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DECEMBER 1972

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I. INTRODUCTION

In this note we sketch the derivation and evaluation of criteria for stability of interchange modes in axisymmetric equilibria of anisotropic, guiding-center plasmas. The present case thus involves two important simplifying restrictions:

- (i) Because of the assumed symmetry we can use an orthogonal curvilinear coordinate system based on the magnetic field lines, the so-called ψ, θ, χ coordinates.¹ The properties of this reference frame are summarized in Appendix A.
- (ii) Because of the restriction to interchange modes, we can obtain useful stability criteria without solving the full Euler-Lagrange minimization problem. Here we define interchanges to be perturbations which leave the shape of magnetic field lines unchanged; that is, the perturbed magnetic field is parallel to the equilibrium field. Since closely related criteria have been derived for scalar pressure plasmas (in closed-line devices),² we include a brief derivation of the scalar-pressure criterion in Appendix B and discuss its correspondence with the present guiding center criteria.

Application of the stability criteria requires detailed knowledge of the plasma equilibrium, which in general necessitates numerical solution of specific model equilibria. In Appendix C we review some aspects of the equilibrium problem, discussed in greater detail in ORNL-TM-4076, in order

to clarify the origins of a crude approximation to the equilibrium magnetic field which allows us to estimate some of the threshold quantities of interest. A more accurate numerical treatment has been carried out by Hedrick for the Canted Mirror Facility (in its axisymmetric limit), and some illustrative cases are included here.

The stability criteria derived in this note are based on the theorem that if $\delta W > 0$ (where $\delta W \equiv \mathcal{E}^{(2)}$ of ORNL-TM-3943), the plasma is stable. However, we avoid the difficult task of computing the complete expression for $\mathcal{E}^{(2)}$, which would require an integration over all space and hence a complete knowledge of the spatial dependence of the perturbation fields, by offering a judgment on stability only if the local value of $\mathcal{E}^{(2)}$ is positive throughout the bulk of the plasma. If there are localized regions in which $\mathcal{E}^{(2)}$ is negative we cannot draw meaningful conclusions without solving the full problem. In this sense, our present criteria are sufficient to guarantee stability against interchanges; but they may be unnecessarily restrictive.

In this note we do not deal with the question of boundary effects, thereby omitting any discussion of line-tying, since our primary interest is in toroidal systems in which line-tying in the usual sense could only occur in the neighborhood of limiting apertures. This is a serious omission from the standpoint of present-day simple-mirror experiments, in which there is strong evidence for a stabilizing effect which may result from the finite length of the experimental plasmas. Within the context of the present theory, interchange modes could be prevented entirely if the field lines passed through superconducting end plates. No effect would be anticipated if the field lines passed through dielectric end

plates. The simple-mirror experiments show little systematic difference between copper and glass end plates, suggesting that the observed "line-tying" is associated with electrostatic rather than electromagnetic boundary conditions.³

II. CRITERIA FOR STABILITY OF INTERCHANGE MODES

In what follows we build upon material derived in ORNL-TM-3943, which should be consulted on any matters of notation.⁴ From Eq. (6.1) of that Memo, the second variation of the total energy is

$$\begin{aligned} \mathcal{E}^{(2)} = \frac{1}{2} \int d^3x \left[\sigma \vec{Q}_\perp^2 + \tau Q_\parallel^2 + \mu_0 \sigma j_\parallel \hat{b} \cdot (\vec{\xi} \times \vec{Q}_\perp) \right. \\ \left. + q \vec{\xi} \cdot \vec{Dp}_\parallel - \left(2 \frac{Q_\parallel}{B} + S \right) \vec{\xi} \cdot \vec{Dp}_\perp \right] + W_K. \end{aligned} \quad (2.1)$$

We shall consider only cases in which $j_\parallel = 0$. The remaining terms in (2.1) will be evaluated in the ψ, θ, χ coordinates discussed in Appendix A. From the definition of the D-operator,

$$\vec{Dp}_\parallel = \hat{\psi} \cdot rB \left(\frac{\partial p_\parallel}{\partial \psi} \right)_B; \quad \vec{Dp}_\perp = \hat{\psi} rB \left(\frac{\partial p_\perp}{\partial \psi} \right)_B, \quad (2.2)$$

where $\left(\right)_B$ indicates that B is held constant. Thus,

$$q \vec{\xi} \cdot \vec{Dp}_\parallel = - \vec{\xi} \cdot \nabla \vec{\xi} \cdot \vec{Dp}_\parallel = - (rB \xi_\psi)^2 \frac{\kappa}{rB} \left(\frac{\partial p_\parallel}{\partial \psi} \right)_B \quad (2.3)$$

and

$$S = rB \xi_\psi \frac{\partial \ln B}{\partial \psi}; \quad (2.4)$$

so that $\mathcal{E}^{(2)}$ becomes

$$\begin{aligned} \mathcal{E}^{(2)} = & \frac{1}{2} \int d\psi d\theta dX J \left\{ \sigma \vec{Q}_\perp^2 + \tau Q_\parallel^2 \right. \\ & - (rB\xi_\psi)^2 \left[\frac{\kappa}{rB} \left(\frac{\partial p_\parallel}{\partial \psi} \right)_B + \frac{\partial \ln B}{\partial \psi} \left(\frac{\partial p_\perp}{\partial \psi} \right)_B \right] \\ & \left. - 2 \frac{Q_\parallel}{B} rB\xi_\psi \left(\frac{\partial p_\perp}{\partial \psi} \right)_B \right\} + W_K. \end{aligned} \quad (2.5)$$

A. "Simple Interchange", the Electrostatic Limit

If we consider a plasma with vanishingly small pressure, any magnetic perturbations must increase the energy; and we need only evaluate electrostatic modes. Then

$$\vec{\xi} = \vec{E} \times \vec{B}/B^2 = -\vec{\nabla}\Phi \times \vec{B}/B^2, \quad (2.6)$$

where Φ is constant along field lines. Evaluating the components of $\vec{\xi}$ and substituting into

$$\vec{Q} = \frac{\hat{\psi}}{rBj} \frac{\partial}{\partial X} (rB\xi_\psi) + \hat{\theta} \frac{r}{j} \frac{\partial}{\partial X} \left(\frac{\xi_\theta}{r} \right) - \hat{\chi} B \left[\frac{\partial}{\partial \psi} (rB\xi_\psi) + \frac{\partial}{\partial \theta} \left(\frac{\xi_\theta}{r} \right) \right], \quad (2.7)$$

one verifies that $\vec{Q} = 0$, in keeping with the electrostatic nature of the perturbation. In what follows we shall neglect the positive (hence stabilizing) kinetic term, W_K . For a detailed discussion of this term and estimates of its magnitude, see ORNL-TM-3943. Then, with $\vec{Q} = 0$, (2.5) becomes

$$\mathcal{E}^{(2)} \approx - \frac{1}{2} \int d\psi d\theta dX J (rB\xi_\psi)^2 \left(\frac{\kappa}{rB} \frac{\partial p_\parallel}{\partial \psi} + \frac{\partial \ln B}{\partial \psi} \frac{\partial p_\perp}{\partial \psi} \right) \quad (2.8)$$

Since $rB\xi_\psi$ is constant along field lines (if $\vec{Q}_\perp = 0$), and since $JdX = d\ell/B$, (2.9) can be written as

$$\mathcal{E}^{(2)} \geq -\frac{1}{2} \int d\psi d\theta (rB\xi_\psi)^2 \int \frac{d\ell}{B} \left(\frac{n}{rB} \frac{\partial p_\parallel}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_\perp}{\partial \psi} \right). \quad (2.9)$$

Clearly the plasma is stable to these modes if

$$\int \frac{d\ell}{B} \left(\frac{n}{rB} \frac{\partial p_\parallel}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_\perp}{\partial \psi} \right) \leq 0, \quad (2.10)$$

which is the usual Rosenbluth-Longmire criterion.⁵ Note that if (2.10) is violated in some narrow range of ψ , it is not possible to reach an unambiguous conclusion regarding stability without knowing $\xi_\psi(\psi)$, that is, the radial structure of the eigenmode.

That the criterion (2.10) relates to the sense of particle drift can be demonstrated by computing the average particle drift from the adiabatic invariant, $J_\parallel$:

$$J_\parallel = \sqrt{\frac{2}{m}} \oint d\ell \sqrt{\epsilon - \mu B} = \sqrt{\frac{2}{m}} \oint dX JB \sqrt{\epsilon - \mu B}. \quad (2.11)$$

The azimuthal drift speed (averaged over a particle bounce period) is⁶

$$\langle \dot{\theta} \rangle = - e^{-1} \left(\frac{\partial J_\parallel}{\partial \psi} \right) / \left(\frac{\partial J_\parallel}{\partial \epsilon} \right). \quad (2.12)$$

Carrying out the indicated differentiations and using the formula (from Appendix A) for $\partial \ell n J / \partial \psi$, we have

$$\frac{\partial J_\parallel}{\partial \epsilon} = \frac{1}{\sqrt{2m}} \oint \frac{d\ell}{\sqrt{\epsilon - \mu B}}, \quad (2.13)$$

and

$$\begin{aligned}
\frac{\partial J_{\parallel}}{\partial \psi} &= \sqrt{\frac{2}{m}} \oint d\chi \frac{\partial}{\partial \psi} \left(J_B \sqrt{\epsilon - \mu B} \right), \\
&= - \sqrt{\frac{2}{m}} \oint \frac{d\ell}{B} \frac{B}{\sqrt{\epsilon - \mu B}} \left[\frac{n}{rB} (\epsilon - \mu B) + \frac{\partial \ell n B}{\partial \psi} \frac{\mu B}{2} \right]
\end{aligned} \tag{2.14}$$

When (2.12) is averaged over the distribution of particles, we obtain

$$\langle \dot{\theta} \rangle \sim e^{-1} \oint \frac{d\ell}{B} \left(\frac{n}{rB} p_{\parallel} + \frac{\partial \ell n B}{\partial \psi} p_{\perp} \right), \tag{2.15}$$

where the pressure components are given by eq. (2.22) of ORNL-TM-3943.

Note that for separable pressure models, $p(B, \psi) = g(\psi) p(B)$, eq. (2.10) becomes proportional to $g'(\psi) \langle \dot{\theta} \rangle$.

B. Generalized Interchange: $\vec{Q}_{\perp} = 0$, $Q_{\parallel} \neq 0$

We define a general interchange to be a disturbance which does not alter the shape of a magnetic line. Since $\vec{Q}$ is the magnetic field perturbation, this definition of general interchange is equivalent to requiring $\vec{Q}_{\perp} = 0$. From eq. (2.7), $\vec{Q}_{\perp} = 0$ only if

$$\frac{\partial}{\partial \chi} (r B \xi_{\psi}) = \frac{\partial}{\partial \chi} \left(\frac{\xi_{\theta}}{r} \right) = 0, \tag{2.16}$$

so that

$$\frac{\partial}{\partial \chi} \left(\frac{Q_{\parallel}}{B} \right) = \frac{\partial}{\partial \chi} \left[\frac{\partial}{\partial \psi} (r B \xi_{\psi}) + \frac{\partial}{\partial \theta} \left(\frac{\xi_{\theta}}{r} \right) \right] = 0. \tag{2.17}$$

Thus, $r B \xi_{\psi}$, ξ_{θ}/r , and $Q_{\parallel}/B$ are all constant along the magnetic field lines and $\mathcal{E}^{(2)}$ can be written as

$$\begin{aligned}
\mathcal{E}^{(2)} &= \frac{1}{2} \int d\psi d\theta \left[\left(\frac{Q_{\parallel}}{B} \right)^2 \int \frac{d\ell}{B} \tau_B^2 - 2 \frac{Q_{\parallel}}{B} r B \xi_{\psi} \int \frac{d\ell}{B} \frac{\partial p_{\perp}}{\partial \psi} \right. \\
&\quad \left. - (r B \xi_{\psi})^2 \int \frac{d\ell}{B} \left(\frac{n}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} \right) \right] + W_K
\end{aligned} \tag{2.18}$$

If we denote $\int d\ell f/B \equiv \langle f \rangle$, we can complete the square in (2.18) to write

$$\begin{aligned} \mathcal{E}^{(2)} = & \frac{1}{2} \int d\psi d\theta \left\{ - (rB\xi_\psi)^2 \left\langle \frac{\kappa}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} \right\rangle \right. \\ & + \left[\frac{Q_{\parallel}}{B} \langle \tau B^2 \rangle^{1/2} - (rB\xi_\psi) \frac{\left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle}{\langle \tau B^2 \rangle^{1/2}} \right]^2 \\ & \left. - (rB\xi_\psi)^2 \frac{\left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle^2}{\langle \tau B^2 \rangle} \right\} + W_K \end{aligned} \quad (2.19)$$

The positive (stabilizing) term in square brackets vanishes if $Q_{\parallel}/B = (rB\xi_\psi) \left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle / \langle \tau B^2 \rangle$, so that

$$\begin{aligned} \mathcal{E}^{(2)} \geq & - \frac{1}{2} \int d\psi d\theta (rB\xi_\psi)^2 \left[\left\langle \frac{\kappa}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} \right\rangle \right. \\ & \left. + \frac{\left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle^2}{\langle \tau B^2 \rangle} \right]. \end{aligned} \quad (2.20)$$

These generalized interchange modes will therefore be stable if

$$\left\langle \frac{\kappa}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} \right\rangle + \frac{\left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle^2}{\langle \tau B^2 \rangle} \leq 0. \quad (2.21)$$

The magnetic field perturbations thus contribute a destabilizing term given by $\left\langle \frac{\partial p_{\perp}}{\partial \psi} \right\rangle^2 / \langle \tau B^2 \rangle$. This is closely related to the scalar pressure criterion,² as one can see by taking the limit of $p_{\parallel} \rightarrow p_{\perp} = p(\psi)$ in criterion (2.21):

$$\frac{\partial p}{\partial \psi} \left\langle \frac{\kappa}{rB} + \frac{\partial \ell n B}{\partial \psi} \right\rangle + \frac{\left(\frac{\partial p}{\partial \psi} \right)^2 \left(\int \frac{d\ell}{B} \right)^2}{\left\langle \left(\frac{1}{\mu_0} + \frac{1}{B} \frac{\partial p}{\partial B} \right) B^2 \right\rangle} \leq 0. \quad (2.22)$$

From Appendix A we have

$$\frac{\partial}{\partial \psi} \int \frac{d\ell}{B} = - \left\langle \frac{\kappa}{rB} + \frac{\partial \ln B}{\partial \psi} \right\rangle \equiv U'(\psi), \quad (\text{A.8})$$

where $U(\psi) \equiv \int d\ell/B$. Note also that $\partial p/\partial B = 0$ so that

$$\langle \tau B^2 \rangle = \left\langle \left(\frac{1}{\mu_0} + \frac{1}{B} \frac{\partial p}{\partial B} \right) B^2 \right\rangle = \frac{1}{\mu_0} \int B d\ell \equiv I. \quad (2.23)$$

With these substitutions, (2.22) becomes

$$- p' U' + \frac{(p')^2 U^2}{I} \leq 0, \quad (2.24)$$

or

$$p' (IU' - U^2 p') \geq 0 \quad (2.25)$$

From Appendix B, Kadomtsev's criterion is

$$\left(p' + \gamma p \frac{U'}{U} \right) (IU' - U^2 p') \geq 0, \quad (2.26)$$

identical but for the γp term, which we have neglected in passing from (2.19) to (2.20).

In Section III we shall investigate the implications of criterion (2.21) and the more complete version in which the kinetic term is estimated as discussed in ORNL-TM-3943.

III. APPLICATION OF THE STABILITY CRITERION

Application of the stability criterion, (2.21), requires detailed knowledge of the spatial variation of the plasma pressure and magnetic field strength. This may be obtained, and (2.21) can be evaluated directly for particular sets of parameters using numerical solution of the non-linear equilibrium equations for specific models of the plasma.

Fortunately, some implications of the stability criterion can be displayed more conveniently using the approximate results derived in Appendix C, where we obtained crude estimates of the local equilibrium properties for a set of pressure models chosen to satisfy "mirror" and "firehose" criteria. In this Section we first discuss the approximate results and then show some corresponding numerical cases to illustrate the rather good agreement with respect to critical plasma pressures. It will become clear in what follows that this agreement stems from the dominant roles played by the gradient in the intensity of the magnetic field, $\partial \ln B / \partial \psi$, and the pressure anisotropy, $p_{\perp} > p_{\parallel}$. We conclude this Section with some comments directed toward the general issue of whether or not a plasma can stabilize itself by its own diamagnetism.

We shall not attempt to carry out the integration along magnetic field lines indicated in (2.21). Instead, we simply estimate $\langle f \rangle \sim (f/B)_{z=0} L$, using a suitable characteristic length L . Note that two of the three terms in (2.21) are proportional to the pressure, which in the model of Appendix C vanishes at the point where $B = B_C$. We denote the corresponding scale length L_p . The integrand of the third term, $\langle \tau B^2 \rangle$, is greater than zero at every point in the trap, leading us to use the half-length of the trap, L_{Total} , for the corresponding scale length.

The first two terms in (2.21), relating to the sense of azimuthal particle drift, were estimated in (C.20). At very low plasma pressure, in a simple-mirror trap,

$$n \sim \partial \ln B_V / \partial \psi < 0.$$

Thus, for all radii greater than the position of the maximum pressure,

that is, where

$$\partial \ln g / \partial \psi < 0,$$

the "particle drift" terms are positive and hence violate criterion (2.21).

In Appendix C we demonstrate that as the plasma pressure is increased, the curvature retains its original (negative) sign in the region where $\partial \ln g / \partial \psi < 0$. By contrast, the gradient of magnetic field strength, $\partial \ln B / \partial \psi$, changes sign at relatively low values of plasma pressure. Since this latter term is multiplied by $p_{\perp}$, whereas the curvature term is multiplied by $p_{\parallel}$, and since $p_{\perp}$ can be significantly greater than $p_{\parallel}$, it is clear that the particle drift terms can be negative, as required for stability, if the plasma pressure exceeds some critical value.

This critical pressure can be estimated from the zeros of the quantity in brackets in (C.20):

$$(sg)_{\text{critical}} = 2 \frac{B_C}{B_V} \left[\frac{X}{2} \left(\frac{B_C}{B_V} - 1 \right) \left(1 + 4 \frac{B_C}{B_V} - \frac{B_C^2}{B_V^2} \right) + \frac{B_C}{B_V} - 5 \frac{B_C^2}{B_V^2} \right]^{-1} \quad (3.1)$$

where

$$X \equiv (\partial \ln g / \partial \psi) (\partial \ln R_V / \partial \psi)^{-1} = R_C / R_p. \quad (3.2)$$

R_C and R_p are the radius of curvature of the (vacuum) magnetic field lines and the pressure gradient scale length respectively. This estimate of the critical pressure, re-expressed in terms of beta, is shown in Fig. 2 as a function of X for $B_C/B_V = 1.3$, a value chosen to match the

the experimental equilibria in the "Canted Mirror Facility".

The dependence of the critical pressure on B_C/B_V reflects the importance of pressure anisotropy, since, on the midplane,

$$\frac{p_\perp}{p_\parallel} = \left(\frac{B_C}{B_V} + 1 \right) \left(\frac{B_C}{B_V} - 1 \right)^{-1} . \quad (3.3)$$

The minimum anisotropy required for a finite critical pressure is

$$\left(\frac{p_\perp}{p_\parallel} \right)_{\min} = (3 + \sqrt{5})(1 + \sqrt{5})^{-1} \cong 1.6 . \quad (3.4)$$

We next estimate the third (de-stabilizing) term in (2.21) using (C.21) to approximate the integrands near the midplane. Note again that since τB^2 does not vanish anywhere in the trap,

$$\left\langle \frac{\partial p_\perp}{\partial \psi} \right\rangle^2 \langle \tau B^2 \rangle^{-1} \sim \left(\frac{L_p}{L_{\text{Total}}} \right) \frac{L_p}{B_0} \left[\left(\frac{\partial p_\perp}{\partial \psi} \right)^2 (\tau B^2)^{-1} \right]_{z=0} \quad (3.5)$$

The geometrical reduction by L_p/L_{Total} in the magnitude of this de-stabilizing term is an important consequence of the axial localization of the plasma pressure which, in turn, is associated with the pressure anisotropy possible in the guiding-center model used here.

The relative importance of the de-stabilizing term can be gauged by comparing it to the dominant stabilizing contributions from the "particle drift" terms. From Appendix C the stabilizing term arising from $\partial \ell m B / \partial \psi$ has the form

$$A \times \frac{sg}{2} \left(\frac{B_C}{B_V} - 1 \right) \left(1 + 4 \frac{B_C}{B_V} - \frac{B_C^2}{B_V^2} \right) \quad (3.6)$$

where

$$A \equiv \frac{sg}{2\mu_0} (B_V)^2 \left(\frac{B_C}{B_V} - 1 \right) \frac{\partial \ell m g}{\partial \psi} \frac{\partial \ell m B_V}{\partial \psi} \quad (3.7)$$

From (C.21),

$$\left(\frac{\partial p_\perp}{\partial \psi} \right)^2 (\tau B^2)^{-1} \sim A \times \frac{sg}{2} \left(\frac{B_C}{B_V} - 1 \right) \left(\frac{B_C}{B_V} + 1 \right)^2 ; \quad (3.8)$$

so that the ratio of de-stabilizing to stabilizing contributions to (2.21) is given roughly by

$$\frac{L_p}{L_{Total}} \left[\frac{\left(\frac{B_C}{B_V} + 1\right)^2}{4 \frac{B_C}{B_V} + 1 - \frac{B_C^2}{B_V^2}} \right] \quad (3.9)$$

Since for $1 \leq B_C/B_V \leq 2$ the quantity in brackets is less than 2, the stabilizing effects of increased plasma pressure should dominate provided $L_p/L_{Total} \leq 0.5$. In the numerical cases to be discussed later, $B_C/B_V = 4/3$, $L_p/L_{Total} \approx 0.4$ so that the ratio given above is around 1/2, in keeping with the observed dominance of the stabilizing term for sufficiently large sg .

To summarize this discussion based on very rough estimates of the properties of the equilibrium, we note the two distinct stabilizing effects which stem from the anisotropy:

(i) Since $p_{\perp}/p_{\parallel} \gg 1$, the reversal of $\partial \ln B / \partial \psi$ by plasma currents can stabilize the interior of the plasma by dominating the curvature term in the expression

$$\frac{\kappa}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ln B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi}.$$

(ii) Because the plasma is localized near the midplane (as a result of its anisotropy), the de-stabilizing effect of the plasma pressure is reduced by a ratio of geometrical lengths, L_p/L_{Total} , allowing the stabilizing effect to dominate in the cases considered here.

Note that near the edge of the plasma where sg decreases toward zero, the stability properties will be those of the vacuum field, hence interchange unstable (for simple mirrors) in the absence of line-tying mechanisms. Similar behavior is expected near the peak pressure, where $\frac{\partial \ln g}{\partial \psi} \rightarrow 0$.

Hedrick has obtained accurate evaluations of the stability criteria of Section II using numerical solution of the equilibrium equations and numerical evaluation of the integrals along field lines which appear in the criteria. The plasma model is that described in Appendix C, and solutions were obtained for a range of values of s . We present in Figure 3 a set of plots of the stability criterion eq. (2.21) versus radius for an annular plasma chosen to model that observed in the Canted Mirror Facility.

Here we show only the region in which $\partial p_{\perp}/\partial \psi$ and $\partial p_{\parallel}/\partial \psi$ are negative, since otherwise the plasma is certain to be stable with respect to interchange. The scale of the stability criterion has been adjusted arbitrarily to make the results more transparent.

The figure clearly demonstrates those features discussed above:

(i) For β in excess of a critical value between 0.10 and 0.20, the stability criterion is satisfied throughout most of the interior of the plasma.

(ii) In addition to the edge, where $g(\psi) \rightarrow 0$, there is a narrow radial shell near the peak of the pressure in which $\partial \ln g/\partial \psi$ is too small to change the sign of $\partial \ln B/\partial \psi$. It is not clear from this type of analysis whether or not an eigenmode of the plasma could be sufficiently localized within this region to exhibit unstable growth.

Appendix A: The ψ , θ , χ Coordinate System

If all currents are in the azimuthal direction, that is, in the direction of the ignorable coordinate, then the vector potential clearly has only azimuthal components, so that (in cylindrical coordinates)

$$\vec{B} = \nabla \times \vec{A} = -\frac{\hat{r}}{r} \frac{\partial}{\partial z} (rA) + \frac{\hat{z}}{r} \frac{\partial}{\partial r} (rA). \quad (\text{A.1})$$

Thus,

$$\hat{\theta} \times \vec{B} = \frac{1}{r} \left[\hat{r} \frac{\partial}{\partial r} (rA) + \hat{z} \frac{\partial}{\partial z} (rA) \right] = \frac{1}{r} \vec{\nabla} \psi. \quad (\text{A.2})$$

We can use the three vectors, $\vec{\nabla} \psi$, $\hat{\theta}$, and $\vec{B}$ as a basis by assigning the following unit vectors and scale factors:

$$\left. \begin{aligned} \hat{x}_1 &= \frac{\vec{\nabla} \psi}{|\vec{\nabla} \psi|}, & h_1 &= \frac{1}{rB} \\ \hat{x}_2 &= \hat{\theta}, & h_2 &= r \\ \hat{x}_3 &= \hat{b} = \frac{\vec{B}}{B}, & h_3 &= \frac{J}{h_1 h_2} = BJ, \end{aligned} \right\} \quad (\text{A.3})$$

where J is a Jacobian to be evaluated in what follows. We denote the variable in the $\hat{x}_3$ -direction as χ .

The Jacobian is simply determined from Ampere's Law:

$$\mu_0 \vec{j} = \nabla \times \vec{B} = -\hat{\theta} \frac{r}{J} \frac{\partial}{\partial \psi} (JB^2) = -r\hat{\theta} \left(\frac{\partial B^2}{\partial \psi} + B^2 \frac{\partial \ln J}{\partial \psi} \right), \quad (\text{A.4})$$

where we have used the standard formula for the curl in orthogonal curvilinear coordinates. We can now form $(\nabla \times \vec{B}) \times \vec{B} = \mu_0 \vec{j} \times \vec{B} (= \nabla \cdot \vec{\mathcal{P}})$ in two ways:

$$\left. \begin{aligned}
\mu_0 \vec{J} \times \vec{B} &= - \hat{\psi} r B \left(\frac{\partial B^2}{\partial \psi} + B^2 \frac{\partial \ln J}{\partial \psi} \right) \\
&= (\nabla \times \vec{B}) \times \vec{B} = - \frac{1}{2} \nabla B^2 + \hat{b} (\vec{B} \cdot \nabla) B + B^2 \vec{\kappa},
\end{aligned} \right\} \quad (A.5)$$

where we have used the standard vector identity to rewrite $(\nabla \times \vec{B}) \times \vec{B}$ and noted that $\vec{\kappa} = (\hat{b} \cdot \nabla) \vec{b}$. Equating the ψ -components gives

$$- r B \left(\frac{\partial B^2}{\partial \psi} + B^2 \frac{\partial \ln J}{\partial \psi} \right) = - \frac{r B}{2} \frac{\partial B^2}{\partial \psi} + B^2 \kappa,$$

whence

$$\frac{\partial \ln J}{\partial \psi} = - \frac{\partial \ln B}{\partial \psi} - \frac{\kappa}{r B} \quad (A.6)$$

or

$$JB = \exp \left(- \int_0^\psi d\psi' \frac{\kappa}{r B} \right). \quad (A.7)$$

We have set the constant of integration equal to unity in order that $JB = 1$ for a uniform magnetic field. For convenience we collect here the useful formulas for gradient, divergence, and curl:

$$\begin{aligned}
\vec{\nabla} \varphi &= r B \frac{\partial \varphi}{\partial \psi} \hat{\psi} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \hat{\theta} + \frac{1}{B J} \frac{\partial \varphi}{\partial \chi} \hat{\chi} \\
\nabla \cdot \vec{A} &= \frac{1}{J} \left[\frac{\partial}{\partial \psi} (r B J A_\psi) + \frac{\partial}{\partial \theta} \left(\frac{J}{r} A_\theta \right) + \frac{\partial}{\partial \chi} \left(\frac{A_\chi}{B} \right) \right] \\
\nabla \times \vec{A} &= \frac{\hat{\psi}}{r B J} \left[\frac{\partial}{\partial \theta} (B J A_\chi) - \frac{\partial}{\partial \psi} (r A_\theta) \right] \\
&\quad + \frac{\hat{\theta} r}{J} \left[\frac{\partial}{\partial \chi} \left(\frac{A_\psi}{r B} \right) - \frac{\partial}{\partial \psi} (B J A_\chi) \right] \\
&\quad + \hat{\chi} B \left[\frac{\partial}{\partial \psi} (r A_\theta) - \frac{\partial}{\partial \theta} \left(\frac{A_\psi}{r B} \right) \right]
\end{aligned}$$

Note the following consequences of A.6:

$$\begin{aligned} \frac{\partial}{\partial \psi} \int \frac{d\ell}{B} &= \frac{\partial}{\partial \psi} \int J d\chi = \int d\chi J \frac{\partial \ell n J}{\partial \psi} \\ &= - \int \frac{d\ell}{B} \left(\frac{\kappa}{rB} + \frac{\partial \ell n B}{\partial \psi} \right) \end{aligned} \quad (A.8)$$

Again, from A.4, $-\hat{\theta} \frac{r}{J} \frac{\partial}{\partial \psi} (JB^2) = \mu_0 \vec{j}$, so that

$$\vec{j} \times \vec{B} = \frac{rB}{\mu_0 J} \frac{\partial}{\partial \psi} (JB^2) \hat{\psi} = \nabla_{\mathbb{M}} \cdot \vec{P} \quad (\text{or } \vec{\nabla} p \text{ for scalar pressure}), \quad (A.9)$$

where

$$(\nabla_{\mathbb{M}} \cdot \vec{P})_{\psi} = rB \left[\frac{\partial p_{\perp}}{\partial \psi} + (p_{\parallel} - p_{\perp}) \frac{\kappa}{rB} \right] \quad (A.10)$$

or, in the case of scalar pressure,

$$(\vec{\nabla} p)_{\psi} = rB \frac{\partial p}{\partial \psi} \quad (A.11)$$

A quantity of interest is

$$\begin{aligned} \frac{\partial}{\partial \psi} \left(\frac{1}{\mu_0} \int \vec{B} \cdot d\vec{\ell} \right) &= \frac{\partial}{\partial \psi} \frac{1}{\mu_0} \int d\chi JB^2 \\ &= \int J d\chi \frac{1}{\mu_0 J} \frac{\partial}{\partial \psi} (JB^2) = - \int J d\chi \frac{(\nabla_{\mathbb{M}} \cdot \vec{P})_{\psi}}{rB} \\ \text{or} \quad &= - \int J d\chi \frac{(\vec{\nabla} p)_{\psi}}{rB} \end{aligned} \quad (A.12)$$

In the scalar-pressure case,

$$\frac{\partial}{\partial \psi} \left(\frac{1}{\mu_0} \int \vec{B} \cdot d\vec{\ell} \right) = - \frac{\partial p}{\partial \psi} \int \frac{d\ell}{B}, \quad (A.13)$$

whereas for tensor pressure,

$$\frac{\partial}{\partial \psi} \left(\frac{1}{\mu_0} \int \vec{B} \cdot d\vec{\ell} \right) = - \int \frac{d\ell}{B} \left[\frac{\partial p_{\perp}}{\partial \psi} + (p_{\parallel} - p_{\perp}) \frac{\kappa}{rB} \right] \quad (A.14)$$

Appendix B: The Kadomtsev Criterion for Interchange
Stability of a Scalar-Pressure Plasma in
Closed-Line Devices

We calculate the energy difference between initial and final states in which the plasma in two arbitrary flux tubes is interchanged adiabatically (together with the flux trapped in the ideal plasma):

$$\begin{aligned} \delta W = W_f - W_i = & \int_1 d^3x \left(\frac{p_{1f}}{\gamma-1} + \frac{B_{1f}^2}{2\mu_0} \right) + \int_2 d^3x \left(\frac{p_{2f}}{\gamma-1} + \frac{B_{2f}^2}{2\mu_0} \right) \\ & - \int_1 d^3x \left(\frac{p_{1i}}{\gamma-1} + \frac{B_{1i}^2}{2\mu_0} \right) - \int_2 d^3x \left(\frac{p_{2i}}{\gamma-1} + \frac{B_{2i}^2}{2\mu_0} \right). \end{aligned} \quad (B.1)$$

For adiabatic interchange, $pV^\gamma = \text{constant}$, so that

$$p_{2f} V_2^\gamma = p_{1i} V_1^\gamma \quad \text{and} \quad p_{1f} V_1^\gamma = p_{2i} V_2^\gamma, \quad (B.2)$$

where V is the volume of the flux tube. The change in energy associated with the plasma itself is therefore

$$\delta W_p = \frac{1}{\gamma-1} \left[p_{2i} \left(\frac{V_2}{V_1} \right)^\gamma V_1 + p_{1i} \left(\frac{V_1}{V_2} \right)^\gamma V_2 - p_{1i} V_1 - p_{2i} V_2 \right] \quad (B.3)$$

We linearize these expressions by setting

$$p_2 = p_1 + \delta p \equiv p + \delta p; \quad V_2 = V_1 + \delta V \equiv V + \delta V. \quad (B.4)$$

Then, the change in plasma energy becomes

$$\delta W_p = \delta p \delta V + \gamma p \frac{(\delta V)^2}{V} + \text{higher order terms.} \quad (B.5)$$

In low-pressure plasmas (in which the magnetic energy is minimized) we recover the usual stability criterion by requiring

$$\delta W_p \cong \delta p \delta V + \gamma p \frac{(\delta V)^2}{V} \geq 0, \quad (\text{B.6})$$

marginally satisfied by requiring

$$pV^\gamma = \text{constant}. \quad (\text{B.7})$$

Keep in mind that V is the volume of a flux tube,

$$V = \oint S \, d\ell = \varphi \oint \frac{d\ell}{B} \equiv \varphi U, \quad (\text{B.8})$$

where φ is the magnetic flux, $\varphi = \vec{B} \cdot \vec{S}$.

Next consider changes in the magnetic energy:

$$\begin{aligned} \delta W_B = W_{Bf} - W_{Bi} = \frac{1}{2\mu_0} & \left[\oint_1 B_{1f}^2 S_1 \, d\ell + \oint_2 B_{2f}^2 S_2 \, d\ell \right. \\ & \left. - \oint_1 B_{1i}^2 S_1 \, d\ell - \oint_2 B_{2i}^2 S_2 \, d\ell \right]. \end{aligned} \quad (\text{B.9})$$

Since in ideal MHD the flux is trapped in the plasma, our interchange satisfies

$$\left. \begin{aligned} \varphi_{2f} &= \varphi_{1i} = \varphi_1 \\ \varphi_{1f} &= \varphi_{2i} = \varphi_2 \end{aligned} \right\} \quad (\text{B.10})$$

Note also that the regions (in space) which we have singled out remain unchanged by our interchange operation:

$$\begin{aligned} S_{1i} &= S_{1f} = S_1 \\ S_{2i} &= S_{2f} = S_2 \end{aligned} \quad (\text{B.11})$$

Thus,

$$\begin{aligned}
\delta W_B &= \frac{1}{2\mu_0} \left\{ \oint_1 B_{1f}^2 s_1^2 \frac{B_{1i} d\ell}{B_{1i} s_1} + \oint_2 B_{2f}^2 s_2^2 \frac{B_{2i} d\ell}{B_{2i} s_2} \right. \\
&\quad \left. - \oint_1 B_{1i}^2 s_1^2 \frac{B_{1i} d\ell}{B_{1i} s_1} - \oint_2 B_{2i}^2 s_2^2 \frac{B_{2i} d\ell}{B_{2i} s_2} \right\} \\
&= \frac{1}{2\mu_0} \left\{ \frac{\varphi_{1f}^2}{\varphi_{1i}} \oint_1 B_{1i} d\ell + \frac{\varphi_{2f}^2}{\varphi_{2i}} \oint_2 B_{2i} d\ell \right. \\
&\quad \left. - \varphi_{1i} \oint_1 B_{1i} d\ell - \varphi_{2i} \oint_2 B_{2i} d\ell \right\}
\end{aligned} \tag{B.12}$$

Using (B.10) and the definition

$$I \equiv \frac{1}{\mu_0} \oint B d\ell, \tag{B.13}$$

we can re-write (B.12) in the form

$$\begin{aligned}
\delta W_B &= \frac{1}{2} \left(\frac{\varphi_2^2}{\varphi_1} I_1 + \frac{\varphi_1^2}{\varphi_2} I_2 - \varphi_1 I_1 - \varphi_2 I_2 \right) \\
&= -\frac{1}{2} (\varphi_2^2 - \varphi_1^2) \left(\frac{I_2}{\varphi_2} - \frac{I_1}{\varphi_1} \right) \\
&= -\frac{1}{2} \delta(\varphi^2) \delta\left(\frac{I}{\varphi}\right)
\end{aligned} \tag{B.14}$$

The change in the total energy of the system is given by (B.6) and (B.14):

$$\delta W = \delta W_p + \delta W_B = \delta p \delta V + \mu_0 \frac{(\delta V)^2}{V} - \frac{1}{2} \delta(\varphi^2) \delta\left(\frac{I}{\varphi}\right) \tag{B.15}$$

The changes in I and p are related by the equilibrium conditions as shown in Appendix A, equation A.13:

$$\delta I = -U \delta p. \tag{B.16}$$

And from (B.8), $V = \varphi U$. Using these two relations we can re-write (B.15) in the form

$$\begin{aligned} \delta W &= \varphi \left[\delta p \delta U + \gamma p \frac{(\delta U)^2}{U} \right] + 2\delta\varphi (U\delta p + \gamma p \delta U) \\ &\quad + (\delta\varphi)^2 \left(\frac{\gamma p U + I}{\varphi} \right), \\ &\equiv A\varphi + B \delta\varphi + C(\delta\varphi)^2. \end{aligned} \quad (B.17)$$

In this form we can minimize δW with respect to $\delta\varphi$ to obtain

$$\begin{aligned} (\delta W)_{\min} &= A\varphi - \frac{B^2}{4C} \\ &= \frac{\varphi}{I + \gamma p U} \left(\delta p + \gamma p \frac{\delta U}{U} \right) (I\delta U - U^2\delta p). \end{aligned} \quad (B.18)$$

For stability, we require $(\delta W)_{\min} \geq 0$, whence

$$\left(\delta p + \gamma p \frac{\delta U}{U} \right) (I\delta U - U^2\delta p) \geq 0, \quad (B.19)$$

for stability.

Appendix C: The Equilibrium

The fundamental equilibrium equations are Maxwell's equations and

$$\vec{j} \times \vec{B} = \nabla \cdot \vec{\Pi}_{\text{mm}}, \quad (C.1)$$

where (see ORNL-TM-3943, eq. 3.2)

$$\begin{aligned} \nabla \cdot \vec{\Pi}_{\text{mm}} &= \vec{\nabla} p_{\perp} + \hat{x}_i \frac{\partial}{\partial x_j} \left[(p_{\parallel} - p_{\perp}) b_i b_j \right] \\ &= \vec{\nabla} p_{\perp} + \hat{b} \frac{\partial}{\partial S} (p_{\parallel} - p_{\perp}) + (p_{\parallel} - p_{\perp}) \left(-\frac{1}{B} \frac{\partial B}{\partial S} \hat{b} + \vec{\kappa} \right). \end{aligned} \quad (C.2)$$

From Maxwell's equations we have

$$\nabla \times \vec{B} = \mu_0 \vec{j} \text{ and } \nabla \cdot \vec{B} = 0 \quad (C.3)$$

For numerical computation it is convenient to use the flux function ψ introduced in (A.2) so that

$$\vec{B} = \frac{1}{r} \nabla \psi \times \hat{e}_\theta \quad (C.4)$$

and (see ORNL-TM-4076, eq. 4.3)

$$\Delta^* \psi + \frac{1}{\sigma} \vec{\nabla} \sigma \cdot \vec{\nabla} \psi = - \frac{r^2}{\sigma} \frac{\partial p_{\parallel}}{\partial \psi} \quad (C.5)$$

where

$$\Delta^* \psi \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \psi$$

One can demonstrate that C.5 is an elliptic differential equation if $\tau = \mu_0^{-1} + B^{-1} \partial p_{\perp} / \partial B \geq 0$. This condition is often cited as the criterion for stability of "mirror" modes, just as $\sigma \geq 0$ is a stability criterion for "firehose" modes. We shall adopt a model for the plasma pressure which guarantees that these criteria are satisfied at every point in the plasma.

Our plasma model is specified by the following expressions:

$$\begin{aligned} \mu_0 p_{\perp} &= s \frac{g(\psi)}{2} (B_C^2 - B^2) \quad , \quad \text{if } B \leq B_C \\ \mu_0 p_{\parallel} &= s \frac{g(\psi)}{2} (B_C - B)^2 \quad , \quad \text{if } B \leq B_C \\ \mu_0 \sigma &= 1 + \text{sg} \left(\frac{B_C}{B} - 1 \right) \quad , \quad \text{if } B \leq B_C \\ \mu_0 \tau &= 1 - \text{sg} \quad , \quad \text{if } B \leq B_C \end{aligned} \quad (C.6)$$

For $B \geq B_C$, $p_\perp = p_\parallel = 0$ and $\mu_0 \sigma = \mu_0 \tau = 1$. Here s and B_C are constants, and $g(\psi)$ is a form factor such that $0 \leq g(\psi) \leq 1$. A normalized plasma pressure can be defined as follows:

$$\beta \equiv \left(\frac{2p_\perp + p_\parallel}{3} \right) / \left(\frac{B^2}{2\mu_0} \right) = \frac{sg}{3} \left(3 \frac{B_C^2}{B^2} - 2 \frac{B_C}{B} - 1 \right). \quad (C.7)$$

In order to evaluate our stability criteria we require estimates of the curvature, $\vec{\kappa}$, and $\partial(\ln B)/\partial\psi$. For the purposes of making analytic estimates an alternate iteration scheme is to be preferred over that implicit in equations (C.4) and (C.5). Here we use

$$\nabla \cdot \vec{B} = 0$$

and (see ORNL-TM-4076, equations 5.4 and 5.8)

$$\sigma \vec{B} = \nabla \varphi_3 + \varphi_1 \nabla g \quad (C.8)$$

$$\varphi_1 = \int_0^\ell \frac{d\ell'}{B} \frac{\partial p_\parallel}{\partial g} \quad (C.9)$$

where $\ell = 0$ is taken as the midplane. This iteration scheme begins by setting $\nabla \varphi_3 = \vec{B}_V / \mu_0$ where $\vec{B}_V$ is the vacuum field. Since $\varphi_1 = 0$ at the midplane we obtain from (C.8)

$$\vec{B} = \frac{\nabla \varphi_3}{\mu_0 \sigma} \approx \frac{\vec{B}_V}{\mu_0 \sigma_V} \quad (C.10)$$

Then

$$\frac{\partial \ln B}{\partial \psi} \approx \frac{\partial \ln B_V}{\partial \psi} - \frac{\partial \ln \mu_0 \sigma_V}{\partial \psi},$$

where

$$\mu_0 \sigma_V = 1 + sg \left(\frac{B_C}{B_V} - 1 \right),$$

and hence

$$\frac{\partial \ln \mu_0 \sigma_V}{\partial \psi} = \frac{1}{\mu_0 \sigma_V} \left[sg \left(\frac{B_C}{B_V} - 1 \right) \frac{\partial \ln g}{\partial \psi} - sg \frac{B_C}{B_V} \frac{\partial \ln B_V}{\partial \psi} \right].$$

This gives an immediate estimate of the plasma pressure necessary to reverse the gradient of the magnetic field strength:

$$\frac{\partial \ln B}{\partial \psi} \cong \frac{1}{\mu_0 \sigma_V} \left\{ \left[1 + sg \left(2 \frac{B_C}{B_V} - 1 \right) \right] \frac{\partial \ln B_V}{\partial \psi} - sg \left(\frac{B_C}{B_V} - 1 \right) \frac{\partial \ln g}{\partial \psi} \right\} \quad (C.11)$$

so that in this sense

$$(sg)_{\text{crit}} = \left[\left(\frac{\partial \ln g}{\partial \psi} \right) \left(\frac{\partial \ln B_V}{\partial \psi} \right)^{-1} \left(\frac{B_C}{B_V} - 1 \right) - \left(2 \frac{B_C}{B_V} - 1 \right) \right]^{-1} \quad (C.12)$$

Alternatively, the ratio of scale lengths must satisfy

$$\left(\frac{\partial \ln g}{\partial \psi} \right) \left(\frac{\partial \ln B_V}{\partial \psi} \right)^{-1} = \frac{1 + sg \left(2 \frac{B_C}{B_V} - 1 \right)}{sg \left(\frac{B_C}{B_V} - 1 \right)}.$$

From (C.7) and (C.10),

$$\beta \cong \frac{sg}{3} \left[3 \frac{B_C^2}{B_V^2} - 2 \frac{B_C}{B_V} - 1 + 2sg \left(\frac{B_C}{B_V} - 1 \right) \frac{B_C}{B_V} \left(3 \frac{B_C}{B_V} - 1 \right) + \text{terms in } (sg)^2 \right] \quad (C.13)$$

In Fig. 1 we display curves of β vs. sg using C.13; and use this estimate of β to plot β_{critical} vs. $(\partial \ln g / \partial \psi) (\partial \ln B_V / \partial \psi)^{-1}$ in Fig. 2 for a value of B_C/B_V similar to that inferred from observations in the Canted Mirror Facility.

To estimate the equilibrium value of the curvature, we recall that

$$\vec{\kappa} \equiv (\hat{b} \cdot \nabla) \hat{b} = \frac{\vec{\nabla}_\perp \vec{B}}{B} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{B^2}$$

But since (see ORNL-TM-4076, eq. 2.9)

$$\sigma(\nabla \times \vec{B}) \times \vec{B} = \vec{\nabla}_\perp p_\perp + \left(\frac{p_\parallel - p_\perp}{B} \right) \vec{\nabla}_\perp \vec{B} ,$$

we obtain

$$\begin{aligned} \vec{\kappa} &= \frac{\vec{\nabla}_\perp \vec{B}}{B} + \frac{1}{\sigma B^2} \left[\vec{\nabla}_\perp p_\perp + \left(\frac{p_\parallel - p_\perp}{B} \right) \vec{\nabla}_\perp \vec{B} \right] \\ &= \frac{\vec{\nabla}_\perp \vec{B}}{B} \left(1 + \frac{p_\parallel - p_\perp}{\sigma B^2} \right) + \frac{\vec{\nabla}_\perp p_\perp}{\sigma B^2} \\ &= \frac{1}{\mu_0 \sigma} \left[\frac{\vec{\nabla}_\perp \vec{B}}{B} + \frac{\vec{\nabla}_\perp (\mu_0 p_\perp)}{B^2} \right] \end{aligned} \quad (C.14)$$

or

$$\frac{\kappa_\psi}{rB} = \frac{1}{\mu_0 \sigma} \left(\frac{\partial \ln B}{\partial \psi} + \frac{\mu_0}{B^2} \frac{dp_\perp}{d\psi} \right) \equiv \frac{\kappa}{rB} .$$

For the pressure model chosen here, we obtain

$$\frac{\kappa}{rB} = \frac{1}{\mu_0 \sigma} \left[\frac{\partial \ln B}{\partial \psi} (1 - sg) + \frac{\partial \ln g}{\partial \psi} \frac{sg}{2} \left(\frac{B_C^2}{B^2} - 1 \right) \right] \quad (C.15)$$

If we retain only terms of sg and larger C.11 becomes

$$\frac{\partial \ln B}{\partial \psi} \simeq \frac{\partial \ln B_V}{\partial \psi} \left[1 + sg \left(\frac{B_C}{B_V} \right) \right] - sg \frac{d \ln g}{d\psi} \left(\frac{B_C}{B_V} - 1 \right) \quad (C.16)$$

while (C.15) reduces to

$$\frac{n}{rB} \approx \frac{\partial \ell n B_V}{\partial \psi} + \frac{1}{2} sg \frac{\partial \ell n g}{\partial \psi} \left(\frac{B_C}{B_V} - 1 \right)^2 \quad (C.17)$$

In this same spirit, C.6 and C.10 can be used to approximate

$$\mu_0 \frac{\partial p_{\perp}}{\partial \psi} \approx \frac{sg}{2} \frac{\partial \ell n g}{\partial \psi} B_V^2 \left(\frac{B_C}{B_V} - 1 \right) \left[1 + \frac{B_C}{B_V} + 2sg \frac{B_C^2}{B_V^2} \right] \quad (C.18)$$

$$\mu_0 \frac{\partial p_{\parallel}}{\partial \psi} \approx \frac{sg}{2} \frac{\partial \ell n g}{\partial \psi} B_V^2 \left(\frac{B_C}{B_V} - 1 \right)^2 \left[1 + 2 sg \frac{B_C}{B_V} \right] \quad (C.19)$$

We may now use C.16 through C.19 to estimate the factor occurring in the simple interchange criterion:

$$\begin{aligned} & \frac{n}{rB} \frac{\partial p_{\parallel}}{\partial \psi} + \frac{\partial \ell n B}{\partial \psi} \frac{\partial p_{\perp}}{\partial \psi} \\ & \approx \frac{sg}{2\mu_0} B_V^2 \left(\frac{B_C}{B_V} - 1 \right) \left(\frac{\partial \ell n g}{\partial \psi} \right) \left(\frac{\partial \ell n B_V}{\partial \psi} \right) \\ & \quad \left\{ 2 \frac{B_C}{B_V} + sg \left[5 \frac{B_C^2}{B_V^2} - \frac{B_C}{B_V} \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \frac{\partial \ell n g}{\partial \psi} \left(\frac{\partial \ell n B_V}{\partial \psi} \right)^{-1} \left(\frac{B_C^2}{B_V^2} - 4 \frac{B_C}{B_V} - 1 \right) \left(\frac{B_C}{B_V} - 1 \right) \right] \right\} \end{aligned} \quad (C.20)$$

The ratio $(\partial p_{\perp}/\partial \psi)^2/(\tau B^2)$ can be approximated to this same order in sg by simply setting $\tau B^2 \approx \mu_0 B_V^2$ and using the lowest order term in C.18 to obtain

$$\begin{aligned} & \frac{sg}{2\mu_0} B_V^2 \left(\frac{B_C}{B_V} - 1 \right) \left(\frac{\partial \ell n g}{\partial \psi} \right) \left(\frac{\partial \ell n B_V}{\partial \psi} \right) \\ & \quad \left[\frac{sg}{2} \frac{\partial \ell n g}{\partial \psi} \left(\frac{\partial \ell n B_V}{\partial \psi} \right)^{-1} \left(\frac{B_C}{B_V} - 1 \right) \left(\frac{B_C}{B_V} + 1 \right)^2 \right] \end{aligned} \quad (C.21)$$

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4. Note the following typographical errors:
p. 28, Eq. 6.4, should read

$$\tau = - B^3 \frac{\partial}{\partial B} \left(\frac{P_{\perp}}{B^2} \right) \text{ [See Eqs. (2.25) and (5.25)] .}$$

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Fig. 1. Plasma pressure, β , as defined in Eq. (C.39) and estimated in Eq. (C.43) versus the parameter $sg(\psi)$ for various values of B_c/B_v .

Fig. 2. Critical values of β versus the ratio of scale lengths, $X \equiv (\partial \ln g / \partial \psi) (\partial \ln B_v / \partial \psi)^{-1}$. The solid curve indicates the conditions for changing the sign of $\partial \ln B / \partial \psi$; the dashed curve indicates for satisfying the stability criterion, Eq. (2.21).

Fig. 3. The radial dependence of $\mathcal{E}^{(3)}$ for interchange modes in a numerically-determined equilibrium. The curves are labelled with s and, in parenthesis, the corresponding values of β .