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LOS ALAMOS ████████ of the ████████ NEW MEXICO
University of California

**The Approximate Doppler Broadening
of an Isolated Nuclear Resonance**

APPROVED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

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The Approximate Doppler Broadening
of an Isolated Nuclear Resonance

by

Joseph J. Devaney

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ABSTRACT

Approximations and bounds to the Doppler broadening of an isolated nuclear resonance are presented.

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It is a pleasure to acknowledge conversations or communications with Professor F. Oberhettinger of Oregon State College; and with G. I. Bell, E. D. Cashwell, G. L. Lamb, and J. E. Young of the Los Alamos Scientific Laboratory.

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I. INTRODUCTION

The narrow, isolated nuclear resonance cross sections of an isotope are effectively broadened by thermal motion of the target nuclei. This note gives a few approximate formulas for calculating such broadening. It is the conclusion of this author, however, that usually these approximate formulas are either too inaccurate or too tedious to merit preference over tables or machine calculation, both of which are currently in widespread use.^{1,2}

¹Examples of tables are:

- M. E. Rose, W. Miranker, and P. Leak, Brookhaven National Laboratory Report BNL-257 (Sept. 1953).
- M. E. Rose, W. Miranker, P. Leak, L. Rosenthal, and J. K. Hendrickson, Westinghouse Electric Corporation, Pittsburgh, Atomic Power Division Report WAPD-SR-506, Vols. I and II (Nov. 1954).
- J. E. Lynn, United Kingdom Atomic Energy Authority, Harwell Report AERE-R3353 (May 1960).
- J. E. Lynn, United Kingdom Atomic Energy Authority, Harwell Report AERE-R3354 (1960).
- Y. Dandeu, G. Olivie, F. Roche, C. E. N. (Saclay), Rapport No. 43 de Section Autonome de Calcul Electronique (1960).
- T. D. Beynon and I. S. Grant, Nucl. Sci. Eng. 17, 547 (1963).
- V. J. Bell, P. A. C. Buckler, and I. C. Pull, United Kingdom Atomic Energy Authority Report AEEW-R-266 (1963).
- D. W. Posener, Australian J. Phys. 12, 184 (1959).
- K. K. Seth and R. H. Tabony, "Tabulation of the Doppler Integrals $\psi(x,t)$ and $\phi(x,t)$," U. S. Atomic Energy Commission, Division of Technical Information Report TID-21304 (Feb. 1964).
- L. Dresner, Oak Ridge National Laboratory Report ORNL-2659 (1959).
- J. L. Cook and D. Elliott, Australian J. Appl. Sci. 11, 16 (1960).

²Methods of calculation will be referred to subsequently and in the Bibliography.

A word about the extent of the various approximations is appropriate, since all practical calculations are in fact approximations to physical reality. We proceed in order from no approximation toward greater and greater approximation.

No approximation at all is, of course, the solution of the simultaneous many-body problem at both atomic and nuclear levels. A presently attainable approximation, however, is conceivably the use of full, multi-level (that is, with complete interference) nuclear cross sections³ together with molecular or crystalline many-body treatments. Such an ambitious undertaking has not yet been successfully attempted, to this author's knowledge; but the many-body problem at the atomic level⁴ has

³Examples of multilevel treatments are:

C. W. Reich and M. S. Moore, Phys. Rev. 111, 929 (1958).

E. Vogt, Phys. Rev. 112, 203 (1958).

D. B. Adler and F. T. Adler, Argonne National Laboratory Report ANL-6792 (Oct. 1963; Reactor Conference).

⁴See, for example, the work of Parks and collaborators:

N. F. Wikner, G. D. Joanou, and D. E. Parks, Nucl. Sci. Eng. 19, 108 (1964).

D. E. Parks, J. R. Beyster, and N. F. Wikner, Nucl. Sci. Eng. 13, 306 (1962).

D. E. Parks, "The Calculations of the Neutron Scattering Kernels in Graphite," General Atomic Report GA-2438 (Oct. 1961).

D. E. Parks, "The Relation of Crystal Symmetry in Graphite to Lattice Vibrations and their Interaction with Slow Neutrons," General Atomic Report GA-2125 (March 1961).

M. S. Nelkin and D. E. Parks, Phys. Rev. 119, 1060 (1960).

Also:

G. Placzek and L. Van Hove, Phys. Rev. 93, 1207 (1954).

V. N. Kascheev and M. A. Krivoglaz, Sov. Phys. - Solid State (English Transl.) 3, 1107 (1961).

R. N. Stuart, "Application of the Debye Model to Scattering of Neutrons from Crystals," University of California Report UCRL-7267, Livermore, California (March 1963).

A. A. Maradudin and V. Ambegaokar, Phys. Rev. 135, A1071 (1964).

V. N. Kashcheev, Phys. Status Solidi, 4, 31 (1964).

been profitably attacked, together with varying complexities of nuclear cross-section laws.

The next level of approximation abandons the many-body problem at the atomic level, substituting for it a velocity distribution of the target nuclei, which is then folded into a cross-section law. Here the investigator may use the actual velocity distribution, if known; assume a Debye solid, or an Einstein solid; or finally, and least accurately, but most simply, choose a Maxwellian velocity distribution. The relative accuracies of these methods have been discussed by a number of authors.⁵ This author's studies indicate that for neutron energies near or above the Debye temperature, a three-dimensional Maxwellian velocity distribution is a useful approximation to the velocity distribution of a Debye solid; and he and his colleagues have used the latter, together with energy-dependent widths, linear level-level interference in scattering, and the use of reaction rates⁶ in an isolated-level Breit-Wigner formalism

⁵Some are:

W. E. Lamb, Jr., Phys. Rev. 55, 190 (1939).

H. E. Jackson and J. E. Lynn, Phys. Rev. 127, 461 (1962).

A. Bernabei, "The Effects of Crystalline Binding on the Doppler Broadening of a Neutron Resonance," Thesis, New York University, (June 1964) (unpublished).

H. E. Jackson, L. M. Bollinger, and R. E. Coté, Phys. Rev. Letters 6, 187 (1961).

P. A. Egelstaff, United Kingdom Atomic Energy Research Establishment Report AERE-M1035 (1962).

C. Le Pipec, A. Michaudon, and P. Ribon, C. R. Acad Sci. Paris, 255, 1913 (1962).

⁶A discussion of this point is to be found in either: J. J. Devaney and B. G. Fagan, Los Alamos Scientific Laboratory Report LA-2144, p. 8 (June 1958); or J. J. Devaney, L. O. Bordwell, and M. J. Devaney, Los Alamos Scientific Laboratory Report LA-2763, p. 10 (Nov. 1962). Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Washington, D.C.

to compute Doppler-broadened cross sections.⁷ Continuing to greater approximation, one may neglect all nuclear level-level interference.

For neutron energies that are large compared to the target temperature, the three-dimensional Maxwellian velocity distribution may be reduced to a Gaussian.⁸ In the same limit, averages over reaction rates may be replaced by averages over cross sections. Some of the approximations offered in this paper are at this level. Others are at the drastic level of all of the above approximations plus the assumption that the intrinsic resonance is so narrow that the energy variation of level widths and similar quantities may be neglected. Such approximations yield the so-called "Doppler-broadened functions" $\psi(x, \xi)$ and $\chi(x, \xi)$ which are defined (after Bethe⁸):

⁷The code used is given in: L. Bordwell, et al., Los Alamos Scientific Laboratory Report LAMS-3075 (May 1964). Results for various isotopes were studied in a series of Los Alamos Scientific Laboratory reports by R. E. Anderson, L. O. Bordwell, D. Coward, B. G. Fagan, M. Goldstein, M. J. Devaney, and J. J. Devaney, as follows: Pu²³⁹, LA-2127 (1957); U²³⁸, LA-2144 (1958); W, LA-2289 (1958); Mo, LA-2373 (1960); Th, LA-2525 (1961); Pu²⁴⁰, LAMS-2574 (1961); and Hf, LA-2763 (1962). Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Washington, D.C. A summary of the derivation of the method used is given in LA-2763.

⁸H. A. Bethe and G. Placzek, Phys. Rev. 51, p. 462 (1937); H. A. Bethe, Rev. Mod. Phys. 9, p. 140 (1937). A simple derivation of the potential-level interference term, χ , is to be found, for example, in G. I. Bell, Los Alamos Scientific Laboratory Report LA-2322, Appendix B (Oct. 1959). Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Washington, D.C.

$$\psi(x, \xi) = \frac{\xi}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{1}{4} \xi^2 (x - y)^2 \right] \frac{dy}{1 + y^2} \quad (1)$$

and

$$\chi(x, \xi) = \frac{\xi}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{1}{4} \xi^2 (x - y)^2 \right] \frac{2y \, dy}{1 + y^2} \quad (2)$$

If Γ is the total neutron width; E_0 , the resonance energy; E , the neutron energy; m , the neutron mass; M , the target mass; and kT , the absolute target temperature in energy units; then $x \equiv 2(E - E_0)/\Gamma$ and $\xi \equiv \Gamma\sqrt{M/4m \, kT \, E}$.

It should be noted that the functions ψ and χ vary from author to author in that ξ may be written as $1/t$, $\chi(x, \xi)$ as $2\varphi(x, \xi)$, etc.

The functions (1) and (2) are those most commonly computed (see Bibliography) and found in tables.¹

II. FORMULAS TO BE APPROXIMATED

For completeness it seems advisable to outline the source of the actual formulas that we shall approximate. If, therefore, we take:

$\sigma_{nac}, \sigma_{na}$ as the center of mass and laboratory Breit-Wigner cross sections for an isolated resonance, neutrons in, process "a" out. (We omit all interference terms.) (c shall mean center of mass; all quantities not so designated are in the laboratory system.)

λ_c, λ the reduced wavelengths.

Γ_{nc}, Γ_n the neutron widths.

Γ_{ac}, Γ_a the widths for process "a".

Γ_c, Γ the total widths.

$E_c, E; E_{oc}, E_o$ the energies and the resonant energies, respectively.

m, v the mass and laboratory velocity of the incoming neutron.

M, V the mass and laboratory velocity of the target nucleus.

$w(V)$ the one-dimensional velocity distribution of the target nucleus; V is thus parallel to v .

kT the temperature, in energy units, of the target nucleus.

$\mu \equiv \frac{mM}{m + M}$ the reduced mass.

$g \equiv \frac{1}{2} \left(\frac{2J + 1}{2I + 1} \right)$ the statistical weight for a compound system; angular momentum, J ; target angular momentum, I ; for $\ell = 0$ neutrons $J = I \pm 1/2$.

$\Delta \equiv 4\sqrt{\frac{mE_n kT}{M}}$ the "Doppler width." E_n is the laboratory neutron energy.

Then, in the center-of-mass system, the Breit-Wigner resonance formula for neutrons in, process "a" out, takes the form:

$$\sigma_{na} = \pi g \lambda_c^2 \frac{\Gamma_{nc} \Gamma_{ac}}{\left(E_c - E_{oc}\right)^2 + \left(\frac{1}{2} \Gamma_c\right)^2} \quad (3)$$

center-of-mass

Taking the one-dimensional Maxwellian velocity distribution of the target, we have:

$$w(V) dV = \sqrt{\frac{M}{2\pi kT}} e^{-\frac{MV^2}{2kT}} dV \quad (4)$$

We explicitly exhibit energy-dependence by writing:

$$\lambda_c^2 = \lambda_{co}^2 \frac{E_{oc}}{E_c} \quad \text{and} \quad \Gamma_{nc} = \Gamma_{nc}^o \sqrt{\frac{E_c}{E_{oc}}}$$

where "o" refers to quantities evaluated at the resonance energy, E_{oc} .

Actually, the quantity measured by physicists, and of practical and physical interest, is the number of reactions per incident neutron.⁶

When the target atoms are in motion, this reaction rate is not the microscopic cross section, nor even the thermal target motion average of that quantity; it is, rather, the relative velocity-weighted average of the microscopic cross section over the target motion. Accordingly, we must integrate over the reaction rate per target nucleus and divide by the incident velocity to get the effective cross section seen by an incoming neutron per target atom.

Our Doppler-broadened (effective) resonance cross section then becomes:

$$\begin{aligned}\bar{\sigma}_{na}(E_n) &= \int_{-\infty}^{\infty} dV w(V) \sigma_{na}(E_c) \sqrt{\frac{E_c m}{E_n \mu}} \\ &= \pi g \lambda_{co}^2 \Gamma_{no}^o \sqrt{\frac{E_{oc} m}{E_n \mu}} \cdot \frac{M}{2\pi kT} \cdot \int_{-\infty}^{\infty} dV \frac{\Gamma_{ac} \exp(-MV^2/2kT)}{\left(E_c - E_{oc}\right)^2 + \frac{1}{4} \left(\sqrt{E_c/E_{oc}} \Gamma_{nc}^o + \Gamma_{ac}^o + \dots\right)^2}\end{aligned}\quad (5)$$

The center-of-mass energy is $E_c = \frac{1}{2} \mu (v - V)^2$; the laboratory incident neutron energy is $E_n = \frac{1}{2} m v^2$. In order to use laboratory system parameters we define

$$E = \frac{m}{\mu} E_c \quad \text{so that} \quad \frac{E_c}{E_{co}} = \frac{E}{E_o}$$

and consequently

$$\Gamma_n = \frac{m}{\mu} \Gamma_{nc}, \quad \Gamma_n^o = \frac{m}{\mu} \Gamma_{nc}^o, \quad \Gamma_a = \frac{m}{\mu} \Gamma_{ac}$$

Now

$$\lambda_{co}^2 = \frac{p_{co}^2}{\hbar^2} = \left(\frac{m}{\mu}\right)^2 \frac{p_o^2}{\hbar^2} = \left(\frac{m}{\mu}\right)^2 \lambda_o^2$$

where p is momentum; thus,

$$E = \frac{1}{2} m(v - V)^2 = E_n - E_n^{\frac{1}{2}} \sqrt{2mV^2} + \frac{1}{2} mV^2$$

By neglecting the last term (which is appropriate, since in omitting the three-dimensional Maxwellian we have already omitted terms of this order),

$$(E_n - E)^2 \approx 4 \frac{m}{M} E_n \left(\frac{1}{2} MV^2 \right) = \frac{\Delta^2}{4} \cdot \frac{MV^2}{2kT}$$

With all this Eq. (5) becomes:

$$\bar{\sigma}_{na}(E_n) = \sqrt{4\pi} g \left(\frac{m}{\mu} \right)^2 \kappa_o^2 \frac{\Gamma_n^o}{\Delta} \sqrt{\frac{E_o}{E_n}} \int_{-\infty}^{\infty} dE e^{-\frac{4(E_n - E)^2}{\Delta^2}} \cdot \frac{\Gamma_a}{(E - E_o)^2 + \frac{1}{4} \left(\sqrt{E/E_o} \Gamma_n^o + \Gamma_a + \dots \right)^2} \quad (6)$$

laboratory system

Equation (6) is the formula to be approximated. As we remarked in Part I, in case the variations of level widths and $\sqrt{E}$ terms are neglected, we get Eq. (6) simplifying to:

$$\bar{\sigma}_{na}(E_n) = \sqrt{4\pi} \, g\left(\frac{m}{\mu}\right)^2 \kappa^2 \frac{\Gamma_n \Gamma_a}{\Delta} \int_{-\infty}^{\infty} \frac{dE \, e^{-\frac{4(E_n - E)^2}{\Delta^2}}}{(E - E_0)^2 + \frac{1}{4} \Gamma^2} \quad (7)$$

laboratory system

Equation (7) is a formula to be approximated, and is effectively the Doppler function, ψ , Eq. (1). We shall often find it convenient to operate on the integral only and so write

$$I(E_n) \equiv \int_{-\infty}^{\infty} dE \frac{e^{-\frac{4(E_n - E)^2}{\Delta^2}}}{(E - E_0)^2 + \frac{1}{4} \Gamma^2}$$

Thus, Eq. (7) becomes:

$$\bar{\sigma}_{na}(E_n) = \sqrt{4\pi} \, g\left(\frac{m}{\mu}\right)^2 \kappa^2 \frac{\Gamma_n \Gamma_a}{\Delta} I(E_n) \quad (7a)$$

laboratory system

Equation (7) is a Gauss transformation⁹ of the function

$$\frac{1}{(E - E_0)^2 + \frac{1}{4} \Gamma^2}$$

⁹W. Magnus and F. Oberhettinger, "Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik," p. 185, Verlag Julius Springer, Berlin (1948).

Before proceeding with the various approximations to Eqs. (6) and (7) offered in this paper, we note that Professor Fritz Oberhettinger has shown¹⁰ that Eq. (5) with the approximations of Eqs. (6) and (7), but not those of Eqs. (5) and (6), is expressible in terms of the error function of a complex argument. That is, if one makes the approximation of no variation of widths, wavelengths, or reaction rate factor over the resonance and converts to the laboratory system, then substituting in Eq. (5)

$$\tilde{x} \equiv \sqrt{\frac{ME}{m kT}}, \quad \tilde{y} \equiv \sqrt{\frac{ME_n}{m kT}}, \quad a \equiv \frac{M\Gamma}{2m kT}, \quad A \equiv \sqrt{\pi} g\lambda^2 \left(\frac{M}{\mu kT}\right)^2 \Gamma_n \Gamma_a,$$

leads to:

$$\bar{\sigma}_{na}(\tilde{y}) = A \int_{-\infty}^{\infty} \frac{d\tilde{x} e^{-(\tilde{x}-\tilde{y})^2}}{(\tilde{x}^2 - \tilde{x}_0^2)^2 + a^2}, \quad (8)$$

for which Professor Oberhettinger finds the solution:

¹⁰F. Oberhettinger, Oregon State University, private communication, April 1959. The author is grateful for Professor Oberhettinger's assistance. Some tables of the error function for complex argument exist, for example: "Tables of the Probability Integral for Complex Argument," Pergamon Press, London (1961).

$$\begin{aligned}
\bar{\sigma}_{na}(\tilde{y}) = & - \frac{\pi A}{4ai\gamma_1} \left\{ (\tilde{y} - \gamma_1) [-(\tilde{y} - \gamma_1)^2]^{-\frac{1}{2}} e^{-(\tilde{y}-\gamma_1)^2} \text{Erfc} \left[\sqrt{-(\tilde{y} - \gamma_1)^2} \right] \right. \\
& - (\tilde{y} + \gamma_1) [-(\tilde{y} + \gamma_1)^2]^{-\frac{1}{2}} e^{-(\tilde{y}+\gamma_1)^2} \text{Erfc} \left[\sqrt{-(\tilde{y} + \gamma_1)^2} \right] \left. \right\} \\
& + \frac{\pi A}{4ai\gamma_2} \left\{ (\tilde{y} - \gamma_2) [-(\tilde{y} - \gamma_2)^2]^{-\frac{1}{2}} e^{-(\tilde{y}-\gamma_2)^2} \text{Erfc} \left[\sqrt{-(\tilde{y} - \gamma_2)^2} \right] \right. \\
& - (\tilde{y} + \gamma_2) [-(\tilde{y} + \gamma_2)^2]^{-\frac{1}{2}} e^{-(\tilde{y}+\gamma_2)^2} \text{Erfc} \left[\sqrt{-(\tilde{y} + \gamma_2)^2} \right] \left. \right\} \quad (9)
\end{aligned}$$

where

$$\gamma_1 \equiv \sqrt{\tilde{x}_0^2 - ia}, \quad \gamma_2 \equiv \sqrt{\tilde{x}_0^2 + ia}, \quad \text{and} \quad \text{Erfc } u \equiv \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-t^2} dt.$$

For ease in mathematical manipulation we define $x \equiv (2/\Delta)(E - E_0)$, $y \equiv (2/\Delta)(E_n - E_0)$, and $c \equiv \Gamma/\Delta$ so that I of (7a) becomes

$$I(y) = \frac{2}{\Delta} \int_{-\infty}^{\infty} dx \frac{e^{-(x-y)^2}}{x^2 + c^2} \quad (10)$$

The solution for this integral is known from optics:¹¹

$$I = \frac{\pi}{\Gamma} \left\{ e^{(c-iy)^2} \text{Erfc}(c - iy) + e^{(c+iy)^2} \text{Erfc}(c + iy) \right\} \quad (11)$$

Expansion of the error functions in Eq. (11) is the content of Approximation β , described in Part IV of this report.

¹¹M. Born, "Optik," pp. 482-483, Verlag Julius Springer, Berlin (1933).
See also M. W. Zemansky, Phys. Rev. 36, 219 (1930).

III. APPROXIMATION α

1. Properties:

These approximations to Eqs. (6) and (7) are based upon the limiting forms of the equations. Thus for $\Delta \ll \Gamma$ - that is to say, for relatively low temperatures - we expect Eqs. (6) and (7) to be very nearly pure Breit-Wigner formulas, (approximation α_0). For $\Delta \gg \Gamma$ - that is, for relatively high temperatures - we expect Eqs. (6) and (7) to be close to pure Maxwellians, or Gaussians, which are approximations α_∞ . In each of these approximations we have two parameters to be determined: a width and a height. We use one to ensure that the total area of the resonance is identical for both the approximation and the approximated.¹² The other we can use to match results at any energy. We take it that the largest cross section is most important, and so our approximation matches Eqs. (6) and (7) there. We also offer matches at arbitrary points.

2. Convenience of Use:

Good to excellent.

¹²Doppler broadening preserves the area of an isolated resonance; see, for example, E. Amaldi, in "Encyclopedia of Physics," S. Flügge, ed., Verlag Julius Springer, Berlin (1959) 38/2, page 348.

3. Accuracy:

α_o : Fair (several per cent) for $\Gamma > \Delta$ and $(E - E_o) \lesssim \Gamma$, otherwise poor, and very poor for both $\Gamma < \Delta$ and $(E - E_o) \gg \Gamma$.

α_∞ : Fair for $\Delta > \Gamma$ and $(E - E_o) \lesssim \Gamma$, otherwise poor, and very poor for both $\Delta < \Gamma$ and $(E - E_o) \gg \Gamma$.

Both α_o and α_∞ are poor for $\Delta \sim \Gamma$.

4. Approximation α_o :

We use Eqs. (7) (constant widths, etc.) and then add the

$$\frac{1}{E_n^2} \text{ dependence}$$

of Eq. (6) external to the integral. For $E_n = E_o$, $I(E_o)$ of Eq. (7a) is easily integrated to give:

$$\bar{\sigma}_{na}(E_o) = 4\pi^{3/2} g\left(\frac{m}{\mu}\right)^2 \kappa_o^2 \frac{\Gamma_o \Gamma_o}{\Delta_o \Gamma_o} e^{\frac{\Gamma_o^2}{\Delta_o^2}} \text{Erfc}\left(\frac{\Gamma_o}{\Delta_o}\right) \quad (12)$$

with all quantities evaluated at E_o , as explicitly indicated by o.

Matching a Breit-Wigner expression to the area of Eq. (7) and to Eq. (12), plus inserting a $\frac{1}{v}$ dependence of $\kappa^2 \Gamma_n$ and any strong energy-dependence of Γ_a yields approximation α_o :

$$\bar{\sigma}_{na}(E_n) \approx \pi g \lambda_o^2 \left(\frac{m}{\mu}\right)^2 \sqrt{\frac{E_o}{E_n}} \frac{\Gamma_T}{\Gamma_o} \frac{\Gamma_n^o \Gamma_a}{(E_n - E_o)^2 + (\Gamma_T/2)^2} \quad (13)$$

approximation α_o

where

$$\Gamma_T \equiv \frac{\Delta_o}{\sqrt{\pi} e^{\Gamma_o^2/\Delta_o^2} \text{Erfc}(\Gamma_o/\Delta_o)} \quad (14)$$

approximation α_o

$$\left(\text{Erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \right)$$

Should the user not require equality at the maximum of Eqs. (6) or (7) which is approximation α_o = Eq. (13) plus Eq. (14), we offer approximation α_{ob} = Eq. (13) plus Eq. (15):

$$\Gamma_T = \Gamma_o \sqrt{1 + \frac{\Delta_o^2}{\Gamma_o^2} \frac{\ln b}{b-1}} \quad (15)$$

approximation α_{ob}

where b is a parameter that matches the original, unbroadened Breit-Wigner expression itself to a Gaussian at $\frac{1}{b}$ of maximum, but with the same area. The product of the two Gaussians is integrable and leads to Eq. (15). In Eq. (15), b acts to vary the point at which approximation and approximated are set equal. In particular for $b = 2$, Eq. (15) becomes:

$$\boxed{\Gamma_T = \sqrt{\Gamma_o^2 + \Delta_o^2} \ln 2} \quad (15a)$$

approximation α_{o2}

Equation (15a) plus Eq. (13), which is approximation α_{o2} , very quickly gives the slight change of a resonance broadened by $\Delta_o \ll \Gamma$ and is most accurate at $\frac{1}{2}$ of maximum.

5. Approximation α_{∞} :

We take a Gaussian, $\text{constant} \cdot \exp \left[\text{constant} \cdot (E_n - E_o)^2 \right]$ and require that the constants be determined by: 1) equal areas, and 2) equal maxima, which latter is given by Eq. (12), to obtain:

$$\boxed{\sigma_{na} \approx \text{Be} \cdot \frac{4(E_n - E_o)^2}{\pi^2 T}} \quad (16)$$

approximation α_{∞}

with

$$B \equiv 4\pi g \left(\frac{m}{\mu} \right)^2 \lambda_o^2 \frac{\Gamma_n^o \Gamma_a^o}{\Gamma_o \Gamma_T} \quad (17)$$

approximation α_{∞}

with Γ_T given by Eq. (14).

Similarly to α_o we may find it advantageous to match at $1/b$ of the maximum and get approximation $\alpha_{\infty b}$ = Eq. (18), Eq. (17), plus Eq. (15)

$$\bar{\sigma}_{na} \approx B \sqrt{\frac{\ln b}{b-1}} e^{-\frac{4 \left(\frac{\ln b}{b-1} \right) (E_n - E_o)^2}{\pi \Gamma_T}} \quad (18)$$

approximation $\alpha_{\infty b}$

where Γ_T is obtained from Eq. (15), not Eq. (14); and for the special case $b = 2$, $\alpha_{\infty 2}$ is:

$$\bar{\sigma}_{na} \approx 4\pi g \frac{m}{\mu}^2 \lambda_o^2 \frac{\Gamma_n^o \Gamma_a^o}{\Gamma_o \sqrt{\Gamma_o^2 + \Delta_o^2} \ln 2} e^{-\frac{4(\ln 2)(E_n - E_o)^2}{\pi \sqrt{\Gamma_o^2 + \Delta_o^2} \ln 2}} \quad (19)$$

approximation $\alpha_{\infty 2}$

IV. APPROXIMATION β

1. Properties:

The approximations are simply the standard expansions of the error functions (Ref. 9, p. 126, for example).

2. Convenience of Use:

Poor, limited to very low or very high temperatures and to energies close to resonance.

3. Accuracy:

β_0 : in principle, perfect. β_ω is an asymptotic expansion in the sense of Poincaré.

4. Approximation β_0 :

We expand $\text{Erfc}(c \pm iy)$ in Eq. (11) to get:

$$\begin{aligned}
I(y) &= \frac{2\sqrt{\pi}}{\Gamma} e^{c^2 - y^2} \cdot \\
&\cdot \left\{ \cos 2cy \left[\sqrt{\pi} - 2 \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{(-1)^{\ell+m} (2\ell)!}{\ell! (2m)! (2\ell+1-2m)!} \cdot y^{2m} c^{2\ell+1-2m} \right] \right. \\
&\quad \left. + 2 \sin 2cy \left[\sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{(-1)^m (2\ell)!}{\ell! (2m)! (2\ell+1-2m)!} y^{2\ell+1-2m} c^{2m} \right] \right\} \quad (20)
\end{aligned}$$

approximation β_0

which, with Eq. (7a), $y = \frac{2}{\Delta} (E_n - E_0)$, and $c \equiv \frac{\Gamma}{\Delta}$, is approximation β_0 .

5. Approximation β_{∞} :

Expanding $\text{Erfc}(c \pm iy)$ asymptotically in Eq. (11) for $c \rightarrow \infty$, we get:

$$\begin{aligned}
I(y) &\rightarrow \frac{\sqrt{\pi}}{\Gamma(c^2 + y^2)} \cdot \\
&\cdot \left\{ \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \frac{(-1)^{\ell+m} (2\ell-1)! (2\ell+1)!}{[2(c^2 + y^2)^2]^{\ell} (2\ell+1-2m)! (2m)!} c^{2\ell+1-2m} y^{2m} \right\} \quad (21)
\end{aligned}$$

approximation β_{∞}

where

$$(2l - 1)! \equiv 1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots \cdot 2l - 1 = \frac{2l!}{l! 2^l}$$

Equation (21) with Eq. (7a) is an asymptotic expression for $\bar{\sigma}$ in the sense of Poincaré and is approximation β_{∞} .

V. APPROXIMATION γ

1. Properties:

These approximations provide bounds on the integral $I(y)$ of Eq. (10) and approach it in the limit of infinite order. They are based upon the type substitution:

$$\exp v \geq \left(1 + \frac{v}{n}\right)^n \quad \text{in Eq. (10) with} \quad \exp v = \lim_{n \rightarrow \infty} \left(1 + \frac{v}{n}\right)^n.$$

2. Convenience of Use:

Poor after the first few orders.

3. Accuracy:

Good for high temperatures and energies close to resonance. Of course, in the limit of infinite order the accuracy to I is perfect for all T, E . Convergence fair to good.

4. Approximation $\bar{\gamma}$:

In Eq. (10) we replace

$$e^{(x-y)^2} \quad \text{by} \quad \left[1 + \frac{(x-y)^2}{n} \right]^n < e^{(x-y)^2}$$

We shall call n the order of the approximation $\bar{\gamma}$, or $\bar{\gamma}_n$. We thereby obtain an upper bound to I and a value for I in the limit $n \rightarrow \infty$:

$$I = \frac{2}{\Delta} \lim_{n \rightarrow \infty} n^n Q_n(y)$$

with

$$Q_n(y) \equiv \int_{-\infty}^{\infty} \frac{dx}{\left[(x-y)^2 + n \right]^n (x^2 + c^2)} \quad (22)$$

where for n not $\rightarrow \infty$, $Q_n(y)$ satisfies $I < \frac{2}{\Delta} n^n Q_n(y)$ and so provides an upper bound to I which approaches I in the limit $n \rightarrow \infty$. Therefore $\frac{2}{\Delta} n^n Q_n(y)$ is approximation $\bar{\gamma}_n$. We shall now give formulas for its evaluation. Writing $k \equiv \sqrt{n}$ in one place in $Q_n(y)$, we define

$$Q_n(y, k) \equiv \int_{-\infty}^{\infty} \frac{dx}{\left[(x-y)^2 + k^2 \right]^n (x^2 + c^2)} \quad (23)$$

where $Q_n(y) = Q_n(y, k) \Big|_{k=\sqrt{n}}$.

Then it is easy to see that the following recursion relation holds:

$$Q_{n+1}(y, k) = - \frac{1}{n} \frac{\partial Q_n(y, k)}{\partial k^2} \quad (24)$$

so that

$$Q_n(y) = Q_n(y, k) \Big|_{\sqrt{n}} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (k^2)^{n-1}} Q_1(y, k) \Big|_{\sqrt{n}} \quad (25)$$

approximation $\bar{\gamma}_n$

where the residue calculus yields easily from Eq. (23):

$$Q_1(y, k) = \frac{(c+k)}{ck} \cdot \frac{\pi}{y^2 + (c+k)^2} \quad (26)$$

approximation $\bar{\gamma}_n$

Approximation $\bar{\gamma}_n$,

$$\underline{I \approx \frac{2}{\Delta} n^n Q_n(y)},$$

is given by Eqs. (25) and (26). Also $\bar{\gamma}_n$ is an upper bound to I , and in the limit $n \rightarrow \infty$, $\gamma_n = I$; moreover $\bar{\gamma}_n$ is an upper bound to $\bar{\gamma}_{n+1}$ for any n .

What follows in this part are detailed calculations based on Eqs. (25) and (26). In Eq. (26) we define

$$f \equiv \frac{k + c}{k} ,$$

$$g \equiv \frac{1}{y^2 + (k + c)^2} ,$$

so that $Q_1(y, k) = \frac{\pi f g}{c}$, and the derivative

$$D \equiv \frac{\partial}{\partial k^2} .$$

Then after considerable manipulation it can be shown that:

$$Q_n(y, k) = \frac{\pi m}{c} (-1)^n \sum_{i=1}^n g^i (i-1)! .$$

$$\cdot (-1)^i \sum_{\substack{\text{possible} \\ \text{distributions} \\ \text{of } D^{n-i} \text{ over } f^i}} \frac{D^{(n-i)} f^i}{\Pi(P_D \text{ on } f + 1)! \Pi \left(\begin{array}{l} \text{number of times} \\ \text{each } (P + 1)! \\ \text{term is repeated} \end{array} \right)!} \quad (27)$$

with $P_D \text{ on } f$ being the power of D on f .

As an example of the notation in the last sum of Eq. (27) for $i = 2$ and for $n = 4$, consider

$$\sum_{\substack{\text{possibilities} \\ \text{[of Eq. (27)]}}} \frac{D^2 f^2}{\Pi(P_D \text{ on } f + 1)! \cdot \Pi(\text{number of repetitions in } \rightarrow)!}$$

$$= \frac{f D^2 f}{1! 3! \cdot 1! 1!} + \frac{(Df)^2}{(2!)^2 \cdot 2!}$$

It is easy to see that:

$$D^{(n)}_f = \frac{(-1)^n (2n-1)\phi_c}{2^n k^{2n+1}} \quad n \geq 1 \quad (28)$$

where

$$(2n-1)\phi \equiv \frac{(2n-1)!}{2^{n-1}(n-1)!}$$

is the odd factorial, sometimes ambiguously written $(2n-1)!!$. Equations (27) and (28) evaluate Eqs. (25) and (26), and we list a few such evaluations:

$$Q_2(k, y) = \frac{\pi}{2ck^3} \cdot \frac{[cy^2 + (2k + c)(k + c)^2]}{[y^2 + (k + c)^2]^2} \quad (29)$$

$$Q_3(k, y) = \frac{\pi}{8ck^5} \cdot \frac{[3cy^4 + 6cy^2(2k + c)(k + c) + (k + c)^3(8k^2 + 9kc + 3c^2)]}{[y^2 + (k + c)^2]^3} \quad (30)$$

$$Q_4(k, y) = \frac{\pi}{16ck^7} \cdot \frac{[5cy^6 + cy^4(23k^2 + 40kc + 15c^2) + cy^2(k + c)^2(47k^2 + 50kc + 15c^2) + (k + c)^2(16k^3 + 29k^2c + 20kc^2 + 5c^3)]}{[y^2 + (k + c)^2]^4} \quad (31)$$

$$Q_5(k, y) = \frac{\pi g}{4! c} \left\{ 4! f^5 g^4 + \frac{5 \cdot 3! c}{k^3} f^3 g^3 + \frac{15c}{2k} f g^2 [c + 2(k + c)] + \frac{3 \cdot 5c}{8k^8} g [2c + 5(k + c)] + \frac{3 \cdot 5 \cdot 7c}{16k^9} \right\} \quad (32)$$

$$Q_6(k, y) = \frac{\pi g}{5! c} \left\{ 5! f^6 g^5 + \frac{6! c}{2^2 k^3} f^4 g^4 + \right. \\
\left. \frac{3 \cdot 5 \cdot 3! c}{2^2 k^6} f^2 g^3 [4k + 7c] + \frac{3 \cdot 5 c}{2^2 k^9} g^2 [15k^2 + 42kc + 28c^2] + \right. \\
\left. \frac{3 \cdot 5 \cdot 7cg(6k + 9c)}{2^4 k^{10}} + \frac{3 \cdot 5 \cdot 7 \cdot 9c}{2^5 k^{11}} \right\} \quad (33)$$

etc.

5. Approximation γ :

If now we make use of the fact that

$$e^{-v} > \left(1 - \frac{v}{n}\right)^n \quad \text{for} \quad \left(\frac{v}{n}\right)^2 < 1 \quad \text{and}$$

$$e^{-v} = \lim_{n \rightarrow \infty} \left(1 - \frac{v}{n}\right)^n,$$

by replacing

$$e^{-(x-y)^2} \quad \text{in Eq. (10) by} \quad \left(1 - \frac{(x-y)^2}{n}\right)^n \quad \text{for} \quad \frac{(x-y)^2}{n} < 1$$

and 0 elsewhere we are led to define:

$$R_n(y) \equiv \int_{x=y-\sqrt{n}}^{y+\sqrt{n}} \frac{dx}{x^2 + c^2} \left[1 - \frac{(x-y)^2}{n} \right]^n \quad (34)$$

approximation γ_n

where R_n now has the property

$$I(y) > \frac{2}{\Delta} R_n(y) \quad \text{and} \quad I(y) = \frac{2}{\Delta} \lim_{n \rightarrow \infty} R_n(y),$$

and so R_n is an approximation to I of Eq. (10).

We shall list only two orders of this approximation:

$$R_1(y) = \frac{(1 + c^2 - y^2)}{c} \arctan \left(\frac{2c}{c^2 + y^2 - 1} \right) + y \ln \left| \frac{(y+1)^2 + c^2}{(y-1)^2 + c^2} \right| - 2 \quad (35)$$

$$R_2(y) = \left\{ \frac{\left[\left(1 + \frac{c^2 - y^2}{2} \right)^2 - y^2 c^2 \right]}{c} \arctan \frac{2\sqrt{2}c}{c^2 + y^2 - 2} + y \left[1 + \frac{c^2 - y^2}{2} \right] \cdot \ln \left| \frac{(y + \sqrt{2})^2 + c^2}{(y - \sqrt{2})^2 + c^2} \right| + \left[\frac{-5\sqrt{2}}{3} + \frac{3\sqrt{2}}{2} y^2 - \frac{\sqrt{2}}{2} c^2 \right] \right\}, \quad (36)$$

etc.

VI. APPROXIMATION δ

1. Properties:

If in Eq. (26) we expand in powers of

$$\frac{1}{\sqrt{y^2 + k^2}} = \frac{1}{\sqrt{y^2 + n}}$$

and apply Eq. (25), it turns out after considerable labor we can take the limit $n \rightarrow \infty$ and sum the resulting expressions into a semiclosed form, the series for which is approximation δ .

2. Convenience of Use:

Good for high temperatures (i.e., $\Gamma \ll \Delta$) and for E_n close to E_0 ; otherwise progressively poorer as these conditions fail to be met.

3. Accuracy:

In principle, perfect (to I), since the series represents the integral I of Eq. (10) exactly. The rate of convergence is fast for $\Gamma < \Delta$ and roughly as the expansion for

$$\cosh y^2 \quad \text{with} \quad y = \frac{2}{\Delta} (E_n - E_0) .$$

4. Approximation δ :

Approximation δ is an exact expression for $I(y)$ of Eq. (10) in terms of y (above in Eq. 3) and $c \equiv \frac{\Gamma}{\Delta}$:

$$I(y) = \frac{2\pi}{\Gamma} e^{c^2 - y^2} \left\{ \cos(2cy) - \frac{2}{\sqrt{\pi}} \int_0^c e^{\left(\frac{c^2 y^2}{s^2} - s^2\right)} ds \right\} \quad (37)$$

approximation δ

When expanded and coupled with Eq. (7a), Eq. (37) is approximation δ . For comparison we put Eq. (37) in the usual notation (see Rose, Miranker, and Leak; BNL 257 of footnote 1),

$$\Psi(x,t) \equiv \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{1+y^2} dy =$$

$$\frac{1}{\sqrt{t}} e^{(1-x^2)/4t} \left\{ \frac{\sqrt{\pi}}{2} \cos\left(\frac{x}{2t}\right) - \int_0^{\frac{1}{2\sqrt{t}}} e^{\left(\frac{x^2}{16t^2 s^2} - s^2\right)} ds \right\} \quad (37a)$$

Equation (37) is the following expansion:

$$I(y) = \frac{2\pi}{\Gamma} e^{-y^2} \left\{ e^{c^2} \cos(2cy) - \frac{2}{\sqrt{\pi}} \cdot \right.$$

$$\left. \begin{aligned} & + \left[\frac{c}{1\phi} + \frac{2c^3}{3\phi} + \frac{2^2 c^5}{5\phi} + \frac{2^3 c^7}{7\phi} + \dots \right] \\ & - \frac{y^2}{2!} \left[2c + \frac{2^2 c^3}{1\phi} + \frac{2^3 c^5}{3\phi} + \dots \right] \\ & + \frac{y^4}{4!} \left[-2^2 c + 2^3 c^3 + \frac{2^4 c^5}{1\phi} + \frac{2^5 c^7}{3\phi} + \dots \right] \\ & - \frac{y^6}{6!} \left[(-3)\phi 2^3 c + (-1)\phi 2^4 c^3 + 2^5 c^5 + \frac{2^6 c^7}{1\phi} + \dots \right] \\ & + \frac{y^8}{8!} \left[(-5)\phi 2^4 c + \dots \right] \\ & + \frac{y^{10}}{10!} \left[(-7)\phi 2^5 c + \dots \right] \end{aligned} \right\} \quad (38)$$

where, for example, $(+7)\phi$ is the odd factorial $1 \cdot 3 \cdot 5 \cdot 7$.

If Eq. (38) is rewritten with factorials expanded for computational purposes, then

$$I(y) = \frac{2\pi}{\Gamma} e^{-y^2} \left\{ e^{c^2} \cos(2cy) - 1.12837916 \cdot \right. \\ \left. + \left[c + (0.6666\cdots)c^3 + (0.2666\cdots)c^5 + (0.07619047619\cdots)c^7 + \right. \right. \\ \left. \left. (0.0169312169\cdots)c^9 + \cdots \right] \right. \\ - y^2 \left[c + 2c^3 + (1.333\cdots)c^5 + (0.5333\cdots)c^7 + (0.1523809523\cdots)c^9 + \cdots \right] \\ - y^4 \left[\frac{c}{6} - \frac{c^3}{3} - (0.666\cdots)c^5 - (0.444\cdots)c^7 - (0.1777\cdots)c^9 - \cdots \right] \\ - y^6 \left[\frac{c}{30} - \frac{c^3}{45} + (0.0444\cdots)c^5 + (0.0888\cdots)c^7 + (0.0592592\cdots)c^9 + \cdots \right] \\ - y^8 \left[\frac{c}{168} - \frac{c^3}{420} + \frac{c^5}{630} - \frac{c^7}{315} - (0.00634920634\cdots)c^9 - \cdots \right] \\ - y^{10} \left[\frac{c}{1080} - \frac{c^3}{3780} + \frac{c^5}{9450} - \frac{c^7}{14175} + \cdots \right] \\ - y^{12} \left[\frac{c}{7920} - \frac{c^3}{35640} + \frac{c^5}{124740} - \cdots \right] \\ - y^{14} \left[\frac{c}{65520} - \frac{c^3}{360360} + \cdots \right] \\ - y^{16} \left[\frac{c}{604800} - \frac{c^3}{3931200} + \cdots \right] \\ - y^{18} \left[\frac{c}{6168960} - \cdots \right] + \cdots \left. \right\}$$

(38a)

VII. BIBLIOGRAPHY

In addition to the papers of footnotes 1, 4, 5, 8, and 11, the following works may profitably be consulted.

- F. T. Adler and Y. D. Naliboff, Reactor Sci. and Tech. 14, 209 (1961) "A Direct Method for the Evaluation of the Resonance Line Shape Functions."
- H. J. Amster, Nucl. Sci. Eng. 11, 343 (1961) "A Differential Equation for Calculating Doppler Broadened Resonances."
- P. A. C. Buckler and I. C. Pull, United Kingdom Atomic Energy Establishment Report AEEW-R226, Winfrith (December 1962) "Doppler Broadening of Cross Sections." (See also V. J. Bell et al., footnote 1.)
- B. E. Clancy and A. Keane, Australian Atomic Energy Commission Report AAEC/E-117, Lucas Heights (March 1964) "Integrals Involving Doppler Broadened Contour Functions" (includes asymptotic series representations).
- L. Dresner, "Resonance Absorption in Nuclear Reactors," Pergamon Press, Inc., New York (1960).
- E. M. Gelbard, Trans. Am. Nucl. Soc. 6, 257 (1963) "The Computation of Doppler-Broadened Cross-Sections."
- D. R. Harris, Bettis Tech. Rev.-Reactor Techn., WAPD-BT-30, page 41 (April 1964), Westinghouse Electric Corp., Pittsburgh, "Differential Equation for Doppler-Broadened Cross Sections."
- G. W. Hinman, G. F. Kuncir, J. B. Sampson, and G. B. West, Nucl. Sci. Eng. 16, 202 (1963) "Accurate Doppler Broadened Absorption".
- A. Keane and M. H. McKay, Australian J. Appl. Sci. 11, 321 (1960) "A Useful Approximation in the Theory of Resonance Absorption" [by a material].

- A. Keane and B. E. Clancy, Aust. Math. Soc. J. 4, 354 (1964). [Extension of ψ and ϕ into the complex domain.]
- A. C. G. Mitchell and M. W. Zemansky, "Resonance Radiation and Excited Atoms," The Macmillan Co., New York, 1934, Chapt. IV, Sect. 2c, and Appendix page 320.
- D. M. O'Shea and H. C. Thacher, Trans. Am. Nucl. Soc. 6, 36 (1963).
[Represents ψ and χ as the real and imaginary parts of the integral,

$$w = (i/\pi) \int_{-\infty}^{\infty} e^{-t^2} / (z - t) dt.]$$
- A. Reichel, Nucl. Sci. Eng. 20, 547 (1964) "On the Evaluation of Doppler-Broadened Cross-Section Functions."
- A. Reichel, Reactor Sci. and Tech. 17, 281 (1963) "Temperature Dependence of Effective Resonance Integrals."
- R. D. Richtmeyer, R. VanNorton, and A. Wolfe, Proc. U. N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958, 16, 180 (1958).
- G. M. Roe, Knolls Atomic Power Laboratory Report KAPL-1241, General Electric Company (Oct. 1954).
- J. B. Sampson and G. W. Hinman, Nucl. Sci. Eng. 18, 531 (1964) "Comment on Doppler Broadened Absorption."
- A. W. Solbrig, Jr., Am. J. Phys. 29, 257 (1960).
- A. W. Solbrig, Jr., Nucl. Sci. Eng. 10, 167 (1961).