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Interactions Between Parametrically Excited Spin Waves in Yttrium Iron Garnet

MASTER

John D. Bierlein

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CHAPTER I

INTRODUCTION

In 1953, Bloembergen and Wang¹ observed that the imaginary part of the microwave susceptibility of ferrites* declined at a value of microwave field much lower than that predicted by the Bloch-Bloembergen equations. They also observed a second absorption peak which appeared at a d.c. magnetic field below that for normal resonance. These effects were first explained by Suhl² as being due to the parametric excitation of certain higher order modes of magnetization, spin waves[†], to relatively large values by a nonlinear interaction involving a mode in which the magnetization is uniform throughout the sample. There are two important nonlinear interactions in which spin waves can be excited to large amplitudes and these are usually referred to as the first and second order Suhl processes.

On resonance, the first order process is important for d.c. fields below a certain critical field and involves a coupling between the uniform mode and certain non-degenerate spin waves. This process has satisfactorily explained both the presence of the subsidiary absorption peak and the shape of the susceptibility decline curve^{2,3}.

Above this critical field, the first order process is forbidden on resonance. Here the Suhl second order process can lead to a decline in the susceptibility and involves a nonlinear interaction between the

* The term "ferrite" shall be used to mean a material that has the properties of a ferrimagnetic insulator.

† By the term "spin waves" we do not mean the uniform mode although this is usually considered as a spin wave of infinite wavelength.

uniform mode and certain degenerate spin waves. There are other processes that also can be of importance here which involve a linear interaction between the uniform mode and degenerate spin waves due to inhomogeneities, surface roughness, voids within the sample, etc.^{4,5}. Including both processes, the shape of the susceptibility decline curve can vary from being very gradual when the linear processes are important, to a curve that declines very sharply at some definite critical driving field when these processes are not present. Experimental data on polycrystalline rare earth garnets and rare earth substituted garnets^{6,7,8,9} have verified the theory for the case when linear scattering is important. However, for well-polished single crystals of yttrium iron garnet (YIG) where the linear processes are less important, discrepancies between theory and experiment have appeared¹⁰. For these samples the single Suhl second order process is not sufficient to describe the susceptibility decline curve. Thus, higher order[†] processes involving interactions between the excited spin waves must be considered. These interactions are in general parametric, just as in the normal Suhl second order process, and as a consequence definite phase relations should exist among the excited spin waves. However, it can be shown (see Chapter II) that the susceptibility decline curve can be explained simply by lumping the effects of higher order spin wave interactions into an effective, power-dependent spin wave relaxation rate. Hence, from susceptibility decline data alone, the importance of specific higher order spin wave interactions

[†] In referring to spin wave interactions the terms "higher order" will henceforth be used in the sense that the spin wave amplitude is not small compared to the uniform mode, e.g., processes where inequality (2.35) is not valid; it will not indicate processes involving greater numbers of spin waves.

is not obvious. This fact is more clearly understood when one considers that only the uniform mode is observed in normal resonance experiments, and any effects of the spin wave modes are detected only from their subsequent influence on the uniform mode.

The parametric effects of the higher order spin wave interactions are more evident when one studies the second harmonic part of the z-component of the magnetization, $M_z(2\omega)$, i.e., the component along the direction of the applied d.c. field. This harmonic content can be observed experimentally as power radiated into a waveguide that is properly coupled to the fundamental waveguide second harmonic fields. The uniform mode contributes to the second harmonic simply from the fact that due to dipole and anisotropy fields the magnetization precesses in an elliptical, rather than a circular, path about the applied d.c. field direction. It is not as obvious that spin waves can contribute to the second harmonic fields because of their short wavelengths; however, it is shown¹¹ that due to single ion anisotropy those spin waves excited by the Suhl second order process can contribute to $M_z(2\omega)$. Therefore, a study of this component of magnetization as a function of applied microwave field can be useful in the study of higher order spin wave interactions.

The course of this investigation has been to perform accurate measurements that show the steady state and transient dependence of both $M_z(2\omega)$ and the uniform mode on applied microwave field for fields above and below Suhl threshold. These data are then compared with a theory developed here which shows that many interesting effects can be explained in terms of the parametric nature of higher order spin wave interactions. This investigation, to our knowledge, has provided the first study of

parametric interactions among spin waves, the fundamental excitations in magnetic insulators.

In Chapter II we present the normal mode theory of ferrimagnetic resonance following closely the approach of Suhl² and Schlömann¹². Since linear processes are ignored, the present theory is applicable only for narrow line-width samples. However, the effects of higher order interactions among spin waves excited by the Suhl processes are considered. It is shown that these interactions can be neglected for the first order process, but that they affect the shape of the susceptibility decline curve when the second order process is important. In addition, an effective relaxation rate method is presented in which the effects of these higher order interactions are included by assuming that they raise the effective temperature of the modes interacting with the spin waves excited by the Suhl process. The result is to make the relaxation rate of the Suhl spin waves power dependent. This method provides results which are in excellent agreement with the experimental curves for the decline of the susceptibility in single crystal YIG spheres ($\Delta H_0 < .4$ gauss) and which can explain the data of Anderson¹⁰ for highly polished single crystals of the same material. Typical curves of the transient response of the uniform and spin wave modes to a fast microwave driving field pulse are presented for cases where zero, one, and two spin wave pairs are excited.

Corrections in the coupling constants for the Suhl second order process (g_{ok}) and the important higher order spin wave interactions ($g_{kk'}$) due to anisotropy fields are also considered. The result of this investigation is that in both interactions there is a dependence on crystal orientation — a result which is in agreement with Schlömann¹².

The coupling constant $g_{kk'}$, also shows a small dependence on the azimuthal angle, $\varphi_{k'}$, of the wave vector, $\vec{k}'$, that defines the interacting spin wave modes.

Using the results of Chapter II, relations for the second harmonic contribution to the z-component of magnetization are derived in Chapter III for cases where zero, one, and two spin wave pairs are excited. The theory presented extends that of Richards¹³ to include higher order spin wave interactions. A simple vector model is presented from which the general characteristics of both the steady-state and transient response of $M_z(2\omega)$ readily can be determined. On the basis of this model, curves for the typical transient response of $M_z(2\omega)$ are presented for the above three cases. Due to radiation damping of the uniform mode, the excited spin waves create a reradiation peak that appears on the trailing edge of $M_z(2\omega)$ after the uniform mode has decayed to a small value. The characteristics of this reradiation peak easily can be determined using the vector model and will depend on the relative contributions to $M_z(2\omega)$ of the uniform and spin wave modes. Again using this model, it may be seen that a power dependent phase shift of $M_z(2\omega)$ relative to the uniform mode can occur.

In Chapter IV experimental results are presented that verify qualitatively the predictions of Chapters II and III. An approximate many spin wave approach is derived which gives much better quantitative agreement than the relatively simple two-spin-wave-pair model presented in Chapter II. The general characteristics of the observed second harmonic line profile can also be reproduced using this method. The spin wave contribution to the second harmonic is experimentally determined by both a phase shift and transient decay method and compared with

that predicted by the many spin wave model. The spin wave linewidth, ΔH_k , can also be determined from transient data and the values obtained using this method are compared with those determined from the susceptibility decline curve. The values of ΔH_k , thus obtained, are then compared to corrected parallel pumping results and to the predicted values for degenerate spin waves using the theory of Sparks, et. al.²¹

CHAPTER II

NORMAL MODE THEORY OF FERROMAGNETIC RESONANCE

The normal mode theory of ferromagnetic resonance is based on the fact that a magnetic system, an insulator in this case, like any electrical or mechanical system oscillates freely in certain normal modes. Theoretically, these modes are determined by solving Maxwell's equations simultaneously with the loss free equation of motion of the magnetization

$$\frac{d\vec{M}(\vec{r},t)}{dt} = -\gamma(\vec{M}(\vec{r},t) \times \vec{H}(\vec{r},t)) \quad (2.1)$$

using appropriate boundary conditions on the magnetic field, $\vec{H}(\vec{r},t)$, and the magnetic induction, $\vec{B}(\vec{r},t) = \vec{H}(\vec{r},t) + 4\pi\vec{M}(\vec{r},t)$. Here $\vec{H}(\vec{r},t)$ is the magnetic field intensity as seen by the magnetization which includes the applied external field $\vec{H}_{\text{ext}}(\vec{r},t)$ and the internal exchange, dipolar, and anisotropy fields, $\vec{H}_{\text{ex}}$, $\vec{H}_{\text{dip}}$, $\vec{H}_a$, all of which depend on the magnetization, $\vec{M}(\vec{r},t)$. The constant γ is the gyromagnetic ratio given by $\gamma = g|e|/2mc$ where g is the spectroscopic splitting factor which is assumed to have its spin only value of $g = 2.0$, e is the electronic charge, m is the mass of the electron, and c is the velocity of light in free space.

A general solution to this problem is, of course, impossible from a practical viewpoint and it has been solved only for a limited number of cases. First, the sample is generally assumed to be small compared to c/ω where ω is the angular frequency of the magnetization. Then Maxwell's equations reduce to their familiar quasi-static form

$$\nabla \cdot (\vec{H}(\vec{r},t) + 4\pi\vec{M}(\vec{r},t)) = 0 \quad (2.2)$$

$$\nabla \times \vec{H}(\vec{r},t) = 0.$$

Even with this assumption a general solution to (2.1) is extremely difficult because of the surface divergence of the magnetization which gives a complicated distribution to the dipolar field. However, if the sample is uniformly magnetized and has an ellipsoidal shape, then this dipolar field takes a simple form and can be described in terms of the well-known demagnetization tensor, $\vec{N}$.

With these assumptions, the solution to (2.1) for small deviations from equilibrium is an infinite set of magnetostatic modes, each being characterized by the spatial variation of the magnetization. For variations of the order of or larger than the sample dimensions an exact solution to (2.1) in the small signal approximation must allow for the surface boundaries, and the modes determined in this case are the well-known "Walker-modes"¹⁴. Except for the uniform mode, these cannot, in principle, be excited directly by a uniform driving field. Because of this and because they apparently are not involved in any of the higher order interactions to be discussed in this paper, the "Walker-modes" will not be considered here. Certain ones of them, however, can lead to a sizeable harmonic content¹³ and they are briefly mentioned in this context in Chapter III.

When the spatial variation of $\vec{M}(\vec{r},t)$ is large (small wavelength) compared to the sample dimensions, the exact surface boundaries become less important and the contribution to the dipolar fields can be calculated in an approximate way. In this case it is convenient to assume periodic boundary conditions so solutions to (2.1) become plane waves which are in general elliptically polarized. These modes are

characterized by a wave vector $\vec{k}$ and are known as spin waves. It is these modes that are of particular interest to us and, therefore, will be discussed extensively in this paper.

The procedure to be used in obtaining solutions to (2.1) is due to Suhl² and Schlömann¹² and will only be outlined here. For a more complete discussion of specific points in the basic theory the reader is referred to these two papers.

A. GENERAL FORMULATION AND LINEAR THEORY

It is convenient first to define a normalized magnetization by

$$\vec{\alpha}(\vec{r}, t) = \frac{\vec{M}(\vec{r}, t)}{|\vec{M}(\vec{r}, t)|}. \quad (2.3)$$

Then the equation of motion (2.1) can be written in terms of $\vec{\alpha}(\vec{r}, t)$ as

$$\frac{d\vec{\alpha}(\vec{r}, t)}{dt} = -\gamma(\vec{\alpha}(\vec{r}, t) \times \vec{H}(\vec{r}, t)). \quad (2.4)$$

The normalized magnetization now can be expanded in a Fourier series in k-space as

$$\vec{\alpha}(\vec{r}, t) = \sum_k \vec{\alpha}_k(t) e^{i\vec{k} \cdot \vec{r}} \quad (2.5)$$

where summation over k extends throughout the first Brillouin zone.

Notice that since $\vec{\alpha}(t)$ is a real quantity the condition

$$\vec{\alpha}_k(t) = \vec{\alpha}_{-k}^*(t) \quad (2.6)$$

on the Fourier components, where $\vec{\alpha}^*(t)$ is the complex conjugate of $\vec{\alpha}(t)$, is a direct consequence of the definition of these components. By making use of the relation

$$\frac{1}{V} \int_V e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d\vec{r} = \delta_{\vec{k}\vec{k}'} \quad (2.7)$$

where the integration is over the volume V of the sample, the Fourier components can be related to the normalized magnetization by

$$\vec{\alpha}_{\vec{k}}(t) = \frac{1}{V} \int \vec{\alpha}(\vec{r}, t) e^{i\vec{k} \cdot \vec{r}} d\vec{r} \quad (2.8)$$

The magnetic field appearing in (2.4) is usually written as a sum of the separate contributions defined under (2.1) as

$$\vec{H}(\vec{r}, t) = \vec{H}_{\text{ext}}(\vec{r}, t) + \vec{H}_{\text{ex}}(\vec{r}, t) + \vec{H}_{\text{dip}}(\vec{r}, t) + \vec{H}_a(\vec{r}, t). \quad (2.9)$$

For normal transverse pumping the external field is

$$\vec{H}_{\text{ext}}(\vec{r}, t) = \hat{k} H_0 + \hat{i} h_0 \cos \omega t \quad (2.10)$$

where ω is the angular frequency of the applied linearly polarized microwave field h_0 , and $\hat{k}$ and $\hat{i}$ are unit vectors in the laboratory z- and x- directions, respectively.

The exchange field can be written in terms of the magnetization as¹⁵

$$\vec{H}_{\text{ex}}(\vec{r}, t) = H_{\text{ex}} a^2 \nabla^2 \vec{\alpha}(\vec{r}, t) \quad (2.11)$$

where H_{ex} is the exchange field constant[†] and a the magnetic lattice spacing. In terms of the Fourier components defined in (2.5) this field becomes

$$\vec{H}_{ex}(\vec{r}, t) = -H_{ex} a^2 \sum_{\vec{k}} k^2 \vec{\alpha}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{r}} \quad (2.12)$$

The dipolar field contribution is determined by solving (2.2) with (2.3) and (2.5) using exact boundary conditions for the uniform mode and periodic boundary conditions for the spin wave modes. This field then can be written as

$$\vec{H}_{dip}(\vec{r}, t) = -4\pi M \sum_{\vec{k}}' \frac{\vec{k}(\vec{k} \cdot \vec{\alpha}_{\vec{k}}(t))}{k^2} e^{i\vec{k} \cdot \vec{r}} - 4\pi \vec{N} \vec{\alpha}_0(t) \quad (2.13)$$

where $\sum_{\vec{k}}'$ indicates the summation includes only those values of $\vec{k}$ greater than some $\vec{k}_{min}$, a cut off wave vector that has the approximate value of inverse sample dimension; and for the samples considered in this paper $\vec{N}$ will have only the three diagonal components N_x , N_y , and N_z which satisfy the relation $N_x + N_y + N_z = 1$.

The last term in $\vec{H}(\vec{r}, t)$, the anisotropy field $\vec{H}_a(\vec{r}, t)$, is discussed in the last section of this chapter and the remainder of this section will be devoted to solving (2.4) for $\vec{\alpha}(\vec{r}, t)$.

In the small amplitude approximation the z-component of $\vec{\alpha}(\vec{r}, t)$ can be expanded in terms of the transverse components as

$$\alpha_z(\vec{r}, t) = 1 - \frac{1}{2} (\alpha_x^2(\vec{r}, t) + \alpha_y^2(\vec{r}, t)) \quad (2.14)$$

[†] For a simple cubic lattice H_{ex} is related to the exchange integral J by $H_{ex} = JS/\mu$ where μ is the Bohr magneton and S the effective spin per lattice point.

Therefore, we need to solve (2.4) only for the transverse components of $\vec{\alpha}(\vec{r}, t)$. However, it will be more convenient to solve this equation for the complex components defined by

$$\alpha^+(\vec{r}, t) = \alpha_x(\vec{r}, t) + i\alpha_y(\vec{r}, t)$$

and

(2.15)

$$\alpha^-(\vec{r}, t) = \alpha_x(\vec{r}, t) - i\alpha_y(\vec{r}, t) .$$

Similarly the Fourier components can be redefined as

$$\alpha_k(t) = \alpha_{kx}(t) + i\alpha_{ky}(t)$$

(2.16)

$$\alpha_k^*(t) = \alpha_{kx}(t) - i\alpha_{ky}(t)$$

and it also will be convenient to use the following notation

$$\omega_H = \gamma H_0$$

$$\omega_M = 4\pi\gamma M$$

$$\omega_{ex} = \gamma H_{ex} \quad (2.17)$$

$$h^+(t) = h_x(t) + ih_y(t)$$

$$k^+ = k_x + ik_y .$$

With this notation and using (2.5), (2.12), and (2.13) the equation of motion for $\alpha_k(t)$ becomes

$$\begin{aligned}
\dot{\alpha}_k(t) = i \{ & [\omega_H - \omega_M N_z \alpha_{oz}(t)] \alpha_k(t) \\
& - [\gamma h^+(t) - \omega_M (N_x \alpha_{ox}(t) + i N_y \alpha_{oy}(t))] \alpha_{kz}(t) \\
& - \omega_{ex} a^2 \sum_{k'} k'^2 [\alpha_{k',z}(t) \alpha_{k-k'}(t) - \alpha_{k'}(t) \alpha_{k-k',z}(t)] \\
& - m_M \sum_{k'} \frac{\vec{k}' \cdot \vec{\alpha}_{k'}(t)}{k'^2} [k'_z \alpha_{k-k'}(t) - k'^+ \alpha_{k-k',z}(t)] \} .
\end{aligned} \tag{2.18}$$

In the small amplitude approximation the Fourier components of $\alpha_z(\vec{r}, t)$ are

$$\alpha_{kz}(t) \cong \delta(k) - \frac{1}{2} \sum_{k'} \alpha_{k+k'}(t) \alpha_{k'}^*(t) . \tag{2.19}$$

With this relation the equations of motion for $\alpha_k(t)$ and $\alpha_o(t)$ to first order become

$$-i\dot{\alpha}_k(t) = A_k \alpha_k(t) + B_k \alpha_{-k}^*(t) \tag{2.20}$$

$$-i\dot{\alpha}_o(t) = A_o \alpha_o(t) + B_o \alpha_o^*(t) - \gamma h^+(t) \tag{2.21}$$

where

$$\begin{aligned}
A_k &= A_{-k} = \omega_i + \omega_{ex} (ak)^2 + \frac{1}{2} \omega_M \sin^2 \theta_k \\
B_k &= B_{-k} = \frac{1}{2} \omega_M \sin^2 \theta_k e^{2i\varphi_k} \\
A_o &= \omega_i + \frac{1}{2} \omega_M (N_x + N_y) \\
B_o &= \frac{\omega_M}{2} (N_x - N_y)
\end{aligned} \tag{2.22}$$

$$\omega_i = \omega_H - \omega_M N_z$$

and where θ_k and φ_k are, respectively, the polar and azimuthal angles of the wave vector $\vec{k}$.

Eqs. (2.20) and (2.21) form what is known as the linear theory of ferrimagnetic resonance and will be used as the starting point of all subsequent discussions of higher order spin wave interactions. The spin wave modes defined by $\alpha_k(t)$ are the normal modes of the sample in the coordinate system defined by the unit vectors given in (2.4). They are not normal modes in time, however, since they are elliptically polarized. The normal modes in space and time can be determined by transforming the $\alpha_k(t)$'s to some new representation defined by

$$\alpha_k(t) = \lambda_k \beta_k(t) - \mu_k \beta_{-k}^*(t) \quad (2.23)$$

where λ_k and μ_k , the transformation constants, are given by

$$\begin{aligned} \lambda_k &= \lambda_{-k} = \cosh \frac{\psi_k}{2} \\ \mu_k &= \mu_{-k} = \sinh \frac{\psi_k}{2} e^{2i\varphi_k} \\ \tanh \psi_k &= \frac{|B_k|}{A_k} \end{aligned} \quad (2.24)$$

With this transformation, which is essentially the Holstein-Primakoff transformation¹⁶, and using Eqs. (2.20) and (2.21), the equations of motion for the normal modes become

$$\dot{\beta}_k(t) = i\omega_k \beta_k(t) \quad (2.25)$$

and

$$\dot{\beta}_0(t) = i\omega_0 \beta_0(t) - i\gamma \hat{h}^+(t) \quad (2.26)$$

where $\hat{h}^+(t)$ is the properly transformed driving field and

$$\omega_k = (A_k^2 - |B_k|^2)^{1/2} \quad (2.27)$$

represents the dispersion relation for the spin wave modes β_k and is represented in Fig. 1.

So far we have neglected processes which can dissipate the energy in a normal mode. Although these are accounted for both phenomenologically^{17,18,19} and microscopically[†], a complete discussion of them is beyond the scope of this paper. For the purpose of the present investigation, dissipation can be conveniently accounted for through the introduction of an imaginary component to the angular frequencies ω_k and ω_o . Therefore, defining Ω_k and Ω_o as

$$\Omega_k = \omega_k + i\eta_k \quad (2.28)$$

$$\Omega_o = \omega_o + i\eta_o$$

Eqs. (2.25) and (2.26) become

$$\dot{\beta}_k(t) = i\Omega_k \beta_k(t) \quad (2.29)$$

$$\dot{\beta}_o(t) = i\Omega_o \beta_o(t) - i\gamma \hat{h}^+ \quad (2.30)$$

where η_k represents the relaxation rate of the mode β_k .

The solution to (2.29) is simply that of an exponentially damped harmonic oscillator which for long times shows that β_k approaches zero. Actually, since β_k is a normal mode of the system, at finite temperatures it is thermally excited to some value β_{kT} , where the amplitude of β_{kT} is given by Bose-Einstein statistics and its phase is assumed to be a random variable.

Since the thermal equilibrium amplitude is usually much smaller than that of any mode which is driven either directly — as is the uniform mode —

† For a recent review see Ref. 20.

Fig. 1 — Spin wave manifold, Eq. (2.27). The value ω_o is the relative position for the resonance frequency of a sphere when $\omega_o/\gamma = 3000$ gauss and $4\pi M = 1750$ gauss. k_c is the maximum wave vector for degenerate spin waves of a sphere (for YIG $k_c = 3.6 \times 10^5 \text{ cm}^{-1}$).

or parametrically, the steady-state results for these modes will not be adversely effected by thermal fluctuations. However, it will be evident in the next section and in Chapter IV that the thermal equilibrium values of β_k must be accounted for when considering the initial transient build up of these modes, especially those that are parametrically driven. For the present discussion of the linear Eqs. (2.29) and (2.30) these effects are ignored.

The driving field h^+ appearing in (2.30) is the linearly polarized field defined in (2.4) which is assumed to be turned on infinitely fast at time $t = 0$ and left on a sufficiently long time to allow the normal modes to attain their steady-state values and then turned off again infinitely fast. Obviously this is an idealized picture of the experimental conditions, but if the response times of the magnetic system are long compared to the actual rise and fall times of the driving field pulse, a condition that exists experimentally, then the use of a rectangular pulse with zero rise and fall times for the actual driving field is a valid approximation.

Being a linearly polarized field, h^+ consists of two counter rotating components. Therefore, the uniform mode also is composed of two counter rotating components but, on resonance, the non-resonant component of β_0 will be much smaller than the resonant one and can be neglected here. This component, however, can contribute to the second harmonic and is discussed briefly in the next chapter.

Then we can write $\beta_0(t)$ as

$$\beta_0(t) = b_0(t) e^{i\omega t} \quad (2.31)$$

where $\omega/2\pi$ is the driving frequency. The equation of motion for the

amplitude of the uniform mode, $b_o(t)$, from (2.30) is

$$\dot{b}_o(t) = i(\Omega_o - \omega) b_o(t) - \frac{1}{2} i\gamma h_o \quad (2.32)$$

On resonance, the transient solutions of b_o to a fast microwave driving field pulse are

$$b_o(t) = -\frac{i\gamma h_o}{2\eta_o} (1 - e^{-\eta_o t}) \quad (2.33)$$

for the initial build up and

$$b_o(t) = -\frac{i\gamma h_o}{2\eta_o} e^{-\eta_o t} \quad (2.34)$$

for the decay after the driving field has been turned off. If the driving field were circularly polarized the factor of 2 in the denominator would not be present. These solutions show that $b_o(t)$ lags h_o by 90° and that in the steady-state region of the pulse $b_o(t)$ increases linearly with h_o while in the transient regions $b_o(t)$ reaches e^{-1} of its steady-state value in a characteristic time $1/\eta_o$.

Of course, the linear theory is valid only in the limit of small amplitudes, and as these become large some discrepancies are to be expected. In fact, Eq. (2.18) in the absence of any modes of large wave vector predicts that the uniform mode will not be linear with increasing field but will gradually decline in a way similar to that predicted by the Bloch-Bloembergen (B-B) equations¹⁸. This result is expected since the transverse component of the magnetization eventually will saturate when the magnetization lies entirely in the transverse plane. For paramagnetic systems this gradual decline in the uniform mode is observed and the experimental curves obtained agree well with the predictions of the B-B equations. However, as was stated in the introduction, the trans-

verse component of the magnetization in ferrites is known to decline at a much lower driving field than is predicted by the B-B equations and for narrow line-width samples this decline is not gradual but has a well defined discontinuity at some critical driving field. This experimental result can be explained by including higher order terms in the equations of motion for the spin wave modes $\alpha_0(t)$ and $\alpha_k(t)$ - the subject of the next section.

B. EFFECTS OF HIGHER ORDER[†] SPIN WAVE INTERACTIONS

In the previous section only the linear terms were included in (2.18) and hence the normal modes were completely decoupled from one another. In this section the effects of some of the higher order terms in the expansion of $\alpha_{kz}(t)$ are considered in a review of the work of Suhl² and Schlömann¹².

To first order, the spin wave modes $\alpha_k(t)$ have a much smaller amplitude than the uniform mode. Therefore, in considering higher order terms in the expansion of $\alpha_k(t)$ at first it will be assumed that

$$|\alpha_k(t)| \ll |\alpha_0(t)| \ll 1. \quad (2.35)$$

The last part of this condition insures that we remain in the small signal region where the expansion (2.14) is valid.

Then, to third order, (2.18) becomes

$$-i\dot{\alpha}_k(t) = A_k \alpha_k(t) + B_k \alpha_{-k}^*(t) + C_k \alpha_k(t) + D_k \alpha_{-k}^*(t) \quad (2.36)$$

where A_k and B_k are given by (2.22) and where C_k and D_k are

[†] See page 2 for a definition of "higher order" in referring to spin wave interactions.

$$\begin{aligned}
C_k = & -\frac{\omega_M}{4} \sin 2\theta_k (\alpha_o e^{-i\varphi_k} + \alpha_o^* e^{i\varphi_k}) + \frac{1}{2} \gamma h^+ \alpha_o^* \\
& + \frac{\omega_M}{4} \left[2N_z - (N_x + N_y) + 3 \cos^2 \theta_k - 1 \right] |\alpha_o|^2 \\
& - \frac{\omega_M}{4} (N_x - N_y) (\alpha_o^*)^2
\end{aligned} \quad (2.37)$$

and

$$\begin{aligned}
D_k = & -\frac{\omega_M}{2} \sin 2\theta_k e^{i\varphi_k} \alpha_o + \frac{1}{2} \gamma h^+ \alpha_o \\
& + \left[\frac{\omega_{ex}}{2} a_k^2 - \frac{\omega_M}{4} (N_x + N_y - 2 \cos^2 \theta_k) \right] (\alpha_o)^2 \\
& - \frac{\omega_M}{4} (N_x - N_y + \sin^2 \theta_k e^{2i\varphi_k}) |\alpha_o|^2,
\end{aligned} \quad (2.38)$$

respectively. Using the transformation (2.23) and including the phenomenological damping term η_k the equation of motion for the normal modes $\beta_k(t)$ becomes

$$-i\dot{\beta}_k(t) = \Omega_k \beta_k(t) + E_k \beta_k(t) + G_k \beta_{-k}^*(t) \quad (2.39)$$

where the coupling terms E_k and G_k , given by

$$E_k = C_k \lambda_k^2 + C_k^* |\mu_k|^2 - D_k \lambda_k \mu_k^* - B_k^* \lambda_k \mu_k \quad (2.40)$$

and

$$G_k = -\lambda_k \mu_k (C_k + C_k^*) + D_k \lambda_k^2 + D_k^* \mu_k^2, \quad (2.41)$$

are functions of the uniform mode α_o .

Since the uniform mode is coupled directly to the driving field h^+ it again will have the form (2.31). Then the factors E_k and G_k can be expanded in terms of $e^{i\omega t}$ as

$$E_k = \sum_{n=-2}^2 E_k^{(n)} e^{in\omega t} \quad (2.42)$$

and

$$G_k = \sum_{n=-2}^2 G_k^{(n)} e^{in\omega t} \quad (2.43)$$

where the terms $E_k^{(n)}$ and $G_k^{(n)}$ are now functions of the uniform mode amplitudes b_o and b_o^* .

Consider first the effect of $E_k^{(n)}$ on the equation of motion for $\beta_k(t)$. The constant term $E_k^{(0)}$ which will lead to a shift in ω_k has no effect on the amplitude of the spin wave modes $\beta_k(t)$ and hence can be ignored. Likewise, the other terms of E_k also will be ignored since they merely provide a rapid modulation to ω_k .

The G_k terms can, however, have rather large effects on these modes since it is through them that the modes β_k and β_{-k}^* are coupled together.

The term $G_k^{(0)}$ will provide a static coupling between β_k and β_{-k} and, can be treated in the same way as were the linear Eqs. (2.20) and (2.21). The $G_k^{(1)}$ and $G_k^{(2)}$ terms will give rise to the Suhl first and second order processes, respectively, and are discussed separately in the next two sections. It can be shown that $G_k^{(-1)}$ and $G_k^{(-2)}$ are much smaller than $G_k^{(1)}$ and $G_k^{(2)}$, respectively, and will lead to much larger threshold fields. Therefore, these components will be ignored.

1. FIRST ORDER PROCESS

Consider the $G_k^{(1)}$ term in (2.43). In this case Eq. (2.39) becomes upon ignoring E_k

$$-i\dot{\beta}_k(t) = \Omega_k \beta_k(t) + G_k^{(1)} e^{i\omega t} \beta_{-k}^*(t) \quad (2.44)$$

The $e^{i\omega t}$ time dependence in (2.44) is eliminated by assuming β_k to be of the form

$$\beta_k(t) = b_k(t) e^{i\omega t/2}$$

in which case Eq. (2.45) becomes

$$b_k(t) = [i(\omega_k - \frac{\omega}{2}) - \eta_k] b_k(t) + iG_k^{(1)} b_{-k}^*(t) . \quad (2.46)$$

The solution to this equation has the form

$$b_k(t) \sim e^{\kappa t} \quad (2.47)$$

where

$$\kappa = -\eta_k \pm (|G_k^{(1)}|^2 - (\omega_k - \frac{\omega}{2})^2)^{1/2} . \quad (2.48)$$

In the presence of the coupling term $G_k^{(1)}$, (2.47) shows that the amplitude of the spin wave modes $b_k(t)$ will be exponentially damped if $\kappa < 0$ and will grow exponentially if $\kappa > 0$. The change in behavior occurs at $\kappa = 0$ in which case from (2.48) we have

$$\eta_k^2 = |G_k^{(1)}|^2 - (\omega_k - \frac{\omega}{2})^2 . \quad (2.49)$$

From (2.37), (2.38), (2.40), and (2.43) the coupling constant $|G_k^{(1)}|$ can be written as

$$|G_k^{(1)}| = |g_k| |b_0| \quad (2.50)$$

where $|b_0|$ is the amplitude of the uniform mode, and where g_k is independent of the spin wave mode amplitudes. From (2.49) and (2.50) the amplitude of the spin wave mode b_k will start to grow exponentially when the uniform mode has reached a value given by

$$|b_o|^2 = \frac{\eta_k^2 + (\omega_k - \frac{\omega}{2})^2}{|g_k|^2} . \quad (2.51)$$

Below this value the spin waves are exponentially damped and the linear theory of the previous section can be used to relate the uniform mode amplitude to the driving field, which from (2.32) in the steady-state is

$$|b_o| = \frac{\gamma h_o}{2 \sqrt{\eta_o^2 - (\omega_o - \omega)^2}} . \quad (2.52)$$

From (2.51) and (2.52) it is evident that the lowest threshold will occur when

$$\omega = \omega_o = 2\omega_k . \quad (2.53)$$

When this is realized the threshold field becomes

$$h_c = \frac{2\eta_k \eta_o}{\gamma |g_k|} . \quad (2.54)$$

Of course, spin waves can be excited when (2.53) is not satisfied; and the case when $\omega \neq \omega_o$ but $\omega = \frac{\omega_k}{2}$ can explain the presence of the subsidiary absorption peak as is discussed elsewhere^{2,12}.

The effect of parametric spin wave growth on the uniform mode can be determined from Eqs. (2.52) and (2.54) with the result that above h_c the uniform mode "sticks" at a value given by

$$|b_o| = \frac{\eta_k}{|g_k|} . \quad (2.55)$$

Experimentally this "sticking" can be observed by a measurement of the imaginary part of the microwave susceptibility as a function of applied driving field h_o . In terms of the uniform mode an expression for this component of the susceptibility is

$$\chi'' = \frac{M |\alpha_o|}{h_o} \quad (2.56)$$

In the linear region, where α_o and h_o are linearly related, χ'' is a constant while in the so-called nonlinear region, where spin waves satisfying the condition (2.53) are excited and where the uniform mode "sticks," the susceptibility should decline inversely with applied field.

The susceptibility has been observed by many experimenters including the present one for applied d.c. magnetic fields where condition (2.53) is satisfied and the results of these experiments agree well with the predictions given above even for fields much larger than the threshold

This is a rather surprising result when the relative amplitudes of b_k and b_o are compared for driving fields above threshold. The dependence of the amplitude b_k on the driving field can be determined using (2.55) and a relation for b_o derived in a manner similar to that used to obtain (2.46). The result of this calculation is in the steady-state region the square of the spin wave mode b_k is linearly related to the driving field and can be written as

$$|b_k|^2 = \frac{\eta_o}{2|g_k|} \left(\frac{h}{h_c} - 1 \right) \quad (2.57)$$

where h_c is given by (2.54). Comparing this result with (2.55), since η_o and η_k are of the same order of magnitude, for large driving fields

the amplitude of the spin wave mode b_k can in fact be greater than that of the uniform mode, a situation which is clearly in contradiction with the assumption (2.35). Therefore, above the threshold field, the other terms that had previously been neglected in the expansion of Eq. (2.18) should now be included.

If these other terms are included the general form of the resultant equation of motion for the spin wave amplitude is

$$\begin{aligned} \dot{b}_k(t) = & [i(\omega_k - \frac{\omega}{2}) - \eta_k] b_k(t) + ig_k b_0 b_{-k}^* \\ & + i \sum_{k' \neq 0, k} h_{k'} b_{k'} b_{k-k'} \end{aligned} \quad (2.58)$$

where the coupling coefficient $h_{k'}$ is independent of spin wave amplitude and where the angular frequencies of the parametrically excited spin waves $b_{k'}$ and $b_{k-k'}$ satisfy the relation

$$\omega_{k'} + \omega_{k-k'} = \omega_k \quad (2.59)$$

In deriving (2.58) we have neglected terms that involve four or more spin waves since they will lead to much larger threshold fields than is predicted by the $h_{k'}$ process.

It is reasonable to assume that when the $h_{k'}$ process begins to be of importance only a single term in the summation over $\vec{k}'$ needs to be included, namely the spin wave $\vec{k}'$ which has maximum parametric coupling to $\vec{k}$. The equation of motion for this spin wave, which is obtained using the same approximations that were used to derive (2.58), predicts that the mode b_k will "stick" above a certain threshold field h_c' . From (2.58), above h_c' , it can be shown that the uniform mode will increase

again. This increase should be present as a deviation in the $1/h$ dependence of the imaginary part of the susceptibility predicted by the single Suhl process. To the knowledge of the present investigator this deviation has never been observed experimentally.

The fact that the above deviation has not been observed can be understood when it is considered that the spin waves excited by the Suhl first order process are not in general z -directed, i.e., the wave vector $\vec{k}$ of any individually directed mode will have a definite azimuthal angle, ϕ_k . Since the interaction $|g_k|$ is independent of this angle, the spin wave mode b_k is excited equally in all azimuthal directions. Therefore, the amplitude of any individually directed mode will be small, and cannot attain the amplitude necessary for a second nonlinear process to be important. However, for extremely high fields, this amplitude can become large, but then the susceptibility will have declined to too small a value for departures from a $1/h$ type dependence to be observed experimentally.

The above discussion where it was reasoned that the spin wave mode b_k is excited equally in all azimuthal directions will no longer be valid when anisotropy fields are considered. Schlömann¹² shows that for cubic crystals the crystalline anisotropy does not influence the first order process when the d.c. field is applied along a $[100]$ direction, but when it is applied along a $[111]$ direction the threshold field should decrease with increasing anisotropy constant and there will be a preferred direction, ϕ_k , for the excited spin waves b_k . However, for this orientation, there are six directions where the coupling constant has the same maximum value and, hence, the amplitude of b_k for each direction will be $1/6$ of the total mode amplitude excited. Then when the fact is

also included that $|h_k|$ is smaller than $|g_k|$ for spin waves satisfying (2.59), it again will be unlikely for a process such as described by h_k to be important except possibly for extremely large fields.

Summarizing the present section, the first order process is adequately described by including only the g_k coupling term even for large applied microwave fields. The effects of the other coupling terms can be neglected because they will result in threshold fields that are much larger than the field given in (2.54). In the next section, where the Suhl second order process is considered, it will be shown that the higher order coupling terms to be considered there can have a rather large effect on the susceptibility decline. One of the reasons for this result is that the spin waves excited by the second order process are z-directed and, hence, this single mode can be parametrically excited to a large amplitude.

2. SECOND ORDER PROCESS

For large applied d.c. magnetic fields the spin waves with $\omega_k = \omega/2$ that are coupled to the uniform mode by the g_k term of the previous section will fall below the spin wave band and, hence, cannot be excited. For spheroids, magnetized along the axis of revolution, this situation occurs whenever the resonance frequency is greater than some cutoff frequency defined by

$$\omega_c = 2\omega_M N_T \quad (2.60)$$

where N_T is the transverse demagnetization factor. In this section it will be assumed that the applied d.c. field is greater than ω_c/γ so that the terms in (2.43) that were important in the last section can be ignored. The remaining terms in (2.43) are $G_k^{(\pm 2)} e^{\pm 2i\omega t}$ and their effects are

considered in the present section as are the effects of other coupling terms that do not involve the uniform mode.

As was stated before, the $G_k^{(-2)}$ term can immediately be ignored since it will result in a much larger threshold field than predicted when $G_k^{(2)}$ is included. With this elimination the equation of motion for the spin wave mode $\beta_k(t)$ from (2.39) and (2.43) becomes

$$-i\dot{\beta}_k(t) = \Omega_k \beta_k(t) + G_k^{(2)} e^{2i\omega t} \beta_{-k}^*(t) \quad (2.61)$$

where Ω_k is given by (2.28) and the coupling coefficient $G_k^{(2)}$ can be obtained from (2.37), (2.38), (2.41), and (2.43) and is given by

$$G_k^{(2)} = e^{-2i\omega t} \frac{\alpha_o^2}{4\omega_k} [\omega_{ex}(ak)^2 + \omega_M \cos^2 \theta_k - \omega_{MT}] [\omega_{ex}(ak)^2 + \omega_i + \frac{1}{2} \omega_M \sin^2 \theta_k + \omega_k] \quad (2.62)$$

In obtaining this equation we have assumed that $\eta_o/\gamma \ll 4\pi M$ which for most materials is a valid approximation, i.e., typically for YIG $\eta_o/\gamma \sim 1$ gauss while $4\pi M = 1750$ gauss.

As in the previous section it will be convenient to define a spin wave variable that is independent of the angular frequency ω by

$$\beta_k(t) = b_k(t) e^{i\omega t} \quad (2.63)$$

By substituting (2.63) into (2.61) the equation of motion for $b_k(t)$ becomes

$$\dot{b}_k(t) = [i(\omega_k - \omega) - \eta_k] b_k(t) + iG_k^{(2)} b_{-k}^*(t) \quad (2.64)$$

which is identical to the Eq. (2.46) except where $\omega/2$ and $G_k^{(1)}$ have been

replaced by ω and $G_k^{(2)}$, respectively.

Therefore, a solution to this equation will be of the form (2.47) and leads to results identical to those discussed in the last section: In the steady-state above a certain threshold field the uniform mode "sticks" and the spin wave mode b_k grows exponentially. This threshold field can be determined from (2.52), (2.48), and (2.62) and is given by

$$|h_c|^2 = \frac{4}{\gamma^2} \left\{ \frac{[\eta_k^2 + (\omega_k - \omega)^2]}{|g_{ok}|^2} \right\}^{1/2} [\eta_o^2 + (\omega_o - \omega)^2] \quad (2.65)$$

where

$$g_{ok} = \frac{\omega_M}{4} \left[(3 \cos^2 \theta_k - 1) \left\{ \frac{1}{2} [(\omega_M/2\omega_k)^2 \sin^4 \theta_k + 1]^{1/2} + 1 \right\} + \frac{\omega_M}{4\omega_k} \sin^4 \theta_k \right] . \quad (2.66)$$

It is immediately obvious from (2.65) that a minimum threshold field occurs when

$$\omega = \omega_o = \omega_k \quad (2.67)$$

and for the polar angle, θ_k , that maximizes $|g_{ok}|$. For driving frequencies greater than the cutoff frequency given in (2.60), $|g_{ok}|$ is a maximum when z-directed spin waves are excited in which case the threshold field takes the familiar form

$$h_c = \Delta H_o \sqrt{\frac{\Delta H_k}{4\pi M}} \quad (2.68)$$

where

$$\Delta H_o = 2\eta_o/\gamma$$

and

$$\Delta H_k = 2\eta_k/\gamma \quad (2.69)$$

are the full intrinsic absorption and spin wave line widths, respectively. This threshold field was first derived by Suhl² and is generally known as the second Suhl or second order threshold field.

In determining the minimum of (2.65) it has been assumed for simplicity that the spin wave relaxation rates are constant for all wave vectors considered. Although this is not correct, even if the dependence of η_k on wave vector as predicted by Sparks, et. al.²¹, (Fig. 3) is included, the result will still hold that the minimum threshold field occurs for z-directed spin waves.

Above this field, since the uniform mode has the constant value given by

$$|b_o| = |b_c| = \sqrt{\frac{\eta_k}{g_{ok}}}, \quad (2.70)$$

the imaginary part of the microwave susceptibility will have a $1/h$ type dependence. Unlike the last section, though, there is experimental evidence[†] that indicates the $1/h$ type dependence is not present for frequencies greater than the cutoff frequency (2.60). The uniform mode is found to increase above threshold at a rate which at first is very much less than linear in h but which becomes nearly linear at about 16 dB above threshold. Above this point relaxation oscillations^{22,23} are present.

These results indicate that other coupling terms similar to those considered in the last section must be considered and their effects on the uniform mode and susceptibility determined.

† See Ref. 10 and Fig. 22 of this paper.

3. EFFECTS OF INTERACTIONS ABOVE THE SECOND SUHL THRESHOLD

First, it will be convenient to determine the equation of motion for the uniform mode. Again considering only the resonant component of the uniform mode so that

$$b_0(t) = b_0 e^{i\omega t} \quad (2.71)$$

from (2.39) and (2.23) with the aid of the inequality (2.35) the equation of motion for the uniform mode can be written as

$$\dot{b}_0(t) = [i(\omega_0 - \omega) - \eta_0] b_0(t) + 2i \sum_{k'} g_{ok'} b_k b_{-k} b_0^* - i\gamma h_0/2 \quad (2.72)$$

where $g_{ok'}$ is given by (2.66).

The equation of motion for the spin wave mode b_k when interactions between $k \neq 0$ spin waves are included is more difficult to determine and is outlined here. Including only the terms of interest from (2.18) and (2.19) the equation of motion for α_k becomes

$$\begin{aligned} \dot{\alpha}_k(t) = i \left\{ A_{kk} \alpha_k + B_{kk} \alpha_k^* + \frac{\omega_{MN} z}{2} \sum_{k'} \alpha_{k'} \alpha_{k'}^* \alpha_k \right. \\ \left. + \frac{\omega_{ex} a^2}{2} \sum_{k' k''} (k^2 - 2\vec{k} \cdot \vec{k}') \alpha_{k'} \alpha_{k''+k-k'} \alpha_{k''}^* \right. \\ \left. + \frac{\omega_{MN}}{2} \sum_{k'} [(\cos^2 \theta_{k-k'}) - \frac{1}{2} \sin^2 \theta_{k'}] \alpha_{k'} \alpha_{k''+k-k'} \alpha_{k''}^* \right. \\ \left. - \frac{1}{2} \sin^2 \theta_{k'} e^{2i\phi_{k-k'}} \alpha_{-k'} \alpha_{k''+k-k'} \alpha_{k''}^* \right\} \end{aligned} \quad (2.73)$$

where

$$\cos^2 \theta_{k-k'} = \frac{k_z^2 - 2k_z k'_z + k_z'^2}{k^2 - 2\vec{k} \cdot \vec{k}' + k'^2} \quad (2.74)$$

In obtaining (2.73) it has been assumed that terms of the form $\alpha_k \alpha_k$, can be lumped into the relaxation rate η_k . (Actually these terms correspond to the three magnon processes which have been shown²¹ to contribute to η_k). The number of terms in the summations on the right-hand side of (2.73) is limited because both energy and momentum must be conserved by each process considered. The possible processes involving the spin wave mode excited by the uniform mode are represented in Fig. 2.

In the process represented in Fig. 2(a), a spin wave b_k with energy corresponding to ω_k scatters into three spin waves with wave vectors $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$, and energies corresponding to ω_1 , ω_2 , and ω_3 satisfying the relations

$$\begin{aligned}\vec{k} &= \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \\ \omega &= \omega_1 + \omega_2 + \omega_3\end{aligned}\tag{2.75}$$

The other diagrams can be interpreted in a similar manner.

Process 2(a) can never satisfy both conservation laws simultaneously and the inequality $\omega_k \geq \omega_c$ as prescribed by the discussion following (2.60). Process 2(b) can satisfy both conservation laws for a large range of different modes $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$. However, since the present model predicts that only the spin wave mode parametrically excited by g_{ok} has a large amplitude, just the interactions involving this mode represented by Figs. 2(c) and 2(d) will be considered. Process 2(d) involves the single initial mode b_k and, hence, will lead to a frequency shift which can be absorbed into ω_k . We therefore, consider only the process represented in Fig. 2(c); so after applying transformation (2.23), the equation of motion for the spin wave amplitude b_k can be written as

Fig. 2 — Possible four-spin-wave interaction involving the mode b_k .

$$\dot{b}_k(t) = [i(\omega_k - \omega) - \eta_k] b_k + ig_{ok} b_o^2 b_{-k}^* + i \sum_{k' \neq 0} g_{k'k} b_{k'} b_{-k'} b_{-k}^* \quad (2.76)$$

where g_{ok} is given by (2.66) and

$$\begin{aligned} g_{kk'} &= j_{kk'} \lambda_k^2 \lambda_{k'}^2 + j_{-kk'} \mu_k^2 (\mu_{k'}^*)^2 \\ &\quad - (\ell_{kk'} \lambda_k^2 + \ell_{-kk'} \mu_k^2) \lambda_{k'} \mu_{k'}^* \\ &\quad + (h_{kk'} + h_{-kk'}) \lambda_{k'} \mu_{k'}^* \lambda_k \mu_k \\ &\quad - (f_{kk'} \mu_{k'}^2 + f_{-kk'}^* \lambda_{k'}^2) \lambda_k \mu_k. \end{aligned} \quad (2.77)$$

Here

$$j_{kk'} = \frac{\omega_{ex} a^2}{2} (k^2 - 2\vec{k} \cdot \vec{k}') + \frac{\omega_M}{2} (\cos^2 \theta_{k-k'} - \frac{1}{2} \sin^2 \theta_{k'}) \quad (2.78)$$

$$\begin{aligned} h_{kk'} &= \frac{\omega_M}{2} (\cos^2 \theta_o + \cos^2 \theta_k - \frac{1}{2} \sin^2 \theta_k - \frac{1}{2} \sin^2 \theta_{k'}) \\ &\quad - \omega_{ex} a^2 \vec{k} \cdot \vec{k}' \end{aligned} \quad (2.79)$$

$$\ell_{kk'} = -\frac{\omega_M}{4} (\sin^2 \theta_{k'} e^{2i\varphi_{k'}} + \sin^2 \theta_k e^{2i\varphi_k}) \quad (2.80)$$

$$f_{kk'} = -\frac{\omega_M}{4} \sin^2 \theta_{k'} e^{2i\varphi_{k'}} \quad (2.81)$$

and λ_k and μ_k are given by (2.24). Notice that the coupling constant $g_{kk'}$ is not equal to $g_{-kk'}$.

When $\vec{k}$ is z-directed as predicted by the g_{ok} coupling term, then $g_{kk'}$ becomes

$$\begin{aligned}
g_{kk'} = & \frac{\omega_M}{6} - \left(\frac{\omega_M}{3}\right) \left(1 - \frac{3}{2} \sin^2 \theta_{k'}\right)^{1/2} \cos \theta_{k'}, \\
& + \frac{1}{2} \omega_M \left[\frac{\left(1 - \left(1 - \frac{3}{2} \sin^2 \theta_{k'}\right)^{1/2} \cos \theta_{k'}\right)^2}{2 - \frac{3}{2} \sin^2 \theta_{k'} - 2 \left(1 - \frac{3}{2} \sin^2 \theta_{k'}\right)^{1/2} \cos \theta_{k'}} \right. \\
& \left. - \frac{1}{2} \sin^2 \theta_{k'} \right] + \frac{\omega_M^2}{16\omega_{k'}} \sin^4 \theta_{k'}. \quad (2.82)
\end{aligned}$$

For $k' = 0$ this result reduces to (2.66). In the case of $\vec{k}$ being z-directed $g_{kk'}$ is independent of azimuthal angle. The spin waves $b_{k'}$ will then be excited in all azimuthal directions equally. This fact has important implications concerning the contribution to the second harmonic component of the magnetization from the spin wave mode $b_{k'}$ and will be discussed in the next chapter.

The spin wave frequency $\omega_{k'}$ in (2.76) is understood to be an effective frequency into which the effects of shift by spin wave interactions has been included. This frequency shift leads to a fine structure in the decline of the susceptibility as has been shown by Schlömann^{12,24}. Although there is some evidence for the fine structure in the samples studied here, the effect is too small to justify an elaborate theoretical treatment.

Equations (2.72) and (2.76) then are of concern to us and can in principle be solved for the uniform mode as a function of applied field when any number of spin waves is included in the summation on the right-hand side of these equations. In practice, however, it is feasible to solve them when only a limited number of spin wave pairs are considered and then solutions can only be obtained in closed form when transient effects are ignored. The transient regions can be analyzed numerically with the aid of a computer and the results of such a computation for

various coupling parameters and spin wave pairs will be given. First, however, let us solve (2.72) and (2.76) simultaneously for the case where two spin wave pairs are coupled both to the uniform mode and to one another.

Assuming that all spin waves are resonant so that

$$\omega_o = \omega_k = \omega_{k'} = \omega \quad (2.83)$$

the equations of motion of concern to us are from (2.72) and (2.76) using (2.83)

$$\begin{aligned} \dot{b}_o = & -\eta_o b_o + 2ig_{ok} b_k b_{-k}^* \\ & + 2ig_{ok'} b_{k'} b_{-k'}^* - i\gamma h_o/2 \end{aligned} \quad (2.84)$$

$$\dot{b}_k = -\eta_k b_k + ig_{ko} b_o^2 b_{-k}^* + 2i\bar{g}_{kk'} b_{k'} b_{-k'}^* b_{-k}^* \quad (2.85)$$

$$\dot{b}_{k'} = -\eta_{k'} b_{k'} + ig_{k'o} b_o^2 b_{-k'}^* + 2i\bar{g}_{k'k} b_k b_{-k}^* b_{-k'}^* \quad (2.86)$$

where

$$\bar{g}_{k'k} = \frac{1}{2} (g_{k-k'} + g_{kk'}) = \bar{g}_{k'-k} \quad (2.87)$$

and where we have used the relation

$$g_{ok} = g_{o-k} \quad (2.88)$$

The equations for b_o^* , b_{-k}^* , and $b_{-k'}^*$, can be obtained directly from (2.84)-(2.86) by taking the complex conjugate and replacing k and k' by its negative. Since we are assuming the spin wave defined by wave vector $\vec{k}$ is the first to be excited the coupling coefficients must be such that $g_{ok} > g_{ok'}$.

Before we determine an expression for the uniform mode as a function

of the applied microwave field it is pertinent to consider the question: Above the threshold field (2.68) what spin wave, if any, will next go unstable and at what field will this instability occur?

Assume for the present that $g_{ok'} = 0$. Then in the steady-state (2.86) becomes

$$\eta_{k'} b_{k'} = 2i\bar{g}_{k'k} b_k b_{-k} b_{-k'}^* \quad (2.89)$$

Multiplying both sides by b_{-k} , taking the magnitude of the resulting equation, and noting from the symmetry of the problem that $|b_{k'}|^2 = |b_{-k'}|^2 = |b_k b_{-k}|$ we obtain the result

$$|b_k|^2 = \frac{\eta_{k'}}{2\bar{g}_{k'k}} \quad (2.90)$$

which shows that if the spin waves $b_{k'}$ are excited then the mode b_k to which they are coupled will be constant above some second threshold field, h_c' , to be determined.

Above the threshold field h_c but below a field where the $g_{kk'}$ process is important the spin wave mode b_k is related to the driving field by

$$|b_k|^2 = \frac{\eta_o}{2g_{ok}} \left(\frac{h_o}{h_c} - 1 \right) \quad (2.91)$$

where in this equation, since b_k is a z-directed spin wave, the coupling constant g_{ok} is given by

$$g_{ok} = \frac{\omega_M}{2} \quad (2.92)$$

Then the minimum threshold for the $g_{kk'}$ process occurs when $\eta_{k'}/2\bar{g}_{k'k}$ is a minimum. For mathematical convenience it will again be assumed that $\eta_{k'}$ is independent of k' since the result is unchanged even if

Fig. 3 — Dependence of degenerate spin wave linewidths on wave vector, Eq. (B-28) in Ref. 11, for YIG sphere ($4\pi M = 1750$ gauss, $H_0 = 4.5 \times 10^9$ gauss, $\omega_0/\gamma = 3000$ gauss). A constant term of .2 gauss has been added to represent the Kasuya-LeCraw³⁵ linewidth at room temperature.

Fig. 4 — Dependence of $\frac{\bar{g}_{kk'}}{g_{ok}}$ on k . Curve (a) neglects anisotropy while curves (b) and (c) include anisotropy and differ only in azimuthal angle ϕ_k . Curve (b) corresponds to $e^{-2i\phi_k} = 1$, curve (c) to $e^{-2i\phi_k} = -1$.

the theoretical dependence of $\eta_{k'}$ on k' shown in Fig. 3, is included.

Thus, a minimum in the expression (2.90) occurs when $g_{kk'}$ is a maximum. The coupling constant $g_{kk'}$ is plotted against $\theta_{k'}$ for $\theta_k = 0$ in Fig. 4 which clearly shows that a maximum is attained when $\sin^2 \theta_{k'} = \frac{2}{3}$, corresponding to $k' \rightarrow 0$. For $\sin^2 \theta_{k'} = 2/3$ the coupling constant $\bar{g}_{kk'}$ is given

$$\bar{g}_{kk'} (k' \rightarrow 0, \theta_k = 0) = \omega_M/2 \quad (2.93)$$

which is the same as (2.92).

Therefore, from (2.90) and (2.91) the threshold field h'_c above which the spin waves k' are excited is given by

$$h'_c = h_c (1 + \eta_{k'}/\eta_0) \quad (2.94)$$

If radiation damping effects are ignored $\eta_{k'}/\eta_0$ for YIG is of the order of unity so that $h'_c = 2h_c$. Hence, the $g_{kk'}$ process, in contrast to the results of the last section, can cause deviations to occur in the susceptibility decline curve at fields close enough to the threshold field (2.90) to be observed experimentally.

Having shown that the $g_{kk'}$ process can be of importance we will now return to the problem of determining a relation between the uniform mode and driving field in the more general case where $g_{ok'} \neq 0$. The steady-state Eqs. (2.84)-(2.86) can be written as

$$\eta_0 b_0 = 2ig_{ok'} b_k b_{-k}^* + 2ig_{ok'} b_{k'} b_{-k'}^* - i\gamma h_0/2 \quad (2.95)$$

$$\eta_k b_k = ig_{ko} b_0^2 + 2i\bar{g}_{kk'} b_{k'} b_{-k'}^* \quad (2.96)$$

and

$$\eta_{k'} b_{k'} = i g_{k'o} b_o^2 b_{-k'}^* + 2i \bar{g}_{k'k} b_k b_{-k} b_{-k'}^* \quad (2.97)$$

Multiplying (2.96) by b_{-k} and (2.97) by $b_{-k'}$, and solving these equations simultaneously for $b_k b_{-k}$ and $b_{k'} b_{-k'}$, give the results

$$b_k b_{-k} = i |b_k|^2 b_o^2 \frac{(g_{ko} \eta_{k'} + 2i g_{k'o} \bar{g}_{kk'} |b_{k'}|^2)}{(\eta_k \eta_{k'} + 4 \bar{g}_{kk'} \bar{g}_{k'k} |b_k|^2 |b_{k'}|^2)} \quad (2.98)$$

and

$$b_{k'} b_{-k'} = i |b_{k'}|^2 b_o^2 \frac{(g_{k'o} \eta_k + 2i g_{ko} \bar{g}_{k'k} |b_k|^2)}{(\eta_k \eta_{k'} + 4 \bar{g}_{kk'} \bar{g}_{k'k} |b_k|^2 |b_{k'}|^2)} \quad (2.99)$$

We define a quantity, A , given by

$$\begin{aligned} A^2 &= (g_{ko} \eta_{k'})^2 + 4 g_{k'o}^2 \bar{g}_{kk'}^2 |b_{k'}|^4 \\ &= (g_{k'o} \eta_k)^2 + 4 g_{ko}^2 \bar{g}_{k'k}^2 |b_k|^4 \end{aligned} \quad (2.100)$$

in terms of which

$$|b_o|^2 = \frac{\eta_k \eta_{k'} + 4 \bar{g}_{kk'} \bar{g}_{k'k} |b_k|^2 |b_{k'}|^2}{A} \quad (2.101)$$

In deriving (2.101) use has been made of the fact that $|b_k b_{-k}| = |b_k|^2$.

It will be convenient to write (2.98) and (2.99) in the simple form

$$b_k b_{-k} = \frac{i |b_k|^2 b_o^2}{|b_o|^2} e^{i \phi_{kk'}} \quad (2.102)$$

and

$$b_{k'} b_{-k'} = \frac{i |b_{k'}|^2 b_o^2}{|b_o|^2} e^{i \Phi_{k'k}} \quad (2.103)$$

where from (2.98) and (2.99) and (2.101).

$$e^{i \Phi_{kk'}} = \frac{g_{ko} \eta_{k'} + 2i g_{k'o} \bar{g}_{kk'} |b_{k'}|^2}{A} \quad (2.104)$$

Now by substituting (2.102) and (2.103) into (2.95) we have

$$b_o [\bar{\eta}_o + i \Delta \omega] = -i \gamma h_o / 2 \quad (2.105)$$

where

$$\bar{\eta}_o = \eta_o + |b_k|^2 \cos \Phi_{kk'} + |b_{k'}|^2 \cos \Phi_{k'k} \quad (2.106)$$

and

$$\Delta \omega = |b_k|^2 \sin \Phi_{kk'} + |b_{k'}|^2 \sin \Phi_{k'k} \quad (2.107)$$

The effects of the higher spin wave interactions can be determined from (2.105).

First, the relaxation rate η_o is replaced by an effective rate $\bar{\eta}_o$ that can easily be shown to depend on driving field. From (2.104) $\cos \Phi_{kk'}$ and $\cos \Phi_{k'k}$ are always positive which indicates that the effective relaxation rate, or equivalently, linewidth ΔH_o , increases as the driving field is increased beyond the threshold field (2.68). We have observed the qualitative features of this increase in linewidth. However, due to the difficulty of interpreting line shape data in the presence of heavy radiation damping these data will not be presented here.

The second feature in (2.105) of interest is the $\Delta \omega$ term. It gives rise to a frequency shift which can be neglected if the restriction is

made that the d.c. field always be adjusted to resonance. Since $\Delta\omega$ is positive the applied field for resonance must increase as incident power increases. This increase has also been observed experimentally; we find about a 1.3 gauss positive resonance field shift for driving fields of 16dB above threshold in a YIG sphere at 8.42 GHz.

Neglecting the $\Delta\omega$ term, (2.105) becomes

$$b_o = -\frac{i\gamma h_o}{2\bar{\eta}_o} \quad (2.108)$$

which again indicates that, at resonance, the uniform mode is 90° out of phase with the driving field. Then from (2.108), (2.100), (2.101) and the real part of (2.104) an expression can be obtained for the uniform mode and driving field that has the form

$$\begin{aligned} b_o &= f_1(x) \\ h &= f_2(x) \end{aligned} \quad (2.109)$$

where f_1 and f_2 are two different functions of a third parameter x .

The result of calculating b_o and h from (2.109) is shown in Fig. 5.

Here the calculation has been presented for various values of the ratios

$g_{ok}, \eta_k / \eta_k, g_{ok}$ and $\frac{g_{ok}\eta_k}{g_{kk}\eta_o}$, and for ease of comparison with the experimental results of Chapter IV the logarithm of the quantities $\left|\frac{b_o}{b_c}\right|^2$ and $\left|\frac{h}{h_c}\right|^2$ are plotted where h_c is the threshold field (2.68) and b_c is the value of b_o that corresponds to this field given by (2.70).

It is obvious from this figure that for large values of driving field the uniform mode and driving field are linearly related. The susceptibility decline curve is plotted in Fig. 6 and shows that when higher order interactions are present the decline is less rapid than the

Fig. 5 — Dependence of uniform mode on driving field, Eq. (2.109), when two spin wave pairs are excited $\frac{\pi}{2}$ above threshold. Here $\alpha = \frac{g_{0k}\eta_k}{g_{0k}\eta_{k'}}$ and $\beta = \frac{g_{0k}\eta_{k'}}{g_{kk'}\eta_0}$.

Fig. 6 - Susceptibility decline curves for the single Suhl process and the two-spin-wave-pair model ($\alpha = \beta = 1$). (χ''_0 is the susceptibility below threshold)

Fig. 7 - Dependence of uniform mode on driving field as predicted by Schlömann's linear scattering theory.

1/h dependence predicted by simple theory.

It is interesting to compare the results obtained here with those of Schlömann¹² for high power levels. He neglects the higher order spin wave interaction $g_{kk'}$, but does include a linear inhomogeneity scattering term that causes a rounding off of the susceptibility decline curve near threshold. This behavior is typical of polycrystalline samples and roughly polished samples of single crystal YIG, but not of highly polished samples of single crystal YIG. For high power levels Schlömann shows that the susceptibility should asymptotically approach a 1/h type dependence. At moderate power levels he also shows that the susceptibility should behave as

$$\frac{\chi''}{\chi''_0} = 1 - C' \left(\frac{h_0}{2\Delta H} \right)^2 \quad (2.110)$$

where χ''_0 is the susceptibility below threshold and the constant C' depends on the relative importance of the linear scattering term, g_{ko} , and $g_{kk'}$, and is large and positive for the case where the linear scattering involves long wavelength spin waves and is small and even can be negative for the case where the linear scattering involves all the degenerate modes. Here b_0 has been expanded in terms of $\frac{h_0}{2\Delta H}$ with only the first term of the expansion being retained. A typical curve obtained from Schlömann's theory showing the dependence of the uniform mode on applied field is given in Fig. 7 and clearly indicates a different type behavior than that given in Fig. 5 or that of the experimental curve in Fig. 22.

C. EFFECTIVE RELAXATION RATE METHOD

Another method of including the effects of the higher order spin waves is to assign a power dependent effective relaxation rate to the first spin wave to be excited by the uniform mode. This method was applied by Schlömann²⁵ in explaining the behavior of the longitudinal susceptibility in parallel pumping experiments and can be applied to the present problem as follows.

It has been shown²¹ that at X-band frequencies the major process that gives rise to spin wave relaxation involves three spin waves where a mode $\vec{k}$ interacts with a second mode $\vec{k}'$ to form a new spin wave $\vec{k}''$. At room temperature the relaxation rate η_k is approximately proportional to the effective absolute temperature of the modes that interact with b_k . Normally it is assumed that these modes are excited to some thermal level which is determined by room temperature and that they essentially retain their thermal values even when involved in parametric couplings of the type discussed in this chapter. However, it has been shown that both the modes b_k and $b_{k'}$ can grow to large amplitudes in which case their effective temperature increases with a consequent increase in relaxation rate.

Defining this effective relaxation rate as $\bar{\eta}_k$, we can write

$$\bar{\eta}_k = \eta_k \left(1 + \frac{T_k - T}{T} \right) \quad (2.111)$$

where η_k is the relaxation rate of mode b_k at room temperature T , and T_k is the effective temperature of the interacting modes. The temperature difference will just be proportional to the power absorbed by these modes. Then (2.111) can be written as

$$\bar{\eta}_k = \eta_k (1 + \tilde{C}P) \quad (2.112)$$

where C is a constant and P is the power absorbed by the interacting modes. If it is assumed that a large number of modes are excited to the same level then P can be approximated by

$$P = P_{ab} - P_{ab_o} \quad (2.113)$$

where the power absorbed by the sample and the power absorbed in the uniform mode are given by

$$P_{ab} = K\gamma |b_o| h_o/2 \quad (2.114)$$

$$P_{ab_o} = K\eta_o |b_o|^2$$

respectively, where the constant of proportionality K depends on the frequency and sample volume. Using (2.113) and (2.114), (2.112) becomes

$$\bar{\eta}_k = \eta_k [1 + C'' |b_o| (\gamma h_o/2 - \eta_o |b_o|)] \quad (2.115)$$

where $C'' = KC$. This relation can be substituted for the relaxation rate η_k in the expression for the uniform mode above threshold, (2.70), whence

$$|b_o|^2 = \frac{\eta_k [1 + C'' |b_o| (\gamma h_o/2 - \eta_o |b_o|)]}{g_{ok}} \quad (2.116)$$

Eq. (2.116) is plotted for $C'' = \frac{1}{2}$ and 1 in Fig. 8 shows the same general characteristic as given in Fig. 6 except that here the initial slope at threshold is not zero.

The effective relaxation rate method generally gives better agreement with experimental data than does the two-spin-wave-pair method, which is perhaps not surprising since it is a gross over simplification to assume that only two spin wave pairs are present. However, it will be shown in the next two chapters that only the two-spin-wave-pair method or an approach based on this method can explain the dependence of the second

Fig. 8 — Dependence of uniform mode on driving field using the effective-relaxation-rate method.

harmonic content of the z-component of the magnetization on applied field.

D. TRANSIENT BEHAVIOR OF THE UNIFORM AND $k \neq 0$ SPIN WAVE MODES ABOVE THE SECOND SUHL INSTABILITY THRESHOLD.

The exact transient behavior of Eqs. (2.84), (2.85), and (2.86) is quite difficult to determine and must be done numerically with aid of a computer. However, some general characteristics can be determined for specific examples which will be discussed here.

Below threshold the response of the uniform mode to a fast microwave pulse is described by Eqs. (2.33) and (2.34). This response is shown in Fig. 9 and is composed of an exponential rise and fall, both occurring in the characteristic time $1/\eta_0$, and a steady-state portion which is linearly related to the driving field. The front and trailing edges of the pulse in this figure and subsequent figures will be slightly exaggerated to show clearly the behavior of the system in these regions.

Next consider the response of the uniform mode and spin wave mode b_k just above Suhl threshold. At first the uniform mode exponentially rises toward a value determined by the linear relation

$$|b_0| = \gamma h_0 / 2\eta_0 . \quad (2.117)$$

However, as soon as $|b_0|$ becomes greater than the critical value $\gamma h_c / 2\eta_0$ the spin wave modes start to increase with the consequence that the uniform mode ceases to grow and then decreases to the value $|b_c|$. The mode b_k starts with zero slope from some initial thermal value and then increases according to (2.47) where κ is given by

$$\kappa = -\eta_k + g_{ko} |b_0|^2 . \quad (2.118)$$

Therefore, b_k increases with an exponential time constant that approaches

Fig. 9 - Pulse response of uniform mode to a rectangular driving pulse below threshold.

Fig. 10 - Pulse response of $|b_o|$ and $|b_k|$ to driving field pulse when a single spin wave pair is excited above threshold. $|b_c|$ is given by Eq. (2.70).

zero as $|b_o|$ approaches $|b_c|$. The decay of the mode b_k is determined by (2.118) while that of $|b_o|$ is determined by Eq. (2.108) which shows that the uniform mode should decay at a faster rate than the initial build up rate η_o . The general character of b_k and b_o as discussed above is shown in Fig. 10.

The inclusion of a second spin wave pair $b_{k'}$ and $b_{-k'}$ in (2.84)-(2.86) can be analyzed in a manner similar to the above. For simplicity, we ignore the $g_{ok'}$ interaction for this qualitative discussion since the general form of the transient solution is not greatly affected by this approximation. The initial transient of b_o is identical to that shown in Fig. 10 up to $t = t_c$. At this point the amplitude of b_k reaches a value, given by (2.90), which is large enough to excite the spin wave pair $b_{k'} - b_{-k'}$. For times beyond t_c the mode b_k reaches a maximum and then decreases to the value given in (2.90). At the same time the uniform mode again increases to some steady-state value above b_c . The general behavior of b_k , $b_{k'}$, and b_o is presented in Fig. 11. The positions of the maxima and minima in b_o and b_k will vary slightly with the inclusion of a nonvanishing $g_{ok'}$ and they can have a damped oscillatory type behavior after the initial overshoot. Also, the mode $b_{k'}$ can have a slight overshoot followed by a damped oscillatory motion. Actual computer solutions have the general forms given in the three figures and will not be given in this section. However, some specific solutions are presented in Chapter IV where comparison can be made with experimental results.

E. EFFECTS OF CRYSTALLINE ANISOTROPY

The effects of anisotropy on the equations of motion for $\vec{\alpha}(\vec{r}, t)$ can

Fig. 11 - Pulse response of $|b_o|$, $|b_k|$, and $|b_{k'}|$ to driving field pulse when two spin wave pairs are excited above threshold. $|b_c|$ is given by Eq. (2.70) and $|b_k|$ by (2.90). t_c is the time the second spin wave pair becomes excited.^c

be determined by including an effective anisotropy field in (2.1). This field can be derived from the anisotropy energy which is usually expressed in terms of an anisotropy constant and direction cosines of magnetization. Although the approach is valid for any type of crystal lattice, we concentrate on a cubic system since it applies to the sample of interest in this paper, YIG. Including only a single anisotropy constant, K_1 , the anisotropy energy for this system is given by²⁶

$$E_a = K_1 (\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2) \quad (2.119)$$

where for YIG, $K_1/M = -43$ gauss²⁷ at room temperature, M being the saturation magnetization at this temperature, and where α_1 , α_2 , and α_3 are the direction cosines of the magnetization with respect to the three principal crystal axes of the cubic system. Since we are assuming only small deviations from the z -direction, the direction cosines given in (2.119) can be written in terms of the direction cosines of the laboratory system α_x and α_y , where α_z is approximated by (2.14). Then the anisotropy energy for a cubic system is to second order in α_x and α_y

$$E_a = K_{xx} \alpha_x^2 + K_{yy} \alpha_y^2 + K_{xy} \alpha_x \alpha_y + \dots \quad (2.120)$$

where the constants in front of each term are products of various direction cosines and K_1 .

The effective anisotropy field can be determined from the expression for the magnetic energy of the magnetization in an applied field,

$$E = -\vec{H} \cdot \vec{M} \quad (2.121)$$

Replacing $\vec{H}$ by $\vec{H}_a$, the effective anisotropy field, and E by E_a , we have

$$\vec{H}_a = -\frac{1}{M} \nabla_{\vec{\alpha}} E(\vec{\alpha}) \quad (2.122)$$

For the cubic system of interest here this field becomes

$$\vec{H}_a = -\frac{1}{M} [(2K_{xx}\alpha_x + K_{xy}\alpha_y) \hat{i} + (2K_{yy}\alpha_y + K_{xy}\alpha_x) \hat{j}] \quad (2.123)$$

Then the equation of motion for α can be written as

$$\dot{\alpha} = A_a \alpha + (B'_a + i C_a) \alpha \quad (2.124)$$

where

$$\begin{aligned} A_a &= \frac{\gamma}{M} (K_{xx} + K_{yy}) \\ B'_a &= \frac{\gamma}{M} (K_{xx} - K_{yy}) \\ C_a &= \gamma K_{xy} / M \end{aligned} \quad (2.125)$$

We have ignored a constant term in (2.124) since it must cancel in the complete equation of motion for α .

The cross coupling term, C_a , can be eliminated by a simple rotation of the coordinate axes in the x-y plane through an angle φ so that α can be written as

$$\alpha = \alpha' e^{i\varphi}.$$

Eq. (2.124) then becomes, with proper choice of φ

$$\dot{\alpha}' = A_a \alpha' + B_a \alpha'$$

where

$$B_a = (B_a'^2 + C_a^2)^{1/2} \quad (2.126)$$

Therefore, to first order in α , the results of the previous sections are valid when anisotropy is included provided the coupling terms A_k

and $B_k e^{-2i\phi_k}$ are replaced by

$$A_k \rightarrow A_k + A_a \quad (2.127)$$

$$B_k e^{-2i\phi_k} \rightarrow B_k e^{-2i\phi_k} + B_a \quad (2.128)$$

One immediate consequence of the anisotropy field is that the applied d.c. field required for resonance in a spherical sample will now depend on crystal orientation. From (2.27), noting that $B_0 = 0$ and $A_0 = \gamma H_0$ for a sphere, an equation for the applied resonance field can be written as

$$H_0 = -\frac{1}{\gamma} A_a + \frac{\omega}{\gamma} \left(1 + \frac{B_a^2}{\omega^2}\right)^{1/2} \quad (2.129)$$

which shows that since $|B_a| \ll \omega$ for YIG, A_a will have a much larger effect on the resonance field than B_a . This explains why for fields applied in a plane within a few degrees of the (111) plane, as in our experiments, the resonance field, when plotted against orientation in this plane, does not reflect the three-fold symmetry of the (111) plane: In the (111) plane A_a is constant while B_a has the three-fold symmetry of this plane and, hence, $\bar{H}_0$ has this same symmetry. The maximum variation in $\bar{H}_0$, however, is only 1.4 gauss for YIG at room temperature and $\omega/\gamma = 3000$ gauss. A few degrees from the (111) plane, say towards the (110), A_a has an angular dependence that reflects the two-fold symmetry of the (110) plane while B_a still has the same characteristic as before. Then $\bar{H}_0$ will reflect the symmetry of A_a . For example, consider YIG when $\bar{H}_0$ is in a plane 1° from the (111) towards the (110) plane and at the frequency given above. B_a is still close to three-fold symmetric but with a maximum variation in $\bar{H}_0$ of only 1.4 gauss. The two-fold symmetry variation in A_a , however, is ≈ 5 gauss so that

$\vec{H}_0$ will exhibit a nearly two-fold symmetry characteristic of the (110) plane even though the applied field is only 1° from the (111) plane.

This is interesting in that any property of the magnetic system that is primarily determined by B_a , as in second harmonic generation, will retain the three-fold symmetry behavior of the (111) plane while the resonance field and other properties primarily determined by A_a will show a different symmetry unless the sample is aligned to within better than 1 degree.

The effect of anisotropy on the coupling coefficients g_{ok} and $g_{kk'}$ can be obtained by expanding the anisotropy energy to fourth order in α . Here the algebra becomes quite lengthy and only the results are given for YIG at room temperature.

The variation of g_{ko} and $g_{kk'}$ when the applied d.c. field lies in the (110) plane is shown in Fig. 12(a), and Fig. 12(b) shows this variation in the (111) plane. Although the maximum value of both g_{ok} and $g_{kk'}$ can vary by as much as 30 percent from their respective values in the absence of anisotropy, the result still holds that the z-directed spin waves have the largest coupling constant g_{ok} while the spin waves with $k' \rightarrow 0$ give the largest coupling constant $g_{kk'}$ with z-directed modes. However, as shown in Fig. 4, for $\vec{H}_0$ in the (111) plane, parallel to the [100] direction, the coupling constant $g_{kk'}$ between z-directed modes and spin wave modes k' is not independent of the azimuthal angle $\phi_{k'}$. The difference amounts to about 6 percent and will have little effect on the mode amplitudes. It is possible, however, for this difference to affect the second harmonic part of the magnetization as is discussed in the next chapter.

In this chapter we have treated the problem of ferromagnetic

Fig. 12 (a) — Dependence of coupling constants g_{ko} , curve (1), and $g_{kk'}$, curve (2), normalized to $\omega_M/2$, on crystal orientation when the d.c. field lies in the (110) plane. (0° corresponds to a [100] direction, both k and k' are z directed)

Fig. 12 (b) — Dependence of coupling constants g_{ko} , curve (1), and $g_{kk'}$, curve (2), normalized to $\omega_M/2$, on crystal orientation when the d.c. field lies in the (111) plane. (90° corresponds to a [110] direction, both k and k' are z directed)

resonance from the normal mode approach. We could equally well have presented a quantum mechanical theory or the canonical theory of Schlömann¹² but the results of these methods do not differ substantially from those presented here. In fact, no sizeable differences are expected to appear until five magnon processes are considered. Comparison with the experimental curves of uniform mode amplitude presented in Fig. 22 indicate that either the effective relaxation rate approach or the two spin wave pair theory will give general agreement, but the effective relaxation rate approach appears to give the better fit. However, it will be shown in the next chapter that the above two approaches give entirely different results when applied to the second harmonic content of the magnetization and that the effective relaxation rate method does not give satisfactory agreement with experimental second harmonic data.

CHAPTER III

SECOND HARMONIC GENERATION

Second harmonic generation in ferrites was first observed by Ayres, et. al.,²⁸ in 1956, who related the existence of a second harmonic component of magnetization to an elliptically polarized uniform mode. The effects of spin waves on the harmonic content of the magnetization was investigated by Richards¹³ who showed that, due to anisotropy, above the second Suhl threshold, spin waves can contribute to the second harmonic magnetization in spite of their short wavelength. In this chapter we shall extend this work to include the effects of the higher order spin wave interactions discussed in the last chapter. The effects of magneto-static modes and inhomogeneities on the second harmonic magnetization are treated by Richards¹³ and will not be repeated here.

A. GENERAL EXPRESSION FOR THE SECOND HARMONIC PART OF THE Z-COMPONENT OF THE MAGNETIZATION

In general, both the z- and transverse components of the magnetization will contain all harmonic components of the driving frequency $\omega/2\pi$. However, since only the second harmonic part of the z-component of the magnetization is of experimental interest here, discussion will be limited to it.

Consider the expansion for the z-component of the normalized magnetization which is

$$\alpha_z(\vec{r}, t) \cong 1 - \frac{1}{2} (\alpha_x^2(\vec{r}, t) + \alpha_y^2(\vec{r}, t)) . \quad (2.14)$$

Using (2.5) this can be written in terms of its Fourier components as

resonance from the normal mode approach. We could equally well have presented a quantum mechanical theory or the canonical theory of Schlömann¹² but the results of these methods do not differ substantially from those presented here. In fact, no sizeable differences are expected to appear until five magnon processes are considered. Comparison with the experimental curves of uniform mode amplitude presented in Fig. 22 indicate that either the effective relaxation rate approach or the two spin wave pair theory will give general agreement, but the effective relaxation rate approach appears to give the better fit. However, it will be shown in the next chapter that the above two approaches give entirely different results when applied to the second harmonic content of the magnetization and that the effective relaxation rate method does not give satisfactory agreement with experimental second harmonic data.

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Using (2.5) this can be written in terms of its Fourier components as

$$\sum_k \alpha_{kz}(t) e^{i\vec{k} \cdot \vec{r}} \cong 1 - \frac{1}{2} \sum_{kk'} \alpha_k^* \alpha_{k'} e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} . \quad (3.1)$$

Experimentally we can observe only the space average of $\alpha_z(\vec{r}, t)$, $\alpha_{oz}(t)$, which from (3.1) is

$$\alpha_{oz} \cong 1 - \frac{1}{2} \sum_k |\alpha_k|^2 . \quad (3.2)$$

It is immediately obvious from (3.2) that if the modes α_k are circularly polarized, the z-component of the magnetization is constant. However, the modes are in general not circularly polarized and the transformation (2.23) must be applied in order to write (3.2) in terms of the normal modes $\beta_k(t)$. Then writing α_{oz} as

$$\alpha_{oz} = \alpha_{oz}(0) + \frac{1}{2} \alpha_{oz}(2\omega) e^{i2\omega t} + \text{C.C.} \quad (3.3)$$

where $\alpha_{oz}(0)$ is the time independent contribution, and using (2.63) with the condition (2.83), the second harmonic component of α_{oz} is

$$\alpha_{oz}(2\omega) = \lambda_o \mu_o^* b_o^2 + \sum_{k>0} \lambda_k \mu_k^* b_k b_{-k} . \quad (3.4)$$

In deriving (3.4) we have ignored an overall phase factor and used the relations $\lambda_k = \lambda_{-k}$ and $\mu_k = \mu_{-k}$. This is the general relation for the second harmonic part of the z-component of the magnetization as will be used in this paper. Including anisotropy the constants $\lambda_o \mu_o^*$ and $\lambda_k \mu_k^*$ can be written as

$$\begin{aligned} \lambda_o \mu_o^* &= \frac{1}{2\omega_o} (B_a + B_o) \\ \lambda_k \mu_k^* &= \frac{1}{2\omega_k} (B_a + |B_k| e^{i2\varphi_k}) \end{aligned} \quad (3.5)$$

where B_a is given in (2.126), B_o and B_k in (2.22) and ω_o and ω_k in (2.27).

Several general comments can be made concerning the second harmonic component of α_{oz} by investigating (3.4) and (3.5). First, note that for an ellipsoid of revolution ($B_o = 0$) below the Suhl instability ($b_k b_{-k} = 0$), (3.4) becomes

$$|\alpha_{oz}(2\omega)| = \frac{|B_a|}{2\omega_o} |b_o|^2 \quad (3.6)$$

which clearly shows that since $b_o \sim h_o$, $\alpha_{oz}(2\omega)$ is proportional to the square of the driving field. Also, since $\alpha_{oz}(2\omega)$ is linearly related to B_a , its amplitude for different crystal orientations will show a symmetry which is characteristic of B_a as was stated in the last chapter. The importance of anisotropy in harmonic generation is also apparent from (3.6) since when $B_a = 0$, $\alpha_{oz}(2\omega)$ vanishes. However, there are other terms neglected in (3.4) which can lead to harmonic generation even when $B_o = B_a = 0$, as discussed below.

For a linearly polarized driving field a contribution results when the non-resonant component of the uniform mode is included. This component will result in a second harmonic z-component of magnetization equal to $|b_o|^2 \frac{\gamma \Delta H_o}{4\omega_o}$, where ΔH_o is the uniform mode intrinsic full line-width. For a polished YIG sphere where $\Delta H_o \sim 1$ gauss and orientated so that the applied field lies close to the (111) plane (this corresponds to the experimental conditions for the samples studied here) the above contribution will be two orders of magnitude less than the anisotropy contribution (3.6) and, therefore, can be neglected.

Off uniform mode resonance some of the magnetostatic modes can also lead to a harmonic content which, in fact, can be larger than that given

by (3.6). Richards¹³ shows that in the Fletcher-Bell²⁹ notation where a particular mode is defined by a set of three indices (n, m, r) the only modes that can contribute to harmonic generation are those for $n \geq 2$ and $m = 0$. Since for a sphere none of these modes (at least up to $n = 5$) is degenerate with the uniform mode, the magnetostatic mode contribution to $\alpha_{oz}(2\omega)$ can be neglected when discussing the effects of higher order spin wave interactions on the second harmonic.

B. EFFECTS OF SPIN WAVES PARAMETRICALLY EXCITED BY THE UNIFORM MODE

It was shown in the last chapter that the first spin waves to go unstable at the second Suhl threshold are z-directed. For these spin waves $B_k = 0$ and since we will be concerned with an ellipsoid of revolution, $B_o = 0$ also. Then (3.4) becomes

$$\alpha_{oz}(2\omega) = \frac{|B_a|}{2\omega} b_o^2 + 2b_k b_{-k} \quad (3.7)$$

where b_k is the z-directed spin wave excited above the second Suhl threshold. The term $b_k b_{-k}$ can be obtained from (2.100), (2.102), and (2.104) upon setting $g_{kk'} = 0$. Then (3.7) becomes

$$|\alpha_{oz}(2\omega)| = \frac{|B_a|}{2\omega} \left| |b_o|^2 + 2i |b_k|^2 \right| \quad (3.8)$$

which shows that the spin wave mode b_k contributes 90° out of phase from b_o . This is expected since the mode b_k is driven parametrically by the uniform mode and $b_k b_{-k}$ would necessarily have this phase relation. Using the relation (2.91) for $|b_k|^2$ and remembering that in this model, above threshold, the uniform mode "sticks" at the amplitude η_k/g_{ok} , $\alpha_{oz}(2\omega)$ can be written in terms of the driving field h_o as

$$|\alpha_{oz}(2\omega)| = \frac{|B_a|}{2\omega} \frac{\eta_k}{g_{ok}} \left[1 + \left(\frac{\eta_o}{\eta_k} \right)^2 \left(\frac{h_o}{h_c} - 1 \right)^2 \right]^{1/2} \quad (3.9)$$

where h_c is given by (2.68). This relation is shown in Fig. 13 for various values of η_o/η_k and has the general characteristics of an initial zero slope and the linear dependence $\alpha_{oz}(2\omega) \sim \frac{h_o}{h_c}$ for large applied fields. The experimental dependence of $\alpha_{oz}(2\omega)$ on applied field, shown in Fig. 24, does not at all agree with (3.9) since experimentally a minimum occurs in $\alpha_{oz}(2\omega)$ while (3.9) shows $\alpha_{oz}(2\omega)$ always to increase with h . In fact, if only the single Suhl coupling term g_{ok} is considered, all spin waves modes that are driven by this coupling will contribute 90° out of phase with the uniform mode and thus cannot give a minimum to $\alpha_{oz}(2\omega)$. The effective relaxation rate method of the last chapter considers only the single coupling term g_{ok} and will lead to this same property.

Therefore, although the effective relaxation rate method satisfactorily explains the experimental susceptibility decline data, it can not be used to explain the dependence of $\alpha_{oz}(2\omega)$ on driving field. It appears then that the effects of the higher order spin wave interactions must be considered in a more explicit manner than just to assign an effective relaxation rate to the first spin wave mode that is excited above the second Suhl threshold.

C. EFFECTS OF HIGHER ORDER SPIN WAVE INTERACTIONS

Consider the two parametrically excited spin wave pairs discussed in the last chapter. Then (3.4) becomes

$$\alpha_{oz}(2\omega) = \frac{1}{2\omega} B_a (b_o^2 + 2b_k b_{-k}) + 2 (B_a + B_{k'}) e^{2i\phi_{k'}} b_{k'} b_{-k'} \quad (3.10)$$

where the modes b_k are z-directed and the modes $b_{k'}$ are excited by the $g_{kk'}$ interaction and have $k' \rightarrow 0$. In this case $B_{k'} \neq 0$ but if it is

Fig. 13 — Dependence of $\alpha_{oz}(2\omega)$ on driving field when a single spin wave pair is excited above threshold, Eq. (3.9).

assumed that the $g_{kk'}$ interaction is azimuthally independent, the modes $b_{k'}$ will be excited equally in all azimuthal directions so that the term $B_{k'} e^{2i\phi_{k'}}$ averages to zero. However, as was shown in the last chapter, $g_{kk'}$ is not entirely azimuthally independent when anisotropy is considered. This azimuthal dependence can lead to an enhancement of the $b_{k'} b_{-k'}$ contribution to $\alpha_{oz}(2\omega)$ but, since this dependence is small (see Fig 4), it will be neglected here. Then (3.10) becomes

$$\alpha_{oz}(2\omega) = \frac{|B_a|}{2\omega} |b_o|^2 + 2b_k b_{-k} + 2b_{k'} b_{-k'} \quad (3.11)$$

Using (2.102), (2.103), (2.104), (2.100) and assuming $g_{ok'} = 0$ for $k' \rightarrow 0$ (3.11) can be written as

$$|\alpha_{oz}(2\omega)| = \frac{|b_o|^2}{2\omega} |B_a| \left| \left(1 - \frac{4|b_k|^2 |b_{k'}|^2 \bar{g}_{k'k}}{|b_o|^2 \eta_{k'}} \right) + i \frac{2|b_k|^2}{|b_o|^2} \right| \quad (3.12)$$

Therefore, the term $b_{k'} b_{-k'}$ in (3.11) contributes to $\alpha_{oz}(2\omega)$ 180° out of phase with b_o^2 and hence can give rise to a possible minimum in $\alpha_{oz}(2\omega)$. In terms of b_o^2 alone, (3.12) becomes, above the threshold field h'_0 ,

$$|\alpha_{oz}(2\omega)| = \frac{|B_a|}{2\omega} |b_o|^2 \left[1 - \frac{g_{ko}}{\bar{g}_{kk'}} \right] + \frac{g_{ko}}{\bar{g}_{kk'}} |b_c|^2 + i \frac{\eta_{k'}}{\bar{g}_{k'k}} \quad (3.13)$$

where $|b_c|$ is given by (2.70) and we have used (2.90) to eliminate $|b_k|^2$.

In this two-spin-wave-pair model the uniform mode increases as the driving field increases. Therefore, it is obvious from the term in brackets in (3.13), that the initial slope of $\alpha_{oz}(2\omega)$ is negative for

driving fields above h'_c provided $g_{ko}/\bar{g}_{kk'} > 1$. The relation (3.12) or (3.13) is shown in Fig. 14 for typical values of the ratios $g_{ko}/\bar{g}_{kk'}$ and $\eta_{k'}/\eta_k$. Also shown is that portion of Fig. 13 when the $g_{kk'}$ process is not important (between h_c (2.68) and h'_c (2.94)).

The effect of including the $g_{ok'}$ term can be determined in a manner similar to that used for (3.12) with the result

$$\begin{aligned} |\alpha_{oz}(2\omega)| = \frac{|B_a|}{2\omega} |b_o|^2 & \left| \left[1 - \frac{g_{ok}}{\bar{g}_{k'k}} \frac{wv}{1+wv} (\alpha \eta_{k'}/\eta_k + 1) \right] \right. \\ & \left. + i \left(\frac{g_{ok}}{\bar{g}_{kk'}} \right) \left(\frac{v \eta_{k'}/\eta_k + w\alpha}{1 + wv} \right) \right| \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} w &= \frac{2|b_{k'}|^2 \bar{g}_{k'k}}{\eta_k} \\ v &= \frac{2|b_k|^2 \bar{g}_{k'k}}{\eta_{k'}} \\ \alpha &= \frac{g_{ok'}}{g_{ok}} \frac{\eta_k}{\eta_{k'}} \end{aligned} \quad (3.15)$$

This equation, which reduces to (3.12) upon setting $g_{ok'} = 0$, is also shown in Fig. 14 for the single ratio $g_{ok'}/g_{ok} = .05$. The general result of including this term is that the minimum in $\alpha_{oz}(2\omega)$ vanishes as $g_{ok'}/g_{ok}$ is increased.

D. USE OF SECOND HARMONIC GENERATION IN THE STUDY OF SPIN WAVE INTERACTIONS

In the last section it was found that the inclusion of the higher order spin wave interactions can greatly alter the dependence of the second harmonic part of the z-component of the magnetization on the

Fig. 14 - Dependence of $\alpha_{oz}(2\omega)$ on driving field when two spin wave pairs are excited above threshold, Eq. (3.14). All curves are for $\eta_o/\eta_k = 1$. The other ratios for g_{ok}/g_{ok} , g_{ok}/g_{kk} , and $\eta_{k'}/\eta_k$ are, respectively:

- (a) 0, 1.2, 1
- (b) 0, 1.2, .5
- (c) .05, 1.2, .5
- (d) 0, 1.6, 1
- (e) 0, 1.6, .5
- (f) .05, 1.6, .5

applied microwave field. There are also interesting effects that occur in the transient response of $\alpha_{oz}(2\omega)$ to a fast microwave driving field pulse, but before discussing them it will be convenient to present the results of the last two sections in terms of a vector diagram. Both transient and steady-state results can be easily interpreted in terms of such a diagram.

1. Vector Diagram Representation and Phase Shift of $\alpha_{oz}(2\omega)$.

First consider the case of one spin wave pair, $b_k b_{-k}$, only. Since $b_k b_{-k}$ is 90° out of phase with the uniform mode's contribution to $\alpha_{oz}(2\omega)$, the second harmonic will consist of the two vectors b_o^2 and $b_k b_{-k}$ with the phase relation shown in Fig. 15(a). The g_{ok} interaction that exists between the modes b_k and b_o is also shown in this figure as is the resultant vector which represents the total amplitude of $\alpha_{oz}(2\omega)$. Since simple theory predicts that b_o sticks above threshold, it can easily be seen that the resultant vector representing $\alpha_{oz}(2\omega)$ can only increase above threshold. The effective relaxation rate method will predict the same behavior, the only difference here being that above threshold the uniform mode can also increase.

Another quantity that can easily be obtained experimentally as will be discussed in the next chapter is the overall phase shift of $\alpha_{oz}(2\omega)$ with respect to the uniform mode. In the model represented in Fig. 15(a) it is evident that this shift in phase approaches 90° when $|b_k|^2 \gg |b_o|^2$.

The inclusion of a second spin wave pair is represented in Fig. 15(b) for the case where $g_{ok'} = 0$ and in Fig. 15(c) we have represented a typical result when $g_{ok'} \neq 0$. In the former case the second spin wave term $b_{k'} b_{-k'}$ will be 180° out of phase with b_o^2 and for the latter this term can be anywhere from 90° to 180° out of phase with b_o^2 depending

Fig. 15 — Vector diagram representation of $\alpha_{oz}(2\omega)$. For an explanation of these diagrams see text.

on the relative strengths of the interactions $g_{ok'}$ and $g_{kk'}$.

From these figures it is evident that for large driving fields when $|b_{k'}|^2$ is large the overall phase shift of $\alpha_{oz}(2\omega)$ can be greater than 90° , and when $g_{ok'} = 0$ this shift can approach 180° . It is also evident that a minimum in $\alpha_{oz}(2\omega)$ does not of necessity imply a shift greater than 90° and, in fact, it is very possible to observe one effect without observing the other. Although Fig. 15(c) is probably more representative of the actual physical situation, in what follows it is assumed that $g_{ok'} = 0$.

2 . Decomposition of $\alpha_{oz}(2\omega)$ to Determine the Spin Wave Contribution.

Experimentally it is possible to measure independently the uniform mode amplitude, second harmonic amplitude, and the phase shift of $\alpha_{oz}(2\omega)$ with respect to the uniform mode. From these three quantities and Eq. (3.7) the amplitude and phase of the $k \neq 0$ spin wave contribution to the second harmonic can be determined. If the assumptions in the derivation of Eq. (3.7) are correct, then both the 90° and 180° phase components of the spin wave contribution to $\alpha_{oz}(2\omega)$ can be determined experimentally. The dependence of these components on incident driving field can aid in determining which spin wave interactions are important above the Suhl threshold, e.g., if the 90° component "sticks" then the $g_{ok'}$ process is not important while if this component continues to increase then the $g_{ok'}$ process must also be included.

E. TRANSIENT RESPONSE OF $\alpha_{oz}(2\omega)$ TO A FAST PULSE DRIVING FIELD

Using the vector relations in the last section the expected transient behavior of $\alpha_{oz}(2\omega)$ can easily be determined. We already know the transient response of the uniform mode and the spin wave modes b_k and

b_k , from the last chapter (Figs. 9-11) and, therefore, they only need to be vector added to obtain $\alpha_{oz}(2\omega)$.

Below threshold only the uniform mode contributes so $\alpha_{oz}(2\omega)$ will have the same form as b_o except with rise and fall times twice as fast as those exhibited by the uniform mode, i.e., $\alpha_{oz}(2\omega) \sim |b_o|^2$. This is represented in Fig. 16(b).

Just above threshold where a single spin wave mode is excited the second harmonic component of M_z has the general characteristics shown in Fig. 16(c). The trailing edge of the response is slightly exaggerated to show a situation that can easily exist experimentally, namely, due to radiation damping, the uniform mode can decay (and rise) at a much faster rate than predicted by the intrinsic relaxation rate η_o . The effective or "loaded" relaxation rate can typically be $> 10 \eta_o$ in which case the uniform mode can decay at a much faster rate than the spin wave modes. Therefore, as depicted in Fig. 16(c), the decay of $\alpha_{oz}(2\omega)$ is rapid at first due to the uniform mode decay and then the remaining spin wave contribution decays at a slower rate. The change in the decay characteristics of the trailing edge of $\alpha_{oz}(2\omega)$ is even more pronounced when a second spin wave mode is excited, as is discussed next.

In Fig. 16(d) we have represented a typical transient response of $\alpha_{oz}(2\omega)$ as predicted by the theory presented in the last two chapters for a case in which two spin waves pairs are excited. Here the steady-state portion of the pulse is less than that represented in Fig. 16(c) since the second excited spin wave contributes 180° out of phase with the uniform mode contribution. Two trailing edges are presented in this figure and they can be explained in terms of the vector diagrams of

Fig. 16 - Pulse response of second harmonic, $a_{oz}(2\omega)$, to rectangular driving field pulse when zero, one, and two spin wave pairs are excited above threshold (corresponding to curves (b), (c), and (d).) The two trailing edges for the response shown in curve (d) are explained in the text.

the last section. Again the loaded uniform mode relaxation rate is assumed to be much larger than the spin wave decay rates so that the "reradiation" character of the trailing edge is due entirely to the spin wave modes and occurs after the uniform mode has decayed to a small value.

The trailing edge denoted by (1) in Fig. 16(d) occurs when in the steady-state the uniform mode's contribution to $\alpha_{oz}(2\omega)$ is greater than that of the spin wave mode's b_k , (see Fig. 15). For this case when the uniform mode initially decays $\alpha_{oz}(2\omega)$ will also decay until the uniform mode's and the spin wave's 180° out of phase contributions are equal in magnitude. Then as the uniform mode continues to decay $\alpha_{oz}(2\omega)$ will increase to a value determined by the spin wave modes alone. After rising to a maximum $\alpha_{oz}(2\omega)$ then decays at a rate determined by the relaxation rate of the spin wave modes. This model also predicts that during the steady-state the phase of $\alpha_{oz}(2\omega)$ with respect to b_o^2 is less than 90° while during the reradiation period the phase shift is greater than 90° .

The curve denoted by (2) in Fig. 16(d) represents the trailing edge of $\alpha_{oz}(2\omega)$ when in the steady-state the uniform mode's contribution is less than that of the 180° out of phase contribution of the spin wave modes. This case can be analyzed in a manner similar to the above and will not be presented here. It is interesting to note, however, that this case predicts the phase of $\alpha_{oz}(2\omega)$ with respect to b_o^2 to be greater than 90° in both the steady-state and reradiation regions of the pulse.

In this chapter we have extended the work of Richards¹³ to include the effects of higher order spin wave interactions on the second harmonic content of the z-component of the magnetization. These interactions can also affect the other harmonic components of M_z and can, of course, have

an effect on the harmonic content of the transverse magnetization. These components are not of interest to us since we have observed only the fundamental transverse and second harmonic z-components of the magnetization. The principle result of this chapter is that a minimum can occur in $\alpha_{oz}(2\omega)$ above the Suhl threshold if the effects of higher order spin wave interactions are explicitly included. It is not possible for an effective relaxation rate model to predict a minimum in $\alpha_{oz}(2\omega)$ when only a single g_{ok} type interaction is included.

In the next chapter experimental results of $\alpha_{oz}(2\omega)$ and b_o as a function of driving field are presented and at this time it appears that both sets of data can not be explained simultaneously by a single theory. One reason for this may be due to the fact that situations can exist where the uniform mode excites spin waves that do not have the correct phase relations to contribute to $\alpha_{oz}(2\omega)$. More will be said about this in the next chapter.

CHAPTER IV

EXPERIMENTAL RESULTS

In this chapter experimental results are presented in an attempt to verify some of the theoretical predictions of the two previous chapters. Of primary interest is the transient and steady-state dependence of the transverse, $M_T(\omega)$, and longitudinal second harmonic, $M_Z(2\omega)$, components of magnetization on applied driving field. These components can be measured using standard microwave techniques which have been discussed by many authors³⁰. Those of importance to this work are reviewed in the next section.

It is also necessary to measure experimentally the uniform and spin wave mode relaxation rates, since they appear as phenomenological parameters. The uniform mode relaxation rate, η_0 , is usually determined from a measurement of the resonance linewidth although it can also be determined from the transient decay of the uniform mode. The linewidth method is used here since, in the presence of radiation damping, the transient decay is very fast (20 nsec). In spite of broadening due to radiation damping, the intrinsic linewidth can be extracted from the experimentally measured width, as discussed in the Appendix.

The linewidth of the first spin wave to go unstable can be measured by observing the field at which the susceptibility declines and using Eq. (2.68). At present, however, there are no experimental means available to measure the spin wave linewidths for the other degenerate modes. In this case, it is necessary either to apply the theory of Sparks, et. al.²¹, (Fig. 3), or to use this theory in correcting the results obtained by parallel pumping techniques³¹ so they apply to

degenerate spin waves. This subject is discussed in section D. 7 of the present chapter.

It was mentioned in Chapter II that experiments were performed in an attempt to observe higher order spin wave interactions for d.c. fields where the first order process is allowed at resonance. These experiments were negative in that no departure from a $1/h$ dependence in the susceptibility decline curve was observed. The experimental results to be discussed here are therefore limited to experimental conditions where the Suhl second order process is of importance and where higher order interactions were seen.

A. MEASUREMENTS OF FUNDAMENTAL FREQUENCY PARAMETERS

The quantities of interest at the fundamental frequency are the uniform mode amplitude, b_o ; the driving field amplitude, h ; uniform and spin wave mode relaxation rates, η_o and η_k , respectively; and the radial frequency ω .

Experimentally, it is possible to observe the amplitude of the space average transverse magnetization, $M_T(\omega)$. Thus, assuming an $e^{i\omega t}$ time dependence we have

$$M_T(\omega) e^{i\omega t} = \frac{1}{V_s} \int_{V_s} [M_x(\vec{r}, t) + iM_y(\vec{r}, t)] d\vec{r} \quad (4.1)$$

where the integration is over the volume, V_s , of the sample. From (2.8), (2.15), (2.16), and (2.25) and using the fact that for YIG spheres magnetized in the (111) plane, the uniform mode amplitude is related to $M_T(\omega)$ by

$$b_o = \frac{M_T(\omega)}{M} \quad (4.2)$$

where a constant phase factor has been ignored.

Usually $M_T(\omega)$ (or equivalently $\chi''h$) is measured by placing the sample in a microwave cavity and observing the change in the cavity Q by monitoring the reflected power as the applied d.c. field is passed through resonance. Because of the relatively large Q 's that can be obtained in microwave cavities, this method is quite sensitive to small changes in absorbed power when the cavity coupling is close to critical and has been used in the study of susceptibility decline^{7,9} and in the determination of ΔH_k ³². In the present case, however, sensitivity is not important when considering fundamental parameters and it is more convenient to place the sample in the center of a shorted waveguide $\frac{\lambda_g}{2}$ from the shorted end as shown in Fig. 17. This method has the advantage that in the presence of heavy radiation damping only a single coupling constant (for the sample) needs to be considered whereas for cavity detection it is necessary to consider both sample and cavity coupling. Also, by eliminating the need for a cavity one is not limited by the cavity ring time in observing fast transient responses.

In a shorted waveguide the transverse magnetization can be determined either by monitoring the reflected power or by placing a small loop near the sample in such a configuration as to couple only to the transverse magnetization and not to the microwave driving fields. The loop coupling gives a direct measure of the transient behavior of the transverse magnetization which is useful in the comparison of experimental and theoretical results. Although this method can also be used to determine the steady-state behavior (and χ''), the reflected power method is used here since problems associated with uncertain detector response as a function of power can more conveniently be eliminated. This method is discussed below.

The power absorbed by the sample is given by

$$P_{ab} = \frac{\omega}{2} \chi'' h^2 V_s \quad (4.3)$$

where h is the amplitude of the linearly polarized driving field at the sample which, due to radiation damping, is not the incident field h_o . The driving field is related to the incident power (see Appendix) by

$$h^2 = C' \frac{P_{inc}}{(1 + \beta)^2} \quad (A.32)$$

where C' is given by (A.33) and β is the sample coupling constant which is related to the reflection coefficient $|\Gamma|$ by (A.21) and (A.24). Therefore, the driving field can be determined from a measurement of the incident and reflected power once the constant C' is determined.

Below threshold the uniform mode amplitude is obtained from its equation of motion and is given by

$$|b_o| = \frac{\gamma h}{2\eta_o} \quad (4.4)$$

Above threshold (4.3) is valid provided the correct definition for the susceptibility is used, $\chi'' = \frac{M_T(\omega)}{h}$. In terms of the low power susceptibility $\chi''_o = \frac{M}{\Delta H_o}$, (4.3) can be written as

$$P_{ab} = \frac{\omega}{2} \chi''_o \frac{\beta}{\beta_o} h^2 V_s \quad (4.5)$$

where β and β_o are the coupling constant on resonance for the driving field h above and below threshold, respectively. In this region the relation between the uniform mode amplitude and driving field must be modified to read

$$|b_o| = \frac{\gamma h_o}{2\eta_o} \frac{\beta}{\beta_o} \quad (4.6a)$$

From the Appendix, this can equally as well be written as

$$|b_o| = \frac{\gamma h_o}{2\eta_L} \frac{\beta}{\beta_o} \quad (4.6b)$$

where h_o is the incident applied field and $\frac{2\eta_L}{\gamma}$ the "loaded" linewidth defined in the Appendix. Therefore, using Eqs. (A.32) and (4.6), the experimental values of the uniform mode and driving field amplitude can be obtained from the incident and reflected power.

Another method used to study the uniform mode which is particularly useful for observing transient behavior is to place a small loop in the x-z plane at the position of the sample as shown in Fig. 18. This loop should be weakly coupled to the sample so as not to interfere with the second harmonic fields to be discussed below. One end of the loop is grounded to the waveguide wall and the other is connected to a detector through an intermediate coaxial line and waveguide through which power is transmitted from the loop. In this configuration the transmitted power detected, P_{trans} , is proportional to the square of the transverse magnetization and can be represented by

$$P_{trans} = K' |M_T(\omega)|^2 \quad (4.7)$$

where K' is a constant, independent of incident power and magnetization, depending only on geometry and frequency. This constant can be determined experimentally since both P_{trans} and $M_T(\omega)$ can be measured, the latter by the reflection method discussed above.

B. MEASUREMENT OF SECOND HARMONIC PARAMETERS

The second harmonic part of the z-component of the magnetization can be observed by placing the sample in a waveguide that normally propagates energy at the second harmonic frequency. The amplitude of the second harmonic microwave fields then can be determined in a manner similar to that used to obtain the relations for the amplitude of the radiation fields of $M_T(\omega)$ (see the Appendix) since their origins are similar. Thus, we can define a second harmonic field $h_{2\omega}$ as

$$h_{2\omega} = -i K 4\pi M_z(2\omega) \quad (4.8)$$

where K is given by (A.11) when the sample is placed in the plane of the short in a shorted waveguide or in the maximum h fields of a cavity.

The radiated power is then obtained from substituting (4.8) into (A.9) and becomes

$$P_{2\omega} = 2\omega K V_s 2\pi M_z^2(2\omega) \quad (4.9)$$

Therefore, the amplitude $\alpha_{oz}(2\omega) = M_z(2\omega)/M$ appearing in Chapter III can be determined from the second harmonic power generated by the sample.

Below threshold (4.9) is commonly written in terms of the absorbed power as

$$P_{2\omega} = C P_{ab}^2 \quad (4.10)$$

where C is the "C-factor" for the system which from (3.6), (4.3), and (4.4) can be written as

$$C = K \frac{4\pi}{\omega V_s} \left(\frac{|B_a|}{\Delta H_0} \right)^2 \quad (4.11)$$

Here $|B_a|$ is given by (2.126) and ΔH_0 is the intrinsic uniform mode

linewidth. Through K , the C -factor depends on the geometry of the sample surroundings and can easily vary over several orders of magnitude. Because of the low powers involved ($P_{2\omega} \approx -70$ dBm at threshold for a .05" diameter YIG sphere, $\Delta H_0 = .4$ gauss, placed in a shorted K_u -band waveguide $\frac{\lambda_g}{2}$ from the short) it is necessary to find an experimental configuration where C is large.

A cavity, resonant at the second harmonic frequency, gives a C -factor considerably larger than can be obtained in a non-resonant waveguide section. Also, through the use of iris coupling, the problem of impedance matching between the input (in this case X-band waveguide) and the propagating structure (K_u -band waveguide) can easily be eliminated. However, in studying transient effects, a cavity system is limited by its ring time given by³³

$$\tau = \frac{Q_L}{\pi f} \quad (4.12)$$

where Q_L is the loaded Q of the cavity, τ is the characteristic relaxation time for the fields in the cavity, and $f = \omega/2\pi$. For the cavity used here where $Q_L \approx 2500$ and $f = 16.84$ GHz this corresponds to a ring time of approximately 50 nanoseconds. The relaxation time of the radiation damped uniform mode's contribution to the second harmonic is of the order of 10 nsec while that for the spin wave mode's is of the order of 100 nsec assuming $\Delta H_0 \approx \Delta H_k \approx .5$ gauss. Therefore, for the present case any effects of spin waves on the trailing edge of the second harmonic, if present, should not be obscured by the 50 nsec cavity ring time. There is an effect, however, which is obscured by this ring time and requires a lower Q to be observed. This is discussed in section D.5 of this chapter.

C. EXPERIMENTAL APPARATUS

1. Second Harmonic Cavity

Of the several experimental arrangements that can be used to observe the second harmonic component of M_z , the one used here is shown in Figs. 17 and 18. The incident X-band waveguide consists of a reduced height section followed by a cavity which is resonant in the TE_{012} mode at twice the fundamental frequency. As shown in Fig. 17 the magnetic field patterns for this mode can couple easily to $M_z(2\omega)$ when the sample is located at the center of the cavity. The reduced height section is designed so as to have little effect on the fundamental microwave fields ($VSWR < 1.1$) but has a cutoff frequency of 22 GHz for the TE_{01} mode which is well above that for the resonant second harmonic fields (≈ 17 GHz) and, hence, acts essentially as a short for these fields. Coupling from the cavity to the second harmonic waveguide is provided by a circular iris which, because of the following K_u -band waveguide, acts as a short for the fundamental frequency.

There are two modes (TE_{110} and TE_{204}) that are close to the TE_{012} mode in frequency that could, at first thought, compete with the TE_{012} and create a coupled mode effect. However, these modes are not of importance here since their field patterns are such that they cannot easily couple to $M_z(2\omega)$. Also, the cavity will not even act as a resonant structure for the TE_{204} mode since the fields for this mode can easily propagate through the reduced height section, i.e., the cutoff frequency of the reduced height section is well below that for the TE_{20} mode.

The resonant frequency for the TE_{012} mode of the cavity shown in Fig. 18 is 16.86 GHz and can be lowered by the insertion of a sapphire

Fig. 17 — Position of sample in waveguide. The iris acts as a short for the fundamental frequencies (X-band) and the reduced height section is an effective short for the second harmonic cavity fields (TE_{012}).

Fig. 18 - Detail of cavity showing orientation of loop and position of sapphire tuning stub.

rod at a position of maximum electric field as shown. The sample is placed in the center of this cavity and is at a position of maximum magnetic field for a fundamental frequency of 8.42 GHz. The loaded Q of the cavity at the second harmonic frequency (16.84 GHz) is approximately 2500.

The C-factor (4.11) can be calculated for this system and is

$$C = 6.1 \times 10^{-6} \text{ watts}^{-1} \text{ (cavity)}$$

for a .05" diameter YIG sphere ($\Delta H_0 \approx .4$ gauss) magnetized in the (111) plane in a direction where $|B_a|$ is maximum ($|B_a|/\gamma = 64.5$ gauss). It is interesting to compare this value of C with that for a shorted K_u -band waveguide when the sample is placed $\lambda_g/2$ from the short and at a frequency of 16.84 GHz. This value is

$$C = 3.7 \times 10^{-8} \text{ watts}^{-1} \text{ (waveguide)}$$

which is two orders of magnitude smaller than for the cavity system.

At threshold, when using the above cavity, the expected second harmonic power is of the order of -50 dBm which, although quite small, can be detected experimentally. Several methods were attempted for observing the second harmonic signal, and the greatest sensitivity resulted through superheterodyne detection using an i-f amplifier. However, the i-f frequency of our amplifier (30 MHz) is too small for observation of rise and fall times smaller than 100 nsec. Straight detection with a video amplifier provided fast response but proved to be too noisy for accurate measurements. With the equipment available, the most sensitive detection method for both transient and steady-state measurements is through superheterodyne detection and video amplification (Hewlett-Packard 462A amplifier) with the i-f envelope being observed

directly on the oscilloscope face (Tektronix Type 581A with Type 82 Plug-In). The i-f frequency used was 80 MHz. With this type detection the minimum detectable signal was -60 dBm which is 10 dB below the expected signal at threshold.

2. Fast-Pulse, High-Power Spectrometer

The spectrometer used in this study is shown in Fig. 19. Its main active components are an X-band klystron (Varian X-13B), a diode switch (Philco L4145 diode in a Philco P901A mount), a 20 watt c.w. traveling wave tube (Varian VA-617G), and a K_u -band klystron (Varian X-12) used as a local oscillator in the superheterodyne detection of the second harmonic signal. To stabilize the klystron frequencies against thermal drift, both were mounted in an oil bath. This along with the use of well-regulated stable klystron power supplies constructed for this purpose eliminated the need for associated lock-in circuitry. A well-regulated power supply with associated protection circuits was also built for the traveling wave tube (TWT).

A fast 20 watt peak pulse (rise and fall times < 5 nsec) of microwave energy is obtained by pulsing the diode switch through a biasing circuit with a fast rise pulser (Hewlett-Packard 212A). The 5 nsec microwave pulse can be obtained from the 212A pulser (20 nsec rise time) because of the non-linear character of the diode switch; i.e., for a typical 20 volt pulse, approximately 18 volts are used to overcome the reverse bias where the transmitted power changed from .1% to 10% of its final value while during the remaining voltage when the diode is forward biased the transmitted power changed by 90%. Since the switching speed of the diode is less than 1 nsec (manufacturers specification), microwave pulses with rise times less than a nanosecond are conceivable.

Fig. 19 - Fast pulse, high power spectrometer used in the detection of the transverse fundamental and longitudinal second harmonic components of magnetization.

Because the diode switch is not a perfect attenuator when reverse biased, there is always some power incident on the sample which, if the klystron is run c.w., is sufficient to cause heating of the samples used there ($\leq .05$ " dia.). To reduce this heating the average power at the sample is lessened by pulsing the reflector of the X-13B klystron with a pulse of longer duration than and properly synchronized with the 212-A pulse. This pulse is obtained from a General Radio 1217-C pulser.

A low pass filter cutoff at 12 GHz is placed following the traveling wave tube so that unwanted harmonics generated by the TWT are attenuated. Power is detected using normal techniques with crystal diodes that were calibrated against thermistors and a power bridge (Sperry 31A1). The crystal detectors are resistance loaded to improve their pulse response.

3. Second Harmonic Phase Shift Measurement

The additional microwave apparatus needed to measure the phase of the second harmonic relative to the uniform mode is shown in Fig. 20 and is inserted at positions A-A' in the high power spectrometer represented in Fig. 19. A certain amount (-20 dB) of incident power is applied to a crystal diode which is used as a harmonic generator. The second harmonic component thus obtained has a constant phase with respect to the fundamental power incident on the sample and, hence, also to the uniform mode if the applied d.c. field is always adjusted to resonance. This harmonic component is mixed with the second harmonic fields generated by the sample through a X- to K_u-band transition and precision phase shifter. Any power dependent phase shift of the second harmonic relative to the

uniform mode can readily be determined using this apparatus.* One

* In this measurement it must be assumed that the phase of the second harmonic signal generated by the crystal diode is not power dependent. This can easily be checked experimentally and for the present case no such dependence was observed. Also, since all attenuators produce a phase shift, it is necessary to set all attenuators between the points A and A' in Fig. 21 at a convenient position and not to change them during the experiment unless, of course, they are calibrated for phase shift.

Fig. 20 — Additional microwave apparatus needed in the phase shift measurements. (Inserted as A-A' in the spectrometer in Fig. 19).

disadvantage of using this method of measuring relative phase is that, in the absence of a microwave delay line, it is not possible to measure the phase of the second harmonic reradiation (which occurs after the incident power is pulsed off) since in this region there is no source for the constant phase harmonic fields. To observe the phase of this reradiation region a 1 μ sec delay is needed which for conventional waveguide is not practical since this requires a 180 meter delay line. To our knowledge lumped parameter delay lines are not commercially available at the frequency of interest. However, it is possible to tap a portion of the power directly from the klystron for use as a harmonic generator source. Then if the phase shift through the TWT were constant and the phase shift of the attenuators following the TWT were calibrated, it would be possible to determine the phase of this reradiation region. However, this method requires a more efficient harmonic generator than was available to us.

D. EXPERIMENTAL RESULTS

Experiments were performed at a fundamental frequency of 8.42 GHz on highly polished YIG spheres ($\Delta H_0 \leq .4$ gauss) to determine the dependence of $M_T(\omega)$ and $M_Z(2\omega)$ on applied field. The data presented here are for a sphere .05" in diameter with the d.c. field applied within 6° of the (111) plane and for three different orientations in this plane. Experiments were also performed on other samples of YIG (.04" and .02" diameter spheres) as a qualitative check and since the results are very similar to those for the .05" sphere they are not presented here.

The sample is mounted on a quartz rod so that its [111] axis is approximately along the axis of the rod. This is done by letting the sample freely rotate on a horizontal magnet pole face so that the easy

axis, $[111]$, is vertical. It was then glued to the quartz rod in this position.

1. VARIATION OF RESONANCE FIELD AND AMPLITUDE OF SECOND HARMONIC WITH ORIENTATION

Fig. 21a shows the dependence of d.c. resonance field on orientation below threshold while Fig. 21b shows the corresponding dependence of $\alpha_{oz}(2\omega)$. Notice that these two curves do not reflect the same symmetry. The $\alpha_{oz}(2\omega)$ curve shows a three-fold symmetry while $\bar{H}_0$ shows an approximate two-fold symmetry. The $\bar{H}_0$ curve is not exactly two-fold symmetric since the quartz sample holder is mounted slightly off axis and hence there is a small rotational "wobble" present which destroys the two-fold symmetry.

The general behavior shown in Fig. 21, as explained in Chapter II, is due to the fact that $\alpha_{oz}(2\omega)$ and $\bar{H}_0$ depend on anisotropy differently when the d.c. magnetic field lies close to but not in the (111) plane. The curves in Fig. 21 can be reproduced theoretically as shown if the sample is mounted so that the quartz rod axis lies 6° from a sample $[111]$ direction towards a $[110]$ direction. X-ray crystallographic analysis reveals that the sample is oriented $4^\circ \pm 3^\circ$ from a $[111]$ direction, in agreement with the above. Also, the ratio of the maximum to the minimum in $\alpha_{oz}(2\omega)$ from Fig. 21b is approximately 3, which agrees with the theoretical value.

2. STEADY-STATE RESULTS

a. Transverse Magnetization

Data were taken on the transverse magnetization as a function of applied field for the three different orientations shown in Fig. 21.

Fig. 21 — Experimental (O) and theoretical (solid curves) dependence of H_0 , (a), and $\alpha_{oz}(2\omega)$, (b), on orientation when the applied d.c. field lies in a plane 6° from the (111) plane towards the (110). 0° corresponds to a [110] direction. The arrows indicate the orientations studied and are, from the left; 180° , 190° , and 200° , respectively. The theoretical scale factor for $\alpha_{oz}(2\omega)$ is adjusted to .4 (the maximum experimental value in arbitrary units, i.e. .4 corresponds to

$$\frac{|B_a|}{2\omega} = \frac{32.2}{H_0}.$$

The curves obtained from these data are all very similar and a typical one is shown in Fig. 22. Here we plot $M_T(\omega)$ in dB above its value at the threshold field h_c vs. the driving field in dB above h_c . Also plotted are several theoretical curves that are discussed below.

The value of h_c can be obtained from this type curve by a simple extrapolation as is shown. However, it is more conventional to plot these data on a susceptibility decline vs. $1/h$ curve, and this, along with the corresponding extrapolation to determine h_c , is shown in Fig. 23. Values of h_c , the intrinsic uniform mode linewidth ΔH_0 , the loaded linewidth ΔH_L , and the z-directed spin wave linewidth obtained from h_c using Eq. (2.68) are given in Table I for the three orientations considered. The average value of ΔH_K for the three orientations (.49 gauss) is in exact agreement with that of Anderson¹⁰ for single crystal YIG (9.0 GHz) with $\Delta H_0 = 1$ gauss.

The initial experimental slope of $M_T(\omega)$ vs. h (Fig. 22) is very close to unity which is predicted by the linear theory of Chapter II. There is a slight bending over near the threshold field which is generally attributed to inhomogeneity scattering. Above threshold, a definite change in dependence appears which cannot be explained by the single Suhl process as is shown. Relaxation oscillations are present above 16 dB above threshold and, therefore, data are presented only up to this value.

The curves labeled as (a) and (c) in Fig. 22 are predicted by the two-spin-wave-pair-model of Chapter II. Curve (a) is presented as one of the better fits to the experimental data using this model while curve (c) results when the coupling constants are determined to give a good fit to the second harmonic data of Fig. 25. The curve labeled (b) which

Fig. 22 - Transverse magnetization, $|M_T(\omega)|$, vs. driving field for the 180° orientation (see Fig. 21). The open circles are the experimental points. The curves shown are obtained as follows:

- (a) Two spin wave pair model with $\eta_o/\eta_k = 1$, $\eta_{k'}/\eta_k = .5$, $g_{ok}/\bar{g}_{kk'} = 1.6$, $g_{k'o}/g_{ko} = .5$
- (b) Effective relaxation rate model with $C'' = 1/2$
- (c) Two spin wave pair model with same parameters as in Fig. 28 ($\eta_o/\eta_k = 1$, $\eta_{k'}/\eta_k = .5$, $g_{ok}/\bar{g}_{kk'} = 1.2$)
- (d) Many spin wave pair model with $\alpha' = 1.7$

Fig. 23 - Susceptibility decline curve for the 180° orientation. ($h_c = 6.45 \times 10^{-3}$ gauss).

TABLE I

EXPERIMENTAL VALUES OF h_c , ΔH_o , ΔH_L , AND ΔH_k

FROM SUSCEPTIBILITY DECLINE DATA

Orientation (See Fig. 21)	180°	190°	200°
h_c (milligauss)	6.5	6.7	6.7
ΔH_o (gauss)	.38	.41	.40
ΔH_L (gauss)	3.8	4.1	3.9
ΔH_k (gauss)	.51	.46	.49

gives the best fit is predicted using the effective relaxation rate method which is presented in Chapter II. Curve (d) is the theoretical curve predicted by an approximate many-spin-wave-pair-method which is presented in section 3 below.

b. Second Harmonic Magnetization

Fig. 24 shows the experimental dependence of the steady-state portion of the second harmonic magnetization on applied field for the three different orientations studied. The dashed curve represents the experimental dependence of the initial spike of the second harmonic pulse and is very nearly the same for all three orientations. Comparison with Fig. 22 shows that the initial spike's and uniform mode's dependence on driving field are nearly the same, reflecting the fact that this portion of the second harmonic pulse is due to the uniform mode, before the spin waves have had a chance to build up.

The three steady-state curves of this figure show a minimum with a folding over at high power levels just before the onset of relaxation oscillations. The general behavior of all three up to approximately 14 dB above threshold is explained in Chapter III in terms of the two-spin-wave-pair-model where the second spin wave pair contributes 180° out of phase with the uniform mode thus giving rise to a minimum. The folding over at high power levels cannot be explained by present theory and may be associated with the onset of relaxation oscillations.

In Fig. 25 we present data for the 180° orientation together with a curve (a) predicted by the two-spin-wave-pair-model which gives one of the better agreements. The other curves and experimental data in this figure are discussed below. As is evident from this figure the two-spin-wave-pair-model shows the general characteristic present in the

Fig. 24 — Experimental dependence of $\alpha_{oz}(2\omega)$ on driving field. The data plotted correspond to the 180° (○), 190° (□), and 200° (+) orientations. The initial spike curve is typical for all orientations.

Fig. 25 — Comparison between experimental and theoretical dependencies of $\alpha_{oz}(2\omega)$ on driving field for the 180° orientation. Curve (a) results from the two-spin-wave-pair model corresponding to curve (c) of Fig. 22. Curve (b) results from the many-spin-wave-pair model, Eq. (4.20), for $\alpha' = 1.7$ and $\beta' = .85$. Curve (c) is the spin wave contribution using Eq. (4.22). The steady-state experimental points are the open circles and the spin wave contribution, obtained from the transient decay of $\alpha_{oz}(2\omega)$, are the open squares.

experimental data but does not give good quantitative agreement for the overall curve. This is not surprising when it is considered that the two-spin-wave-pair-model considerably over simplifies the actual physical situation as explained below.

When the second spin wave pair is excited, the uniform mode increases beyond the constant value (2.70) it had between the fields (2.68) and (2.94). Above this value spin waves other than those that are either z-directed or with $k \rightarrow 0$ can be excited parametrically by the uniform mode. An approximate threshold condition for the excitation of these other spin waves can be defined in analogy with (2.70). This condition is that spin waves, $\vec{q}$, are excited provided the uniform mode is greater than the critical value given by

$$|b_c|_q = \frac{\eta_q}{g_{oq}}. \quad (4.13)$$

When the dependence of g_{oq} from (2.66) is used in (4.13) together with the assumption that η_q is approximately constant, we find that these excited spin waves lie within a cone about the z-direction, i.e., a large number of spin waves are excited propagating in all azimuthal directions and with polar directions θ_q up to that predicted by (4.13) through g_{oq} . It is then impossible to treat the problem exactly as was done in the two spin wave case. Therefore, an approximate method for handling this problem was developed which is presented in the next section.

3. MANY-SPIN-WAVE-PAIR MODEL

The dependence of the second harmonic magnetization when many spin wave pairs are excited can be determined by including more terms in

(3.11) in which case an expression for $\alpha_{oz}(2\omega)$ becomes

$$\alpha_{oz}(2\omega) = \frac{|B_a|}{2\omega} b_o^2 + 2 \sum_{k>0} b_k b_{-k} \quad (4.14)$$

where the summation over k includes all degenerate spin waves excited from $k=0$ to $k = k_c$ (see Fig. 1). Using the form developed in Chapter II, Eq. (2.102), the term $b_k b_{-k}$ in (4.14) can be written as

$$b_k b_{-k} = i \frac{|b_k|^2 b_o^2}{|b_o|^2} e^{i\Phi_k} \quad (4.15)$$

where the phase factor Φ_k includes the effects of all the other excited spin waves as well as being dependent on η_k and g_{ok} .

This phase factor is determined by substituting (4.15) into an equation obtained from (2.76) in the steady-state for the degenerate spin wave pair $b_k b_{-k}$. The result of this substitution is

$$e^{i\Phi_k} = \frac{g_{ok}|b_o|^2}{\eta_k} + i \frac{2}{\eta_k} \sum_{k'>0} \bar{g}_{k'k} |b_{k'}|^2 e^{i\Phi_{k'}} \quad (4.16)$$

This equation cannot be solved exactly. However, an approximate solution can be obtained if we assume that Φ_k is independent of k , whence we get

$$\tan \Phi_k = 2 \sum_{k'>0} \frac{\bar{g}_{k'k}}{g_{ok}} |b_{k'}|^2 \quad (4.17a)$$

It follows that Φ_k will in fact be independent of k provided $\frac{\bar{g}_{k'k}}{g_{ok}}$ is independent of k for all those spin waves k' excited to appreciable amplitude. Examination of Fig. 4 shows that this is not an unreasonable approximation since, at least for k a z-directed spin wave, $\frac{\bar{g}_{k'k}}{g_{ok}}$ changes by less than 15% as k' is varied over the maximum

allowable range consistent with $\omega_{k'} = \omega_k$. Thus we may use (4.17a) with some degree of confidence. The basic meaning of (4.17a) is that the large number of spin wave phases is replaced by a single, effective phase, which is dependent on the level of excitation.

The summation in the right-hand side of (4.17a) can be determined in an approximate manner by (1) using the conservation of energy equation

$$\sum_k \eta_k |b_k|^2 = \gamma \frac{h}{2} |b_o| \quad (4.18)$$

where h is the linearly polarized driving field corrected to include radiation damping and (2) assuming that summations of the form $\sum_k f(\vec{k}) |b_k|^2$ can be replaced by $\langle f(\vec{k}) \rangle \sum_k |b_k|^2$ where $\langle f(\vec{k}) \rangle$ is some average of $f(\vec{k})$ over the region where the summation is performed. Eq. (4.18), which can easily be derived from (2.72) and (2.76), simply states that the rate of energy absorbed through the uniform mode, $\gamma \frac{h}{2} |b_o|$, equals the rate dissipated by the normal modes. Then (4.17a) becomes

$$\tan \Phi_k = \left\langle \frac{\bar{g}_{k'k}}{\eta_{k'}} \right\rangle_{k'} \frac{1}{g_{ok}} \left(\frac{\gamma h}{2} |b_o| - \eta_o |b_o|^2 \right) \quad (4.17b)$$

from which $e^{i\Phi_k}$ can readily be determined. Here $\left\langle \frac{\bar{g}_{k'k}}{\eta_{k'}} \right\rangle_{k'}$ is some average of $\frac{\bar{g}_{k'k}}{\eta_{k'}}$ with respect to k' .

Then using (4.17b) and (4.18), the expression (4.14) for the second harmonic becomes

$$\alpha_{oz}(2\omega) = \alpha_{oz}(2\omega)_c F(2\omega) \quad (4.19)$$

where

$$F(2\omega) = y^2 \left[1 - \frac{\alpha' \beta' (x-y)^2}{\sqrt{1 + \alpha'^2 y^2 (x-y)^2}} \right] + i \frac{\beta' y (x-y)}{\sqrt{1 + \alpha'^2 y^2 (x-y)^2}} \quad (4.20)$$

Here x and y are the driving field and uniform mode amplitudes, respectively, normalized to their values at threshold, $\alpha_{oz}(2\omega)_c$ is the amplitude of the second harmonic at threshold and

$$\alpha' = \left\langle \frac{\bar{g}_{k'k}}{g_{ok}} \frac{g_{ok'}}{\eta_{k'}} \right\rangle_{kk'} |b_c|^2 \beta'$$

and

$$\beta' = \left\langle \frac{\eta_o}{\eta_k} \right\rangle_k \quad (4.21)$$

where $|b_c|$ is given by (2.70).

By using the experimental data relating the uniform mode and driving field (i.e., y as a function of x) along with (4.19) and (4.20), a curve for the dependence of $\alpha_{oz}(2\omega)$ on driving field is obtained which agrees well with experiment. In Fig. 25 a comparison is made for the parameters α' and β' which give the best fit* to the experimental data. All three orientations studied were compared and the results are tabulated in Table II. The value of β' , which is relatively independent of orientation, agrees well with the average measured value of the ratio $\eta_o/\eta_k = .83$ from Table I considering that β' is some average of this ratio. Thus, α' is really the only adjustable parameter in our theory in the sense that it cannot be measured independently. The values of α' appearing in Table II can be compared to (4.21) in an approximate manner by assuming all coupling constants and relaxation rates to be independent of k . Then for $k' \rightarrow 0$ when using the ratio η_o/η_k from Table I, $\alpha' \approx \frac{\bar{g}_{kk'}}{g_{ok}} \frac{\eta_o}{\eta_k} = .83$ which is up to a factor of two different from the values appearing in Table II. This discrepancy is

* A best fit here means the values of α' and β' that minimizes a weighted average deviation of the theoretical and experimental positions of the extrema.

TABLE II

VALUES OF α' AND β' THAT GIVE THE BEST AGREEMENT
BETWEEN Eq. (4.19) AND EXPERIMENT

Orientation (See Fig. 21)	180°	190°	200°
α'	1.7	1.45	1.3
β'	.85	.95	.95

not too serious since the above expression for α' oversimplifies the averaging required in (4.21).

The spin wave contribution to $\alpha_{oz}(2\omega)$ can be obtained by setting the uniform mode contribution equal to zero in (4.20), i.e., the first term in the brackets. Then the spin wave contribution is given by (4.19) where $F(2\omega)$ is replaced by $F_k(2\omega)$, the spin wave part.

We find

$$F_k(2\omega) = \beta' y(x-y) \quad (4.22)$$

This is also plotted in Fig. 25 for the value of β' obtained for the 180° orientation. The experimental points (squares) are determined from the second harmonic decay transient as is discussed in section D.7. A possible reason for the discrepancy between (4.22) and the experimental data is also discussed in this section.

A relation between the uniform mode and driving field can also be obtained in the many spin wave approximation. From (2.74) in the steady-state and using (4.15), (4.1/b), (4.18) and the same averaging process used to obtain (4.20) we have

$$x = y + \frac{1}{\alpha'y} \sqrt{y^4 - 1} \quad (4.23)$$

This is plotted as curve (d) in Fig. 22 and shows generally poor agreement with experimental data. If (4.23), instead of the experimental dependence of the uniform mode on driving field, is used in (4.20), it can be shown that $\alpha_{oz}(2\omega)$ has a minimum only at threshold. Thus, the second harmonic data above threshold cannot be explained in this way. Since (4.19) and (4.20) do reproduce these data when the experimental dependence of x on y is used, the fault seems to lie in the expression (4.23) and not necessarily in the method used to obtain (4.19)

and (4.20).

Equation (4.23) was obtained assuming that the only interactions affecting the uniform mode are those involving spin wave pairs of the form $b_k b_{-k}$. However, it is possible for processes to exist that affect the uniform mode but have little, if any, effect on the second harmonic. This is easily understood when it is considered that, in order for spin waves to contribute to the second harmonic, a definite phase relation must exist between the spin wave pairs b_k and b_{-k} (see (4.14)). A definite phase relation does exist between these pairs when they are parametrically excited by the four particle processes discussed in Chapter II which are represented in Fig. 2(c). This is no longer true, however, when pairs are formed from spin waves excited by different processes as, for example, the process represented by Fig. 2(b). Including these processes the equation of motion for spin wave modes, b_k , when $\omega = \omega_k$, is

$$\dot{b}_k = -\eta_k b_k + i g_{ok} b_o^2 b_{-k}^* + i \sum_{k_1, k_2} g_{k_1 k_2 k} b_{k_1} b_{k_2} b_{k_1+k_2-k}^* \quad (4.24)$$

Upon setting $\dot{b}_k = 0$ and multiplying the resultant equation by b_{-k} , (4.24) becomes

$$\eta_k b_k b_{-k} = i g_{ok} b_o^2 |b_k|^2 + i \sum_{k_1, k_2} g_{k_1 k_2 k} b_{k_1} b_{k_2} b_{k_1+k_2-k}^* b_{-k} \quad (4.25)$$

We assume a random phase approximation so that the quantity $b_{k_1+k_2-k}^* b_{-k}$ is zero unless $k_1 + k_2 = 0$, i.e., if the spin waves b_k and b_{-k} are created by different processes their phases will not be coherent. If $k_1 + k_2 = 0$, however, the $g_{k_1 k_2 k}$ processes in Eq. (4.25) are represented by Fig. 2(c) and, here, the modes b_k and b_{-k} have a coherent phase

since they are then excited by the same process. Therefore, using (4.15) for $b_k b_{-k}$ and the random phase approximation, Eq. (4.25) reduces to (4.16).

The equation for the uniform mode when these processes are included becomes

$$\dot{b}_0 = -\eta_0 b_0 + i \sum_{k_1, k_2} g_{0k_1 k_2} b_{k_2} b_{k_1 - k_2} b_{k_1}^* - i\gamma h/2 \quad (4.26)$$

The terms $b_{k_2} b_{k_1 - k_2} b_{k_1}^*$ cannot be averaged to zero, as was done above, since this phase is well defined by the scattering process of Fig. 2(b). Therefore, processes of the type represented in Fig. 2(b) can have an effect on the uniform mode without affecting the term $b_k b_{-k}$ in (4.14) and, hence, the spin wave contribution to $\alpha_{oz}(2\omega)$ has been accounted for in a correct manner. Thus, Eq. (4.20) is likely to be a valid expression for the second harmonic when the experimental dependence of the uniform mode on driving field is used.

4. PHASE SHIFT OF $M_z(2\omega)$ WITH RESPECT TO THE UNIFORM MODE

The phase of the second harmonic relative to the uniform mode was measured for the 200° orientation, and these data are presented in Fig.

26. The theoretical dependence of steady-state phase on applied field using the many-spin-wave-model of the last section with the constants α' and β' from Table II for this orientation shows good agreement with experimental data for incident power levels below where relaxation oscillations are present. The phase Φ_s is obtained from Eq. (4.20) by the relation

$$\tan \Phi_s = \frac{\text{Im} [F(\omega)]}{\text{Re} [F(2\omega)]} \quad (4.27)$$

where Im and Re stand for the imaginary and real parts, respectively.

Fig. 26 - Dependence of phase shift of $\alpha_{\frac{1}{2}}(2\omega)$ relative to the uniform mode on driving field for the 200° orientation. O - steady-state region of the pulse, □ - initial spike. Theoretical curve is predicted by the many-spin-wave-pair model.

The phase of the initial spike of the $\alpha_{oz}(2\omega)$ pulse, also given in this figure, shows a general negative trend above threshold. (A negative phase shift is defined as occurring when the uniform mode lags the driving field by less than 90° .) This is due to the fact that the initial spike of the pulse is not on resonance but the steady-state portion is since the d.c. field is always adjusted to satisfy this condition. The sign of this phase shift with respect to threshold can be determined from the general steady-state equation of motion for the uniform mode which from (2.32) becomes

$$b_o = \frac{\gamma h_o / 2}{(\omega_o - \omega) + i \eta_o} \quad (4.28)$$

which is valid up to threshold. Above threshold this equation holds provided we replace η_o by an effective relaxation rate, $\tilde{\eta}_o$, e.g., see (2.106), and $\omega_o - \omega$ is replaced by $\omega_o - \omega - \Delta\omega$ where $\Delta\omega$ represents a frequency shift due to spin wave excitation. Resonance is defined in the steady-state above threshold by

$$\omega_o = \gamma H_o = \omega + \Delta\omega \quad (4.29)$$

which shows that H_o must increase for increasing $\Delta\omega$ when ω is fixed. The amplitude of the initial spike approximately obeys (4.28) since the spin waves have not yet built up. However, the applied d.c. field is still given (4.29). Therefore, by substituting (4.29) into (4.28) we have

$$b_o = \frac{\gamma h_o / 2}{\Delta\omega + i \eta_o} \quad (4.30)$$

for the approximate amplitude of the initial spike. In terms of a phase Φ_p this can be written as

$$b_o = \gamma h_o / 2 c \quad i \quad (-\pi/2 + \Phi_p) \quad (4.31)$$

where

$$\Phi_p = \tan^{-1} \frac{\Delta\omega}{\eta_o} . \quad (4.32)$$

When $\Delta\omega > 0$ as it is since a positive resonance d.c. field shift is observed, b_o in (4.31) will lag the driving field by less than 90° and hence give the initial spike a negative phase shift as is experimentally observed.

From the experimental data of Figs. 22, 24, and 26, and using Eq. (4.14), the spin wave contribution to the second harmonic can be determined as was discussed in Chapter III. The results of this analysis are shown in Fig. 27. Here we have plotted the total spin wave contribution along with the components that are 90° and 180° out of phase with the uniform mode. Curve (a) in Fig. 27 is the total spin wave contribution to $\alpha_{oz}(2\omega)$ using Eq. (4.22) with the constants α' and β' taken from Table II for the 200° orientation. Curves (b) and (c) are the spin wave contributions that are 90° , $\alpha_{oz}(2\omega)_{k_\perp}$, and 180° , $\alpha_{oz}(2\omega)_{k_\parallel}$, out of phase with the uniform mode which are obtained from (4.20) by setting the uniform mode contribution to zero as was done in obtaining (4.22) and are

$$\alpha_{oz}(2\omega)_{k_\perp} = \frac{\beta' y(x-y)}{\sqrt{1 + \alpha'^2 y^2 (x-y)^2}} \alpha_{oz}(2\omega)_c \quad (4.33)$$

and

$$\alpha_{oz}(2\omega) = \frac{\beta' \alpha' y^2 (x-y)^2}{\sqrt{1 + \alpha'^2 y^2 (x-y)^2}} \alpha_{oz}(2\omega)_{k_\parallel} \quad (4.34)$$

Fig. 27 - Spin wave contributions to $\alpha_{oz}(2\omega)$ for the 200° orientation. ○-total spin wave contribution, □-spin wave contribution 90° out of phase with the uniform mode, + -contribution 180° out of phase with the uniform mode. The curves are predicted by the many-spin-wave-pair model.

For the data presented, the agreement between (4.22), (4.33), and (4.34) and the corresponding experimental data is excellent. The agreement between (4.22) and the total spin wave contribution is not as good, however, when this contribution is obtained from the transient decay of the second harmonic signal as was mentioned in section D.3. This point is discussed in section D.7. of the present chapter.

It is also apparent from Fig. 27 that the experimentally determined spin wave contribution which is 90° out of phase with the uniform mode does not "stick" above a second threshold although this component does increase at a slower rate than the 180° component. As discussed in Chapter III, section D, we expect the 90° component to "stick" only if the coupling term g_{ok} is zero.

5. SECOND HARMONIC LINE PROFILE

Fig. 28(a) shows a series of second harmonic line shapes for the 180° orientation as incident power increases from threshold, curve (a), to just below where relaxation oscillations occur, curve (e). For power levels up to and just above threshold a single resonance is observed in the second harmonic as shown in curve (a). Then two peaks appear, curves (b) and (c), corresponding to power levels between where the maximum and minimum occur in the $\alpha_{oz}(2\omega)$ vs. h curve (Fig. 25). For power levels above the minimum in this figure, three peaks appear with the middle one increasing for further increase in incident power and the two outer ones remaining essentially independent of power, curves (d) and (e). The corresponding line shapes for the transverse magnetization are similar to curve (a) in Fig. 28(a) for all power levels in that only a single line appears, but there is some broadening of this line as power increases from (a) to (e) due to spin wave excitation.

Fig. 28 (a) — Second harmonic line profile. Driving field increases from threshold, curve (a), to just below where relaxation oscillations occur, curve (e). Experimental curves are on the left. The theoretical curves on the right are predicted by Eq. (4.35) with the assumed uniform mode and spin wave amplitude line shapes shown in Fig. 28 (b).

It is difficult to obtain an expression for the second harmonic line shape which is valid above threshold when starting from the spin wave equations of motion. This difficulty comes from the fact that the effective uniform mode relaxation rate is power dependent near resonance (when b_o is above b_c , (2.70)) and is independent of power far from resonance. An approximate relation can be obtained from Eq. (4.20), however, if it is assumed that (1) the uniform mode contribution to $\alpha_{oz}(2\omega)$ has a Lorentzian line shape and (2) that the spin wave amplitude ($\sum_{k \neq 0} |b_k|^2$) is zero for d.c. fields where the uniform mode amplitude is less than b_c (as the applied field passes through resonance) and increases from zero for $b_o = b_c$ (corresponding to the d.c. field H_c) to its steady value at resonance (refer to Fig. 28(b)). The functional dependence of the spin wave amplitude on d.c. field from H_c to H_o is assumed, for simplicity, to be linear as shown in Fig. 28(b) although, the general characteristics of the line shape can be reproduced using other types of curves.

With these assumptions, the equation describing the lineshape of $\alpha_{oz}(2\omega)$ is given by (4.19) where $F(2\omega)$ is replaced by $F_{\Delta}(2\omega)$ and becomes

$$F_{\Delta}(2\omega) = y^2 \left(L - \frac{f_s^2 \alpha' \beta' (x-y)^2}{\sqrt{1 + f_s^2 \alpha'^2 y^2 (x-y)^2}} \right) + i \frac{f_s y (x-y)}{\sqrt{1 + f_s^2 \alpha'^2 y^2 (x-y)^2}} \quad (4.35)$$

Here L is the Lorentzian lineshape function (assumed for the uniform mode) given by

Fig. 28 (b) — Line shape functions assumed for the uniform mode and spin wave amplitudes.

$$L = \frac{1}{4 \left(\frac{H-H_0}{\Delta H_L} \right)^2 + 1} \quad (4.36)$$

where H is the applied d.c. field, f_s is the assumed linear line shape function for the spin wave amplitude given by

$$f_s = 1 - \frac{|H - H_0|}{|H_c - H_0|} \quad |H - H_0| \leq |H_c - H_0| \quad (4.37)$$

$$= 0 \quad |H - H_0| > |H_c - H_0|$$

and x and y are the values of the driving field and uniform mode at resonance normalized to their values at threshold. The line shapes predicted by (4.35) for α' and β' from Table II for the 180° orientation are also shown in Fig. 28(a) and show the general characteristics experimentally observed.

6. TRANSIENT RESULTS

Figs. 29(a)-(f) show actual oscillographs of the transverse magnetization (upper trace) and the 80 MHz beat envelope of the second harmonic z-component magnetization with a local oscillator (lower trace). Presented is a typical series for the driving fields indicated and for the 180° orientation. Similar pulse shapes were also observed for the other two orientations and are not presented here.

The characteristics to note are; (1) the presence of a minimum following the initial spike of the transverse magnetization for high power levels; (2) the presence of two, Fig. 29(d), then three, Fig. 29(e) spikes on the second harmonic pulse as the incident power increases above threshold with the presence of only a single initial spike on the uniform

Fig. 29 — Oscillographs of $M_T(\omega)$ and $\alpha_{oz}(2\omega)$ pulse responses for various levels of driving field. In (a) through (f), $\alpha_{oz}(2\omega)$ is the upper trace (using 80 MHz i-f detection) and $M_T(\omega)$ the lower trace. In (g) the second harmonic is the lower trace (video detected) while the upper trace is $M_T(\omega)$. Trace (h) is an expansion of the second harmonic response in (e).

mode; and (3) the presence of a reradiation spike on the second harmonic pulse after the transverse magnetization (uniform mode) has decayed to a small value.

The general features of the transverse magnetization can be explained in terms of the two spin wave model of Chapter II and closely resembles the pulse shapes shown in Figs. 10 and 11. Actual computer solutions to (2.84) through (2.86) for the case when two spin wave pairs are excited and analogous equations for three spin wave pairs have been performed for various values of the coupling constants and spin wave relaxation rates. The ones that best represent the experimental data of Fig. 29 are presented in Figs. 30 and 31. Notice that these figures show little difference in the uniform mode response between the two and three spin wave pair cases but the second harmonic shows a marked difference. In these figures we have plotted $M_T(\omega)$ and $M_z(2\omega)$ normalized to their values at threshold on a normalized time scale $\eta_k t$.

The agreement with the experimentally observed pulse shapes for the transverse magnetization for the corresponding driving field is extremely good considering that only two or three spin wave pairs are included whereas probably a large number are actually excited.

The theoretical pulse shapes shown in Figs. 30 and 31 for the second harmonic resemble the experimental data of Fig. 29 for driving fields just above threshold. However, for larger driving fields a broad "hump" appears on the second harmonic signal that does not appear on the transverse magnetization and can not be explained by including two or three spin wave pairs. It will be noticed, however, that the three spin wave case gives much better agreement than when two spin wave pairs are included. Therefore, it seems that it might be possible

Fig. 30 - Computer results for the transient response of $|b_o|^2$ and $|\alpha_{oz}(2\omega)|^2$ when two spin wave pairs are excited above threshold. The coupling constants are: $\eta_o/\eta_k = .8$, $\eta_k/\eta_k = .5$, $g_{ok}/g_{ok} = .5$, $g_{kk}/g_{ok} = .8$. Initial conditions on the spin wave mode amplitudes are $|b_k|^2/|b_c|^2 = |b_k|^2/|b_c|^2 = .1$. Driving field corresponds to 14 dB above threshold. $\alpha_{oz}(2\omega)_c$ is the value of $\alpha_{oz}(2\omega)$ at threshold.

Fig. 31 - Computer results for the transient response of $|b_o|^2$ and $|\alpha_{oz}(2\omega)|^2$ when three spin wave pairs are excited above threshold. The coupling constants are : $\eta_o/\eta_k = .8$, $\eta_{k'}/\eta_k = .5$, $\eta_{k''}/\eta_k = .4$, $g_{ok'}/g_{ok} = .5$, $g_{ok''} = 0$, $\bar{g}_{k'k''}/g_{ok} = \bar{g}_{kk''}/g_{ok} = .2$, $\bar{g}_{kk'}/g_{ok} = .7$. Initial conditions on all spin wave mode amplitudes are the same as in Fig. 30. Driving field corresponds to 14 dB above threshold. $\alpha_{oz}(2\omega)_c$ is the value of $\alpha_{oz}(2\omega)$ at threshold.

to explain the second harmonic response if a much larger number of spin wave pairs is considered. However, computer time for this is likely to be prohibitive since, on the GE 625 computer, it required 12 minutes to treat two spin wave pairs and 20 minutes for three spin wave pairs.

The reradiation spike can be reproduced theoretically as is shown in Figs. 30 and 31 and the general character is similar to that experimentally observed (Fig. 29). In Chapter III it was argued that this reradiation portion can have the two distinct shapes shown in Fig. 16 depending on the relative amplitudes of the uniform and spin wave modes. The 80 MHz signal shown in Fig. 29 only shows the characteristic represented by (1) of Fig. 16(d). The shape shown in Fig. 16 (d-2) cannot be observed by the detection system used here because of the response time of the second harmonic cavity Q . At high power levels the Q of the second harmonic cavity can be reduced by a factor of 10 and it is still possible to observe the second harmonic signal. This is accomplished experimentally by removing the iris. Then, because of the mismatch between the X- and K_u -band waveguides a cavity is still present but the loaded Q is now about 300. With this system the signal detected using straight video amplification is shown in Fig. 29(g) and definitely shows the existence of a minimum as predicted by theory. At higher power levels the minimum is much less pronounced although it does not seem to vanish.

7. SPIN WAVE CONTRIBUTION TO SECOND HARMONIC FROM TRANSIENT DATA

Since the uniform mode decays due to radiation damping at a much faster rate than the spin waves, the trailing edge of $\alpha_{oz}(2\omega)$ is due entirely to the spin wave contribution. Therefore, if the spin wave

decay is exponential it is possible to determine the spin wave contribution to the second harmonic by extrapolating the trailing edge of $\alpha_{oz}(2\omega)$, plotted on semi-log paper, back to the time when the driving field was turned off. This procedure is shown in Fig. 32 for the trailing edge of Fig. 29(h). The points $\alpha_{oz}(2\omega)_s$ so obtained are plotted in Fig. 25 and compared with the theoretical result, Eq. (4.22), for β' from Table II. Notice that above about 8 dB above threshold the experimentally obtained data have the same slope as the theoretical curve but are about 8 dB greater. This deviation is not at all surprising when one considers the conditions under which the transient decay method of obtaining $\alpha_{oz}(2\omega)_s$ is valid.

It must be assumed that the relaxation rates of all the spin waves that contribute to the second harmonic are equal. If this were not true, as it generally is not, then deviations are expected. However in this case, when only 90° and 180° spin wave components contribute to $\alpha_{oz}(2\omega)$ (see Fig. 15(b)) then the experimental points should fall below the theoretical curve which is directly opposite to what is observed. The data of Fig. 25 can be understood if it is assumed (1) that the spin wave relaxation rates are not equal and (2) that there are spin wave components that contribute to $\alpha_{oz}(2\omega)$ with a phase relative to the uniform mode greater than 180° . For example, consider the case where, in the steady-state, there are 270° and 90° phase components that exactly cancel one another. Then the steady-state spin wave contribution to $\alpha_{oz}(2\omega)$ is due entirely to the 180° component. If now the 270° component decays much faster than either the 90° or 180° component, then the reradiation portion will be greater than it would have been if all components decayed equally, thus leading to a larger extrapolated spin

Fig. 32 - Extrapolation procedure to find the spin wave contribution, $\alpha_{oz}(2\omega)_s$. The value 1 on the $\alpha_{oz}(2\omega)_s$ axis corresponds to 6.7 dB above threshold.

wave contribution. It is reasonable to expect there to be some spin waves nearly 270° out of phase with the uniform mode, namely those which are excited parametrically by coupling to the spin waves which make up the 180° component.

8. MEASUREMENT OF SPIN WAVE LINE WIDTH

Although the spin wave contribution to $\alpha_{oz}(2\omega)$ apparently cannot accurately be obtained from the transient decay of this component, this decay should give a measure of the lifetime of the spin wave modes that contribute most to the second harmonic. This decay rate was measured for several power levels using a semi-log plot identical to Fig. 32 and the results are tabulated in Table III. When compared to the value of ΔH_k obtained from the susceptibility decline curve ($\Delta H_k = .49$ gauss), the values in Table III indicate that those spin waves that contribute most to $\alpha_{oz}(2\omega)$ have a larger linewidth than the z-directed modes. This result agrees with the theory of Sparks, et. al.²¹ (Fig. 3) along with the many-spin-wave-model presented above.

In the many-spin-wave-model the principle spin wave pairs that contribute to $\alpha_{oz}(2\omega)$ are close to but not exactly z-directed. Fig. 3 shows that for these spin waves the value of ΔH_k is larger than that for the z-directed mode. Thus an average relaxation rate for the spin wave modes contributing to $\alpha_{oz}(2\omega)$ will necessarily be larger than the z-directed value in agreement with experiment. For example, if the average wave vector excited is k and k_c is the wave vector of the z-directed spin wave, then Fig. 3 shows our data to be consistent with $\frac{k}{k_c} = .92$.

A parallel pumping experiment was also performed in order to check

TABLE III

DECAY TIMES OF $\alpha_{oz}(2\omega)$ AND CORRESPONDING SPIN WAVE
LINEWIDTHS FOR DRIVING FIELDS ABOVE THRESHOLD

Driving field (dB above threshold)	Decay time (nsec)	ΔH_k (gauss)
9.6	84	.68
12.2	91	.63
14	95	.60
15.6	90	.63

the values obtained for ΔH_k from the susceptibility decline curve. This is a rather poor check since corrections must be made for the facts that (1) parallel pumping measures ΔH_k for $\theta_k = \pi/2$ while transverse pumping measures ΔH_k for $\theta_k = 0$, and (2) different frequencies are involved. From parallel pumping with $\frac{\omega_k}{2\pi} = 4.02$ GHz, and $k = 3.6 \times 10^5$ cm⁻¹, the spin wave linewidth was measured to be .73 gauss. The corresponding value of ΔH_k from transverse pumping but corrected* to $\theta_k = \pi/2$ and $\frac{\omega_k}{2\pi} = 4.02$ GHz is .75 gauss which is in excellent agreement. It is also interesting to note that for YIG, the theory of Sparks, et. al.,²¹ predicts $\Delta H_k = .51$ gauss for z-directed spin waves degenerate with the uniform mode at $\frac{\omega_k}{2\pi} = 8.42$ GHz while the measured value is .49 gauss, again in excellent agreement.

E. CONCLUSIONS

The purpose of this study was to investigate the effects of higher order spin wave interactions in YIG by studying the dependence of $M_T(\omega)$ and $M_z(2\omega)$ on applied driving field. The comparison between experiment and theory indicates that, for applied d.c. fields where the first order process is forbidden on resonance, these higher order interactions are important and must be considered above the second Suhl threshold. This is especially true in considering harmonic generation since it has been shown that the minimum occurring here is a direct consequence of these interactions.

Theoretically, the steady-state transverse magnetization data is best explained using the effective relaxation rate method presented in

* We have used Eq. (B.29) from Ref. 21 for this correction where θ_1 in (B.29) corresponds to our θ_k .

Chapter II, a method that was first used by Schlömann to explain parallel pumping susceptibility data. Although this method is reasonably correct from a physical viewpoint, i.e., a large number of spin waves are excited above threshold thus raising the effective temperature of these modes as is assumed in the theory, in a sense it is an oversimplification since only a single spin wave pair is considered explicitly and phase relations are completely ignored. For this reason the effective relaxation rate method is unable to explain the second harmonic data.

At the expense of destroying the agreement for the transverse data, the two-spin-wave-pair model presented in Chapter II and applied to second harmonic generation in Chapter III can be used to explain the general features of the second harmonic magnetization data. But, again, it is an oversimplification to assume that only two spin wave pairs are excited and, hence, an approximate many-spin-wave-pair method was developed (section D. of this chapter). This method provides very good agreement with the second harmonic data when the experimental dependence of the uniform mode on driving field is included. Because of this agreement, it is believed that the interactions and spin wave pairs affecting the second harmonic magnetization are properly included in the theory. However, the dependence of the uniform mode on driving field, as predicted by the many spin wave model, is in poor agreement with the experimental data. This reflects the fact that the uniform mode is probably involved in interactions other than just those that parametrically excite the spin wave pairs contributing to the second harmonic.

The spin wave contribution to the second harmonic can be obtained from either the phase shift or transient extrapolation methods as discussed above. Because of the assumption concerning the equality of

spin wave relaxation rates necessary in the transient method, deviations are expected and observed here. The phase shift method, however, is in excellent agreement with experiment and has the added advantage of being able to separate the 90° and 180° spin wave components contributing to the second harmonic. Hence, this technique is quite useful in the study of spin wave interactions.

The spin wave linewidths obtained using susceptibility decline, second harmonic decay, and parallel pumping techniques agree well with one another and with current theories. The spin wave linewidth obtained from susceptibility decline data ($\Delta H_k = .49$ gauss) also agrees with the result of Anderson¹⁰ who used the same technique.

The experimental evidence presented here that is not well understood is the appearance of a broad "hump" on the second harmonic magnetization pulse which is absent on the transverse magnetization data. It is believed that this hump is due to interactions involving the many spin wave pairs excited at these power levels and can probably be explained when solving for the transient response of the second harmonic when say five to ten spin wave pairs are included. Due to the computer times involved in this calculation only up to three spin wave pairs have been included when determining the transient response of both the uniform mode and second harmonic magnetization. This effect can be of particular interest to device applications since it shows that a large amount of second harmonic power can be obtained over a sizeable portion of the pulse well above Suhl threshold.

When using either faster computer techniques or a faster computer it would be interesting to see if this hump can be obtained theoretically when including more spin wave pairs. Also, it would be interesting to

study experimentally the phase shift of the second harmonic reradiation using either of the techniques discussed previously and to compare these results with those predicted in Chapter III.

Relaxation oscillations were always present for driving fields above about 16 dB above threshold and several effects observed in both the transverse and second harmonic data near this power level seem to be associated with the onset of these oscillations. Although these effects have not been studied here, several attempts have been made toward understanding the oscillations.^{22,23} A study of the second harmonic magnetization, which seems to be more sensitive to these oscillations than the transverse magnetization, would be useful in future investigations concerning relaxation oscillations.

APPENDIX

RADIATION DAMPING

When a magnetic material whose magnetization is excited to some amplitude on resonance is placed in a waveguide structure which is initially unexcited, the magnetization relaxes to its original equilibrium direction even in the absence of intrinsic relaxation mechanisms due to a process known as radiation damping. The origin of this damping is in the precessing magnetization of the material given by

$$\vec{M}_T = \vec{M}_T(\omega) e^{i\omega t} \quad (\text{A.1})$$

which can radiate microwave energy when properly coupled to the propagating structure.

A. DETERMINATION OF RADIATION FIELDS

The average energy radiated per second is determined by integrating the Poynting vector over the surface of the sample. Thus,

$$P_{\text{rad}} = \int \vec{n} \cdot \vec{S} dA = \int_V \nabla \cdot \vec{S} d\vec{r} \quad (\text{A.2})$$

where $\vec{S}$ is the time average of the Poynting vector which in terms of complex fields, is given by

$$\vec{S} = \frac{c}{8\pi} \text{Re} (\vec{E}_r^* \times \vec{H}_r) \quad (\text{A.3})$$

where $\vec{E}_r$ and $\vec{H}_r$ are the complex electric and magnetic radiation field vectors, respectively. From Maxwell's equations and (A.3), P_{rad} becomes

$$P_{\text{rad}} = \frac{i\omega}{4} \int_{V_s} (\vec{H}_r \cdot \vec{M}_T^* - \vec{H}_r^* \cdot \vec{M}_T) d\vec{r} \quad (\text{A.4})$$

where the integration is over the volume, V_s , of the sample. Assuming

the sample dimensions are small compared to the wavelength of the radiation fields, both $\vec{H}_r$ and $\vec{M}_T$ can be pulled out of the integrand and (A.4) becomes

$$P_{\text{rad}} = \frac{i\omega}{4} (\vec{H}_r \cdot \vec{M}_T^* - \vec{H}_r^* \cdot \vec{M}_T) V_s . \quad (\text{A.5})$$

It is convenient to assume that the magnetic sample is placed in the center of a shorted waveguide an integral number of half wavelengths, λ_g , from the short. Then a simple relation exists between the radiation fields and transverse magnetization which can be determined by equating (A.5) to the microwave energy propagated in the waveguide.

The waveguide fields for the TE_{10} mode (assumed to be the dominant propagating mode) are

$$H_{rx} = -h_r \sin \frac{\pi x}{b} e^{i(ky + \omega t)} \quad (\text{A.6})$$

$$H_{ry} = ih_r \cos \frac{\pi x}{b} e^{i(ky + \omega t)} \quad (\text{A.7})$$

$$E_{rz} = \frac{\omega}{kc} h_r \sin \frac{\pi x}{b} e^{i(ky + \omega t)} \quad (\text{A.8})$$

and $E_{rx} = E_{ry} = H_{rz} = 0$. Here k is the propagation constant, $\frac{2\pi}{\lambda_g}$, c the velocity of light, and h_r is the maximum amplitude of the magnetic field in the waveguide. The directions x , y , z , and dimensions a and b are defined in Fig. A.1.

In terms of these fields and (A.2), the radiated power is

$$P_{\text{rad}} = \frac{\omega ab}{16 \pi k} |h_r|^2 . \quad (\text{A.9})$$

By comparing (A.9) and (A.5) where the radiation field can be written as

$$h_r = -iK4\pi M_T(\omega) \quad (\text{A.10})$$

Fig. A-1 — Position of sample in shorted waveguide.

where

$$K = \frac{2k}{ab} V_s \quad . \quad (\text{shorted waveguide}) \quad (\text{A.11a})$$

Equation (A.10) is also valid for a sample in a microwave cavity at a position of maximum microwave h field if K is replaced by

$$K = Q_L F \quad (\text{cavity}) \quad (\text{A.11b})$$

where Q_L is the loaded Q of the cavity and F its filling factor defined as

$$F = \frac{\int_{V_s} H^2 d\vec{r}}{\int_{V_c} H^2 d\vec{r}}$$

where V_c is the volume of the cavity.

The effect of this field on the magnetization can be determined from Eq. (2.32) where h_r is substituted for h_o . With this substitution and (4.2) the equation for the uniform mode amplitude, on resonance, becomes

$$\dot{b}_o(t) = - (\eta_o + \eta_r) b_o(t) \quad (\text{A.12})$$

where

$$\eta_r = \frac{K\omega_M}{2} \quad (\text{A.13})$$

which shows the desired result that even when intrinsic relaxation processes are ignored ($\eta_o = 0$) the transverse magnetization decays. The rate of decay, η_r , is determined by the dimensions of the waveguide and sample, the sample saturation magnetization, and in general by the position of the sample in the waveguide. For example, if the sample is constrained to the electrical plane of the short in a shorted waveguide, the general relation for K is

$$K = \frac{2k}{ab} V_s \cos^2 \frac{\pi x}{a}$$

where x is measured from the center of the waveguide.

Radiation damping can be accounted for in a convenient way by assigning an additional relaxation rate to the uniform mode or, equivalently, in the presence of an externally applied microwave field, h_o , by substituting for the driving field a field composed of the incident and radiation fields. Thus,

$$h = h_o + h_r. \quad (A.14)$$

When the phase is properly included (A.14) becomes, on resonance,

$$h = h_o - |h_r| \quad (A.15)$$

which shows the expected result that the radiation field subtracts from the incident field. Then the equation of motion for the uniform mode will have one of the following equivalent forms

$$\dot{b}_o \sim -\eta_o b_o - i\gamma h/2 \quad (A.16)$$

or

$$\dot{b}_o \sim -\eta_L b_o - i\gamma h_o/2 \quad (A.17)$$

where

$$\eta_L = \eta_o + \eta_r. \quad (A.18)$$

Experimentally it is possible to measure the loaded linewidth $\Delta H_L = \frac{2\eta_L}{\gamma}$ and incident field h_o using standard microwave techniques. Determination of the intrinsic linewidth ΔH_o and driving field, h , from these is the subject of the next section.

B. DETERMINATION OF DRIVING FIELD AND ΔH_o IN THE PRESENCE OF WAVEGUIDE RADIATION DAMPING

Consider the experimental arrangement in Fig. A-1. The power absorbed by the sample is given by

$$P_{ab} = \frac{\omega}{2} \chi'' V_s h^2 \quad (A.19)$$

where h is the linearly polarized driving field (A.14) consisting of the incident and radiation fields. In terms of the incident power (A.19) can be written as

$$P_{ab} = (1 - |\Gamma|^2) P_{inc} \quad (A.20)$$

where Γ is the reflection coefficient. In terms of the voltage standing wave ratio, r , related to Γ by

$$r = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad (A.21)$$

(A.20) can be written as

$$P_{ab} = \frac{4r}{(1+r)^2} P_{inc} \quad (A.22)$$

or, equivalently,

$$P_{ab} = \frac{4 \cdot 1/r}{(1+1/r)^2} P_{inc} \quad (A.23)$$

Which of these two forms is to be used depends on the sign of Γ .

This sign can be accounted for by defining a sample coupling coefficient β as

$$\begin{aligned} \beta &= r && \text{for overcoupled sample} \\ \beta &= \frac{1}{r} && \text{for undercoupled sample} \end{aligned} \quad (A.24)$$

where the overcoupled ($\beta > 1$) or undercoupled ($\beta < 1$) case can be determined experimentally by a simple microwave technique³⁴. Then

(A.20) becomes

$$P_{ab} = \frac{4\beta}{(1+\beta)^2} P_{inc} \quad (A.25)$$

The sample coupling constant β may be related to the susceptibility as follows:

Substituting (A.15) for h , (A.19) can be written as

$$P_{ab} = \frac{\omega}{2} \chi'' h_o^2 \left[\frac{1}{1 + \frac{|h_r|}{h}} \right]^2 \quad (A.26)$$

Then using (A.10) along with the definition for susceptibility,

$|M_T(\omega)| = \chi'' h$, we have

$$\frac{|h_r|}{h} = 4\pi K \chi'' \quad (A.27)$$

Upon relating the field h_o to the incident power by an equation similar to (A.9), Eq. (A.26) becomes

$$P_{ab} = \frac{16 \pi K \chi''}{(1 + 4\pi K \chi'')^2} P_{inc} \quad (A.28)$$

Comparing this to (A.25), we have

$$\beta = 4\pi K \chi'' , \quad (4.29)$$

the desired result.

In the low power region where $\chi'' = \chi''_o = \frac{M}{\Delta H_o}$, the coupling coefficient β written as β_o , is

$$\beta_o = \frac{\eta_r}{\eta_o} \quad (A.30)$$

where we have used Eq. (A.13) for η_r . Then the intrinsic linewidth is related to the loaded (measured) linewidth from (A.18) and (A.30) and is

$$\Delta H_o = \frac{\Delta H_L}{1 + \beta_o} \quad (A.31)$$

From (A.19), (A.25), and (A.29), the driving field h can be determined from

$$h^2 = c' \frac{P_{inc}}{(1+\beta)^2} \quad (A.32)$$

where

$$c' = \frac{8 \beta_o \Delta H_o}{M \omega V_s} \quad (A.33)$$

In (A.33) the constant K has been experimentally determined by (A.29) through a measurement of the low power coupling constant β_o and linewidth ΔH_o .

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