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FORCED VIBRATION OF CONTINUOUS PANELS

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FORCED VIBRATION OF CONTINUOUS PANELS

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Robert L. Dineen⁽¹⁾ and Cyrus O. Varan⁽²⁾

SUMMARY

Analytical solution is formulated for the transient as well as the steady-state response of continuous plates subjected to periodic time dependent loads. The solution is applicable to systems with any given number of uniform panels.

INTRODUCTION

The application of dynamic loads to a structural system will normally produce a transient and a steady-state response. The transient motion results from the reaction of the system to the initial conditions and the external loads. The transient response in many instances result in stresses and deflections well above those corresponding to the steady-state motion.

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Considerable effort has been devoted to determining steady-state motion, due in large to the fact that the time variable may be eliminated in the governing equation. Such a simplification is not possible if transient effects are to be determined. Hence less work has been done on this aspect of the forced motion problem for many structures.

The purpose of this work is to derive a method for determining the complete solution, i.e., the transient and steady-state response.

A considerable amount of work has been done in investigating the response of single panel plates to time dependent loads. As in the case of the beams, the normal mode method has been generally favored and is included in several texts.^{1,2} Single panel plates have been examined for such conditions as moving loads,³ random loads⁴ and sonic booms.⁵

1) Meirovitch, L., Analytical Methods in Vibrations (Macmillan, Toronto: 1969).

2) Nowacki, W., Dynamics of Elastic Systems (John Wiley & Sons, New York: 1963).

3) Raske, T. F. and Schlack, A. L. Jr., "Dynamic Response of Plates Due to Moving Loads," Journal of the Acoustical Society of America, Vol. 42, No. 3, August 1966, pp. 625-635.

4) Lin, Y. K., "Nonstationary Response of Continuous Structures to Random Loading," Journal of the Acoustical Society of America, Vol. 35, No. 2, Feb. 1963, pp. 222-227.

5) Crocker, M. J., "Multimode Response of Panels to Normal and to Traveling Sonic Booms," Journal of the Acoustical Society of America, Vol. 42, No. 5, 1967, pp. 1070-1079.

The problem of ribbed plate under conditions of static loading⁶ and free vibration⁷ has also been examined (Reference⁷ contains a comprehensive bibliography on the natural frequency problem), and field solutions are now available for both of these cases.

Little attention, however, has been devoted even to determining the natural frequencies of continuous plates. Only five papers were found in the literature on this problem. As might be expected, then, only one paper containing a closed form solution has apparently been written dealing with the dynamic response problem.

Veletsos and Newmark⁸ considered a rectangular continuous plate over rigid supports which was hinged along the edges normal to the supports. These edges could also be subjected to in-plane forces. Each span was uniform in geometry and stiffness with constant in-plane forces. These values, however, may vary from span to span. Only small linearly elastic deflections are considered. The method of analysis was an extension of Holzer's method in which the plate is assumed to vibrate at some given frequency and with a specified number

6) Dean, D. and Omid'varan, C., "Analysis of Ribbed Plates," Journal of the Structural Division, ASCE, ST3, Paper No. 6474, March 1969, pp. 411-440.

7) Bacchus, C., Dynamic Flexural and Membrane Stiffness Matrices of Full Size Panels, Ph.D. Dissertation, University of New Mexico, 1971.

8) Veletsos, A. S. and Newmark, N. M., "Determination of the Natural Frequencies of Continuous Plates Hinged Along Two Opposite Edges," Journal of Applied Mechanics, Vol. 23, March 1956, pp. 97-102.

of half-sine waves. As a result of this steady-state condition, a finite amount of rotation is assumed to exist at the left-most support of the plate. The moment at the right-most support that would cause this rotation is then calculated. The natural frequency corresponding to the assumed mode shape was determined by repeating this procedure until the exciting moment vanished. The main computational work was determining the value of the exciting moment at the right support. The authors used a dynamic three-slope equation and the plate distortions to determine this value.

Dill and Pister⁹ investigated rectangular plate systems under conditions of steady-state vibration assuming that each panel of the system was uniform, the results being valid for the linearly elastic small deflection theory. For loads of constant shape which vary harmonically with time, the time variable in the governing partial differential is eliminated.

The solution was obtained by dividing the problem into three parts. The first part considered the case of arbitrary moments and displacements along two opposite edges with the other two edges simply supported with no normal load on the plate. The second part assumed the same condition with the edges interchanged. In the third part of the solution the panel was simply-supported on all edges and subjected

9) Dill, E. H. and Pister, K. S., "Vibration of Rectangular Plates and Plate Systems," Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 123-132.

to the normal load. Such an arrangement permitted the solution to be expressed in a series form which satisfied the boundary conditions at the simply supported edges. For the first and second parts, the boundary conditions on the other two edges were then expressed as sine series expansions. The governing partial differential equation was then reduced to a homogeneous ordinary differential equation which was solved using these boundary conditions. The solution of the third part was in the form of a double sine series, since all edges were simply-supported. The total deflection equation for the plate with arbitrary edge conditions was the sum of these three deflections. By operating on the total deflection equation, expressions for the slopes and shears at the edges were determined.

For the most general condition on a panel (all four edges connected to adjacent panels), a seven moment equation was derived by satisfying the force and compatibility conditions at adjoining edges. Several examples are worked for the cases of free and forced vibrations.

While a closed form solution resulted from this work, it is valid only for the special case of harmonic loads. The method is not applicable for other types of loads nor does it give the transient response for periodic loads.

A particular case, a two-panel continuous plate simply-supported at its outer edges, was examined by Ungar.¹⁰ He determined the natural

10) Ungar, E. E., "Free Oscillations of Edge-Connected Simply Supported Plate Systems," Journal of Engineering for Industry, Transactions of the ASME, Vol. 83, 1961, pp. 434-440.

frequencies and mode shapes using the governing partial differential equation for plate vibration. The time variable was eliminated by assuming harmonic motion. The solution was assumed to be an infinite sine series such that the boundary conditions on two opposite simply-supported edges were satisfied. Substitution into the governing equation reduced it to an ordinary differential solution. The solution for this equation is a function with eight arbitrary constants. By considering the plate as two panels with a common boundary and satisfying the force and compatibility condition at this boundary and the support conditions at the other edges, a set of eight homogeneous linear equations with eight unknown constants was obtained. Setting the determinant of this matrix equal to zero yielded eventually two frequency equations. The solution of these transcendental equations was done using a graphical plot.

Dickinson and Warburton¹¹ used the Bolotin edge effect method to determine the natural frequencies of rectangular plate systems. Using this procedure, approximate frequency equations were obtained for two two-panel systems. Good agreement was found between the results for these two systems when compared with series solutions.

More recently, Bacchus¹² determined the natural frequencies of a continuous uniform plate system, simply supported on its outer edges

11) Dickinson, S. M. and Warburton, G. B., "Natural Frequencies of Plate Systems Using Edge Effect Method," Journal of Mechanical Engineering Science, Vol. 9, No. 4, Oct. 1967, pp. 318-24.

12) Bacchus, Stiffness Matrices of Full Size Panels, pp. 80-101.

with rigid equally spaced interior supports. The force-deformation relationships for the plate edges at the interior supports were expressed first as the coefficients of a half-range Fourier sine series. Using the force and compatibility requirements at adjacent plate edges and difference operator notation, the force-deformation relations were reduced to a difference equation in terms of the edge slope. Five closed form solutions for this equation were then examined. The result was a single frequency equation for the system. The model equation in terms of the slopes at the support points was also determined. The corresponding mode shapes and frequency factors for a four span beam were calculated.

Finally, the previously cited reference by Pilkey for determining the dynamic response of a continuous beam is applicable to plate problems according to the author. No attempt was made, however, to demonstrate its use.

ANALYTIC SOLUTION

General Comments and Assumptions

The response of a continuous plate system to time-dependent normal loads will be obtained by first determining the moments in the plate at each support location. Once these values are known, each panel of the system is isolated, and its response is calculated. The response of the system, then, is obtained panel-by-panel.

The moments at the support locations are determined by considering a typical panel of the system subjected to an arbitrary normal load and unknown edge moments. Using the Laplace transform method, the governing equation of motion for the panel is reduced to a partial differential equation in spatial variables only. By satisfying the transformed force and compatibility conditions at adjacent panel edges, a difference equation for the moments at the supports is obtained. The solution of this equation and its subsequent inversion give the moments at each support location. The deflection of the plate may then be found by determining the deflection of each panel subjected to the appropriate edge moments and normal loads.

The assumptions made in deriving the governing equations are:

1. The classical theory of plates is used in which no account is taken of the effects of shear deformation and rotatory inertia.
2. All panels are of equal length and width, and are uniform in stiffness.

3. The plate deflections are small and linearly elastic.
4. The normal load on the panels is separable in the time and spatial variables.

Response of a Typical Panel

A continuous plate system of N panels, simply-supported around its outside edges, is shown in Figure 1. Consider a typical panel of this system, the n^{th} . This panel may be represented as a plate simply-supported along its edges and subjected to a normal load, $F(x,y,t)$, and edge moments, $M(y,t)$ and $M'(y,t)$, as illustrated in Figure 2. The governing equation for the panel is,

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = F(x,y,t) , \quad 1$$

where

$$D = \text{plate stiffness} = Eh^3/12(1 - \nu^2)$$

$$\rho = \text{mass density of the panel material}$$

$$h = \text{panel thickness}$$

$$\nabla^4 w = \text{biharmonic operator} = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$E = \text{modulus of elasticity}$$

$$\nu = \text{Poisson's ratio}$$

and $w(x,y,t) = \text{panel deflection} .$

The boundary conditions for the panel as shown in Figure 5.2 are:

$$w(x,0,t) = w(x,b,t) = 0 \quad 2a$$

Note: the outer edges and internal support edges are simply supported.

Figure 1 Continuous Plate System

Figure 2 Free Body Diagram of the n^{th} Panel

Figure 3 Free Body Diagram of the n^{th} Support

$$D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \bigg|_{y=0, b} = 0 \quad 2b$$

$$w(0, y, t) = 0$$

$$M(y, t) = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \bigg|_{x=0} = -D \frac{\partial^2 w}{\partial x^2} \bigg|_{x=0} \quad 3b$$

$$w(a, y, t) = 0 \quad 4a$$

$$M'(y, t) = D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \bigg|_{x=a} = D \frac{\partial^2 w}{\partial x^2} \bigg|_{x=a} \quad 4b$$

Assume that the plate is at rest when the normal load is applied.
The initial conditions are, therefore,

$$w(x, y, 0) = 0 \quad 5a$$

$$\frac{\partial w}{\partial t} \bigg|_{t=0} = \dot{w}(x, y, 0) = 0 \quad 5b$$

The solution of 1 may be obtained by first using the Laplace transform to eliminate the time variable. The equation becomes, then,

$$\nabla^4 \bar{w} + \frac{\rho h}{D} \left[p^2 \bar{w} - p w(x, y, 0) - \dot{w}(x, y, 0) \right] = \frac{\bar{F}(x, y, p)}{D}, \quad 6$$

where $\mathcal{L}_t w(x, y, t) = \bar{w}(x, y, p)$.

Letting

$$q^4 = -\frac{\rho h p^2}{D},$$

and using the initial conditions 5, Equation 6 takes the form

$$\nabla^4 \bar{w} - q^4 \bar{w} = \frac{\bar{F}}{D}. \quad 7$$

The solution of this equation is in two parts:

- (a) a homogeneous solution, $\bar{u}(x,y,p)$,
- (b) a particular solution, $\bar{v}(x,y,p)$.

Homogeneous Solution

The governing homogeneous equation is,

$$\nabla^4 \bar{u} - q^4 \bar{u} = 0. \quad 8$$

Assume a solution of the form,

$$\bar{u}(x,y,p) = \sum_{j=1}^{\infty} X_j(x,p) \sin \frac{j\pi}{b} y, \quad 9$$

noting that the above equation satisfies the boundary conditions at $y = 0, b$. Substituting 9 into 8 gives,

$$\sum_{j=1}^{\infty} \left[X_j^{IV} - 2\beta_j^2 X_j^{II} + (\beta_j^4 + q^4) X_j \right] \sin \beta_j y = 0, \quad 10$$

where the superscripts indicate differentiation with respect to x

and $\beta_j = \frac{j\pi}{b}$.

The governing equation for $X_j(x,p)$ is, then,

$$X_j^{IV} - 2\beta_j^2 X_j^{II} + (\beta_j^4 - q^4) X_j = 0. \quad 11$$

The solution is,

$$X_j = A_j \cosh \varphi_j x + B_j \sinh \varphi_j x + C_j \cos \psi_j x + D_j \sin \psi_j x, \quad 12$$

where $\varphi_j = \left(q^2 + \beta_j^2 \right)^{\frac{1}{2}}, \quad 13$

and $\psi_j = \left(q^2 - \beta_j^2 \right)^{\frac{1}{2}}. \quad 14$

From 9 and 12, the homogeneous solution is,

$$\bar{u}(x,y,p) = \sum_{j=1}^{\infty} (A_j \cosh \omega_j x + B_j \sinh \omega_j x + C_j \cos \psi_j x + D_j \sin \psi_j x) \sin \beta_j y . \quad 15$$

Particular Solution

Assume that the normal load function may be separated such that,

$$F(x,y,t) = H(x)Y(y)T(t) . \quad 16$$

Now expand the function of "y" in a Fourier sine series. Then,

$$F(x,y,t) = \sum_{j=1}^{\infty} H(x)Y_j T(t) \sin \beta_j y . \quad 17$$

The Laplace transform of the load function with respect to time is,

$$\bar{F}(x,y,p) = \sum_{j=1}^{\infty} H(x)Y_j \bar{T}(p) \sin \beta_j y . \quad 18$$

Now take the transformed particular solution to be of the form,

$$\bar{v}(x,y,p) = \sum_{j=1}^{\infty} \bar{v}_j(x,p) \sin \beta_j y . \quad 19$$

Substituting the last two equations into 7 gives the governing equation for $\bar{v}_j(x,p)$:

$$\bar{v}_j^{IV} - 2\beta_j^2 \bar{v}_j^{II} + (\beta_j^4 - q^4) \bar{v}_j = \frac{H(x)Y_j \bar{T}(p)}{D} . \quad 20$$

For a given function $H(x)Y_j \bar{T}(p)$ this equation may be solved for $\bar{v}_j(x,p)$ using techniques from the theory of ordinary differential equations. In the two specific cases examined in the following sections, the method of undetermined coefficients was used.

Uniform Load in the "x" Direction

Assume that the load is uniform in the x-direction so that

$$F(x,y,t) = F_0 Y(y) T(t) , \quad 21$$

or

$$F(x,y,t) = \sum_{j=1}^{\infty} F_0 Y_j T(t) \sin \beta_j y . \quad 22$$

Therefore,

$$\bar{F}(x,y,p) = \sum_{j=1}^{\infty} F_0 Y_j \bar{T}(p) \sin \beta_j y . \quad 23$$

Take the particular solution to be of the form,

$$\bar{v}(x,y,p) = \sum_{j=1}^{\infty} \bar{v}_j(p) \sin \beta_j y . \quad 24$$

Substituting the coefficient of the above series into 20 and letting $H(x) = F_0$ give,

$$\bar{v}_j(p) = \frac{F_0 Y_j \bar{T}(p)}{D(\beta_j^4 - q^4)} . \quad 25$$

The particular solution is, then,

$$\bar{v}(x,y,p) = \sum_{j=1}^{\infty} \frac{F_0 Y_j \bar{T}(p)}{D(\beta_j^4 - q^4)} \sin \beta_j y . \quad 26$$

Linearly Varying Load in the "x" Direction

For a load that varies linearly with the distance x,

$$F(x,y,t) = F_0 x Y(y) T(t) ,$$

or
$$F(x,y,t) = \sum_{j=1}^{\infty} F_{0xY_j} T(t) \sin \beta_j y , \quad 27$$

and
$$\bar{F}(x,y,t) = \sum_{j=1}^{\infty} F_{0xY_j} \bar{T}(p) \sin \beta_j y . \quad 28$$

Using the method of undetermined coefficients, the form of the solution is taken as,

$$\bar{v}(x,y,p) = \sum_{j=1}^{\infty} (C_{0j} + C_{1j}x) \sin \beta_j y , \quad 29$$

where
$$\bar{v}_j(x,p) = (C_{0j} + C_{1j}x) .$$

Substitute this expression into 20 to determine the values of the coefficients. The resulting equation is,

$$(\beta_j^4 - q^4)(C_{0j} + C_{1j}x) = \frac{F_{0xY_j} \bar{T}(p)}{D} . \quad 30$$

Equating coefficients of like terms gives,

$$C_{0j} = 0 ,$$

and

$$C_{1j} = \frac{F_{0Y_j} \bar{T}(p)}{D(\beta_j^4 - q^4)} . \quad 31$$

Therefore,

$$\bar{v}(x,y,p) = \sum_{j=1}^{\infty} \frac{F_{0xY_j} \bar{T}(p)}{D(\beta_j^4 - q^4)} \sin \beta_j y . \quad 32$$

Total Solution

The total solution for a typical panel is the sum of the homogeneous and the particular solutions; i.e.,

$$\bar{w}(x,y,p) = \bar{u}(x,y,p) + \bar{v}(x,y,p) ,$$

or,

$$\begin{aligned} \bar{w}(x,y,p) = \sum_{j=1}^{\infty} \left[A_j \cosh \varphi_j x + B_j \sinh \varphi_j x + C_j \cos \psi_j x + \right. \\ \left. + D_j \sin \psi_j x + \bar{v}_j(x,p) \right] \sin \beta_j y . \end{aligned} \quad 33$$

The value of $\bar{v}_j(x,p)$ depends on the specific load condition as illustrated in the previous sections. The coefficients contained in the homogeneous part of the solution are determined using the remaining boundary conditions at $x = 0,a$.

Evaluation of the Coefficients in the Total Solution

To determine the values of the coefficients in Equation 33 , the transformed boundary conditions at $x = 0,a$ must be used. From Equations 3 and 4 , these are,

$$\bar{w}(0,y,p) = 0 , \quad 34a$$

$$\left. \frac{\partial^2 \bar{w}}{\partial x^2} \right|_{x=0} = - \frac{\bar{M}}{D} , \quad 34b$$

$$\bar{w}(a,y,p) = 0 , \quad 35a$$

$$\text{and} \quad \left. \frac{\partial^2 \bar{w}}{\partial x^2} \right|_{x=a} = \frac{\bar{M}'}{D} . \quad 35b$$

Since the solution is expressed in terms of a sine series expansion in the y-direction, expand the unknown moments at the panel edges $x = 0, a$ in the same manner in order to equate the Fourier coefficients of the series. Therefore, let

$$M(y, t) = \sum_{j=1}^{\infty} M_j(t) \sin \beta_j y, \quad 36a$$

and

$$M'(y, t) = \sum_{j=1}^{\infty} M'_j(t) \sin \beta_j y. \quad 36b$$

The corresponding transformed expressions are,

$$\bar{M}(y, p) = \sum_{j=1}^{\infty} \bar{M}_j(p) \sin \beta_j y, \quad 37$$

and

$$\bar{M}'(y, p) = \sum_{j=1}^{\infty} \bar{M}'_j(p) \sin \beta_j y. \quad 38$$

Using Equation 33, the boundary conditions give the equations shown in matrix form on the following page from which the expressions for A_j , B_j , C_j and D_j may be determined. The solution of this set of equations gives,

$$A_j = \frac{-\bar{D}v_j^{II}(0, p) - \bar{M}_j - \psi_j^2 \bar{D}v_j(0, p)}{2q^2 D}, \quad 39$$

$$B_j = \frac{\left[\bar{M}_j + \psi_j^2 \bar{D}v_j(0, p) + \bar{D}v_j^{II}(0, p) \right] \cosh \varphi_j a - \psi_j^2 \bar{D}v_j(a, p) + \bar{M}'_j - \bar{D}v_j^{II}(a, p)}{2q^2 D \sinh \varphi_j a},$$

40

$$\begin{bmatrix}
 1 & 0 & 1 & 0 \\
 \varphi_j^2 & 0 & -\psi_j^2 & 0 \\
 \cosh \varphi_j a & \sinh \varphi_j a & \cos \psi_j a & \sin \psi_j a \\
 \varphi_j^2 \cosh \varphi_j a & \varphi_j^2 \sinh \varphi_j a & -\psi_j^2 \cos \psi_j a & -\psi_j^2 \sin \psi_j a
 \end{bmatrix}
 \begin{bmatrix}
 A_j \\
 B_j \\
 C_j \\
 D_j
 \end{bmatrix}
 =
 \begin{bmatrix}
 -\bar{v}_j(0,p) \\
 -\frac{\bar{M}_j}{D} - \bar{v}_j^{II}(0,p) \\
 -\bar{v}_j(a,p) \\
 \frac{\bar{M}_j'}{D} - \bar{v}_j^{II}(a,p)
 \end{bmatrix}$$

$$C_j = \frac{Dv_j^{II}(0,p) + \bar{M}_j - D\phi_j^2 \bar{v}_j(0,p)}{2q^2 D}, \quad 41$$

$$D_j = \frac{-\bar{M}_j' + Dv_j^{II}(a,p) - D\phi_j^2 \bar{v}_j(a,p) + \left[D\phi_j^2 \bar{v}_j(0,p) - \bar{M}_j - Dv_j^{II}(0,p) \right] \cos \psi_j a}{2q^2 D \sin \psi_j a}.$$

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Difference Equation for Moments at the Supports

Continuity in the plate system is maintained by requiring that the force and compatibility conditions between the edges of adjacent panels be satisfied. From these conditions and the expression for deflection derived in the previous section, a difference equation for the moments at the support locations is obtained.

Force Condition at the n^{th} Support

Figure 3 is a diagram of the n^{th} support with the adjacent n^{th} and $n+1^{\text{st}}$ panels removed. Since the effect of rotatory inertia is neglected and no torsional resistance is offered by the support, the forces acting represent an equilibrium condition. The only forces that need be considered are the edge moments, since the shearing forces are balanced by reactions at the supports. Using the series expansion for the moments, the equilibrium condition reduces to,

$$M_{jn}'(t) = -M_{j,n+1}(t). \quad 43$$

The corresponding transformed expression is,

$$\bar{M}'_{jn}(p) = -\bar{M}_{jn+1}(p) . \quad 44$$

Using Boole's difference operator, E , 44 becomes,

$$\bar{M}_{jn}(p) = -E^{-1}\bar{M}'_{jn}(p) . \quad 45$$

Compatibility Condition at the n^{th} Support

The compatibility condition requires that the slope at the edges of adjacent panels must be equal; i.e.,

$$\left. \frac{\partial w_n(x,y,t)}{\partial x} \right|_{x=a} = \left. \frac{\partial w_{n+1}(x,y,t)}{\partial x} \right|_{x=0} . \quad 46$$

The transformed compatibility condition is, then,

$$\left. \frac{\partial \bar{w}_n(x,y,p)}{\partial x} \right|_{x=a} = \left. \frac{\partial \bar{w}_{n+1}(x,y,p)}{\partial x} \right|_{x=0} . \quad 47$$

Difference Equation for Moments

Using Equation 33 with a second subscript, "n", to denote the panel number, Equation 47 becomes,

$$\begin{aligned} & A_{jn}\varphi_j \sinh \varphi_j a + \varphi_j (\cosh \varphi_j a - E) B_{jn} - C_{jn}\psi_j \sin \psi_j a + \\ & + \psi_j (\cos \psi_j a - E) D_{jn} = \bar{v}'_{jn+1}(0,p) - \bar{v}'_{jn}(a,p) . \end{aligned} \quad 48$$

By substituting for the coefficients in 48 the relations given in 39 through 42, and making use of the force requirement 45, Equation 48 takes the form,

$$\left[k_{1j}\Delta\nabla_n + 2(k_{1j} + k_{2j}) \right] \bar{M}'_{jn} = \bar{G}_{jn} , \quad 49$$

where $\Delta \nabla_n = E - 2 - E^{-1}$ = the central difference operator.

The expressions for k_{1j} , k_{2j} and $\bar{G}_{jn}$ are given in Appendix C.

Finally, letting $\bar{M}'_{jn} = \bar{M}_{jn}$,

where $\bar{M}_{jn}$ now represents the j^{th} Fourier coefficient for the moment at the n^{th} support, the governing difference equation for the moment coefficients is,

$$\left[k_{1j} \Delta \nabla_n + 2(k_{1j} + k_{2j}) \right] \bar{M}_{jn} = \bar{G}_n . \quad 50$$

Solution of the Difference Equation

The solution of the difference equation derived in the preceding section depends on the boundary conditions at the first and last supports. One of the most practical cases is a continuous plate system simply-supported at its ends. For this support condition, the solution may be expressed as a finite sine series.

Continuous Plate System Simply-Supported at $n = 0, N$

Assume that the time and the discrete variables in the expressions for end moments are separable and may be written as the finite series,¹

$$M_{jn}(t) = \sum_{r=1}^{N-1} \Phi_{jr}(t) \sin \frac{r\pi}{N} n , \quad 51$$

where n is the discrete variable and N is the total number of panels.

Assume also that the external load term may be represented in a similar manner as,

¹ Wah, Thein and Calcote, Lee R., Structural Analysis By Finite Difference Calculus (New York: Van Nostrand Reinhold, 1970), p.22.

$$G_{jn}(t) = \sum_{r=1}^{N-1} \Psi_{jr}(t) \sin \frac{r\pi}{N} n . \quad 52$$

For the general case where a moment and external load exist for every value of the discrete variable, the coefficients of Equations 51 and 52 are given respectively as,

$$\Phi_{jr}(t) = \frac{2}{N} \sum_{s=1}^{N-1} M_{js}(t) \sin \frac{s\pi}{N} r , \quad 53$$

and

$$\Psi_{jr}(t) = \frac{2}{N} \sum_{s=1}^{N-1} G_{js}(t) \sin \frac{s\pi}{N} r . \quad 54$$

Transforming 51 and 52 with respect to time gives,

$$\bar{M}_{jn} = \sum_{r=1}^{N-1} \bar{\Phi}_{jr}(p) \sin \frac{r\pi}{N} n , \quad 55$$

and

$$\bar{G}_{jn} = \sum_{r=1}^{N-1} \bar{\Psi}_{jr}(p) \sin \frac{r\pi}{N} n . \quad 56$$

Substituting 55 and 56 into the difference equation 50 results in the expression,

$$- 4k_{1j} \sum_{r=1}^{N-1} \sin^2 \frac{r\pi}{2N} \bar{\Phi}_{jr}(p) \sin \frac{r\pi}{N} n + 2(k_{1j} + k_{2j}) \cdot \sum_{r=1}^{N-1} \bar{\Phi}_{jr}(p) \sin \frac{r\pi}{N} n = \sum_{r=1}^{N-1} \bar{\Psi}_{jr}(p) \sin \frac{r\pi}{N} n . \quad 57$$

From which,

$$\bar{\Phi}_{jr}(p) = \frac{\bar{\Psi}_{jr}(p)}{2(k_{1j} + k_{2j}) - 4k_{1j} \sin^2 \frac{r\pi}{2N}} \quad 58$$

Therefore,

$$\Phi_{jr}(t) = \mathcal{L}^{-1} \bar{\Phi}_{jr}(p) = \mathcal{L}^{-1} \frac{\bar{\Psi}_{jr}(p)}{2(k_{1j} + k_{2j}) - 4k_{1j} \sin^2 \frac{r\pi}{2N}}, \quad 59$$

and

$$M_{jn} = \sum_{r=1}^{N-1} \mathcal{L}^{-1} \frac{\bar{\Psi}_{jr}(p)}{2(k_{1j} + k_{2j}) - 4k_{1j} \sin^2 \frac{r\pi}{2N}}. \quad 60$$

Application of the Discrete Method

A two-panel continuous plate system will be examined to illustrate the application of the discrete method derived in the preceding sections. The external normal loading in this example is taken as uniform and equal in both panels and is assumed to vary sinusoidally with time.

The solution procedure is basically the same as that followed for the elastic beams. In the case of the plate, however, the results are the Fourier series coefficients for the support moments. The actual values of the moments are obtained by expanding the series. It is also necessary to expand the loading in the y-direction in terms of a Fourier series. Hence some additional steps are involved in the plate solution as compared to the beam solution.

Solution Procedure

The general procedure for determining the deflection response of a continuous plate is, first, to solve for the transformed moments at the support locations. The transformed response of each panel is then found using the appropriate transformed support moments and

normal loads. The actual deflection and support moment relations are obtained by inverting these transformed expressions.

The following procedure may be used to determine the internal plate moments at the support locations:

1. Express the loading in each span as a function of x and a Fourier sine series expansion in the y -direction. The load function, then, should have the form of Equation 17. Use a Laplace transform on this expression with respect to time to obtain the form of Equation 18.
2. If the load is uniform or linearly varying in the x -direction, the particular solutions are given by Equations 26 and 32, respectively. For other forms of $H(x)$, Equation 20 will have to be solved to obtain the particular solution.
3. Determine the transformed load coefficient, $\bar{G}_{jn}$, for each panel using the relationships in Appendix C.
4. Take the Laplace transform of Equation 54 with respect to time and expand the result to obtain the values of the coefficients $\bar{\Psi}_{jr}(p)$ in terms of the transformed load coefficients, $\bar{G}_{js}(p)$. Note that "N" in this expression is the number of panels in the plate system.
5. Determine the expressions for the coefficients $\bar{\Phi}_{jr}(p)$ from Equation 58 along with the results from Step 4.
6. Expand Equation 55 to obtain the transformed moment coefficients in terms of $\bar{\Phi}_{jr}(p)$.

7. Invert these results to obtain the moment coefficients,
 $M_{jn}(t)$.
8. Substitute the results from Step 7 into Equation 36a to obtain the final expression for the moments at the support locations.

The moment in the plate at a support location may be considered as the edge moment acting on simply-supported panels adjacent to the support. Using the appropriate support moments and loading for a particular panel, Equation 33 may be used to determine the transformed deflection coefficients for the panel. Inversion gives the coefficients in real time. The total deflection, then, is given in the form of a Fourier sine series.

Two-Panel Continuous Plate

For the two-panel continuous plate shown in Figure 4, the moment at the interior support location and the resulting deflection of a panel will be determined using the discrete method. The load is uniform and equal in both panels and varies sinusoidally with time. The plate is assumed to be simply supported on its outer edges and to be initially at rest.

Support Moment

The external force acting on the panels is,

$$F(x,y,t) = F_0 \sin \omega t .$$

This expression has the form of Equation 16 where,

$$H(x) = F_0$$

$$Y(y) = 1$$

and

$$T(t) = \sin \omega t .$$

The expansion of the second function in a Fourier sine series gives the coefficient,

$$Y_j = \frac{2}{j\pi} [1 - (-1)^j] . \quad 61$$

The transformation of the time function is,

$$\mathcal{L}T(t) = \bar{T}(p) = \frac{\omega}{p^2 + \omega^2} .$$

For a uniform spatial loading, the transformed particular solution is given by Equation 26 as

$$\bar{v}(x, y, p) = \sum_{j=1}^{\infty} \frac{2F_0 \omega [1 - (-1)^j]}{j\pi D (\beta_j^4 - q^4) (p^2 + \omega^2)} \sin \beta_j y \quad 62$$

where $\beta_j = \frac{j\pi}{b} .$

Since there is only one interior support, only one load function needs to be determined. Using the results from Appendix C and noting that $\bar{v}(x, y, p)$ in this case is independent of x , the transformed load function is,

$$\bar{G}_{j1} = \frac{4\psi_j \varphi_j F_0 \omega [1 - (-1)^j]}{j\pi (\beta_j^4 - q^4) (p^2 + \omega^2)} (k_{3j} - k_{4j}) . \quad 63$$

Transforming Equation 54 and expanding results in the relationship,

$$\bar{\Psi}_{j1}(p) = \bar{G}_{j1}(p) .$$

From Equation 58 and with $N = 2$,

$$\bar{\Phi}_{j1}(p) = \frac{\bar{\Psi}_{j1}(p)}{2k_{2j}} . \quad 64$$

Finally 55 gives the expression for the transformed moment coefficient,

$$\bar{M}_{j1}(p) = \frac{2\psi_j \varphi_j F_0 \omega [1 - (-1)^j] (k_{3j} - k_{4j})}{j\pi (\beta_j^4 - \omega^4) (p^2 + \omega^2) k_{2j}} . \quad 65$$

The inversion of the last expression is done using the inversion integral.² The form of the integral for this case is,

$$M_{j1}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} 2\psi_j \varphi_j F_0 \omega [1 - (-1)^j] (k_{3j} - k_{4j})}{j\pi (\beta_j^4 - \mu^4) (\lambda^2 + \omega^2) k_{2j}} d\lambda . \quad 66$$

The poles for determining the residues are the roots of the expressions,

$$\beta_j^4 - \mu^4 = 0$$

$$\lambda^2 + \omega^2 = 0$$

and

$$k_{2j} = 0 .$$

There are no poles corresponding to the first expression since the numerator becomes zero at $\mu^2 = \pm \beta_j^2$. The second equation has roots

²Thomson, W. T., Laplace Transformation (New Jersey: Prentice-Hall, 1960), p. 144.

at $\lambda = \pm i\omega$. Finally, let the roots of the third relationship, the transcendental equation,

$$\psi_j \tanh \varphi_j a = \varphi_j \tan \psi_j a \quad 67$$

be designated as $\pm c_m$ where

$$u_m^2 = \frac{c_m}{b^2}.$$

For the first set of poles the form of the residue is

$$R = \frac{e^{\lambda t} 2F_0 \omega \psi_j \varphi_j [1 - (-1)^j] (k_{3j} - k_{4j})}{j\pi (\beta_j^4 - u^4) 2\lambda k_{2j}}. \quad 68$$

The form of the residue for the second set of poles is,

$$R_m = \frac{-4 e^{\lambda t} F_0 \omega \psi_j \varphi_j [1 - (-1)^j] (k_{3j} - k_{4j})}{j\pi (\lambda^2 + \omega^2) i S U_j}$$

where

$$U_j = \varphi_j \cos \psi_j a \sinh \varphi_j a - \psi_j \sin \psi_j a \cosh \varphi_j a +$$

$$- 2a (\beta_j^2 \cos \psi_j a \cosh \varphi_j a + \psi_j \varphi_j \sin \psi_j a \sinh \varphi_j a).$$

69

Since the second set of poles corresponds to the roots of a transcendental equation, the sum of the residues for this case will be an infinite series.

The final form of the moment coefficient is the sum of all the residues. Therefore,

$$M_{j1}(t) = \frac{2F_0 \psi_{1j} \varphi_{1j} [1 - (-1)^j] P_{1j}}{j\pi (\beta_j^4 - \omega^2 S^2) Q_{1j}} \sin \omega t + \\ + \sum_{m=1}^{\infty} \frac{8F_0 \omega [1 - (-1)^j] P_{2j}}{j\pi S (\omega^2 - a_m^2/b^4 S^2) Q_{2j}} \sin \frac{c_m}{b^2 S} t . \quad (70)$$

The values of ψ_{1j} , φ_{1j} , P_{1j} , Q_{1j} , P_{2j} and Q_{2j} are given in Appendix D. Finally, the moment at the support location is,

$$M_1(t) = \sum_{j=1}^{\infty} M_{j1}(t) \sin \beta_j y . \quad (71)$$

Panel Deflection

The transformed expression for the panel deflection is given by Equation 33, which may be written as,

$$\bar{w}(x, y, p) = \sum_{j=1}^{\infty} \bar{w}_j(x, p) \sin \beta_j y$$

where $\bar{w}_j(x, p) = A_j \cosh \varphi_j x + B_j \sinh \varphi_j x + C_j \cos \psi_j x + D_j \sin \psi_j x + \bar{v}_j(x, p) . \quad (72)$

Since the geometry and loading of the plate are symmetric about the interior support, the panel deflection will also be symmetric about this line. Consider, then, the second panel. Since the right edge is simply-supported, $\bar{M}_j' = 0$. Using the relationships given by Equations 39 through 42 for the coefficients in 72 and using the particular solution,

$$\bar{v}_j(x,p) = \frac{2F_0\omega[1 - (-1)^j]}{j\pi D(\beta_j^4 - q^4)(p^2 + \omega^2)}, \quad 73$$

the equation for the deflection becomes,

$$\bar{w}_j(x,p) = \frac{F_0\omega[1 - (-1)^j] [\psi_j \varphi_j (k_{3j} - k_{4j}) \Gamma_j(x,p) + k_{2j} Z_j(x,p)]}{q^2 D j \pi (\beta_j^4 - q^4) (p^2 + \omega^2) k_{2j} \sin \psi_j a \sinh \varphi_j a} \quad 74$$

where $\Gamma_j(x,p)$ and $Z_j(x,p)$ are given in Appendix D.

The general form of the inversion integral is,

$$w_j(x,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} f(x,\lambda) d\lambda}{\mu^2 D j \pi (\beta_j^4 - \mu^4) (\lambda^2 + \omega^2) k_{2j} \sin \psi_j a \sinh \varphi_j a} \\ = \Sigma R \quad 75$$

where $f(x,\lambda)$ is the numerator of Equation 74. The poles for the residues correspond to the roots of the expressions,

$$\lambda^2 + \omega^2 = 0$$

$$k_{2j} = 0$$

and $\sin \psi_j a \sinh \varphi_j a = 0$.

The poles are

$$\lambda = \pm i\omega \text{ with } \mu^2 = \mp \omega^2,$$

$$\lambda = \pm i \frac{c_m}{b^2 S} \text{ with } \mu^2 = \mp \frac{c_m^2}{b^2 S^2}$$

and $\lambda = \mp i \frac{\xi_n^2}{S} \text{ with } \mu^2 = \pm \xi_n^2$.

The values of c_m are the roots of the transcendental equation of the preceding section. The values of ξ_n may be determined from the expression,

$$\xi_n^2 = \left(\frac{n^2 \pi^2}{a^2} + \beta_j^2 \right)^{\frac{1}{2}}, \quad n = 1, 2, \dots$$

There are no poles corresponding to either

$$q^2 = 0$$

or

$$\beta_j^4 - q^4 = 0.$$

The forms of the residues for these poles are, respectively,

$$R_1 = \frac{e^{\lambda t} f(x, \lambda)}{\mu^2 D_j \pi (\beta_j^4 - \mu^4) 2\lambda k_{2j} \sin \psi_j a \sinh \varphi_j a} \quad 76$$

$$\text{and} \quad R_m = \frac{e^{\lambda t} f(x, \lambda)}{\mu^2 D_j \pi (\beta_j^4 - \mu^4) (\lambda^2 + \omega^2) z \sin \psi_j a \sinh \varphi_j a} \quad 77$$

$$\text{where} \quad z = \frac{\partial k_{2j}}{\partial \lambda} = \frac{1S}{2\psi_j \varphi_j} \left(\varphi_j \cos \psi_j a \sinh \varphi_j a - \psi_j \sin \psi_j a \cosh \varphi_j a + \right. \\ \left. - 2\beta_j^2 a \cos \psi_j a \cosh \varphi_j a - 2\psi_j \varphi_j a \sin \psi_j a \sinh \varphi_j a \right)$$

$$\text{and} \quad R_n = \frac{2\psi_j \varphi_j e^{\lambda t} f(x, \lambda)}{\mu^2 D_j \pi (\beta_j^4 - \mu^4) (\lambda^2 + \omega^2) k_{2j} i a S g_j}, \quad 78$$

$$\text{where} \quad g_j = \psi_j \sin \psi_j a \cosh \varphi_j a + \varphi_j \cos \psi_j a \sinh \varphi_j a.$$

The final form of the Fourier series coefficient for the deflection is, then;

$$\begin{aligned}
 w_j(x,t) = & \frac{F_0 [1 - (-1)^j]}{\omega S D_j \pi (\beta_j^4 - \omega^2 S^2) \bar{k}_{2j} \sin \bar{\psi}_j a \sinh \bar{\varphi}_j a} \cdot \\
 & \cdot \left[\bar{\psi}_j \bar{\varphi}_j (\bar{k}_{3j} - \bar{k}_{4j}) \bar{\Gamma}_j(x,p) + \bar{k}_{2j} \bar{Z}_j(x,p) \right] \sin \omega t + \\
 & - \sum_{m=1}^{\infty} \frac{4b^2 \psi_j' \varphi_j' F_0 \omega [1 - (-1)^j]}{c_m S P' D_j \pi (\beta_j^4 - c_m^2/b^4) (\omega^2 - c_m^2/b^4 S^2) \sin \psi_j' a \sinh \varphi_j' a} \cdot \\
 & \cdot \left[\psi_j' \varphi_j' (k_{3j}' - k_{4j}') \Gamma_j'(x,y) + k_{2j}' Z_j'(x,p) \right] \sin \frac{c_m}{b^2 S} t + \\
 & - \sum_{n=1}^{\infty} \frac{4\psi_j'' \varphi_j'' F_0 \omega [1 - (-1)^j]}{aS \xi_n^2 D_j \pi (\beta_j^4 - \xi_n^4) (\omega^2 - \xi_n^4/S^2) k_{2j}'' P''} \cdot \\
 & \cdot \left[\psi_j'' \varphi_j'' (k_{3j}'' - k_{4j}'') \Gamma_j''(x,p) + k_{2j}'' Z_j''(x,p) \right] \sin \frac{\xi_n^2}{S} t .
 \end{aligned} \tag{79}$$

The expressions for the bar, prime and double-prime quantities are listed in Appendix D.

$$F(x,y,t) = F_0 \sin \omega t$$

Figure 4 Two-Panel Continuous Plate with Uniform Load

Numerical Results

Consider a two-panel continuous plate whose panels are square in plan with the following properties and dimensions:

$$E = 2.9 \times 10^7 \text{ psi}$$

$$\nu = 0.3$$

$$\rho = 7.346 \times 10^4 \text{ lbs.-sec.}^2/\text{in.}^4$$

$$a = b = 100 \text{ in.}$$

$$h = 2.0 \text{ in.}$$

Take the load to be,

$$F_0 = 50 \text{ psi}$$

with a forcing frequency

$$\omega = 150 \text{ radians/sec.}$$

The transcendental equation, 67, was solved numerically for this case to obtain its roots. These values are listed in Table 1.

Using these parameters, the equations for the moment at the interior support and the deflection of the second span were programmed to obtain numerical results. The data shown in the following plots are based on the first three terms of the Fourier series expansion for the load.

The moment in the plate at the interior support location is plotted in Figure 5 for the first three 4 millisecond time intervals after the load is applied. Since the loading and the geometry of the plate are symmetrical about the interior support, each panel is equivalent to a plate simply-supported on three edges and clamped on

Table 1

Roots of the Transcendental Equation, 67,

For a Square Panel

Indices		Roots
j	m	$c_{jm} = u^2 b^2$
1	1	23.64
	2	58.64
	3	113.22
	4	187.43
	5	281.32
2	1	51.67
	2	86.13
	3	140.84
	4	215.29
	5	309.40
3	1	100.26
	2	133.79
	3	188.11
	4	262.51
	5	356.72
4	1	168.95
	2	201.72
	3	255.47
	4	329.30
	5	423.68
5	1	257.54
	2	289.76
	3	342.94
	4	416.64
	5	510.51

Figure 5 Variation of Moment Across the Two-Panel Plate at the Interior Support

the fourth. The maximum moment for a uniform static load on such a plate occurs at the center of the fixed edge. Figure 6 shows the variation in the plate moment at this point for the first .032 sec. of the loading time. Both the resultant and steady-state moments are included as well as the moment corresponding to the uniform static load of 50 psi.

The deflections of the second panel of the plate along the line $y = 50$ in. are shown in Figure 7. The deflections are plotted for three time intervals after the loading is applied.

Discussion of the Results

The result of the previous derivation represents a closed form solution for the continuous plate problem. The method is exact inasmuch as the original theory is exact. No approximations were made in the derivation or in the inversion of the transformed results.

As in the beam problems previously examined, the dynamic character of the loads causes a substantial increase in the moment at the interior support when compared to the static value. There is also a significant difference between the steady-state and the resultant moments, again indicating that the transient effect can be important.

Figure 6 Resultant and Steady-State Moment in the Two-Panel Plate Along the Interior Support at $y = 50$ in.

DISCUSSIONS AND CONCLUSIONS

Discussions

The response of two classes of continuous structural systems to time-dependent loads was obtained using the discrete mechanics method. The structures in this instance were limited to uniform beams and plates continuous over equally spaced supports. The response of the system was obtained by first determining that of a typical element. Then, by satisfying the force and compatibility conditions at the junction of adjacent elements, a difference equation for the moments at the support locations was obtained and solved in closed form. The response of each element was then determined using the appropriate support moments and the applied normal loads.

Two solutions for the element response for a continuous beam system were examined. In the first solution, the response of a span was obtained using a modified separation of variables technique. The result was an integral equation which was subsequently evaluated using a numerical method.

In the second solution, the Laplace transform was applied to the governing beam equation to eliminate the time variable. This resulted in an algebraic expression for the transformed moments. The final expressions in real time were obtained by inversion.

The second method was also used to determine the response of a continuous plate system. The results in this case yielded the Fourier series coefficients for the support moments.

Numerical Solution

The first method of solution, the integral equation form, is easy to formulate and evaluate. The calculations required to formulate the integral equation are simplified considerably since the shape functions and the necessary frequencies correspond to the natural modes and frequencies of a simply-supported beam. Evaluation of the moments at discrete time intervals is done using a computer program.

There are some limitations in this method with the most serious being the accuracy of the results and the total length of time for which values may be determined. For the two-span beam problem, the best value for the maximum moment at the interior support was about 3% over the exact value. There was also a slight phase shift between the exact and the numerical solutions. Neither of these discrepancies could be reduced beyond a certain limit by either decreasing the time interval or using more terms of the infinite series. Considerable error was noted between the exact and the numerical values near the end of the time corresponding to the half period of the forcing function. The method, then, appears to be applicable for only short time periods.

The deflection, obtained by numerically integrating the moments, had errors as high as 12%. This reflects the relatively large percentage differences between the exact and the approximate values of the moments in the initial part of the response.

Analytic Solution

The second method, using the Laplace transform technique, resulted in a closed form solution for the continuous beam and plate problems. The resultant solution could be separated readily into its steady-state and transient components, so that long-time effects on a damped system could be determined.

The method as derived assumed that the systems were initially at rest and that the forcing functions were sinusoidal. These assumptions were only made for convenience and are not limiting factors of the method. Any arbitrary initial conditions of displacement and velocity could be used. The only limitations on the forcing function are that it must be expressed in functional form and separable in the time and spatial variables, and it must be Laplace transformable. It should also be noted that the normal loads on the elements may have different time and spatial functions. For example, the two-span beam could have had a uniform load in one span that varied sinusoidally with time at a given frequency, while the load in the second span could have varied linearly with distance and time.

Impulsive load effects may be treated by considering two time eras: the time during which the forcing function acts and the time following, when the forcing function vanishes. The response during the first time period may be solved using the present method. This corresponds to the forced motion response of the system. The second time period is one of free vibration whose initial conditions correspond to the terminal conditions of the first time period.

There is substantially more effort involved in obtaining the final expressions for the moments and deflections using the analytical method when compared to the numerical method. This is due primarily to the inversion which requires the determination of the poles and their corresponding residues.

Conclusions

The numerical solution is the easier of the two methods to use. Once the program for the solution of the integral equation is written, only two quantities must be determined for any particular problem. Due to the simplicity of the shape function, these calculations are straightforward. The method should be quite satisfactory for beam problems where information in the initial time stage is required.

The analytic method provides a closed form solution that gives mathematically exact results for the moments at the supports and the deflection of any element for both beam and plate systems. The solution may also be divided, in the case of a periodic forcing function, into its transient and steady-state components. The steady-state solution, then, represents the long term effect for a damped system.

Appendix C:

Coefficients of the Difference Equation 50

$$k_{1j} = \varphi_j \sin \psi_j a - \psi_j \sinh \varphi_j a$$

$$k_{2j} = \psi_j \cos \psi_j a \sinh \varphi_j a - \varphi_j \sin \psi_j a \cosh \varphi_j a$$

$$k_{3j} = \psi_j \sin \psi_j a + \varphi_j \sinh \varphi_j a$$

$$k_{4j} = \psi_j \cos \psi_j a \sinh \varphi_j a + \psi_j \sin \psi_j a \cosh \varphi_j a$$

$$\begin{aligned} \bar{G}_{jn} = & -2q^2 D \sin \psi_j a \sinh \varphi_j a \left[\bar{v}'_{jn+1}(0, p) - \bar{v}'_{jn}(a, p) \right] \\ & + Dk_y \left[\bar{v}''_{jn}(0, p) + \bar{v}''_{jn+1}(a, p) \right] + D\psi_j \varphi_j k_{3j} \cdot \\ & \cdot \left[\bar{v}_{jn}(0, p) + \bar{v}_{jn+1}(a, p) \right] + Dk_{2j} \left[\bar{v}''_{jn+1}(0, p) + \bar{v}''_{jn}(a, p) \right] + \\ & - D\psi_j \varphi_j k_{4j} \left[\bar{v}_{jn+1}(0, p) + \bar{v}_{jn}(a, p) \right] \end{aligned}$$

where

$$\bar{v}'_{jn}(x, p) = \frac{d}{dx} \bar{v}_{jn+1}(x, p)$$

and

$$\bar{v}''_{jn}(x, p) = \frac{d^2}{dx^2} \bar{v}_{jn}(x, p) .$$

Appendix D:

Coefficients for Equation 70

$$\psi_{1j} = \left(uS - \beta_j^2 \right)^{\frac{1}{2}}$$

$$\varphi_{1j} = \left(uS + \beta_j^2 \right)^{\frac{1}{2}}$$

$$P_{1j} = \varphi_{1j} \sinh \varphi_{1j} a (1 - \cos \psi_j a) + \psi_{1j} \sin \psi_j a (1 - \cosh \varphi_{1j} a)$$

$$Q_{1j} = \psi_{1j} \sinh \varphi_{1j} a \cos \psi_j a - \varphi_{1j} \cosh \varphi_{1j} a \sin \psi_j a$$

$$\psi_{2j} = \left(\frac{c_m}{b^2} - \beta_j^2 \right)^{\frac{1}{2}}$$

$$\varphi_{2j} = \left(\frac{c_m}{b^2} + \beta_j^2 \right)^{\frac{1}{2}}$$

$$P_{2j} = \varphi_{2j} \sinh \varphi_{2j} a (1 - \cos \psi_{2j} a) + \psi_{2j} \sin \psi_{2j} a (1 - \cosh \varphi_{2j} a)$$

$$Q_{2j} = \varphi_{2j} \sinh \varphi_{2j} a \cos \psi_{2j} a - \psi_{2j} \cosh \varphi_{2j} a \sin \psi_{2j} a + \\ - 2a \left(\beta_j^2 \cos \psi_{2j} a \cosh \varphi_{2j} a + \varphi_{2j} \psi_{2j} \sinh \varphi_{2j} a \sin \psi_{2j} a \right)$$

Coefficients for Equation 74

$$\Gamma_j(x, p) = \sin \psi_j a \sinh \varphi_j a (\cos \psi_j x - \cosh \varphi_j x) + \\ + \sin \psi_j a \cosh \varphi_j a \sinh \varphi_j x - \cos \psi_j a \sinh \varphi_j a \sin \psi_j x$$

$$Z_j(x, p) = \sin \psi_j a \sinh \varphi_j a \left(2q^2 - \psi_j^2 \cosh \varphi_j x - \varphi_j^2 \cos \psi_j x \right) + \\ + \psi_j^2 (\cosh \varphi_j a - 1) \sin \psi_j a \sinh \varphi_j x + \varphi_j^2 (\cos \psi_j a - 1) \cdot \\ \cdot \sinh \varphi_j a \sin \psi_j x$$

Coefficients for Equation 79

$$\bar{\psi}_j = (\omega S - \beta_j^2)^{\frac{1}{2}}$$

$$\bar{\phi}_j = (\omega S + \beta_j^2)^{\frac{1}{2}}$$

$$\bar{\psi}_j' = \left(\frac{c_m}{b^2} + \beta_j^2 \right)^{\frac{1}{2}}$$

$$\bar{\phi}_j' = \left(\frac{c_m}{b^2} - \beta_j^2 \right)^{\frac{1}{2}}$$

$$\bar{\psi}_j'' = \frac{n\pi}{a}$$

$$\bar{\phi}_j'' = \left(\frac{n^2 \pi^2}{a^2} + 2\beta_j^2 \right)^{\frac{1}{2}}$$

The bar, prime and double-prime values for k_{2j} , k_{3j} and k_{4j} are obtained by substituting for ψ_j and ϕ_j in the expressions of Appendix C the appropriate values given above. To obtain the corresponding values of Γ_j and Z_j make the same substitutions into the expressions on the preceding page. Note that q^2 in Z_j has the following values:

$$\bar{q}^2 = \omega S$$

$$q'^2 = \frac{c_m}{b^2}$$

$$q''^2 = \frac{n^2 \pi^2}{a^2}$$