamic model simulating the behavior of the heating zone (reactor core), riser, steam separator, and downcomer with length along the flow path and time as independent variables. Except for pressure variation effects, the simplifying assumptions and the governing equations employed in this dynamic model are identical with those of Ref. 1; thus they will not be repeated here.
2. A steam-drum lumped-parameter dynamic model simulating the effects of system pressure transients and water level.
3. A steam-main dynamic model simulating the steam flow from drum to turbine admission valve. A pressure control system with both normal and turbine speed-governor override modes of operation is included in this model.

The simplifying assumptions and the governing equations for parts 2 and 3 will be presented and discussed in the following sections.

SIMPLIFYING ASSUMPTION

Although the time variation of pressure is considered in this analysis, the effect of spatial pressure variation on coolant thermodynamic properties is still neglected. This assumption is easily justified because the spatial pressure variation is small in comparison with the system operating pressure.³ Unlike Ref. 1, where the coolant thermodynamic properties were treated as constants, the saturated liquid and vapor enthalpies and specific volumes are accurately evaluated from Steam Tables⁴ as functions of the time-varying system pressure.

The steam drum and steam main to turbine models are based on a lumped-parameter representation. In contrast with the recirculation loop, spatial effects in the steam drum and main do not play an important role and are therefore ignored. The frictional pressure drop in the steam main, being small, is neglected, and steam-main volume is combined with that of the steam drum. Critical or choking flow is assumed to occur at the throat of the turbine admission valve. This assumption is in agreement with the current design practice at full power and is also considered to be valid around the full-power operating point in the range of disturbances studied.

The steam separator and steam dryer units shown in Fig. 1 are assumed to operate with 100% effectiveness. The liquid and vapor parts of the two-phase mixture leaving the riser enter the water-level element and the steam drum, respectively. The water-level element is defined as the segment at the top of the downcomer containing the downcomer-steam-drum liquid interface. Moreover, only saturated dry vapor leaves the steam drier to enter the steam main.

The water-level controller design is not within the scope of this analysis. Consequently the feedwater mass flow rate is assumed to remain constant during the dynamic operation.

In contrast with Ref. 1, the pressure variation may result in two-phase flow in the water-level element. It is assumed that no steam enters the downcomer from the water-level element. The large flow cross section in this element reduces the fluid velocity, thus forcing the steam bubbles to either remain within the water-level element or to rise to the steam drum. The determination of the steam bubble trajectories during the transient operation involves the ballistics of the vapor particles and hence is not within the scope of this analysis. However, it can be shown that the mathematical formulation presented for system

Fig. 1 – Schematic diagram of typical direct-cycle boiling-water reactor.

pressure variation is unaffected by the location of steam bubbles in the water-level element. The steam bubble trajectories will only affect the system water level. With the bubble trajectories unknown, the water-level position may be bracketed by considering the following limiting cases:

1. The water-level lower bound may be obtained by assuming that the steam bubbles formed in the water-level element enter the dome instantaneously.
2. The water-level upper bound may be calculated by assuming that the steam bubbles formed in the water-level element stay within that element.

MATHEMATICAL FORMULATION

Because details of the recirculation-loop dynamic model are fully described in Ref. 1, it will suffice to state here that the differential equations for the conservation of mass, energy, and momentum for the two-phase flow of coolant in two-dimensional form (length along the flow path and time) and the momentum integral equation around the closed recirculation loop are developed. The neutron-kinetics equations for the reactor core are derived. The equations of state and the slip-ratio and void-fraction correlations, together with the heat-transfer and two-phase pressure-drop relationships, are established. The recirculation boundary equations, initial conditions, and the power control equations are determined. In this manner a set of partial multi-variable integro-differential equations is obtained. These equations are first expressed spatially in finite difference form and then integrated numerically in time on an IBM 7094 with a "Modified-Euler" technique to obtain the transient behavior of the recirculation loop.

Because boiling occurs in the water-level element, a new set of water-level equations must be derived for this element. In conformity with Ref. 1 and for analytical convenience, the coolant enthalpy is taken as a main dependent variable when a subcooled flow persists in the water-level element, and the void fraction is chosen as a main dependent variable when two-phase flow conditions occur in that element. The liquid-phase mass flow rate into the water-level element, the vapor-phase mass flow rate into the steam drum, the water level, and pressure constitute the coupling between the model presented in this paper and the recirculation-loop model.

Steam-drum Model

The application of the continuity and energy equations to the steam drum, in a lumped-parameter form, gives

$$\frac{d(M_f + M_g)}{dt} = m_{gr} - m_t \quad (1)$$

and

$$\frac{d(M_f h_f + M_g h_g)}{dt} - \frac{V}{J} \frac{dp}{dt} = (m_{gr} - m_t) h_g \quad (2)$$

where

$$m_{gr} = (\rho_g \alpha \gamma u_f A)_r \quad (3)$$

$$V = M_f v_f + M_g v_g \quad (4)$$

and

$$\frac{dV}{dt} = -A \frac{dy}{dt} \quad (5)$$

Water-level Model

The application of the continuity and energy equations to the element at the top of the downcomer yields

$$\frac{d}{dt} \{ [\rho_g \alpha + \rho_f (1 - \alpha)] y A \} = m_{fr} - m_d \quad (6)$$

and

$$\frac{d}{dt} \{ [\rho_g \alpha h_g + \rho_f (1 - \alpha) h_f] y A \} - \frac{y A}{J} \frac{dp}{dt} = m_{fr} h_f - m_d h_f \quad (7)$$

where

$$m_{fr} = [\rho_f (1 - \alpha) u_f A]_r \quad (8)$$

and

$$m_d = (\rho_f u_f A)_d \quad (9)$$

These equations are valid when two-phase flow occurs in the water-level element. These equations will also be valid under subcooled flow conditions in the water-level element if the void fraction α in that element is set equal to zero and the saturated liquid enthalpy h_f in the water-level element is substituted by the subcooled enthalpy h in that element.

Explicit Equations

The preceding steam-drum and water-level equations are not ideally suited to numerical solution. However, they can be employed to develop a set of explicit equations for the time derivative of (1) system pressure, (2) water level, (3) steam and liquid masses in the drum, (4) the water-level element enthalpy (if subcooled conditions persist within this element), and (5) the water-level element void fraction (if

a two-phase flow occurs within this element) as shown later. The derivation of these explicit equations are presented in the Appendix.

For a subcooled flow in the water-level element,

$$\frac{dp}{dt} = \frac{h_{fg}[v_g(m_{gr} - m_t) + v_f(m_{fr} - m_d)]}{v_{fg}[M_h - (V/J)] - h_{fg}[M_v + (Ay/v_f)(dv_f/dp)]} \quad (10)$$

$$\frac{dy}{dt} = \frac{1}{\rho_f} \left(\frac{m_{fr} - m_d}{A} + \frac{y}{v_f^2} \frac{dv_f}{dp} \frac{dp}{dt} \right) \quad (11)$$

$$\frac{dh}{dt} = \frac{1}{\rho_f y} \left[\frac{m_{fr}}{A} (h_f - h) + \frac{y}{J} \frac{dp}{dt} \right] \quad (12)$$

$$\frac{dM_g}{dt} = -\frac{1}{v_{fg}} \left[A \frac{dy}{dt} + M_v \frac{dp}{dt} + v_f(m_{gr} - m_t) \right] \quad (13)$$

and

$$\frac{dM_f}{dt} = m_{gr} - m_t - \frac{dM_g}{dt} \quad (14)$$

For a two-phase flow in the water-level element,

$$\frac{dp}{dt} = \frac{h_{fg}[v_g(m_{gr} - m_t) + v_f(m_{fr} - m_d)]}{v_{fg}\{M_h + Ay[C_1 - (B_1 C_2/B_2)] - [(V + Ay)/J]\} - h_{fg}M_v} \quad (15)$$

$$\frac{dy}{dt} = \frac{1}{\rho_f} \left[\frac{m_{fr} - m_d}{A} - \frac{yv_{fg}}{v_f h_{fg}} \left(C_1 - \frac{1}{J} - \frac{B_1 C_2}{B_2} \right) \frac{dp}{dt} \right] \quad (16)$$

$$\frac{d\alpha}{dt} = \frac{1}{B_2 y} \left(\frac{m_{fr} - m_d}{A} - \rho \frac{dy}{dt} - C_2 y \frac{dp}{dt} \right) \quad (17)$$

where

$$B_1 = \rho_g h_g - \rho_f h_f \quad (18)$$

$$B_2 = \rho_g - \rho_f \quad (19)$$

$$C_1 = - \left[\frac{\alpha h_g}{v_g^2} \frac{dv_g}{dp} + \frac{(1-\alpha)h_f}{v_f^2} \frac{dv_f}{dp} - \alpha \rho_g \frac{dh_g}{dp} - (1-\alpha) \rho_f \frac{dh_f}{dp} \right] \quad (20)$$

$$C_2 = - \left(\frac{\alpha}{v_g^2} \frac{dv_g}{dp} + \frac{1-\alpha}{v_f^2} \frac{dv_f}{dp} \right) \quad (21)$$

Equations for steam and liquid masses in the drum are identical with those of the subcooled case.

Steam-main Model

The application of the energy equation to the steam flow from the drum to the turbine admission valves gives⁵

$$m_t = \frac{A_v}{v_v'} \left[2gJ'(h_g - h_v) \right]^{1/2}_{s=\text{const}} \quad (22)$$

where

$$v_v' = \left[\frac{X^3 v_g^2}{\alpha^2} + \frac{(1-X)^3 v_f^2}{(1-\alpha)^2} \right]^{1/2}_v \quad (23)$$

Since the steam-main mass is lumped with that of the steam drum, no energy-storage term is needed in the above equation. Furthermore, in view of the critical choking flow at the turbine admission valve, the application of the momentum equation to a control volume containing the critical flow cross section yields⁵

$$m_t = A_v \left[-g \left(\frac{\partial p}{\partial v_v^*} \right)_{s=\text{const}} \right]^{1/2} \quad (24)$$

where

$$v_v^* = \left[\frac{X^2 v_g}{\alpha} + \frac{(1-X)^2 v_f}{1-\alpha} \right]_v \quad (25)$$

Since the amount of steam condensation in the steam main to the turbine admission valve is very small, v_v and v_v^* will both be approximately equal to v_g .

Pressure-control-system Model

With the pressure control system operative, the pressure regulator adjusts the turbine admission-valve flow cross-sectional area to maintain

a reasonably constant upstream pressure. Under this operating condition, the valve area is given by:

$$A_v = (A_v)_0 + b \int_{t_0}^t \delta(t - \tau) \quad (26)$$

where

$$\delta(t) = \frac{|P(t) - P_0|}{P(t) - P_0} \quad (27)$$

Under the large-load-change condition, when the operation of the pressure regulator is overridden by the turbine speed governor, the valve area is given by

$$A_v = A_v(t) \quad (28)$$

$A_v(t)$ is a known function of time (generally a step or ramp).

METHOD OF SOLUTION

In Ref. 1 the solution of the recirculation-loop space-dependent model is performed in three numerical integration and iterative loops: (1) a thermal loop; (2) a hydrodynamic loop; and (3) a neutron-kinetics loop. In the thermal loop the updated values of coolant enthalpy or void fraction are obtained by a "Modified-Euler" integration procedure. In the hydrodynamic loop the updated coolant velocities are computed with the use of an iterative method. Finally, in the neutron-kinetics loop, updated values of the thermal-neutron flux in the reactor core are determined.

In the present study the explicit differential equations for the steam drum and water-level element (Eqs. 10 to 17) are incorporated in the thermal loop. In this manner the updated values of the system pressure, the water level, the steam and liquid masses in the drum, and the water-level element enthalpy or void fraction are obtained by a Modified-Euler integration procedure. The steam-main equations (Eqs. 22 and 24) are incorporated in the hydrodynamic loop to determine critical pressure and mass flow rate at the turbine admission valve by a trial and error technique.

After each time increment the updated values of thermodynamic properties are calculated from the Steam Tables⁴ and the updated turbine admission valve flow area is calculated from Eq. 26 or Eq. 28. The updated values thus obtained are used as the initial quantities for the next time increment, and the three main iterative loops are continued until the final problem time is reached.

VERIFICATION OF THE MATHEMATICAL MODEL

For verification of the correctness of the mathematical formulation and numerical techniques, this analysis is applied to the transient study of a typical 110-Mw(e) direct-cycle forced-circulation boiling-water nuclear plant. The nuclear plant was assumed to be operating at 100% rated power followed by a ramp increase in utility demand load at a rate of 10%/sec up to 25% overload. The plant transient response for this overload operation has already been determined under the assumption of constant system pressure.¹ Since the mathematical model developed in Ref. 1 has been experimentally verified, it is believed that a comparison of these two studies would provide a verification of the mathematical model given in this paper. Although, unlike its predecessor, the present analysis treats the system pressure as variable, the results of both studies should be in agreement if the pressure-regulator control parameters are properly selected.

Figures 2 and 3 show the transient response of several plant variables for both constant system pressure¹ and present analyses. For both cases the plant power was controlled by a uniform variation in reactivity with a reactivity controller gain setting of -6.0×10^{-5} sec⁻¹ and a negligible measurement time delay. The pressure regulator gain setting for the analysis in this paper was taken equal to 0.0858 sq ft/sec. An examination of Figs. 2 and 3 reveals that the results of both analyses are in good agreement. However, the effect of system pressure variation combined with the nuclear-thermal-hydraulic interaction renders the system transient behavior more susceptible to oscillation. An increase in the pressure-regulator gain setting from 0.0858 sq ft/sec would tend to introduce undesirable oscillations of larger amplitude. Conversely, a further reduction in the pressure-regulator gain setting would increase the system power and flow stability. However, this increased system stability is accompanied by an increased lag in plant response, as measured by steam flow to turbine.

APPLICATION TO STEAM GENERATORS

The mathematical model presented in this paper is also applied to the dynamic analysis of a typical 400-Mw(t) vertical shell and tube steam generator with constant tube temperature. The plant transient response due to a hypothetical large-load-change condition when the operation of the pressure regulator is overridden by the turbine speed governor is studied. The turbine admission flow area is assumed to have a 10% step increase while the pressure regulator remains inoperative.

Fig. 2 - Dynamic analysis of 110 Mw(e) boiling-water reactor.

Fig. 3 - Dynamic analysis of 110 Mw(e) boiling-water reactor.

Figure 4 shows the transient response of several plant variables obtained from the present analysis. Initially, as compared with steam discharge to steam drum, the steam discharge to turbine undergoes a sharp rise with a subsequent sharp fall in system pressure. Later, as the steam flow to steam drum and turbine tend to approach each other, the system pressure levels off in a damped oscillatory fashion. Part c of Fig. 4 brackets the water-level position with two limiting cases. The lower curve, labelled "liquid level," is calculated on the assumption that steam bubbles formed in the water-level element enter the dome instantaneously. The upper curve, labelled "two-phase mixture level," is obtained by assuming that the steam bubbles stay within the element.

The gradual fall in both water-level curves after the initial transient is due to the constant feedwater-flow-rate assumption. One useful feature of the analysis in this paper is the determination of the spatial distribution of all dependent variables during the system transient operation. As an example, the spatial variation of coolant enthalpy around the recirculation loop is plotted in Fig. 5 for various times.

CONCLUSION

The space-dependent mathematical model, consisting of the recirculation-loop dynamic model presented in Ref. 1, combined with the steam-drum and steam-main-to-turbine models presented in this study, provides a powerful tool for the stability and transient analyses of boiling-water systems. The digital computer program developed has many applications in the design of boiling-water reactors and steam generators, a few of which have been demonstrated.

APPENDIX

DERIVATION OF THE EXPLICIT EQUATIONS

Differentiating Eq. 4 with respect to time and combining the result with Eq. 5 yields

$$v_f \frac{dM_f}{dt} + v_g \frac{dM_g}{dt} + M_v \frac{dp}{dt} = -A \frac{dy}{dt} \quad (\text{A.1})$$

where

$$M_v = M_f \frac{dv_f}{dp} + M_g \frac{dv_g}{dp} \quad (\text{A.2})$$

Fig. 4 – Dynamic analysis of 400 Mw(t) steam generator.

Fig. 5 — Dynamic analysis of 400 Mw(t) steam generator.

Eliminating dM_f/dt and dM_g/dt between Eqs. 1, 2, and A.1, one obtains

$$\left[M_v h_{fg} - \left(M_h - \frac{V}{J} \right) v_{fg} \right] \frac{dp}{dt} = -A h_{fg} \frac{dy}{dt} - (m_{gr} - m_t) v_g h_{fg} \quad (A.3)$$

where

$$M_h = M_f \frac{dh_f}{dp} + M_g \frac{dh_g}{dp} \quad (A.4)$$

As mentioned previously, the liquid enthalpy or the void fraction is taken as a main dependent variable when a subcooled or two-phase flow persists in the water-level element respectively. The derivation of the explicit equations are, therefore, divided into two parts as follows:

A Subcooled Flow in the Water-level Element

Under subcooled flow conditions Eq. 6 reduces to

$$\frac{dy}{dt} = \frac{1}{\rho_f} \left(\frac{m_{fr} - m_d}{A} + \frac{y}{v_f^2} \frac{dv_f}{dp} \frac{dp}{dt} \right) \quad (A.5)$$

Substituting Eq. A.5 into Eq. A.3 and solving for dp/dt yields

$$\frac{dp}{dt} = \frac{h_{fg}[v_g(m_{gr} - m_t) + v_f(m_{fr} - m_d)]}{v_{fg}[M_h - (V/J)] - h_{fg}[M_v + (Ay/v_f)(dv_f/dp)]} \quad (A.6)$$

Expanding Eq. 7 and substituting Eq. A.5 into this equation, one obtains

$$\frac{dh}{dt} = \frac{1}{\rho_f y} \left[\frac{m_{fr}}{A} (h_f - h) + \frac{y}{J} \frac{dp}{dt} \right] \quad (A.7)$$

Employing Eqs. A.1 and 1, one may find

$$\frac{dM_g}{dt} = -\frac{1}{v_{fg}} \left[A \frac{dy}{dt} + M_v \frac{dp}{dt} + v_f(m_{gr} - m_t) \right] \quad (A.8)$$

and

$$\frac{dM_f}{dt} = m_{gr} - m_t - \frac{dM_g}{dt} \quad (A.9)$$

Two-phase Flow in the Water-level Element

Under two-phase flow conditions, Eq. 6 reduces to

$$\frac{dy}{dt} = \frac{1}{\rho} \left[\frac{m_{fr} - m_d}{A} + y \left(\frac{\alpha}{v_g^2} \frac{dv_g}{dp} + \frac{(1-\alpha)}{v_f^2} \frac{dv_f}{dp} \right) \frac{dp}{dt} - y(\rho_g - \rho_f) \frac{d\alpha}{dt} \right] \quad (A.10)$$

where

$$\rho = \rho_g \alpha + \rho_f (1 - \alpha) \quad (A.11)$$

In a similar manner Eq. 6 may be reduced to

$$\frac{dy}{dt} = \frac{1}{(\rho h)} \left\{ \frac{(m_{fr} - m_d)h_f}{A} + y \left[\frac{1}{J} + \frac{\alpha h_g}{v_g^2} \frac{dv_g}{dp} + \frac{(1-\alpha)h_f}{v_f^2} \frac{dv_f}{dp} - \alpha \rho_g \frac{dh_g}{dp} - (1-\alpha)\rho_f \frac{dh_f}{dp} \right] \frac{dp}{dt} - y(\rho_g h_g - \rho_f h_f) \frac{d\alpha}{dt} \right\} \quad (A.12)$$

where

$$(\rho h) = \alpha \rho_g h_g + (1 - \alpha) \rho_f h_f$$

Combining Eqs. A.3, A.10, and A.12 in a manner similar to that for the subcooled case will yield explicit equations for dp/dt , dy/dt , and da/dt (see Eqs. 15, 16, and 17). Equations A.8 and A.9 are used for the calculation of dM_g/dt and dM_f/dt as before.

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Author Index

A

Adler, F. T., 45
Aizerman, M. A., 45, 56, 59
Akcasu, A. Z., 28, 31–32, 42, 531,
540
Albrecht, R. W., 312
Avery, R., 45, 48, 547–548

B

Bailey, R. E., 484
Bailly du Bois, B., 228, 231–232
Baldwin, G. C., 547
Barandiaran, M., 8
Barnard, E., 296
Barss, W. M., 343
Beck, G. P., 420
Becker, M., 245
Behrens, D. J., 264
Berkovitz, L. D., 140
Bethe, H. A., 34
Beyster, J. R., 438, 473
Bogoliubov, N. N., 88–89, 96, 103
Bryson, A. R., Jr., 140

C

Campe, K. M., 484
Canosa, J., 361

Chang, S. S. L., 140
Chernick, J., 38, 100, 106, 343
Chezem, C. G., 46, 48, 55, 65
Christensen, H., 562
Cockrell, R. G., 323
Coffer, C. O., 398
Cohn, C. E., 540, 543
Corngold, N., 232, 236, 248
Cox, C. W., 431

D

Daitch, P. B., 225
Dalfes, A., 31
Davison, B., 238, 343
Dee, J. B., 398
Demuth, H. B., 518
Denham, W. F., 140
Dickerman, C. E., 371
Dreyfus, S. E., 140

F

Fenech, H., 24–25
Foulke, L. R., 22
Fultz, S. C., 249

G

Gage, S. J., 45
Garelis, E., 246–248, 262

- Ghatak, A. K., 243
 Gozani, T., 246, 248
 Gyftopoulos, E. P., 1, 39, 47, 64
- H**
- Hapke, B., 343
 Henry, A. F., 47, 65, 248, 343
 Higgins, T. J., 56
 Hill, R. F., 56
 Hobson, R. L., 225
 Honeck, H. C., 243
 Hornyik, K., 457
 Horowitz, J., 228, 231–232
 Hudson, J. L., 562
- I**
- Imhoff, D. H., 545
- J**
- Jenkins, A. L., 288
 Johnson, S. L., 22, 24
 Jones, A. B., 562
- K**
- Kaplan, S., 342
 Keepin, G. R., 315, 431
 King, J. S., 245–246, 273
 Kirchenmayer, A., 562
 Kliger, I., 110
 Koppel, J. U., 232
 Krasin, A. K., 315
 Krasovskii, N. N., 72, 77, 82
 Krylov, N. M., 88–89, 96, 103
 Kurstedt, H. A., 420
 Kyong, S. H., 17
- L**
- LaSalle, J., 40
 Lee, R. R., 225
 Lefschetz, S., 40
 Lellouche, G. S., 40
 Leonard, B. E., 277
 Letov, A. M., 45, 56–57
 Liu, T. C., 182–205
 Long, R. L., 386
 Lur'e, A. I., 45, 57
- M**
- Malkin, I. G., 93, 95, 98, 100, 107–108
 Margulies, R. S., 39
 Maurette, C., 228, 231–232
 Melsa, J. L., 207
 Michael, P., 232
 Miley, G. H., 420
 Mohler, R. R., 137, 168–170, 205
 Moss, L. I., 437
 Murray, H. S., 77
- N**
- Nahavandi, A. N., 566
 Neill, J. M., 473
 Neustadt, L. W., 206
 Noble, L. D., 32
 Nordsieck, A., 469
- O**
- Orndoff, J. D., 393
 Osborn, R. K., 531
- P**
- Perez, R. B., 323
 Perry, J. E., 168, 205
 Pluta, P. R., 544
 Pontryagin, L. S., 206
 Popov, V. M., 7–8
 Powers, P. N., 45
 Purohit, S. N., 296

Q

Quisenberry, K. S., 245

R

Raievski, V., 482

Rajagopal, V., 299

Ramanna, R., 277, 279

Randall, D., 343

Razumikhin, B. S., 77, 83

Roberts, J. J., 206

Robinson, L. E., 371

Russell, J. L., Jr., 246-248, 473

S

St. John, D. S., 343

Scalettar, R., 342, 398, 401, 419

Schilling, F. P., 518

Schultz, D. G., 77

Seale, R. L., 46, 48, 55, 65

Secker, P. A., 164

Shapiro, A. H., 504

Shapiro, C., 236

Sheff, J. R., 312-313

Shen, C. N., 182

Shoptaugh, J. R., Jr., 398

Shotkin, L. M., 34, 39, 42, 88

Simmons, B. E., 245-247, 273

Sjostrand, N. G., 245, 248, 256

Smets, H. B., 32

Smith, H. P., Jr., 206

Solberg, K. O., 562

Swen, G. E., 299

U

Uhrig, R., 295

V

von Hollen, R. F., 566

W

Wall, I., 24-25

Ward, A. G., 342-343

Weaver, L. E., 164

Weintraub, L., 518

Welton, T. A., 7, 31

West, G. B., 398

Whittemore, W. L., 398

Wintner, A., 40

Wollman, W., 232

Wyman, M. E., 457

Z

Zivi, S. M., 562

Zubarev, T. N., 421

Subject Index

A

Absolute stability, 2
Absorber, oscillating, 22–23
Absorption cross section, 251, 282, 316, 354
 control of, for nuclear rocket, 182–205
Absorption lifetime, 236
Accelerator booster, 473–483
Adaptive control, 165, 174, 180
Adiabatic method in space-dependent dynamics, 342–360
Adiabatic model, 38–43, 99, 430–431, 469
Adiabatic treatment of the xenon problem, 342–360
Adjoint flux, 300, 334
Adjoint function, 47, 324, 326
Adjoint operator, in fluctuation theory, 537
Adjoint variables (auxiliary, costate), 140, 148, 154, 157, 166, 191–193
Adjoint vector (costate), 139–140, 167
Aizerman's method, 59–63, 65, 72–74

American Nuclear Society, 24
Angle and energy representations, 226–227
Angular flux, 324–325
Area-ratio method, extrapolated, 246, 248, 252, 262, 264, 268–275
 of Sjostrand, 245, 248, 252, 255–256, 262–264, 268–275
Argonaut reactor, 323, 337
Assembly, moderating, 225–244
 subcritical, 245, 312–313, 476
Asymptotic spectra, 232, 242
Asymptotic stability, 1, 13
 in coupled systems, 58–61, 71, 80–81, 85–87
 criteria for, 2–8, 30–31, 58, 71
Atomics International, 438, 440, 451
ATPR (Advanced TRIGA Prototype Reactor), 398–419
Auxiliary controller, 110, 112, 126–129
Auxiliary variables (adjoint, costate), 140, 148, 154, 157, 166, 191–193

B

Bang-bang control (dither), 153, 162, 187, 199

- Basic probability distribution, 531
- Beryllium, energy-dependent diffusion
 - in, 234-236, 240-241
- Beryllium-moderated reactors, space-
 - dependent spectral density in, 312-322
- Boiling-water reactors, dynamic anal-
 - ysis for, 576-581
 - excursions in, 278, 544-545
 - with integral superheat, 544-565
 - oscillations in, 567, 579-580
 - pressure effects in, 566-583
- Boltzmann equation (see Transport equation)
- Boundedness, 12, 29, 32-33, 39
 - (See also Stability)
- Break frequencies, 302, 313, 317
 - in fluctuation spectra, 540, 542
- Brookhaven research reactor, 475
- Buckling, 225, 236-242, 315-316, 336
 - in compressible media, 491, 495-499, 508, 512-513
- Burst calculations, 24-25, 361, 370
 - Nordheim-Fuchs model for, 398, 401, 405, 419, 434
 - (See also Excursion; Pulse)
- Burst reactor, WSMR (FBR), 386-397
- Burst rod (pulse rod), 387, 393, 421, 426-427, 432, 435
- Burst width, in accelerator booster, 476, 479-483
 - in FBR reactor, 392-393
 - in pulsed neutron source, 474-476, 479-482
 - in SORA pulsed reactor, 476, 479-483
 - in TRIGA reactor, 398-400, 409, 423, 428, 459, 469
- Burst yield in FBR reactor, 391, 393, 395
- Bursts, neutron, in FBR reactor, 391-395
 - in pulsed neutron experiments, 225, 261, 281, 285, 293
 - (See also Excursion; Pulse)
- C**
- Calculus of variations, 114, 123, 132-136, 200
- Canonical coordinates, 91-93, 99
- Centers, 88-89, 93, 95, 98-100, 107-108
- Čerenkov radiation, 411
- Characteristic equation, 4, 31, 56, 91, 94, 100
- Characteristics, equation of, 504-505
- Chemical shim control, 300, 304
- Choppers, 226, 475-476, 481
- Circulating-fuel reactor, 30
- Cladding penetration, 371-373, 379
- Coherence of fuel failure, 373-376
- Computer, on-line process, 518-531
- Constraint boundary, 137, 141-143, 152, 156, 162
- Constraints, on changes in neutron density, 111
 - for control vectors, 208, 219-220
 - in input-output formulation, 14, 17, 19
 - for nuclear rocket control, 137, 140, 168-170, 188
 - in optimal xenon problem, 206
 - on propellant flow system, 143, 147, 157-162
 - on rate of temperature rise, 170-171
 - on reactivity, 169-170, 188
 - on reactor power, 148, 153, 169-170
 - on thermal stresses, 111, 154, 162

- Continuum eigenfunctions, 233-236, 239, 345, 537
- Continuum eigenvalues, 225-226, 228, 236, 238, 242, 537
- Control, adaptive, 165, 174, 180
 automatic, 518-531
 bang-bang (dither), 153, 162, 187, 199
 chemical shim, 300, 304
 continuous, 212
 dither (bang-bang), 153, 162, 187, 199
 neutronic, 219-223
 of nuclear rocket, 137-205
 optimal (see Optimal control)
 of reactor by on-line computer, 526-529
 suboptimal, 138, 161-162, 168, 208-223
- Control drum, 183, 446
- Control law, 113, 122-124, 135
 for nuclear rocket, 170-171, 182, 191, 193-197, 199, 204-205
- Control matrix, 207
 nonsingular, 210-214
 singular, 215-219
- Control policy, 215-216
- Control region, 208, 220-223
- Control space, 215
- Control system, model-following, 164-165
 for nuclear rocket, 161, 189-190
- Control variables, in optimal control, 152, 165, 169-171, 174, 200
 in suboptimal control, 214, 215
- Control vector, in input-output
 formulation for optimal control, 14, 17
 in optimal nuclear rocket control, 138, 165, 167, 191
 in suboptimal control, 208, 210-215, 219-220
- Control-weighting matrix, 166, 172, 177
- Controller, auxiliary, 110, 112, 126-129
- Controller, optimal, 113-114
 program, for optimal control, 122-123, 125
 for suboptimal control, 221
 (See also Optimal control)
- Controller-computer, 207
- Coolant blockage, 373, 375
- Coolant vaporization, 371-372
- Correlation function, 539, 541
- Cost functional (see Performance index)
- Costate variables (adjoint, auxiliary), 140, 148, 154, 157, 166, 191-193
- Costate vector (adjoint), 139-140, 167
- Counting intensity in pulsed-source experiment, 473-475
- COUPKIN code, 334-336
- Coupled-core dynamics (see Coupled reactors)
- Coupled reactors, 45-87, 278, 323
 boiling-water superheat reactors as, 544-565
 computer codes for, 334-335
 coupling for, 48-49, 68-70, 324, 544, 547-548
 dynamic equations for, 47, 49-50, 78-79, 324-333
 fast-thermal systems as, 46, 323
 in nuclear rocket, 46, 48, 323
 parameters for, 65-68, 331-333, 335-340

- stability of, 45-87, 559
 - with time delay, 52-55, 81-87
 - without time delay, 51-52, 79-81
 - Coupling, for coupled reactors, 48-49, 68-70, 324, 544, 547-548
 - for multiregion reactors, 324, 332, 544, 547-548, 550-558
 - Coupling parameters, 65-68, 331, 333, 335-340
 - Critical decay constant, 247-248
 - Critical systems, fluctuations in, 532
 - Criticality, 247
 - in compressible media, 484-517
 - for coupled systems, 550
 - in fluctuation theory, 538
 - Cross section, absorption, 251, 282, 316, 354
 - control of, for nuclear rocket, 182-205
 - for crystalline effects, 226
 - detector, 535
 - double differential, 482
 - for quantum-level effects, 226
 - removal, 284
 - scattering, 227
 - total, 227, 283, 482
 - xenon, 90, 105, 351
 - Cross-section calculations for UFTR, 335-336
 - Cross sections in compressible media, 486, 489, 499, 501
 - Crystal spectrometers, 228
- D
- DDK code, 391
 - Decay constant, critical, 247-248
 - prompt, 246-247, 249, 251, 264-265, 278
 - Delayed eigenfunctions, 246, 249-252, 256
 - Delayed flux in pulsed neutron measurements, 245, 248-249, 251-254, 258, 265-268
 - Delayed modes, 246, 248-251, 255, 273
 - Delayed-neutron background in accelerator booster, 477
 - Delayed-neutron fraction, room-return effect on, 393
 - Delayed-neutron heating, 364-366
 - Delayed-neutron precursors, in accelerator booster, 477
 - in coupled reactors, 548, 559
 - in nuclear rocket, 144, 169, 195, 197-199, 202-203
 - in pulsed neutron measurements, 249, 252-253
 - in SORA pulsed reactor, 477
 - in space-dependent dynamics, 324-325, 327-328, 332
 - Delayed-neutron spectrum, 325
 - Delayed-neutron tail, 363-364
 - Delayed neutrons, 47, 112
 - in compressible media, 502
 - effect on spectral density, 313-315
 - in excursion calculations, 361, 363-367
 - in fluctuation theory, 533-536, 539-540
 - one-group model for, 166, 249, 290, 363, 533, 559
 - in pulsed neutron measurements, 262-264, 274, 279-280, 288, 291-294
 - stability effect of, 1-2, 32, 65
 - (See also Dynamic equations)
 - Delayed photoneutrons, 315
 - Describing function, 56
 - Detailed balance, 227
 - Detector cross section, 535

- Detector position, effect on spectral density, 312, 318–320
effect on transfer function, 23, 299, 302, 307
in pulsed neutron reactivity measurement, 260, 268–274
- Detector size, effect on spectral density, 316
- Deuterium–tritium reaction, 279
- Die-away, neutron, 277–278, 294, 297, 476–479
(See also Pulsed neutron)
- Diffusion, energy-dependent, 234–236, 240–241
(See also Thermalization)
time-dependent, 234, 238, 240, 249, 282–285, 333–334
in compressible media, 484–485, 491–497, 509–516
- Diffusion coefficient of hydrogen, 438, 440
- Diffusion cooling, 242, 279, 296–297
- Diffusion time, 482
- Dimensionality of subsets in state space, 218
- Disassembly, 437–445
- Dissociation of hydrogen in SNAP reactor, 443
- Dither control (bang–bang), 153, 162, 187, 199
- Doppler effect, 302, 310, 361–363, 365
- Doublet density, 532–534, 539–540, 543
- Doublet theory, 532–533, 539–540
- Dresden reactor, 363
- Dynamic equations, for accelerator booster, 476–477
for boiling reactors, 547–549, 566–567, 571, 575
for coupled reactors, 47, 49–50, 78–79, 324–333
for excursion calculations, 362, 367–370
normalized, 91
for nuclear rockets, 143–144, 165, 168–169, 183–186
for optimal reactor control, 112, 221
for pulsed neutron reactivity measurements, 252–253
for reactor stability, 2, 28–29, 38, 90
for SNAP reactor excursions, 443–445
for SORA pulsed reactor, 476–477
with space dependence (see Space-dependent dynamics)
for steam system, 571–576, 579–583
for TRIGA reactor, 430–432, 467–469
two-energy group, 183
for xenon concentration, 90, 104–106, 351
- Dynamic parameters, for coupled systems, 65–68, 331–333, 335–340
measurement of, 299–311
- Dynamic programming, 1, 24–26, 190
- Dynamic reactivity effect, 484–517
- DYSTAC analog computer, 172
- E**
- EBR-I reactor, 34–36
- EBR-II reactor, 371–373, 376–381
- Edgerton, Germeshausen & Grier, Inc., 449
- Eigenfunctions, continuum, 233–236, 239, 345, 537

- delayed, 246, 249–252, 256
 - in fluctuation theory, 537, 543
 - prompt, 246, 249–252, 256
 - in pulsed neutron experiments, 225–226, 232–237, 239–240, 278, 283–284, 296
 - singular, 226–227
 - in space-dependent dynamics, 23, 344, 354–355
 - for space-dependent spectral density, 314
 - Eigenvalue-buckling ($\lambda - B^2$) plane, 225, 237–242
 - Eigenvalue-eigenfunction analysis, 226
 - Eigenvalue spectrum, 537, 540
 - Eigenvalues, continuum, 225–226, 228, 236, 238, 242, 537
 - in coupled systems, 334–335, 549
 - for equilibrium states, 9–11, 91–92
 - in fluctuation theory, 537, 540, 543
 - in pulsed neutron experiments, 225–232, 237, 243, 250
 - in space-dependent dynamics, 345–346, 349–350, 354–355
 - Eigenvector scalar products, 207–213, 215, 219, 223
 - Eigenvectors, for equilibrium states, 9
 - in space-dependent dynamics, 345, 350
 - Electron accelerator, 473–483
 - Energy groups, 225, 325
 - Energy mesh points, 228–232
 - Energy release, in SNAP reactor, 441
 - in TRIGA reactor, 398–400, 405–408
 - Energy resolution, 474, 476, 481–482
 - Equilibrium points, 51–53, 89–91, 98–99
 - Equilibrium states, 1, 9, 13, 31
 - eigenvalues for, 9–11, 91–92
 - at infinity, stability of, 1, 8–13
 - Erdmann–Weierstrass corner conditions, 134
 - Error vector, 165–166, 174, 177
 - Error-weighting matrix, 166, 177
 - Euler–Lagrange equations, 133–134
 - Excursion calculations, 24–25, 361–370
 - Excursion data, correlation of, 24
 - Excursions, 39, 41–42, 361–370
 - in boiling superheat reactors, 544–559
 - in FBR reactor, 386–397
 - heat-transfer effects in, 364–367
 - maximum temperature in, 361, 363–369, 374–376, 378, 392–393
 - Nordheim–Fuchs model for, 398, 401, 405, 419, 434
 - quenching of, 38, 43
 - in SNAP reactor, 437–456
 - in TREAT reactor, 372
 - in uranium–zirconium hydride reactors, 398–472
 - (See also Burst; Pulse)
 - Expansion of core vessel in SNAPTRAN-3 reactor, 449–452
 - Expansions in small parameters, 38, 42–43, 94, 107
- F
- FAIM code, 334
 - Fast burst reactor, WSMR, 386–397
 - Fast reactor pulsing, 421, 473–483
 - (See also Pulsed reactors)
 - Fast reactor safety, 371–385, 479, 483

- Fast-thermal systems, 46, 323
- FBR (Fast Burst Reactor), 386-397
- Feedback, in boiling superheat reactors, 551, 559
 disassembly, 437-438, 455
 Doppler, 361-363, 365
 expansion, 455
 inherent, 24
 linear, 28-37, 39
 as convolution integral, 3, 28
 moderator, 364-367
 nonlinear, 3, 31, 39, 46, 49
 in coupled reactors, 55-56, 62-64
 optimal (see Optimal control)
 positive, 31
 in coupled reactors, 58
 proportional to power, 58
 in Saxton reactor, 300, 307-310
 in SNAP reactor, 437-445, 448, 451-452, 455
 spectral effects on, 437-438
 temperature (see Reactivity coefficient, temperature)
 temperature dependent, 373
 in TRIGA reactor, 420-436, 457, 472
 (See also Reactivity coefficient)
- Feedback control, 166, 168, 175, 178-180, 186, 547
- Feedback gains, 165, 167-168, 174, 180
- Feedback kernel, 28-30, 34
- Feedback loop, compensating, 185-186
- Feedback network, 124-125, 127-128
- Feedback transfer function, 2-3, 6-7, 28, 301-303, 308-309
- Fermi age, 182, 184-185, 277, 279-289, 291-296
- Film boiling, 399, 414-417, 419
- Finite escape time, 28-29, 32-36, 38-44
- Fission-foil burst-yield measurements, 391
- Fission-product release, 437, 455, 519
- Floquet theory, 491-492
- Fluctuation of amplitude in pulsed neutron experiments, 258
- Fluctuation spectra, 531-543
- Fluctuation theory, 531-543
- Flux tilting, 46, 357
- Flux-wire measurements, 303, 309
- Focus (spiral point), 9, 98-99
- FOMENT code, 258
- Four-group calculation, 258-259
- FPR (Flexible Plastic Reactor), 246, 251, 258-259
- Free gas, thermalization in, 228, 230-231, 233, 236-237
- Free-gas kernel, 228, 236-237
- Frequency domain for boiling-reactor stability analyses, 547
- Frequency response (see Transfer function)
- Fuchs-Nordheim model, 398, 401, 405, 419, 434
- Fuel, fast reactor, ceramic, 372, 376-379
 metallic, 373
 thorium based, 373, 381-383
 UC, 373, 377, 379
 UO₂, 371-373, 376-377
 US, 371, 373, 376, 378
- Fuel burnup, 278
- Fuel cycles, 25

- Fuel disintegration in SNAP reactor, 439-445, 452-455
- Fuel failure, in fast reactor accidents, 371-385
 - coherence of, 373-376
 - effect of irradiation on, 372, 378-379
- Fuel-failure calculations, 373-376
- Fuel-failure threshold, 371-374, 379-381
- Fuel loading, computer controlled, 527-528
 - optimal, 1, 24-26
- Fuel vaporization, 455
- G**
- GAM-1 code, 335-336
- Garelis-Russell method, 248, 252, 257, 261-262, 264, 268-275
- Gas-cooled reactors, 519-520
- General Atomic Division, 398, 475, 479
- General Electric Co., 361, 544, 563
- General Public Utilities Corp., 299
- Generation time, neutron, 39, 245, 247, 289
 - in coupled systems, 339, 548
 - effective, 90
 - subcritical, 245
 - (See also Neutron lifetime)
- Global stability (stability in the large), 28, 31-32, 71
- Godiva reactor, 479
- Green's function in nuclear rocket feedback control, 186
- H**
- Hafnium sleeves in pile oscillator, 303
- Hamiltonian function, 114, 139, 148, 167, 191, 342-350
- Hamilton-Jacobi equation, 207-209, 223
- Hamilton's equations, 115, 139, 167
- Hardening of neutron spectrum, 437
- Harmonics, higher, in pulsed neutron measurements, 246, 248, 251, 261, 273-274, 277-278, 294
- Harwell accelerator booster, 475-476
- Heat-exchange dynamics for nuclear rocket, 147-157, 168-169
- Helium-cooled graphite reactor (UHTREX), 519-520
- Hermite polynomials, 228, 231-234
- HPRR (Health Physics Research Reactor), 386
- I**
- IBR pulsed reactor, 421, 475, 479-480
- Importance, 327, 329, 438
 - (See also Adjoint; Weighting)
- Impulse response, 30
- Index of performance (see Performance index)
- Inertial effects, 371
- Inhour equation, 291, 314
- Input matrix, 168
- Input-output formulation for optimal control, 1, 13-21
- Instrumentation, reactor, 303-305, 411-414
- Integral Thermal Superheat Reactor (SBWR), 544-546, 550, 559
- Interlocks in automatic reactor control, 526-528, 530

- Inverse multiplication, 258–259
- Inverse period, in FBR reactor, 391–394
in TRIGA reactor, 408
- Ion chambers, 411
- Isotope production with accelerator booster, 481, 483
- J**
- Jet pumps, 544, 547, 562–564
- Joint Nuclear Research Institute (Dubna, Russia), 421
- Junction points, 142
- K**
- KARE system of diffusion-theory programs, 258
- Kernel, energy exchange, 236–237, 242–243
feedback, 28–30, 34
free gas, 228, 236–237
scattering, 234, 242–243
symmetric, 227
- KESIT-I code, 434
- Kinetic equations (see Dynamic equations)
- Kinetic parameters, for coupled systems, 65–68, 331–333, 335–340
measurement of, 299–311
- KIWI reactor, 46
(See also Nuclear rocket)
- KIWI-TNT reactor, 440–442
- Knolls Atomic Power Laboratory (KAPL), 246
- L**
- Lagrange stability, 7, 28, 32, 34, 37, 39
- Laguerre polynomials, 296
- Lambda (λ) – B^2 plane, 225, 237–242
- Laplace transformability, 32–34, 36
- Legendre–Clebsch condition, 135
- Lethargy, 282
- Liapunov's first method, 30
- Liapunov's second method, 31, 100
in coupled systems, 55, 61, 63, 77–82, 86
- Liapunov function (V-function), 4–6, 32, 100
in coupled systems, 49, 57–58, 61, 63, 70–73, 79
in suboptimal control, 208–218, 223
- Lifetime, absorption, 236
neutron (see Neutron lifetime)
region, 324
- Limit cycles, 13, 88–89, 93–95, 100–103, 106
- Linear accelerator, 473–483
- Linear stability, 4, 30, 35
- Liouville equation, 531
- Local stability, 28–31
at infinity, 8, 12–13
- Local transfer function, 299–300, 306–308, 311
- Los Alamos Scientific Laboratory (LASL), 386–387, 390–391, 393–396
- Lur'e–Letov method, 45, 57–61, 63, 70–72
- M**
- Marshak boundary condition, 241
- Massachusetts Institute of Technology, 1
- Matrix–Ricatti equation, 167
- Maximum principle, Hamilton's equations for, 139

- in nuclear rocket control, 137-139, 142, 154, 164-166, 182, 190, 199-202
 - for solution of synthesis problem, 114-124
- Maxwellian distribution, 233, 296
- Mayer's problem, 123, 132-136
- Measurement matrix, 166
- Meltdown accidents, 371-375, 378
- Melting of fuel, 455
- Minimum-time trajectory for xenon problem, 206
- Mixed Spectrum Superheat Reactor (MSSR), 544-547, 550, 559
- Modal distortion, 24
- Modal-shape difference, 246, 248, 251-252, 274-275
- Mode shapes in pulsed neutron measurements, 245
- Model-following control system, 164-165, 180
- Moderating assembly, 225-244
- Moderating material, 225-226
- Modes, delayed, 246, 248-251, 255, 273
 - higher, 22-23, 348-349
 - in pulsed neutron measurements, 246, 248, 251, 261, 273-274, 277-278
 - for space-dependent spectral density, 313
 - prompt, 246, 248-251, 255, 273
- Modular cores, 323
- Monte Carlo technique, 242
- Movable singularities, 39
- Multigroup theory, in pulsed neutron measurements, 225, 251
 - in space-dependent dynamics, 313, 334, 344, 351, 354
 - for TRIGA reactor, 464
- N
- National Reactor Testing Station, 445
- Neutron balance, time dependent, 500-501
- Neutron bursts, in FBR reactor, 391-394
 - in pulsed neutron experiments, 225, 261, 281, 285, 293
 - (See also Excursion; Pulse)
- Neutron diffraction, 482
- Neutron diffusion (see Diffusion)
- Neutron-dynamics equations (see Dynamics equations)
- Neutron fluctuations, 531-543
- Neutron generation time, 39, 245, 247, 289
 - in coupled systems, 339, 548
 - effective, 90
 - subcritical, 245
 - (See also Neutron lifetime)
- Neutron generators, 261, 281
- Neutron lifetime, 301, 367, 408, 443
 - in accelerator booster, 479
 - in coupled systems, 46, 559
 - for nuclear rocket, 147, 171, 184
 - in space-dependent dynamics, 314, 316, 327, 332
 - thermal, 289, 293
 - for TRIGA reactor, 402, 458
- Neutron moderation (see Moderating; Thermalization)
- Neutron source, pulsed, 253, 261, 281, 295, 473
- Neutron spectroscopy, 476, 482

- Neutron transport, 226, 237–238, 243, 284, 323–324, 532
(See also Transport equation)
- Neutronic control, 219–223
(See also Control)
- Neutrons, delayed (see Delayed neutrons)
- Nodes, 9, 12
- Noise, 125, 174
space dependence of, 312–322, 531–543
(See also Fluctuations)
- Noise source, 540, 543
- Nonleakage probability, time dependent, 499–501
- Nordheim–Fuchs model, 398, 401, 405, 419, 434
- Nuclear rocket, 137–205
control of absorption cross section in, 182–205
control system for, 161, 189–190
coupled reactors in, 46, 48, 323
design conditions for, 147
design parameters for 171
dynamic equations for, 143–144, 165, 168–169, 183–186
heat exchange dynamics for, 144–145, 168–169
optimal control of, 137–205
optimal heat exchange for, 147–150
optimal start-up of, 157–158, 164–181, 184, 190, 195, 198–199
propellant for, 146–147, 169–172
pump performance in, 159
specific impulse of, 146–147, 161
start-up of, 157–158, 164–181, 184, 190, 195, 198–199
thrust of, 146–147, 161
- Nuclear superheat reactors, 46, 544–565
- Nucleate boiling, 415
- Nyquist method, 48
- ### O
- On-line process computer, 518–531
- One-group model (see Delayed neutrons)
- One-group theory, in pulsed neutron measurements, 249
reactivity effect in, 300
for space-dependent spectral density, 312–313, 320
in space–time dynamics, 344, 354
- Operation monitoring by on-line computer, 524–526
- Optimal control, approximate time-optimal method for, 207–224
input–output formulation for, 1, 13–21
of nuclear rocket, 137–205
synthesis of, 110–136, 138, 164–181
of xenon, 206
- Optimal controller, 113–114
- Optimal heat-exchange process, 147–157
- Optimal refueling, 1, 24–26
- Optimal start-up, of heat-exchange process, 150
of nuclear rocket, 157–158, 164–181, 184, 190, 195, 198–199
- Optimal trajectories, 162, 176
for eigenvector scalar product solutions, 210–211
minimum time, in xenon problem, 206
on phase-constraint boundaries, 142–143
small perturbations about, 164–165, 175

- for start-up of nuclear rocket, 157-158
- Optimization of pulsed reactor operation, 420, 426, 435, 481
- Oscillating absorber, 22-23
- Oscillating flux in compressible media, 491-492
- Oscillation tests, 22, 306, 311
- Oscillations, 22-23, 98, 100-101, 107
 - in boiling-water reactors, 567, 579-580
 - in pulsed reactors, 424-426, 435
 - relaxation, 100
 - xenon, 46, 101, 278, 343
- P**
- Parameter space, stability regions in, 14, 53-55
- Peak fission rate in FBR reactor, 393-394
- Peak neutron yield in accelerator booster, 475-476, 482
- Peak power, in accelerator booster, 479-480
 - in TRIGA reactor, 398, 400, 406, 408, 434, 459, 469
- Performance index, 110, 113
 - coolant flow rate as, 148
 - in input-output formulation for optimal control, 14, 17
 - for optimal control of nuclear rocket, 138, 164-166, 170, 182, 190
 - propellant consumption as, 151
- Perturbation equations, 182
- Perturbation series, 42
- Perturbations in optimal control, 127, 164-165, 175, 184
- Perturbed kinetics equations for nuclear rocket, 184-189
- Phase-constraint boundary, 137, 141-143, 152, 156, 162
- Phase plane, 8-9, 368
- Phase points in fluctuation theory, 535
- Phase space, in neutron transport, 325
 - stability regions in, 82, 85
 - xenon-iodine, 206
- Phillips Petroleum Co., 445
- Photoneutrons, from accelerator pulse, 473
 - delayed, 315
- Pile oscillator, 22-23
- Plutonium-239, absorption rate in, 423
 - as pulsed reactor fuel, 477
- Point-reactor model, for accelerator booster, 476-479
 - for analysis of power history, 446, 452
 - for boiling-water reactors, 547
 - comparison with coupled equations, 329, 332
 - derivation of, 327-333
 - in excursion calculations, 24-25, 361, 373, 467
 - fluctuation effects in, 532, 539
 - limitations of, 24-25, 278, 343, 437, 455, 532, 539
 - for SORA pulsed reactor, 476-477
 - (See also Dynamic equations)
- Polyethylene moderator, 258

- Polynomial expansions, in
thermalization, 228, 232–234
- Pontryagin's method (see Maximum principle)
- Popov's criterion, 7
- Power coefficient of reactivity,
90, 299, 310–311, 424
- Power cost, dependence on
refueling policies, 26
- Power reactor stability, 300
- Power spectrum, 299, 305–306,
311
(See also Noise; Spectral density)
- Power transients (see Excursions)
- Precursors, delayed neutron (see Delayed neutron)
- Pre-Hamiltonian function, 166–167
- Pressure in boiling systems,
566–583
- Pressure transients, 371–372
- Pressurized water reactors, 300
- Process computer for UHTREX
reactor, 521–524
- Program controller for optimal
control, 122–123, 125
- Projective transformation, 10
- Prompt decay constant, 246–247,
249, 251, 264–265, 278
- Prompt eigenfunctions, 246,
249–252, 256
- Prompt flux in pulsed neutron
measurements, 245, 248, 251–
254, 258, 265–268
- Prompt-jump approximation, 548–
550
- Prompt modes, 246, 248–251, 255,
273
- Prompt reactivity coefficient,
33–35
in SNAP reactor, 437, 443, 455
in TRIGA reactor, 402–405,
418–419, 427, 459
- Propagation of disturbances, 22
- Pulse rod (burst rod), 387, 393,
421, 426–427, 432, 435
- Pulse width, in accelerator booster,
476, 479–483
in FBR reactor, 392–393
in pulsed neutron source,
474–476, 479–482
in SORA pulsed reactor, 476,
479–483
in TRIGA reactor, 398–400,
409, 423, 428, 459, 469
- Pulsed neutron curves, rise portion
of, 277–298
- Pulsed neutron data, typical,
263, 281
- Pulsed neutron experiments, 225–
244
- Pulsed neutron measurements,
higher modes in, 246, 248, 251,
261, 273–274, 277–278
- Pulsed neutron reactivity meas-
urements, 245–276
- Pulsed neutron source, 253, 261,
281, 284, 295, 473
- Pulsed reactors, comparison with
accelerator booster, 480–483
FBR, 386–397
Nordheim–Fuchs model for, 398,
401, 405, 419, 434
oscillations in, 424–426, 435
revolving-reflector, 421, 475–483

- SORA, 475-483
- TRIGA, 398-436, 457-472
- WSMR (FBR), 386-397
- (See also Burst; Excursion)
- Pumping power, failure of, 374
- PUNS code, 264
- Q**
- Quenching of excursions, 38, 43
- R**
- Radiation-damage studies with
 accelerator booster, 481, 483
- Ramp (see Reactivity insertion
 rate)
- RAT code, 416-417
- Reactivity, compensated, 446-448,
 450, 452
- Doppler (see Doppler effect)
- from fuel failure, 372-373,
 375-376, 378
- from local physical change, 300
- oscillating, 22-23
- of reflecting materials, 387
- subcritical, 247, 254, 260, 268,
 273-275
- in uranium-zirconium hydride
 reactor excursions, 437-456
- Reactivity calibration for FBR
 reactor, 393
- Reactivity coefficient, delayed,
 427-430
- energy, 450
- expansion, 438, 443, 450-452,
 455
- isothermal, 302
- positive, 33, 35, 38-39, 41,
 427
- power, 90, 299, 310-311, 424
- prompt, 33-35
- in SNAP reactor, 437, 443, 455
- in TRIGA reactor, 402-405,
 418-419, 427, 459
- temperature, 38, 90, 243
- dynamic, 393
- positive, 38-39, 41, 43
- in SNAP reactor, 437-438, 443,
 448-449, 451, 455
- static, 394
- in TRIGA ATPR reactor, 398,
 402-405, 408, 418-419
- in TRIGA Mark II reactor,
 457-458, 469
- in TRIGA series pulsing,
 420-421, 423, 427-430
- in UHTREX reactor, 519
- (See also Feedback)
- Reactivity computation, by on-line
 computer, 525
- from power history, 446, 452
- Reactivity feedback (see Reactivity
 coefficient)
- Reactivity insertion, large, in
 TRIGA reactor, 398-419
- Reactivity insertion rate, 371,
 373, 544, 549-559
- Reactivity measurements, in FBR
 reactor, 390-391
- by pulsed neutrons, 245-276
- Reactivity modulator, 299, 303-306
- Reactivity step, 35, 363-367,
 398-399, 402
- Reactor dynamics (see Dynamic
 equations)

- Reactor excursions (see Excursions)
- Reactor instrumentation, 303-305, 411-414
- Reactor pulsing (see Pulsed reactors)
- Reactor safety, 24, 46, 364, 528
 of computer-controlled systems, 528, 530
 in pulsed reactors, 479, 483
 sodium effect on, 373, 379-381
 TREAT experiments for, 371-385
- Reactor stability (see Stability)
- Reactors, pulsed (see Pulsed reactors)
 zoned, 46
- Reciprocal period, in FBR reactor, 391-394
 in TRIGA reactor, 408
- Refractory fuel elements, 519
- Regeneration of neutrons, 279-281, 288-293
- Regularity conditions, 140
- Relaxation oscillations, 100
- Relaxation time, 227, 241, 356-357
- Removal cross section, 284
- Rensselaer Polytechnic Institute, 205, 475
- Repetition rate in pulsed reactors, 420-431, 424-426, 435, 475, 479-480, 483
- Rethermalization, 243
- Revolving-reflector pulsed reactors, 421, 475-483
- Riccati equation, 167-168, 179
- Rocket, nuclear (see Nuclear rocket)
- Rossi alpha, 393
- Routh-Hurwitz criterion, 60, 62, 98
- ROVER, 46, 86
 (See also Nuclear rocket)
- S**
- Saddle points, 9, 12
- Safety, reactor (see Reactor safety)
- Safety block, 387-388, 390
- SAIM code, 334-336
- Sandia Corporation, 393, 397
- Saxton Nuclear Experimental Corp., 299
- Saxton reactor, 299-311
- Scattering cross section, 227
- Scattering kernel, 234, 242-243
- Series pulsing of TRIGA reactor, 420-436
- Shape function, 65, 67
- Shock effects, 371, 386-387, 392-397, 480
- Shutdown, automatic, 523
- Simulation of boiling superheat reactors, 545-547, 562-564
- Single-temperature problem, 40, 42
- Singlet density, 531-535, 538-541
- Singlet theory, 532-533, 539-540, 543
- Singular eigenfunctions, 226-227
- Singular points (see Equilibrium)
- Singularities, double, 11-13
 at infinity, 8-13
 movable, 39
- Singularity, unstable star type, 11

- Sjostrand area-ratio method, 245, 248, 252, 255–256, 262–264, 268–275
- Slowing down of neutrons (see Thermalization)
- Small-parameter expansions, 38, 42–43, 94, 107
- SNAP reactor, 437–456
 - energy release in, 441
 - fuel disintegration in, 439–445, 452–455
 - power pulses in, 441, 451–453
 - pressures in, 438–441, 444–445
- SNAP-10A/2 reactor, 445
- SNAPTRAN reactors, 440
- SNAPTRAN-1 reactor, 445–449, 455
- SNAPTRAN-3 reactor destructive experiment, 448–455
- Sodium, effect on reactor safety, 373, 379–381
- SORA pulsed reactor, 475–483
- Space-dependent dynamics, 1, 22–24, 278
 - adiabatic method in, 342–360
 - in boiling systems, 544–583
 - in compressible media, 484–517
 - eigenfunctions in, 23, 344, 354–355
 - eigenvalues in, 345–346, 349–350, 354–355
 - higher modes in, 22–23, 313, 348–349
 - modal distortion in, 24
 - in SNAP reactor excursions, 455
 - spectral density in, 312–322, 531–543
 - xenon effect in, 46, 300, 342–360
- Space-dependent energy balance, 443
- Space-dependent heat transfer in TRIGA, 467–469
- Space-dependent kinetics (see Space-dependent dynamics)
- Space-dependent noise, 312–322, 531–543
- Space-dependent transfer function (see Transfer function)
- Spatial correction factors for pulsed neutron reactivity measurements, 245–246, 265, 268–272
- Spatial coupling, 323–341
- Spatial dependence of reactivity measurements, 245–276
- Spatial effects in reactor dynamics (see Space-dependent dynamics)
- Spectra, asymptotic, 232, 242
 - fluctuation, 531–543
- Spectral coupling, 323–341
- Spectral density, 312–322, 531–543
 - detector-position effect on, 312, 318–320
 - detector-size effect on, 316
 - (See also Power spectrum)
- SPERT reactor, 366
- Spherical harmonics, 225–226, 237–242
- Spiral (focus), 9, 98–99
- Stability, absolute, 2
 - asymptotic, 1, 13
 - in coupled systems, 58–61, 71, 80–81, 85–87
 - criteria for, 2–8, 30–31, 58, 71

- of coupled reactors, 45–87, 559
 - with time delay, 52–55, 81–87
 - without time delay, 51–52, 79–81
- delayed-neutron effect on, 1–2, 32, 65
- of equilibrium states at infinity, 1, 8–13
- global (stability in the large), 28, 31–32, 71
- Lagrange, 7, 28, 32, 34, 37, 39
- in the large (global stability), 28, 31–32, 71
- linear, 4, 30, 35
- local, 28–31
 - at infinity, 8, 12–13
- of oscillating flux in compressible media, 511
- of power reactors, 300
- structural, 9, 13, 14
- Stability margin, 4
- Stability regions, in parameter space, 14, 53–55
 - in phase space, 82, 85
- Start-up, optimal, of heat-exchange process, 150
 - of nuclear rocket, 157–158, 164–181, 184, 190, 195, 198–199
- Start-up accident, 41
- State space, 165, 214–215
- State variables, in input–output formulation for optimal control, 20
 - in optimal reactor control, 138, 169, 174, 193, 200
 - in stability studies, 8
- State vector, in input–output formulation for optimal control, 14, 17
 - in optimal reactor control, 112, 138–139, 165–167
 - in suboptimal control, 208
- Steam-flow dynamics, 566–568, 571–576, 579–583
- Steam-formation dynamics, 562
- Steam generator, 300, 566
- Steam system for boiling-water reactors, 566–570
- Stochastic processes, 539
- Structural stability, 9, 13–14
- Subcritical assembly, 245, 312–313, 476
- Subcritical reactivity, 247, 254, 260, 268, 273–275
- Subcritical system, time-dependent neutron balance in, 500–501
- Subcriticality, effect on spectral density, 318–319
- Subcriticality measurements, 24, 245–276
- Suboptimal control, 138, 161–162, 168, 208–223
- Sulfur-threshold flux measurements, 391
- Superheat reactors, 46, 544–565
- SWAK code, 258
- Switching function, 149, 156–157
- Switching time, 170, 172
- Sylvester inequalities, 71
- Sylvester's theorem, 58, 80, 83
- Synthesis of optimal control systems, 110–136, 138, 164–181
- System matrix, 168

T

- Telegrapher's equation, 238, 484, 487, 491-493, 502-509
- Temperature, maximum, in excursions, 361, 363-369, 374-376, 378, 392-393
- Temperature coefficient (see Reactivity)
- Temperatures in TRIGA reactor (see TRIGA)
- TEMPEST code, 335-336
- TER-9 code, 361, 363-367
- Terminal-condition matrix, 166
- Terminal conditions, 17-18, 154, 166, 171, 192
- Terminal manifold, 156
- Terminal phase, 140
- Terminal state, 169, 208, 219-222
- Thermal cutoff energy, 243
- Thermal-shock effects, 371, 386-387, 392-397, 480
- Thermal stresses, 111, 154, 162, 170-171
- Thermalization, 243
- in accelerator booster, 473-474
 - energy-dependent diffusion theory for, 234-236, 240-241
 - energy transients in, 249
 - in free gas, 228, 230-231, 233, 236-237
 - polynomial expansions in, 228, 232-234
 - slowing-down data, 279-294
 - slowing-down density, 282-283, 286
 - slowing-down theory, 277, 280-281, 285-287, 297
 - slowing-down time, 286, 292-293, 474, 481-482
 - time-dependent, 225-226
- Thermalization delay time, 455
- Three-group calculation, 240-241
- Time delay, in coupled reactor stability, 52-55, 81-87
- in pulsed neutron analyzer, 261
- Time-dependent diffusion (see Diffusion)
- Time of flight, 226, 228, 473-475, 481
- Time-optimal control, 206-224
- (See also Optimal control)
- Time-optimal solution of xenon problems, 206
- Time-varying gain, 125, 165, 167, 172, 179-180, 182, 190
- Trajectories, effect of delay on, 86
- fuel expulsion, 371-372
 - near singularities at infinity, 11-16
 - optimal (see Optimal trajectories)
 - in power-temperature phase plane, 368
 - steam bubble, 569-570
 - temperature-composition, 439, 441
- Transfer function, 22, 124-125, 278, 301
- for boiling superheat reactors, 559-562
 - detector-position effect on, 23, 299, 302, 307
 - feedback, 2-3, 6-7, 28, 301-303, 308-309
 - local, 299-300, 306-308, 311

- power, 302-303
 - power-to-void, 562-563
 - space-dependent, 22-24
 - zero power, 3, 30, 301, 303, 306-307
 - (See also Transmittance)
 - Transient response (see Dynamic equations; Excursions)
 - Transition matrix, 129-132
 - Transition time, 208, 210, 212
 - Transmittance, 112, 124-125, 127-128
 - (See also Transfer function)
 - Transport equation, 227, 238, 282, 323-324, 333, 340, 532
 - (See also Neutron transport)
 - Transversality, 115, 134
 - TREAT reactor, 371-385, 438, 441
 - TRIGA reactor, 364, 398-436, 457-472, 475
 - cladding strength in, 414-415, 418-419
 - cladding temperatures in Mark II, 464-466
 - coolant flow in Mark II, 457, 459, 461-464, 467, 469-471
 - coolant temperatures in Mark II, 457, 459, 461-466, 469-471
 - dynamics of, 398-436, 457-472
 - energy release in, 398-400, 405-408
 - fuel-element specifications for, 399-400
 - fuel temperatures, in ATPR, 398, 400, 402, 405-406, 410-411, 413-419
 - in Mark II, 459-461, 466, 469-471
 - large reactivity insertions in, 398-419
 - oxidation of cladding in, 417
 - power distribution in, 459-465
 - pressures in, 399-400, 402, 410-415, 419
 - series pulsing of, 420-436
 - temperatures in series pulsing of, 420-424, 426, 429-435
 - varying heat capacity in, 402, 405, 419
 - Two-group theory, in pulsed neutron experiments, 277, 280-291, 293-297
 - reactivity effect in, 300
 - reactor dynamic equations of, 183
 - Two-phase flow, 566, 569, 571, 573, 579, 582
 - Two-point boundary value problems, 207, 212
 - Two-temperature problem, 34, 39, 63-64, 88-90, 94, 98-100, 108
- U**
- Ultra High Temperature Reactor Experiment (UHTREX), 518-531
 - University of Arizona, 172, 214
 - University of Florida Training Reactor (UFTR), 323, 335-339
 - University of Illinois, 459
 - Upscattering, 228
 - Uranium-233 as a pulsed reactor fuel, 478
 - Uranium-molybdenum alloy fast assembly, 386-387
 - Uranium-zirconium hydride, 398-419, 459

Uranium-zirconium hydride reactors,
excursions in, 437-456
(See also Burst; Pulse)

V

V-function (see Liapunov function)
Vallecitos Atomic Laboratory, 559
Van der Pol equation, 40, 89
Vaporization, of coolant, 371-372
of fuel, 455
Variance, 539
Variational method, 340
Varying gain, 125, 165, 167, 172,
179-180, 182, 190

W

Wave number, 356
Waves, 240, 242, 278
Weighted-neutron population, 327,
331-332, 339
Weighting, importance, 327, 329, 438
Weighting function, exponential,
182, 188
in optimal control, 113, 123, 135,
167-168, 175, 190
in space-dependent dynamics, 47,
65, 255, 467
(See also Adjoint)
Weighting matrix, control, 166, 172,
177

error, 133-134

Welton's criterion, 7, 32
Westinghouse Electric Corp., 299,
583
White Sands Missile Range (WSMR),
386-397
White Sands Missile Range Fast
Burst Reactor (WSMR FBR),
386-397
WKB approximation, 342, 344, 354
WSMR (White Sands Missile Range),
386-397
WSMR burst reactor (FBR), 386-397

X

Xenon, 46, 88-90, 99-106, 300
adiabatic treatment for, 342-360
optimal control of, 206
Xenon oscillations, 46, 101, 278,
343

Z

Zirconium hydride, 398-419, 459
Zirconium hydride reactors, excursions in, 437-456
(See also Burst; Pulse)
Zirconium-hydrogen phase diagram,
401
Zoned reactors, 46

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