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AN INFORMAL REPORT

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Self-Consistent Bilocal Field Theory

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scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

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Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.95

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ISSUED: March 1972

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SELF-CONSISTENT BILOCAL FIELD THEORY

by

Marvin Rich

ABSTRACT

A non-relativistic formulation of quantum field theory is outlined in which the field operators are assumed to depend on a pair of spatial positions so as to describe extended particles with internal as well as center of mass motion. The perturbation expansion of the S-matrix is obtained in the interaction picture for a simplified version of the theory in which the fermion fields are bilocal and the boson fields are local. Comparison of corresponding self-energy matrix elements of the bilocal and local theory indicates that, while the local result is definitely divergent, the bilocal term is likely finite. Within the bilocal framework, a self-consistent equation is derived for the internal fermion motion in terms of a dynamical potential calculated from the fermion self-energy diagrams. Examples of this equation are given for scalar, pseudoscalar, and vector meson interactions using the lowest order dynamical potential. Relativistic extensions of this theory are discussed, and some applications which might serve as tests of its validity are considered.

I. INTRODUCTION

Since its beginnings¹ almost fifty years ago, many attempts have been made to cure the various diseases of local quantum field theory, principal of which is the problem of divergent self-energy and vertex corrections which arise in perturbation solutions. Most successful of these was the invariant renormalization procedures² developed in the late nineteen forties, which provided a means of sweeping the divergence problem under the rug for some classes of interacting fields by combining the infinities with the unknown bare masses and coupling constants to give the observed parameters. These methods have met with great success when applied to quantum electrodynamics. For strongly interacting fields, this technique has been largely irrelevant due to the fact that perturbation expansions in powers of the coupling strength are no longer expected to be applicable. Some of the other

ideas which have been sporadically pursued in relation to strong interaction field theory, but which at best have had meager success in application to physical problems, are those of strong coupling³ perturbation theory, wherein expansions are attempted in powers of the reciprocal of the coupling constant, nonlocal field theories⁴ where a short range interaction at a distance is allowed via an arbitrarily chosen interaction kernel, and non-polynomial Lagrangian theories⁵ in which the Lagrangian for the field may contain arbitrary functions of products of the field operators.

Still another variation of field theory was introduced by Yukawa⁶ in 1953. In this, the field operators were assumed to be bilocal, that is, functions of two space-time points, rather than of a single point as in local or conventional nonlocal field theories. The basic feature here was that a two point field equation, similar, say, to a two

particle Schrödinger equation, and containing an "interaction" term, could be introduced for the "free" field so that its particles would have a space-time extension and not be simply points. Thus the particles would possess an intrinsic form factor determined by the interaction, which, in principle, could remedy the divergence difficulties of perturbation theory. Here again, there was no unique way of specifying the "interaction" of the bilocal field equation, and there were other difficulties which marred the formulation.

In this report, we will consider a form of bilocal quantum field theory similar that of Yukawa, but within a non-relativistic framework and from a somewhat different point of view. The discussion will be in terms of perturbation theory, though the summing of some classes of Feynman diagrams, the use of which remains applicable, will be made to obtain non-perturbative equations. The resultant theory can be looked upon as a composite, having aspects related to Yukawa's ideas, to strong-coupling theory and to nonlocal field theory. The details of the theory will be outlined in terms of the example of a neutral fermion field in interaction with a neutral scalar meson field. The Fermi field will be treated non-relativistically, and only it will be assumed bilocal. The usual local relativistic formulation will be used for the mesons. We will be primarily interested in the one fermion problem. This problem is usually sidestepped in local field theory where particle structure enters only indirectly (at least in perturbation theory) through the modification of one particle propagators and vertex functions. The picture of the bilocal fermion particle which we will use is that of a "bare" fermion moving in an effective potential produced by the surrounding cloud of mesons, which was itself generated by the "bare" fermion. Since the fermions are treated non-relativistically, no particle anti-particle pairs will be involved. The meson cloud may be considered to have a mass distribution which can be described in terms of appropriate moments, and which is capable of undergoing rotations and vibrations. In the following, μ total mass, M , will be ascribed to the distribution and other possible moments will be ignored. The particles of the bilocal fields may therefore be regarded as composite entities, with one coordinate

describing the location of the "bare" point fermion and the other, the center of mass of its meson cloud. Although it is possible to think in terms of such a division, the "bare" particle should actually be inseparable from its self-generated potential. It might be noted that this picture is quite different from that of the usual point particle whose surrounding field is assumed to rigidly follow the particle motion.

If it is assumed that only the "bare" fermion interacts with the external meson field (locally, through a Yukawa type interaction), then the interaction will be effectively spread out over a small spatial volume due to the internal motion of the bare fermion in its self-consistent potential. As stated above, this internal motion will provide a natural form factor at the interaction vertices and will also modify the one particle propagators in a manner which could aid the convergence of the typically divergent terms which arise in perturbation theory.

While it is possible to express the above picture in a Hamiltonian formalism using an arbitrary potential to specify the internal fermion motion, we will be interested in one specific choice which makes the "free" bilocal field propagator identical to the exact one fermion propagator of the interacting fields. This choice identifies the "bare" fermion potential with the self-energy parts of the perturbation expansion. Its effect is then that of a generalized counter term which entirely removes self-energy diagrams from the perturbation expansion, transferring their effect to the "free" fermion field equation, and thereby producing a partial separation of the interacting meson field into internal and external parts. With this identification, the "free" fermion equations become highly non-linear, resembling Hartree equations in form, and may be regarded as self-consistent equations for the internal fermion motion and for its meson cloud.

In section II, the S-matrix expansion is developed for the example of the non-relativistic bilocal fermion field in interaction with a neutral scalar meson field. A derivation of the self-consistent potential for the internal fermion motion is given in section III using the same non-relativistic model. In section IV we consider the

lowest order self-consistent equations for a neutral scalar, a neutral pseudo-scalar, and the electromagnetic interaction, and attempt to put them into a tractable form. Section V deals with the question of how such self-consistent bilocal field theories might be made relativistically invariant. The final section is devoted to speculations on possible physical applications of bilocal field theory, particularly in areas which have not proven accessible to local field theory. No attempt has

been made to consider internal consistency, or its lack, in this formulation. A similar approach, using the idea of a self-consistent potential as a generalized mass counter term, can also be applied to local field theory. This will be discussed separately in an addendum to this report. In the following, units in which $\hbar$ and c are unity will be used. Four-vectors will be given in terms of Minkowsky space, with pure imaginary time components.

II. NON-RELATIVISTIC FORMULATION

We will work in the Hamiltonian formalism using the specific model of a neutral scalar meson field in interaction with a non-relativistic neutral fermion field. The Fermi field, whose particles will at times be called "nucleons," will be treated as spinless for simplicity, and will be represented in the Schrödinger picture by a bilocal operator $f(\vec{x}, \vec{y})$ which obeys the anti-commutation relations

$$\begin{aligned} [f(\vec{x}, \vec{y}), f^\dagger(\vec{x}', \vec{y}')]_+ &= \delta(\vec{x} - \vec{x}') \delta(\vec{y} - \vec{y}') \quad , \\ [f(\vec{x}, \vec{y}), f(\vec{x}', \vec{y}')]_+ &= 0 \quad . \end{aligned} \quad (\text{II.1})$$

The $\vec{x}$ coordinates of the field operator will be assumed to give the position of a bare fermion of mass m which can interact locally with the meson field. The $\vec{y}$ coordinates can be considered those of the center of mass of a "potential" in which the bare fermion moves. This potential will be treated as a particle of mass M so that it can propagate through space.

The total Hamiltonian H can be written as the sum of a free Hamiltonian H_0 plus an interaction term H_I ,

$$H = H_0 + H_I \quad .$$

These will be chosen as follows:

$$\begin{aligned} H_0 = \int d^3x \int d^3y \left\{ \frac{1}{2m} \nabla_x f^\dagger(\vec{x}, \vec{y}) \cdot \nabla_x f(\vec{x}, \vec{y}) + \frac{1}{2M} \nabla_y f^\dagger(\vec{x}, \vec{y}) \cdot \nabla_y f(\vec{x}, \vec{y}) \right. \\ \left. + f^\dagger(\vec{x}, \vec{y}) \int d^3x' \int d^3y' V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') f(\vec{x}', \vec{y}') \right\} + \frac{1}{2} \int d^3x \left\{ \pi^2(\vec{x}) + [\nabla_x \cdot \phi(\vec{x})]^2 + u^2 \phi^2(\vec{x}) \right\} \quad . \end{aligned} \quad (\text{II.2})$$

and

$$H_I = -g \int d^3x \int d^3y f^\dagger(\vec{x}, \vec{y}) f(\vec{x}, \vec{y}) \phi(\vec{x}) - \int d^3x \int d^3y f^\dagger(\vec{x}, \vec{y}) \int d^3x' \int d^3y' V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') f(\vec{x}', \vec{y}') \quad . \quad (\text{II.3})$$

$\phi(\vec{x})$ is the local operator for the meson field which obeys the usual commutation relations

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= \delta(\vec{x} - \vec{y}) \quad , \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad . \end{aligned} \quad (\text{II.4})$$

Before dividing the Hamiltonian as above, an unspecified nonlocal potential $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ has been added and subtracted. It, thus, has no net effect. In the separation into a free plus an interaction Hamiltonian, one of these terms was retained in H_0 and the other in H_I . It will be assumed that V is translationally invariant,

$$V(\vec{x} + \vec{a}, \vec{y} + \vec{a} | \vec{x}' + \vec{a}, \vec{y}' + \vec{a}) = V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$$

that it is Hermitian, and that it is explicitly independent of time.

The field equations associated with the unperturbed Hamiltonian H_0 and the total Hamiltonian H are, respectively

$$\left[-\frac{1}{2m} \nabla_{\vec{x}}^2 - \frac{1}{2M} \nabla_{\vec{y}}^2 \right] f_0(\vec{x}, \vec{y}) + \int d^3x' \int d^3y' V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') f(\vec{x}', \vec{y}') = i \dot{f}_0(\vec{x}, \vec{y}) \quad (II.5)$$

$$\nabla^2 \phi_0(\vec{x}) - \ddot{\phi}_0(\vec{x}) - \mu^2 \phi_0(\vec{x}) = 0$$

and

$$\left[-\frac{1}{2m} \nabla_{\vec{x}}^2 - \frac{1}{2M} \nabla_{\vec{y}}^2 \right] f(\vec{x}, \vec{y}) = i \dot{f}(\vec{x}, \vec{y}) - g f(\vec{x}, \vec{y}) \phi(\vec{x}) \quad (II.6)$$

$$\nabla^2 \phi(\vec{x}) - \ddot{\phi}(\vec{x}) - \mu^2 \phi(\vec{x}) = g \int d^3y f^\dagger(\vec{x}, \vec{y}) f(\vec{x}, \vec{y})$$

where, for any Schrödinger representation operator A

$$\dot{A}_0 = i [H_0, A] \quad \text{and} \quad \dot{A} = i [H, A]$$

In effect, nothing has been done so far. The $\vec{y}$ coordinates could have been dispensed with at this point by also dropping the $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ terms, except for the original assumption that the Fermi field should be intrinsically bilocal.

For the purpose of making perturbation expansions, we will want explicit expressions of the non-interacting field operators defined by Eq. (II.5). This can be done in the usual way in terms of the eigenfunctions of the free field equations. For the meson field one has

$$\phi_0(\vec{x}) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[e^{i\vec{k} \cdot \vec{x}} a(\vec{k}) + e^{-i\vec{k} \cdot \vec{x}} a^\dagger(\vec{k}) \right] \quad (II.7)$$

where $a^\dagger(\vec{k})$ is the creation operator for a meson of momentum $\vec{k}$. The a 's obey the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

The fermion field may be similarly expanded in terms of the eigenstates of the Schrödinger equation

$$\left[-\frac{1}{2m} \nabla_{\vec{x}}^2 - \frac{1}{2M} \nabla_{\vec{y}}^2 \right] \psi(\vec{x}, \vec{y}) + \int d^3x' \int d^3y' V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') \psi(\vec{x}', \vec{y}') = E \psi(\vec{x}, \vec{y})$$

Transforming to center of mass and relative coordinates by

$$\vec{R} = \alpha \vec{x} + \beta \vec{y} \quad \vec{r} = \vec{x} - \vec{y} \quad , \quad \alpha = m/(m+M) \quad \beta = M/(m+M) \quad (II.8)$$

and letting

$$\psi(\vec{x}, \vec{y}) = e^{i\vec{P} \cdot \vec{R}} \psi(\vec{r})$$

and

$$\epsilon = E - \vec{P}^2 / 2(m + M) \quad ,$$

one obtains the equation for the relative motion

$$-\frac{1}{2m_r} \nabla^2 \psi(\vec{r}) + \int d^3 r' V(\vec{r} | \vec{r}') \psi(\vec{r}') = \epsilon \psi(\vec{r}) \quad , \quad (II.9)$$

where

$$m_r = mM / (m + M) \quad .$$

It has been assumed that $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ may be written as

$$V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') = \delta(\vec{R} - \vec{R}') V(\vec{r} | \vec{r}') \quad .$$

If we let "a" represent the complete set of quantum numbers which designate an eigenstate of Eq. (II.9), then $f_0(\vec{x}, \vec{y})$ may be written as

$$f_0(\vec{x}, \vec{y}) = (2\pi)^{-3/2} \int d^3 p \sum_a e^{i\vec{P} \cdot \vec{R}} \psi_a(\vec{r}) b_{\vec{P}, a} \quad , \quad (II.10)$$

The operators $b_{\vec{P}, a}^\dagger$, which create states with center of mass momentum $\vec{P}$ and internal quantum numbers a, satisfy the anti-commutation relations

$$[b_{\vec{P}, a}, b_{\vec{P}', a'}^\dagger]_+ = \delta_{a, a'} \delta(\vec{P} - \vec{P}') \quad ,$$

$$[b_{\vec{P}, a}, b_{\vec{P}', a'}]_+ = 0 \quad .$$

Time dependent free field operators can be obtained, in the usual way, by means of the transformation

$$\begin{aligned} f_0(\vec{x}, \vec{y}, t) &= e^{iH_0 t} f_0(\vec{x}, \vec{y}) e^{-iH_0 t} \\ &= (2\pi)^{-3/2} \int d^3 p \sum_a e^{i(\vec{P} \cdot \vec{R} - E_{\vec{P}, a} t)} \psi_a(\vec{r}) b_{\vec{P}, a} \\ \phi_0(\vec{x}, t) \equiv \phi(\vec{x}) &= e^{iH_0 t} \phi_0(\vec{x}) e^{-iH_0 t} \\ &= (2\pi)^{-3/2} \int \frac{d^3 k}{\sqrt{2\omega_k}} \left[e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} a(\vec{k}) + e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} a^\dagger(\vec{k}) \right] \quad . \end{aligned} \quad (II.11)$$

It is now possible to develop the perturbation expansion for the S-matrix in an identical manner to that made in local field theory, so as to examine the effect of a bilocal fermion field having explicit internal structure. To do so, we transform to the interaction representation where the field functions obey time dependent free field equations similar to Eq. (II.5). The state vectors satisfy the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi_1(t)\rangle = H_I(t) |\psi_1(t)\rangle \quad , \quad (II.12)$$

where

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}.$$

If $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ had been explicitly time dependent, this transformation would not have been possible, at least not with useful results, since neither the free field equations for the field operators nor the Schrödinger equation, Eq. (II.12), would have resulted. In the interaction representation, the field operators are just those given by Eq. (II.11).

Proceeding in the usual manner, we define a time displacement operator $U(t, t')$ which transforms an interaction representation state vector at time t' into the corresponding state vector at time t :

$$|\psi_I(t)\rangle = U(t, t') |\psi_I(t')\rangle.$$

From this relation and Eq. (II.12), one obtains an equation for $U(t, t')$

$$i \frac{\partial}{\partial t} U(t, t') = H_I(t) U(t, t'),$$

and the boundary condition

$$U(t, t) = 1.$$

The corresponding integral equation for U is

$$U(t, t') = 1 - i \int_{t'}^t d\tau H_I(\tau) U(\tau, t').$$

Using the definition of the S-matrix

$$S = \lim_{\substack{t \rightarrow \infty \\ t' \rightarrow -\infty}} U(t, t'),$$

one obtains the familiar perturbation expansion for S by iteration of the equation for U

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_n T [H_I(t_1) \dots H_I(t_n)],$$

where T is the Wick time ordering operator. For the present example

$$\begin{aligned} H_I(t) = & -g \int d^3x \int d^3y f^\dagger(\vec{x}, \vec{y}, t) f(\vec{x}, \vec{y}, t) \phi(x) \\ & - \int d^3x \int d^3y f^\dagger(\vec{x}, \vec{y}, t) \int d^3x' \int d^3y' V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') f(\vec{x}', \vec{y}', t). \end{aligned}$$

In the following, we will replace $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ by

$$V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \equiv V(\vec{x}, \vec{y} | \vec{x}', \vec{y}') \delta(t - t')$$

and write

$$H_I(t) = \int d^3x \int d^3y H_I(\vec{x}, \vec{y}, t)$$

with

$$\begin{aligned} H_I(\vec{x}, \vec{y}, t) = & -g f^\dagger(\vec{x}, \vec{y}, t) f(\vec{x}, \vec{y}, t) \phi(x) \\ & - f^\dagger(\vec{x}, \vec{y}, t) \int d^3x' \int d^3y' \int_{-\infty}^{\infty} dt' V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') f(\vec{x}', \vec{y}', t'). \end{aligned} \quad (II.13)$$

The S-matrix then becomes

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{i=1}^n \left\{ \int d^3x_i \int d^3y_i \int_{-\infty}^{\infty} dt_i \right\} T [H_I(\vec{x}_1, \vec{y}_1, t_1) \dots H_I(\vec{x}_n, \vec{y}_n, t_n)] \quad (II.14)$$

The form of Eq. (II.14) is essentially the same as that obtained in relativistic perturbation theory. However, this S-matrix is non-relativistic due to the choice of $f(\vec{x}, \vec{y}, t)$, and the fact that the Hamiltonian density $H_I(\vec{x}, \vec{y}, t)$ appearing in the expansion is a function of two spatial positions rather than a single point.

To illustrate the consequences of the bilocal Fermi field, we will consider the form taken by the second order self-energy term of S resulting from the scalar interaction, Eq. (II.13). This is represented by the Feynman diagram in Figure 1, which shows a free fermion of center of mass momentum $\vec{P}$ and internal quantum numbers a' emit a meson of momentum $\vec{q}$, which is then reabsorbed to leave the Fermi particle in state $(\vec{P}, a)$. The doubled line for the fermion is used to indicate bound particle-potential states and propagators, as opposed to "free particle"- "free potential" states for which single fermion lines will be used. The procedures for calculation, via the use of Wick's theorem for the expansion of time ordered products of operators in terms of normal products, are identical to those of relativistic local quantum field theory.

Figure 1. Feynman diagram for the second order fermion self energy.

If we call the S-matrix element in question M_1 , we obtain

$$M_1 = (-i)^2 g^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \int d^3x \int d^3y \int d^3x' \int d^3y' \left[(2\pi)^{-3/2} e^{-i(\vec{P} \cdot \vec{R} - E_{\vec{P}, a} t)} \psi_a(\vec{r}) \right] \cdot \left[\frac{1}{2} \Delta_F(x-x') \right] \cdot \left[-\frac{1}{2} S_F(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \right] \cdot \left[(2\pi)^{-3/2} e^{i(\vec{P}' \cdot \vec{R}' - E_{\vec{P}', a'} t')} \psi_{a'}(\vec{r}') \right] \quad (II.15)$$

The propagators Δ_F and S_F for the intermediate meson and nucleon are given by

$$\Delta_F(x-x') = -\frac{2i}{(2\pi)^4} \int d^4q \frac{e^{iq \cdot (x-x')}}{q^2 + \mu^2 - i\epsilon} \quad (II.16)$$

and

$$S_F(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = -2 \langle 0 | T [f(\vec{x}, \vec{y}, t) f^\dagger(\vec{x}', \vec{y}', t')] | 0 \rangle = -\frac{2i}{(2\pi)^4} \int dE \int d^3P e^{i[\vec{P} \cdot (\vec{R} - \vec{R}') - E(t-t')]} G_{E, \vec{P}}(\vec{r}, \vec{r}') \quad (II.17)$$

with

$$G_{E, \vec{P}}(\vec{r}, \vec{r}') = \sum_a \frac{\psi_a(\vec{r}) \psi_a^*(\vec{r}')}{E - E_{\vec{P}, a} + i\epsilon} \quad (II.18)$$

Equation (II.18) is simply the Green's function for the internal fermion motion.

Inserting equations (II.16) and (II.17) into Eq. (II.15), and performing the integrations over the center of mass and time coordinates gives

$$M_1 = - \frac{g^2}{(2\pi)^3} \delta(\vec{P}-\vec{P}') \delta(E_{\vec{P},a} - E_{\vec{P}',a}) \int dE \, d^3Q \int d^3r \, d^3r' \cdot \quad (II.19)$$

$$\cdot \psi_a^*(\vec{r}) \frac{e^{i\beta(\vec{P}-\vec{Q}) \cdot (\vec{r}-\vec{r}')}}{(\vec{P}-\vec{Q})^2 - (E_{\vec{P},a} - E)^2 + \mu^2 - i\epsilon} G_{E,Q}(\vec{r},\vec{r}') \psi_a(\vec{r}')$$

This result is to be contrasted with that obtained for the corresponding Feynman diagram when the free fermion field is local, and so has no internal structure. The free nucleon field equation would then be

$$-\frac{1}{2m} \nabla^2 f(\vec{x},t) = i \frac{\partial}{\partial t} f(\vec{x},t)$$

with solution

$$f(\vec{x},t) = (2\pi)^{-3/2} \int d^3p \, e^{i(\vec{p} \cdot \vec{x} - E_{\vec{p}} t)} b_{\vec{p}},$$

where $E_{\vec{p}} = \vec{p}^2/2m$. The S-matrix element, M_1' , corresponding to Eq. (II.19) is

$$M_1' = - \frac{g^2}{(2\pi)^3} \delta(\vec{P}-\vec{P}') \delta(E_{\vec{P}} - E_{\vec{P}'}) \int dE \, d^3Q \frac{1}{(\vec{P}-\vec{Q})^2 - (E_{\vec{P}} - E)^2 + \mu^2 - i\epsilon} \frac{1}{E - E_Q + i\epsilon} \quad (II.20)$$

The differences between M_1 and M_1' are seen to be changes at external particle vertices due to the presence of the internal fermion wave functions $\psi_a^*(\vec{r})$ and $\psi_a(\vec{r}')$; the appearance of the factor $e^{i\beta(\vec{P}-\vec{Q}) \cdot (\vec{r}-\vec{r}')}$ in the meson propagator, and the change in the nucleon propagator from $1/(E - E_Q + i\epsilon)$ to $G_{E,Q}(\vec{r},\vec{r}')$. There is also the necessity of integrating over the internal coordinates.

The E integrals in both Equations (II.19) and (II.20) can be performed by contour integration to give

$$M_1 = \frac{-ig^2}{2(2\pi)^2} \delta(\vec{P}-\vec{P}') \delta(E_{\vec{P},a} - E_{\vec{P}',a}) \int d^3Q \int d^3r \, d^3r' \psi_a^*(\vec{r}) \frac{e^{i\beta(\vec{P}-\vec{Q}) \cdot (\vec{r}-\vec{r}')}}{\sqrt{(\vec{P}-\vec{Q})^2 + \mu^2}} G_{E,Q}(\vec{r},\vec{r}') \psi_a(\vec{r}') \quad (II.19')$$

where the E appearing in $G_{E,Q}$ is now

$$E \equiv E_{\vec{P},a} - \sqrt{(\vec{P}-\vec{Q})^2 + \mu^2},$$

and

$$M_1' = \frac{-ig^2}{2(2\pi)^2} \delta(\vec{P}-\vec{P}') \delta(E_{\vec{P}} - E_{\vec{P}'}) \int d^3Q \frac{1}{\sqrt{(\vec{P}-\vec{Q})^2 + \mu^2}} \frac{1}{(E_{\vec{P}} - E_Q) - \sqrt{(\vec{P}-\vec{Q})^2 + \mu^2} + i\epsilon} \quad (II.20')$$

In M_1' , the $\vec{Q}$ integration results in a logarithmic divergence. The situation for M_1 is considerably more complicated, and questions of convergence cannot be answered immediately since the integrals over $\vec{r}$ and $\vec{r}'$ must first be performed. (We have assumed above that interchange of the order of integration is permissible.)

Although it is not possible to make definite statements about the behavior of the integrals in Eq. (II.19') without explicit knowledge of the internal states $\psi_a(r)$ and their Green's function $G_{E,Q}(\vec{r},\vec{r}')$, which in turn depend on a specification of $V(\vec{x},\vec{y}|\vec{x}',\vec{y}')$, some estimates can be attempted. For instance, if it is assumed that only the lowest state in the sum for $G_{E,Q}$, Eq. (II.18), is important for virtual

particle propagation, and that this is a non-degenerate bound s-state with energy $\epsilon = -b^2/2m_r$, then

$$\psi_a(\vec{r}) \sim e^{-br}$$

and

$$G_{E,\vec{Q}} \sim \frac{e^{-br} e^{-br'}}{E - E_{\vec{Q},a} + i\epsilon}$$

The integrals over $\vec{r}$ and $\vec{r}'$ in Eq. (II.19) can now be done, giving

$$\begin{aligned} & \int d^3r d^3r' \psi_a^*(\vec{r}) G_{E,\vec{Q}}(\vec{r},\vec{r}') e^{i\beta(\vec{P}-\vec{Q}) \cdot (\vec{r}-\vec{r}')} \psi_a(\vec{r}') \\ & \sim \int d^3r d^3r' \frac{e^{-2br} e^{-2br'} e^{i\beta(\vec{P}-\vec{Q}) \cdot (\vec{r}-\vec{r}')}}{(E_{\vec{P},a} - E_{\vec{Q},a}) - \sqrt{(\vec{P}-\vec{Q})^2 + \mu^2} + i\epsilon} \\ & \sim \frac{(16\pi b)^2}{[(E_{\vec{P},a} - E_{\vec{Q},a}) - \sqrt{(\vec{P}-\vec{Q})^2 + \mu^2} + i\epsilon] [\beta^2(\vec{P}-\vec{Q})^2 + 4b^2]} \end{aligned}$$

For large $\vec{Q}$, this factor varies like $1/Q^{10}$, and the $\vec{Q}$ integral in Eq. (II.19') will converge rapidly for large internal momenta.

A somewhat different sort of estimate of the convergence properties of a self-energy term like Eq. (II.19') can be made using a one-dimensional model for the meson and nucleon fields with an arbitrary choice for the internal potential of the bilocal Fermi field which will permit analytic evaluation of all quantities. In analogy with the development above, we can adopt the Hamiltonian $H = H_0 + H_I$ where

$$H_0 = \int dx dy f^\dagger(x,y) \left[-\frac{1}{2m} \frac{\partial^2}{\partial x^2} - \frac{1}{2M} \frac{\partial^2}{\partial y^2} + V(x-y) \right] f(x,y) + \frac{1}{2} \int dx \left[\pi^2(x) + \left(\frac{d\phi(x)}{dx} \right)^2 + \mu^2 \phi^2(x) \right]$$

and

$$H_I = -g \int dx \int dy f^\dagger(x,y) f(x,y) \frac{d\phi(x)}{dx} - \int dx \int dy f^\dagger(x,y) V(x-y) f(x,y)$$

The potential $V(x-y)$ has been chosen to be local, and a derivative of the meson field has been used in H_I so as to insure self-energy divergence in the corresponding local theory. Deleting all intermediate steps, the second order self-energy terms equivalent to Eqs. (II.19') and (II.20') are now, respectively

$$\begin{aligned} M_1 = & -i \frac{g^2}{2} \delta(P-P') \delta(E_{P,a} - E_{P',a}) \int dQ \frac{(P-Q)^2}{\sqrt{(P-Q)^2 + \mu^2}} \\ & \cdot \int dz dz' \psi_a^*(z) e^{i\beta(P-Q)(z-z')} \sum_b \frac{\psi_b(z) \psi_b^*(z')}{(E_{P,a} - E_{Q,b}) - \sqrt{(P-Q)^2 + \mu^2} + i\epsilon} \psi_a(z') \end{aligned}$$

and

$$M_1' = -i \frac{g^2}{2} \delta(P-P') \delta(E_P - E_{P'}) \int dQ \frac{(P-Q)^2}{\sqrt{(P-Q)^2 + \mu^2}} \frac{1}{(E_P - E_Q) - \sqrt{(P-Q)^2 + \mu^2} + i\epsilon}$$

where $E_{P,a} = E_P + \epsilon_a$ and $E_P = P^2/2(M+m)$. It is seen that the Q integral in M_1' will diverge logarithmically.

Figure 2. The potential $V(x-y)$.

$$V(z) = \begin{cases} 0 & |z| < a/2 \\ \infty & |z| > a/2 \end{cases}.$$

The equation for the Green's function is

$$-\frac{1}{2m_r} \frac{d^2}{dz^2} G_E(z, z') + V(z) G_E(z, z') - E G_E(z, z') = -\delta(z-z'),$$

which has the solution

$$G_E(z, z') = \begin{cases} \frac{2m_r}{k \sin ka} \sin k(z - \frac{a}{2}) \sin k(z' + \frac{a}{2}) & z > z' \\ \frac{2m_r}{k \sin ka} \sin k(z + \frac{a}{2}) \sin k(z' - \frac{a}{2}) & z < z' \end{cases},$$

where $k = \sqrt{2m_r E}$. The integrals over z and z' in M_1 may now be evaluated in order to determine the behavior of the Q integrand for large Q . This factor will be defined as

$$F_{E, \gamma}(Q) \equiv - \int_{-a/2}^{a/2} dz \int_{-a/2}^{a/2} dz' \psi_1^*(z) e^{i\gamma(z-z')} G_E(z, z') \psi_1(z'),$$

where

$$\gamma = \beta(P - Q)$$

and

$$E = \frac{P^2 - Q^2}{2(m + M)} - \sqrt{(P-Q)^2 + \mu^2}.$$

The lowest internal state with $n = 1$ has been chosen for the free fermions. It is sufficient to do only the z' integration of $F_{E, \gamma}$ to find the large Q behavior in question:

The potential $V(x-y)$ that will be chosen for the evaluation of M_1 is an infinite square well, shown in Figure 2. This potential, though unrealistic, is easy to treat. The solutions of the internal equation

$$-\frac{1}{2m_r} \frac{d^2 \Psi(z)}{dz^2} + V(z) \Psi(z) = \epsilon \Psi(z)$$

are

$$\Psi_n(z) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} (z - \frac{a}{2}) \quad n = \pm 1, \pm 2, \dots,$$

with eigenvalues

$$\epsilon_n = \frac{1}{2m_r} \left(\frac{n\pi}{a} \right)^2.$$

$$\begin{aligned}
F_{E,\gamma} = & \frac{im_r}{ka \sin ka} \int_{-a/2}^{a/2} dz e^{i\gamma z} \sin \frac{\pi}{a} (z - \frac{a}{2}) \sin k(z + \frac{a}{2}) \\
& \cdot \left\{ \frac{e^{\frac{1}{2} \frac{a}{2} (k-\pi/2)}}{k - \pi/2 - \gamma} \left[e^{iz(k-\pi/2-\gamma)} - e^{-\frac{1}{2} \frac{a}{2} (k-\pi/2-\gamma)} \right] - \frac{e^{-\frac{1}{2} \frac{a}{2} (k-\pi/2)}}{k - \pi/2 + \gamma} \left[e^{-iz(k-\pi/2+\gamma)} - e^{\frac{1}{2} \frac{a}{2} (k-\pi/2+\gamma)} \right] \right. \\
& - \frac{e^{\frac{1}{2} \frac{a}{2} (k+\pi/2)}}{k + \pi/2 - \gamma} \left[e^{iz(k+\pi/2-\gamma)} - e^{-\frac{1}{2} \frac{a}{2} (k+\pi/2-\gamma)} \right] + \frac{e^{-\frac{1}{2} \frac{a}{2} (k+\pi/2)}}{k + \pi/2 + \gamma} \left[e^{-iz(k+\pi/2+\gamma)} - e^{\frac{1}{2} \frac{a}{2} (k+\pi/2+\gamma)} \right] \Big\} \\
& + \frac{im_r}{ka \sin ka} \int_{-a/2}^{a/2} dz e^{i\gamma z} \sin \frac{\pi}{a} (z - \frac{a}{2}) \sin k(z - \frac{a}{2}) \\
& \cdot \left\{ \frac{e^{-i(ka+\pi)/2}}{k - \pi/2 - \gamma} \left[e^{\frac{1}{2} \frac{a}{2} (k-\pi/2-\gamma)} - e^{-iz(k-\pi/2-\gamma)} \right] - \frac{e^{i(ka+\pi)/2}}{k - \pi/2 + \gamma} \left[e^{-\frac{1}{2} \frac{a}{2} (k-\pi/2+\gamma)} - e^{-iz(k-\pi/2+\gamma)} \right] \right. \\
& - \frac{e^{-i(ka-\pi)/2}}{k + \pi/2 - \gamma} \left[e^{\frac{1}{2} \frac{a}{2} (k+\pi/2-\gamma)} - e^{iz(k+\pi/2-\gamma)} \right] + \frac{e^{i(ka-\pi)/2}}{k + \pi/2 + \gamma} \left[e^{-\frac{1}{2} \frac{a}{2} (k+\pi/2+\gamma)} - e^{-iz(k+\pi/2+\gamma)} \right] \Big\} .
\end{aligned}$$

The z integration will bring one additional factor of k or γ into the various denominators and leave factors of $e^{\pm ika}$ in the numerators. For large Q , k will be purely imaginary, producing an increasing exponential $e^{|k|a}$ in some numerators. This will be cancelled by a similar term from the denominator factor $\sin ka$. The net result is that

$$\lim_{Q \rightarrow \infty} F_{E,\gamma} \sim \frac{1}{Q^3} .$$

Thus for large momenta, the Q integrand of M_1 will vary like $1/Q^2$, and this integral will indeed converge.

The convergence of the second order self-energy integrals in the above examples is only indicative of what may hold true. Other types of normally divergent S-matrix terms have not been examined. In a more realistic situation with no approximations, convergence may or may not occur, though it is suspected that divergent integrals will not be present in the bilocal formulation. It has, of course, been assumed that internal states $\psi_a(\vec{r})$ exist and are bound. Unbound internal states would seem to be uninterpretable and contrary to the assumed physical picture.

III. THE SELF-CONSISTENT POTENTIAL

In the discussion of the previous section, the potential $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$, which determines the internal fermion structure, was left largely arbitrary. As such, its use may be considered merely as a device which can conceivably lead to the elimination of divergent terms in the perturbation expansion of the S-matrix. However, if the point of view is taken that divergences should not occur in the S-matrix expansion, then it

plausible that the unrenormalized self-energy terms provide explicit information about particle structure. Turning the argument around, we will assume that the self-energy terms are the manifestation of the particle structure as determined by the potential V . This assumption can be made explicit by choosing the potential in such a way that its contributions to the S-matrix via the second term of the interaction Hamiltonian, Eq. (II.3), exactly cancel the fermion self-energy parts produced by the first term. In this sense the potential V becomes self-consistent since it is determined by the boson-fermion interaction itself, and thereby has its origin in the nature of the states which it must determine. The "free" fermion equations which result with the use of this self-consistent potential will have many similarities to Hartree equations. However, they will be considerably more complicated, and are, in principle, exact.

There are two equivalent ways of obtaining the self-consistent "free" fermion equations. The first is to simply note that in the original theory, before the $V(\vec{x}, \vec{y} | \vec{x}', \vec{y}')$ term was added and subtracted from the total Hamiltonian, the exact one-fermion propagator is the sum of single internal fermion lines containing all possible self-energy insertions. If the potential V is chosen to exactly reproduce the interacting one-fermion propagator, i.e. the Green's function of the "free" fermion equation is made identical to the true field theoretic propagator, then no fermion self-energy terms can appear in the perturbation expansion. Otherwise multiple insertions of the true fermion propagator would occur in the expansion terms. As stated before, this may be accomplished by using the second term in Eq. (II.3) as a generalized counter term to exactly cancel the ordinary irreducible self-energy contributions, thereby identifying the potential.

A more formal derivation can be made in a manner similar to that of the Bethe-Salpeter equation.⁷ The one-fermion Green's function is first defined, in the usual way, as the vacuum expectation value of the time-ordered product of a pair of field operators in the Heisenberg representation

$$K(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = - \langle 0 | T \{ f(\vec{x}, \vec{y}, t) f^\dagger(\vec{x}', \vec{y}', t') \} | 0 \rangle \quad (III.1)$$

We consider only the time ordering $t > t'$, since in this non-relativistic theory containing no antiparticle states, $K = 0$ for $t < t'$. Inserting a complete set of one-fermion states between the two f -operators then gives

$$K(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = - \sum_{\vec{P}, a} \langle 0 | f(\vec{x}, \vec{y}, t) | \vec{P}, a \rangle \langle \vec{P}, a | f^\dagger(\vec{x}', \vec{y}', t') | 0 \rangle \quad t > t' \quad (III.2)$$

The intermediate states have been defined in terms of their center of mass momentum $\vec{P}$ and a set of additionally needed quantum numbers "a". A one-fermion amplitude can now be defined as

$$\chi_{\vec{P}, a}(\vec{x}, \vec{y}, t) \equiv \langle 0 | f(\vec{x}, \vec{y}, t) | \vec{P}, a \rangle \quad (III.3)$$

so that

$$K(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = - \sum_{\vec{P}, a} \chi_{\vec{P}, a}(\vec{x}, \vec{y}, t) \chi_{\vec{P}, a}^*(\vec{x}', \vec{y}', t') \quad (III.4)$$

The amplitudes $\chi_{\vec{P}, a}$ can be written in terms of center of mass and relative coordinates $\vec{R}$ and $\vec{r}$, defined by Eq. (II.8), as

$$\chi_{\vec{P}, a} = (2\pi)^{-3/2} e^{i(\vec{P} \cdot \vec{R} - E_{\vec{P}, a} t)} \psi_{\vec{P}, a}(\vec{r}) \quad (III.5)$$

where $E_{\vec{p},a}$ is the total energy associated with state $|\vec{p},a\rangle$.

We now consider a function $F(\vec{x},\vec{y},t,t')$ defined by

$$F(\vec{x},\vec{y},t,t') = \int d^3x' d^3y' K(\vec{x},\vec{y},t|\vec{x}',\vec{y}',t') \chi_{\vec{p},a}^*(\vec{x}',\vec{y}',t') \quad (III.6)$$

where $\chi_{\vec{p},a}^*$ is a particular amplitude, which will be assumed to be a non-degenerate "bound" state for simplicity, i.e. a state for which $\psi_{\vec{p},a}(\vec{r})$ is localized. Inserting $K(\vec{x},\vec{y},t|\vec{x}',\vec{y}',t')$, as given by Eq. (III.4), into the integral for $F(\vec{x},\vec{y},t,t')$ results in

$$\begin{aligned} F(\vec{x},\vec{y},t,t') &= - \int d^3x' d^3y' \sum_{\vec{Q},b} \chi_{\vec{Q},b}^*(\vec{x},\vec{y},t) \chi_{\vec{Q},b}^*(\vec{x}',\vec{y}',t') \chi_{\vec{p},a}(\vec{x}',\vec{y}',t') \\ &= - \sum_{\vec{Q},b} (2\pi)^{-3} \int d^3x' d^3r' \chi_{\vec{Q},b}^*(\vec{x},\vec{y},t) e^{-i(\vec{Q}\cdot\vec{x}' - E_{\vec{Q},b}t')} \psi_{\vec{Q},b}^*(\vec{r}') e^{i(\vec{p}\cdot\vec{x}' - E_{\vec{p},a}t')} \psi_{\vec{p},a}(\vec{r}') \\ &= - \sum_{\vec{Q},b} \delta(\vec{Q}-\vec{p}) \chi_{\vec{p},b}^*(\vec{x},\vec{y},t) e^{i(E_{\vec{p},b} - E_{\vec{p},a})t'} \int d^3r' \psi_{\vec{p},b}^*(\vec{r}') \psi_{\vec{p},a}(\vec{r}') \quad (III.7) \end{aligned}$$

The limit of $F(\vec{x},\vec{y},t,t')$ as $t' \rightarrow -\infty$ is now taken in the form

$$\lim_{t' \rightarrow -\infty}^L F(\vec{x},\vec{y},t,t') \equiv \lim_{t' \rightarrow -\infty} \frac{1}{|t'|} \int_{2t'}^{t'} d\tau F(\vec{x},\vec{y},t,\tau) \quad (III.8)$$

to give

$$\begin{aligned} \lim_{t' \rightarrow -\infty}^L F(\vec{x},\vec{y},t,t') &= - \sum_{\vec{Q},b} \delta(\vec{Q}-\vec{p}) \delta_{b,a} N_{\vec{p},a} \chi_{\vec{p},a}^*(\vec{x},\vec{y},t) \\ &= - N_{\vec{p},a} \chi_{\vec{p},a}^*(\vec{x},\vec{y},t) \quad (III.9) \end{aligned}$$

where

$$N_{\vec{p},a} \equiv \int d^3r' |\psi_{\vec{p},a}(\vec{r}')|^2$$

The $\delta_{b,a}$ in Eq. (III.9) results from the assumption that for non-degenerate states, $E_{\vec{p},b} = E_{\vec{p},a}$ only if $b = a$ so that

$$\lim_{t \rightarrow -\infty} \frac{1}{|t'|} \int_{2t'}^{t'} d\tau e^{i(E_{\vec{p},b} - E_{\vec{p},a})\tau} = \begin{cases} 0 & b \neq a \\ 1 & b = a \end{cases}$$

The integral equation for the one fermion propagator will be of the same form as in conventional field theory, namely

$$\begin{aligned} K(\vec{x},\vec{y},t|\vec{x}',\vec{y}',t') &= K_0(\vec{x},\vec{y},t|\vec{x}',\vec{y}',t') - \int d^3x_1 d^3y_1 dt_1 \int d^3x_2 d^3y_2 dt_2 \\ &\quad \cdot K_0(\vec{x},\vec{y},t|x_1,y_1,t_1) G(\vec{x}_1,\vec{y}_1,t_1|\vec{x}_2,\vec{y}_2,t_2) K(\vec{x}_2,\vec{y}_2,t_2|\vec{x}',\vec{y}',t') \quad (III.10) \end{aligned}$$

$$\left(-\frac{1}{2m} \nabla_x^2 - \frac{1}{2M} \nabla_y^2 - i \frac{\partial}{\partial t}\right) \chi_{\vec{p},a}^+(\vec{x}, \vec{y}, t) = i \int d^3x' d^3y' dt' \bar{G}(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \chi_{\vec{p},a}^+(\vec{x}', \vec{y}', t') \quad (III.13)$$

In general, the interaction kernel $\bar{G}(1|1')$ will be expressible in an integral form similar to

$$\bar{G}(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = (2\pi)^{-4} \int d^3Q dE e^{i\vec{Q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \bar{g}(\vec{Q}, E | \vec{r}, \vec{r}') \quad (III.14)$$

If this is inserted into the previous equation and Eq. (III.5) is used to represent $\chi_{\vec{p},a}^+$, we obtain an equation in the relative coordinate, $\vec{r}$, for the "internal" function $\psi_{\vec{p},a}^+(\vec{r})$:

$$\left(-\frac{1}{2m_r} \nabla_r^2 - \epsilon_{\vec{p},a}\right) \psi_{\vec{p},a}^+(\vec{r}) = i \int d^3r' \bar{g}(\vec{p}, E_{\vec{p},a} | \vec{r}, \vec{r}') \psi_{\vec{p},a}^+(\vec{r}') \quad (III.15)$$

where $\epsilon_{\vec{p},a} = E_{\vec{p},a} - \vec{p}^2/2(M+m)$. Comparing this with Eq. (II.9) of the previous section, it is seen that if the potential $V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t')$ is chosen to be

$$V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = i \delta(\vec{R} - \vec{R}') \delta(t - t') \bar{g}(\vec{p}, E_{\vec{p},a} | \vec{r}, \vec{r}') \quad (III.16)$$

in

$$\left(-\frac{1}{2m} \nabla_x^2 - \frac{1}{2M} \nabla_y^2 - E_{\vec{p},a}\right) \chi_{\vec{p},a}^+(\vec{x}, \vec{y}, t) + \int d^3x' d^3y' dt' V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \chi_{\vec{p},a}^+(\vec{x}', \vec{y}', t') = 0 \quad (III.17)$$

then the same equation for the relative motion, Eq. (III.15), will result. Therefore, the selection of Eq. (III.16) as the potential for the bilocal field theory will result in a "free" fermion field equation whose Green's function is identical to the interacting fermion Green's function.

In using Eq. (III.16) for our potential one important factor must be taken into account. This result was obtained formally in terms of a bilocal theory in which the "bare" fermion was not bound to its "potential" cloud. The Feynman diagrams in Fig. 3 for the Green's function equation contain the "free particle" - "free potential" single fermion lines. When Eq. (III.17) is used as the "free" nucleon equation, the fermion propagators in the S-matrix expansion become the doubled line true propagators. With respect to the diagrams in Fig. 3, this means an explicit summing of terms to turn all "free particle" - "free potential" fermion lines into doubled lines. An example of this is shown in Fig. 4.

Figure 4. An example of the summation of uncrossed fermion self-energy parts to give a "completely crossed" diagram.

Therefore, for our purposes, the total interaction term $\bar{G}(1|1')$ is a sum of proper irreducible self-energy parts whose meson lines are completely crossed and overlapping. Terms similar to those to the left of the equality sign in Fig. 4 do not occur.

A rather unusual feature of Eq. (III.16) is that this potential is explicitly dependent on the total energy and momentum. This raises some questions (which have not been looked into) about the consistency of adding and subtracting this potential in the total Hamiltonian in the Schrödinger representation and then transforming to the interaction representation. However, it should be clear from the above, that once in the interaction representation, it is perfectly consistent to add the potential $V(1|1')$ of Eq. (III.16) to the fermion field equation and to subtract it from the interaction Hamiltonian as was done above.

The problem of the meaning of the coupling between the internal motion and the center of mass motion is still present, however. This coupling does not interfere with the invariance properties of our equations. Further, it is not unique to these equations, since such coupling also occurs in the Bethe-Salpeter equation. Although Eq. (III.16) is non-relativistic, this potential is closely related to a similar relativistic result and the momentum dependence may in some way be connected with the fact that unitary representations of the Lorentz group must be infinite. These questions require further investigation. On the other hand, these problems can be largely avoided by setting the center of mass momentum, $\vec{P}$, equal to zero in Eq. (III.16). Such a restricted definition of the self-consistent potential will not produce a complete cancellation of the fermion self-energy terms of the S-matrix. A further simplifying modification of Eq. (III.16) could be made by also setting the energy $E_{\vec{P},a}$ to zero.

A seeming defect of the bilocal formulation of field theory given above is that in addition to the usual unspecified parameters of the bare fermion mass, m , and the unrenormalized coupling constant, g , an additional quantity M , the mass of the dynamical potential, appears. However, this should not be an arbitrary constant, but should be determined by the theory. The self-consistent potential has been assumed to result from the virtual meson cloud formed by the bare fermion. It is therefore reasonable to take this mass equal to the energy of the virtual mesons in this cloud, which is just the expectation value of the free meson Hamiltonian with respect to the true one fermion state. This state may be written in terms of the time development operator $U(t,t')$ defined in Section II, and is just

$$|\psi_{\vec{P},a}(0)\rangle = U(0,-\infty)|\phi_{\vec{P},a}^+\rangle,$$

where

$$|\phi_{\vec{P},a}^+\rangle = b_{\vec{P},a}^+|0\rangle$$

is the non-interacting one fermion state with center of mass momentum $\vec{P}$ and internal quantum numbers "a". The mass M , which in general will be state dependent, is then

$$\begin{aligned} M &= \lim_{\vec{P}' \rightarrow \vec{P} = 0} \langle \psi_{\vec{P},a}(0) | H_M | \psi_{\vec{P}',a}(0) \rangle / \langle \psi_{\vec{P},a}(0) | \psi_{\vec{P}',a}(0) \rangle \\ &= \lim_{\vec{P}' \rightarrow \vec{P} = 0} \langle \phi_{\vec{P},a}^+ | U^+(0, -\infty) H_M U(0, -\infty) | \phi_{\vec{P}',a}^+ \rangle / \delta(\vec{P}' - \vec{P}) \end{aligned} \quad (III.18)$$

where H_M is the free meson Hamiltonian. This prescription, with the limit $\vec{P}' \rightarrow \vec{P}$, has been used to account explicitly for the δ -function from the center of mass motion which arises in such matrix elements. Just as with $S = U(\infty, -\infty)$, $U(0, -\infty)$ cannot be given in closed form. However, its perturbation expansion is

$$\begin{aligned}
U(0, -\infty) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^0 dt_1 \dots dt_n T [H_I(t_1) \dots H_I(t_n)] \\
&= \sum_{n=0}^{\infty} U_n(0, -\infty)
\end{aligned}
\tag{III.19}$$

Due to the presence of H_m in Eq. (III.18), terms in $U(0, -\infty)$ formed only from the counterterm of $H_I(t)$, i.e. from the potential $V(1|1')$, do not contribute to M . To second order in the coupling strength

$$\begin{aligned}
M &= \lim_{\vec{P}' \rightarrow \vec{P}=0} \langle \phi_{\vec{P},a}^+ | \{1 + U_1^+ + U_2^+\} H_m \{1 + U_1 + U_2\} | \phi_{\vec{P},a} \rangle / \delta(\vec{P}' - \vec{P}) \\
&= \lim_{\vec{P}' \rightarrow \vec{P}=0} \langle \phi_{\vec{P},a}^+ | U_1^+(0, -\infty) H_m U_1(0, -\infty) | \phi_{\vec{P},a} \rangle / \delta(\vec{P}' - \vec{P})
\end{aligned}
\tag{III.20}$$

For the scalar meson interaction this becomes

$$\begin{aligned}
M &= \lim_{\vec{P}' \rightarrow \vec{P}=0} \frac{g^2}{\delta(\vec{P}' - \vec{P})} \langle \phi_{\vec{P},a}^+ | \int d^3x d^3y \int_{-\infty}^0 dt f^+(\vec{x}, \vec{y}, t) \varepsilon(\vec{x}, \vec{y}, t) \phi(\vec{x}, t) H_m \\
&\quad \cdot \int d^3x' d^3y' \int_{-\infty}^0 dt' f^+(\vec{x}', \vec{y}', t') \varepsilon(\vec{x}', \vec{y}', t') \phi(\vec{x}', t') | \phi_{\vec{P},a} \rangle \\
&= \frac{g^2}{2(2\pi)^3} \int d^3r d^3r' \int d^3q \psi_{0,a}^*(\vec{r}) e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} \\
&\quad \cdot \sum_b \frac{\psi_{-q,b}^*(\vec{r}) \psi_{-q,b}^*(\vec{r}')}{[E_{0,a} - E_{-q,b} - \sqrt{q^2 + \mu^2} + i\epsilon]^2} \psi_{0,a}(\vec{r}')
\end{aligned}
\tag{III.21}$$

In the limit that only the lowest state contributes significantly to the b sum, the internal state functions are taken independent of center of mass momenta, and the mass M on the right of Eq. (III.21) is allowed to go to infinity, this becomes

$$M = \frac{g^2}{8\pi} \int d^3r d^3r' |\psi(\vec{r})|^2 \frac{e^{-\mu|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} |\psi(\vec{r}')|^2
\tag{III.22}$$

The mass M must be found simultaneously with the solution of the equation for the internal fermion motion, and, therefore, represents a second consistency condition.

A possible objection to the above definition for the mass M is that although the center of mass energy has been removed in that prescription, the kinetic energy of the relative motion of the dynamical potential with respect to the bare fermion has not. This latter is

$$\begin{aligned}
&\lim_{\vec{P}' \rightarrow \vec{P}=0} \left[-\frac{1}{2M} \int d^3x d^3y \chi_{\vec{P},a}^*(\vec{x}, \vec{y}, t) \nabla_y^2 \chi_{\vec{P},a}(\vec{x}, \vec{y}, t) \right] / \delta(\vec{P}' - \vec{P}) \\
&= -\frac{1}{2M} \int d^3r \psi_{0,a}^*(\vec{r}) \nabla_r^2 \psi_{0,a}(\vec{r})
\end{aligned}$$

One might reason that the necessity to subtract such a term from the right hand side of Eq. (III.18) would imply that this kinetic energy appears twice in the total Hamiltonian, once in the bare fermion term and again in the free meson part. This would seem inconsistent, and argue against this subtraction. However, perhaps the definition, Eq. (III.18), should not be taken too literally. The question of the selection of M is therefore another matter that bears further study.

IV. EXAMPLES FOR SIMPLE INTERACTIONS

It is now of interest to examine the forms taken by the self-consistent one fermion equation for several specific interactions, and also to set down a few of the simpler elements of the S-matrix and again contrast them with those of the corresponding local theory. The boson-fermion interactions which will be considered are of the Yukawa type for scalar, pseudoscalar and vector mesons. For the latter two cases, the spin of the Fermi particles will have to be taken into account explicitly. However, we will continue to use only isotopic spin zero fields for simplicity. The scalar meson case will be treated in some detail, but equations relating to other interactions will merely be written down.

In the previous section, the self-consistent potential which specified the internal fermion motion was related to the sum of its proper irreducible self-energy parts appearing in the perturbation series for the S-matrix. To obtain a tractable one fermion equation, this sum must be arbitrarily truncated at some point. Here, we will consider only the lowest order potential in powers of the coupling constant, though higher terms could be kept to give "better" approximations. Before proceeding with these examples, we wish to note the following. If Eq. (III.16), which satisfies the original criteria of Section II, is used for the potential $V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t')$, then we have

$$\int d^3x' d^3y' dt' V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \chi_{\vec{p}, a}(\vec{x}', \vec{y}', t') = -i(2\pi)^{-3/2} e^{i\vec{p} \cdot \vec{R} - iE_{\vec{p}, a} t} \int d^3x' \bar{g}(\vec{p}, E_{\vec{p}, a} | \vec{r}, \vec{r}') \psi_{\vec{p}, a}(\vec{r}').$$

On the other hand, if one makes the definition

$$\begin{aligned} U(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') &= -i \bar{G}(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \\ &= -i(2\pi)^{-4} \int d^3Q dE e^{i\vec{Q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \bar{g}(\vec{Q}, E | \vec{r}, \vec{r}') \end{aligned} \quad (IV.1)$$

then, assuming interchange of integration order allowable, one likewise obtains

$$\begin{aligned} \int d^3x' d^3y' dt' U(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') \chi_{\vec{p}, a}(\vec{x}', \vec{y}', t') \\ = -i(2\pi)^{-4} \int d^3r' \int d^3Q dE \int d^3x' dt' e^{i\vec{Q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \bar{g}(\vec{Q}, E | \vec{r}, \vec{r}') (2\pi)^{-3/2} e^{i\vec{p} \cdot \vec{R}' - iE_{\vec{p}, a} t'} \psi_{\vec{p}, a}(\vec{r}') \\ = -i(2\pi)^{-3/2} \int d^3x' \bar{g}(\vec{p}, E_{\vec{p}, a} | \vec{r}, \vec{r}') \psi_{\vec{p}, a}(\vec{r}'). \end{aligned}$$

Therefore, for the purpose of obtaining the one fermion equation, it is equivalent, and more convenient, to use Eq. (IV.1) rather than Eq. (III.16) for the self-consistent potential.

The second order fermion self-energy for the scalar meson interaction was considered in Section II. It may be written in the form

$$S_2 = (-1)^2 g^2 \int d^3x d^3y dt \int d^3x' d^3y' dt' \langle \vec{P}, a | f^\dagger(\vec{x}, \vec{y}, t) \frac{1}{(2\pi)^4} \int d^3Q dE e^{i\vec{Q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \cdot G_{E,Q}(\vec{r}, \vec{r}') \frac{(-1)}{(2\pi)^4} \int d^4q \frac{e^{iq \cdot (x-x')}}{q^2 + \mu^2 - i\epsilon} f(\vec{x}', \vec{y}', t') | \vec{P}', a' \rangle$$

This matrix element is identically cancelled by the first order S-matrix element arising from the second term in the interaction Hamiltonian, Eq. (II.3), the Feynman diagram for which is given in Figure 5.

This latter term is simply

$$S_1 = i \int d^3x d^3y dt \langle \vec{P}, a | f^\dagger(\vec{x}, \vec{y}, t) \int d^3x' d^3y' dt' \cdot V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') f(\vec{x}', \vec{y}', t') | \vec{P}', a' \rangle$$

Using S_1 and Eq. (IV.1), the second order self-consistent potential for the internal motion of a fermion of center of mass momentum $\vec{P}$ and internal quantum numbers "a" may be written as

$$V_2(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = -ig^2(2\pi)^{-8} \int d^3Q dE \int d^4q e^{i\vec{Q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \frac{e^{iq \cdot (x-x')}}{q^2 + \mu^2 - i\epsilon} \sum_b \frac{\psi_{Q,b}^*(\vec{r}) \psi_{Q,b}^*(\vec{r}')}{E - E_{Q,b}^+ + i\epsilon} \quad (IV.2)$$

The equation for the internal motion is immediately found from this to be

$$-\frac{1}{2m_r} \nabla_r^2 \psi_{P,a}^*(\vec{r}) - ig^2(2\pi)^{-4} \int d^3r' \int d^4q \frac{e^{i\vec{Q} \cdot (\vec{r} - \vec{r}')}}{q^2 + \mu^2 - i\epsilon} \sum_b \frac{\psi_{Q,b}^*(\vec{r}) \psi_{Q,b}^*(\vec{r}')}{E - E_{Q,b}^+ + i\epsilon} \psi_{P,a}^*(\vec{r}') = (E_{P,a}^+ - \vec{P}^2/2(m+M)) \psi_{P,a}^*(\vec{r}) \quad (IV.3)$$

where now

$$\vec{Q} = \vec{P} - \vec{q} \quad \text{and} \quad E = E_{P,a}^+ - q_0$$

Only those solutions of this equation having unit norm are to be considered physical.

Equation (IV.3) is not yet in a particularly tractable form for finding the internal state function $\psi_{P,a}^*(\vec{r})$. This is primarily because of the appearance of the Green's function

$$G_{E,Q}(\vec{r}, \vec{r}') = \sum_b \frac{\psi_{Q,b}^*(\vec{r}) \psi_{Q,b}^*(\vec{r}')}{E - E_{Q,b}^+ + i\epsilon}$$

within the interaction term as a "source" for the mesonic potential. Since this is a sum over all internal states, Eq. (IV.3), in a sense, is not a single equation, but an infinite set of coupled equations for the internal excitations. The internal state cannot be easily found unless some rather drastic approximations

Figure 5. Feynman diagram for the first order fermion self-energy due to the self-consistent potential acting as a counter term.

are made. (However, see Appendix B.) What will be assumed here is that only the lowest internal state contributes significantly to the off-shell propagation. The lowest two or three such states could have been retained without prohibitive complication. Under this assumption, and with the performance of the q_0 integration as in Eq. (II.19'), the internal equation becomes

$$-\frac{1}{2m_r} \nabla_r^2 \psi_{\vec{P},a}(\vec{r}) - \frac{g^2}{2(2\pi)^3} \int d^3 r' \int d^3 q \frac{e^{i\beta \vec{q} \cdot (\vec{r} - \vec{r}')}}{\sqrt{q^2 + \mu^2}} \frac{\psi_{\vec{Q},a}(\vec{r}) \psi_{\vec{Q},a}^*(\vec{r}') \psi_{\vec{P},a}(\vec{r}')}{\sqrt{q^2 + \mu^2} - (\vec{P}^2 - \vec{Q}^2)/2(m+M) - i\epsilon} = \epsilon_{\vec{P},a} \psi_{\vec{P},a}(\vec{r}) \quad (IV.4)$$

where "a" designates the solution of this equation with lowest energy $\epsilon_{\vec{P},a} = E_{\vec{P},a} - \vec{P}^2/2(m+M)$, and $\vec{Q} = \vec{P} - \vec{q}$ as before. As a final simplification, we will go to the center of mass system $\vec{P} = 0$, and further assume that all $\psi_{\vec{Q},a}(\vec{r})$ are independent of their center of mass momenta, so that for the lowest internal state we have

$$-\frac{1}{2m_r} \nabla^2 \psi(\vec{r}) - \frac{g^2}{2(2\pi)^3} \psi(\vec{r}) \int d^3 r' \int d^3 q' \frac{e^{i\beta \vec{q}' \cdot (\vec{r} - \vec{r}')}}{\sqrt{q'^2 + \mu^2}} \frac{|\psi(\vec{r}')|^2}{\sqrt{q'^2 + \mu^2} + q'^2/2(m+M)} = \epsilon \psi(\vec{r}) \quad (IV.5)$$

The ground state solution of this equation will be spherically symmetric. Thus, making the usual substitution

$$\psi(\vec{r}) = Y_{0,0}(\Omega) \frac{\phi(r)}{r},$$

one obtains the one-dimensional equation for $\phi(r)$

$$-\frac{1}{2m_r} \frac{d^2}{dr^2} \phi(r) - \frac{g^2 Y_{0,0}^2}{2(2\pi)^3} \phi(r) \int \frac{d^3 r'}{r'^2} \int d^3 q' \frac{e^{i\beta \vec{q}' \cdot (\vec{r} - \vec{r}')}}{\sqrt{q'^2 + \mu^2}} \frac{\phi^2(r')}{\sqrt{q'^2 + \mu^2} + q'^2/2(m+M)} = \epsilon \phi(r)$$

The angular integrals in the above equation can now be done to give the final form

$$-\frac{1}{2m_r} \frac{d^2}{dr^2} \phi(r) - \frac{g^2 \phi(r)}{2(2\pi)^2 \beta^2 r} \int_0^\infty dq \int_0^\infty \frac{dr'}{r'} \phi^2(r') \{ \cos \beta q |r - r'| - \cos \beta q (r + r') \} \left[\sqrt{q^2 + \mu^2} \left(\sqrt{q^2 + \mu^2} + q^2/2(m+M) \right) \right]^{-1} = \epsilon \phi(r) \quad (IV.6)$$

A point of interest is that if the potential mass M is allowed to go to infinity, then $\beta = 1$ and the $q^2/2(m+M)$ term in Eq. (IV.5) vanishes. The $d^3 q$ integration may then be performed in that equation to give

$$-\frac{1}{2m} \nabla^2 \psi(\vec{r}) - \frac{g^2}{8\pi} \psi(\vec{r}) \int d^3 r' \frac{|\psi(\vec{r}')|^2}{|\vec{r}' - \vec{r}|} = \epsilon \psi(\vec{r}) \quad (IV.7)$$

Except for a factor of two in the interaction term, this is just the equation which one would obtain in a local nonrelativistic field theory with scalar meson interaction if the fields were treated "classically" and, with the neglect of translational invariance, one solved for the "meson field" produced by fermion distribution as a source and used this as a "self-consistent" potential for the motion of the Fermi particle. In the present bilocal formulation, such an equation is seen to merely be a particular limit of an exact equation for the internal fermion motion in the "one internal meson" approximation.

Before going on to other types of interactions, let us write down the next approximation to the self-consistent potential for the scalar meson case. The only irreducible Feynman diagram for the self-energy

in fourth order which contains no second order self-energy inserts is shown in Fig. 6.

Figure 6. Fourth order self-energy contribution to the self-consistent potential.

The S-matrix element corresponding to this diagram is

$$S_4 = (-i)^4 (-g)^4 \int d^3x d^3y dt \int d^3x_1 d^3y_1 dt_1 \int d^3x_2 d^3y_2 dt_2 \int d^3x' d^3y' dt' \langle \vec{P}, a | r^\dagger(\vec{x}, \vec{y}, t) \\ \cdot [-\frac{1}{2} S_F(\vec{x}, \vec{y}, t | \vec{x}_1, \vec{y}_1, t_1)] [\frac{1}{2} \Delta_F(x_1 - x_2)] [-\frac{1}{2} S_F(\vec{x}_1, \vec{y}_1, t_1 | \vec{x}_2, \vec{y}_2, t_2)] [\frac{1}{2} \Delta_F(x_1 - x')] \\ \cdot [-\frac{1}{2} S_F(\vec{x}_2, \vec{y}_2, t_2 | \vec{x}', \vec{y}', t')] r(\vec{x}', \vec{y}', t') | \vec{P}', a' \rangle$$

where the propagators $S_F(1|1')$ and $\Delta_F(x)$ are as defined in Section II. Again, comparing this with

$$S_1 = i \int d^3x d^3y dt \langle \vec{P}, a | r^\dagger(\vec{x}, \vec{y}, t) \int d^3x' d^3y' dt' V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') r(\vec{x}', \vec{y}', t') | \vec{P}', a' \rangle,$$

provides the identification

$$V_4(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = i g^4 \int d^3x_1 d^3y_1 dt_1 \int d^3x_2 d^3y_2 dt_2 \cdot [-\frac{1}{2} S_F(\vec{x}, \vec{y}, t | \vec{x}_1, \vec{y}_1, t_1)] [\frac{1}{2} \Delta_F(x - x_2)] \\ \cdot [-\frac{1}{2} S_F(\vec{x}_1, \vec{y}_1, t_1 | \vec{x}_2, \vec{y}_2, t_2)] [\frac{1}{2} \Delta_F(x_1 - x')] [-\frac{1}{2} S_F(\vec{x}_2, \vec{y}_2, t_2 | \vec{x}', \vec{y}', t')] \quad (IV. 8)$$

This potential will contain an internal Green's function $G_{E,Q}(\vec{r}', \vec{r}'')$ in each of the three fermion propagators $S_F(1|1')$ of Eq. (IV.8). The non-linear structure of $V_4(1|1')$ will therefore be considerably more complex than that of the second order self-consistent potential, Eq. (IV.2), since it will contain six powers of the internal fermion state functions.

For the case of the interaction of neutral pseudoscalar mesons with a non-relativistic Fermi field, the interaction Hamiltonian in the interaction representation will be

$$H_I(t) = -\frac{f}{\mu} \int d^3x d^3y r^\dagger(\vec{x}, \vec{y}, t) [\vec{\sigma} \cdot \nabla_x \phi(\vec{x}, t)] r(\vec{x}, \vec{y}, t) \\ - \int d^3x d^3y r^\dagger(\vec{x}, \vec{y}, t) \int d^3x' d^3y' dt' V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') r(\vec{x}', \vec{y}', t') \quad (IV.9)$$

The field equations will be the same as in the scalar meson case, "free" equations similar to Eq. (II.5). Following procedures identical to those above, we obtain the second order self-consistent potential

$$V_2(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = -1 \left(\frac{f}{\mu}\right)^2 (2\pi)^{-8} \int d^3q \int d^4q \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{x}') - iE(t-t')}}{q^2 + \mu^2 - i\epsilon} \times \vec{\sigma} \cdot \vec{q} \sum_b \frac{\psi_{\vec{Q},b}^*(\vec{r}) \psi_{\vec{Q},b}^*(\vec{r}')}{E - E_{\vec{Q},b} + i\epsilon} \vec{\sigma} \cdot \vec{q} \quad (IV.10)$$

The resultant equation for the internal fermion motion is

$$-\frac{1}{2m_r} \nabla^2 \psi_{\vec{P},a}(\vec{r}) - 1 \left(\frac{f}{\mu}\right)^2 (2\pi)^{-4} \int d^3r' \int d^4q \frac{e^{i\vec{q} \cdot (\vec{r} - \vec{r}')}}{q^2 + \mu^2 - i\epsilon} \vec{\sigma} \cdot \vec{q} \sum_b \frac{\psi_{\vec{Q},b}^*(\vec{r}) \psi_{\vec{Q},b}^*(\vec{r}')}{E - E_{\vec{Q},b} + i\epsilon} \vec{\sigma} \cdot \vec{q} \psi_{\vec{P},a}(\vec{r}') = \epsilon_{\vec{P},a} \psi_{\vec{P},a}(\vec{r}) \quad (IV.11)$$

where $\vec{Q} = \vec{P} - \vec{q}$ and $E = E_{\vec{P},a} - q_0$. Under the approximations of internal propagation only in the lowest internal state and independence of $\psi_{\vec{Q},b}$ from $\vec{Q}$, and after integration over q_0 one obtains

$$-\frac{1}{2m_r} \nabla^2 \psi_s(\vec{r}) - \frac{(f/\mu)^2}{2(2\pi)^3} \int d^3r' \int d^3q \frac{e^{i\vec{q} \cdot (\vec{r} - \vec{r}')}}{\sqrt{q^2 + \mu^2}} \left[\sqrt{q^2 + \mu^2} + q^2/2(m+M) \right]^{-1} \vec{\sigma} \cdot \vec{q} \sum_s \psi_s(\vec{r}) \psi_s(\vec{r}') \times \vec{\sigma} \cdot \vec{q} \psi_s(\vec{r}') = \epsilon \psi_s(\vec{r}) \quad (IV.12)$$

where the subscript s on $\psi_s(\vec{r})$ is used to enumerate a possible set of degenerate states. It can be shown that this equation cannot have a bound s-wave solution as was possible in the scalar meson case.

The vector boson case which will be considered is that of the electromagnetic interaction. The fermion field must therefore be charged. However, the dynamical potential will be chargeless. The interaction Hamiltonian will be taken as

$$H_I(t) = \int d^3x \int d^3y \bar{r}^\dagger(\vec{x}, \vec{y}, t) \left\{ \frac{ie}{m} \vec{\lambda}(\vec{x}, t) \cdot \nabla_x - \frac{e}{2m} \vec{\sigma} \cdot [\nabla_x \times \vec{\lambda}(\vec{x}, t)] \right\} r(\vec{x}, \vec{y}, t) - \int d^3x \int d^3y \bar{r}^\dagger(\vec{x}, \vec{y}, t) \int d^3x' \int d^3y' d\tau V(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', \tau) r(\vec{x}', \vec{y}', \tau) \quad (IV.13)$$

The Coulomb gauge has been used to eliminate longitudinal photons. The bare fermion is assumed to have no anomalous magnetic moment. The free field equation for $\vec{\lambda}(x)$ is

$$\square^2 \vec{\lambda}(x) = 0$$

and the field function can be expanded in terms of creation and annihilation operators as

$$\vec{\lambda}(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1}^2 \left[e^{ik \cdot x} a_{\lambda, \vec{k}} + e^{-ik \cdot x} a_{\lambda, \vec{k}}^\dagger \right] \vec{e}_\lambda(\vec{k})$$

Here, the $\vec{\epsilon}_\lambda(\vec{k})$ are unit vectors such that $\vec{\epsilon}_1(\vec{k})$, $\vec{\epsilon}_2(\vec{k})$, and $\vec{k}$ form an orthogonal triad. From these,

$$v_2(\vec{x}, \vec{y}, t | \vec{x}', \vec{y}', t') = -\frac{1}{2} \left(\frac{e}{m}\right)^2 (2\pi)^{-6} \int d^3q \int dE \int d^4q \sum_\lambda e^{i\vec{q} \cdot (\vec{R} - \vec{R}') - iE(t-t')} \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}}{q^2 - 1\epsilon} \vec{\epsilon}_\lambda(\vec{q}) \cdot [i\alpha\vec{q} + \vec{\nabla}_x - \vec{\sigma} \times \vec{q}/2] \sum_b \frac{\psi_{Q,b}^*(\vec{r}) \psi_{Q,b}^*(\vec{r}')}{E - E_{Q,b}^* + 1\epsilon} \vec{\epsilon}_\lambda(\vec{q}) \cdot [\alpha\vec{\nabla}_{R'} + \vec{\nabla}_{x'} + \vec{\sigma} \times \vec{q}/2] \quad (IV.14)$$

where $\alpha = m/(M+m)$. The internal fermion state function therefore satisfies the equation

$$-\frac{1}{2m_x} \nabla^2 \psi_{\vec{P},a}(\vec{r}) - \frac{1}{2(2\pi)^3} \left(\frac{e}{m}\right)^2 \int d^3q \sum_\lambda \frac{e^{i\vec{q} \cdot (\vec{r} - \vec{r}')}}{|\vec{q}|} \vec{\epsilon}_\lambda(\vec{q}) \cdot [i\alpha\vec{P} + \vec{\nabla}_x - \vec{\sigma} \times \vec{q}/2] \sum_b \frac{\psi_{Q,b}^*(\vec{r}) \psi_{Q,b}^*(\vec{r}')}{|\vec{q}| + (E_{Q,b}^* - E_{\vec{P},a}^*) - 1\epsilon} \cdot \vec{\epsilon}_\lambda(\vec{q}) \cdot [i\alpha\vec{P} + \vec{\nabla}_{x'} + \vec{\sigma} \times \vec{q}/2] \psi_{\vec{P},a}(\vec{r}') = \epsilon_{\vec{P},a} \psi_{\vec{P},a}(\vec{r}) \quad (IV.15)$$

with $\vec{Q} = \vec{P} - \vec{q}$. The integration over q_0 has already been performed. In the center of mass frame, and with the center of mass dependence omitted from $\psi(\vec{r})$, this becomes

$$-\frac{1}{2m_x} \nabla^2 \psi_a(\vec{r}) - \frac{1}{2(2\pi)^3} \left(\frac{e}{m}\right)^2 \int d^3q \sum_\lambda \frac{e^{i\vec{q} \cdot (\vec{r} - \vec{r}')}}{|\vec{q}|} \vec{\epsilon}_\lambda(\vec{q}) \cdot [\vec{\nabla}_x - \vec{\sigma} \times \vec{q}/2] \sum_b \frac{\psi_b(\vec{r}) \psi_b^*(\vec{r}')}{|\vec{q}| + q^2/4 + (M+m) + (\epsilon_b - \epsilon_a) - 1\epsilon} \cdot \vec{\epsilon}_\lambda(\vec{q}) \cdot [\vec{\nabla}_{x'} + \vec{\sigma} \times \vec{q}/2] \psi_a(\vec{r}') = \epsilon_a \psi_a(\vec{r}) \quad (IV.16)$$

We will now compare the bilocal scattering amplitudes for the two processes of meson-nucleon scattering and nucleon-nucleon scattering in second order perturbation theory, with those obtained from a similar non-relativistic local field theory. In order to avoid complications due to spin dependent interactions, only the neutral scalar meson case will be considered.

The lowest order Feynman diagrams for meson-nucleon scattering are shown in Figure 7.

Figure 7. Feynman diagrams for second order meson-nucleon scattering.

The corresponding scattering amplitude in the center of mass system, omitting energy and momentum δ -functions, will be

$$S_2 = -ig^2(2\pi)^{-2} (2\omega_q)^{-1} \int d^3r d^3r' \psi_{\vec{P},a}^*(\vec{r}) \left\{ e^{-i\vec{q}\cdot\vec{r}'} \sum_b \frac{\psi_{0,b}(\vec{r}') \psi_{0,b}^*(\vec{r})}{E_{\vec{P},a} - E_{0,b} + \omega_q + i\epsilon} e^{i\vec{q}\cdot\vec{r}} + e^{i\vec{q}\cdot\vec{r}} \right. \\ \left. \cdot \sum_b \frac{\psi_{(\vec{P}-\vec{q}),b}(\vec{r}') \psi_{(\vec{P}-\vec{q}),b}^*(\vec{r})}{E_{\vec{P},a} - E_{(\vec{P}-\vec{q}),b} - \omega_q + i\epsilon} e^{-i\vec{q}\cdot\vec{r}'} \right\} \psi_{\vec{P},a}(\vec{r}) .$$

State dependent "form factors" can now be defined by

$$F_{\vec{P},a;\vec{Q},b}(\vec{q}) = \int d^3r \psi_{\vec{P},a}^*(\vec{r}) \psi_{\vec{Q},b}(\vec{r}) e^{i\vec{q}\cdot\vec{r}} , \quad (IV.17)$$

so that S_2 becomes

$$S_2 = -ig^2(2\pi)^{-2} (2\omega_q)^{-1} \sum_b \left\{ \frac{F_{\vec{P},a;0,b}(-\vec{q}) F_{0,b;\vec{P},a}(-\vec{q})}{E_{\vec{P},a} - E_{0,b} + \omega_q} + \frac{F_{\vec{P},a;(\vec{P}-\vec{q}),b}(\vec{q}) F_{(\vec{P}-\vec{q}),b;\vec{P},a}(\vec{q})}{E_{\vec{P},a} - E_{(\vec{P}-\vec{q}),b} - \omega_q} \right\} . \quad (IV.18)$$

If the $\vec{P}$ dependence of $\psi_{\vec{P},a}(\vec{r})$ is disregarded, and if only the lowest internal state (assumed to be an s-state) is retained in the nucleon propagator, Eq. (IV.18) becomes simply

$$S_2 = -ig^2(2\pi)^{-2} (2\omega_q)^{-1} |F(\vec{q})|^2 \left\{ \frac{1}{E_{\vec{P}} + \omega_q} + \frac{1}{E_{\vec{P}} - E_{(\vec{P}-\vec{q})} - \omega_q} \right\} .$$

This is to be compared with

$$S_2' = -ig^2(2\pi)^{-2} (2\omega_q)^{-1} \left\{ \frac{1}{E_{\vec{P}} + \omega_q} + \frac{1}{E_{\vec{P}} - E_{(\vec{P}-\vec{q})} - \omega_q} \right\}$$

obtained from local field theory. The only difference, within the set of approximations made, is the appearance of the form factor $|F(\vec{q})|^2$ due to the internal nucleon distribution. Because the form factor will differ depending on the internal fermion state, the effective coupling constant will change if other than the lowest state is specified for the initial or final fermion.

For the case of nucleon-nucleon scattering due to one scalar meson exchange, the Feynman diagram is given in Figure 8. Omitting the exchange term, the S-matrix element in the center of mass system is

Figure 8. Feynman diagram for nucleon-nucleon scattering due to the exchange of a single meson.

$$T_2 = ig^2(2\pi)^{-2} \int d^3r d^3r' \frac{e^{i\vec{q} \cdot (\vec{r} - \vec{r}')}}{q^2 + \mu^2 - i\epsilon} \psi_{\vec{p}_1, a}^*(\vec{r}') \psi_{\vec{p}_2, a}^*(\vec{r}) \psi_{\vec{p}_1, a}(\vec{r}') \psi_{\vec{p}_2, a}(\vec{r}) \quad , \quad (\text{IV.19})$$

where $\vec{q} = \vec{p}_2 - \vec{p}_2'$ is the momentum transfer. The energy and momentum conserving δ -functions have again been omitted. Under the same approximations as for the previous example, Eq. (IV.19) becomes

$$T_2 = ig^2(2\pi)^{-2} \frac{|F(\vec{q})|^2}{q^2 + \mu^2 - i\epsilon} \quad .$$

The corresponding matrix element in local field theory is identical if the form factor $|F(\vec{q})|^2$ is set equal to unity. Thus, here again the only changes between the bilocal and the local S-matrix element is the appearance of a form factor. With spin dependent interactions the form factors are more complicated, but, in general, only modifications of propagators or the appearance of form factors mark the difference between local and bilocal calculations.

V. RELATIVISTIC INVARIANCE

The extension of the bilocal field theory to a relativistically invariant form appears not so much difficult as it does ambiguous. What is needed for this extension is an invariant two body equation that can be used for the fermion field equation, and a consistent procedure, say, for developing a perturbation solution for the S-matrix in the interaction representation. In the non-relativistic limit, the theory would be expected to reduce to that discussed in the previous sections. Since consistency would seem difficult to prove, and since more than one sort of relativistic two body equation already exists, there are probably numbers of different prescriptions which might appear to fit these requirements. In this section, we will consider only one such possible relativistic form of bilocal field theory. We will also mention a simple, but rather ad hoc, semi-relativistic prescription which is probably adequate for calculations. This section should be regarded as speculative since nothing will actually be proved.

Perhaps the best known, though not the best understood, relativistic two body equation in quantum mechanics is the Bethe-Salpeter equation.⁷ In the ladder approximation, this may be written as an integro-differential equation. This approximate form of the Bethe-Salpeter equation may be used as one starting point for the formulation of a relativistic bilocal field theory. If we assume that the bilocal fermion field describes bare Fermi particles, which obey a Dirac-like equation, in interaction with a dynamical potential with boson-like properties, then the bilocal field equation will have a form like

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \left(\frac{\partial^2}{\partial y_\nu^2} - M^2 \right) f(x, y) = (\text{interaction term}) \quad ,$$

where m is the mass of the bare fermion and M that of its self-consistent potential, as before. The development of a relativistic bilocal field theory with this type of field equation can be made with the usual Lagrangian formalism. Again, using the example of a Fermi field in interaction with a neutral scalar meson field, one can write the Lagrangian for the system as

)

$$L = \int d^4x \int d^4y \left\{ - \frac{\partial \bar{f}(x,y)}{\partial y_\nu} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \frac{\partial f(x,y)}{\partial y_\nu} - M^2 \bar{f}(x,y) \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) f(x,y) \right. \\ \left. + \frac{1}{2} \delta^4(x-y) \left[\left(\frac{\partial \phi(x)}{\partial x_\mu} \right)^2 + \mu^2 \phi^2(x) \right] + g \bar{f}(x,y) f(x,y) \phi(x) \right\} \quad (V.1)$$

As discussed more fully in Appendix A, field equations may be obtained from Eq. (V.1) in the same manner as with a local Lagrangian, giving

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \left(\frac{\partial^2}{\partial y_\nu^2} - M^2 \right) f(x,y) = -g f(x,y) \phi(x) \quad (V.2)$$

and

$$\left(\frac{\partial^2}{\partial x_\mu^2} - \mu^2 \right) \phi(x) = g \int d^4y \bar{f}(x,y) f(x,y).$$

It is also possible to derive a conserved energy-momentum tensor and to define a Hamiltonian (Eqs. (A.8) and (A.9)). However, the energy and momentum conservation laws hold only with respect to a center of mass time

$$T \equiv X_0 = \alpha x_0 + \beta y_0 \quad \alpha + \beta = 1$$

with no restraints whatsoever with respect to the relative time

$$t \equiv x_0 = x_0 - y_0$$

Because the center of mass time alone determines the development of the bilocal system, the Schrödinger equation for the state vectors in the Schrödinger representation must have the form

$$i \frac{\partial}{\partial T} |\Psi_s(T)\rangle = H |\Psi_s(t)\rangle$$

In the interaction picture this will become

$$i \frac{\partial}{\partial T} |\Psi_1(T)\rangle = H_I(T) |\Psi_1(T)\rangle \quad (V.3)$$

as before. Equation (V.3) determines the choice of the time ordering operator which appears in the perturbation expansion of the S-matrix. With two time coordinates, T and t, now available, it would have been uncertain which to use in arranging time ordered products of operators if there were not a clear-cut Lagrangian formalism which leads to the construction of a S-matrix expansion in simple analogy with the non-relativistic situation.

If one proceeds as in Section II by adding and subtracting a fermion potential to the Hamiltonian, and in the interaction picture, associating one of these terms with the free fermion Hamiltonian and the other with the interaction term, the field equations and interaction Hamiltonian for the present example will be

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \left(\frac{\partial^2}{\partial y_\nu^2} - M^2 \right) f(x,y) + \int d^4x' d^4y' V(x,y|x',y') f(x',y') = 0 \quad (V.4a)$$

$$\left(\frac{\partial^2}{\partial x_\mu^2} - \mu^2 \right) \phi(x) = 0 \quad (V.4b)$$

and

$$H_I(T) = -g \int d^4x \int d^4z \bar{f}(x,y) f(x,y) \phi(x) - \int d^4x \int d^4z \bar{f}(x,y) \int d^4x' d^4y' V(x,y|x',y') f(x',y') \quad (V.5)$$

with

$$X = \alpha x + \beta y \quad z = x - y \quad \alpha + \beta = 1$$

The S-matrix expansion in powers of $H_I(T)$ becomes

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\tau_1 \dots d\tau_n T_X [H_I(\tau_1) \dots H_I(\tau_n)] \quad (V.6)$$

In terms of the Hamiltonian density

$$H_I(x,y) = -g \bar{f}(x,y) f(x,y) \phi(x) - \bar{f}(x,y) \int d^4x' d^4y' V(x,y|x',y') f(x',y')$$

Eq. (V.6) may be written as

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 d^4y_1 \dots \int d^4x_n d^4y_n T_X [H_I(x_1,y_1) \dots H_I(x_n,y_n)] \quad (V.7)$$

where the time ordering is with respect to the center of mass times of the $\Psi_I(x,y)$. To apply Eq. (V.7), the field operator $F(x,y)$ must be expanded in terms of the solutions of the free field equation, Eq. (V.4a). The assumption must be made that Eq. (V.4a) indeed has a complete set of normalizable solutions which can serve as a suitable basis for this expansion. This is a major assumption, because equations like Eq. (V.4a) are far from completely understood, and this lack will be compounded when the equation is made non-linear by the self-consistency requirements. The expansion of the Fermi field operator will be written as

$$f(x,y) = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3Q \{ e^{iQ \cdot X} u_{Q,a}^+(z) b_{Q,a} + e^{-iQ \cdot X} v_{Q,a}^-(z) c_{Q,a}^\dagger \} \quad (V.8)$$

where $u_{Q,a}^+(z)$ and $v_{Q,a}^-(z)$ are the respective internal positive and negative energy spinor functions, "a" represent the set of internal quantum numbers including spin orientations, and $b_{Q,a}^\dagger$ and $c_{Q,a}^\dagger$ are creation operators assumed to obey the anticommutation relations

$$\{b_{Q,a}, b_{Q',a'}^\dagger\}_+ = \{c_{Q,a}, c_{Q',a'}^\dagger\}_+ = \delta_{a,a'} \delta(\vec{Q}-\vec{Q}')$$

with all others being zero. Possible restrictions on the normalization of the internal functions $u_{Q,a}^+(z)$ and $v_{Q,a}^-(z)$ have not yet been investigated. We will simply assume the following reasonable choice which will be consistent with the approximations and interpretation discussed later:

$$\int d^4z u_{Q,a}^{\dagger+}(z) u_{Q,a}^+(z) = \int d^4z v_{Q,a}^{\dagger-}(z) v_{Q,a}^-(z) = 1$$

With this normalization, the factor $\sqrt{m_a/E_{Q,a}}$, where m_a is the mass of state "a" and $E_{Q,a} = \sqrt{\vec{Q}^2 + m_a^2}$, which is usually seen in the local analog of Eq. (V.8), has been omitted. The expansion for the meson field is still that given by Eq. (II.7).

Within the framework of these assumptions we will again look at the second order self-energy of the fermion illustrated in Fig. (1). This may now be written as

$$S_2 = -g^2 \int d^4x d^4y \int d^4x' d^4y' (2\pi)^{-3/2} e^{-iP \cdot X} \bar{u}_{P,a}(x) \left[-\frac{1}{2} S_B(x,y|x',y') \right] \left[\frac{1}{2} A_B(x,x') \right] \\ \cdot (2\pi)^{-3/2} e^{iP' \cdot X'} \bar{u}_{P',a'}(x') ,$$

where the propagators are now defined by

$$-\frac{1}{2} S_B(x,y|x',y') \equiv \langle 0 | T_X [f(x,y) \bar{f}(x',y')] | 0 \rangle \\ = \frac{1}{(2\pi)^4} \sum_a \int d^4Q \frac{e^{iQ \cdot (X-X')}}{Q^2 + m_a^2 - i\epsilon} \{ E_{Q,a} [u_{Q,a}^+(x) \bar{u}_{Q,a}^-(x') - v_{Q,a}^-(x') \bar{v}_{Q,a}^-(x')] \\ + Q_0 [u_{Q,a}^+(x) \bar{u}_{Q,a}^-(x') + v_{Q,a}^-(x) \bar{v}_{Q,a}^-(x')] \} \quad (V.9)$$

and

$$\frac{1}{2} A_B(x,x') \equiv -\langle 0 | T_X [\phi(x) \phi(x')] | 0 \rangle \quad (V.10) \\ = \frac{-1}{(2\pi)^4} \int d^4q \frac{e^{iq \cdot (X-X')}}{q^2 + \mu^2 - i\epsilon} e^{i\vec{q} \cdot (\vec{x} - \vec{x}')} \{ \cos \beta \omega_q (x_0 - x_0') - i \frac{q_0}{\omega_q} \sin \beta \omega_q (x_0 - x_0') \} .$$

The fermion propagator in Eq. (V.9) reduces to the usual Feynman propagator for a Dirac particle if the sum over "a" is made a spin sum and the spinor functions $u_{Q,a}^+(x)$ and $v_{Q,a}^-(x)$ are replaced by free particle spinors of momentum $\vec{Q}$ with the normalization $u_Q^\dagger u_Q = v_Q^\dagger v_Q = 1$. The meson propagator is similar to that of the non-relativistic bilocal theory, and would reduce to it if the relative times x_0 and x_0' were zero, or at least equal to one another. The appearance of the relative times in Eq. (V.10) is due to the propagation of the meson not with respect to its own time x_0 but with respect to the center of mass time $T-X_0$. This represents a form of microscopic non-causal behavior.

Proceeding as in Section IV, we can identify the lowest order self-consistent potential by specifying the first order self-energy correction due to $V(x,y|x',y')$.

$$S_1 = i \int d^4x d^4y \int d^4x' d^4y' \bar{T}(x,y) V(x,y|x',y') f(x',y') ,$$

and imposing the requirement $S_1 + S_2 = 0$. The resulting potential is

$$V_2(x,y|x',y') = -1 \frac{g^2}{(2\pi)^4} \sum_a \int d^4Q \int \frac{d^4q}{2\omega_q} e^{iQ \cdot (X-X')} \left[\frac{u_{Q,a}^+(x) \bar{u}_{Q,a}^-(x')}{Q_0 - E_{Q,a} + i\epsilon} + \frac{v_{-Q,a}^-(x) \bar{v}_{-Q,a}^-(x')}{Q_0 + E_{Q,a} - i\epsilon} \right. \\ \left. \times \frac{e^{iq \cdot (X-X')} e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}}{q^2 + \mu^2 - i\epsilon} \right] [2\omega_q \cos \beta \omega_q (t-t') - 2iq_0 \sin \beta \omega_q (t-t')] . \quad (V.11)$$

The equation for the fermion expansion functions becomes

$$\begin{aligned}
\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \left(\frac{\partial^2}{\partial y_\nu^2} - M^2 \right) \chi_{P,a}(x,y) = & i \frac{g^2}{(2\pi)^4} \sum_b \int d^4 Q \int \frac{d^4 q}{2\omega_q} \int d^4 x' d^4 y' e^{iQ \cdot (X-X')} \left[\frac{u_{Q,b}^+(z) \bar{u}_{Q,b}^+(z')}{Q_0 - E_{Q,b} + i\epsilon} \right. \\
& + \left. \frac{v_{-Q,b}^-(z) \bar{v}_{-Q,b}^-(z')}{Q_0 + E_{Q,b} - i\epsilon} \right] \frac{e^{iQ \cdot (X-X')} e^{i\beta Q \cdot (z-z')}}{q^2 + \mu^2 - i\epsilon} \left[2\omega_q \cos \beta \omega_q (t-t') \right. \\
& \left. \left. - 2i q_0 \sin \beta \omega_q (t-t') \right] \chi_{P,a}(x',y') \quad (V.12)
\end{aligned}$$

where $\chi_{P,a}(x,y)$ represents one of the positive or negative energy solutions $e^{iP \cdot X} u_{P,a}^+(z)$ or $e^{iP \cdot X} v_{P,a}^-(z)$.

Several disadvantages of a Bethe-Salpeter-like equation for the bilocal field lie in the mathematical complexity of such equations and the difficulty of interpretation of their elementary solutions. The non-relativistic interpretation in terms of probability amplitude distributions over all space at a given time will no longer apply because of the necessary retention of the relative time t . This, of course, does not disqualify the use of such an equation. A more simple description, and one closer to the non-relativistic formalism, would be in terms of complete sets of two-body functions specified on arbitrary space-like surfaces, and an invariant prescription which describes their dynamical development from one to the next of a sequence of such surfaces. However it is not clear that this can be done.

An approximate semi-relativistic prescription of this latter sort can be obtained if it is assumed that the mass M of the dynamical potential is large compared to that of the bare fermion, and that effects due to the relative time are negligible except for $t \ll 1$. We can then replace the equation for the internal fermion motion with a Dirac equation

$$(\gamma \cdot \nabla_z - \gamma_4 c_{P,a} + m_r) \chi_{P,a}(x,y) = - \int d^4 x' d^4 y' V(x,y|x',y') \chi_{P,a}(x',y') \quad (V.13)$$

and take for $\chi_{P,a}(x,y)$ a form such as

$$\chi_{P,a}(x,y) = e^{iP \cdot X} \psi_{P,a}(\vec{x}) \sqrt{\delta(\tau)}$$

In the above equation, $c_{P,a}$ will be an eigenvalue specifying the binding of the system with respect to the reduced mass $m_r = m\mu/(m + \mu)$, so that the binding energy will be

$$\text{binding energy} = m_r - c_{P,a}$$

The mass of a bound system in state "a" will then be

$$m_a = m + M - (m_r - c_{0,a})$$

Within these approximations, the resulting equation for the internal amplitude $\psi_{P,a}(\vec{x})$ obtained from the reduction of Eq. (V.12) will be

$$\begin{aligned}
(\gamma \cdot \nabla_z - \gamma_4 \epsilon_{P,a} + m_a) \psi_{P,a}(\vec{z}) = & - \frac{\pi g^2}{(2\pi)^4} \int d^3 q \sum_b \int d^3 z' \frac{1}{\sqrt{q^2 + \mu^2}} \left[\frac{\bar{u}_{\vec{P}-\vec{q},b}(\vec{z}) \bar{u}_{\vec{P}-\vec{q},b}(\vec{z}')}{E_{P,a} - E_{P-q,b} - \sqrt{q^2 + \mu^2} - i\epsilon} \right. \\
& \left. + \frac{\bar{v}_{\vec{q}-\vec{P},b}(\vec{z}) \bar{v}_{\vec{q}-\vec{P},b}(\vec{z}')}{E_{P,a} + E_{P-q,b} + \sqrt{q^2 + \mu^2} - i\epsilon} \right] e^{i\vec{q} \cdot (\vec{z} - \vec{z}')} \psi_{P,a}(\vec{z}') \quad (IV.14)
\end{aligned}$$

The integrations over Q and q_0 have already been carried out. This equation can be reduced to a tractable form if the further approximations are made, as in the non-relativistic case, that only the lowest internal state contributes significantly to the right hand side of Eq. (V.14), that all solutions shall be independent of their center of mass momenta, that m_a is sufficiently large that terms like $q^2/2m_a$ can be neglected compared to $\sqrt{q^2 + \mu^2}$, and finally, taking $\psi_{P,a}(\vec{z})$ to be a positive energy state, that the negative energy states in the potential can be neglected because of the larger energy denominators associated with them. If all this is done, the significantly simpler equation

$$(\vec{\gamma} \cdot \nabla_z - \gamma_4 \epsilon + m_r) \psi_r(\vec{z}) = \frac{g^2}{8\pi} \int d^3 z' \frac{e^{-\mu|\vec{z}-\vec{z}'|}}{|\vec{z}-\vec{z}'|} \sum_r \psi_r(\vec{z}) \bar{\psi}_r(\vec{z}') \psi_r(\vec{z}') \quad (V.15)$$

is obtained, where the subscript r distinguishes among a possible set of degenerate ground states. Equation (V.15) will have spherically symmetric s-wave solutions, for which r is just the magnetic quantum number for the total angular momentum $j = 1/2$.

As seen from the above discussion, an invariant bilocal field theory is indeed possible. A thorough investigation of its content and of the various approximations made here has yet to be made. There is, in addition, the question of other possible invariant forms of such a theory with the likelihood that their content would be different as well.

VI. SUMMARY

In the preceding sections the basic elements of a non-relativistic bilocal quantum field theory have been developed, and indications given for how to make this theory relativistically invariant. A great many questions have been left unanswered in this treatment, and many things taken for granted. Among the more important of the former are perhaps those of whether the approximations made in Section III to obtain tractable equations for the fermion internal motion are sufficiently good, whether these equations always have bound states, and if the theory is indeed completely finite. Even if these are assumed so, there remain many other more fundamental questions pertaining to internal consistency, convergence of perturbation series, and of interpretation, aside from the need to investigate the relativistic formulation more fully. However, within the framework of the present approximations it is possible to attempt direct calculations of various properties of the interacting α -fermion system, some of which are not accessible to local field theory. This is perhaps the best justification for the proposed formalism. In the following, we would like to enumerate some further questions which immediately come to mind with regard to the present theory, and also to outline several calculations pertaining to the electromagnetic properties of the interacting fermions.

The list of questions and speculations that arise concerning the bilocal formalism is quite large. All problems that have been treated by local field theory are legitimately subject to reinvestigation within this new formulation in order to see whether significant differences in results occur and whether such differences lead to better or worse agreement with experiment. Concerning the framework of the theory itself, the present formalism does not incorporate a direct means for treating possible rotations and vibrations of the dynamical potential associated with the "bare" fermion. As such "collective" states undoubtedly exist and are no doubt important, their treatment and their effects on the "bare" fermion motion require further investigation. Related to this are problems of the appropriate treatment of angular momentum and charge which might be carried by the self-consistent potential, and whether the boson field should be treated bilocally. Another problem is that of the exact significance of the dependence of the internal fermion motion on the center of mass state of the particle-potential system. Relativistically, it is clear that the internal state function must depend on the center of mass momentum, since it represents a spatially extended body which must "distort" when viewed from different Lorentz frames and cannot be rigid. Questions concerning "renormalizable" and "non-renormalizable" interactions come up in a new context. Some renormalizations will still be appropriate here. However, if the bilocal theories are completely finite, there would seem to be no distinction between renormalizable and non-renormalizable interactions, although it may be that one type will lead to bound bare fermion-potential internal states, while the other will not. The interpretation, properties, and decays of excited states of the internal fermion equation require investigation. Such states should lead to resonances in scattering cross sections which might be identifiable with known resonant states. There is a possibility here that an approximate SU_3 invariance can arise naturally in this theory, as it does in the nuclear shell model. Microscopic causality should be looked into. The relativistic bilocal formulation considered in Section V is not fully causal, though microscopically causal forms may exist. However, macroscopic causality should be valid in all formulations. Still other matters for consideration are whether para-statistics might be more appropriate for bilocal fields, the correspondence with local field theory, the effects of higher order terms in the self-consistent potential on internal states, the treatment of four-fermion type interactions, the applicability and utility of Green's function and dispersion techniques to this theory, and many more.

Among the many applications of the present theory which come to mind are those of the electromagnetic properties of the nucleon. As a first step in such calculations, the self-consistent equations for the internal motion of an isotopic-spin one-half nucleon field in interaction with an isotopic-spin one pseudo-scalar meson field must be derived, simplified, and solved, using the lowest order potential which provides bound states. The principal difficulty in this connection will be the non-spherical nature of the self-consistent potential. With this done, the electromagnetic interaction can be added and its effects calculated. Two problems which would appear subject to relatively easy treatment are those of the electromagnetic mass difference of the neutron and proton, and the calculation of the electromagnetic form factors of the nucleons. The first of these would require the addition of second order electromagnetic corrections to the mesonic dynamical potential. These are shown in Figure 9 in terms of self-energy diagrams.

Fig. 9. Feynman diagrams representing electromagnetic corrections to the meson-nucleon dynamical potential.

Here, the doubled solid line represents the nucleon, the dashed line the pion, and the wavy line the photon. These contributions can probably be treated by perturbation theory, with mass changes appearing as eigenvalue shifts in the internal motion equations. (The changes in the form factors due to such corrections to the internal wave functions that would appear in scattering calculations can be thought of as effective electromagnetic corrections to the strong coupling constants.)

The electromagnetic form factors for the nucleon, and thereby the neutron and proton magnetic moments, can be obtained by the calculation of electron nucleon scattering. The lowest order Feynman diagrams for this process are given in Figures 10a through 10c.

Figure 10. Diagrams a through c show the lowest order contributions to electron-nucleon scattering in the present theory. Diagram d shows some additional terms if a $\lambda\phi^4$ meson-meson interaction were included.

The single solid line represents the electron. It is possible that these terms will be insufficient to describe the scattering adequately, and that classes of terms such as in diagram 10d involving a $\lambda\phi^4$ interaction for the pion, or perhaps the explicit use of ρ and ω mesons, will be needed. Calculations of this sort are presently under investigation.

ACKNOWLEDGMENTS

I would like to thank Dr. William Visscher for his patience in listening to these ideas and for his valuable advice on many occasions.

Appendix A. The Energy-Momentum Tensor in Relativistic Bilocal Field Theory

We want to consider the Lagrangian for a bilocal field in interaction with a local field. The bilocal field functions will be represented by $\eta_\sigma(x, y)$ and the local functions by $\phi_\rho(x)$, where x and y are a pair of independent four-vectors. For the present discussion all quantities will be assumed to be C-numbers as opposed to quantum mechanical operators. The Lagrangian L can be written in terms of its Lagrangian density $\mathcal{L}(x, y)$ as

$$L = \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \mathcal{L}(x, y) \quad (A.1)$$

where Ω_x and Ω_y are fixed volumes for the respective x and y coordinates. The Lagrangian density will be assumed to be a function of the set of bilocal field functions and its first and second derivatives, and to also depend on the set of local operators and its first derivatives

$$\mathcal{L}(x, y) = \mathcal{L}(\eta_\sigma(x, y), \frac{\partial \eta_\sigma}{\partial x_\mu}, \frac{\partial \eta_\sigma}{\partial y_\mu}, \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\nu}, \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial y_\nu}, \frac{\partial^2 \eta_\sigma}{\partial y_\mu \partial y_\nu}, \phi_\rho(x), \frac{\partial \phi_\rho}{\partial x_\mu}) \quad (A.2)$$

Note that ϕ_ρ has been chosen to be a function of x only.

Following a procedure analogous to that used in local field theory, we first assume that L is invariant for infinitesimal variations of the functional forms of the field quantities about their physical values, provided these variations, and those of the first derivatives of the bilocal functions, vanish on the surfaces of Ω_x and Ω_y . Using the symbol δ for the functional variations, we have

$$\begin{aligned} \delta L = \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \left\{ \frac{\partial \mathcal{L}}{\partial \eta_\sigma} \delta \eta_\sigma + \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial x_\mu}} \delta \frac{\partial \eta_\sigma}{\partial x_\mu} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial y_\mu}} \delta \frac{\partial \eta_\sigma}{\partial y_\mu} + \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\nu}} \delta \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\nu} + \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial y_\nu}} \delta \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial y_\nu} \right. \\ \left. + \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial y_\mu \partial y_\nu}} \delta \frac{\partial^2 \eta_\sigma}{\partial y_\mu \partial y_\nu} + \frac{\partial \mathcal{L}}{\partial \phi_\rho} \delta \phi_\rho + \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_\rho}{\partial x_\mu}} \delta \frac{\partial \phi_\rho}{\partial x_\mu} \right\} \quad (A.3) \end{aligned}$$

From the fact that δ commutes with $\frac{\partial}{\partial x_\mu}$ and $\frac{\partial}{\partial y_\mu}$, and that the variations and appropriate derivatives vanish on the surfaces of the integration regions, integration by parts gives

$$\begin{aligned} \delta L = \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \left\{ \left[\frac{\partial \mathcal{L}}{\partial \eta_\sigma} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial x_\mu}} - \frac{\partial}{\partial y_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial y_\mu}} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\nu}} + \frac{\partial^2}{\partial x_\mu \partial y_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial y_\nu}} \right. \right. \\ \left. \left. + \frac{\partial^2}{\partial y_\mu \partial y_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial y_\mu \partial y_\nu}} \right] \delta \eta_\sigma + \left[\frac{\partial \mathcal{L}}{\partial \phi_\rho} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_\rho}{\partial x_\mu}} \right] \delta \phi_\rho \right\} \end{aligned}$$

Because the variations and the volumes Ω_x and Ω_y are arbitrary, $\delta L = 0$ implies that the field functions satisfy the Euler-Lagrange equations

In such a mixed Lagrangian containing both local and bilocal fields, a factor of $\delta^4(x-y)$ may be present in some of the local terms. This can be treated consistently by merely ignoring all derivations of the δ -function.

$$\frac{\partial \mathcal{L}}{\partial \eta_\sigma} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial x_\mu}} - \frac{\partial}{\partial y_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \eta_\sigma}{\partial y_\mu}} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\nu}} + \frac{\partial^2}{\partial x_\mu \partial y_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial y_\nu}} + \frac{\partial^2}{\partial y_\mu \partial y_\nu} \frac{\partial \mathcal{L}}{\partial \frac{\partial^2 \eta_\sigma}{\partial y_\mu \partial y_\nu}} = 0 \quad (\text{A.4})$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x_\mu}} = 0 \quad (\text{A.5})$$

To obtain the energy-momentum tensor, infinitesimal Lorentz transformations, which allow for possible displacements of the volumes of integration, must be considered. Representing such variations by δ , one can write

$$\delta L = \int_{\Omega_x} \delta(d^4x) \int_{\Omega_y} d^4y \mathcal{L}(x,y) + \int_{\Omega_x} d^4x \int_{\Omega_y} \delta(d^4x) \mathcal{L}(x,y) + \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \delta \mathcal{L}(x,y) \quad .$$

The first two terms on the right give the changes in the Jacobians due to the Lorentz transformation $x'_\mu = x_\mu + \delta x_\mu$. The variation $\delta \mathcal{L}(x,y)$ combines both the effects of a change of functional form and a displacement of position. Therefore

$$\begin{aligned} \delta L &= \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \left\{ \mathcal{L}(x,y) \left[\frac{\partial}{\partial x_\mu} \delta x_\mu + \frac{\partial}{\partial y_\mu} \delta y_\mu + \bar{\mathcal{L}}(x,y) \right] + \frac{\partial \mathcal{L}(x,y)}{\partial x_\mu} \delta x_\mu + \frac{\partial \mathcal{L}(x,y)}{\partial y_\mu} \delta y_\mu \right\} \\ &= \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \left\{ \frac{\partial}{\partial x_\mu} \left[\mathcal{L}(x,y) \delta x_\mu \right] + \frac{\partial}{\partial y_\mu} \left[\mathcal{L}(x,y) \delta y_\mu \right] + \bar{\mathcal{L}}(x,y) \right\} \quad . \end{aligned}$$

If $\bar{\mathcal{L}}(x,y)$ is expanded as in Eq. (A.3), and the equations of motion, Eq. (A.4) and (A.5), used to replace the terms $\frac{\partial \mathcal{L}}{\partial \eta_\sigma}$ and $\frac{\partial \mathcal{L}}{\partial \phi}$, the resulting relation may be written as

$$\begin{aligned} \delta L &= \int_{\Omega_x} d^4x \int_{\Omega_y} d^4y \left\{ \frac{\partial}{\partial x_\mu} \left[\mathcal{L} \delta x_\mu + \frac{\partial \mathcal{L}}{\partial \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma + \frac{\partial \mathcal{L}}{\partial \phi} \bar{\mathcal{L}} \phi \right] + \frac{\partial}{\partial y_\mu} \left[\mathcal{L} \delta y_\mu + \frac{\partial \mathcal{L}}{\partial \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma \right] - \frac{1}{2} \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma \right. \right. \\ &\quad \left. \left. - \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial}{\partial x_\nu} \bar{\mathcal{L}} \eta_\sigma \right] - \frac{1}{2} \frac{\partial}{\partial y_\mu} \left[\frac{\partial \mathcal{L}}{\partial y_\nu} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma - \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial}{\partial y_\nu} \bar{\mathcal{L}} \eta_\sigma \right] - \frac{1}{2} \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial y_\nu} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial}{\partial y_\nu} \left[\frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial}{\partial x_\mu} \bar{\mathcal{L}} \eta_\sigma \right] - \frac{1}{2} \frac{\partial}{\partial y_\nu} \left[\frac{\partial \mathcal{L}}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \bar{\mathcal{L}} \eta_\sigma \right] + \frac{1}{2} \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial}{\partial y_\nu} \bar{\mathcal{L}} \eta_\sigma \right] \right\} \quad (\text{A.6}) \end{aligned}$$

For an infinitesimal translation one has

$$\eta'_\sigma(x',y') = \eta_\sigma(x,y) \quad \text{and} \quad \phi'_\rho(x') = \phi_\rho(x) \quad ,$$

where the prime indicates the transformed function or variable. Therefore

$$\begin{aligned} \delta \eta_\sigma(x,y) &= \eta'_\sigma(x',y') - \eta_\sigma(x,y) = \bar{\mathcal{L}} \eta_\sigma + \frac{\partial \eta_\sigma}{\partial x_\mu} \delta x_\mu + \frac{\partial \eta_\sigma}{\partial y_\mu} \delta y_\mu = 0 \quad , \\ \delta \phi_\rho(x) &= \phi'_\rho(x') - \phi_\rho(x) = \bar{\mathcal{L}} \phi_\rho + \frac{\partial \phi_\rho}{\partial x_\mu} \delta x_\mu = 0 \quad . \end{aligned} \quad (\text{A.7})$$

In addition

$$\delta x_\mu = \delta y_\mu = \text{constant} \equiv a_\mu.$$

Using Eq. (A.7) to replace $\bar{\partial}\eta_\sigma$ and $\bar{\partial}\phi_\rho$ in Eq. (A.6), and then transforming to center of mass and relative coordinates by the relations

$$X = \alpha x + \beta y \quad z = x - y \quad \alpha + \beta = 1$$

and

$$\int_{\Omega_x} d^4x \int_{\Omega_y} d^4y + \int_{\Omega} d^4x \int_{\omega} d^4z,$$

and making use of the fact that for the infinitesimal translations

$$\delta x_\mu = a_\mu \quad \text{and} \quad \delta z_\mu = 0,$$

we obtain

$$\begin{aligned} \delta L = & \int_{\Omega} d^4x \int_{\omega} d^4z \left\{ \frac{\partial}{\partial x_\mu} \left[\mathcal{L} \delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} - \frac{\partial \mathcal{L}}{\partial \phi_\rho} \frac{\partial \phi_\rho}{\partial x_\nu} \right] - \frac{\partial}{\partial z_\mu} \left[\frac{\partial \mathcal{L}}{\partial \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\mu} \right] + \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial x_\lambda} \frac{\partial^2 \eta_\sigma}{\partial x_\mu \partial x_\lambda} \right] \right. \\ & - \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\nu} + \frac{1}{2} \frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial^2 \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\mu} + \frac{\partial}{\partial z_\mu} \left[\frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} \right] - \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\nu} \\ & \left. + \frac{1}{2} \frac{\partial}{\partial x_\lambda} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\mu} \right\} \delta x_\mu. \quad (A.8) \end{aligned}$$

All terms in the integrand of Eq. (A.8) are in the form of four vector divergences, either with respect to x_μ or z_μ . If the "relative" volume ω is allowed to expand over all "relative" space-time, and it is assumed that the bilocal fields $\eta_\sigma(X, z)$ and their derivatives vanish as $z \rightarrow \infty$, then the terms $\frac{\partial}{\partial z_\mu} \left[\dots \right]$ in Eq. (A.8) do not contribute to δL . Further, assuming $\delta L = 0$, and using the fact that the terms $\frac{\partial \mathcal{L}}{\partial x_\mu} \delta x_\mu$ and the center of mass volume Ω are arbitrary, one obtains

$$\frac{\partial Q_{\mu\nu}}{\partial x_\mu} = 0 \quad (A.9)$$

with

$$\begin{aligned} Q_{\mu\nu}(X) \equiv & \int d^4z \left\{ \frac{\partial \mathcal{L}}{\partial \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial \phi_\rho} \frac{\partial \phi_\rho}{\partial x_\nu} - \mathcal{L} \delta_{\mu\nu} - \frac{\partial}{\partial x_\lambda} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\nu} - \frac{1}{2} \frac{\partial}{\partial x_\lambda} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial \eta_\sigma}{\partial x_\nu} \right. \\ & \left. + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial^2 \eta_\sigma} \frac{\partial^2 \eta_\sigma}{\partial x_\lambda \partial x_\nu} \right\}. \quad (A.10) \end{aligned}$$

$Q_{\mu\nu}(X)$ is the energy-momentum tensor. It obeys a conservation law, but only with respect to the center of mass coordinates, and not with respect to the internal coordinates. However, this still permits the definition of a conserved energy-momentum four vector

$$P_\mu = \int Q_{\mu 4} d^3x$$

for which, in particular

$$\frac{dP_0}{dX_0} = - \int \sum_{i=1}^3 \frac{\partial Q_{1,4}}{\partial X_1} d^3X = 0$$

Therefore P_0 can be identified as the Hamiltonian for the bilocal system. A quantum mechanical bilocal system will be expected to develop in time like

$$i \frac{\partial}{\partial X_0} |\Psi\rangle = P_0 |\Psi\rangle$$

This equation provides no restrictions for the relative time variable, x_0 . Therefore, the behavior of this system with respect to the relative time need not be causal, though macroscopic causality is still retained.

Appendix B. The One Fermion Green's Function Equation

In Section IV, the equations for the internal fermion motion were approximated by retaining only the lowest (or several lowest) energy states in the expansion of the internal Green's function

$$G_{E,\vec{Q}}(\vec{r},\vec{r}') = \sum_b \frac{\psi_{\vec{Q},b}(\vec{r}) \psi_{\vec{Q},b}^*(\vec{r}')}{E - E_{\vec{Q},b} + i\epsilon}$$

This approximation results in a non-linear integro-differential equation for the functions $\psi_{\vec{Q},b}(\vec{r})$ (or when more than one term from $G_{E,\vec{Q}}$ is kept, a set of such coupled equations). However, such an approximation is not necessary in principle since it is possible to obtain an explicit equation for $G_{E,\vec{Q}}$ from Eq. (III.10) of Section III, shown again below

$$K(1|1') = K_0(1|1') - \int d2 \int d3 K_0(1|2) \bar{G}(2|3) K(3|1') \quad (B.1)$$

As before, $K(1|1')$ is an abbreviation for $K(\vec{x},\vec{y},t|\vec{x}',\vec{y}',t')$ and $d2$ for $d^3x_2 d^3y_2 dt_2$.

Equation (B.1) can be written in differential form as

$$\left(-\frac{1}{2m} \nabla_x^2 - \frac{1}{2M} \nabla_y^2 - i \frac{\partial}{\partial t}\right) K(1|1') = -i \delta(\vec{x}-\vec{x}') \delta(\vec{y}-\vec{y}') \delta(t-t') + \int d2 \bar{G}(1|2) K(2|1') \quad (B.2)$$

where use has been made of the equation for $K_0(1|1')$

$$\left(-\frac{1}{2m} \nabla_x^2 - \frac{1}{2M} \nabla_y^2 - i \frac{\partial}{\partial t}\right) K_0(1|1') = -i \delta(\vec{x}-\vec{x}') \delta(\vec{y}-\vec{y}') \delta(t-t')$$

Writing $K(1|1')$ in terms of amplitude functions

$$\chi_{\vec{p},a}^*(\vec{x},\vec{y},t) = (2\pi)^{-3/2} e^{i\vec{p}\cdot\vec{R}} \psi_{\vec{p},a}(\vec{r})$$

as in Section III, i.e.

$$K(1|1') = -\theta(t-t') \int d^3P \sum_a \chi_{P,a}^{*}(1) \psi_{P,a}^{*}(1') \quad , \quad (B.3)$$

and disposing of the theta function $\theta(t-t')$ by using a Fourier integral representation for the time dependence of the right hand side of Eq. (B.3), gives

$$K(1|1') = \frac{1}{(2\pi)^4} \int d^3P \, dE \, e^{i\vec{P} \cdot (\vec{R}-\vec{R}') - iE(t-t')} G_{E,\vec{P}}(\vec{r},\vec{r}') \quad , \quad (B.4)$$

where

$$G_{E,\vec{P}}(\vec{r},\vec{r}') = \sum_a \frac{\psi_{P,a}^{*}(\vec{r}) \psi_{P,a}^{*}(\vec{r}')}{E - E_{P,a} + i\epsilon} \quad . \quad (B.5)$$

This result was used before in Section II.

We can now substitute Eq. (B.4) into Eq. (B.2) to obtain

$$\begin{aligned} \frac{1}{(2\pi)^4} \int d^3P \, dE \, e^{i\vec{P} \cdot (\vec{R}-\vec{R}') - iE(t-t')} &= \frac{1}{2m_r} \nabla_r^2 + \vec{P}^2/2m - E \, | G_{E,\vec{P}}(\vec{r},\vec{r}') \\ &= -1 \, \delta(\vec{R}-\vec{R}') \, \delta(\vec{r}-\vec{r}') \, \delta(t-t') - (2\pi)^{-4} \int d^2 \bar{G}(1|2) \int d^3P \, dE \, e^{i\vec{P} \cdot (\vec{R}_2-\vec{R}') - iE(t_2-t')} G_{E,\vec{P}}(\vec{r}_2,\vec{r}') \quad . \end{aligned}$$

where $\bar{G} = G + m$. Using the form, Eq. (III.14), for the kernel $\bar{G}(1|2)$

$$\bar{G}(1|1') = (2\pi)^{-4} \int d^3Q \, dE \, e^{i\vec{Q} \cdot (\vec{R}-\vec{R}') - iE(t-t')} \bar{g}(\vec{Q},E|\vec{r},\vec{r}') \quad ,$$

and Fourier integrating out the dependence on the center of mass and time coordinates, then gives the differential equation for the internal Green's function

$$\left[-\frac{1}{2m_r} \nabla_r^2 + \vec{P}^2/2m - E \right] G_{E,\vec{P}}(\vec{r},\vec{r}') - i \int d^3r_2 \, \bar{g}(\vec{P},E|\vec{r},\vec{r}_2) G_{E,\vec{P}}(\vec{r}_2,\vec{r}') = -\delta(\vec{r}-\vec{r}') \quad . \quad (B.6)$$

Since $\bar{g}(\vec{P},E|\vec{r},\vec{r}')$ will be a function of $G_{E,\vec{P}}(\vec{r},\vec{r}')$, this equation is highly non-linear. However, if Eq. (B.6) could be solved first, one could either project out the states $\psi_{P,a}^{*}(\vec{r})$ from $G_{E,\vec{P}}(\vec{r},\vec{r}')$, or use this Green's function to provide an exact linear Schrödinger equation for the $\psi_{P,a}^{*}(\vec{r})$.

As a specific example of Eq. (B.6), we will again consider the case of the neutral scalar meson interaction. To second order in the coupling constant, the reduced interaction kernel $\bar{g}(\vec{P},E|\vec{r},\vec{r}')$ is found from Eqs. (III.15) and (IV.3) to be

$$\bar{g}(\vec{P},E|\vec{r},\vec{r}') = g^2 (2\pi)^{-4} \int d^4q \, \frac{e^{i\vec{q} \cdot (\vec{r}-\vec{r}')}}{q^2 + \mu^2 - i\epsilon} G_{E-q_0,\vec{P}-\vec{q}}(\vec{r},\vec{r}') \quad . \quad (B.7)$$

Therefore, the internal Green's function equation becomes

$$\left[-\frac{1}{2m} \nabla_{\vec{r}}^2 + \vec{p}^2/2m - E \right] G_{E,\vec{p}}(\vec{r},\vec{r}') = i g^2 (2\pi)^{-4} \int d^3 r_2 \int d^4 q \frac{e^{i\vec{q} \cdot (\vec{r}-\vec{r}_2)}}{q^2 + \mu^2 - i\epsilon} G_{E-q_0, \vec{p}-\vec{q}}(\vec{r},\vec{r}_2) G_{E,\vec{p}}(\vec{r}_2,\vec{r}') \\ = -\delta(\vec{r}-\vec{r}') \quad (B.8)$$

Because of the form, Eq. (B.5), for $G_{E,\vec{p}}(\vec{r},\vec{r}')$, the q_0 integration can be performed to give

$$\left[-\frac{1}{2m} \nabla_{\vec{r}}^2 + \vec{p}^2/2m - E \right] G_{E,\vec{p}}(\vec{r},\vec{r}') = \frac{g^2}{2(2\pi)^3} \int d^3 r_2 \int d^3 q \frac{e^{i\vec{q} \cdot (\vec{r}-\vec{r}_2)}}{\sqrt{q^2 + \mu^2}} G_{E-\sqrt{q^2 + \mu^2}, \vec{p}-\vec{q}}(\vec{r},\vec{r}_2) G_{E,\vec{p}}(\vec{r}_2,\vec{r}') \\ = -\delta(\vec{r}-\vec{r}') \quad (B.9)$$

In the limit $M = \infty$, this equation becomes independent of all center of mass momenta, and simplifies to

$$\left[-\frac{1}{2m} \nabla_{\vec{r}}^2 - E \right] G_{E,0}(\vec{r},\vec{r}') = \frac{g^2}{2(2\pi)^3} \int d^3 r_2 \int d^3 q \frac{e^{i\vec{q} \cdot (\vec{r}-\vec{r}_2)}}{\sqrt{q^2 + \mu^2}} G_{E-\sqrt{q^2 + \mu^2}, 0}(\vec{r},\vec{r}_2) G_{E,0}(\vec{r}_2,\vec{r}') = -\delta(\vec{r}-\vec{r}') \quad (B.10)$$

As a final comment, it should be mentioned that even if the second order Green's function from Eq. (B.9) or Eq. (B.10) were available, its use in obtaining $\psi_{F,a}(\vec{r})$ might, in fact, be a poorer approximation than that used in Section IV. This may be so because, in this approximation, $G_{E,\vec{p}}(\vec{r},\vec{r}')$ will contain elements due to "free particle"-free potential" states. Intuitively, such states must be unphysical, and so would be expected to gradually vanish as higher approximation are added to the interaction kernel $\bar{G}_{F,E}(\vec{r},\vec{r}')$.

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