Possible Zeroth Order Approximation to Elastic Scattering of Two Pions*

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ABSTRACT

A possible strong-interaction Born term for pion-pion scattering is suggested and discussed.

The purpose of this paper is to point out the existence of a class of narrow resonance approximations, related to Euler B function models, one of which can provide a realistic Born term for $\pi\pi$ scattering.

Much theoretical progress in understanding hadron interactions has stemmed from the Euler B function model for a two-body spinless process $p_1 + p_2 \rightarrow (-p_3) + (-p_4)$ [$s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$] namely⁽¹⁾

$$T(s, t) = B(-\alpha_s, -\alpha_t) = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} \quad (1)$$

where $\alpha_s = \alpha(0) + \alpha'_s s$. This model has several special properties when $\alpha(0)=1$ exactly and the simplest generalization to N particles is theoretically satisfactory (ghost free) only in this case. Physically this value of $\alpha(0)$ is unfortunately unacceptable because (i) observed Regge trajectories have a lower intercept and (ii) it implies, at least for the simplest form, a negative squared mass particle at $\alpha_s=0$. This second difficulty can be overcome by rather easy modifications (multiplication⁽²⁾ by $\alpha_s \alpha_t (\alpha_s + \alpha_t)^{-1}$) but the first difficulty is less trivial to overcome.

We first list four (inter-related) properties of Eq. (1) taken with $\alpha(0)=1$. They are

(A) Two-unit spacing of linear trajectories, which follows immediately from

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$$B(-\alpha_s, -\alpha_t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{n-\alpha_s} (\alpha_t+1)(\alpha_t+2) \dots (\alpha_t+n) \quad (2)$$

and the fact that the residues have parity $(-1)^n$ under the interchange $\alpha_t \leftrightarrow \alpha_u$.

(B) The Fairlie-Jones summability condition⁽⁴⁾, namely

$$B(-\alpha_s, -\alpha_t) + B(-\alpha_t, -\alpha_u) + B(-\alpha_u, -\alpha_s) = \int_{-\infty}^{\infty} dx |x|^{\alpha_t-1} |1-x|^{\alpha_s-1} \quad (3)$$

which follows by the changes of variables $x=(1-y)^{-1}$, $x=1-1/y$ in the x ranges $(1, \infty)$, $(-\infty, 0)$ respectively.

(C) The supplementary condition which states that the following quantity vanishes

$$\alpha(s) + \alpha(t) + \alpha(u) + 1 = \gamma = 0 \quad (4)$$

(D) The Plante identities⁽⁵⁾, arising from closure of the integration contour in Eq. (3) and taking proper account of the branch points at $x=0$ and $x=1$.

Exploiting the analyticity properties of the integrand then yields

$$B(-\alpha_s, -\alpha_t) e^{+i\pi\alpha_t} B(-\alpha_t, -\alpha_u) e^{-i\pi\alpha_s} B(-\alpha_u, -\alpha_s) = 0 \quad (5)$$

and cyclic permutations thereof. The exponent signs correspond to closure in the upper, lower half x -plane respectively.

To discuss a simple hadronic process such as pion-pion scattering we need to introduce isospin through

$$\begin{aligned} T^{abcd}(s, t, u) &= \langle \pi^c(-p_s) \pi^d(-p_t) | T | \pi^a(p_1) \pi^b(p_2) \rangle \\ &= T_s(stu) \delta_{ab} \delta_{cd} + T_t(stu) \delta_{ac} \delta_{bd} + T_u(stu) \delta_{ad} \delta_{bc} \end{aligned} \quad (6)$$

and then avoid unwanted double-charged resonances by writing

$$T_s(stu) = A_4(st) - A_4(tu) + A_4(us) \quad (7)$$

and cyclic permutations where $A_4(st)$ has poles in the same channels as

$B(-\alpha_s, -\alpha_t)$ of Eq. (1). Now, however, we wish a physically acceptable ρ -f trajectory intercept $\alpha_f(0)$ and a physically acceptable external pion mass μ .

What properties of $B(-\alpha_s, -\alpha_t)$ with $\alpha(0) = 1$ should be retained and which abandoned?

Firstly, the two unit spacing of trajectories (A) is desirable phenomenologically. Certainly a ρ' near the $f'(1260)$ and strongly coupled to $\pi\pi$ is ruled out experimentally. The situation with respect to the $T \approx 0$ s-wave phase shift δ_0^0 is less certain; recent experiments, however, indicate that any structure is very broad and ill-defined; if it is parametrized as an ϵ -resonance one finds $600 \text{ MeV} \lesssim \Gamma_\epsilon < \infty$ (infinity means no resonance pole!) - see e.g. Protopopescu et. al.⁽⁶⁾ It becomes desirable therefore not to include any ^(ϵ) on a footing with well-defined (narrow) resonances ρ -f-g-etc. in the Born term. The assumption of no odd daughters may be regarded as our fundamental one.

Secondly, the summability condition (B) is desirable theoretically since it will allow a multiparticle extension (see below). The extension will possess ^{the} generalization of projective invariance which is a necessary condition, in known dual models, for a consistent factorization.

Thirdly, the supplementary condition (C) of the B function model must be relaxed since for physical trajectories $\gamma = 3\alpha_s(0) + 4\alpha_t' \mu^2 + 1 \approx 5/2$.

Fourthly, the Plahte identities (D) must be discarded since it is simple to derive from them a contradiction unless $\gamma = 0$ (modulo 2). Note that this fact, taken together with the retention of summability, implies that we are forced to introduce new singularities of the integrand in the x-plane.

We now proceed to formulate rules for the construction of an amplitude satisfying such properties. Inspired by Eq. (1) and the fact (explained later) that the new singularities must be at the fixed points $x = e^{\pm i\pi/3}$ we write

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$$A_4(-\alpha_s, -\alpha_t) = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{1}{2}\gamma} (x-e^{-i\pi/3})^{\frac{1}{2}\gamma} \phi_4(\alpha_s \alpha_t \alpha_u, x) \quad (8)$$

where ϕ_4 is to be determined.

Define $v = \frac{1}{2}(s-u)$ and make a Laplace transform by writing $(1-x) = e^{-w}$.

That is

$$A_4(-\alpha_s, -\alpha_t) = \int_0^\infty dw w^{-\alpha_t-1} e^{\alpha'vw} \left[\sum_{n=0}^\infty \frac{1}{(2n+1)!} \left(\frac{w}{2}\right)^{2n} \right]^{-\alpha_t-1} \left[1 + 2 \sum_{n=1}^\infty \frac{w^{2n}}{(2n)!} \right]^{\frac{1}{2}\gamma} \phi_4(\alpha_s \alpha_t \alpha_u, 1-e^{-w}) \quad (9)$$

and use $(|v| \rightarrow \infty, \text{Re } v < 0)$

$$\int_0^\infty dw w^{-\alpha_t-1+r} e^{\alpha'vw} = (\alpha'v)^{\alpha_t-r} \Gamma(r-\alpha_t) \quad (10)$$

to find that the odd terms are missing providing ϕ_4 is symmetric under $x \rightarrow \frac{x}{x-1}$ and $\alpha_s \rightarrow \alpha_u$. Taken together with the $x \rightarrow (1-x)$, $\alpha_s \rightarrow \alpha_t$ crossing symmetry this implies that

$$\begin{aligned} \phi_4(\alpha_s \alpha_t \alpha_u, x) &= \phi_4(\alpha_t \alpha_u \alpha_s, \frac{1}{1-x}) = \phi_4(\alpha_u \alpha_s \alpha_t, 1-\frac{1}{x}) \\ &= \phi_4(\alpha_t \alpha_s \alpha_u, 1-x) = \phi_4(\alpha_s \alpha_u \alpha_t, \frac{1}{x}) = \phi_4(\alpha_u \alpha_t \alpha_s, \frac{x}{x-1}) \end{aligned} \quad (11)$$

[Property I of ϕ_4]

To ensure Regge behaviour in $\text{Re } v < 0$ ($\text{Im } v \neq 0$) we invoke Watson's Lemma⁽⁷⁾ to deduce analyticity of the integrand in $\text{Re } w > 0$ or, equivalently $|1-x| < 1$. Consider now the summability

$$\begin{aligned}
& A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s) = \\
& \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{1}{2}\gamma} (x-e^{-i\pi/3})^{\frac{1}{2}\gamma} \phi_4(\alpha_s \alpha_t \alpha_u, x) \quad (12) \\
& = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{1}{2}\gamma} (x-e^{-i\pi/3})^{\frac{1}{2}\gamma} \phi_4(\alpha_s \alpha_t \alpha_u, x) \\
& + \int_0^1 d\left(\frac{1}{1-x}\right) \left(\frac{1}{1-x}\right)^{-\alpha_u-1} \left(1-\frac{1}{1-x}\right)^{-\alpha_t-1} \left(\frac{1}{1-x}-e^{i\pi/3}\right)^{\frac{1}{2}\gamma} \left(\frac{1}{1-x}-e^{-i\pi/3}\right)^{\frac{1}{2}\gamma} \phi_4(\alpha_t \alpha_u \alpha_s, \frac{1}{1-x}) \\
& + \int_0^1 d\left(1-\frac{1}{x}\right) \left(1-\frac{1}{x}\right)^{-\alpha_s-1} \left(1-(1-\frac{1}{x})\right)^{-\alpha_u-1} \left(1-\frac{1}{x}e^{i\pi/3}\right)^{\frac{1}{2}\gamma} \left(1-\frac{1}{x}e^{-i\pi/3}\right)^{\frac{1}{2}\gamma} \phi_4(\alpha_u \alpha_s \alpha_t, 1-\frac{1}{x}) \\
& \quad (13)
\end{aligned}$$

Full Regge behaviour is seen to imply analyticity of ϕ_4 for $|x| < 1$, $|1-x| < 1$, $|\frac{1}{x}| < 1$, $|\frac{1}{1-x}| < 1$, which leaves only the fixed points $x = e^{\pm i\pi/3}$. Since we have already taken into account the branch points of order $\frac{\gamma}{2}$ to ensure the summability requirement we deduce that $\phi_4(\alpha_s \alpha_t \alpha_u, x)$ is analytic everywhere in the x -plane, including at the point at infinity, except possibly for poles (simple or multiple poles) at the fixed points $x = e^{\pm i\pi/3}$ [Property II of ϕ_4].

We note that the fixed-point singularities give rise to an extra piece in the asymptotic expansion for $|v| \rightarrow \infty$, which falls exponentially at complex v but which for real v has slightly altered local duality properties (oscillatory with period $3/\alpha'$ instead of $1/\alpha'$); see Ref. 8 for a detailed discussion of the simplest case $\phi_4 = 1$.

For the multiparticle extension we introduce four z_i ($i=1,2,3,4$) variables lying in order on an arbitrary circle in the z -plane and identify

$$x = (z_2, z_1; z_3 z_4) \quad (14)$$

where

$$(a,b;c,d) = (a-c)(b-d)(a-d)^{-1}(b-c)^{-1} \quad (15)$$

is an anharmonic ratio. Recalling that

$$B(-\alpha_s, -\alpha_t) = \int \prod_{i=1}^4 dz_i (dV_3)^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \left[\prod_{i=1}^4 \prod_{j=i+1}^4 (z_i - z_{i+1})^{-2} \right]^{\frac{\alpha_p(0)-1}{2}} \quad (16)$$

$$\left[\prod_{i < j} (z_i - z_j)^{(-1)^{ij}} \right]^{2(\alpha_p(0) - \alpha_\pi(0))}$$

where $dV_3 = dz_a dz_b dz_c (z_a - z_b)^{-1} (z_b - z_c)^{-1} (z_c - z_a)^{-1}$ is the usual Haar measure, and making use of the identities

$$\frac{\gamma}{2} = \left(\frac{\alpha_p(0)-1}{2} \right) + 2(\alpha_p(0) - \alpha_\pi(0)) \quad (17)$$

$$(x - e^{i\pi/3})(x - e^{-i\pi/3}) = \frac{1}{16} \prod_{i=1}^4 (z_i - z_{i+1})^2 \left[\sum_{q_1 q_2 q_3 q_4} \prod_{k=1}^4 (z_{q_k} - z_{q_{k+1}})^2 \right]$$

$$= \frac{1}{16} \prod_{i < j} (z_i - z_j)^{-(-1)^{i-j}} \left[\sum_{q_1 q_2 q_3 q_4} (-1)^P \prod_{k < \ell} (z_{q_k} - z_{q_\ell})^{(-1)^{k-\ell}} \right] \quad (19)$$

where the sums are over all 4! permutations of $q_1, q_2, q_3, q_4 (=1, 2, 3, 4)$ and

$P =$ parity of permutation we may make the multiparticle generalization

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i \cdot p_j} \left[\sum_{q_1 \dots q_N} \prod_{k=1}^N (z_{q_k} - z_{q_{k+1}})^2 \right]^{\frac{1}{2}(\alpha_p(0)-1)}$$

$$\left[\sum_{q_1 \dots q_N} (-1)^P \prod_{k < \ell} (z_{q_k} - z_{q_\ell})^{(-1)^{k-\ell}} \right]^{2(\alpha_p(0) - \alpha_\pi(0))} \Phi_N(\{p_i, z_i\}) \quad (20)$$

Here, the function Φ_N , like ϕ_4 , must be totally symmetric in all $N!$ permutations of the argument pairs $\{p_i, z_i\}$ ($i=1, 2, \dots, N$). This ensures that the

integrand has the higher symmetry than projective invariance necessary (from experience) to allow ghost-free factorization.

The explicit construction of ϕ_4 is facilitated by introducing the following auxiliary function with simple poles at the fixed points

$$f(x) = f(x^{-1}) = x(x - e^{i\pi/3})^{-1}(x - e^{-i\pi/3})^{-1} \quad (21)$$

and then the properties I and II dictate that ϕ_4 (i) be invariant under an S_3 permutation group acting simultaneously on the sets of three objects $\alpha_s, \alpha_t, \alpha_u$ and $f(x), f(1-x), f(1-\frac{1}{x})$ and (ii) be analytic in these quantities.

These general rules lay the basis for construction of attractive mesonic Born terms.

To conclude we cite explicit examples. We note that the trivial case $\phi_4 = 1$ gives rise to the Mandelstam amplitude, ⁽⁸⁾ which for $\alpha(0) = 1$ and $\alpha'_\mu = -1$ becomes simply ⁽¹⁾ $B(-\alpha_s, -\alpha_t)$.

As a more interesting non-trivial example, consider the $A(-\alpha_s, -\alpha_t)$ constructed out of the linear form

$$\phi_4(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s (f(x) + \alpha_t f(1-x) + \alpha_u f(1-\frac{1}{x})) \quad (22)$$

$$= \frac{1}{8} \sum_{q_1 q_2 q_3 q_4} \alpha_{q_1 q_2} f((z_{q_1} z_{q_2}; z_{q_3} z_{q_4})) \quad (23)$$

This has the properties that for physical intercept $\alpha(0)$ and physical pion mass the resonance widths in the four-point function seem to be positive, at least for all the lowest levels. In Table I we give the widths up to $\alpha_s = 8$ with normalizations $\alpha_s = 0.48 + 0.9s$ and $\Gamma_\rho = 112$ MeV for direct comparison to Table I on page 1348 of Reference 3. The widths along the parent coincide exactly with those of the well-known Lovelace-Shapiro model, $A_4^{(LS)} = \alpha_s \alpha_t (\alpha_s + \alpha_t)^{-1} B(-\alpha_s, -\alpha_t)$. The ϵ and ρ' are, of course absent by construction.

Because of the encouraging absence of ghosts in the four-pion amplitude we conjecture that this model has a ghost-free multiparticle extension for physically-acceptable masses. This conjecture is lent weight by the fact that by a simple extrapolation to the unphysical values $\alpha(0) = 1$ and $\alpha'_{\mu}{}^2 = -\frac{1}{2}$ we find

$$A_4(-\alpha_s, \alpha_t) \rightarrow -3 \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \quad (24)$$

where we used the appropriate unphysical supplementary condition

$\gamma = \alpha_s + \alpha_t + \alpha_u + 1 = 2$. This is the four-pion Born term for the ghost-free Neveu-Schwarz dual pion model. (10,11)

Further details of this approach will be given elsewhere.

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TABLE I

Partial widths for $\pi\pi$ decay predicted by the model given by Equation of the text. The widths are normalized to $\Gamma(\rho \rightarrow \pi\pi) = 112\text{MeV}$ and the trajectory is $\alpha_\rho(t) = 0.48 + 0.90t$. Also $\alpha_\pi(0) = \alpha_\rho(0) - 1/2$. The table is designed to be compared directly with Table I on Page 1348 of J. A. Shapiro, Phys. Rev. 179, 1345(1969).

	Mass (MeV)							
J^P	760	1300	1670	1980	2240	2480	2690	2890
8^+								5
7^-							6	
6^+						13		34
5^-					14		32	
4^+				34		71		12
3^-			38		69		8	
2^+		96		149		13		47
1^-	<u>112</u>		141		16		24	
0^+		259		68		15		54