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**MAXIMUM LIKELIHOOD
NONLINEAR CORRELATED FIELDS
(BNW PROGRAM LIKELY)**

BERNARD H. DUANE

SEPTEMBER, 1967

**AEC RESEARCH &
DEVELOPMENT REPORT**

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PACIFIC NORTHWEST LABORATORY

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MAXIMUM LIKELIHOOD
NONLINEAR CORRELATED FIELDS
(BATTELLE-NORTHWEST PROGRAM LIKELY)

By

Bernard H. Duane

Theoretical Physics Section
Reactor Physics Department

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MAXIMUM LIKELIHOOD NONLINEAR CORRELATED FIELDS
(BATTELLE-NORTHWEST PROGRAM LIKELY)

Bernard H. Duane

ABSTRACT

Battelle-Northwest program Likely constructs maximum likelihood multidimensional vector fields satisfying any statistical-or-deterministic linear-or-nonlinear vector field theory cast as a variational extremal subject to correlated-statistic or implicit-anholonomic constraints.

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TABLE OF CONTENTS

ABSTRACT	iii-iv
LIST OF FIGURES.	vii-viii
LIST OF TABLES	vii-viii
SUMMARY.	1
Analysis Logic.	1
Input	2
Numerics	3
Output.	3
Availability	4
STATISTICAL PHYSICS.	5
Correlated Measurement Error	5
Statistical Physics of Transmutation Measurements	6
Maximum Likelihood Nonlinear Theory	10
Domain of Convergence	15
Maximum Likelihood Nonlinear Statistics	16
Orthogonal Resolution of Statistical Physics into Statistical Theory and Physical Theory	17
Principle of Maximum Likelihood	23
Propagation of Correlated Error	27
Orthogonal Resolution of Statistical Theory	29
Orthogonal Resolution of Physical Theory	35
NUMERICS	43
Numeric Sequence: Program Likely	43
Numeric Sequence: Program Learn	46
BITS	51-52
Input Logic: Program Likely	51-52
Sample Input: Program Likely	55-56
Sample Output: Program Likely.	61-62
Input Logic: Program Learn	75-76
Sample Input: Program Learn	79-80
Sample Output: Program Learn	83-84
ACKNOWLEDGEMENTS	95-96

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LIST OF FIGURES

1	Maximum Likelihood Nonlinear Vector Fields (Battelle-Northwest Program Likely)	9
2	Orthogonal Resolution of Statistical Physics into Statistical Theory and Physical Theory	22
3	Orthogonal Resolution of Statistical Theory into Radial Probability and Angular Probability	32
4	Orthogonal Synthesis of Statistical Theory and Dubious Physical Theory into Statistical Theory for Simplified Physical Theory	39
5	Sample Output: Program Likely	73
6	Sample Output: Program Likely	74
7	Sample Output: Program Learn	93
8	Sample Output: Program Learn	94

LIST OF TABLES

I	Misfit-Gradient Numeric Sequence: Program Likely	13
II	Output Numeric Sequence: Program Likely	43
III	Output Numeric Sequence: Program Likely	45
IV	Misfit-Gradient Numeric Sequence: Program Learn	46
V	Output Numeric Sequence: Program Learn	47
VI	Output Numeric Sequence: Program Learn	48

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SUMMARY

ANALYSIS LOGIC

The physical theory, to which the vector measurements $\underline{y}_m \pm U_m^{1/2}$ at phase-space coordinates $\underline{x}_m$ are to be fitted by variation of parameters $\underline{p}$, is cast as a set of implicit (or explicit) equations.

Theory: $q(\underline{y}, \underline{x}, \underline{p}) = 0.$

Implicit-theory flexibility plays a key role here: both by direct acceptance of any physical conservation or balance law with no need for intricate (or unfeasible) transformation; and also by free acceptance of implicit anholonomic constraints, implied merely by providing fewer field equations q than there are field variables $\underline{y}$.

The statistically weighted and projectively constrained construction of the mean square misfit Q forms the foundation of the analysis logic.

Mean Square Misfit	Measurement: m
$Q(\underline{p}) = \sum_{m=1}^M q(\underline{y}_m, \underline{x}_m, \underline{p})$	$\underline{x}$ -span: I
$\circ \left[\sum_{i=1}^J \sum_{j=1}^J \{ \partial q(\underline{y}_m, \underline{x}_m, \underline{p}) / \partial y_m^i \} \right]$	$\underline{y}$ -span: J
$U_m^{ij} \{ \partial q(\underline{y}_m, \underline{x}_m, \underline{p}) / \partial y_m^j \}]^{-1}$	$\underline{p}$ -span: K
$\circ q(\underline{y}_m, \underline{x}_m, \underline{p})$	q -span: L

This is the quantity to be minimized in achieving a best fit, and the optimality of its construction merits critical scrutiny in the context of each specific application.

The maximum-likelihood theory is constructed by solving the minimal-misfit conditions.

Best theory: $\partial Q / \partial \underline{p} = 0$, and $\partial^2 Q / \partial \underline{p} \partial \underline{p}$ positive definite.

These nonlinear partial differential equations are solved by iterating second-order analytic continuation to solid convergence.

Iterative algorithm: $\underline{p} = \underline{p} - (\partial Q / \partial \underline{p}) \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1}$.

INPUT

Input data, fed in either on free-format cards or via flexible input linkage, include: estimates and bounds for the theory parameters $\underline{p}$ (25 parameter limit); vector measurements $\underline{y}_m$ (100 vector limit, each with up to 5 components); corresponding phase-space coordinates $\underline{x}_m$ (if any); correlated measurement-uncertainty tensors U_m (100 tensor limit, each with up to 5 x 5 components); range and resolution of graphical output (up to 25 phase-space cross plots of 5-vector fields); and a theory subroutine for calculating the vector-misfit fields

$$q(\underline{y}, \underline{x}, \underline{p})$$

$$\partial q(\underline{y}, \underline{x}, \underline{p}) / \partial \underline{y}$$

and their first and second parametric gradients

$$\partial q(\underline{y}, \underline{x}, \underline{p}) / \partial \underline{p}$$

$$\partial^2 q(\underline{y}, \underline{x}, \underline{p}) / \partial \underline{y} \partial \underline{p}$$

$$\partial^2 q(\underline{y}, \underline{x}, \underline{p}) / \partial \underline{p} \partial \underline{p}$$

$$\partial^3 q(\underline{y}, \underline{x}, \underline{p}) / \partial \underline{y} \partial \underline{p} \partial \underline{p} \quad .$$

NUMERICS

The computational numerics have been quite fully optimized with balanced emphasis upon reliability, versatility, and speed. Numeric techniques typifying the state of the art include:

- Full use of numeric rounding, to minimize accrual of truncation bias.
- No iterative peripheral transmission, to enhance both reliability and speed.
- Self-evaluative sequence of ill-conditioned-matrix inversion logic, based upon use of two entirely different matrix-inversion methods. The iterative loops use a biggest-residual-pivot one-sided single-sweep matrix-inversion method designed for both reliability and speed. The saddle-decision logic uses a more versatile matrix-inversion method designed for both reliability and firm saddle detection, involving biggest-residual-blade two-sided single-sweep saddle-preserving transformation to principal blades, followed by blade inversion and inverse dual transformation. Output includes an evaluative cross-check assessing the adequacy of both matrix-inversion methods.
- Two-stage recovery logic for failure of second-order analytic continuation, branching first to cubic-expansion refinement of amplitude of last change, and then to reversing-and-halving of last change.
- Computation of Student and Fisher angular probability by angular Fourier transform.

OUTPUT

Output-processing versatility includes an exhaustive nonlinear statistical analysis of the residual misfit between measurements and theory, based upon nonlinear generalization of the small-sample angular-probability decision theory of

Student and Fisher, and culminating with probabilistically-uncoupled assessments of both the intrinsic quality of the measurements and the intrinsic relevance of the theory, as well as the adequacy of the agreement. Output logic is in harmony with remote-transmission needs, with numeric tabulation normally limited to a few pages having high information density, and with all bulky detail normally compressed to graphical form.

AVAILABILITY

The source deck consists of a 2000-card, 11-link overlay chain, formulated in the machine-independent dialectal intersection of IBM-7090 Fortran-IV-12 and Univac-1108 Fortran-IV-CSC, and designed for both modular growth versatility and freedom from obsolescence. Both program Likely and its scalar field predecessor program Learn are expected to be made available for unrestricted external distribution through the Argonne Code Center after completion of journal documentation. The theoretical physics development work was performed by Pacific Northwest Laboratory for the United States Atomic Energy Commission under the Plutonium Recycle Program.

STATISTICAL PHYSICS

CORRELATED MEASUREMENT ERROR

Any experimental physicist confronted with grim evidence that his measurements contain significant correlated error quite certainly would emerge (eventually) from perusal of dusty tomes on advanced mathematical statistics⁽¹⁾ with the following two fundamental equations at the top of his list of relevant ideas:

$$\underline{y} = \sum_{m=1}^M \underline{y}_m / M \quad . \quad (1)$$

$$<(\delta \underline{y})(\delta \underline{y})> = \sum_{m=1}^M (\underline{y}_m - \underline{y})(\underline{y}_m - \underline{y}) / M(M-1) \quad . \quad (2)$$

These equations differ sufficiently from their more familiar uncorrelated-error limit to merit brief discussion.

In the context of a motivating example, consider measurement of the concentrations (nuclei/volume) of four plutonium isotopes

$$N(^{239}\text{Pu}), N(^{240}\text{Pu}), N(^{241}\text{Pu}), N(^{242}\text{Pu})$$

within a fuel element of a nuclear reactor. Best accuracy in such measurements is obtainable by combining ion-separation techniques with isotope-separation techniques, and the effect of measurement errors may be so intricately interwoven as to defy unraveling. A set of measured quantities containing correlated error, exemplified by

(1) *M. G. Kendall. The Advanced Theory of Statistics, Hafner Publishing Company, New York, 1951-1952. vol. I-II.*

$$y^1 = N(^{239}\text{Pu}), y^2 = N(^{240}\text{Pu}), \dots, y^J = N(^{242}\text{Pu}),$$

is viewed as a statistical state vector

$$\underline{y} = (y^1, y^2, \dots, y^J)$$

Repetitive measurements of the statistical state,

$$\underline{Y}_1, \underline{Y}_2, \dots, \underline{Y}_M,$$

are averaged as in Equation 1 to form the best estimate of the mean state. The best estimate of the mean square uncertainty of the mean is the mean square correlated error tensor $\overline{(\delta \underline{y})(\delta \underline{y})}$, formed as in Equation 2 by dividing the average dyadic square of the statistical vector fluctuations by the fluctuational degrees of freedom $M - 1$.

Let's simply accept these fundamentals as manifestly firm, for now. Our goal here is to extend these basic ideas far beyond the bounds of existing knowledge. The exposition is swift, in a deliberate effort to convey realistically the incredible pace at which new ideas are spawned and developed to maturity by bold inductive inference, in an environment of compelling need.

STATISTICAL PHYSICS OF TRANSMUTATION MEASUREMENTS

Isotopic-transmutation measurement work on the Plutonium Recycle Test Reactor accrues rather precise measurements of isotope concentrations as an implicit function of rather imprecise measurements of neutron energy-spectrum exposure time. The approach to the analysis of such data has been to eliminate the imprecisely-known spectral exposure from direct consideration by casting the analysis theory in terms of interrelated isotope concentrations. This approach appears to have a useful range of

good applicability, limited principally by spectral-shift effects. But analysis progress on such data has been impeded by chronic lack of availability, in advanced statistics texts, of any statistical analysis method having sufficient versatility to span this sort of implicitly-imprecise correlated-error vector-measurement circumstance.

The slippery intricacy of this statistical theory-development problem is illustrated in Figure 1, which is a three-dimensional projection of a four-dimensional plot of plutonium-isotope concentration measurements in comparison with theoretical predictions. (The fourth axis $y^4 = N(^{242}\text{Pu})$ is not shown in the figure but is present in both measurements and theory.) Each measurement of the four plutonium-isotope concentrations (distilled from the raw data by either direct or indirect use of Equations 1 and 2) appears in such a plot as a four-vector $\underline{y}$ terminated by an ellipsoidal four-by-four measurement-uncertainty tensor $U = \langle (\delta \underline{y})(\delta \underline{y}) \rangle$. The simplest relevant nuclear transmutation theory predicts that such measurements, as an implicit function of exposure time, should cluster with vortex symmetry about a four-dimensional matrix-exponential corkscrew spiral,

$$q(\underline{y}, \underline{p}) = 0, \quad (3)$$

as shown in Figure 1.⁽²⁾ Statistical analysis logic strives to evaluate the theory parameters $\underline{p}$ by optimal minimization of the residual misfit between measurement and theory.

(2) A J -dimensional corkscrew spiral has the explicit matrix-exponential form

$$\underline{y}(t) = \underline{y}(0) \cdot e^{tH} + \underline{Q} \cdot (e^{tH} - 1)/H, \quad (4)$$

where $\underline{y}(t)$ is the isotope-concentration vector at exposure time t . The theory parameters $\underline{p}$ consist of the initial

Formulation of the relevant statistical physics as a set of precise mathematical equations suitable for electronic computer evaluation appears beneath Figure 1. The intricately-weighted construction of the mean square misfit Q , appearing as the second of these four equations, forms the foundation of the analysis logic. This is the quantity to be minimized in achieving a best fit, and the optimality of its construction merits critical scrutiny prior to application to a specific set of measurements.

The construction of this weighting logic can be verified at a glance by the following inductive line of thought, which first penetrated this baffling maze. The experimental physicist who made the measurements would form a contribution to the mean square misfit by inverse-variance weighting logic, which consists of contracting the dyadic square of any measurable discrepancy $\underline{dy}$ upon the matrix inverse of the mean square measurement-uncertainty tensor $U = \langle (\underline{\delta y})(\underline{\delta y}) \rangle$, in the form

$$(\underline{dy}) \cdot U^{-1} \cdot (\underline{dy}) = \sum_{i=1}^J \sum_{j=1}^J (dy^i) U_{ij}^{-1} (dy^j), \quad (5)$$

so as to assess discrepancy variance as a dimensionless-ratio fraction of the mean square measurement scatter. Transformed from the measurement coordinates $\underline{y}$ to the misfit coordinate $\underline{q}$, this expression becomes (formally)

(2 contd)

concentrations $\underline{y}(0)$, the source-sink vector $\underline{Q}$, and the transmutation-rate matrix H . The implicit form, Equation 3, consists of any nonredundant set of $J-1$ equations obtainable by algebraic elimination of the imprecisely-measurable exposure time t from the J -fold set, Equation 4.

The physical theory of nuclear transmutation is developed comprehensively in B.H. Duane, "Transmutation Logic for S_n Transport Analysis," pp. 11-19, HW-84369, Physics Research Quarterly Report, General Electric Hanford Laboratories, July-Aug-Sept., 1964.

MAXIMUM LIKELIHOOD NONLINEAR VECTOR FIELDS
(BATTELLE-NORTHWEST PROGRAM LIKELY)

Theory:

$$\underline{q}(\underline{y}, \underline{p}) = 0$$

Mean Square Misfit: $Q = \sum_{m=1}^M \underline{q}_m \cdot \left[\sum_{i=1}^J \sum_{j=1}^J (\partial \underline{q}_m / \partial y_m^i) U_m^{ij} (\partial \underline{q}_m / \partial y_m^j) \right]^{-1} \cdot \underline{q}_m$

Best Theory:

$$\partial Q / \partial \underline{p} = 0 \text{ and } \partial^2 Q / \partial \underline{p} \partial \underline{p} \text{ Positive Definite}$$

Tangentially-Averaged Normal Probability:

$$dP = e^{-Q/2\sigma^2} (Q/2\sigma^2)^{(L \cdot M - K - 1)/2} d(Q/2\sigma^2)^{1/2} / \left[\{(L \cdot M - K)/2 - 1\} ! / 2 \right]$$

FIGURE 1. Maximum Likelihood Nonlinear Vector Fields
(Battelle-Northwest Program Likely)

$$(d\mathbf{q})^T \left[\sum_{i=1}^J \sum_{j=1}^J (\partial \mathbf{q} / \partial y^i) U^{ij} (\partial \mathbf{q} / \partial y^j) \right]^{-1} (d\mathbf{q}) \quad (6)$$

Since the theory has been cast in the implicit form $\mathbf{q}(\underline{y}, \underline{p}) = 0$, the misfit differentials

$$d\mathbf{q} = \sum_{j=1}^J [\partial \mathbf{q}(\underline{y}, \underline{p}) / \partial y^j] dy^j$$

span only the $(J-1)$ -dimensional vortex subspace normal to the theory curve. Consequently, the formal transformation, Equation 6, is in fact a degenerate form which ignores tangential discrepancies. But this is precisely what is needed here: transformation of the imprecisely-known spectral exposure time to an ignorable coordinate for which the data contain little significant information. Replacement of $d\mathbf{q}$ by $\mathbf{q}$, to simplify the notation, and summing the normal mean square misfits, Equation 6, over all measurements m , now completes the construction of the statistically-weighted, projectively-constrained mean square misfit Q appearing beneath Figure 1.

Mentally distilling the essence of this tenuous but firm line of reasoning leads to the far more general abstract formulation appearing in the "Analysis Logic" section at the beginning of the Summary. At this vast level of generality, potential applications range so broadly through all fields of scientific endeavor as to truly stagger the imagination.

MAXIMUM LIKELIHOOD NONLINEAR THEORY

The preceding section, in résumé, completes the first (and creatively most elusive) of four major phases of this theory-development work, by casting statistical physics as a variational extremal of the statistically weighted and projectively constrained functional form

$Q(\underline{p}) = \sum_{m=1}^M q(\underline{y}_m, \underline{x}_m, \underline{p})$ $\circ \left[\sum_{i=1}^J \sum_{j=1}^J \{ \partial q(\underline{y}_m, \underline{x}_m, \underline{p}) / \partial y_m^i \} \right]$ $U_m^{ij} \{ \partial q(\underline{y}_m, \underline{x}_m, \underline{p}) / \partial y_m^j \}]^{-1}$ $\circ q(\underline{y}_m, \underline{x}_m, \underline{p}) \quad (7)$	Measurement: m $\underline{x}$ -span: I $\underline{y}$ -span: J $\underline{p}$ -span: K $\underline{q}$ -span: L
--	--

The second theory-development phase, into which we now plunge, constructs the maximum-likelihood nonlinear theory

$$\boxed{q(\underline{y}, \underline{x}, \underline{p}) = 0} \quad , \quad (8)$$

by solving the minimal-misfit conditions

$$\boxed{\partial Q(\underline{p}) / \partial \underline{p} = 0, \text{ and}} \quad (9)$$

$$\boxed{\partial^2 Q(\underline{p}) / \partial \underline{p} \partial \underline{p} \text{ positive definite.}} \quad (10)$$

The unprecedented versatility of this analysis imposes no restrictions whatever on the complexity of the physical theory, Equation 8, beyond mere existence of the defining expressions, Equations 7, 9, and 10, over the domain of application. The implicit (or explicit) theory Equation 8 may contain both arbitrarily intricate nonlinearity in all variables and arbitrarily severe degeneracy (implied by providing fewer field equations $q = 0$ than there are field variables $\underline{y}$).

These nonlinear partial differential equations are solved by iterating second-order analytic continuation to solid convergence. The basic logic here is quite simple. Solving the differential expansion of the extremal conditions, Equation 9,

$$\partial Q / \partial \underline{p} + (d\underline{p}) \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p}) \doteq 0 \quad ,$$

for the parameter differentials $d\underline{p}$ by the matrix inversion

$$d\underline{p} = - (\partial Q / \partial \underline{p}) \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1}$$

generates the recursive algorithm

$$\underline{p} = \underline{p} - (\partial Q / \partial \underline{p}) \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1} \quad . \quad (11)$$

After each such parameter change, the mean square misfit Q is computed from Equation 7 and compared with its previous least value. A drop in mean square misfit, indicative of a convergent trend, closes the iterative loop by recursive traverse of the refinement algorithm, Equation 11. A rise in mean square misfit, indicative of a divergent trend, branches the logic first to cubic expansion refinement of the amplitude of the last parameter change and then to reversing-and-halving of the last change, to provide two-stage recovery effort. The occurrence of any spill, division by zero, or noninvertible matrix aborts the run. Solid agreement of the mean square misfit with its previous least value terminates the iteration. Output includes the normal principal blades of the misfit second gradient $\partial^2 Q / \partial \underline{p} \partial \underline{p}$, to provide decisive assessment of the positive-definite requirement, Equation 10.

The tensor calculus detail of Table I helps to convey these ideas to an electronic computer.

TABLE I. *Misfit-Gradient Numeric Sequence: Program Likely
(Deducible mentally from Equation 7)*

Step-1: Evaluate mean square misfit Q. (Measurements: M, y-span: J, p-span: K, q-span: L)

$$W_m = \left[\sum_{i=1}^J \sum_{j=1}^J (\partial q_m / \partial y_m^i) U_m^{ij} (\partial q_m / \partial y_m^j) \right]^{-1} \quad , \quad \text{q-space statistical weight} \\ \text{(LxL dyad).} \quad (12)$$

$$Q = \sum_{m=1}^M q_m \circ W_m \circ q_m \quad . \quad \text{(On this page, open dot } \circ \text{ means} \\ \text{q-space inner product)} \quad (13)$$

Step-2: Evaluate misfit gradient $\partial Q / \partial p$. (Matrix identity: $dW = -W \circ (dW^{-1}) \circ W$)

$$\partial W_m / \partial p = -W_m \circ \sum_{i=1}^J \sum_{j=1}^J [(\partial^2 q_m / \partial p \partial y_m^i) U_m^{ij} (\partial q_m / \partial y_m^j) \\ + (\partial q_m / \partial y_m^i) U_m^{ij} (\partial^2 q_m / \partial y_m^j \partial p)] \circ W_m \quad . \quad (14)$$

$$\partial Q / \partial p = \sum_{m=1}^M [(\partial q_m / \partial p) \circ W_m \circ q_m + q_m \circ (\partial W_m / \partial p) \circ q_m \\ + q_m \circ W_m \circ (\partial q_m / \partial p)] \quad . \quad (15)$$

TABLE I. (contd)

Step-3: Evaluate misfit second gradient $\partial^2 Q / \partial p \partial p$.

$$\begin{aligned} \partial^2 W_m / \partial p \partial p = & (\partial W_m / \partial p) \circ W_m^{-1} \circ (\partial W_m / \partial p) + (\partial W_m / \partial p) \circ W_m^{-1} \circ (\partial W_m / \partial p) - W_m \circ \sum_{i=1}^J \sum_{j=1}^J \\ & [(\partial^3 q_m / \partial p \partial p \partial y_m^i) U_m^{ij} (\partial q_m / \partial y_m^j) + 2(\partial^2 q_m / \partial p \partial y_m^i) U_m^{ij} (\partial^2 q_m / \partial y_m^j \partial p) \\ & + (\partial q_m / \partial y_m^i) U_m^{ij} (\partial^3 q_m / \partial y_m^j \partial p \partial p)] \circ W_m \quad . \end{aligned} \quad (16)$$

$$\begin{aligned} \partial^2 Q / \partial p \partial p = & \sum_{m=1}^M [(\partial^2 q_m / \partial p \partial p) \circ W_m \circ q_m + 2(\partial q_m / \partial p) \circ (\partial W_m / \partial p) \circ q_m \\ & + 2(\partial q_m / \partial p) \circ W_m \circ (\partial q_m / \partial p) + q_m \circ (\partial^2 W_m / \partial p \partial p) \circ q_m + 2q_m \circ (\partial W_m / \partial p) \\ & \circ (\partial q_m / \partial p) + q_m \circ W_m \circ (\partial^2 q_m / \partial p \partial p)], \text{ triangle storage.} \end{aligned} \quad (17)$$

DOMAIN OF CONVERGENCE

Precise mathematical delineation of the domain of convergence for the iterative second-order analytic continuation method,

$$\underline{p} = \underline{p} - (\partial Q / \partial \underline{p}) \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1} \quad ,$$

has not yet progressed significantly beyond the rudimentary observation that one iteration generates the exact solution of the variational extremal

$$\partial Q / \partial \underline{p} = 0$$

if the third gradient $\partial^3 Q / \partial \underline{p} \partial \underline{p} \partial \underline{p}$ vanishes. But the rapid terminal convergence rate inferable from this observation does provide an unusually firm empirical convergence decision, by enhancing the practicality of driving the run to solid convergence. Further, the clean simplicity and vast generality of this iterative logic offers hope of stimulating creative effort to evolve a domain-of-convergence theorem.

Meanwhile, several relevant rules of thumb have emerged from accrued experience.

Rule-of-Thumb 1: If the principal blades of the misfit second gradient $\partial^2 Q / \partial \underline{p} \partial \underline{p}$ are positive-definite and well above the level of truncation error (in an appropriate dimensionless-ratio sense), then the run is likely to converge quickly and firmly within half a dozen iterations. Both program Likely and program Learn output the principal blades of the misfit second gradient, to provide a firm basis for positive-definiteness decisions.

Rule-of-Thumb 2: If any nonpositive-definite principal blades of the misfit second gradient lie well above truncation-error level (in an appropriate dimensionless-ratio sense),

then it is likely that better parameter estimates are needed to get away from a saddle point.

Rule-of-Thumb 3: If any principal blades of the misfit second gradient lie within the realm of truncation-error smallness (in an appropriate dimensionless-ratio sense), then it is likely that parameter indeterminacy is present which needs to be alleviated either by simplification of the theory or by extension of the measurements.

Rule-of-Thumb 4: If numeric trends seem intricately interwoven beyond comprehension, try calling for graphical output each iteration, to discern what is happening.

MAXIMUM-LIKELIHOOD NONLINEAR STATISTICS

Summarized briefly, the preceding sections construct the physical theory which best fits the measurements. The third major phase of this work, which we now begin, constructs the statistical theory which best fits the residual discrepancies.

Statistical physics theorizes that, if the entire experiment were rerun many times, then individual runs would cluster asymptotically about their grand mean with the normal probability distribution

$$dP = e^{-Q(\underline{P})/2\sigma^2} [Q(\underline{P})/2\sigma^2]^{(L \cdot M - 1)/2} d[Q(\underline{P})/2\sigma^2]^{1/2} d\Omega(L \cdot M - 1)/\pi^{L \cdot M/2} \quad (18)$$

The volume element of the $(L \cdot M)$ -dimensional phase space spanned by the mean square misfit function Q of Equation 7 appears here in local polar geodesic coordinates, as the product of the radial phase element $[Q(\underline{P})/2\sigma^2]^{1/2} d[Q(\underline{P})/2\sigma^2]^{1/2}$ and the $(L \cdot M - 1)$ -dimensional solid angle element $d\Omega(L \cdot M - 1)$. The probability element dP has been

normalized to integrate to unity.⁽³⁾ The rerun measurement variance σ^2 and the true theory parameters $\underline{P}$ remain to be determined, by seeking the best statistical fit to residual discrepancies.

ORTHOGONAL RESOLUTION OF STATISTICAL PHYSICS INTO STATISTICAL THEORY AND PHYSICAL THEORY

The true mean square misfit $Q(\underline{P})$ appearing in the statistical theory, Equation 18, can be related to the observed mean square misfit $Q(\underline{p})$ appearing in the variational extremal Equation 7 by the second-order Taylor expansion

$$Q(\underline{P}) = Q(\underline{p}) + (1/2) [\partial^2 Q(\underline{p}) / \partial p \partial p] : (\underline{P} - \underline{p}) (\underline{P} - \underline{p}) + \dots, \quad (23)$$

(3) *The tabulated integrals* (R. S. Burington, Handbook of Mathematical Tables and Formulas, Handbook Publisher Inc., Sandusky, Ohio, 1943),

$$\int_0^\infty e^{-t} t^z dt = z! \quad , \text{ and} \quad (19)$$

$$\int_0^\pi \sin^n \theta d\theta = \pi^{1/2} [(n-1)/2]! / (n/2)! \quad , \text{ yield} \\ \text{respectively :} \quad (20)$$

$$\int_0^\infty e^{-Q/2\sigma^2} [Q/2\sigma^2]^{(L \cdot M - 1)/2} d[Q/2\sigma^2]^{1/2} = \underbrace{[(L \cdot M - 2)/2]! / 2}_{\text{and}} \quad , \quad (21)$$

$$\int d\Omega (L \cdot M - 1) = \int_0^\pi \sin^{L \cdot M - 2} \theta_{L \cdot M - 2} d\theta_{L \cdot M - 2} \dots \\ \int_0^\pi \sin^2 \theta_2 d\theta_2 \int_0^\pi \sin \theta_1 d\theta_1 \int_{-\pi}^\pi d\theta_0 \quad , \\ = 2\pi^{L \cdot M / 2} / [(L \cdot M - 2)/2]! \quad . \quad (22)$$

with first-order interaction $[\partial Q(\underline{p})/\partial \underline{p}] \cdot (\underline{P} - \underline{p})$ vanishing over the entire rerun phase space by virtue of the variational extremal condition $\partial Q(\underline{p})/\partial \underline{p} = 0$ of Equation 9.* This observation strongly suggests that the variational extremal formulation of the physical theory has generated some sort of deep-seated functional orthogonality in the statistical theory.

Pursuing this intriguing thought leads to returning briefly to the iterative algorithm, Equation 11, to calculate the measurement-space gradient of the theory parameters. With full mathematical rigor for any first-order variation whatsoever, the refinement algorithm

$$\underline{p} = \underline{p} - (\partial Q/\partial \underline{p}) \cdot (\partial^2 Q/\partial \underline{p} \partial \underline{p})^{-1} \text{ yields}$$

$$\delta \underline{p} = - [\delta(\partial Q/\partial \underline{p})] \cdot (\partial^2 Q/\partial \underline{p} \partial \underline{p})^{-1} - (\partial Q/\partial \underline{p}) \cdot \delta[(\partial^2 Q/\partial \underline{p} \partial \underline{p})^{-1}];$$

and since the last term vanishes ($\partial Q/\partial \underline{p} = 0$), the parameter gradient in measurement space can be read off the first term as

$$\partial \underline{p}/\partial y_m^j = - (\partial^2 Q/\partial y_m^j \partial \underline{p}) \cdot (\partial^2 Q/\partial \underline{p} \partial \underline{p})^{-1} . \quad (24)$$

This surprisingly simple result, evaluable numerically by referring to Equation 7, proves to be the analysis wedge needed to crack nonlinear statistics.

* It is crucially important here to grasp that validity of the normal statistical theory, Equation 18, hinges upon construing the term "rerun" in the comprehensive sense: "Go back to the laboratory. Start over. Make a new set of measurements. Construct the new physical-theory parameters which best fit the new data."

For now the concept of rerun gradient, describing functional variations inducible by rerun-variation of both the measurements and the physical theory and needed to investigate functional orthogonality, can be cast as the explicit differential operator

$$\nabla_j^m = \partial/\partial y_m^j + (\partial \underline{p}/\partial y_m^j) \cdot \partial/\partial \underline{p} \quad . \quad (25)$$

To grasp the significance and power of this concept, consider the rerun gradient of the misfit gradient $\partial Q/\partial \underline{p}$. Referring to Equation 7 for relevant functional structure yields, mentally,

$$\begin{aligned} \nabla_j^m(\partial Q/\partial \underline{p}) &= \partial^2 Q/\partial y_m^j \partial \underline{p} + (\partial \underline{p}/\partial y_m^j) \cdot (\partial^2 Q/\partial \underline{p} \partial \underline{p}) \quad , \\ &= \partial^2 Q/\partial y_m^j \partial \underline{p} - \partial^2 Q/\partial y_m^j \partial \underline{p} \quad , \text{ using (24),} \\ &= 0, \end{aligned}$$

in manifest harmony with rerun satisfaction of the variational extremal $\partial Q/\partial \underline{p} = 0$.

Returning again to the expansion, Equation 23, and forming the inner product of its rerun gradients to study functional orthogonality, yields through second-order terms:

$$\begin{aligned} &\sum_{m=1}^M \sum_{i=1}^J \sum_{j=1}^J [\nabla_i^m Q(\underline{p})] U_m^{ij} [\nabla_j^m \{ (1/2) (\partial^2 Q(\underline{p})/\partial \underline{p} \partial \underline{p}) : (\underline{p} - \underline{p})(\underline{p} - \underline{p}) \}] \\ &= \sum_{m=1}^M \sum_{i=1}^J \sum_{j=1}^J [2 \underline{q}_m \circ W_m \circ \partial \underline{q}_m / \partial y_m^i] U_m^{ij} [(\partial \underline{p}/\partial y_m^j) \cdot (\partial^2 Q/\partial \underline{p} \partial \underline{p}) \cdot (\underline{p} - \underline{p})] \quad , \\ &= \sum_{m=1}^M \sum_{i=1}^J \sum_{j=1}^J [2 \underline{q}_m \circ W_m \circ \partial \underline{q}_m / \partial y_m^i] U_m^{ij} [(\partial^2 Q/\partial y_m^j \partial \underline{p}) \cdot (\underline{p} - \underline{p})] \quad , \end{aligned}$$

$$\begin{aligned}
&= 4 \sum_{m=1}^M \underline{q}_m \circ W_m \circ \left[\sum_{i=1}^J \sum_{j=1}^J (\partial \underline{q}_m / \partial y_m^i) U_m^{ij} (\partial \underline{q}_m / \partial y_m^j) \right] \circ W_m \\
&\quad \circ (\partial \underline{q}_m / \partial \underline{p}) \cdot (\underline{P} - \underline{p}) \quad , \\
&= 4 \sum_{m=1}^M \underline{q}_m \circ W_m \circ W_m^{-1} \circ W_m \circ (\partial \underline{q}_m / \partial \underline{p}) \cdot (\underline{P} - \underline{p}) = 4 \sum_{m=1}^M \underline{q}_m \circ W_m \\
&\quad \circ (\partial \underline{q}_m / \partial \underline{p}) \cdot (\underline{P} - \underline{p}) \quad , \\
&= 2 [\partial Q(\underline{p}) / \partial \underline{p}] \cdot (\underline{P} - \underline{p}) = 0 \quad , \text{ since } \partial Q / \partial \underline{p} = 0 \quad . \quad (26)
\end{aligned}$$

What has been proven here is that the mean square misfit $Q(\underline{p})$ and its second variation $(1/2)[\partial^2 Q(\underline{p}) / \partial \underline{p} \partial \underline{p}] : (\underline{P} - \underline{p})(\underline{P} - \underline{p})$ possess second-order functional orthogonality over the entire rerun phase space.

Since the statistical physics underlying the normal distribution, Equation 18, becomes a bit dubious beyond second order, the probability element dP can be factored without further ado into functionally orthogonal subspaces spanning respectively the mean square misfit $Q(\underline{p})$ and its second variation

$$\delta^2 Q(\underline{p}) = (1/2)[\partial^2 Q(\underline{p}) / \partial \underline{p} \partial \underline{p}] : (\underline{P} - \underline{p})(\underline{P} - \underline{p}),$$

(27)

in the statistically uncoupled form

$$\begin{aligned}
 dP = & e^{-Q(\underline{p})/2\sigma^2} [Q(\underline{p})/2\sigma^2]^{(L \cdot M - K - 1)/2} d[Q(\underline{p})/2\sigma^2]^{1/2} \\
 & \cdot d\Omega(L \cdot M - K - 1) \pi^{-(L \cdot M - K)/2} e^{-\delta^2 Q(\underline{p})/2\sigma^2} [\delta^2 Q(\underline{p})/2\sigma^2]^{(K-1)/2} \\
 & \cdot d[\delta^2 Q(\underline{p})/2\sigma^2]^{1/2} d\Omega(K-1)/\pi^{K/2} .
 \end{aligned} \tag{28}$$

The volume elements for each normal subspace appear here in local polar geodesic coordinates. The probability elements have been normalized to integrate to unity within each normal subspace.

What now has been accomplished is an orthogonal resolution of statistical physics into statistical theory, involving only the mean square misfit $Q(\underline{p})$, and physical theory, involving only its second variation $\delta^2 Q(\underline{p})$. The sketch in Figure 2 strives to convey this idea geometrically, and foreshadows in spirit the central theme to pursue: explicit further orthogonal resolution of these statistically normal subspaces into normal sub-subspaces.

To clarify these concepts before proceeding, however, let us calculate a few things of interest from the factored probability distribution, Equation 28. Since the observed mean square misfit $Q(\underline{p})$ appears only in the first factor, its expectation value falls out neatly as

$$\langle Q(\underline{p}) \rangle = \frac{\int_0^\infty Q e^{-Q/2\sigma^2} (Q/2\sigma^2)^{(L \cdot M - K - 1)/2} d(Q/2\sigma^2)^{1/2}}{\int_0^\infty e^{-Q/2\sigma^2} (Q/2\sigma^2)^{(L \cdot M - K - 1)/2} d(Q/2\sigma^2)^{1/2}} ,$$

FIGURE 2. Orthogonal Resolution of Statistical Physics into Statistical Theory and Physical Theory

$$\begin{aligned}
 &= 2 \sigma^2 [(L \cdot M - K)/2]! / [(L \cdot M - K - 2)/2]! \quad , \text{ from Equation 21,} \\
 &= \sigma^2 (L \cdot M - K) \quad . \quad (29)
 \end{aligned}$$

Since the misfit second variation $\delta^2 Q(\underline{p})$ appears analogously in the second factor, its expectation value falls out mentally as

$$\langle \delta^2 Q(\underline{p}) \rangle = \sigma^2 K \quad . \quad (30)$$

These results reveal a general property of orthogonal resolution worth remembering. The expectation value of each normal component of the mean square misfit is equal to its degrees of freedom times the rerun measurement variance.

PRINCIPLE OF MAXIMUM LIKELIHOOD

The rerun measurement variance σ^2 and the true theory parameters $\underline{p}$ remain to be evaluated, by seeking the best statistical fit to residual discrepancies. The fundamental idea here has its roots in the old adage, "Don't put all your eggs in one basket." Paraphrased axiomatically, this thought emerges as the thermodynamicist's Principle of Maximum Entropy, the quantum mechanician's Principle of Equipartition of States, the experimentalist's Principle of Equipartition of Ignorance, and the statistician's Principle of Maximum Likelihood. Phrased adroitly in transformation-invariant language, the Principle of Maximum Likelihood approaches the subtle uniqueness of a definitive axiom inaccessible to experimental disproof.

PRINCIPLE OF MAXIMUM LIKELIHOOD

Maximizing the rerun probability element $dP(\underline{s})$ with respect to variation of its statistical-theory parameters $\underline{s}$, by solving the variational extremal conditions

$$\partial dP(\underline{s}) / \partial \underline{s} = 0 \quad , \text{ and} \quad (51)$$

$$\partial^2 dP(\underline{s}) / \partial \underline{s} \partial \underline{s} \text{ negative definite} \quad , \quad (52)$$

subject both to the normalization constraint $\int dP = 1$ and to all relevant physical-theory constraints, yields the maximum likelihood statistical fit to residual discrepancies.

Glancing back at the functional dependence of the probability element dP of Equation 28 upon its statistical-theory parameters $\underline{s} = (\sigma^2, \underline{p})$ leads to shifting the physical-theory factor to $\underline{p}$ -space Cartesians, to obtain

$$\begin{aligned} dP = & e^{-Q/2\sigma^2} [Q/2\sigma^2]^{(L \cdot M - K - 1)/2} d(Q/2\sigma^2)^{1/2} \\ & \cdot d\Omega(L \cdot M - K - 1) \pi^{-(L \cdot M - K)/2} e^{-(1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) : (\underline{p} - \underline{p}) (\underline{p} - \underline{p}) / 2\sigma^2} \\ & \cdot |(1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\pi\sigma^2|^{1/2} \prod_{k=1}^K dp^k \quad . \end{aligned} \quad (35)$$

Performing the maximum likelihood variation with respect to $\underline{p}$ now gives

$$\partial dP / \partial \underline{P} = (1/2\sigma^2) (\partial^2 Q / \partial \underline{p} \partial \underline{p}) \cdot (\underline{p} - \underline{P}) \, dP = 0, \text{ or } \boxed{\underline{P} \doteq \underline{p}} \quad , \quad (34)$$

$$\text{and } \partial^2 dP / \partial \underline{P} \partial \underline{P} = - (1/2\sigma^2) (\partial^2 Q / \partial \underline{p} \partial \underline{p}) \, dP \quad . \quad (35)$$

The positive-definite misfit minimum, Equation 10, implies negative-definite second variations, Equation 35, and thus establishes the likelihood extremal as a maximum. The maximum likelihood estimates ($\doteq$) of the true theory parameters $\underline{P}$ coalesce neatly with the best-fit parameters $\underline{p}$. The symmetry of Equation 35 yields $\langle \underline{p} \rangle = \underline{P} + \langle \underline{p} - \underline{P} \rangle = \underline{P}$, in display of the unbiased character of these parameter estimates.

Performing the variation with respect to σ^2 reveals two perplexing alternatives. Construing the Principle of Maximum Likelihood as referring to detailed uncoupled maximization of the statistical-theory probability element and of the physical-theory probability element yields

$$\partial dP / \partial (1/2\sigma^2) = [-Q + (L \cdot M - K)\sigma^2] \, dP = 0, \text{ or } \boxed{\sigma^2 \doteq Q(\underline{p}) / (L \cdot M - K)} \quad , \quad (36)$$

$$\text{and } \partial^2 dP / \partial (1/2\sigma^2) \partial (1/2\sigma^2) = - 2(L \cdot M - K)\sigma^4 \, dP \quad . \quad (37)$$

Gross coupled maximization of the complete probability element yields, in contrast,

$$\partial dP / \partial (1/2\sigma^2) = [-Q + (L \cdot M)\sigma^2] \, dP = 0 \quad , \text{ or } \sigma^2 \doteq Q(\underline{p}) / (L \cdot M) \quad , \quad (38)$$

$$\text{and } \partial^2 dP / \partial (1/2\sigma^2) \partial (1/2\sigma^2) = - 2(L \cdot M)\sigma^4 \, dP \quad ,$$

$$\partial^2 dP / \partial (1/2\sigma^2) \partial \underline{P} = 0 \quad .$$

The negative-definite second variations establish both extrema as maxima. Comparative scrutiny shows that the detailed probability maximum, Equation 36, is both lower and broader than the gross probability maximum, Equation 38. Referring back to Equation 29 resolves the dilemma by revealing that the variance estimate Equation 36 is unbiased. It follows that the Principle of Maximum Likelihood must be construed in the Detailed sense. According to Equation 36 then, the maximum likelihood estimate of the rerun measurement variance σ^2 is given by dividing the observed mean square misfit $Q(\underline{p})$ by its degrees of freedom $(L \cdot M - K)$.

Now that all parameters of the statistical theory have been evaluated numerically, the maximum likelihood probability distribution, Equation 28 or 33, can be employed to calculate the numeric expectation value of anything connected with the problem. The mean square uncertainty in the physical-theory parameters falls out of Equation 33 quite simply as the correlated error tensor

$$\begin{aligned}
 \langle (\underline{p} - \underline{P})(\underline{p} - \underline{P}) \rangle &= \int (\underline{p} - \underline{P})(\underline{p} - \underline{P}) e^{-(1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) : (\underline{p} - \underline{P})(\underline{p} - \underline{P}) / 2\sigma^2} \\
 &\quad \cdot \left| (1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\pi\sigma^2 \right|^{(1/2)} \prod_{k=1}^K dp^k, \\
 &= \left| (1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\pi\sigma^2 \right|^{(1/2)} \left[\partial / \partial \{ -(1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\sigma^2 \} \right] \\
 &\quad \cdot \int e^{-(1/2)(\partial^2 Q / \partial \underline{p} \partial \underline{p}) : (\underline{p} - \underline{P})(\underline{p} - \underline{P}) / 2\sigma^2} \prod_{k=1}^K dp^k,
 \end{aligned}$$

$$\begin{aligned}
&= \left| (1/2) (\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\pi\sigma^2 \right|^{1/2} \\
&\quad \cdot [\partial / \partial \{ -(1/2) (\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\sigma^2 \}] \\
&\quad \cdot \left| (1/2) (\partial^2 Q / \partial \underline{p} \partial \underline{p}) / 2\pi\sigma^2 \right|^{-1/2}, \\
&= 2\sigma^2 [\partial | \partial^2 Q / \partial \underline{p} \partial \underline{p} |] / [| \partial^2 Q / \partial \underline{p} \partial \underline{p} | \partial (\partial^2 Q / \partial \underline{p} \partial \underline{p})] \quad , \text{ or}
\end{aligned}$$

$$\boxed{ \langle (\delta \underline{p}) (\delta \underline{p}) \rangle = \langle (\underline{p} - \underline{P}) (\underline{p} - \underline{P}) \rangle = 2\sigma^2 (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1} } \quad . \quad (39)$$

This key result states that the mean square uncertainty in the physical-theory parameters, $\langle (\delta \underline{p}) (\delta \underline{p}) \rangle$, is equal to twice the measurement variance σ^2 times the matrix inverse of the mean-square-misfit second gradient $\partial^2 Q / \partial \underline{p} \partial \underline{p}$.

The root-mean-square uncertainty in each physical-theory parameter can be read off the diagonal elements of this tensor as

$$\boxed{ \delta p^k = \langle (\delta p^k) (\delta p^k) \rangle^{1/2} } \quad . \quad (40)$$

PROPAGATION OF CORRELATED ERROR

Experimentalists fight this gory battle for the best physical-theory parameters $\underline{p}$ and their mean square uncertainty

$\langle (\delta \underline{p})(\delta \underline{p}) \rangle$, motivated by hope that these sanguinolent numbers will be useful. Suppose the work is used later to calculate some function $f(\underline{p})$ and its mean square uncertainty $\langle (\delta f)(\delta f) \rangle$. Second-order Taylor expansion about the true value of the function $f(\underline{P})$ yields

$$f(\underline{p}) = f(\underline{P}) + [\partial f(\underline{P}) / \partial \underline{P}] \cdot (\underline{p} - \underline{P}) \\ + (1/2) [\partial^2 f(\underline{P}) / \partial \underline{P} \partial \underline{P}] : (\underline{p} - \underline{P})(\underline{p} - \underline{P}) \quad ,$$

$$\partial f(\underline{p}) / \partial \underline{p} = \partial f(\underline{P}) / \partial \underline{P} + [\partial^2 f(\underline{P}) / \partial \underline{P} \partial \underline{P}] \cdot (\underline{p} - \underline{P}) \quad ,$$

$$\partial^2 f(\underline{p}) / \partial \underline{p} \partial \underline{p} = \partial^2 f(\underline{P}) / \partial \underline{P} \partial \underline{P} \quad .$$

Mentally forming expectation values with the maximum likelihood probability distribution, Equation 33, now gives, with second-order reliability,

$$\langle f(\underline{p}) \rangle = f(\underline{P}) + (1/2) [\partial^2 f(\underline{P}) / \partial \underline{P} \partial \underline{P}] : \langle (\delta \underline{p})(\delta \underline{p}) \rangle \quad , \quad (41)$$

$$\langle (\delta f)(\delta f) \rangle = \langle [f(\underline{p}) - f(\underline{P})]^2 \rangle = [\partial f(\underline{p}) / \partial \underline{p}] \cdot \langle (\delta \underline{p})(\delta \underline{p}) \rangle \\ \cdot [\partial f(\underline{p}) / \partial \underline{p}] \quad . \quad (42)$$

The last term of Equation 41 displays quantitatively the amount by which statistical resolution effects can be expected

to distort the apparent mean value of any nonlinear function of the physical-theory parameters. The structure of the term reveals that such distortion vanishes only for linear functions or at points of full functional inflection. It is to be recalled, in contrast, that the preceding section has established that there is no such distortion in the physical-theory parameters themselves, basically because the minimum misfit formulation of the physical theory and the maximum likelihood formulation of the statistical theory do not permit it.

The last equation displays the mean square uncertainty of any function of the physical-theory parameters as the tensor contraction of the mean square parameter uncertainty upon the dyadic square of the parameter gradient of the function.

ORTHOGONAL RESOLUTION OF STATISTICAL THEORY

In brief essence, the preceding sections construct the physical theory which best fits the measurements and the statistical theory which best fits the residual discrepancies, by nonlinear generalization of the early nineteenth-century least-squares fitting methods of Gauss and Legendre. The fourth (and perhaps most subtle) major phase of this theory-development flight, into which we now soar, constructs probabilistically-uncoupled assessments of both the intrinsic quality of the measurements and the intrinsic relevance of the theory, as well as the adequacy of the agreement, by nonlinear generalization of the early twentieth-century small-sample regression theory of Fisher and Student.

Orthogonal resolution of the statistical-theory subspace into normal sub-subspaces quantifies the experimentalist's intuitive urge to identify unreliable measurements by scrutinizing residual discrepancies for lack of randomness. The most lucid exposition path here swoops through the breathtaking

sequence of transformation maneuvers which first unraveled this tangled skein. The local physical-theory misfit

$$q_m = q(y_m, x_m, p) \quad ,$$

for, say, the measurement m depicted in Figure 1, tensor-transforms outward to the enveloping measurement-space differential

$$\delta y_m^j = q_m \circ W_m \circ \sum_{i=1}^J (\partial q_m / \partial y_m^i) U_m^{ij} \quad , \quad (43)$$

which in turn projects inward into the physical-theory subspace of Figure 2 in accordance with Equation 24 as the parameter differential

$$\begin{aligned} \delta p_m &= \sum_{j=1}^J (\partial p / \partial y_m^j) \delta y_m^j \quad , \\ &= - q_m \circ W_m \circ \sum_{i=1}^J \sum_{j=1}^J (\partial q_m / \partial y_m^i) U_m^{ij} (\partial^2 Q / \partial y_m^j \partial p) \\ &\quad \cdot (\partial^2 Q / \partial p \partial p)^{-1} \quad . \end{aligned} \quad (44)$$

Consequently, the mean square misfit element

$$Q_m = q_m \circ W_m \circ q_m$$

(45)

projects into the physical-theory subspace as the misfit second-variation wobble

$$\delta^2 Q_m = (1/2) \delta p_m \cdot (\partial^2 Q / \partial p \partial p) \cdot \delta p_m \quad , \text{ or}$$

$$\begin{aligned}
 \delta^2 Q_m &= (1/2) \underline{q}_m \circ W_m \circ \left[\sum_{i=1}^J \sum_{j=1}^J (\partial \underline{q}_m / \partial y_m^i) U_m^{ij} (\partial^2 Q / \partial y_m^j \partial \underline{p}) \right] \\
 &\quad \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p})^{-1} \\
 &\quad \cdot \left[\sum_{j=1}^J \sum_{i=1}^J (\partial^2 Q / \partial \underline{p} \partial y_m^j) U_m^{ji} (\partial \underline{q}_m / \partial y_m^i) \right] \circ W_m \circ \underline{q}_m \quad , \quad (46)
 \end{aligned}$$

leaving as the orthogonal residue spanned by the statistical-theory subspace

$$Q_m - \delta^2 Q_m \quad . \quad (47)$$

By construction, then, this last function, Equation 47, evaluable numerically from Equations 45 and 46, spans the measurement-m sub-subspace within the statistical-theory subspace.

Paralleling the logic of Student and Fisher, we next shift the statistical-theory probability element of Equation 33 to display the probability angle ψ_m defined as in Figure 3 by the local/total mean square misfit ratio

$$\cos^2 \psi_m = (Q_m - \delta^2 Q_m) / Q \quad , \quad 0 \leq \psi_m \leq \pi/2 \quad . \quad (48)$$

Fisher's ingenious succession of Cartesian and polar transformations then yields, mentally,

FIGURE 3. Orthogonal Resolution of Statistical Theory into Radial Probability and Angular Probability

$$\begin{aligned}
dP &= e^{-Q/2\sigma^2} [Q(\sin^2\psi_m)/2\sigma^2]^{(L\cdot M-K-L-1)/2} \\
&\cdot d[Q(\sin^2\psi_m)/2\sigma^2]^{1/2} d\Omega(L\cdot M-K-L-1) \\
&\cdot [Q(\cos^2\psi_m)/2\sigma^2]^{(L-1)/2} d[Q(\cos^2\psi_m)/2\sigma^2]^{1/2} \\
&\cdot d\Omega(L-1)/\pi^{(L\cdot M-K)/2} \\
&\cdot e^{-(1/2)(\partial^2 Q/\partial \underline{p} \partial \underline{p}) : (\underline{p}-\underline{P})(\underline{p}-\underline{P})/2\sigma^2} |(1/2)(\partial^2 Q/\partial \underline{p} \partial \underline{p})/2\pi\sigma^2|^{1/2} \\
&\cdot \prod_{k=1}^K dp^k, \text{ or}
\end{aligned}$$

$$\begin{aligned}
dP &= e^{-Q/2\sigma^2} [Q/2\sigma^2]^{(L\cdot M-K-1)/2} d[Q/2\sigma^2]^{1/2} d\Omega(L\cdot M-K-L-1) \\
&\cdot [\sin^{L\cdot M-K-L-1}\psi_m] [\cos^{L-1}\psi_m] d\psi_m d\Omega(L-1)/\pi^{(L\cdot M-K)/2} \\
&\cdot e^{-(1/2)(\partial^2 Q/\partial \underline{p} \partial \underline{p}) : (\underline{p}-\underline{P})(\underline{p}-\underline{P})/2\sigma^2} |(1/2)(\partial^2 Q/\partial \underline{p} \partial \underline{p})/2\pi\sigma^2|^{1/2} \\
&\cdot \prod_{k=1}^K dp^k.
\end{aligned}$$

(49)

What we have here now includes an orthogonal resolution of statistical theory (first two lines) into radial probability (first line except $d\Omega$) displaying Pearson's radial (χ^2) distribution, and angular probability (second line except $d\Omega$)

displaying Fisher's angular (F) distribution. The two solid angle elements $d\Omega(L \cdot M - K - L - 1)$ and $d\Omega(L - 1)$ span further orthogonal subspaces of the statistical theory. The multidimensional Gaussian distribution in the last two lines, it will be recalled, spans the physical theory.

Calculating the probability angle ψ_m for each measurement from Equations 45, 46, and 48, and then integrating the angular probability factor of Equation 49, provides finally as a quantitatively decisive assessment of the intrinsic reliability of each measurement its

Probability of Less Error

$$P(\psi'_m > \psi_m) = \frac{\int_{\psi_m}^{\pi/2} (\sin^{L \cdot M - K - L - 1} \psi'_m) (\cos^{L-1} \psi'_m) d\psi'_m}{\int_0^{\pi/2} (\sin^{L \cdot M - K - L - 1} \psi'_m) (\cos^{L-1} \psi'_m) d\psi'_m} .$$

(50)

These glowingly informative angular probabilities range from 0 to 1 and can be expected to hover around 1/2 with a mean square dispersion of 1/12, since for any probability P,

$$\langle P \rangle = \int_0^1 P dP = 1/2, \text{ and} \tag{51}$$

$$\langle (P - \langle P \rangle)^2 \rangle = \int_0^1 (P - 1/2)^2 dP = 1/12 . \tag{52}$$

Measurements whose Probability of Less Error is P may be viewed as bad data with confidence P or as good data with confidence $1-P$.

It is interesting to observe, in curious contrast, that analogous logic applied to the radial probability factor of Equation 49 yields

$$P(Q' < Q) = \int_0^{(Q/2\sigma^2)^{1/2}} e^{-Q'/2\sigma^2} (Q'/2\sigma^2)^{(L \cdot M - K - 1)/2} d(Q'/2\sigma^2)^{1/2} \\ = \int_0^\infty e^{-Q'/2\sigma^2} (Q'/2\sigma^2)^{(L \cdot M - K - 1)/2} d(Q'/2\sigma^2)^{1/2},$$

which degenerates to a mathematical constant independent of measurement fluctuation by virtue of the maximum likelihood condition $\sigma^2 \doteq (Q/L \cdot M - K)$ of Equation 36. In short, maximum likelihood formulation of the statistical theory bleeds its radial probability quite dry.

ORTHOGONAL RESOLUTION OF PHYSICAL THEORY

Copernicus' heliocentric theory of the solar system displaced the mathematically equivalent geocentric theory of Ptolemy because the heliocentric theory is simpler, in the intrinsic sense of fitting all relevant data with far fewer parameters. The art of mathematical physics consists, in essence, of evolving mathematical representations of physical science which optimally combine both realism and simplicity. This thought kicks around informally among physicists as the Principle of Least Astonishment. Orthogonal resolution of the physical-theory subspace into normal sub-subspaces

sharpens this loose philosophy into a quantitative axiom describable more sharply as the Principle of Maximum Simplicity.

In hot pursuit of simplicity now, envision the rather common circumstance, in electronic computer work of this sort, that a subset of the physical-theory parameters $\underline{p}_s$ seems to be a bit shaky (like 1 ± 2 perhaps). Consider the sensitivity of the mean square misfit to simplification of the theory by setting the dubious parameters to zero with the parameter shift $\delta \underline{p}_s = -\underline{p}_s$, and then shifting the remaining parameter coset $\underline{p}_c$ to achieve the minimum misfit for the simplified theory. Since the first variation vanishes at the minimum misfit $\partial Q / \partial \underline{p} = 0$, the misfit worth $\delta^2 Q$ of any parameter shift $\delta \underline{p}$ is given by the second variation

$$\begin{aligned} \delta^2 Q &= (1/2) \delta \underline{p} \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p}) \cdot \delta \underline{p} \quad , \\ &= (1/2) \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_s) \cdot \delta \underline{p}_s + \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_c) \cdot \delta \underline{p}_c \\ &\quad + (1/2) \delta \underline{p}_c \cdot (\partial^2 Q / \partial \underline{p}_c \partial \underline{p}_c) \cdot \delta \underline{p}_c \quad . \end{aligned} \quad (53)$$

Minimum misfit for the simplified theory yields

$$\begin{aligned} \partial \delta^2 Q / \partial \underline{p}_c &= \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_c) + \delta \underline{p}_c \cdot (\partial^2 Q / \partial \underline{p}_c \partial \underline{p}_c) = 0 \quad , \text{ or} \\ \delta \underline{p}_c &= - \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_c) \cdot (\partial^2 Q / \partial \underline{p}_c \partial \underline{p}_c)^{-1} \quad , \text{ so} \quad (54) \\ \delta^2 Q_s &= (1/2) \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_s) \cdot \delta \underline{p}_s \\ &\quad - \delta \underline{p}_s \cdot (\partial^2 Q / \partial \underline{p}_s \partial \underline{p}_c) \cdot (\partial^2 Q / \partial \underline{p}_c \partial \underline{p}_c)^{-1} \cdot (\partial^2 Q / \partial \underline{p}_c \partial \underline{p}_s) \cdot \delta \underline{p}_s \end{aligned}$$

$$+ (1/2) \delta p_s \cdot (\partial^2 Q / \partial p_s \partial p_c) \cdot (\partial^2 Q / \partial p_c \partial p_c)^{-1} \\ \cdot (\partial^2 Q / \partial p_c \partial p_s) \cdot \delta p_s \quad , \text{ or}$$

$$\delta^2 Q_s = (1/2) \delta p_s \cdot [\partial^2 Q / \partial p_s \partial p_s - (\partial^2 Q / \partial p_s \partial p_c) \\ \cdot (\partial^2 Q / \partial p_c \partial p_c)^{-1} \cdot (\partial^2 Q / \partial p_c \partial p_s)] \cdot \delta p_s \quad . \quad (55)$$

This expression gives the minimum misfit worth $\delta^2 Q_s$ of any parameter-subset shift δp_s . Its functionally orthogonal coset $\delta^2 Q_c$ is obtainable simply by subtracting it from the functionally complete set $\delta^2 Q$ of Equation 53. After simplification,

$$\delta^2 Q_c = (1/2) \delta p \cdot (\partial^2 Q / \partial p \partial p_c) \cdot (\partial^2 Q / \partial p_c \partial p_c)^{-1} \\ \cdot (\partial^2 Q / \partial p_c \partial p) \cdot \delta p \quad . \quad (56)$$

Mentally adding the last two equations verifies that they sum identically to

$$\delta^2 Q = (1/2) \delta p \cdot (\partial^2 Q / \partial p \partial p) \cdot \delta p = \delta^2 Q_s + \delta^2 Q_c \quad . \quad (57)$$

The corresponding orthogonal resolution of the probability distribution, Equation 49, now can be formed mentally from its last two lines to give

$$\begin{aligned}
dP = & e^{-Q/2\sigma^2} [Q/2\sigma^2]^{(I \cdot M - K - 1)/2} d[Q/2\sigma^2]^{(1/2)} d\Omega(I \cdot M - K - I - 1) \\
& \cdot [\sin^{I \cdot M - K - I - 1} \psi_m] [\cos^{I - 1} \psi_m] d\psi_m d\Omega(I - 1) / \pi^{(I \cdot M - K)/2} \\
& \cdot e^{-\delta^2 Q_S / 2\sigma^2} [\delta^2 Q_S / 2\sigma^2]^{(K_S - 1)/2} d[\delta^2 Q_S / 2\sigma^2]^{(1/2)} d\Omega(K_S - 1) / \pi^{K_S/2} \\
& \cdot e^{-\delta^2 Q_C / 2\sigma^2} [\delta^2 Q_C / 2\sigma^2]^{(K_C - 1)/2} d[\delta^2 Q_C / 2\sigma^2]^{(1/2)} d\Omega(K_C - 1) / \pi^{K_C/2}
\end{aligned}
\tag{58}$$

The first two lines here, it is to be recalled, span the statistical theory. The last two lines span the physical theory with an orthogonal resolution cleanly relevant to theory-simplification logic. The third line spans any theory simplification we may wish to consider, involving both shifting and then discarding a K_S -fold parameter subset $\underline{p}_S$. The last line spans the residual simplified theory, involving the K_C -fold parameter coset $\underline{p}_C$. The last two phase elements appear in polar geodesic coordinates, with $K_S + K_C = K$.

Paralleling the angular probability logic of the preceding section, we next shift the first and third lines of Equation 58 to display the theory-simplification probability angle ψ_S , defined as in Figure 4 by the dubious/simplified mean square misfit ratio

FIGURE 4. *Orthogonal Synthesis of Statistical Theory and Dubious Physical Theory into Statistical Theory for Simplified Physical Theory*

$$\cos^2 \psi_s = (\delta^2 Q_s) / (Q + \delta^2 Q_s) \quad , \quad 0 \leq \psi_s \leq \pi/2 \quad . \quad (59)$$

Fisher's succession of Cartesian and polar transformations to the radial-angular basis $[(Q_s = Q + \delta^2 Q_s)^{1/2}, \psi_s]$ then yields

$$\begin{aligned} dP = & e^{-Q_s/2\sigma^2} [(Q_s \sin^2 \psi_s)/2\sigma^2]^{(L \cdot M - K - 1)/2} d[(Q_s \sin^2 \psi_s)/2\sigma^2]^{1/2} \\ & \cdot d\Omega(L \cdot M - K - L - 1) [\sin^{L \cdot M - K - L - 1} \psi_m] [\cos^{L-1} \psi_m] d\psi_m \\ & \cdot d\Omega(L-1)/\pi^{(L \cdot M - K)/2} \\ & \cdot [(Q_s \cos^2 \psi_s)/2\sigma^2]^{(K_s - 1)/2} d[(Q_s \cos^2 \psi_s)/2\sigma^2]^{1/2} \\ & \cdot d\Omega(K_s - 1) \pi^{-K_s/2} e^{-\delta^2 Q_c/2\sigma^2} [\delta^2 Q_c/2\sigma^2]^{(K_c - 1)/2} \\ & \cdot d[\delta^2 Q_c/2\sigma^2]^{1/2} d\Omega(K_c - 1)/\pi^{K_c/2} \quad , \quad \text{or} \end{aligned}$$

$$\begin{aligned} dP = & e^{-(Q + \delta^2 Q_s)/2\sigma^2} [(Q + \delta^2 Q_s)/2\sigma^2]^{(L \cdot M - K_c - 1)/2} \\ & \cdot d[(Q + \delta^2 Q_s)/2\sigma^2]^{1/2} d\Omega(L \cdot M - K - L - 1) \\ & \cdot [\sin^{L \cdot M - K - L - 1} \psi_m] [\cos^{L-1} \psi_m] d\psi_m d\Omega(L-1)/\pi^{(L \cdot M - K)/2} \\ & \cdot [\sin^{L \cdot M - K - 1} \psi_s] [\cos^{K_s - 1} \psi_s] d\psi_s d\Omega(K_s - 1)/\pi^{K_s/2} \\ & \cdot e^{-\delta^2 Q_c/2\sigma^2} [\delta^2 Q_c/2\sigma^2]^{(K_c - 1)/2} d[\delta^2 Q_c/2\sigma^2]^{1/2} \\ & \cdot d\Omega(K_c - 1)/\pi^{K_c/2} \quad . \end{aligned} \quad (60)$$

The first two lines here (except $d\Omega$) display Pearson's radial (χ^2) probability distribution for the statistics of the simplified physical theory. The third line (except $d\Omega$) displays Fisher's angular (F) probability distribution for dubious measurements. The fourth line displays Fisher's angular (F) probability distribution for dubious aspects of the physical theory. The last two lines display Pearson's radial (χ^2) probability distribution for the simplified physical theory. The solid angle elements $d\Omega$ span further orthogonal subspaces.

Calculating the probability angle ψ_s from Equations 55 and 59 for any proposed theory simplification (such as discarding a dubious parameter subset p_s by setting it to zero with the shift $\delta p_s = -p_s$) and then integrating angular doubts provides finally as a quantitatively decisive assessment of the intrinsic relevance of each dubious aspect of the physical theory its

$$\begin{aligned}
 & \text{Probability of More Worth} \\
 P(\psi'_s < \psi_s) &= \int_0^{\psi_s} (\sin^{L \cdot M - K - 1} \psi'_s) (\cos^{K_s - 1} \psi'_s) d\psi'_s \\
 & \quad + \int_0^{\pi/2} (\sin^{L \cdot M - K - 1} \psi'_s) (\cos^{K_s - 1} \psi'_s) d\psi'_s .
 \end{aligned}$$

(61)

These scintillating numeric jewels permit casting theory-simplification logic as the quantitative axiom:

PRINCIPLE OF MAXIMUM SIMPLICITY

A physical-theory parameter subset whose Probability of More Worth is P may be discarded with confidence P or kept with confidence $1-P$.

Exhaustive assessment of the intrinsic relevance of each K_s -fold subset within a K -fold set of physical-theory parameters leads to looping through Equations 55, 59, and 61 a total of

$K + K(K-1)/2! + K(K-1)(K-2)/3! + \dots + K!/(K-k)!k! + \dots + 1 = 2^K - 1$ times.

NUMERICSNUMERIC SEQUENCE: PROGRAM LIKELY

The numeric computational sequences shown in Tables I, II, and III evolved progressively over a two year period as the analysis logic was formulated, programmed, debugged, and optimized.⁽⁵⁾ Consequently, these brief tables depicting the optimal calculational sequence constitute the firmest available algebraic record of the precise meaning of most of the numeric output of the program. Output labelling is in general harmony with these tables, within the limitations of machine characters.

TABLE II. *Output Numeric Sequence: Program Likely*

DOUBT Link

$$\text{Statistical Weight : } W_m = \left[\sum_{i=1}^J \sum_{j=1}^J (\partial q_m / \partial y_m^i) U_m^{ij} (\partial q_m / \partial y_m^j) \right]^{-1}$$

$$\text{Mean Square Misfit : } Q_m = q_m \circ W_m \circ q_m \quad , \quad Q = \sum_{m=1}^M Q_m$$

$$\text{Measurement Variance: } \sigma^2 = Q / (L \cdot M - K)$$

$$\text{Parameter Variance : } <(\delta p)(\delta p)> = 2 \sigma^2 (\partial^2 Q / \partial p \partial p)^{-1}$$

$$\text{Parameter Doubt : } \delta p^k = \pm <(\delta p^k)(\delta p^k)>^{1/2}$$

(5) *Dr. R. P. Matsen provided unusually exhaustive cross checks by formulating a double-precision generalization of the analysis (BNW program Double Likely).*

TABLE II. (contd)MISFIT Link

Theory Error: $\delta y_m^j = q_m \circ W_m \circ \sum_{i=1}^J (\partial q_m / \partial y_m^i) U_m^{ij}$

Theory Resolution Error: $\delta^2 q_m = (1/2) (\partial^2 q_m / \partial p \partial p) : < (\delta p) (\delta p) >$

$$\delta^2 y_m^j = (\delta^2 q_m) \circ W_m \circ \sum_{i=1}^J (\partial q_m / \partial y_m^i) U_m^{ij}$$

Theory Wobble: $q_m \circ (\partial^2 Q / \partial q_m \partial p) = q_m \circ W_m \circ \sum_{i=1}^J \sum_{j=1}^J (\partial q_m / \partial y_m^i) U_m^{ij}$
 $\cdot (\partial^2 Q / \partial y_m^j \partial p)$

$$\delta^2 Q_m = (1/2) q_m \circ (\partial^2 Q / \partial q_m \partial p) \cdot (\partial^2 Q / \partial p \partial p)^{-1}$$

$$\cdot (\partial^2 Q / \partial p \partial q_m) \circ q_m$$

Probability of Less Error: $\cos^2 \psi_m = (Q_m - \delta^2 Q_m) / Q$

$$P(\psi_m' > \psi_m) = \int_{\psi_m}^{\pi/2} (\sin^{L \cdot M - K - L - 1} \psi_m') \cdot (\cos^{L-1} \psi_m') d\psi_m'$$

$$/ \int_0^{\pi/2} (\sin^{L \cdot M - K - L - 1} \psi_m') \cdot (\cos^{L-1} \psi_m') d\psi_m'$$

Expectations: $\langle \cos^2 \psi_m \rangle = (L \cdot M - K) / L \cdot M$

$$\langle (\cos^2 \psi_m - \langle \cos^2 \psi_m \rangle)^2 \rangle = 2 (L \cdot M - K) K / (L \cdot M)^2 (L \cdot M + 2)$$

TABLE III. *Output Numeric Sequence: Program Likely*WORTH Link

$$\text{Subset Worth: } \delta^2 Q_S = (1/2) \underline{p}_S \cdot [\partial^2 Q / \partial \underline{p}_S \partial \underline{p}_S - (\partial^2 Q / \partial \underline{p}_S \partial \underline{p}_C) \cdot (\partial^2 Q / \partial \underline{p}_C \partial \underline{p}_C)^{-1} \cdot (\partial^2 Q / \partial \underline{p}_C \partial \underline{p}_S)] \cdot \underline{p}_S$$

$$\text{Coset Worth: } \delta^2 Q_C = (1/2) \underline{p} \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p}_C) \cdot (\partial^2 Q / \partial \underline{p}_C \partial \underline{p}_C)^{-1} \cdot (\partial^2 Q / \partial \underline{p}_C \partial \underline{p}) \cdot \underline{p}$$

$$\text{Theory Worth: } \delta^2 Q = \delta^2 Q_S + \delta^2 Q_C = (1/2) \underline{p} \cdot (\partial^2 Q / \partial \underline{p} \partial \underline{p}) \cdot \underline{p}$$

$$\text{Probability of More Worth: } \cos^2 \psi_S = (\delta^2 Q_S) / (Q + \delta^2 Q_S)$$

$$P(\psi'_S < \psi_S) = \int_0^{\psi_S} (\sin^{L \cdot M - K - 1} \psi'_S) \cdot (\cos^{K_S - 1} \psi'_S) d\psi'_S$$

$$/ \int_0^{\pi/2} (\sin^{L \cdot M - K - 1} \psi'_S) \cdot (\cos^{K_S - 1} \psi'_S) d\psi'_S$$

$$\text{Expectations: } \langle P \rangle = 1/2, \quad \langle (P - \langle P \rangle)^2 \rangle = 1/12$$

$$\langle \cos^2 \psi_S \rangle = K_S / (L \cdot M - K_C)$$

$$\langle (\cos^2 \psi_S - \langle \cos^2 \psi_S \rangle)^2 \rangle = 2 K_S (L \cdot M - K) / (L \cdot M - K_C)^2 (L \cdot M - K_C + 2)$$

NUMERIC SEQUENCE: PROGRAM LEARN

Program Likely evolved as a vector field generalization of its scalar field predecessor Program Learn. Most of the intricate matrix algebra of Program Likely degenerates to scalar algebra in Program Learn, but otherwise the analysis logic is quite similar in both programs, and some of the subroutines are in fact identical. The analysis logic of Program Learn is obtainable as a special case of that of Program Likely, by setting the span of the vector fields $\underline{q}$ and $\underline{y}$ to one ($J=1=L$), switching the parameter notation from p^k to a^j , and particularizing the implicit vector theory $\underline{q}(\underline{y}, \underline{x}, \underline{p}) = 0$ to the explicit scalar theory $y = y(\underline{x}, \underline{a})$. Tables (IV, V, and VI) present numeric computational sequences and output labelling for program Learn.

TABLE IV. *Misfit-Gradient Numeric Sequence: Program Learn*

Step-1: Evaluate mean square misfit Q. Iterative Algorithm

$$Q = \sum_{m=1}^M W_m [y_m - y(\underline{x}_m, \underline{a})]^2$$

$$\underline{a} = \underline{a} - (\partial Q / \partial \underline{a}) \cdot (\partial^2 Q / \partial \underline{a} \partial \underline{a})^{-1}$$

Step-2: Evaluate misfit gradient $\partial Q / \partial \underline{a}$.

$$\partial Q / \partial \underline{a} = \sum_{m=1}^M 2 W_m [y_m - y(\underline{x}_m, \underline{a})] [-\partial y(\underline{x}_m, \underline{a}) / \partial \underline{a}]$$

Step-3: Evaluate misfit second gradient $\partial^2 Q / \partial \underline{a} \partial \underline{a}$

$$\begin{aligned} \partial^2 Q / \partial \underline{a} \partial \underline{a} = & \sum_{m=1}^M 2 W_m [\{\partial y(\underline{x}_m, \underline{a}) / \partial \underline{a}\} \{\partial y(\underline{x}_m, \underline{a}) / \partial \underline{a}\} \\ & - \{y_m - y(\underline{x}_m, \underline{a})\} \{\partial^2 y(\underline{x}_m, \underline{a}) / \partial \underline{a} \partial \underline{a}\}] \end{aligned}$$

TABLE V. *Output Numeric Sequence: Program Learn*DOUBT Link

$$\text{Mean Square Misfit} : Q_m = W_m [y_m - y(\underline{x}_m, \underline{a})]^2, \quad Q = \sum_{m=1}^M Q_m$$

$$\text{Measurement Variance: } \sigma^2 = Q/(M-J)$$

$$\text{Parameter Variance} : <(\delta \underline{a})(\delta \underline{a})> = 2 \sigma^2 (\partial^2 Q / \partial \underline{a} \partial \underline{a})^{-1}$$

$$\text{Parameter Doubt} : \delta a^j = \pm <(\delta a^j)(\delta a^j)>^{1/2}$$

MISFIT Link

$$\begin{aligned} \text{Theory Variance: } <[\delta y_m(\underline{a})]^2> = [\partial y_m(\underline{a}) / \partial \underline{a}] \cdot <(\delta \underline{a})(\delta \underline{a})> \\ &\cdot [\partial y_m(\underline{a}) / \partial \underline{a}] \end{aligned}$$

$$\text{Theory Doubt} : \delta y_m(\underline{a}) = \pm <[\delta y_m(\underline{a})]^2>^{1/2}$$

Theory Resolution Error:

$$\delta^2 y_m(\underline{a}) = -(1/2) [\partial^2 y_m(\underline{a}) / \partial \underline{a} \partial \underline{a}] : <(\delta \underline{a})(\delta \underline{a})>$$

$$\text{Misfit Variance: } <[\delta \{y_m - y_m(\underline{a})\}]^2> = \sigma^2 / W_m - <[\delta y_m(\underline{a})]^2>$$

$$\text{Misfit Doubt} : \delta [y_m - y_m(\underline{a})] = \pm <[\delta \{y_m - y_m(\underline{a})\}]^2>^{1/2}$$

TABLE V. (contd)

Probability of Less Error:

$$\cos^2 \psi_m = [W_m - (1/2)(\partial^2 Q / \partial y_m \partial \underline{a}) \cdot (\partial^2 Q / \partial \underline{a} \partial \underline{a})^{-1} \\ \cdot (\partial^2 Q / \partial \underline{a} \partial y_m)] [y_m - y_m(\underline{a})]^2 / Q$$

$$P(\psi'_m > \psi_m) = \int_{\psi_m}^{\pi/2} \sin^{M-J-2} \psi'_m d\psi'_m / \int_0^{\pi/2} \sin^{M-J-2} \psi'_m d\psi'_m$$

$$\text{Expectations: } \langle \cos^2 \psi_m \rangle = (M-J)/M$$

$$\langle (\cos^2 \psi_m - \langle \cos^2 \psi_m \rangle)^2 \rangle = 2 (M-J)J/M^2(M+2)$$

TABLE VI. Output Numeric Sequence: Program LearnWORTH Link

$$\text{Subset Worth: } \delta^2 Q_s = (1/2) \underline{a}_s \cdot [\partial^2 Q / \partial \underline{a}_s \partial \underline{a}_s - (\partial^2 Q / \partial \underline{a}_s \partial \underline{a}_c) \\ \cdot (\partial^2 Q / \partial \underline{a}_c \partial \underline{a}_c)^{-1} \cdot (\partial^2 Q / \partial \underline{a}_c \partial \underline{a}_s)] \cdot \underline{a}_s$$

$$\text{Coset Worth: } \delta^2 Q_c = (1/2) \underline{a} \cdot (\partial^2 Q / \partial \underline{a} \partial \underline{a}_c) \cdot (\partial^2 Q / \partial \underline{a}_c \partial \underline{a}_c)^{-1} \\ \cdot (\partial^2 Q / \partial \underline{a}_c \partial \underline{a}) \cdot \underline{a}$$

$$\text{Theory Worth: } \delta^2 Q = \delta^2 Q_s + \delta^2 Q_c = (1/2) \underline{a} \cdot (\partial^2 Q / \partial \underline{a} \partial \underline{a}) \cdot \underline{a}$$

TABLE VI. (contd)

Probability of More Worth: $\cos^2 \psi_s = (\delta^2 Q_s) / (Q + \delta^2 Q_s)$

$$P(\psi'_s < \psi_s) = \int_0^{\psi_s} (\sin^{M-J-1} \psi'_s) (\cos^{J_s-1} \psi'_s) d\psi'_s$$

$$\div \int_0^{\pi/2} (\sin^{M-J-1} \psi'_s) (\cos^{J_s-1} \psi'_s) d\psi'_s$$

Expectations: $\langle \cos^2 \psi_s \rangle = J_s / (M - J_c)$

$$\langle (\cos^2 \psi_s - \langle \cos^2 \psi_s \rangle)^2 \rangle = 2 J_s (M - J) / (M - J_c)^2 (M - J_c + 2)$$

BITS

INPUT LOGIC: PROGRAM LIKELY

51

BNWL-390

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C          MAXIMUM LIKELIHOOD NONLINEAR CORRELATED FIELDS
C  BATTELLE-NORTHWEST PROGRAM LIKELY (05SEP65) BNWL-284 (14NOV66)
-----
C NAMELIST  DEFINITION
C$INPUT    INPUT LIST
-----
C          INTEGER INPUT
C  IL=      ITERATION LIMIT (MINIMUM 0, NO MAXIMUM, TRY 7)          PRESET 0
C  IX=      NUMBER OF PRECISE X-COORDINATES (MINIMUM 0, MAXIMUM 5-JY) PRESET 0
C  JY=      NUMBER OF IMPRECISE Y-COORDINATES (MINIMUM 1, MAXIMUM 5-IX) PRESET 1
C  KP=      NUMBER OF PARAMETER P-COORDINATES (MINIMUM 1, MAXIMUM 25) PRESET 1
C  LQ=      NUMBER OF MISFIT Q-COORDINATES (MINIMUM 1, MAXIMUM JY)   PRESET 1
C  NM=      NUMBER OF MEASUREMENT VECTORS 2(M) (MINIMUM 1, MAXIMUM 100) PRESET 1
C  NC=      NUMBER OF CONSTANTS OR CONTROLS C (MINIMUM 0, MAXIMUM 25) PRESET 0
C  LOTS=     NUMBER OF SETS OF DOUBIOUS-PARAMETER SUBSETS (MIN 0, MAX 10) PRESET 0
C  MAP=      THEORY-MEASUREMENT HYPER-SPACE PLOTS (NONE 0, MAXIMUM 25) PRESET 0
C           MINUS MAP MEANS MORE MAPS LATER (B5 END-OF-FILE BYPASS)
C           PLUS MAP MEANS LAST MAP CASE (B5 END-FILE, REWIND, UNLOAD)
C           (UNLIMITED PLOT SETS BY 0-ITERATION CHANGE-CASE SEQUENCE)
C  LAG(1)=   ITERATIONS TO LAG BEFORE PARAMETER VARIATION LAG(K),K=1,KP
C           GOOD GUESS OR LINEAR LAG=0, BAD GUESS NON-LINEAR LAG=1,
C           EXPONENTIAL LAG=2, EXPLOSIVE LAG=3 (MIN 0, NO MAX)
C  LOT(1,1)= DOUBIOUS-PARAMETER SUBSETS (LOT(K,L),K=1,KP),L=1,LOTS
C           (0 NULL SUBSET, 1 SUBSET-1, 2 SUBSET-2, ..., KP SUBSET-KP)
C           (FOR WORTH EVALUATION BY THEORY REGRESSION TO P(K(LOT))=0.)
-----
C          EXPONENTIAL INPUT
C  RATE=     USE DP(K)=RATE*DP(K) CONVERGENCE-RATE CONTROL OPTION PRESET1.
C  HALF=     USE DP(K)=-HALF*DP(K) IF QUADRIC AND CUBIC EXPANSIONS FAIL PRESET.5
C           (HALF=-.5 AIDS QUARTIC RECOVERY AT RISK OF QUINTIC FAILURE)
C  BAD=      BAD-FIT MULTIPLE OF ROOT-MEAN-SQUARE MISFIT (TRY 3. OR -3.) PRESET3.
C           MINUS BAD MEANS (DISCARD BAD DATA, SET BAD PLUS, RUN REFIT)
C  C(1)=     THEORY CONSTANTS OR MULTIPLE-THEORY CONTROLS C(N),N=1,NC
C  P(1)=     PARAMETER ESTIMATES P(K),K=1,KP
C           CLOSE ENOUGH TO AVOID SADDLE POINTS AND MULTIPLE SOLUTIONS
C  PMIN(1)=  PARAMETER MINIMA (MULTIPLE-SOLUTION LIMIT) PMIN(K),K=1,KP
C  PMAX(1)=  PARAMETER MAXIMA (MULTIPLE-SOLUTION LIMIT) PMAX(K),K=1,KP
C           TO BYPASS PMIN=-1.+38 PMAX=1.+38 TO FIX PMIN=PMAX
C  Z(1,1)=   MEASUREMENTS (Z(M,J1),M=1,MM),J1=1,JY+IX PACKED VECTORS
C           IMPRECISE(Y(M,J),M=1,MM),J=1,JY PRECISE(X(M,I),M=1,MM),I=1,IX
C  U(1,1)=   MEAN SQUARE CORRELATED MEASUREMENT UNCERTAINTY U(M,J1J2)
C           ((U(M,J1,J2),M=1,MM),J2=J1,JY),J1=1,JY PACKED TENSORS
C  UGEN=     NOT -0. GENERATE UNCERTAINTY U(M,J,J)=Y(M,J)**UGEN PRESET-0
C  UEXP=     NOT -0. EXPONENTIATE U(M,J1,J2)=U(M,J1,J2)**UEXP PRESET-0
C  YEXP=     NOT -0. DO U(M,J1,J2)=U(M,J1,J2)*(Y(M,J1)*Y(M,J2))**YEXP PRESET-0
C  WSUM=     NOT -0. NORMALIZE INVERSE-UNCERTAINTY SUM TO WSUM PRESET-0
C  WLOG=     NOT -0. NORMALIZE UNCERTAINTY DETERMINANT TO 1/EXP(WLOG) PRESET-0
-----
C          1 MIL BY 1 MIL PLOTTER GRID HAS 20000 MIL BY 20000 MIL SPAN
C  ZDZSUS(1,1,1)= ((Z-SCALE(Z1,UZ,Z2)=(S1,US,S2)MILS),J=1,JY+IX),M=1,MAP

```



```

C      DZ.NE.0.  INDEPENDENT-VARIABLE(Z=(Y OR X)) DATA RESOLUTION
C      DZ=0.     DEPENDENT-VARIABLE(Z=(Y OR X)) DATA FLAG
C      DATA TRANSFORM FLEXIBLY VIA SUBROUTINE PLOT(ZP)
C      THEN TRANSLATE AND SCALE LINEARLY TO MATCH ENDS
C      DS.NE.0.  INDEPENDENT-VARIABLE(X-AXIS) CROSS-PLOT RESOLUTION
C      DS=0.     DEPENDENT-VARIABLE(Y-AXIS) CROSS-PLOT FLAG
C      DS=-0.    SELECTIVE CROSS-PLOT BYPASS FLAG
C      OFF-SCALE POINTS PLOT PROJECTED TO NEAREST BOUNDARY

C      PLOTTER GRID, RANGE, SYMBOL, AND CURVE ALTERATION OPTIONS
C MESH Y= (10 9 8 7 6 5 4 3 2 1)=(1 2 3 4 5 6 7 8 9 10)MIL Y-GRID    PRESET10
C MESH X= (10 9 8 7 6 5 4 3 2 1)=(1 2 3 4 5 6 7 8 9 10)MIL X-GRID    PRESET10
C      2-10 MIL PLOTTER GRID HAS 30000 MIL SPAN
C MARK= (0 1 2 3 4 5 6 7 8 9 10 11)=( * )SYMBOL PRESET 1
C      (0 1 2 3 4 5 6 7 8 9 10 11)=(. 1 2 3 4 5 6 7 8 9 0 -)NUMBER
C      MINUS MARK SUPERPOSES PLOTS, WITH CYCLIC MARKS
C LINE= (2 4 8)=(LINES POINTS ARCS) CURVE-SEGMENTATION MODE        PRESET 2
C      MINUS LINE OUTPUTS BOTH PLOTS AND EQUIVALENT NUMERIC TABLES
C ERR=   ROOT-MEAN-SQUARE-ERROR MULTIPLE FOR ERROR-CURVE PLOTS      PRESET1.
C      PLUS ERR PLOTS INPUT MEASUREMENT UNCERTAINTY
C      MINUS ERR PLOTS MAXIMUM-LIKELIHOOD MEASUREMENT UNCERTAINTY
C F1=    CONVERGED-PLOT FRACTION OF ROOT-MEAN-SQUARE ERROR          PRESET1.-4
C MVS1=  PLOT-ITERATION LIMIT (PLOT AND WRITE DUBIOUS POINT)        PRESET64

C $      END OF CASE

C      RUN TERMINATES AT SOLID 36-BIT CONVERGENCE ON LOCAL SECOND-ORDER
C      MINIMUM OF MEAN SQUARE RESIDUAL MISFIT QQ, AT OCCURRENCE OF NULL
C      DETERMINANT, AT SPILL OR DIVISION BY ZERO, OR AT ITERATION LIMIT.
C      TO TERMINATE RUN AT END OF THIS ITERATION, DEPRESS SENSE-SWITCH 1.
C      AT END OF RUN, PARAMETERS P(K) ARE BEST VALUES, UGEN=UEXP=YEXP=-0.,
C      ALL OTHER INPUT IS UNDISTURBED (FOR +BAD). FULL OVERLOAD ACCESS.
C      MACHINE TIME T=(2.76 CYCLES=1.00-7 MINUTES)*IL*MM*(KP**3+3*LQ**3)

C      TO ASSESS EFFECT OF VARYING NUMBER OF PARAMETERS, DEFINE PARAMETERS
C      IN RETENTION ORDER, AND VARY PARAMETER SPAN IN CHANGE-CASE SEQUENCE.

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SAMPLE INPUT: PROGRAM LIKELY


```

IDENT CUS189 XP BH DUANE 3702-C BLDG EXT 3039 28NOV66 LIKELY*0
$JOB B DUANE 3702C BLDG ...LIKELY... LIKELY 1
$EXECUTE IBJOB LIKELY 2
$IBJOB FLOCS,MAP LIKELY 3
$IBFIC THEOR. NODECK LIKELY 4
SUBROUTINE THEORY
COMMON /KEEP/ IL,IX,JY,KP,LAG(25),LINE,LOT(25,10),LOTS,LQ,MAP,MARK INPUT
1, MESHX,MESHY,MM,MOST,NC, BAD,C(25),ERR,FIT,HALF,P(25),PMAX(25) INPUT
2, PMIN(25),RATE,U(101,15),UEXP,UGEN,WLOG,WSUM,YEXP,Z(101,5) INPUT
3, ZDZSDS(6,5,25), M,MAX,Q(5),QP(25,5),QY(5,5),QYP(25,5,5) THEORY
C 2, PMIN(25),RATE,U(101,28),UEXP,UGEN,WLOG,WSUM,YEXP,Z(101,7) INPUT U
C 3, ZDZSDS(6,7,25), M,MAX,Q(7),QP(25,7),QY(7,7),QYP(25,7,7) THEORY U
COMMON /MAX2/ QPP(325,5),QYPP(325,5,5) THEORY
C COMMON /MAX2/ QPP(325,7),QYPP(325,7,7) THEORY U
COMMON /MAX3/ QYY(15,5),QYYP(25,15,5) THEORY
C COMMON /MAX3/ QYY(28,7),QYYP(25,28,7) THEORY U

C MAX=U MAX=1 MAX=2 MAX=3 THEORY
C Q QY QP QYP QPP QYPP QYY QYYP THEORY
C PLOT QUIESCENT FIT PROBABILITY THEORY

C TRIANGULAR TRANSMUTATION QL=(Y1/P11)**PL-SUM(YJ*PJL,J=1,L)
C P(1) P(2) P(3) P(4) P(5) P(6) P(7) P(8) P(9) P(10) P(11) P(12) P(13)
C P11 P2 P12 P22 P3 P13 P23 P33 P4 P14 P24 P34 P44
C 1/N11 H2/H1 1/N12 1/N22 H3/H1 1/N13 1/N23 1/N33 H4/H1 1/N14 1/N24 1/N34 1/N44

ETH=FMPR(P(1),Z(M,1))
TH=ALOG(ETH)

C ENERGY-CONVERSION EXPOSURE Z(M,JY+1)=1.-SUM(C(J)*Z(M,J))/SUM(C(J)*Z(1,J))
ENERGY=0.
SOURCE=0.
DO 10 J=1,JY
ENERGY=FADR(ENERGY,FMPR(C(J),Z(M,J)))
10 SOURCE=FADR(SOURCE,FMPR(C(J),Z(1,J)))
Z(M,JY+1)=FADR(1.,-FDPR(ENERGY,SOURCE))

K=1
DO 600 L=1,LQ
K=K+1

C INITIALIZE MISFII FIELDS Q QY QP QYP QPP QYPP QYY QYYP
DO 400 J1=1,JY
K12=0
DO 400 K1=1,KP

IF(MAX.LI.1) GO TO 300
QYP(K1,J1,L)=0.
IF(MAX.LI.2) GO TO 300
DO 100 K2=K1,KP
K12=K12+1

```

```

      QPP(K12,L)=0.
100 QYPP(K12,J1,L)=0.

      IF(MAX.LT.3) GO TO 300
      DO 200 J2=J1,JY
      J12=((J1-1)*(JY+JY-J1))/2+J2
      QYY(J12,L)=0.
200 QYPP(K1,J12,L)=0.

300 QP(K1,L)=0.
400 QY(J1,L)=0.

      ETHL=EXP(FMPR(P(K),TH))
      Q(L)=ETHL
      PETHL=FMPR(P(K),ETHL)
      QY(1,L)=FDPR(PETHL,Z(M,1))
      QP(1,L)=FDPR(PETHL,P(1))
      QP(K,L)=FMPR(TH,ETHL)

      IF(MAX.LT.1) GO TO 500
      QYP(1,1,L)=FMPR(P(K),FDPR(PETHL,ETH))
      QYP(K,1,L)=FMPR(FADR(1.,FMPR(P(K),TH)),FDPR(ETHL,Z(M,1)))
      IF(MAX.LT.2) GO TO 500
      QPP(1,L)=FMPR(FADR(P(K),-1.),FDPR(QP(1,L),P(1)))
      KK=((K-1)*(KP+KP-K))/2+K
      QPP(KK,L)=FMPR(TH,QP(K,L))
      QPP(K,L)=FMPR(QYP(K,1,L),FDPR(Z(M,1),P(1)))
      QYPP(1,1,L)=FMPR(P(K),FDPR(QPP(1,L),Z(M,1)))
      QYPP(KK,1,L)=FMPR(FADR(2.,FMPR(P(K),TH)),FDPR(QP(K,L),Z(M,1)))
      QYPP(K,1,L)=FMPR(FADR(2.,FMPR(P(K),TH)),FDPR(PETHL,ETH))

      IF(MAX.LT.3) GO TO 500
      QYY(1,L)=FMPR(FADR(P(K),-1.),FDPR(QY(1,L),Z(M,1)))
      QYYP(1,1,L)=FMPR(QYPP(1,1,L),FDPR(P(1),Z(M,1)))
      QYYP(K,1,L)=FDPR(FADR(FMPR(P(K),QYPP(KK,1,L)), -QYP(K,1,L)), Z(M,1))

500 JL=L+1
      DO 600 J=1,JL
      K=K+1
      Q(L)=FADR(Q(L),-FMPR(P(K),Z(M,J)))
      QY(J,L)=FADR(QY(J,L),-P(K))
      QP(K,L)=-Z(M,J)
      IF(MAX.GT.0) QYP(K,J,L)=-1.
600 CONTINUE

C      IF ZERO DIVISOR, SPILL, OR NIL MATRIX, SET RUN BAD, DIAGNOSE WHERE
      CALL SPILL(6HTHEORY)
      RETURN
      END
= GET 4PLOTBASIC LIKELY 5
= GET LIKELY LIKELY*6

```

\$DATA	GO GO GO
\$INPUT	PRTR*00
IL=25, IX=1, JY=2, KP=4, LG=1, MM=41, NC=2, LOTS=2, MAP=1,	PRTR 01
LAG(1)= 4*0,	PRTR 02
LOT(1,1)= 1,2,3,4, LOT(1,2)= 1, 2,2,2,	PRTR 03
C(1)= 239.12700, 240.12910,	PU AMU04
PMIN(1)= -1.+38, 0.,-1.+38,-1.+38,	PRTR 05
P(1)= 8.3291688-01,	0 MODE06
P(2)= 4.3000000-01, 7.3985345-01, 1.3890529+00,	1 MODE07
PMAA(1)= 4*1.+38,	PRTR 08
Z(1,1)=	PU239
.12006+01, .11606+01, .11600+01, .11298+01, .11316+01, .11222+01,	PU239
.11006+01, .10972+01, .10782+01, .10745+01, .10706+01, .10591+01,	PU239
.10005+01, .10381+01, .10039+01, .99583-00, .99290-00, .96877-00,	PU239
.95258-00, .90772-00, .86458-00, .86700-00, .84828-00, .85735-00,	PU239
.84315-00, .80932-00, .80436-00, .79360-00, .77521-00, .78419-00,	PU239
.69305-00, .68842-00, .59239-00, .59235-00, .57041-00, .56024-00,	PU239
.54489-00, .46521-00, .37046-00, .35018-00, .32476-00,	
Z(1,2)=	PU240
.80437-01, .89809-01, .91243-01, .96798-01, .97507-01, .98126-01,	PU240
.10510+00, .10677+00, .11118+00, .11153+00, .11316+00, .11586+00,	PU240
.11012+00, .12021+00, .13056-00, .13181-00, .13325-00, .13860-00,	PU240
.14190-00, .15306-00, .15998-00, .15935-00, .16032-00, .16451-00,	PU240
.16099-00, .17349-00, .17332-00, .17894-00, .17845-00, .18139-00,	PU240
.19537-00, .19992-00, .21651-00, .21662-00, .22034-00, .21928-00,	PU240
.22128-00, .22899-00, .23973-00, .24175-00, .24182-00,	
Z(1,3)=	EXPOSE
.00000+00, .22619-01, .21843-01, .40226-01, .38402-01, .44785-01,	BU
.55450-01, .56647-01, .67296-01, .69841-01, .71459-01, .78007-01,	BU
.76085-01, .90556-01, .10785+00, .11288+00, .11365+00, .12635-00,	BU
.13714-00, .15985-00, .18841-00, .18532-00, .20029-00, .18758-00,	BU
.19591-00, .21592-00, .22097-00, .22259-00, .23741-00, .22742-00,	BU
.28307-00, .28386-00, .33817-00, .33829-00, .35060-00, .35913-00,	BU
.36006-00, .41771-00, .47463-00, .48688-00, .50428-00,	
U(1,1)=	239 239
.28517-06, .30191-06, .38660-06, .52129-06, .48482-06, .59092-06,	ERR99
.78064-06, .86106-06, .10280-05, .10652-05, .11119-05, .13592-05,	ERR99
.12869-05, .15360-05, .20424-05, .22640-05, .22127-05, .25581-05,	ERR99
.28836-05, .34921-05, .43013-05, .42011-05, .50808-05, .44127-05,	ERR99
.45727-05, .51178-05, .51671-05, .53066-05, .53283-05, .51256-05,	ERR99
.66896-05, .60419-05, .63466-05, .64336-05, .65006-05, .73352-05,	ERR99
.65565-05, .60104-05, .58420-05, .46563-05, .43348-05,	
U(1,2)=	239 240
-.28265-06, -.19155-06, -.27419-06, -.20831-06, -.19881-06, -.20995-06,	ERR90
-.21946-06, -.26916-06, -.23323-06, -.21740-06, -.22844-06, -.31249-06,	ERR90
-.27810-06, -.20197-06, -.23950-06, -.30169-06, -.24490-06, -.22357-06,	ERR90
-.21163-06, -.13760-06, -.21039-07, -.35537-07, -.28818-06, -.15712-06,	ERR90
-.71132-07, -.19079-08, .10005-06, .53423-08, .32832-06, .22241-06,	ERR90
.24194-06, .67339-06, .11353-05, .10854-05, .11368-05, .70274-06,	ERR90
.11086-05, .17001-05, .15625-05, .20502-05, .20615-05,	
U(1,3)=	240 240

.28435-06, .19084-06, .27050-06, .21580-06, .20591-06, .22115-06,	ERR00
.24301-06, .29011-06, .27423-06, .26445-06, .27786-06, .36386-06,	ERR00
.33212-06, .29749-06, .38452-06, .45102-06, .40931-06, .44077-06,	ERR00
.47379-06, .53370-06, .59767-06, .59185-06, .84980-06, .71389-06,	ERR00
.70994-06, .78732-06, .74192-06, .85616-06, .70347-06, .73544-06,	ERR00
.11098-05, .92075-06, .11769-05, .12122-05, .13163-05, .16459-05,	ERR00
.14143-05, .16038-05, .23367-05, .21101-05, .22274-05,	
ZDZSDS(1,1,1)= 1.3, -0.025, 0.05, 0., 196.85, 9842.5,	Y1 PU239
ZDZSDS(1,2,1)= -0.02, 0., 0.36, 0., 0., 14960.6,	Y2 PU240
ZDZSDS(1,3,1)= -0.375, 0., 0.575, 0., 0., 14960.6,	X1 EXPOS
\$	END CASE

SAMPLE OUTPUT: PROGRAM LIKELY


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000000000
MAXIMUM LIKELIHOOD NONLINEAR CORRELATED FIELDS
INPUT ----- INPUT
ITERATE LIMIT X-COORDINATES Y-COORDINATES P-COORDINATES Q-COORDINATES MEASUREMENTS CONSTANTS SETS(SUBSETS)
IL 25 IX 1 JY 2 KP 4 LQ 1 MM 41 NC 2 LOTS 2
UGEN(ERATOR) UEXP(ONENT) YEXP(ONENT) WSUM (W=1/U) LOG(DET(W)) RATE(CONVERGE) HALF MOVE RAD (REFIT)
-0. -0. -0. -0. -0. 1.0000000E 00 5.0000000E-01 3.0000000E 00
HYPER PLOTS RMS PLOT FIT PLOT-FIT TRYS ERR(OF CURVE) POINT SYMBOL CURVE SEGMENT PLOTTER GRID PLOTTER GRID
MAP 1 1.0000000E-04 MOST 64 1.0000000E 00 MARK 1 LINE 2 MESHX 10 MESHY 10
CONSTANTS C(N),N=1, 2
2.3912700E 02 2.4012910E 02
PARAMETER-VARIATION DELAY LAG(K),K=1, 4
0 0 0 0
DOUBIOUS-PARAMETER SUBSETS (LOT(K,L),K=1, 4),L=1, 2
1 2 3 4
1 2 2 2
PARAMETER MINIMA PMIN(K),K=1, 4
-1.0000000E 38 0. -1.0000000E 38 -1.0000000E 38
PARAMETER ESTIMATES P(K),K=1, 4
8.3291687E-01 4.3000000E-01 7.3985345E-01 1.3890529E 00
PARAMETER MAXIMA PMAX(K),K=1, 4
1.0000000E 38 1.0000000E 38 1.0000000E 38 1.0000000E 38
MEASUREMENT VECTORS ((IMPRECISE Y(M,J),J=1, 2),(PRECISE X(M,I),I=1, 1)),M=1, 41
M= 1 M= 2 M= 3 M= 4 M= 5 M= 6 M= 7
Y(M, 1)= 1.2006000E 00 1.1606000E 00 1.1600000E 00 1.1298000E 00 1.1316000E 00 1.1222000E 00 1.1006000E 00
Y(M, 2)= 8.0437000E-02 8.9808999E-02 9.1242999E-02 9.6797999E-02 9.7507000E-02 9.8125999E-02 1.0510000E-01
X(M, 1)= -0. 2.2619000E-02 2.1843000E-02 4.0225999E-02 3.8402000E-02 4.4784999E-02 5.5450000E-02
M= 8 M= 9 M= 10 M= 11 M= 12 M= 13 M= 14
Y(M, 1)= 1.0972000E 00 1.0782000E 00 1.0745000E 00 1.0706000E 00 1.0591000E 00 1.0605000E 00 1.0381000E 00
Y(M, 2)= 1.0677000E-01 1.1118000E-01 1.1153000E-01 1.1316000E-01 1.1586000E-01 1.1612000E-01 1.2021000E-01
X(M, 1)= 5.6647000E-02 6.7295999E-02 6.9840999E-02 7.1459000E-02 7.8006999E-02 7.6684999E-02 9.0555999E-02
M= 15 M= 16 M= 17 M= 18 M= 19 M= 20 M= 21
Y(M, 1)= 1.0039000E 00 9.9583000E-01 9.9289999E-01 9.6876999E-01 9.5257999E-01 9.0771999E-01 8.6457999E-01
Y(M, 2)= 1.3056000E-01 1.3181000E-01 1.3325000E-01 1.3860000E-01 1.4190000E-01 1.5306000E-01 1.5998000E-01
X(M, 1)= 1.0785000E-01 1.1288000E-01 1.1365000E-01 1.2635000E-01 1.3714000E-01 1.5985000E-01 1.8841000E-01

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	M= 22	M= 23	M= 24	M= 25	M= 26	M= 27	M= 28
Y(M, 1)=	8.6700000E-01	8.4827999E-01	8.5734999E-01	8.4315000E-01	8.0931988E-01	8.0435998E-01	7.9359999E-01
Y(M, 2)=	1.5935000E-01	1.6032000E-01	1.6451000E-01	1.6899000E-01	1.7349000E-01	1.7332000E-01	1.7894000E-01
X(M, 1)=	1.8532000E-01	2.0029000E-01	1.8758000E-01	1.9591000E-01	2.1592000E-01	2.2097000E-01	2.2259000E-01

	M= 29	M= 30	M= 31	M= 32	M= 33	M= 34	M= 35
Y(M, 1)=	7.7520999E-01	7.8419000E-01	6.9305000E-01	6.8842000E-01	5.9238999E-01	5.9234999E-01	5.7041000E-01
Y(M, 2)=	1.7845000E-01	1.8139000E-01	1.9537000E-01	1.9992000E-01	2.1651000E-01	2.1662000E-01	2.2034000E-01
X(M, 1)=	2.3741000E-01	2.2742000E-01	2.8307000E-01	2.8386000E-01	3.3817000E-01	3.3829000E-01	3.5060000E-01

	M= 36	M= 37	M= 38	M= 39	M= 40	M= 41
Y(M, 1)=	5.6023999E-01	5.4488999E-01	4.6521000E-01	3.7046000E-01	3.5017999E-01	3.2476000E-01
Y(M, 2)=	2.1928000E-01	2.2128000E-01	2.2299000E-01	2.3973000E-01	2.4175000E-01	2.4182000E-01
X(M, 1)=	3.5913000E-01	3.6606000E-01	4.1770999E-01	4.7463000E-01	4.8687999E-01	5.0427999E-01

MEAN SQUARE CORRELATED MEASUREMENT UNCERTAINTY TENSORS ((U(M, J1, J2), J1=1, 2), J2=1, 2), M=1, 41

	J2= 1 1=M	J2= 2 1=M	J2= 1 2=M	J2= 2 2=M	J2= 1 3=M	J2= 2 3=M	J2= 1 4=M
U(M, 1, J2)=	2.8517000E-07	-2.8265000E-07	3.0191000E-07	-1.9155000E-07	3.8659999E-07	-2.7419000E-07	5.2128999E-07
U(M, 2, J2)=	-2.8265000E-07	2.8435000E-07	-1.9155000E-07	1.9084000E-07	-2.7419000E-07	2.7050000E-07	-2.0831000E-07

	J2= 2 4=M	J2= 1 5=M	J2= 2 5=M	J2= 1 6=M	J2= 2 6=M	J2= 1 7=M	J2= 2 7=M
U(M, 1, J2)=	-2.0831000E-07	4.8482000E-07	-1.9881000E-07	5.9092000E-07	-2.0995000E-07	7.8063999E-07	-2.1946000E-07
U(M, 2, J2)=	2.1580000E-07	-1.9881000E-07	2.0591000E-07	-2.0995000E-07	2.2115000E-07	-2.1946000E-07	2.4301000E-07

	J2= 1 8=M	J2= 2 8=M	J2= 1 9=M	J2= 2 9=M	J2= 1 10=M	J2= 2 10=M	J2= 1 11=M
U(M, 1, J2)=	8.6105999E-07	-2.3916000E-07	1.0280000E-06	-2.3323000E-07	1.0652000E-06	-2.1740000E-07	1.1119000E-06
U(M, 2, J2)=	-2.0916000E-07	2.9011000E-07	-2.3323000E-07	2.7422999E-07	-2.1740000E-07	2.6445000E-07	-2.2844000E-07

	J2= 2 11=M	J2= 1 12=M	J2= 2 12=M	J2= 1 13=M	J2= 2 13=M	J2= 1 14=M	J2= 2 14=M
U(M, 1, J2)=	-2.2844000E-07	1.3592000E-06	-3.1249000E-07	1.2869000E-06	-2.7810000E-07	1.5360000E-06	-2.0197000E-07
U(M, 2, J2)=	2.7706000E-07	-3.1249000E-07	3.6386000E-07	-2.7810000E-07	3.3212000E-07	-2.0197000E-07	2.9748999E-07

	J2= 1 15=M	J2= 2 15=M	J2= 1 16=M	J2= 2 16=M	J2= 1 17=M	J2= 2 17=M	J2= 1 18=M
U(M, 1, J2)=	2.0424000E-06	-2.3950000E-07	2.2640000E-06	-3.0169000E-07	2.2127000E-06	-2.4490000E-07	2.5581000E-06
U(M, 2, J2)=	-2.3950000E-07	3.8452000E-07	-3.0169000E-07	4.5102000E-07	-2.4490000E-07	4.0931000E-07	-2.2357000E-07

	J2= 2 18=M	J2= 1 19=M	J2= 2 19=M	J2= 1 20=M	J2= 2 20=M	J2= 1 21=M	J2= 2 21=M
U(M, 1, J2)=	-2.2457000E-07	2.0836000E-06	-2.1163000E-07	3.4921000E-06	-1.3760000E-07	4.3012999E-06	-2.1039000E-08
U(M, 2, J2)=	4.4076999E-07	-2.1163000E-07	4.7379000E-07	-1.3760000E-07	5.3370000E-07	-2.1039000E-08	5.9766999E-07

	J2= 1 22=M	J2= 2 22=M	J2= 1 23=M	J2= 2 23=M	J2= 1 24=M	J2= 2 24=M	J2= 1 25=M
U(M, 1, J2)=	4.2010999E-06	-3.5536999E-06	5.0807999E-06	-2.8818000E-07	4.4126999E-06	-1.5712000E-07	4.5727000E-06
U(M, 2, J2)=	-3.5536999E-06	5.9184999E-07	-2.8818000E-07	8.4979999E-07	-1.5712000E-07	7.1388999E-07	-7.1131999E-08

	J2= 2 25=M	J2= 1 26=M	J2= 2 26=M	J2= 1 27=M	J2= 2 27=M	J2= 1 28=M	J2= 2 28=M

00H00000S							
U(M, 1,J2)=	-7.1131999E-08	5.1177999E-06	-1.9079000E-09	5.1671000E-06	1.0005000E-07	5.3065999E-06	-5.3422999E-09
U(M, 2,J2)=	7.0993999E-07	-1.9079000E-09	7.8731999E-07	1.0005000E-07	7.4191999E-07	-5.3422999E-09	8.5616000E-07
U2= 1 29=M	J2= 2 29=M	J2= 1 30=M	J2= 2 30=M	J2= 1 31=M	J2= 2 31=M	J2= 1 32=M	
U(M, 1,J2)=	5.3282999E-06	3.2832000E-07	5.1255999E-06	2.2241000E-07	6.6895999E-06	2.4193999E-07	6.0418999E-06
U(M, 2,J2)=	3.2832000E-07	7.0346999E-07	2.2241000E-07	7.3543999E-07	2.4193999E-07	1.1698000E-06	6.7339000E-07
U2= 2 32=M	J2= 1 33=M	J2= 2 33=M	J2= 1 34=M	J2= 2 34=M	J2= 1 35=M	J2= 2 35=M	
U(M, 1,J2)=	6.7339000E-07	6.3465999E-06	1.1353000E-06	6.4335999E-06	1.0854000E-06	6.5005999E-06	1.1368000E-06
U(M, 2,J2)=	9.2074999E-07	1.1353000E-06	1.1769000E-06	1.0854000E-06	1.2122000E-06	1.1368000E-06	1.3163000E-06
U2= 1 36=M	J2= 2 36=M	J2= 1 37=M	J2= 2 37=M	J2= 1 38=M	J2= 2 38=M	J2= 1 39=M	
U(M, 1,J2)=	7.3352000E-06	7.0274000E-07	6.5564999E-06	1.1886000E-06	6.0104000E-06	1.7001000E-06	5.8419999E-06
U(M, 2,J2)=	7.0274000E-07	1.6459000E-06	1.1886000E-06	1.4143000E-06	1.7001000E-06	1.6038000E-06	1.5625000E-06
U2= 2 39=M	J2= 1 40=M	J2= 2 40=M	J2= 1 41=M	J2= 2 41=M			
U(M, 1,J2)=	1.5625000E-06	4.6562999E-06	2.0502000E-06	4.3347999E-06	2.0615000E-06		
U(M, 2,J2)=	2.3367000E-06	2.0502000E-06	2.1101000E-06	2.0615000E-06	2.2274000E-06		

UNCERTAINTY NORMALIZATION
SUM(W=1/U) LOG(ABS(DET(W)))
1.1832597E 09 1.1312437E 03

HYPER-SPACE MAPS ((Y1,S1,Y2)=(S1,DS,S2)MILS),J=1, 2), X-SCALES ((X1,DX,X2)=(S1,DS,S2)MILS),I=1, 1),M=1, 1							
DATA	RESOLUTION DZ	DATA	PLOT	RESOLUTION DS	PLOT		
END POINT Z1	(0=DEPENDENT)	END POINT Z2	END POINT S1	(0=DEPENDENT)	END POINT S2		
1.3000000E 00	-2.5000000E-02	4.9999999E-02	0.	1.9685000E 02	9.8425000E 03		
-2.0000000E-02	0.	3.6000000E-01	0.	0.	1.4960600E 04		
-3.7500000E-01	0.	5.7499999E-01	0.	0.	1.4960600E 04		

OUTPUT OUTPUT

PARTITION P	ENTROPY S=LOG(DP/DV)	ITERATIVE LOGIC	PARTITION P	ENTROPY S=LOG(DP/DV)
NORMAL	SN=-(YM-Y)**2	QQ(M)=Q(M) (1/QY(M)U(M)YQ(M))Q(N)	SYMMETRIC	SS=-LOG(EXP((YM-Y)**2)-1)
EXPONENTIAL	SE=-Y+LOG(Y**YM/FACT(YM))	QQ=SUM(QQ(M))	SKEW	SK=-LOG(EXP((YM-Y)**2)+1)
BINOMIAL	S8=SE(Y,YM)+SE(1-Y,N-YM)	DP=-(DQQ/DP)/(DQQQ/DP/DP)		
SEQUENTIAL REFINEMENT TRAJECTORY SEEKING SOLID 36-BIT CONVERGENCE ON LOCAL SECOND-ORDER ENTROPY-MAXIMUM MISFIT-MINIMUM				

GOOD	TRY TAG	PARAMETER	*	MISFIT	MISFIT GRADIENT	DQQQ/DP/DP	NEXT MOVE	NEXT TREND
OR BAD	IT K	P(K)	PIN	QQ=SUM(QQ(M))	DQQ/DP(K)	BLADE PIVOTS	DP(K)	DQQ=(DQQQ/DP)/DP/2
	0	1	8.3291687E-01*		1.3812300E 05	1.2266184E 08	-1.3833307E-03	
	0	2	4.3000000E-01*		-3.5179200E 04	2.6873877E 06	-1.6738656E-03	
	0	3	7.3985345E-01*		-2.9791261E 05	-1.0396839E 05	1.1179211E-03	
GOOD	0	4	1.3890529E 00*	5.6424773E 02	-3.3933045E 04	1.0328534E 05	7.7825372E-03	-3.6465626E 02

000000000S

4

1	1	8.3153354E-01		2.0184832E 03	1.2143731E 03	-1.3915412E-01	
1	2	4.2832613E-01		-7.5925933E 01	2.7141380E 06	-5.3344837E-02	
1	3	7.4097137E-01		-3.5004221E 03	-3.4986255E 03	-6.4302806E-02	
GOOD	1	4	1.3966354E 00	2.1492447E 02	-9.4848819E 02	1.6877393E 03	6.0207908E-02 -5.4424762E 01

2	1	6.9237942E-01		2.1950440E 05	1.1048695E 08	7.9024801E-03	
2	2	3.7496129E-01		-1.2432338E 05	2.7078722E 06	7.9734145E-04	
2	3	6.7666856E-01		-4.8440373E 05	-1.6469610E 05	8.0597216E-03	
BAD	2	4	1.4570433E 00	1.2052078E 03	-5.3203862E 04	1.6616298E 05	-2.6334858E-03 -1.0642733E 03

3	1	8.0790473E-01		8.7391258E 03	1.1953888E 08	-6.2194099E-02	
3	2	4.1926801E-01		-2.0969814E 03	2.7088170E 06	-2.0841813E-02	
3	3	7.3005255E-01		-1.8383493E 04	-8.5292422E 03	-2.7440873E-02	
GOOD	3	4	1.4070589E 00	1.9693451E 02	-2.4297942E 03	6.9194378E 03	1.8607646E-02 -2.0264560E 01

4	1	7.4571063E-01		3.6219768E 04	1.1607637E 08	-1.1611413E-02	
4	2	3.9842620E-01		-1.4376659E 04	2.7340891E 06	-4.4589316E-03	
4	3	7.0261167E-01		-7.9369962E 04	-2.9085158E 04	-4.2155034E-03	
GOOD	4	4	1.4256666E 00	1.9311469E 02	-8.4241806E 03	2.7891156E 04	4.4842646E-03 -2.9782876E 01

5	1	7.3409922E-01		1.8183696E 03	1.1536527E 08	-7.8726933E-02	
5	2	3.9396726E-01		1.4537535E 02	2.7493095E 06	-2.8789614E-02	
5	3	6.9839618E-01		-3.9141990E 03	-3.4725176E 03	-3.2854299E-02	
GOOD	5	4	1.4301508E 00	1.6086982E 02	-4.5771332E 02	1.7175307E 03	2.8472705E-02 -1.5887017E 01

6	1	6.5537228E-01		6.0825184E 04	1.1043572E 08	2.8209275E-04	
6	2	3.6517765E-01		-3.9205166E 04	2.8141656E 06	-5.0015844E-04	
6	3	6.6554188E-01		-1.3279808E 05	-4.7690058E 04	1.3398207E-03	
BAD	6	4	1.4586235E 00	2.1993360E 02	-1.4245121E 04	4.6982417E 04	1.3036453E-03 -7.9864532E 01

7	1	7.0954899E-01		7.4218747E 03	1.1382095E 08	-3.0958880E-02	
7	2	3.8496951E-01		-3.0717049E 03	2.7659508E 06	-1.1685750E-02	
7	3	6.6815089E-01		-1.6213464E 04	-7.6576406E 03	-1.2529664E-02	
GOOD	7	4	1.4390298E 00	1.5315944E 02	-1.7521965E 03	6.2914878E 03	1.2340670E-02 -6.1758815E 00

8	1	6.7859011E-01		1.0172132E 04	1.1174168E 08	-1.5375486E-02	
8	2	3.7330376E-01		-5.5110852E 03	2.7922194E 06	-6.1420280E-03	
8	3	6.7562122E-01		-2.2263100E 04	-9.7271680E 03	-5.9195234E-03	
GOOD	8	4	1.4513704E 00	1.4596128E 02	-2.3766763E 03	8.5291193E 03	7.1710131E-03 -3.9042346E 00

9	1	6.6321462E-01		2.8040900E 03	1.1058423E 08	-2.2324069E-02	
9	2	3.6716173E-01		-1.4005577E 03	2.8074067E 06	-9.0220876E-03	
9	3	6.6970170E-01		-6.1278903E 03	-4.1721875E 03	-8.6831983E-03	
GOOD	9	4	1.4585415E 00	1.4101250E 02	-6.5904740E 02	2.7059065E 03	1.0802386E-02 -1.9361718E 00

10	1	6.4089055E-01		5.7162063E 03	1.0885002E 08	-5.9831538E-03	
10	2	3.5813964E-01		-3.8086434E 03	2.8300021E 06	-2.5502898E-03	

66

BNWL-390

00H00M00S									
	10	3	6.6101850E-01		-1.2492757E 04	-6.3635390E 03	-2.1756969E-03		
GOOD	10	4	1.4693438E 00	1.3906815E 02	-1.3415531E 03	5.1722137E 03	3.3705450E-03	-9.1455448E-01	
	11	1	6.3490740E-01		4.7577738E 02	1.0833092E 08	-7.3807992E-03		
	11	2	3.5558935E-01		-2.5111930E 02	2.8365832E 06	-3.1114019E-03		
	11	3	6.5884281E-01		-1.0374146E 03	2.4577813E 03	-2.7899126E-03		
GOOD	11	4	1.4727144E 00	1.3806221E 02	-1.1272402E 02	-5.3335861E 02	4.0238112E-03	-1.4478413E-01	
	12	1	6.2752660E-01		6.6370548E 02	1.0770382E 08	-6.9187498E-04		
	12	2	3.5247795E-01		-4.7570440E 02	2.8446231E 06	-3.0128096E-04		
	12	3	6.5605289E-01		-1.4494880E 03	-2.5100000E 03	-2.4738492E-04		
GOOD	12	4	1.4767382E 00	1.3791307E 02	-1.5620445E 02	7.6154274E 02	4.1573221E-04	-1.1119143E-02	
	13	1	6.2683473E-01		6.3250809E 00	1.0764070E 08	-9.1028796E-05		
	13	2	3.5217667E-01		-3.4583626E 00	2.8453927E 06	-4.7283108E-05		
	13	3	6.5580551E-01		-1.3775395E 01	2.4135312E 03	-3.4206528E-05		
GOOD	13	4	1.4771539E 00	1.3790198E 02	-1.5054331E 00	-6.7200165E 00	7.8933449E-05	-2.9931473E-05	
	14	1	6.2674369E-01		2.5186539E-02	1.0762865E 08	-1.8462640E-03		
	14	2	3.5212939E-01		-1.9767761E-02	2.8454344E 06	-7.9691357E-08		
	14	3	6.5577130E-01		-5.2982330E-02	2.4128320E 03	-6.8023559E-04		
GOOD	14	4	1.4772328E 00	1.3790171E 02	-6.7729950E-03	2.4078369E-02	-1.5320750E-03	-4.1111150E-08	
	15	1	6.2489743E-01		-3.9748663E 01	1.0785229E 08	-4.6669934E-06		
	15	2	3.5212931E-01		3.0004076E 01	2.8529672E 06	8.0901920E-08		
	15	3	6.5509107E-01		8.6119224E 01	2.4102031E 03	-2.3607876E-06		
BAD	15	4	1.4757008E 00	1.3790172E 02	9.5705853E 00	4.1100372E 01	-5.6406953E-06	-3.4679911E-05	
	16	1	6.2666698E-01		-4.8916817E-02	1.0763793E 08	-8.6638585E-05		
	16	2	3.5212938E-01		5.9692383E-02	2.8457465E 06	2.6604031E-08		
	16	3	6.5574304E-01		9.0331078E-02	2.4127695E 03	-3.1920147E-05		
GOOD	16	4	1.4771692E 00	1.3790150E 02	1.8940926E-02	1.0257111E 00	-7.2035060E-05	-4.0600696E-09	
	17	1	6.2658035E-01		-1.6642570E-01	1.0764842E 08	1.6642064E-04		
	17	2	3.5212941E-01		1.0774040E-01	2.8460990E 06	-8.1971434E-10		
	17	3	6.5571113E-01		3.7310886E-01	2.4127383E 03	6.1320634E-05		
BAD	17	4	1.4770972E 00	1.3790219E 02	3.4806252E-02	2.5204468E-01	1.3816830E-04	-4.1834766E-09	
	18	1	6.2666681E-01		-1.8426037E-01	1.0763795E 08	-5.2882469E-05		
	18	2	3.5212938E-01		1.5280342E-01	2.8457465E 06	8.4120466E-08		
	18	3	6.5574299E-01		3.8895416E-01	2.4129375E 03	-1.9484493E-05		
BAD	18	4	1.4771690E 00	1.3790179E 02	4.9426079E-02	1.5488586E 00	-4.4199057E-05	-3.0818086E-09	
	19	1	6.2666698E-01		-4.8916817E-02	1.0763793E 08	-8.6638585E-05		
	19	2	3.5212938E-01		5.9692383E-02	2.8457465E 06	2.6604031E-08		
	19	3	6.5574304E-01		9.0331078E-02	2.4127695E 03	-3.1920147E-05		
GOOD	19	4	1.4771692E 00	1.3790150E 02	1.8940926E-02	1.0257111E 00	-7.2035060E-05	-4.0600696E-09	

MAXIMUM-LIKELIHOOD STATISTICAL ANALYSIS										
BEST FIT			ROOT MEAN		MEASUREMENT	MISFIT	GRADIENT	NORMAL BLADES OF		CURVATURE TENSOR
TAG	PARAMETER	*	SQUARE	DOUBT	VARIANCE	DOQ/DP(K)	DDQ/DP(K1)/DP(K1)	DDQ/DP(K1)/DP(K2)	BY SADDLE-LEG	NEXT MOVE DP(K)
K	P(K)	PIN	DP(K)	I*I=-1	QQ/(M*LG-KP)	SHOULD BE 0	K1	MINUS=SADDLE	NONZERO=SADDLE	K2 MATRIX INVERSE
1	0.2666698E-01		2.6963519E 00			-4.8916817E-02	3	1.0763793E 0P	0.	2 -8.6676794E-05
2	3.5212938E-01		1.6949595E-02			5.9692383E-02	2	2.8457463E 06	0.	4 2.7007672E-08
3	6.5574304E-01		9.9351726E-01			9.0331078E-02	4	2.4127695E 03	0.	1 -3.1934469E-05
4	1.4771692E 00		2.2389011E 00		3.7270676E 00	1.8940926E-02	1	1.0252838E 00		-7.2068069E-05

NON-LINEAR MAXIMUM-LIKELIHOOD THEORY-REGRESSION ANALYSIS (I*I=-1)										
PARAMETER	BEST THEORY	MISFIT-WORTH	NORMAL BLADES	THEORY WORTH	WORTH RATIO	FREEDOM	PROBABILITY	THEORY		
SUBSET	TAG	PARAMETER	* THEORY	SUBSET	THEORY COSET	SUBSET+COSET	SUBSET/THEORY	SET COSET	OF MORE	WORTH SUBSET
LOT(K)	K	P(K)	PIN	DOQ*COB(A)**2	DOQ*SIN(A)**2	DOQ=P.QQPP.P/2	(EXPECT KS/KP)	KS	KC	(EXPECT 1/2) SEEMS
1	1	6.2666698E-01		1.2500000E-01	2.2874111E 07	2.2874111E 07	5.4646931E-09	1	3	8.5569201E-01 BAD
2	2	3.5212938E-01		1.6086641E 03	2.2872503E 07	2.2874112E 07	7.0326843E-05	1	3	6.6409861E-09 GOOD
3	3	6.5574304E-01		1.5000000E 00	2.2874110E 07	2.2874111E 07	6.5576317E-08	1	3	5.2972422E-01 BAD
4	4	1.4771692E 00		1.6406250E 00	2.2874110E 07	2.2874112E 07	7.1724097E-08	1	3	5.1114265E-01 BAD
1	1	6.2666698E-01		1.2500000E-01	2.2874111E 07	2.2874111E 07	5.4646931E-09	1	3	8.5569201E-01 BAD
2	2	3.5212938E-01								
2	3	6.5574304E-01								
2	4	1.4771692E 00		1.1695219E 05	2.2757159E 07	2.2874112E 07	5.1128625E-03	3	1	5.3386435E-05 GOOD

MEAN SQUARE CORRELATED PARAMETER DOUBT (DPDP(K1,K2),K1=1, 4),K2=1, 4

	K2= 1	K2= 2	K2= 3	K2= 4
K1= 1	7.2703139E 00	-1.2089155E-04	2.6788716E 00	6.0350047E 00
K1= 2	-1.2089155E-04	2.8728877E-04	-3.9857344E-05	-1.0381511E-03
K1= 3	2.6788716E 00	-3.9857344E-05	9.6707656E-01	2.2236825E 00
K1= 4	6.0350047E 00	-1.0381511E-03	2.2236825E 00	5.0126781E 00

MEAN SQUARE CORRELATED DOUBT PROPAGATES TO FUNCTION F HAVING PARAMETER GRADIENT DF/DP AS DFOF=(DF/DP).(DPDP).(DF/DP)

HYPER-SPACE CROSS PLOTS

DO PLOT 1 TIME 00H00M00S

NON-LINEAR MULTIPLE-FIELD MAXIMUM-LIKELIHOOD THEORY-MEASUREMENT CORRELATION ANALYSIS (I*I=-1)										
TAG	J	MEASUREMENT	BEST THEORY	RESOLUTION	BEST THEORY	MEAN SQUARE	MEAN SQUARE	PROBABILITY	FIT	
M	I	Y(M,J)	X(M,I)	ERROR DY(M,J)	ERROR DDY(M,J)	L	MISFIT Q(M,L)	MISFIT QWQ(M)	WORBLE DDQ(M)	OF LESS ERROR IS
1	Y1	1.2006000E 00	-1.3013823E-03	-1.7311808E 00	Q1	-1.4362335E-03	6.1043640E 00	1.2617188E 00	7.4009002E-01	GOOD
1	Y2	8.0437000E-02	1.3162370E-03	1.7509415E 00						
1	X1	0.								

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7

2	Y1	1.1606000E-00	1.5638167E-04	-1.2939409E-00	Q1	2.1941215E-04	1.9774260E-01	3.9184570E-02	1.6016279E-01	GOOD
2	Y2	8.9808999E-02	-1.8768124E-04	1.6148784E-00						
2	X1	2.3871832E-02								
3	Y1	1.1600000E-00	-1.2731266E-03	-1.4405599E-00	Q1	-1.6681720E-03	8.2829721E-00	1.1367188E-00	8.3072419E-01	GOOD
3	Y2	9.1242999E-02	1.4606306E-03	1.6527233E-00						
3	X1	2.3216277E-02								
4	Y1	1.1298000E-00	5.8529667E-04	-6.5708114E-01	Q1	1.6657989E-03	8.8838559E-00	1.1289063E-00	8.4829558E-01	GOOD
4	Y2	9.6797999E-02	-1.2781654E-03	1.4349277E-00						
4	X1	4.2431377E-02								
5	Y1	1.1316000E-00	-2.4144925E-05	-6.9148431E-01	Q1	-6.5336004E-05	1.4416350E-02	1.6021729E-03	4.5804262E-02	GOOD
5	Y2	9.7507000E-02	5.0442401E-05	1.4446153E-00						
5	X1	4.0470988E-02								
6	Y1	1.1222000E-00	6.7234476E-04	-4.8512911E-01	Q1	2.5856383E-03	2.0110765E-01	1.9843750E-00	9.7471271E-01	GOOD
6	Y2	9.8125999E-02	-1.9226975E-03	1.3873188E-00						
6	X1	4.7321767E-02								
7	Y1	1.1006000E-00	3.3455276E-05	-1.4606797E-01	Q1	4.2265840E-04	4.5009329E-01	2.6611328E-02	2.5893869E-01	GOOD
7	Y2	1.0510000E-01	-2.9462105E-04	1.2916190E-00						
7	X1	5.8713264E-02								
8	Y1	1.0972000E-00	-1.2692165E-04	-3.0506293E-01	Q1	-7.7007524E-04	1.3004772E-00	7.2265625E-02	4.2696463E-01	GOOD
8	Y2	1.0677000E-01	5.9349326E-04	1.3303505E-00						
8	X1	6.0057923E-02								
9	Y1	1.0782000E-00	1.5322627E-05	1.4108740E-01	Q1	-1.9978359E-04	8.2421875E-02	3.7841797E-03	1.1316337E-01	GOOD
9	Y2	1.1118000E-01	1.3139648E-04	1.2098701E-00						
9	X1	7.1429715E-02								
10	Y1	1.0745000E-00	-9.9521714E-05	2.7886617E-01	Q1	6.5570883E-04	8.8592803E-01	5.5664063E-02	3.5666271E-01	GOOD
10	Y2	1.1153000E-01	-4.1892491E-04	1.1758542E-00						
10	X1	7.4042953E-02								
11	Y1	1.0706000E-00	4.4731309E-05	2.6656432E-01	Q1	-3.0792318E-04	1.8635260E-01	1.1840820E-02	1.6791788E-01	GOOD
11	Y2	1.1316000E-01	1.9725191E-04	1.1754702E-00						
11	X1	7.5809166E-02								
12	Y1	1.0591000E-00	3.2135788E-06	1.1032041E-01	Q1	-5.3249300E-05	4.4423917E-03	2.0980835E-04	2.6334397E-02	GOOD
12	Y2	1.1586000E-01	3.5247727E-05	1.2100352E-00						
12	X1	8.2667961E-02								
13	Y1	1.0605000E-00	1.0064144E-04	1.9321349E-01	Q1	-9.5262564E-04	1.5209896E-00	7.2265625E-02	4.5968708E-01	GOOD
13	Y2	1.1612000E-01	6.1981365E-04	1.1899309E-00						

69

BNWL-390

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13	X1	8.1371620E-02							
14	Y1	1.0381000E-00	-4.9652509E-04	7.4559399E-01	Q1	1.2088045E-03	2.2991158E-00	1.2304688E-01	5.4763499E-01 GOOD
14	Y2	1.2021000E-01	-6.9590757E-04	1.0049915E-00					
14	X1	9.5647595E-02							
15	Y1	1.0039000E-00	7.7094661E-04	7.9610296E-01	Q1	-1.7328393E-03	3.5437376E-00	1.7382813E-01	6.5136108E-01 GOOD
15	Y2	1.3056000E-01	9.8634400E-04	1.0210677E-00					
15	X1	1.1422657E-01							
16	Y1	9.9563000E-01	2.6411175E-04	6.7668622E-01	Q1	-6.9816597E-04	5.1097237E-01	1.9897461E-02	2.7807530E-01 GOOD
16	Y2	1.3181000E-01	4.0694446E-04	1.0477651E-00					
16	X1	1.1954489E-01							
17	Y1	9.9289999E-01	8.2767067E-04	8.2978536E-01	Q1	-1.7823763E-03	3.4693469E-00	1.2890625E-01	6.4921889E-01 GOOD
17	Y2	1.3325000E-01	1.0073354E-03	1.0099089E-00					
17	X1	1.2070300E-01							
18	Y1	9.6876999E-01	6.3017390E-04	9.6166976E-01	Q1	-1.1608563E-03	1.2896598E-00	5.3710938E-02	4.2817588E-01 GOOD
18	Y2	1.3860000E-01	6.3619351E-04	9.7065393E-01					
18	X1	1.3534164E-01							
19	Y1	9.3257999E-01	2.2632912E-04	1.0423480E-00	Q1	-3.8237870E-04	1.2553085E-01	5.7678223E-03	1.3941832E-01 GOOD
19	Y2	1.4190000E-01	2.0561676E-04	9.4695618E-01					
19	X1	1.4539036E-01							
20	Y1	9.0771999E-01	1.0240253E-03	1.1654528E-00	Q1	-1.4955923E-03	1.5691007E-00	7.8125000E-02	4.6556718E-01 GOOD
20	Y2	1.3306000E-01	7.7036076E-04	9.0106168E-01					
20	X1	1.7165376E-01							
21	Y1	8.6457999E-01	-2.0794915E-03	1.3435256E-00	Q1	2.6342030E-03	3.9761523E-00	1.9628906E-01	6.7946553E-01 GOOD
21	Y2	1.5998000E-01	-1.3222164E-03	8.5426251E-01					
21	X1	1.9969765E-01							
22	Y1	8.6700000E-01	-2.1499493E-03	1.3215304E-00	Q1	2.7715117E-03	4.4939232E-00	2.0019531E-01	7.1073983E-01 GOOD
22	Y2	1.5935000E-01	-1.3986642E-03	8.5974290E-01					
22	X1	1.9850277E-01							
23	Y1	8.4827999E-01	-4.2911738E-03	9.7538949E-01	Q1	7.4374564E-03	2.6210474E-01	1.0976563E-00	9.9245633E-01 GOOD
23	Y2	1.6032000E-01	-4.0953372E-03	9.3067558E-01					
23	X1	2.1235192E-01							
24	Y1	8.5734999E-01	1.0902748E-03	1.0909170E-00	Q1	-1.6958900E-03	1.5341312E-00	6.7138672E-02	4.6224341E-01 GOOD
24	Y2	1.6451000E-01	9.0765523E-04	9.0818983E-01					
24	X1	2.0196995E-01							
25	Y1	8.4315000E-01	2.5973071E-03	1.1798222E-00	Q1	-3.7136860E-03	7.0511180E-00	3.3398438E-01	8.1698970E-01 GOOD

70

BNWL-390

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25 Y2 1.6899000E-01 1.9476466E-03 8.8471506E-01

25 X1 2.0956091E-01

26 Y1 8.6931996E-01 -2.7903607E-04 1.1993454E 00 Q1 3.8635650E-04 6.7471077E-02 3.1127930E-03 1.0243453E-01 GOOD

26 Y2 1.7349000E-01 -2.0273377E-04 8.7138487E-01

26 X1 2.3243568E-01

27 Y1 8.0435998E-01 -1.7386141E-03 1.3184775E 00 Q1 2.1878816E-03 2.1626689E 00 1.0400391E-01 5.3507851E-01 GOOD

27 Y2 1.7332000E-01 -1.1141479E-03 8.4491373E-01

27 X1 2.3643975E-01

28 Y1 7.9359999E-01 1.9099689E-03 1.1365650E 00 Q1 -2.7750134E-03 3.2502498E 00 1.5087891E-01 6.3101395E-01 GOOD

28 Y2 1.7694000E-01 1.4793467E-03 8.8031471E-01

28 X1 2.4043269E-01

29 Y1 7.7520999E-01 -3.2424120E-03 1.4837978E 00 Q1 3.5788193E-03 5.5205051E 00 2.6757813E-01 7.5970994E-01 GOOD

29 Y2 1.7845000E-01 -1.7566252E-03 8.0386964E-01

29 X1 2.5516847E-01

30 Y1 7.8419000E-01 2.9056216E-03 1.3648368E 00 Q1 -3.5010353E-03 5.3616239E 00 2.5292969E-01 7.5301953E-01 GOOD

30 Y2 1.8139000E-01 1.7679663E-03 8.3039695E-01

30 X1 2.4585634E-01

31 Y1 6.9305000E-01 -1.6876681E-03 1.0661542E 00 Q1 2.4626143E-03 1.8582066E 00 7.3730469E-02 5.0350931E-01 GOOD

31 Y2 1.9537000E-01 -1.3544462E-03 8.6590880E-01

31 X1 3.0602723E-01

32 Y1 6.8842000E-01 2.6274705E-03 1.5848412E 00 Q1 -2.9800199E-03 2.9457013E 00 1.3427734E-01 6.0754244E-01 GOOD

32 Y2 1.9992000E-01 1.5276935E-03 8.0518997E-01

32 X1 3.0607478E-01

33 Y1 5.9238999E-01 2.4277239E-03 1.2738998E 00 Q1 -2.8389655E-03 2.1690969E 00 1.1181641E-01 5.3492712E-01 GOOD

33 Y2 2.1651000E-01 1.5333482E-03 8.0459399E-01

33 X1 3.6801648E-01

34 Y1 5.9234999E-01 2.4243229E-03 1.2368977E 00 Q1 -2.9919893E-03 2.3791891E 00 1.1914063E-01 5.5629954E-01 GOOD

34 Y2 2.1662000E-01 1.6279132E-03 8.1050740E-01

34 X1 3.6796149E-01

35 Y1 5.7041000E-01 2.7114539E-03 1.1677521E 00 Q1 -3.4132078E-03 2.9394277E 00 1.5820313E-01 6.0496204E-01 GOOD

35 Y2 2.2034000E-01 1.8957710E-03 8.1645882E-01

35 X1 3.6216843E-01

36 Y1 5.6023999E-01 -2.5676193E-04 8.8104253E-01 Q1 4.2568892E-04 4.1106125E-02 2.0523071E-03 7.9877980E-02 GOOD

36 Y2 2.1928000E-01 -2.4976829E-04 8.5704485E-01

36 X1 3.9093593E-01

71

BNWL-390

00H00H00S											10
37	Y1	5.448899E-01	-6.129014E-04	1.103701E-00	Q1	8.032433E-04	1.561478E-01	9.216308E-03	1.542526E-01	GOOD	
37	Y2	2.212800E-01	-4.553606E-04	8.200046E-01							
37	X1	4.013476E-01									
38	Y1	4.652100E-01	-3.566586E-03	1.066979E-00	Q1	4.573140E-03	4.653371E-00	4.453125E-01	7.058002E-01	GOOD	
38	Y2	2.289900E-01	-2.696672E-03	8.067361E-01							
38	X1	4.574889E-01									
39	Y1	3.704600E-01	-4.630572E-04	6.415851E-01	Q1	9.112730E-04	1.497150E-01	2.688598E-02	1.411736E-01	GOOD	
39	Y2	2.397300E-01	-5.895163E-04	8.167995E-01							
39	X1	5.230162E-01									
40	Y1	3.501799E-01	3.396137E-04	8.218304E-01	Q1	-5.115121E-04	5.205760E-02	1.240539E-02	8.048523E-02	GOOD	
40	Y2	2.417500E-01	3.310455E-04	8.010960E-01							
40	X1	5.372599E-01									
41	Y1	3.247600E-01	-4.421032E-04	7.586707E-01	Q1	7.022246E-04	9.687997E-02	2.845764E-02	1.056018E-01	GOOD	
41	Y2	2.418200E-01	-4.643820E-04	7.971136E-01							
41	X1	5.570431E-01									
SUM							1.379015E-02	1.017369E-01			
GOOD RUN SOLID CONVERGENCE TRUE MINIMUM NO SADDLE NO SPILL NO DIVISION BY ZERO CAVEAT EMPTOR MULTIPLE SOLUTIONS

FIGURE 5. Sample Output: Program Likely

FIGURE 6. Sample Output: Program Likely

INPUT LOGIC: PROGRAM LEARN


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C BATTLE NORTHWEST LABORATORY PROGRAM LEARN (SCALAR) BNWC-8-6B
C NAMELIST DEFINITION
C $INPUT INPUT LIST
C
C INTEGER INPUT
C $I= ITERATION LIMIT (MINIMUM 0, NO MAXIMUM, TRY 100) PRESET 0
C $JJ= PARAMETER SPAN (MINIMUM 1, MAXIMUM 100) PRESET 1
C $KK= COORDINATE SPAN (MINIMUM 0, MAXIMUM 4) PRESET 1
C $LL= LOGIC-CONTROL SPAN (TYPICAL 0, MAXIMUM 100) THEORY OPTION PRESET 0
C $MM= MEASUREMENT SPAN (MINIMUM 1, TYPICAL 10*$JJ, MAXIMUM 1000) PRESET 1
C $MAP= THEORY-MEASUREMENT HYPER-SPACE PLOTS (NONE 0, MAXIMUM 25) PRESET 0
C
C -MAP MEANS MORE MAPS LATER (BS END-OF-FILE BYPASS)
C
C +MAP MEANS LAST MAP CASE (BS END-OF-FILE, REWIND, UNLOAD)
C
C (UNLIMITED PLOT SETS BY 0-ITERATION CHANGE-CASE SEQUENCE)
C
C $N= MULTIPLE-SOLUTION LIMIT (MINIMUM 1, NO MAXIMUM) DEAD PRESET 1
C $LAG(1)= $LAG(J),J=1,$JJ ITERATIONS TO LAG BEFORE PARAMETER VARIATION
C
C GOOD GUESS OR LINEAR LAG=0, BAD GUESS NON-LINEAR LAG=1,
C EXPONENTIAL LAG=2, EXPLOSIVE LAG=3 (MIN 0, NO MAX)
C $LOT(1)= $LOT(J),J=1,$JJ PARAMETER SUBSETS OF DUBIOUS WORTH
C
C (0 NULL, 1 SET-1, 2 SET-2, ...) OTHER THAN OBVIOUS $LOT(J)=J
C
C (FOR WORTH EVALUATION BY THEORY REGRESSION TO A(J($LOT))=0.)
C
C EXPONENTIAL INPUT
C $B/F= BAD-FIT MULTIPLE OF ROOT-MEAN-SQUARE MISFIT PRESET 3.
C
C MINUS BAD MEANS (DISCARD BAD DATA, SET BAD PLUS, RUN REFIT)
C $HALF= USE DA(J)=HALF*DA(J) IF QUADRIC AND CUBIC EXPANSION FAIL PRESET 5
C
C (HALF=-.5 AIDS QUARTIC RECOVERY AT RISK OF QUINTIC FAILURE)
C $A(1)= A(J),J=1,$JJ PARAMETER ESTIMATES CLOSE ENOUGH TO AVOID
C
C SADDLE POINTS AND MULTIPLE SOLUTIONS
C $AMIN(1)= AMIN(J),J=1,$JJ PARAMETER MINIMA (MULTIPLE-SOLUTION LIMIT)
C $AMAX(1)= AMAX(J),J=1,$JJ PARAMETER MAXIMA (MULTIPLE-SOLUTION LIMIT)
C
C TO BYPASS AMIN=1.+38 AMAX=1.+38 TO FIX AMIN=AMAX
C $C(1)= C(L),L=1,$LL LOGIC CONTROLS THEORY OPTION
C $YM(1)= YM(M),M=1,$MM IMPRECISE MEASUREMENTS
C $W(1)= W(M),M=1,$MM STOCHASTIC WEIGHTS (INVERSE MEAN SQUARE DOUBT)
C $UGEN= NOT -0. GENERATE UNCERTAINTY W(M)=1/(YM(M)**UGEN) PRESET-0
C $UEXP= NOT -0. EXPONENTIAL UNCERTAINTY W(M)=W(M)**UEXP PRESET-0
C $YEXP= NOT -0. DO W(M)=W(M)/((YM(M)*YM(M))**YEXP) PRESET-0
C $WSUM= NOT -0. NORMALIZE STOCHASTIC-WEIGHT SUM TO WSUM PRESET 1.
C $WLOG= NOT -0. NORMALIZE STOCHASTIC-WEIGHT PRODUCT TO EXP(WLOG) PRESET-0
C $X(1,1)= (X(M,K),M=1,$MM,K=1,$KK PRECISE MEASUREMENT COORDINATES
C
C (SUCH AS MEASUREMENT TIME, POSITION, OR PHASE)
C
C
C 1 MIL BY 1 MIL PLOTTER GRID HAS 20000 MIL BY 20000 MIL SPAN
C $ZDZS(1,1,1)= (Y-SCALE(Y1,DY,Y2)=(S1,DS,S2)MILS,
C
C (X-SCALE(X1,DX,X2)=(S1,DS,S2)MILS),K=1,$KK),M=1,$MAP
C
C DZ=0. Y-DATA IMPLICIT RESOLUTION INCREMENT
C
C DZ.NE.0. X-DATA EXPLICIT RESOLUTION INCREMENT
C
C DATA TRANSFORM FLEXIBLY VIA SUBROUTINE PLOT(Z)

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C          THEN TRANSLATE AND SCALE LINEARLY TO MATCH ENDS
C          DS=0.    Y-PLOT  IMPLICIT RESOLUTION INCREMENT
C          DS.NE.0. X-PLOT  EXPLICIT RESOLUTION INCREMENT (MILS)
C          OFF-SCALE POINTS PLOT PROJECTED TO NEAREST MARGIN

C          PLOTTER GRID, RANGE, SYMBOL, AND CURVE ALTERATION OPTIONS  PRESET
C MESHX= (10 9 8 7 6 5 4 3 2 1)=(1 2 3 4 5 6 7 8 9 10)MIL X-GRID  MESHX=10
C MESHY= (10 9 8 7 6 5 4 3 2 1)=(1 2 3 4 5 6 7 8 9 10)MIL Y-GRID  MESHY=10
C          2-10 MIL PLOTTER GRID HAS 30000 MIL SPAN
C MARK= (0 1 2 3 4 5 6 7 8 9 10 11)=( * )SYMBOL MARK=01
C          (0 1 2 3 4 5 6 7 8 9 10 11)=(. 1 2 3 4 5 6 7 8 9 0 -)NUMBER
C          MINUS MARK SUPERPOSES PLOTS, WITH CYCLIC MARKS
C LINE= (2 4 6)=(LINES POINTS ARCS) CURVE-SEGMENTATION MODE  LINE=2
C          MIDS LINE OUTPUTS BOTH PLOTS AND EQUIVALENT NUMERIC TABLES
C ER= ROOT-MEAN-SQUARE-ERROR MULTIPLE FOR ERROR-CURVE PLOTS  ERR=-1.
C          PLUS ERR PLOTS INPUT MEASUREMENT UNCERTAINTY
C          MINUS ERR PLOTS MAXIMUM-LIKELIHOOD MEASUREMENT UNCERTAINTY
C ER= CONVERGED-PLOT FRACTION OF ROOT-MEAN-SQUARE ERROR  DEAD FIT=1,-4
C MS= PLOT-ITERATION LIMIT (PLOT AND WRITE DUBIOUS POINT)  DEAD MOST=64

C $      END OF CASE

C  RUN TERMINATES AT SOLID 36-BIT CONVERGENCE ON LOCAL SECOND-ORDER
C  MINIMUM OF MEAN SQUARE RESIDUAL MISFIT  $\sigma$ , AT OCCURRENCE OF NULL
C  DETERMINANT, AT SPILL OR DIVISION BY ZERO, OR AT ITERATION LIMIT.
C  TO TERMINATE RUN AT END OF THIS ITERATION, DEPRESS SENSE-SWITCH 1.
C  AT END OF RUN, PARAMETERS A(J) ARE BEST VALUES, UGEN=UEXP=YEXP=-0.,
C  WEIGHTS W(M) ARE NORMALIZED, OTHER INPUT IS UNDISTURBED (FOR +BAD).
C  CHANGE-CASE OVERLOAD FLEXIBILITY ON ALL INPUT.
C  MACHINE TIME IS(2.76 CYCLES=1.00-7 MINUTES)*NN*II*MM*JJ**3

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SAMPLE INPUT: PROGRAM LEARN

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10 RUN 50 ,CUS189,15,150	BH DUANE..05MAY66...BH DUANE....	LEARN*0
10 TEL BHW JOB B	IAPES U9233R1,PLOT	LEARN 1
10 RX MSG A=U9233,H=PLOT		LEARN 2
10N XOT CUR		LEARN 3
10 A		LEARN 4
10 PEF A		LEARN 5
10 IF FOR THEORY,THEORY		LEARN 6
SUBROUTINE THEORY		
COMMON /FIT/ II,JJ,KK,LL,LINE,LL,MAP,MARK,MESHX,MESHY,MM,NN, BAD		INPUT
1, A(100),AMAX(100),AMIN(100),C(100),LAG(100)		INPUT
2, w(1001),YM(1001), X(1001,4), ZDZSDS(6,5,25)		INPUT
3, GM,DGM,DQT,DDGM,DDQT, M,MAX,Y, DY(100), DDY(101,100)		THEORY
C THEORY QUARTER-WAVE COSINE TRANSFORM		
Y=0.		
DO 200 J2=1,JJ		
DY(J2)=COS(FMPR(FLOAT(J2+J2-1),FMPR(1.5707963,X(M,1))))		
IF(MAX,LI,2) GO TO 200		
DO 100 J1=1,J2		
100 DDY(J1,J2)=0.		
200 Y=FAUR(Y,FMPR(A(J2),DY(J2)))		
C IF ZERO DIVISOR, SPILL, OR NIL MATRIX, SET RUN BAD, DIAGNOSE WHERE		
CALL SPILL(6nTHEORY)		
RETURN		
END		-
10N XOT LEARN		GO GO GO
9INPUT I1=10, JJ=10, KK=1, LL=0, MM=101, NN=1, MAP=2, BAD=2.,		SQUARE 1
LAG(1)=10*0, AMIN(1)=10*-1.+38, AMAX(1)=10*1.+38,		SQUARE 2
A(1)=10*1., w(1)=101*1., YM(1)=100*1., 0.,		SQUARE 3
X(1,1)= 0.0, .01, .02, .03, .04, .05, .06, .07, .08, .09,		SQUARE 4
.10, .11, .12, .13, .14, .15, .16, .17, .18, .19,		SQUARE 5
.20, .21, .22, .23, .24, .25, .26, .27, .28, .29,		SQUARE 6
.30, .31, .32, .33, .34, .35, .36, .37, .38, .39,		SQUARE 7
.40, .41, .42, .43, .44, .45, .46, .47, .48, .49,		SQUARE 8
.50, .51, .52, .53, .54, .55, .56, .57, .58, .59,		SQUARE 9
.60, .61, .62, .63, .64, .65, .66, .67, .68, .69,		SQUARE10
.70, .71, .72, .73, .74, .75, .76, .77, .78, .79,		SQUARE11
.80, .81, .82, .83, .84, .85, .86, .87, .88, .89,		SQUARE12
.90, .91, .92, .93, .94, .95, .96, .97, .98, .99,		SQUARE13
1.0,		SQUARE14
LOI(1)= 1,1, 2,2, 3,3, 4,4, 5,5,		SQUARE15
ZDZSDS(1,1,1)= -.1, 0., 1.3, 0., 0., 7000.,		SQUARE16
ZDZSDS(1,2,1)= 0., .005, 1., 0., 50., 10000.,		SQUARE17
ZDZSDS(1,1,2)= .80, 0., 1.50, 0., 0., 7000.,		SQUARE18
ZDZSDS(1,2,2)= .75, .00125, 1., 0., 50., 10000.,		SQUARE19
MARK=-1,		MOSAIC
5		END CASE
10AN XOT CUR		LEARN 7
10 A		LEARN*8
10 MSC LABEL TAPE H	PLOT TAPE H ON HENSEN-LEHNER PLOTTER	PLOT 1
10 MSC BH DUANE	CUS189 3702C BLDG 300 AREA 05MAY66	PLOT 2
10 MSC 1 PLOT	KE-46-1320 (10X71N 20X20/IN) HOLES AT TOP	PLOT 3
10 MSC SYMBOL HEAD	SCRATCH TAPE	PLOT 4
10 END		END RUN

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SAMPLE OUTPUT: PROGRAM LEARN

85

TIME 1.9108072-02M								2
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	
9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	9.9009901-03	

COORDINATES (X(M),K),M=1, 101,K=1, 1

0.0000000	9.9999999-03	2.0000000-02	3.0000000-02	3.9999999-02	4.9999999-02	5.9999999-02	6.9999999-02
7.9999999-02	9.0000000-02	9.9999999-02	1.1000000-01	1.2000000-01	1.3000000-01	1.3999999-01	1.5000000-01
1.6000000-01	1.6999999-01	1.8000000-01	1.9000000-01	2.0000000-01	2.0999999-01	2.2000000-01	2.3000000-01
2.3999999-01	2.5000000-01	2.6000000-01	2.7000000-01	2.7999999-01	2.9000000-01	3.0000000-01	3.0999999-01
3.1999999-01	3.3000000-01	3.4000000-01	3.4999999-01	3.6000000-01	3.7000000-01	3.8000000-01	3.8999999-01
4.0000000-01	4.1000000-01	4.1999999-01	4.3000000-01	4.4000000-01	4.5000000-01	4.5999999-01	4.7000000-01
4.8000000-01	4.8999999-01	5.0000000-01	5.1000000-01	5.2000000-01	5.2999999-01	5.4000000-01	5.5000000-01
5.5999999-01	5.6999999-01	5.8000000-01	5.9000000-01	5.9999999-01	6.1000000-01	6.2000000-01	6.3000000-01
6.3999999-01	6.5000000-01	6.6000000-01	6.6999999-01	6.8000000-01	6.9000000-01	7.0000000-01	7.0999999-01
7.2000000-01	7.3000000-01	7.3999999-01	7.5000000-01	7.6000000-01	7.7000000-01	7.7999999-01	7.9000000-01
8.0000000-01	8.0999999-01	8.1999999-01	8.3000000-01	8.4000000-01	8.4999999-01	8.6000000-01	8.7000000-01
8.8000000-01	8.8999999-01	9.0000000-01	9.1000000-01	9.1999999-01	9.3000000-01	9.4000000-01	9.5000000-01
9.5999999-01	9.7000000-01	9.8000000-01	9.8999999-01	1.0000000+00			

THEORY-MEASUREMENT HYPER-MAPS (Y-SCALE (Y1,DY,Y2)=(S1,DS,S2)MILS, X-SCALE ((X1,DX,X2)=(S1,DS,S2)MILS),K=1, 1),M=1, 2							
DATA		RESOLUTION DZ	DATA		RESOLUTION DS	DATA	
END POINT Z1	(D=IMPLICIT)		END POINT Z2	END POINT S1	(D=IMPLICIT)	END POINT S2	
-9.9999999-02	0.0000000		1.3000000+00	0.0000000	0.0000000	7.0000000+03	
0.0000000	4.9999999-03		1.0000000+00	0.0000000	5.0000000+01	1.0000000+04	
8.0000000-01	0.0000000		1.5000000+00	0.0000000	0.0000000	7.0000000+03	
7.5000000-01	1.2500000-03		1.0000000+00	0.0000000	5.0000000+01	1.0000000+04	

OUTPUT

OUTPUT

TIME 1.0358073-01M

3

PARTITION P	ENTROPY S=LOG(DP/DV)	ITERATIVE LOGIC	PARTITION P	ENTROPY S=LOG(DP/DV)
NORMAL	SN=-(YM-Y)**2	Q(M)=-W(M)*S(YM(M),Y(X(M),A))	SYMMETRIC	SS=-LOG(EXP((YM-Y)**2)-1)
EXPONENTIAL	SE=-Y+LOG(Y**YM/FACT(YM))	Q=SUM(Q(M))	SKEW	SK=-LOG(EXP((YM-Y)**2)+1)
BINOMIAL	SB=SE(Y,YM)+SE(1-Y,N-YM)	DA=-(DQ/DA)/(DDQ/DA/DA)		

SEQUENTIAL REFINEMENT TRAJECTORY SEEKING SOLID 36-BIT CONVERGENCE ON LOCAL SECOND-ORDER ENTROPY-MAXIMUM MISFIT-MINIMUM

GOOD OR BAD	TRY I	TAG J	PARAMETER A(J)	* PIN	MISFIT g=SUM(Q(I))	ENTROPY-GRADIENT -DQM/DA(J)	EQUIPARTITION = +DQT/DA(J)	DETERMINANT DDQ/DA/DA BLADE	NEXT MOVE DA(J)	NEXT TREND DQ=(DQ/DA)DA/2
	0	1	1.0000000+00*			1.2705083+00	1.0891090+00	9.9918246-01	2.7349959-01	
	0	2	1.0000000+00*			-4.1023231-01	1.0891089+00	9.9961920-01	-1.4240483+00	
	0	3	1.0000000+00*			2.6189801-01	1.0891089+00	9.9980590-01	-7.4519677-01	
	0	4	1.0000000+00*			-1.7000797-01	1.0891089+00	9.9943957-01	-1.1814218+00	
	0	5	1.0000000+00*			1.4973798-01	1.0891089+00	9.9952853-01	-8.5847835-01	
	0	6	1.0000000+00*			-1.0441676-01	1.0891089+00	9.9971165-01	-1.1151747+00	
	0	7	1.0000000+00*			1.0653557-01	1.0891089+00	9.9990201-01	-9.0211280-01	
	0	8	1.0000000+00*			-7.3752026-02	1.0891089+00	1.0000001+00	-1.0842033+00	
	0	9	1.0000000+00*			8.3614703-02	1.0891089+00	9.9926658-01	-9.2526280-01	
GOOD	0	10	1.0000000+00*		5.3777131+00	-5.5954894-02	1.0891089+00	9.9935226-01	-1.0662282+00	-5.3622822+00
	1	1	1.2734998+00			1.2705083+00	1.2705082+00	9.9918246-01	2.9749745-08	
	1	2	-4.2404833-01			-4.1023231-01	-4.1023229-01	9.9961920-01	-1.9163316-08	
	1	3	2.5480323-01			2.6189801-01	2.6189796-01	9.9980590-01	5.2325001-08	
	1	4	-1.8142182-01			-1.7000797-01	-1.7000801-01	9.9943957-01	3.3512288-08	
	1	5	1.4152165-01			1.4973798-01	1.4973799-01	9.9952853-01	-1.3519501-08	
	1	6	-1.1517485-01			-1.0441676-01	-1.0441676-01	9.9971165-01	-3.5060064-10	
	1	7	9.7887188-02			1.0653557-01	1.0653556-01	9.9990201-01	7.1744870-09	
	1	8	-8.4203318-02			-7.3752026-02	-7.3752055-02	1.0000001+00	2.9749742-08	
	1	9	7.4737191-02			8.3614703-02	8.3614769-02	9.9926658-01	-6.7135741-08	
GOOD	1	10	-6.6228151-02		1.5430651-02	-5.5954894-02	-5.5954877-02	9.9935226-01	-1.7282043-08	-5.4707171-15
	2	1	1.2734996+00			1.2705083+00	1.2705083+00	9.9918246-01	1.4832117-08	
	2	2	-4.2404835-01			-4.1023231-01	-4.1023231-01	9.9961920-01	-2.1805672-10	
	2	3	2.5480328-01			2.6189801-01	2.6189802-01	9.9980590-01	-3.9805994-09	
	2	4	-1.8142178-01			-1.7000797-01	-1.7000798-01	9.9943957-01	5.4257582-09	
	2	5	1.4152163-01			1.4973798-01	1.4973797-01	9.9952853-01	5.4257583-09	
	2	6	-1.1517485-01			-1.0441676-01	-1.0441676-01	9.9971165-01	1.6632150-09	
	2	7	9.7887196-02			1.0653557-01	1.0653556-01	9.9990201-01	4.4851223-09	
	2	8	-8.4203288-02			-7.3752026-02	-7.3752023-02	1.0000001+00	-3.0399639-09	
	2	9	7.4737124-02			8.3614703-02	8.3614701-02	9.9926658-01	1.6632150-09	
GOOD	2	10	-6.6228168-02		1.5430651-02	-5.5954894-02	-5.5954889-02	9.9935226-01	-4.4509176-09	-1.7535540-16

87

BNWL-390

TIME 2.6412760-01M

4

NON-LINEAR MAXIMUM-LIKELIHOOD STATISTICAL ANALYSIS

TAG	BEST FIT PARAMETER A(J)	* PIN	ROOT MEAN SQUARE DOUBT DA(J) I*I=-1	MEASUREMENT VARIANCE Q/(NM-JJ)	MISFIT GRADIENT DQ/DA(J) SHOULD BE 0	NORMAL BLADES OF CURVATURE TENSOR DDQ/DA(J1)/DA(J1) J1 MINUS=SADDLE	DDQ/DA(J1)/DA(J2) NONZERO=SADDLE	NEXT MOVE DA(J) BY SADDLE-LEG J2 MATRIX INVERSE
1	1.2734996+00		1.8423155-02		-2.8474460-08	8	1.0000001+00	7 2.8712206-08
2	-4.2404835-01		1.8423155-02		1.2136296-08	7	9.9990201-01	3 -1.2304657-08
3	2.5480328-01		1.8423155-02		1.0259101-08	3	9.9980590-01	6 -1.0408691-08
4	-1.8142178-01		1.8423155-02		3.0559009-09	6	9.9971165-01	2 -3.1334583-09
5	1.4152163-01		1.8423155-02		9.8953275-10	2	9.9961920-01	5 -1.0464265-09
6	-1.1517465-01		1.8423155-02		-5.5297556-10	5	9.9952853-01	4 5.1150698-10
7	9.7887196-02		1.8423155-02		-5.4715162-09	4	9.9943957-01	10 5.4792327-09
8	-8.4203288-02		1.8423155-02		4.0454292-09	10	9.9935226-01	9 -4.1328820-09
9	7.4737124-02		1.8423155-02		-2.3865096-09	9	9.9926658-01	1 2.3633761-09
10	-6.6228168-02		1.8423155-02	1.6956759-04	1.2805648-09	1	9.9918246-01	-1.3403688-09

MEAN SQUARE CORRELATED PARAMETER DOUBT (DADA(J1,J2),J1=1, 10),J2=1, 10

J1=	J2= 1	J2= 2	J2= 3	J2= 4	J2= 5	J2= 6	J2= 7
1	3.3941267-04	-3.1138799-06	-3.1138776-06	-3.1138777-06	-3.1138784-06	-3.1138767-06	-3.1138772-06
2	-3.1138799-06	3.3941264-04	-3.1138745-06	-3.1138781-06	-3.1138781-06	-3.1138785-06	-3.1138796-06
3	-3.1138776-06	-3.1138745-06	3.3941265-04	-3.1138745-06	-3.1138753-06	-3.1138773-06	-3.1138779-06
4	-3.1138777-06	-3.1138781-06	-3.1138745-06	3.3941265-04	-3.1138745-06	-3.1138789-06	-3.1138754-06
5	-3.1138784-06	-3.1138781-06	-3.1138753-06	-3.1138745-06	3.3941265-04	-3.1138819-06	-3.1138741-06
6	-3.1138767-06	-3.1138785-06	-3.1138775-06	-3.1138789-06	-3.1138819-06	3.3941265-04	-3.1138802-06
7	-3.1138772-06	-3.1138796-06	-3.1138779-06	-3.1138754-06	-3.1138741-06	-3.1138802-06	3.3941264-04
8	-3.1138811-06	-3.1138737-06	-3.1138758-06	-3.1138771-06	-3.1138813-06	-3.1138724-06	-3.1138731-06
9	-3.1138789-06	-3.1138801-06	-3.1138728-06	-3.1138774-06	-3.1138811-06	-3.1138737-06	-3.1138824-06
10	-3.1138764-06	-3.1138770-06	-3.1138787-06	-3.1138776-06	-3.1138751-06	-3.1138794-06	-3.1138779-06
J1=	J2= 8	J2= 9	J2= 10				
1	-3.1138811-06	-3.1138789-06	-3.1138764-06				
2	-3.1138737-06	-3.1138801-06	-3.1138770-06				
3	-3.1138758-06	-3.1138728-06	-3.1138787-06				
4	-3.1138771-06	-3.1138774-06	-3.1138776-06				
5	-3.1138813-06	-3.1138811-06	-3.1138751-06				
6	-3.1138724-06	-3.1138737-06	-3.1138794-06				
7	-3.1138731-06	-3.1138824-06	-3.1138779-06				
8	3.3941264-04	-3.1138737-06	-3.1138722-06				
9	-3.1138737-06	3.3941266-04	-3.1138855-06				
10	-3.1138722-06	-3.1138855-06	3.3941265-04				

88

MEAN SQUARE CORRELATED DOUBT PROPAGATES TO FUNCTION F HAVING PARAMETER GRADIENT DF/DA AS DFOF=(DF/DA).(DADA).(DF/DA)

THEORY-MEASUREMENT HYPER-SPACE PLOTS

BNWL-390

TIME 2.0349935+00M

5

JO PLOT 1 TIME 2.0352376+00M

JO PLOT 2 TIME 2.4398274+00M

NON-LINEAR MAXIMUM-LIKELIHOOD THEORY-MEASUREMENT CORRELATION ANALYSIS (I*I=-1)									
TAG M K	COORDINATE X(M,K)	MEASUREMENT Y(M)	BEST THEORY Y(X(M,K),A)	RANDOM RMS DOUBT DY	RESOLUTION ERROR DY	THEORY ERROR Y-YM	DOUBT(ERROR) RMS D(Y-YM)	PROBABILITY OF LESS ERROR	FIT IS
1 1	0.00000	1.0000000+00	9.7157262-01	5.56021-02	0.00000	-2.86274-02	1.18374-01	1.55588-01	GOOD
2 1	1.00000-02	1.0000000+00	9.7267113-01	5.46976-02	0.00000	-2.71289-02	1.18796-01	1.48058-01	GOOD
3 1	2.00000-02	1.0000000+00	9.7721929-01	5.23169-02	0.00000	-2.27807-02	1.19955-01	1.25742-01	GOOD
4 1	3.00000-02	1.0000000+00	9.8393699-01	4.84604-02	0.00000	-1.60110-02	1.21564-01	8.97429-02	GOOD
5 1	4.00000-02	1.0000000+00	9.9281563-01	4.39938-02	0.00000	-7.48697-03	1.23251-01	4.26164-02	GOOD
6 1	5.00000-02	1.0000000+00	1.0019495+00	3.96268-02	0.00000	1.94949-03	1.24660-01	1.12285-02	GOOD
7 1	6.00000-02	1.0000000+00	1.0113636+00	3.69393-02	0.00000	1.13636-02	1.25546-01	6.58433-02	GOOD
8 1	7.00000-02	1.0000000+00	1.0193185+00	3.60137-02	0.00000	1.98185-02	1.25815-01	1.14812-01	GOOD
9 1	8.00000-02	1.0000000+00	1.0264664+00	3.69869-02	0.00000	2.64664-02	1.25532-01	1.52579-01	GOOD
10 1	9.00000-02	1.0000000+00	1.0306320+00	3.90957-02	0.00000	3.06320-02	1.24891-01	1.75345-01	GOOD
11 1	1.00000-01	1.0000000+00	1.0318788+00	4.13840-02	0.00000	3.18788-02	1.24152-01	1.81302-01	GOOD
12 1	1.10000-01	1.0000000+00	1.0300530+00	4.31127-02	0.00000	3.00530-02	1.23562-01	1.70280-01	GOOD
13 1	1.20000-01	1.0000000+00	1.0283009+00	4.36885-02	0.00000	2.53009-02	1.23289-01	1.43355-01	GOOD
14 1	1.30000-01	1.0000000+00	1.0160361+00	4.36498-02	0.00000	1.80561-02	1.23374-01	1.02644-01	GOOD
15 1	1.40000-01	1.0000000+00	1.0069995+00	4.26151-02	0.00000	8.99953-03	1.23735-01	5.14155-02	GOOD
16 1	1.50000-01	1.0000000+00	9.9899573-01	4.12110-02	0.00000	-1.00426-03	1.24209-01	5.76478-03	GOOD
17 1	1.60000-01	1.0000000+00	9.9600987-01	3.99596-02	0.00000	-1.09901-02	1.24618-01	6.32116-02	GOOD
18 1	1.70000-01	1.0000000+00	9.8801518-01	3.93106-02	0.00000	-1.99848-02	1.24824-01	1.14864-01	GOOD
19 1	1.80000-01	1.0000000+00	9.7269830-01	3.94612-02	0.00000	-2.71017-02	1.24776-01	1.55266-01	GOOD
20 1	1.90000-01	1.0000000+00	9.6837241-01	4.02769-02	0.00000	-3.16276-02	1.24515-01	1.80413-01	GOOD
21 1	2.00000-01	1.0000000+00	9.6696832-01	4.13840-02	0.00000	-3.30937-02	1.24152-01	1.88085-01	GOOD
22 1	2.10000-01	1.0000000+00	9.6867598-01	4.23562-02	0.00000	-3.13240-02	1.23824-01	1.77737-01	GOOD
23 1	2.20000-01	1.0000000+00	9.7354453-01	4.28725-02	0.00000	-2.64555-02	1.23646-01	1.50250-01	GOOD
24 1	2.30000-01	1.0000000+00	9.8107158-01	4.27984-02	0.00000	-1.89284-02	1.23671-01	1.07833-01	GOOD
25 1	2.40000-01	1.0000000+00	9.9055246-01	4.22015-02	0.00000	-9.44753-03	1.23876-01	5.40322-02	GOOD
26 1	2.50000-01	1.0000000+00	1.0010831+00	4.13197-02	0.00000	1.08312-03	1.24173-01	6.21521-03	GOOD
27 1	2.60000-01	1.0000000+00	1.0116452+00	4.04861-02	0.00000	1.16452-02	1.24448-01	6.68814-02	GOOD
28 1	2.70000-01	1.0000000+00	1.0212036+00	4.00132-02	0.00000	2.12036-02	1.24600-01	1.21601-01	GOOD
29 1	2.80000-01	1.0000000+00	1.0268061+00	4.00681-02	0.00000	2.88061-02	1.24583-01	1.64646-01	GOOD
30 1	2.90000-01	1.0000000+00	1.0336777+00	4.06023-02	0.00000	3.36777-02	1.24410-01	1.91730-01	GOOD
31 1	3.00000-01	1.0000000+00	1.0352976+00	4.13840-02	0.00000	3.52976-02	1.24152-01	2.00357-01	GOOD
32 1	3.10000-01	1.0000000+00	1.0334548+00	4.21088-02	0.00000	3.34548-02	1.23908-01	1.89734-01	GOOD
33 1	3.20000-01	1.0000000+00	1.0262726+00	4.25198-02	0.00000	2.82726-02	1.23768-01	1.60595-01	GOOD
34 1	3.30000-01	1.0000000+00	1.0202029+00	4.24873-02	0.00000	2.02029-02	1.23779-01	1.15145-01	GOOD
35 1	3.40000-01	1.0000000+00	1.0099678+00	4.20399-02	0.00000	9.98782-03	1.23931-01	5.71436-02	GOOD
36 1	3.50000-01	1.0000000+00	9.9859250-01	4.13495-02	0.00000	-1.40749-03	1.24163-01	8.07427-03	GOOD
37 1	3.60000-01	1.0000000+00	9.8711623-01	4.06762-02	0.00000	-1.28838-02	1.24386-01	7.39389-02	GOOD
38 1	3.70000-01	1.0000000+00	9.7668759-01	4.02766-02	0.00000	-2.33124-02	1.24516-01	1.33498-01	GOOD

89

BNWL-390

TIME	2.8619140+00									6
39	1	3.80000-01	1.0000000+00	9.6805570-01	4.03012-02	0.00000	-3.16443-02	1.24508-01	1.80496-01	6000
40	1	3.90000-01	1.0000000+00	9.6298657-01	4.07297-02	0.00000	-3.70134-02	1.24368-01	2.10235-01	6000
41	1	4.00000-01	1.0000000+00	9.6117499-01	4.13840-02	0.00000	-3.88250-02	1.24152-01	2.19896-01	6000
42	1	4.10000-01	1.0000000+00	9.6318081-01	4.20100-02	0.00000	-3.68192-02	1.23941-01	2.08456-01	6000
43	1	4.20000-01	1.0000000+00	9.6889591-01	4.23792-02	0.00000	-3.11041-02	1.23816-01	1.76496-01	6000
44	1	4.30000-01	1.0000000+00	9.7784644-01	4.23648-02	0.00000	-2.21536-02	1.23821-01	1.26218-01	6000
45	1	4.40000-01	1.0000000+00	9.6922950-01	4.19779-02	0.00000	-1.07705-02	1.23952-01	6.16237-02	6000
46	1	4.50000-01	1.0000000+00	1.0019834+00	4.13617-02	0.00000	1.98336-03	1.24159-01	1.13791-02	6000
47	1	4.60000-01	1.0000000+00	1.0148837+00	4.07467-02	0.00000	1.48837-02	1.24363-01	8.53590-02	6000
48	1	4.70000-01	1.0000000+00	1.0266584+00	4.03696-02	0.00000	2.66584-02	1.24485-01	1.52406-01	6000
49	1	4.80000-01	1.0000000+00	1.0361099+00	4.03789-02	0.00000	3.61099-02	1.24482-01	2.05401-01	6000
50	1	4.90000-01	1.0000000+00	1.0422338+00	4.07691-02	0.00000	4.22338-02	1.24355-01	2.39027-01	6000
51	1	5.00000-01	1.0000000+00	1.0443217+00	4.13840-02	0.00000	4.43217-02	1.24152-01	2.50069-01	6000
52	1	5.10000-01	1.0000000+00	1.0520400+00	4.19863-02	0.00000	4.20400-02	1.23950-01	2.37210-01	6000
53	1	5.20000-01	1.0000000+00	1.0354727+00	4.23524-02	0.00000	3.54727-02	1.23825-01	2.00811-01	6000
54	1	5.30000-01	1.0000000+00	1.0251278+00	4.23498-02	0.00000	2.51278-02	1.23826-01	1.42998-01	6000
55	1	5.40000-01	1.0000000+00	1.0119018+00	4.19775-02	0.00000	1.19018-02	1.23952-01	6.80813-02	6000
56	1	5.50000-01	1.0000000+00	9.9700557-01	4.13677-02	0.00000	-2.99443-03	1.24157-01	1.71756-02	6000
57	1	5.60000-01	1.0000000+00	9.8165312-01	4.07456-02	0.00000	-1.81419-02	1.24363-01	1.03952-01	6000
58	1	5.70000-01	1.0000000+00	9.6795865-01	4.03528-02	0.00000	-3.20434-02	1.24491-01	1.82709-01	6000
59	1	5.80000-01	1.0000000+00	9.5673385-01	4.03500-02	0.00000	-4.32661-02	1.24492-01	2.44951-01	6000
60	1	5.90000-01	1.0000000+00	9.4941661-01	4.07444-02	0.00000	-5.05834-02	1.24363-01	2.84450-01	6000
61	1	6.00000-01	1.0000000+00	9.4669859-01	4.13840-02	0.00000	-5.31014-02	1.24152-01	2.97477-01	6000
62	1	6.10000-01	1.0000000+00	9.4963974-01	4.20240-02	0.00000	-5.03603-02	1.23937-01	2.82318-01	6000
63	1	6.20000-01	1.0000000+00	9.5760304-01	4.24244-02	0.00000	-4.23969-02	1.23800-01	2.38885-01	6000
64	1	6.30000-01	1.0000000+00	9.7023632-01	4.24340-02	0.00000	-2.97637-02	1.23797-01	1.68981-01	6000
65	1	6.40000-01	1.0000000+00	9.8650144-01	4.20389-02	0.00000	-1.34986-02	1.23932-01	7.71752-02	6000
66	1	6.50000-01	1.0000000+00	1.0049522+00	4.13710-02	0.00000	4.85219-03	1.24156-01	2.84010-02	6000
67	1	6.60000-01	1.0000000+00	1.0239535+00	4.06724-02	0.00000	2.38535-02	1.24387-01	1.36425-01	6000
68	1	6.70000-01	1.0000000+00	1.0413346+00	4.02162-02	0.00000	4.13346-02	1.24535-01	2.34416-01	6000
69	1	6.80000-01	1.0000000+00	1.0555621+00	4.01973-02	0.00000	5.55621-02	1.24541-01	3.11493-01	6000
70	1	6.90000-01	1.0000000+00	1.0649186+00	4.06409-02	0.00000	6.49186-02	1.24397-01	3.60162-01	6000
71	1	7.00000-01	1.0000000+00	1.0681705+00	4.13840-02	0.00000	6.81705-02	1.24152-01	3.76167-01	6000
72	1	7.10000-01	1.0000000+00	1.0646092+00	4.21455-02	0.00000	6.46092-02	1.23895-01	3.57217-01	6000
73	1	7.20000-01	1.0000000+00	1.0541510+00	4.26380-02	0.00000	5.41510-02	1.23727-01	3.02084-01	6000
74	1	7.30000-01	1.0000000+00	1.0373847+00	4.26684-02	0.00000	3.73847-02	1.23716-01	2.11208-01	6000
75	1	7.40000-01	1.0000000+00	1.0155601+00	4.22000-02	0.00000	1.55601-02	1.23877-01	8.88773-02	6000
76	1	7.50000-01	1.0000000+00	9.9051429-01	4.13729-02	0.00000	-9.48571-03	1.24156-01	5.43732-02	6000
77	1	7.60000-01	1.0000000+00	9.6453962-01	4.04758-02	0.00000	-3.54604-02	1.24451-01	2.01736-01	6000
78	1	7.70000-01	1.0000000+00	9.4020089-01	3.98610-02	0.00000	-5.97891-02	1.24649-01	3.34155-01	6000
79	1	7.80000-01	1.0000000+00	9.2611542-01	3.98066-02	0.00000	-7.98846-02	1.24667-01	4.36217-01	6000
80	1	7.90000-01	1.0000000+00	9.0671194-01	4.03786-02	0.00000	-9.32880-02	1.24483-01	4.99249-01	6000
81	1	8.00000-01	1.0000000+00	9.0198871-01	4.13840-02	0.00000	-9.80113-02	1.24152-01	5.19569-01	6000
82	1	8.10000-01	1.0000000+00	9.0728958-01	4.24511-02	0.00000	-9.27104-02	1.23791-01	4.94208-01	6000
83	1	8.20000-01	1.0000000+00	9.2311970-01	4.31779-02	0.00000	-7.68803-02	1.23539-01	4.17990-01	6000
84	1	8.30000-01	1.0000000+00	9.4601592-01	4.32675-02	0.00000	-5.09841-02	1.23508-01	2.84719-01	6000
85	1	8.40000-01	1.0000000+00	9.8348736-01	4.26188-02	0.00000	-1.65126-02	1.23733-01	9.41844-02	6000

TIME	3.0069498+00M								7
86 1	8.50000-01	1.0000000+00	1.0240336+00	4.13740-02	0.00000	2.40336-02	1.24155-01	1.37196-01	GOOD
87 1	8.60000-01	1.0000000+00	1.0672431+00	3.99209-02	0.00000	6.72431-02	1.24630-01	3.72755-01	GOOD
88 1	8.70000-01	1.0000000+00	1.1089682+00	3.88206-02	0.00000	1.08968-01	1.24977-01	5.70464-01	GOOD
89 1	8.80000-01	1.0000000+00	1.1445673+00	3.86151-02	0.00000	1.44567-01	1.25041-01	7.06282-01	GOOD
90 1	8.90000-01	1.0000000+00	1.1819175+00	3.95470-02	0.00000	1.69197-01	1.24749-01	7.80330-01	GOOD
91 1	9.00000-01	1.0000000+00	1.1781394+00	4.13840-02	0.00000	1.78139-01	1.24152-01	8.01773-01	GOOD
92 1	9.10000-01	1.0000000+00	1.1871290+00	4.35032-02	0.00000	1.67129-01	1.23423-01	7.69585-01	GOOD
93 1	9.20000-01	1.0000000+00	1.1326743+00	4.51646-02	0.00000	1.32674-01	1.22827-01	6.55857-01	GOOD
94 1	9.30000-01	1.0000000+00	1.0723312+00	4.56667-02	0.00000	7.23312-02	1.22641-01	3.92790-01	GOOD
95 1	9.40000-01	1.0000000+00	9.8491665-01	4.45032-02	0.00000	-1.50813-02	1.23068-01	8.55934-02	GOOD
96 1	9.50000-01	1.0000000+00	8.7865447-01	4.13745-02	0.00000	-1.29346-01	1.24155-01	6.48758-01	GOOD
97 1	9.60000-01	1.0000000+00	7.3120178-01	3.61985-02	0.00000	-2.68798-01	1.25762-01	9.52182-01	BAD
98 1	9.70000-01	1.0000000+00	5.8981861-01	2.90987-02	0.00000	-4.30381-01	1.27592-01	9.98945-01	BAD
99 1	9.80000-01	1.0000000+00	3.9021486-01	2.03800-02	0.00000	-6.09785-01	1.29271-01	9.99999-01	BAD
100 1	9.90000-01	1.0000000+00	1.9832343-01	1.04950-02	0.00000	-8.01677-01	1.30446-01	1.00000+00	BAD
101 1	1.00000+00	0.0000000	3.9858690-07	2.68964-03	0.00000	3.96587-07	1.30868-01	5.15580-06	GOOD

NON-LINEAR MAXIMUM-LIKELIHOOD THEORY-REGRESSION ANALYSIS (I*I=-1)										
PARAMETER	BEST THEORY	MISFIT-NORTH	NORMAL BLADES	THEORY WORTH DW	WORTH RATIO	FREEDOM	PROBABILITY	THEORY		
SUBSET	TAG	PARAMETER	THEORY SUBSET	THEORY COSET	SUBSET+COSET	SUBSET+COSET	SUBSET/THEORY	SET COSET	OF MORE WORTH	SUBSET
LOG(J)	J	A(J)	PI. DW*COS(F)**2	DW*SIN(F)**2	A.DWG/DADA.A/2	(EXPECT JS/JJ)	(EXPECT JS/JJ)	JS	JC	(EXPECT 1/2) SEEMS
1	1	1.2734996+00	8.1023771-01	1.6443059-01	9.7466830-01	8.3129584-01	1	9	6.6934644-07	GOOD
2	2	-4.2404835-01	8.9835003-02	8.8483333-01	9.7466833-01	9.2169318-02	1	9	-3.2492545-09	GOOD
3	3	2.5480328-01	3.2435818-02	9.4223252-01	9.7466834-01	3.3278826-02	1	9	-6.8234346-08	GOOD
4	4	-1.8142178-01	1.8443478-02	9.5822485-01	9.7466832-01	1.6870344-02	1	9	1.4296719-07	GOOD
5	5	1.4152163-01	1.0906000-02	9.6466233-01	9.7466833-01	1.0266056-02	1	9	-3.5091949-07	GOOD
6	6	-1.1517465-01	6.6271780-03	9.6804114-01	9.7466832-01	6.7994187-03	1	9	-5.1338222-07	GOOD
7	7	9.7887196-02	4.7670351-03	9.6986129-01	9.7466832-01	4.9114504-03	1	9	9.0979128-07	GOOD
8	8	-8.4203268-02	3.5421988-03	9.7112613-01	9.7466833-01	3.6342607-03	1	9	1.5381971-05	GOOD
9	9	7.4737124-02	2.7965356-03	9.7187781-01	9.7466834-01	2.8630617-03	1	9	1.0474947-04	GOOD
10	10	-6.6228168-02	2.1912923-03	9.7247703-01	9.7466832-01	2.2482441-03	1	9	5.2672366-04	GOOD
1	1	1.2734996+00								
1	2	-4.2404835-01	8.9519773-01	7.9470558-02	9.7466829-01	9.1846406-01	2	8	3.2306780-04	GOOD
2	3	2.5480328-01								
2	4	-1.8142178-01	4.8459622-02	9.2620670-01	9.7466832-01	4.9719091-02	2	8	1.9478079-02	GOOD
3	5	1.4152163-01								
3	6	-1.1517465-01	1.6455149-02	9.5818317-01	9.7466832-01	1.6913599-02	2	8	6.8342378-02	GOOD
4	7	9.7887196-02								
4	8	-8.4203268-02	8.2543721-03	9.6841395-01	9.7466832-01	8.4689037-03	2	8	1.3139693-01	GOOD
5	9	7.4737124-02								
5	10	-6.6228168-02	4.9368704-03	9.6973145-01	9.7466832-01	5.0651798-03	2	8	1.9703914-01	GOOD

TIME 3.2632650+00M						8
OLD RUNNING-TIME THEORY			RUN 117	NEW RUNNING-TIME THEORY		
TIME T(MIN)	I RMS DOUBT DT	TIME T(MIN)	TIME T(MIN)	I RMS DOUBT DT	AGE TT(MIN)	
1.2978614+00	4.9664349-02	3.2620768+00	1.3374051+00	5.5451359-02	1.0259828+02	
$TIME(MIN) \quad T = T1 + T2*(I+3)*MM*JJ**2 + T3*(I+2+JJ)*JJ**3 + T4*MAP*MM + T5*MM*(MW-JJ-1)$						
T1 MINIMUM		T2 CORRELATE	T3 INVERT	T4 PLOT	T5 PROBABILITY	
3.0464847-01		6.3110072-05	-2.3177711-06	2.1879631-04	7.7261022-05	
GOOD RUN SOLID CONVERGENCE TRUE MINIMUM NO SADDLE NO SPILL NO DIVISION BY ZERO CAVEAT EMPTOR MULTIPLE SOLUTIONS

FIGURE 7. Sample Output: Program Learn

FIGURE 8. Sample Output: Program Learn

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